Stochastic Geometry and Architecture of Communication Networks *

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This paper proposes a new approach for communication networks planning; this approach is based on stochastic geometry. We first summarize the state of the art in this domain, together with its economic implications, before sketching the main expectations of the proposed method. The main probabilistic tools are point processes and stochastic geometry. We show how several performance evaluation and optimization problems within this framework can actually be posed and solved by computing the mathematical expectation of certain functionals of point processes. We mainly analyze models based on Poisson point processes, for which analytical formulae can often be obtained, although more complex models can also be analyzed, for instance via simulation.

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1 Introduction

This paper proposes a new approach based on stochastic geometry to model telecommunications network architectures for purposes of strategic planning and economic analysis. First we summarize the state-of-the-art in these two fields, before stating the basics and the advantages of the proposed method. Probabilistic models, based on geometric constructions related to stochastic point processes, are then introduced (§2). In §3, we show how several classes of performance evaluation or optimization problems can be reduced to the calculation of moments of functionals of the underlying point processes. We mainly summarize here analytical results obtained for the homogeneous Poisson model. More complex models can be analyzed by space transformation or by simulation (see [1]). The models introduced in the paper seem generic and should find applications for other types of public networks.

1.1 Situation of the problem

The modeling approach presented in this paper seems relevant in two research domains : strategic planning and economical modeling of telecommunications network. We first briefly survey the shortcomings of current methodology in these domains and then explain why our approach should improve existing practice.

1.1.1 Current methods for strategic planning

Current strategic planning methods use a detailed geographical description of the network both for the present and for the future. The description of the future results from an extrapolation of the current network state for a few evolution scenarios. To meet the forecast demand, strategic planning has to choose between a few potential architectures. Their costs and performances are evaluated via detailed calculations of future network configurations. The analytical results, synthesized into discriminating indicators, help to choose the best architecture among those tested. This process usually requires the development of specific software optimization programs adapted to each network architecture.

This approach uses a considerable amount of parameters out of which only some statistical characteristics are actually relevant for strategic planning. But the classical analytical network models are not meant to identify such structural parameters, which can only be empirically perceived by experts.

As structural parameters are not formally identified, the sensitivity of strategic planning choices to input data and hypothesis can only be evaluated on empirical grounds. It is in particular impossible, with this methodology, to know to what extend the conclusions of a study stay valid for different networks, different evolution scenarios or different costs or performances hypothesis.

1.1.2 Network modeling for econometrical analysis

The econometrics of telecommunication networks are based on global statistics on the network and the demand. Until now, one is not really able to integrate the spatial characteristics of networks. The production and cost of the essential network function which consists in connecting pairs of distant subscribers, is not identified. Another specificity of networks is their ability to share the resources needed to serve different connections requests. Operation Research methods, applied to network design problems, model correctly these spatial characteristics, but only on detailed geographical network description, and have not been translated into macroscopic laws, allowing for instance the expression of the network cost function [4].

1.1.3 The stochastic geometry approach

A macroscopic network model, catching the essential spatial characteristics of networks economy through a minimum number of structural parameters, is thus an important research subject. The proposed approach models the space where the network operates by stochastic geometry methods. The objectives of the model are:

- to limit the parameters to those which determine the network optimal architecture ;

- to provide analytical expressions for costs or performance indicators, in terms of these parameters; such expressions constitute then the network production function and lead to more efficient comparisons between strategic alternatives;
- to integrate natural constraints on the spatial repartition of network elements.

The basic principle consists in a stochastic model of the spatial characteristics. The network description is reduced to the parameters of the stochastic processes and the decision variables are functionals of these processes : the comparison between decision variables is then based on the parameters of the stochastic processes. By this way, the production function of telecommunication services can be analytically expressed.

2 Stochastic modeling

The spatial structures observed in communication networks are usually far from being regular. For example, the zones served by telephone concentrators or the cells handled by base stations in mobile communications are far from the regular hexagonal shape which is often taken as a reference model.

Ignoring these structural fluctuations of the geometric objects may cause important bias on the evaluation of key system characteristics.

The stochastic model which we propose in the present paper allows one to take these fluctuations into account. It is based on a representation of the main objects of the telecommunication network (subscribers, stations, cables, mobiles) as a realization of a family of stochastic processes belonging to simple parametric classes.

In what follows, we first introduce a basic cell model and show that some of the key characteristics of the network can be expressed in terms of simple functionals of these processes. In the following sections, we will show how these characteristics can be analytically evaluated.

2.1 The basic cell model

The simplest probabilistic model of a telephone network is that with a single hierarchical level.

The subscribers and the stations are represented through their coordinates in the plane $\mathbb{R}^2$. Each station serves subscribers located in a certain area. Equivalently, the plane is sub-divided into cells, each of which is served by a unique station.

In this paper, we take the assumption that subscribers are served by the their closest station; they are linked to this station, either by a cable or by a radio channel as in mobile communication networks.

Thus the cell served by a station x_i is a convex polygon $\mathcal{C}_i$ known as the *Voronoi cell* with *nucleus* x_i, constructed with respect to the set $\{x_j\}$ of stations. The Voronoi cell $\mathcal{C}_i$ is the intersection of half planes H_{ij} bounded by the bisectors of the segments $[x_i, x_j]$ and containing x_i. The system of all the cells creates a tessellation of the plane called the *Voronoi tessellation* (see Figure 1).

The main idea of our stochastic modeling consists in considering the configuration of the subscribers and the stations as realizations of stochastic point processes. In many applications these processes can be taken independent and Poisson. The main advantage of a Poisson process is its simplicity. The distribution of a Poisson process Π is completely defined through the intensity measure Λ representing the mean density of points. More accurately, the number of the process points in a Borel set B follows the Poisson distribution with the parameter $\Lambda(B)$ and the number of points in disjoint Borel subsets are independent. A natural technical assumption is that the measure Λ is locally finite, e.g. $\Lambda(B) < \infty$ for any bounded Borel B. This guarantees that the number of the points in such a set is almost surely finite. The properties of Poisson processes are presented in detail in [7].

