HOMOGENEOUS PROJECTIVE TRANSFORMATION AND SCALING OF A GENERAL QUADRATIC ALGEBRAIC MATRIX RICCATI EQUATION

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Abstract: The matrix formulation and solution to the problems of homogeneous projective transformation and scaling of a generalized quadratic algebraic matrix Riccati equation is presented. The method presented is independent from the numerical algorithms used for the numerical solution of the equation under study. In the present paper, the proposed method is applied to probability-1 homotopy algorithms for the numerical solution of the equation under study. The Advantages and disadvantages of the proposed method are discussed through a numerical example. *Copyright © 2000 IFAC*

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1. INTRODUCTION

This paper presents a matrix formulation and solution to the problems of homogeneous projective transformations (Morgan, 1986) and scaling of the following general quadratic algebraic matrix Riccati equation.

$$\begin{aligned} A_1XB_1 + A_2XB_2 + C_1XD_1XE_1 \\ +C_2XD_2XE_2 + G = 0 \end{aligned} \tag{1}$$

Where,

$A_1, A_2, C_1, C_2 \in C^{n\times n}$, $B_1, B_2, E_1, E_2 \in C^{p\times p}$, $D_1, D_2 \in C^{p\times n}$, $G \in C^{n\times p}$, $0 \in C^{n\times p}$ are the constant matrix coefficients of the equation and $X \in C^{n\times p}$ is the equation unknown matrix. $C^{m\times l}$ and C^m denote the set of complex matrices with dimension $m\times l$ and $m\times 1$ respectively and m, n, l, p are non-zero physical numbers. In the sequel 0, denotes the zero matrix, vector or number of compatible dimensions.

The importance of equation (1) is briefly discussed in (Tsachouridis, 2000). Homogeneous projective transformations and scaling can be useful when homotopy algorithms are used. In (Tsachouridis, 2000), a global convergence probability-1 homotopy algorithm for the numerical of equation (1) is presented. A major drawback of the homotopy algorithm in (Tsachouridis, 2000) is when (1) has solutions at infinity and the algorithm tries to compute such solutions. In cases like this, the computation of a solution at infinity requires infinite time. Hence, a decision of when to stop the numerical process must be taken. This is really a tricky point since a finite solution with an arbitrarily large norm, might demand large computational time. Therefore, when no knowledge about the geometry of the equation's solutions is available, there is always the danger to stop the algorithm's numerical process on its way to compute a finite solution. This is possible by incorrectly assuming that this finite solution might be at infinity.

One way to avoid this hazard is to transform the original equation in to an equation with no solutions at infinity. With such a transformation, the finite solutions of the transformed equation should correspond, to the finite solutions and to the solutions at infinity (if any) of the original equation. Such transformations providing mathematical formulas for implementing the correspondence of solutions between the original equation and the transformed equation, under certain hypothesis, are the homogeneous projective transformations. For scalar polynomial systems, homogeneous projective transformations are presented in (Morgan, 1986). Their applications for the general quadratic algebraic matrix Riccati equation under study, is a subject of the present paper.

The homogeneous projective transformations are independent of any numerical method and therefore they can be used along with any numerical method for the equation's solution in general. In the present paper the numerical method in (Tsachouridis, 2000), for the solution of equation (1) is used.

Now, when one solves equation (1) with the method (Tsachouridis, 2000), scaling can be useful in some cases. The purpose of scaling is to prevent arithmetic problems on a computer by transforming the coefficients and the unknown matrices of the original equation so that they do not have extreme values. This kind of transformation differs from a homogeneous projective transformation. Scaling assigns predetermined values to the coefficients and the unknown matrices of the original equation. The homogeneous projective transformation changes the structure of the original equation in to a homogeneous equation.

There is no a rigorous mathematical theory to guide the scaling of the equation (1), in its general form, and prove its efficiency. Nevertheless, it will be shown via a numerical example that there are cases in which the proposed scaling affects the curvature and arc length of the homotopy paths associated with the original equation, when using the homotopy algorithm of (Tsachouridis, 2000). Usually, a good practice is to scale equations by scaling the coefficients and the variables in a way such that, scaling minimizes the magnitudes of the equation's coefficients. There are several approaches of doing this. The one followed in the present paper is to consider equation (1) in magnitude form and then to equivalently transform this magnitude equation in to a magnitude exponential vector equation. Then the problem of scaling is to minimize the sum of the squares of the coefficient exponents of the resulted vector equation. Thus, the problem of scaling formulates an optimization problem. More specifically the problem of scaling ends up with the minimization of the Frobenious norm of a matrix function. The variables of this matrix function, are actually the scale factors for the equations (that the exponential magnitude vector equation provide) and the scale factors for the respective equation's unknowns. As in the case of homogeneous transformations, the scaling method presented is independent from the numerical method used for the solution of the equation. Therefore, it can be used with any numerical method.

