IT Tools & Optimization BE 2.5 Report – 25/09/2019

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Introduction: We examine some constrained optimization problems & study KKT conditions.

1. a. The Lagrangian is:

$$\mathcal{L}(x,\lambda) = \sum_{i=1}^{n} \beta_i x_i^2 + \lambda\left(\sum_{i=1}^{n} x_i - \alpha\right)$$

The first order KKT necessary conditions of optimality are:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 2\beta_i x_i + \lambda = 0, \quad i = 1,2,\dots,n$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} x_i - \alpha = 0$$

b. From each of the first n equations we obtained:

$$x_i = -\frac{\lambda}{2\beta_i}, \quad i = 1,2,\dots,n$$

From the last equation we obtained:

$$\sum_{i=1}^{n} x_i = \frac{-\lambda}{2}\left(\sum_{i=1}^{n} \frac{1}{\beta_i}\right) = \alpha \rightarrow \lambda = -\frac{2\alpha}{\sum_{i=1}^{n} \frac{1}{\beta_i}}$$

$$x_i = -\frac{\alpha}{\beta_i \sum_{i=1}^{n} \frac{1}{\beta_i}}, \; i = 1,2,\dots,n$$

Which gives the location of the optimal point.

c. QUADRAPROG was used and found: $x_1 = 0.96, x_2 = 0.48, x_3 = 0.32, x_4 = 0.24$

2. a. The Lagrangian and first order KKT optimality conditions are:

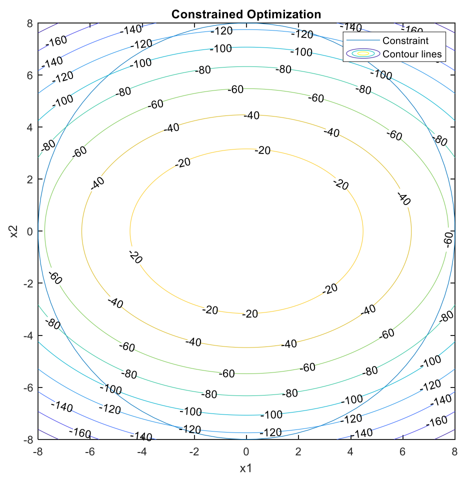
$$\mathcal{L}(x,\mu) = (-x_1^2 - 2x_2^2) + \mu(x_1^2 + x_2^2 - 64)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2x_1 + 2\mu x_1 = 2x_1(\mu - 1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -4x_2 + 2\mu x_2 = 2x_2(\mu - 2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} \rightarrow \mu(x_1^2 + x_2^2 - 64) = 0$$

b. The optimization problem is graphically represented using Matlab.



The optimal points must lie on/inside the circle. From the contour lines we know that:

$$(x_1, y_1) = (0,8) \text{ or } (x_2, y_2) = (0,-8) \text{ which gives } f_{min} = -128$$

c. Using the equations from (a) we saw that the 2 optimal points were satisfied with:

$$\mu = 2 > 0$$

3. a. Lagrangian and KKT conditions:

$$\mathcal{L}(x, \mu_1, \mu_2, \mu_3, \mu_4)$$
$$= x_2 + \mu_1(-x_1^2 - x_2^2 - 2x_1) + \mu_2(x_1^2 - x_2^2 + 4x_1) + \mu_3(x_1 - 4x_2 + 2) + \mu_4(-x_1 - 1)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2\mu_1 x_1 - 2\mu_1 + 2\mu_2 x_1 + 4\mu_2 + \mu_3 - \mu_4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - 2\mu_1 x_2 - 2\mu_2 x_2 - 4\mu_3 = 0$$

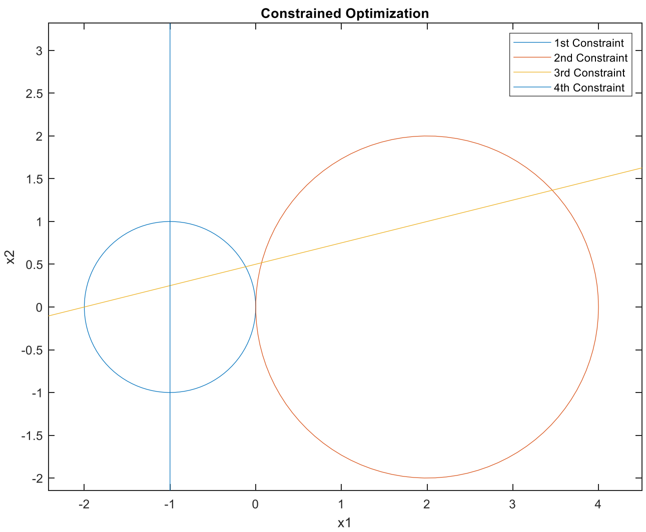
$$\frac{\partial \mathcal{L}}{\partial \mu_1} \to \mu_1(-x_1^2 - x_2^2 - 2x_1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu_2} \to \mu_2(x_1^2 - x_2^2 + 4x_1) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu_3} \to \mu_3(x_1 - 4x_2 + 2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu_4} \to \mu_4(-x_1 - 1) = 0$$

The problem was solved graphically in Matlab using plot and plotting the constraints. The following graph was obtained:



The optimal point must lie on/outside each circle, and on/above the yellow line, and on/to the right of the blue line. From the graph we saw that it is the intersection of the yellow line and the smaller circle, and by solving we achived:

$$x_1 = -0.1279 \qquad x_2 = 0.4744\ (= f_{min})$$

b. The KKT conditions were verified with the values found above and the following values for the Lagrangian multipliers:

$$\mu_1 = \frac{3000}{2693} > 0, \mu_3 = \frac{26163}{13465} > 0, \mu_2 = \mu_4 = 0$$

Conclusion: Graphical solution should be used only for problems with 2 variables (or 3, if functions are simple). Otherwise it would be very difficult to find the optimal point, like in BE1 question 5.b. But if usable, this approach is very illustrative.