In the basic cell model, we consider two independent Poisson processes Π_0 and Π_1 in the plane representing the subscribers and the stations, respectively. Thus the parameters of the model reduce to the intensity measures Λ_0 and Λ_1 of these processes. One may for instance suppose that these two measures are proportional, i.e. $\Lambda_0 = \alpha\Lambda_1$ for some $\alpha >> 1$, which reflects the policy to set more stations in the places where there are more subscribers.

The boundaries of the Voronoi cells which now become a random closed set, have Lebesgue measure 0. If the intensity measures are diffuse, i.e. they have density with respect to the Lebesgue measure, each Π_0-point (each subscriber) lies with probability 1 in a unique Voronoi cell constructed with respect to the process Π_1 (in other words, belongs to the zone of service of a unique station). The type of connections between stations and between the station and its subscribers are the subject of further specification. For example, the stations can be directly linked if their cells are neighbours in the Voronoi tessellation. In this case, these links form the so called *Delaunay triangulation*. Another possibility is to connect directly all the stations within a specified distance R. Similarly, the subscribers can be connected to their station either via direct cables represented by segments (star network), or via intermediate concentrators (multi-level hierarchical network as described in the next section), or by some spanning graph, or by some mixture of these architectures.

A typical configuration of the described model in the homogeneous star network case is shown in Figure 1.

Here are the main distributions of interest for this model:

- the distribution of the geometric characteristics of the cells; e.g. the surface, the boundary length, the number of adjacent cells etc.;
- the distribution of the number of subscribers within a cell;
- the total length of connections between subscribers and the station to which they are connected;
- the distribution of geometric characteristics of the connections between the stations.

2.2 Hierarchical model

Real telecommunications networks have complicated structures, where the stations play different roles depending on their level of hierarchy.

Here we use the term *stations* in a broad sense, including for instance the concentration nodes where cables meet. The level of a station corresponds more or less to the minimal number of stations standing between this station and a network subscriber. We will call 0-level stations the subscribers themselves. The 1-st level stations are the stations directly connected to 0-level stations. The 2-nd level stations serve the 1-st level stations to which they are directly connected etc. The hierarchical structure of existing systems is well seen in the following chain: subscribers, distribution points, remote distribution points, remote concentrators, local exchange etc., though in reality it is rarely observed in a pure form. Nevertheless, the following model

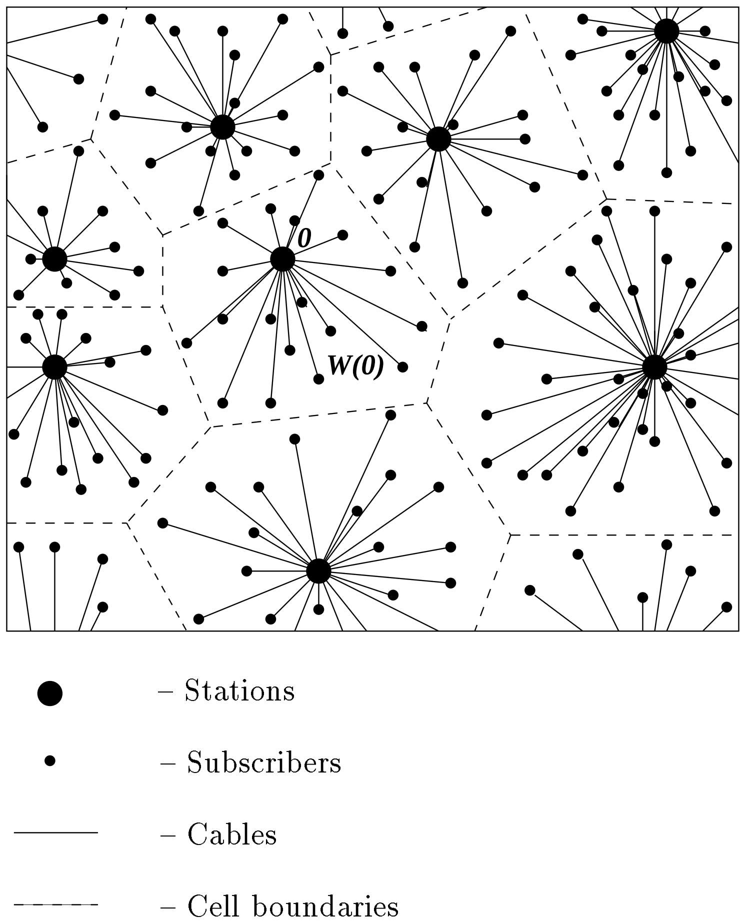

Figure 1: The Basic model with a single hierarchical level of stations (the connections between stations are not specified)

seems reasonable as the first approach to the description of such multi-level systems.

In this model there are N levels of stations, where N can take any integer value. The case $N = \infty$ is not excluded. The stations of level i are represented by a realization of a homogeneous Poisson process Π_i. The processes Π_0, $\Pi_1, \ldots, \Pi_N$ are supposed to be independent with decreasing intensities: $\lambda_0 > \lambda_1 > \ldots > \lambda_N$.

Except for stations of level N, in the pure hierarchical model, stations with the same level have no direct connection between them. As in the basic model the stations of level j ($j = 0, \ldots, N-1$) are connected to their closest stations of level $j+1$. The structure of the basic model repeats itself at each level: the cells of the stations of level j form the Voronoi tessellation of the plane with respect to the process Π_j and the stations of level $j-1$ contained in a cell of a station $x^{(j)}$ of level j are directly connected to the latter. Thanks to that similarity, the main characteristics of this model can be expressed by the characteristics of the basic model.