Both scaling and homogeneous projective transformation contributes to smooth numerical operations and sometimes reduce the computational times. However, the reader should bear in mind that scaling and homogeneous projective transformations are not effective always, but they work well together in many cases. A numerical example in this paper shows, that combining scaling and the homogeneous projective transformation shorts the arc lengths of the homotopy paths, see (Tsachouridis, 2000), and provide solutions closer to unit sphere.

The structure of this paper will be as follows. Section 2 will present the mathematical formulation and solution to the problem of scaling. Next in section 3, the concept of homogeneous projective transformation will be introduced in section 2. The numerical algorithm of (Tsachouridis, 2000), adapted for the solution of the scaled and homogeneous projective transformed equation will be comprehensively discussed in the section 4. In section 5, the proposed method will be illustrated with a numerical example. The numerical example has been particularly chosen to show both positive and negative effects of the scaling and the homogeneous projective transformation. Finally, conclusions will be given in section 6. Because of the limited space, proofs of theorems and other theoretical results such as perturbation and error analysis, relevant to the concept of the present paper, are omitted. Details of all these and other aspects, subject to the practical implementation of the proposed method, will be reported in another paper.

2. SCALING OF EQUATION (1)

In order to scale equation (1), let $u_X \in \Re^{n\times p}$ and $e_o \in \Re^{np\times 1}$ be the scaling factors for the variable and coefficient matrices of equation (1) respectively. Where, $\Re^{m\times l}$ is the set of $m\times l$ real matrices. Furthermore, let $\Re$ be the set of real numbers, and $*$ the Hadamard product matrix operator. Now, let $10^{(\cdot)}: \Re^{n\times p} \to \Re^{n\times p}$, $\left[10^{Y}\right]_{ij} := 10^{[Y]_{ij}}$, be the element to element matrix exponent with base 10 matrix operator of $Y \in \Re^{n\times p}$, and $|\cdot|: C^{n\times p} \to \Re^{n\times p}$, $|Z|_{ij} := |Z_{ij}|$, be the element to element magnitude matrix operator of $Z \in C^{n\times p}$. Moreover, for a general real matrix $W \in \Re^{n\times p}$ with nonnegative elements, define the element to element logarithm with base 10 matrix operator, $log_{10}(\cdot): \Re^{n\times p} \to \Re^{n\times p}$,

$$\left[log_{10}(W)\right]_{ij} := \begin{cases} log_{10}(W)_{ij}, & (W)_{ij} > 0 \\ 0 & (W)_{ij} = 0 \end{cases}, \quad i = 1,\ldots,n,$$

$j = 1,\ldots,p$. Also, let $U_{m\times l}$ and $I_{m\times l}$ denote the $m\times l$ unity and identity matrices respectively.

Now let the scaled equation of equation (1) be

$$10^{vec^{-1}(e_o)} * \left[A_1 \left(10^{u_X} * \hat{X} \right) B_1 + A_2 \left(10^{u_X} * \hat{X} \right) B_2 \right] + 10^{vec^{-1}(e_o)} * \left[C_1 \left(10^{u_X} * \hat{X} \right) D_1 \left(10^{u_X} * \hat{X} \right) E_1 + C_2 \left(10^{u_X} * \hat{X} \right) D_2 \left(10^{u_X} * \hat{X} \right) E_2 + G \right] = 0 \quad (2)$$

Where, $vec^{-1}(\cdot)$ is the inverse of $vec(\cdot)$ operator, see (Tsachouridis, 2000), $\hat{X} \in C^{n \times p}$ is the new scaled variable. Applying in equation (2) the magnitude operator, as defined previously, and after many algebraic manipulations the resulted magnitude equation in vector form is shown below.

$$\left(\sum_{i=1}^{2} 10^{S_{L_i}} \right) vec\left(|\hat{X}| \right) + \left(\sum_{i=1}^{2} 10^{S_{Q_i}} \right) vec\left(|\hat{X}^T| \otimes |\hat{X}| \right) + 10^{S_C} = 0 \quad (3)$$

Where,

$$S_{Q_i} := U_{np \times 1} \left[vec\left(u_X^T \oplus u_X \right) \right]^T + e_o U_{1 \times (np)^2} + U_{np \times 1} \left(\left[vec\left(log_{10}\left(|D_i| \right) \right) \right]^T \otimes U_{1 \times np} \right) + U_{1 \times np} \otimes \left(log_{10}\left(|E_i| \right)^T \oplus log_{10}\left(|C_i| \right) \right) \quad (4)$$

$$S_{L_i} := U_{np \times 1} \left[vec(u_X) \right]^T + e_o U_{1 \times np} + \left(log_{10}\left(|B_i| \right)^T \oplus log_{10}\left(|A_i| \right) \right) \quad (5)$$

$$S_C := e_o + vec\left(log_{10}\left(|G| \right) \right) \quad (6)$$

$\otimes$, $\oplus$ are the Kronecker product and sum matrix operators respectively.

