1. A professor has 15 students and during lecture will (uniformly) at random choose a student to answer a question. The professor asks 8 questions during the lecture. What is the probability no student will have to answer more than one question?

Total outcomes $= 15^8$, 15 students to choose from for each of 8 questions

No student answer more than 1 question = 8 students out of 15 answer 1 each

So $_{15}P_8 = \frac{15!}{(15-8)!} = \frac{15!}{7!}$

$\therefore$ P(No student answer more than 1 question) $= \dfrac{\frac{15!}{7!}}{15^8}$

2. An integer from the range 00000 - 99999 is generated uniformly at random. We are interested only in even integers that start with 2 odd digits where all digits are unique. If we randomly generate 8 of these numbers in succession, what is the probability we get exactly 5 numbers that meet our criteria?

ODD	ODD	__	__	EVEN
5	4	$(10-3) = 7$	$(10-3)-1 = 6$	5

ODD: 1, 3, 5, 7, 9 (5)
Even: 0, 2, 4, 6, 8 (5)

Total outcomes $= 10^5$

P(pick one of these numbers) $= \frac{5 \times 4 \times 7 \times 6 \times 5}{10^5}$

P(exactly 5 meeting criteria) $= \binom{8}{5}\left(\frac{4 \times 5^2 \times 6 \times 7}{10^5}\right)^5 \cdot \left(1 - \frac{4 \times 5^2 \times 6 \times 7}{10^5}\right)^3$

Binomial probability

$= 6.4347 \times 10^{-6}$

P(Not picking a number meeting the criteria)

3. You roll 3 six-sided, fair dice. Let A be the event that at least 2 dice show 4 or above. Let B be the event that all 3 dice show the same value. Are A and B independent?

4 and above $= \frac{3}{6}$

$P(A) = P($2 or 3 dices show 4 or above$) = {}_3C_2 \cdot \frac{3}{6} (\frac{3}{6})^2 + {}_3C_3 \cdot \frac{3}{6} \cdot (\frac{3}{6})^2 = \frac{1}{2}$

$P(B) = P($6 ways of same value, total outcome $= 6^3) = \frac{3!}{6^3} = \frac{1}{36}$

$P(A \cap B) = P($all 3 dices are the same with values $\geq 4)$

$= P(\text{All } 3 = 4) + P(\text{All } 3 = 5) + P(\text{All } 3 = 6) = \frac{1}{6^3} + \frac{1}{6^3} + \frac{1}{6^3} = \frac{1}{72}$

$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{36} = \frac{1}{72} = P(A \cap B)$ $\therefore$ A and B are independent.

4. In poker, a flush is any 5-card hand where all the cards of the same suit. For this problem we will not distinguish between an ordinary flush and special flushes (like straight and royal flushes), meaning we will call any hand that has all 5 cards from the same suit a flush. Poker-player Paul loves a flush. What is the expected number of hands of poker he has to play to get a flush. (We assume each hand is dealt from a new deck containing of randomly ordered cards).

Let X be number of pokerhands he has to play to get a flush.

$P(\text{Getting a flush}) = (\underbrace{4}_{\text{4 suits}} \times \underbrace{{}_{13}C_5}_{\text{13 in each suit}}) / \underbrace{{}_{52}C_5}_{\text{Total outcomes}} = 0.001980792 = p$

P is constant as each hand is dealt from a new deck.

$P(X=1) = p$
$P(X=2) = (1-p)^1 p$
$P(X=3) = (1-p)^2 p$ $\quad \therefore$ X is Geometric

$\Rightarrow E(X) = 1/p = \frac{1}{0.001980792} = 504.8486$

5. A basketball team has a superstar. When their superstar plays, they win 70% of the time. When their superstar does not play they win 50% of the time. Entering a 5 game stretch, the superstar had been recovering from an injury and said the chance they would play the next 5 games was 75%. You go on a trip to the jungle (no internet access). When you return you find out the team won 4 of the 5 games. What is the probability the superstar played those 5 games? You may assume the superstar doesn't get injured during those games (either they play all or none of the 5).

A = Super star plays, B = Win

$P(B|A) = 0.70$, $P(B|\bar{A}) = 0.50$, $P(A) = 0.75$

C = winning 4/5 games, Two cases:

1. A, So $P(C|A) = {}_5C_4 \times 0.70^4 \times (1-0.70) = 5 \times 0.70^4 \times 0.30$
2. $\bar{A}$, So $P(C|\bar{A}) = {}_5C_4 \times 0.50^4 \times 0.50 = 5 \times 0.50^5$

By Law of Total Probability, $P(C) = P(C|A) \cdot P(A) + P(C|\bar{A}) \cdot P(\bar{A})$

$= (5 \times 0.70^4 \times 0.30) \times 0.75 + (5 \times 0.50^5) \times (1-0.75)$

$= 0.309175$

$P(\text{Play 5 games}) = P(A|C)$

Using Baye's theorem $P(A|C) = \frac{P(C|A) \cdot P(A)}{P(C)} = \frac{(5 \times 0.70^4 \times 0.30) \cdot 0.75}{0.3097175}$

$= 0.8737$