Figure 2 shows a typical configuration of a hierarchical star network model.

2.3 Modeling requests for communications

The simplest model of requests is based on the hierarchical cell model. Fix the configurations of all processes

$$\Pi_i = \{x_1^{(i)}, x_2^{(i)}, \ldots\}\; i = 0, 1, \ldots, N,$$

and denote by $\{V^{(i)}(x_k^{(i)})\}_{k=1}^{\infty}$ the Voronoi cell of station $x_k^{(i)}$ of level i, with respect to the process Π_i. For almost all $x \in \mathbb{R}^2$ there exists a unique cell $V^{(i)}(x)$ (a cell of level i) which contains x. Let $n^{(i)}(x)$ denote its nucleus, i.e. the i-level station which is the closest to x.

In our model, a station of level i calls a station of a level j located at the distance r of it with intensity $f_{ij}(r)$, independently of all other stations and of ongoing communications. The conditions

$$\sum_{j=0}^{N} \int_0^{\infty} r f_{ij}(r) dr < \infty,$$

for each $i = 0, 1 \ldots$, ensure that the number of calls made by each station in a finite time is almost sure finite.

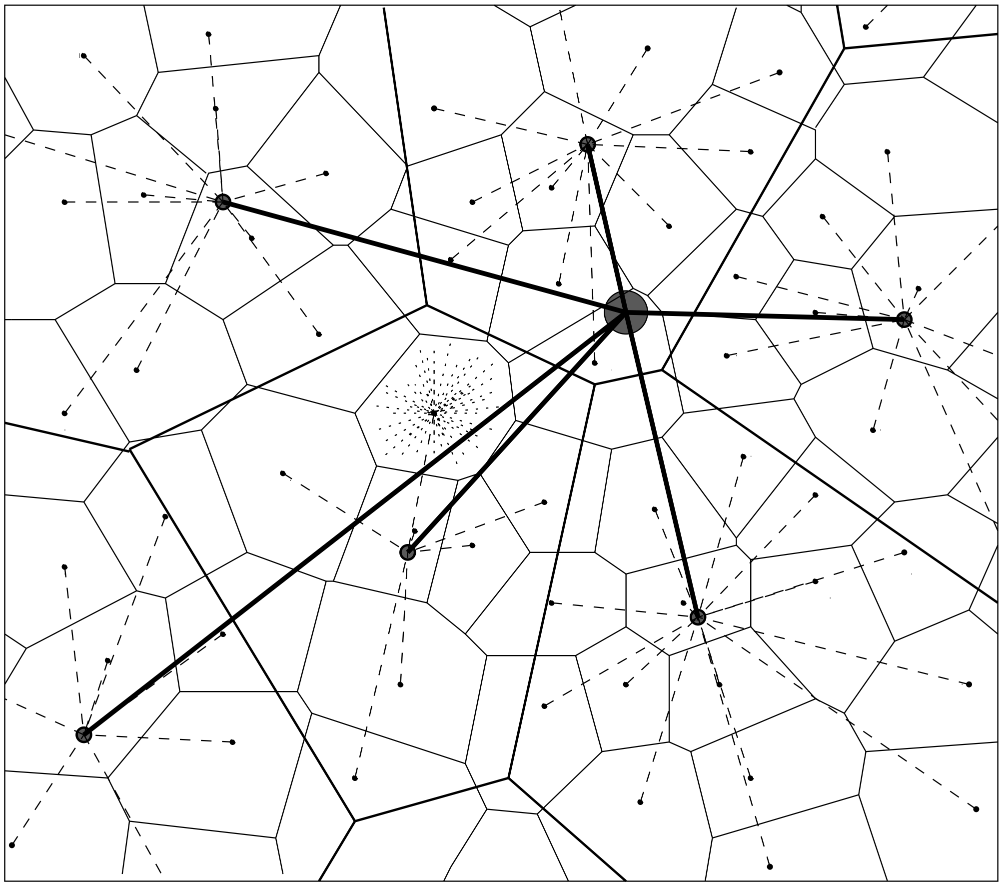

Figure 2: Hierarchical star network model with 3 levels of stations.

We now describe the circuit switching scheme. Given the availability of the required resources, a communication request from $x^{(i)}$ to $x^{(j)}$ is routed as follows. Let $h = h(x^{(i)}, x^{(j)})$ be the minimal hierarchy level such that $V^{(h)}(x^{(i)}) = V^{(h)}(x^{(j)})$ or $h = N$ if there is no such i. The communication is established via direct links from $x^{(i)}$ to $n^{(i+1)}(x^{(i)})$, then from $n^{(i+1)}(x^{(i)})$ to $n^{(i+2)}(x^{(i)})$ etc., to $n^{(h)}(x^{(i)})$; after that via links from $n^{(h)}(x^{(i)})$ to $n^{(h)}(x^{(j)})$ (via connections between stations of the highest level N if $h = N$) and finally via links from $n^{(h)}(x^{(j)})$ to $n^{(h-1)}(x^{(j)})$, etc. and from $n^{(j+1)}(x^{(j)})$ to $x^{(j)}$. In the case when $V^{(h)}(x^{(i)}) = V^{(h)}(x^{(j)})$ for a certain h, there exists a unique route of the described form between $x^{(i)}$ and $x^{(j)}$. The variable h will be called the *height* of communication between $x^{(i)}$ and $x^{(j)}$.

2.4 Mobile communications

Wireless communication systems can be integrated within our framework by adding:

1. a model for the road system;
2. a model of traffic on the roads;
3. a model for the radio communication handling (allocation, capturing, release, handover).

The basic model can be used as follows: the emission/reception stations (antennas) are represented by a point process (equivalent to the first level stations) with respect to which a random process of radio telephone users plays the same role as the process of fixed subscribers in the basic model. In contrast, one of the main peculiarities of mobile communication systems is the phenomenon of *handover* which consists in rerouting the call when a mobile with communication in progress moves from one cell to another.

As a model for the road system, we propose the so-called Poisson line process, where each line is characterized by the following parameters:

- p is the distance from the line to the origin, taken positive if the line is above 0 and negative otherwise (for vertical lines, p is the abscissa of the intersection point of the line with the abscissa axis).
- α $(0 \leq \alpha < \pi)$ is the angle between the line and the abscissa axis (inclination);

- σ is the type of the road represented by the line (highway, one-way road, etc.);
- τ is the parameter defining the traffic characteristics of the road.

The road system is modeled by a realization of a Poisson process in the phase space $\mathbb{R} \times [0, \pi) \times \mathcal{S} \times \mathcal{T}$, where the part $E = \mathbb{R} \times [0, \pi) \times \mathcal{S}$ represents the position and the type of the road and $\mathcal{T}$ is the space of traffic parameters on a road, i.e. the parameters allowing one to characterize the positions and the velocities of the vehicles on a road.

In the homogeneous case (i.e. when the distribution of the process is left invariant by shifts in the plane), the intensity measure takes the form:

$$\lambda_r \, dp \, \mathcal{O}(d\alpha) \, Q(\alpha, d\sigma) \, T(\alpha, \sigma, d\tau),$$

where

- λ_r is the density of roads,
- the probability measure $\mathcal{O}(d\alpha)$ on $[0, \pi)$ gives the orientation distribution of a typical road,
- $Q(\alpha, d\sigma)$ is the distribution of the type of a road having inclination α,
- $T(\alpha, \sigma, d\tau)$ is the distribution of the traffic parameters on a road of type σ and inclination α.

If the orientation distribution $\mathcal{O}$ is uniform, we obtain an isotropic road model. If, in contrast, $\mathcal{O}$ has two atoms $\{0\}$ and $\{\pi/2\}$, we obtain the Manhattan type model, where the masses of the atoms represent the frequencies of the roads in each direction.

Using measure Q, one can take into account different characteristics on roads of different directions (in Manhattan the *avenues* are larger than the *streets*).

The simplest traffic model on a road is a Poisson marked process. For each road $D_i = (p_i, \alpha_i, \sigma_i)$ of the line process, let $x^{(i)}$ denote the local coordinates and consider a set of independent Poisson processes associated with the lines of the process. The process Θ_i on a fixed line D_i is defined on the phase space $\mathbb{R} \times \mathbb{R}$ and driven by the intensity measure $\lambda_t^{(i)} \, dx^{(i)} \, \mathcal{V}^{(i)}(dv)$.

Here the parameter $\lambda_t^{(i)}$ gives the density of the traffic and $\mathcal{V}^{(i)}(dv)$ is the velocity distribution of a typical vehicle (which can be positive or negative with respect to the local coordinates). All these parameters depend on the inclination and the type of the road D_i. A realization of this process has the following form

$$\Theta_i(0) = \sum_j \delta_{(x_j^{(i)}, v_j)},$$

where the first term gives the position of the j-th vehicle at a fixed time, say, 0 and the second one gives its velocity. After a time t the process becomes

$$\Theta_i(t) = \sum_j \delta_{(x_j^{(i)} + v_j t, v_j)}.$$

An important property of a homogeneous Poisson process is that the distribution of the process $\Theta_i(t)$ coincides with that of $\Theta_i(0)$ (this result is known as Barlett's Theorem (cf [7], pp. 59-60 and 49]).

When using this traffic model, one can take for space $\mathcal{T}$ the space of counting measures on $\mathbb{R}^2$. Then $T(\alpha, \sigma, \cdot\,)$ is the distribution of the Poisson process with intensity measure $\lambda_t(\alpha, \sigma)\, dx\, \mathcal{V}(\alpha, \sigma)(dv)$ (in the previous notations $\lambda_t(\alpha_i, \sigma_i) = \lambda_t^{(i)}$ and $\mathcal{V}(\alpha_i, \sigma_i)(dv) = \mathcal{V}^{(i)}(dv)$). For this model, here are the most important distributions of interest:

- the distribution of the number of vehicles in a fixed cell;
- the distribution of the number of cell boundary crossings in a specified time unit (the handover effect explained earlier);
- the distribution of the sojourn time of a typical vehicle in a cell etc.

3 Analytical results

3.1 Basic cell model

Random Voronoi and Delaunay tessellations are well known objects in stochastic geometry. The first two moments of the distributions of the main geometric characteristics can be found, e.g. in [10], [8].

From this, one obtains the moments of the distribution of the surface of a typical cell, its perimeter etc. A typical cell here means the cell of a station "chosen at random" among all the stations of Π_1. An appropriate mathematical definition is provided by Palm theory. By the ergodic theorem (cf. [5], p. 485), the empirical distribution of some characteristic Z of a point, obtained through an observation of the process in a large square ("window") of size R, converges as R grows to infinity, to the Palm distribution of that characteristic. On the other hand, Slivnyak's Theorem (cf. [11], pp. 114–115]) states that the Palm distribution of Z coincides with the distribution of the characteristic Z of point 0, where the point at 0 is added to each configuration of the homogeneous Poisson process. The Voronoi cell of point 0 is denoted $\mathcal{C}_0$.

Many interesting characteristics of a typical Π_1-point (station) fall into the following class of functionals:

$$S_f \stackrel{def}{=} \sum_{x_i \in \Pi_0} f(x_i) \mathbf{1}\{x_i \in \mathcal{C}_0\},$$

where $f(x) : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is a given non-negative function. Often in practice the function $f(x)$ depends only on the distance $|x|$ and has the form $f(x) = D|x|^\beta$ for some non-negative parameters β, D.

The most interesting cases are:

- $\mathcal{N}$, the number of Π_0-particles (subscribers) in a typical cell (here $\beta = 0, D = 1$);
- $\mathcal{L}$, the sum of the distances between all Π_0-particles (subscribers) in a typical cell and its nucleus (their corresponding station).

The variable $\mathcal{L}$ corresponds to the values $\beta = 1, D = 1$ and gives an estimate of the total cable length for connecting subscribers to stations. A better estimate to the total cable length can be obtained by adjusting the parameter β relating the geographic distance to the corresponding cable length; this parameter can be estimated through the analysis of real data.

In the rest of this subsection, we present analytical results obtained in [6].

Theorem 1 *Let f, f_1 and f_2 be non-negative functions defined on $\mathbb{R}^2$ for which the integrals below exist. Then*

$$\text{(i)} \quad \mathbf{E}S_f = \lambda_0 \int f(x) e^{-\lambda_1 \pi |x|^2} dx; \tag{1}$$

(ii) $$\mathbf{cov}(S_{f_1}, S_{f_2}) = \lambda_0 \int f_1(x) f_2(x) e^{-\lambda_1 \pi |x|^2}\, dx$$
$$+ \quad \lambda_0^2 \int\int f_1(x_1) f_2(x_2) [e^{-\lambda_1 A(x_1,x_2)} - e^{-\lambda_1 \pi (|x_1|^2 + |x_2|^2)}]\, dx_1 dx_2, \qquad (2)$$

where $A(x_1, x_2)$ is the area of the union of two discs of radii x_1 and x_2 centered in x_1, x_2, respectively.

In particular, the first two moments of the distribution of the variables $\mathcal{N}$ and $\mathcal{L}$ are given by:

$$\mathbf{E}\mathcal{N} = \frac{\lambda_0}{\lambda_1} \qquad (3)$$

$$\mathbf{E}\mathcal{L} = \frac{\lambda_0}{2\lambda_1^{3/2}} \qquad (4)$$

$$\mathbf{var}\mathcal{N} = \frac{\lambda_0}{\lambda_1} + 0.280\frac{\lambda_0^2}{\lambda_1^2} \qquad (5)$$

$$\mathbf{var}\,\mathcal{L} = \frac{\lambda_0}{\pi\lambda_1^2} + 0.147\frac{\lambda_0^2}{\lambda_1^3} \qquad (6)$$

$$\mathbf{cov}\,(\mathcal{N}, \mathcal{L}) = \frac{\lambda_0}{2\lambda_1^{3/2}} + 0.197\frac{\lambda_0^2}{\lambda_1^{5/2}}. \qquad (7)$$

The proof of this theorem is based on the Palm theory observations, in particular, on Campbell's Theorem (cf. [11], p.99).

The next result concerns the tail decay of the distribution of S_f, when $f(x) = D|x|^\beta$. It is based on large deviation techniques.

Theorem 2 *There exist positive constants A_1, A_2, C_1 and C_2 depending on λ_0 and λ_1 only, such that for all positive x*

$$A_1 x^{-1/(2+\beta)} \exp(-C_1 x^{2/(2+\beta)}) < \mathbf{P}\{S_f > x\} < A_2 \exp(-C_2 x^{2/(2+\beta)}). \qquad (8)$$

In particular, for all $n \in \mathbb{N}$,

$$(1 + 4\lambda_1/\lambda_0)^{-n} < \mathbf{P}\{\mathcal{N} \geq n\} < \binom{6}{n+6} (1 + 0.347\lambda_1/\lambda_0)^{-n} \qquad (9)$$

The constants $A_i,\ C_i\ (i=1,2)$ are the unique solutions of certain equations. In the limiting case, when $\kappa \stackrel{def}{=} \lambda_1/\lambda_0$ vanishes, the decay rate constants C_i become:

$$C_1 = \frac{\lambda_1}{\lambda_0^{2/(2+\beta)}}\frac{\pi}{2}\left(\frac{2+\beta}{2\pi D}\right)^{2/(2+\beta)}(1+o(1));$$
$$C_2 = \frac{\lambda_1\,\mu}{\lambda_0^{2/(2+\beta)}}\left(\frac{2+\beta}{\pi D}\right)^{2/(2+\beta)}(1+o(1)).$$

More complete results can be found in [6]. These results are useful for statistical estimation of the number of subscribers and the total cable length in a cell.

3.2 Traffic in the hierarchical model

We use the notations of Section 2.3. Replacing the processes Π_0 and Π_1 by Π_i and Π_{i+1}, one obtains the characteristics of the number of stations of level i in a cell of the level $i+1$ and the length of cables connecting them by using the results of the Section 3.1.

Let $Q(\rho)$ be the probability that two fixed points of the plane, distant of ρ, belong to the same Voronoi cell of a tessellation constructed with respect to a homogeneous Poisson process with intensity 1. Simple scaling considerations give that the same probability for a Poisson process with intensity λ is $Q(\rho\sqrt{\lambda})$. Using the independence between the processes $\Pi_0,\ldots,\Pi_N$, we obtain that the probability $H_{ij}(r,h)$ to have a communication of a height h between a station $x^{(i)}$ of level i and a station $x^{(j)}$ of level j distant of r from it, can be expressed by the following formulae:

$$H_{ij}(r,h) = \begin{cases} 0, & \text{if } h < i\vee j \text{ or } h=i=j<N; \\ \exp(-\lambda_{i\vee j}\pi r^2), & \text{if } h = i\vee j < N,\ i\neq j; \\ 1, & \text{if } h = i\vee j = N; \end{cases}$$

$$H_{ij}(r,h) = \begin{cases} (1-\exp(-\lambda_{i\vee j}\pi r^2))\, Q(r\sqrt{\lambda_h})\prod_{m=i\vee j+1}^{h-1}(1-Q(r\sqrt{\lambda_m})), \\ \qquad \text{if } h = i\vee j+1,\ldots,N-1,\ i\neq j; \\ \\ (1-\exp(-\lambda_{i\vee j}\pi r^2))\prod_{m=i\vee j+1}^{N-1}(1-Q(r\sqrt{\lambda_m})), \\ \qquad \text{if } h=N,\ i\neq j; \end{cases}$$

and

$$H_{ii}(r,h) = \begin{cases} Q(r\sqrt{\lambda_h})\prod_{m=i+1}^{h-1}(1-Q(r\sqrt{\lambda_m})), & \text{if } h = i+1,\ldots,N-1; \\ \prod_{m=i+1}^{N-1}(1-Q(r\sqrt{\lambda_m})), & \text{if } h = N, \end{cases}$$

where $i \vee j = \max\{i,j\}$.

The term $1-\exp(-\lambda_{i\vee j}\pi r^2)$ above is the probability that the station $x^{(\min\{i,j\})}$ of the lower level does not belong to the cell of the station $x^{(\max\{i,j\})}$ of the higher level (that is always true if $i=j$). The other terms give the probability that the least level cell containing the two stations has level h.

This distribution allows one to express many characteristics of the model. In [1], the following characteristics were obtained:

- The mean number of calls of height h given by a fixed station of level i by unit of time:

$$M_i(h) = 2\pi \sum_{j=0}^{N} \lambda_j \int_0^{+\infty} r f_{ij}(r) H_{ij}(r,h)\, dr.$$

- In a system without capacity constraints, the mean number of communications of level h for a station of level i:

$$2\pi \sum_{j=0}^{N} \lambda_j \int_0^{+\infty} r f_{ij}(r)\, \overline{T}_{ij}(r,h)\, H_{ij}(r,h)\, dr,$$

  where $\overline{T}_{ij}(r,h)$ is the mean duration of a communication of height h from a i-level station to a j-level station at distance r of it.

The function $1-Q(\rho)$ is known as the linear contact distribution function of a planar Voronoi tessellation (cf. [9]). Its numerical values as well as explicit upper and lower bounds are given in [1].

3.3 Mobile communication models

We use the notations of the model for mobile communications introduced in Section 2.4.

We are interested here in some characteristic τ of the traffic, expressed as a non-negative random variable, the distribution of which, denoted by T, depends on the position and the type of the road under consideration. More specifically,

- τ_D will be a family of random variables with distributions T_D indexed by $D \in E = \mathbb{R} \times [0, \pi) \times \mathcal{S}$ – the phase space of the road process.

- $\phi_D(\eta)$ will denote the characteristic function of the distribution T_D.

Given a realization $\Delta = \sum_i D_i$ of the road process, consider the system of independent random variables $\tau_i \stackrel{def}{=} \tau_{D_i}$ associated with the road system. This is just another representation of the Poisson process introduced in Section 2.4.

The main quantity of interest is the additive functional $\Sigma \stackrel{def}{=} \sum_i \tau_i$. Using the independence of the system $\{\tau_i\}$, one obtains the following expression for the characteristic function of the variable Σ:

$$\Phi(\eta) = \mathbf{E} \prod_{D_i \in \mathrm{supp}\, \Delta} \phi_{D_i}(\eta).$$

Now using the explicit expression for the Poisson process generating functional (cf. [5, p.225]), we obtain the following theorem:

Theorem 3

$$\Phi(\eta) = \exp\left\{\int_E (\phi_D(\eta) - 1)\, M(dD)\right\}, \tag{10}$$

where $M(dD) = \lambda_r\, dp\, \mathcal{O}(d\alpha)\, Q(\alpha, d\sigma)$ *is the intensity measure of the road process* Δ. *In particular, if the integral below exist, then*

$$\mathbf{E}\Sigma = \int_E \mathbf{E}^T \tau_D\, M(dD), \tag{11}$$

$$\mathbf{var}\Sigma = \int_E \mathbf{E}^T \tau_D^2\, M(dD),. \tag{12}$$

where $\mathbf{E}^T$ *denotes expectation with respect to the distribution* T_D.

Typical examples of such additive functional are the number of communicating vehicles within a cell, and the number of the cell boundary crossings per unit time.

For Z a domain of $\mathbb{R}^2$, let

$$hit(Z) \stackrel{def}{=} \{(p, \alpha) \in \mathbb{R} \times [0, \pi) : \text{ the line } (p, \alpha) \text{ intersects } Z\}.$$

For $D = (p, \alpha, \sigma)$ with $(p, \alpha) \in hit(Z)$, let τ_D be the number $\mathcal{N}(Z)$ of vehicles in communication on road (p, α) and within Z. We take $\tau_D = 0$ if $(p, \alpha) \notin$

$hit(Z)$. As a direct application of the last theorem, we obtain the distribution of the number of the vehicles in communication inside Z.

Consider, for example, the traffic model described in Section 2.4. The positions of communicating vehicles on a road (p, α, σ) at a fixed instant are given by a homogeneous Poisson process with intensity $\lambda_t(\alpha, \sigma)$. Let $l(p, \alpha)$ be the length of the intersection of the road (p, α) with Z. The number of the vehicles in this part of the road has Poisson distribution with parameter $\lambda_t(\alpha, \sigma) l(p, \alpha)$. So taking

$$\phi_D(\eta) = \exp\{\lambda_t(\alpha, \sigma) l(p, \alpha)(e^\eta - 1)\}$$

in (10), we obtain the characteristic function of $\mathcal{N}(Z)$.

In the case of an isotropic road system, Formula (10) becomes

$$\Phi(\eta) = \exp\left\{\frac{\lambda_r}{\pi} \int_{\mathcal{S}} Q(d\sigma) \int_{hit(Z)} (\exp\{\lambda_t(\sigma) l(p, \alpha)(e^\eta - 1)\} - 1)\, d\alpha dp\right\} \quad (13)$$

and the moments of the distribution of $\mathcal{N}(Z)$ are given by

$$\mathbf{E}\mathcal{N}(Z) = \lambda_r |Z| \int_{\mathcal{S}} \lambda_t(\sigma) Q(d\sigma) \quad (14)$$

$$\mathbf{var}\mathcal{N}(Z) = \mathbf{E}\mathcal{N}(Z) + \frac{\lambda_r}{\pi} \int_{\mathcal{S}} \lambda_t^2(\sigma) Q(d\sigma) \int_{hit(Z)} l^2(p, \alpha)\, d\alpha dp. \quad (15)$$

Here we have used the following elementary fact that $\int_{hit(Z)} l(p, \alpha)\, d\alpha dp$ equals π times the area $|Z|$ of the domain Z.

A similar calculation can be done within the framework of the basic cell model, when we take the Voronoi cell $\mathcal{C}_0$ (a random set!) as the domain Z. In [1] we obtained the following expressions

$$\mathbf{E}\mathcal{N}(\mathcal{C}_0) = \frac{\lambda_r \overline{\lambda_t}}{\lambda_1},$$

$$\mathbf{var}\mathcal{N}(\mathcal{C}_0) = \frac{\lambda_r \overline{\lambda_t}}{\lambda_1} + 0.672 \frac{\lambda_r \overline{\lambda_t^2}}{\lambda_1^{3/2}} + 0.280 \frac{\lambda_r \overline{\lambda_t}}{\lambda_1^2},$$

$$\text{where } \overline{\lambda_t^k} \stackrel{def}{=} \int_{\mathcal{S}} \lambda_t^k(\sigma) Q(d\sigma).$$

Another problem of this type is the number of boundary crossings. The same machinery can be used to find the distribution of the total number of crossings of a fixed line segment in the plane during a time unit. For this take τ_D to be the number of crossings of an intersection point of the line $D = (p, \alpha, \sigma)$ with the segment and apply Theorem 3. Unfortunately, this is the only situation when Theorem 3 is directly applicable. The problem with general domain Z is that, with positive probability, there are more than one intersection point, and the number of crossings in these points are strongly dependent (in fact, it can be shown that these variables are associated). This does not matter for computing the first moment of the total number $\mathcal{F}(Z)$ of the boundary crossings since the expectation is always additive regardless the dependence of the summands. Nevertheless, lower and upper bounds can be derived for higher moments and general domains Z as it is shown bellow.

Consider, for example, an isotropic road system with a homogeneous Poisson process as the traffic model on the roads. As above $\lambda_t(\sigma)$ is the intensity of communicating vehicles on a road of type σ and each vehicle is uniformly marked by a velocity with distribution $\mathcal{V}_\sigma(dv)$. It can be shown that the number of crossings of a fixed point on such a road follows the Poisson distribution with parameter $\lambda_t(\sigma)\mathbf{E}_\sigma|V|$, where the last term is the mean of the absolute value of the velocity on a road of type σ (expectation with respect to the measure $\mathcal{V}_\sigma$). Now, a direct application of Formula (14) leads to

Theorem 4 *The mean number of crossings of a fixed domain Z equals*

$$\mathbf{E}\mathcal{F}(Z) = \frac{2}{\pi}|\partial Z|\lambda_r\overline{M_1},$$

where $|\partial Z|$ is the length of the boundary of Z and

$$\overline{M_1} = \int \lambda_t(\sigma)\mathbf{E}_\sigma|V|\, Q(d\sigma).$$

For a typical Voronoi cell in the basic model, this number is

$$\mathbf{E}\mathcal{F}(\mathcal{C}_0) = \frac{8\lambda_r}{\pi\sqrt{\lambda_1}}\overline{M_1}.$$

The following result can also be found in [1]:

Theorem 5 *The variance of the number of boundary crossings of a convex domain Z is bounded by*

$$\frac{2}{\pi}\lambda_r|\partial Z|\overline{M_2} \leq \mathbf{var}\mathcal{F}(Z) \leq \frac{4}{\pi}\lambda_r|\partial Z|\overline{M_2},$$

where

$$\overline{M_2} = \int \left(\lambda_t(\sigma)\mathbf{E}_\sigma|V| + (\lambda_t(\sigma)\mathbf{E}_\sigma|V|)^2\right) Q(d\sigma).$$

For the basic cell model

$$\frac{\lambda_r}{\pi}\left[\frac{0,945\overline{M_1}}{\lambda_1} + \frac{8\overline{M_2}}{\sqrt{\lambda_1}}\right] \le \mathbf{var}\mathcal{F}(\mathcal{C}_0) \le \frac{\lambda_r}{\pi}\left[\frac{0,945\overline{M_1}}{\lambda_1} + \frac{16\overline{M_2}}{\sqrt{\lambda_1}}\right].$$

3.4 Example of architecture optimization

Consider the following optimization problem: for connecting each subscriber to one concentrator, one can either use a direct link, or an indirect link via some distribution point. A distribution point is a location where several links originating from subscribers can be grouped into one further link to some concentrator with some economy of scale. The general assumption is still that connections are always to the closest point (i.e. each subscriber is connected to the closest distribution point, and each distribution point to the closest concentrator)

What is the optimal intensity of such distribution points? Consider 3 independent homogeneous Poisson processes Π_i ($i = 0, 1, 2$) in the plane $\mathbb{R}^2$ with intensities λ_i, representing *subscribers* (process Π_0), *distribution points* (process Π_1) and *concentrators* (process Π_2), respectively. Each point of the process Π_0 is connected to a point of process Π_1 according to the least distance principle. Similarly, each points of Π_1 is connected to the closest point of the process Π_2.

In a typical example, these connections represent cables. Each cable has an associated cost which consists of two parts:

1. the cost of the cable itself, and

2. the cost of the civil engineering (trench, post supports etc.)

The cost of a cable connecting a point of Π_i to the closest point of Π_{i+1}, is given by a polynomial function $C_{i,i+1}r^{\beta_{i,i+1}}$, where r is the distance between the points and $C_{i,i+1}$, $\beta_{i,i+1}$ are some non-negative parameters.

The cost of the civil engineering is given in this case by the function $E_{i,i+1}r^{\varepsilon_{i,i+1}}$, where $E_{i,i+1}$, $\varepsilon_{i,i+1} > 0$.

Note, that the cables from a distribution point to a concentrator share the same civil engineering, and it is reasonable to assume that the cost of civil engineering does not depend on the number of cables which are grouped there.

Our goal here is to find the optimal ratio between intensities of the processes which minimizes the average cost of the connections of a typical concentrator.

More precisely, the cost function is the following:

$$G \stackrel{def}{=} \mathbf{E}_2 \left\{ D_1 \mathcal{N}_1(0) + \sum_{y_i \in \Pi_1 \cap V_2(0)} \Big[E_{12}|y_i|^{\varepsilon_{12}} + C_{12}|y_i|^{\beta_{12}} \mathcal{N}_0(y_i) \right.$$
$$\left. + \sum_{x_j \in \Pi_0 \cap V_1(y_i)} \left(E_{01}|x_j - y_i|^{\varepsilon_{01}} + C_{01}|x_j - y_i|^{\beta_{01}} \right) \Big] \right\}. \tag{16}$$

Here and below,

- $\mathbf{E}_k$ denotes the expectation with respect to the Palm distribution of the process Π_k;
- $V_k(x)$ is the Voronoi cell constructed with respect to the process Π_k, with nucleus $x \in \Pi_k$;
- D_1 is the cost of maintenance and installation of a distribution point;
- $\mathcal{N}_k(x)$ is the number of Π_k-points in $V_{k+1}(x)$ $(k = 0, 1)$.

The following result is proved in [3]: Let

$$\begin{aligned}
A_1 &= \frac{E_{12}}{\pi^{\varepsilon_{12}/2} \lambda_2^{\varepsilon_{12}/2+2}} \Gamma\left(\frac{\varepsilon_{12}}{2} + 1\right) + \frac{D_1}{\lambda_2}; \\
A_2 &= \frac{C_{12}\lambda_0}{\pi^{\beta_{12}/2}\, \lambda_2^{\beta_{12}/2+2}} \Gamma\left(\frac{\beta_{12}}{2} + 1\right); \\
A_3 &= \frac{E_{01}\lambda_0}{\pi^{\varepsilon_{01}/2}\, \lambda_2} \Gamma\left(\frac{\varepsilon_{01}}{2} + 1\right); \\
A_4 &= \frac{C_{01}\lambda_0}{\pi^{\beta_{01}/2}\, \lambda_2} \Gamma\left(\frac{\beta_{01}}{2} + 1\right).
\end{aligned}$$

Theorem 6 *The cost function G has a unique minimal value in λ_1, attained in the point λ_1^* which solves the equation*

$$A_3 \varepsilon_{01} \lambda_1^{-(\varepsilon_{01}/2+1)} + A_4 \beta_{01} \lambda_1^{-(\beta_{01}/2+1)} = 2A_1.$$

In particular case when $\varepsilon_{01} = \beta_{01} \stackrel{def}{=} \beta$ we obtain

$$\lambda_1^* = \left[\frac{\beta(A_3 + A_4)}{2A_1}\right]^{2/(2+\beta)}$$

and the minimal value of the cost function is equal to

$$G(\lambda_1^*) = \frac{\beta + 2}{\beta} A_1^{\beta/(2+\beta)} \left[\frac{\beta}{2}(A_3 + A_4)\right]^{2/(2+\beta)} + A_2.$$

For $\beta = 1$ we obtain respectively

$$\lambda_1^* = \lambda_2 \left[\frac{\lambda_0(C_{01} + E_{01})}{2E_{12} + 4D_1\lambda_2^{3/2}}\right]^{2/3} \quad \text{and}$$

$$G(\lambda_1^*) = \frac{3\lambda_0^{2/3}}{2^{5/2}\lambda_2^{3/2}}(E_{12} + 2D_1\lambda_2^{3/2})^{1/3}(C_{01} + E_{01})^{2/3} + \frac{C_{12}\lambda_0}{\pi^{\beta_{12}/2}\,\lambda_2^{\beta_{12}/2+2}}\Gamma\left(\frac{\beta_{12}}{2} + 1\right).$$

4 Perspectives

The proposed approach to the modeling of telecommunication networks can be used for several variations on the basic models introduced here. As an illustration of this assertion, consider again hierarchical model. The main drawback of the model is that it only contains hierarchical links between stations. However, real systems may have connections to several stations of the higher level, for reliability reasons. These auxiliary connections can easily be taken into consideration. It is sufficient to connect the lower level stations not only to their closest, but to the second closest, third closest etc. stations of the next level. This construction leads to a thinner tessellation than the Voronoi tessellation, which was also studied in the literature. One can derive geometric characteristics of these auxiliary connections as we do in [1].

Unfortunately, analytical methods are not always possible for complex models (as e.g. non-homogeneous, non-Poisson models etc. or even many complex phenomena of Poisson models). For such situations simulations are sometimes the only possible way of studying the system behavior. In connection with this, we would quote the stochastic gradient methods developed by the authors in [2] and [1].

Some non-homogeneous models can also be treated by change of space methods described in [1]. This is important since real systems are rather far from being homogeneous.

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