Now define the scalar function below.

$$J_s(u_X, e_o) := \sum_{i=1}^{2} \left\| S_{L_i} \right\|_F^2 + \sum_{i=1}^{2} \left\| S_{Q_i} \right\|_F^2 + \left\| S_C \right\|_F^2 \quad (7)$$

The problem of scaling is to minimize function (7) with respect to $u_X \in \Re^{n \times p}$ and $e_o \in \Re^{np \times 1}$. It is apparent from equations (2)-(6) that, the minimization of (7), minimizes the magnitude of the equation's coefficients. Hence, the problem of scaling ends up as an optimization (minimization) problem. To this minimization an additional constraint can be, the minimization of the variation between the elements of the coefficient matrices in equation (3). In other words the minimization of the difference between the nonzero elements of the matrices S_{Q_i}, S_{L_i}, S_C, $i = 1,2$. It can be proven that the above minimization problem has an analytic solution. More specifically is equivalent with the solution of a linear $2np \times 2np$ algebraic system with unknowns the elements of the scaling factors $u_X \in \Re^{n \times p}$ and $e_o \in \Re^{np \times 1}$. The structure of such a system and its derivation (i.e. the solution to the minimization problem) is not reported here because of the limited space. All these have been proved by the author and will be presented in another paper.

Finally, once the $u_X \in \Re^{n \times p}$ and $e_o \in \Re^{np \times 1}$ are determined and the scaled equation (2) is solved, the solution to the original equation (1) is recovered from $X = 10^{u_X} * \hat{X}$. Also note that, equation (1) can be recovered from (2) by setting $u_X = 0$ and $e_o = 0$.

3. HOMOGENEOUS PROJECTIVE TRANSFORMATION OF EQUATION (2)

As it was stated in section 1, equation (1) can have solutions at infinity. This is an immediate consequence for the scaled equation (2) too. The concept of an homogeneous projective transformation, see (Morgan, 1986), is to transform the equation (1) or (2) into an equation with no solutions to infinity, and to provide analytic formulas for implementing the correspondence of solutions between the original equation and the transformed equation. Because, as stated in the end of section 2, equation (1) can be recovered from (2) with $u_X = 0$ and $e_o = 0$, a homogeneous projective transformation will be developed for equation (2) next. Also note that the equivalent homogeneous projective transformation for equation (1) can be obtained by setting $u_X = 0$ and $e_o = 0$. The synthesis of such a transformation is synopsized to the theorem 1, below. The proof of theorem 1 will be reported in another more detailed paper.

Theorem 1; Suppose that equation (2) has a finite number of finite solutions and a finite number of solutions at infinity. Also, define $A_a := \left[a_{11} \quad a_{12} \quad \dots \quad a_{1(np)} \right]$ and $L_X := A_a vec\left(\bar{\hat{X}} \right) + a_{np+1}$. Then for almost all $A_a \in C^{1 \times np} - \{0\}$, $a_{1 \times np} \in C - \{0\}$, the equation

$$10^{vec^{-1}(e_o)} * \left\{ \left[A_1 \left(10^{u_X} * \bar{\hat{X}} \right) B_1 + A_2 \left(10^{u_X} * \bar{\hat{X}} \right) B_2 \right] L_X + \left[C_1 \left(10^{u_X} * \bar{\hat{X}} \right) D_1 \left(10^{u_X} * \bar{\hat{X}} \right) E_1 + C_2 \left(10^{u_X} * \bar{\hat{X}} \right) D_2 \left(10^{u_X} * \bar{\hat{X}} \right) E_2 + G L_X^2 \right] \right\} = 0 \quad (8)$$

has no solutions at infinity.

Furthermore the correspondence of solutions between equation (2) and (8) is given by $\hat{X} = \frac{1}{L_X}\hat{\bar{X}}$ when $L_X \neq 0$ for finite solutions, and for $L_X = 0$, $\hat{\bar{X}}$ correspond to a solution $\hat{X}$ at infinity.

Remark 3.1; In terms of equation (1) the correspondence of solutions between equation (1) and (8) is given by $X = \frac{1}{L_X}\left(10^{u_X} * \hat{\bar{X}}\right)$ when $L_X \neq 0$ for finite solutions. For $L_X = 0$, $\hat{\bar{X}}$ correspond to a solution X at infinity.

From the above all it is apparent that choosing $A_a \in C^{1\times np} - \{0\}$, $a_{1\times np} \in C - \{0\}$ in random, equation (8) will have no solutions at infinity under the satisfaction of conditions of theorem 1. Now the utility of the above result in probability-1 homotopy algorithms, for the numerical solution of equation (1), is shown in the next section.

4. PROBABILITY-1 HOMOTOPY ALGORITHMS WITH HOMOGENEOUS PROJECTIVE TRANSFORMATION AND SCALING

In this section the probability-1 theorem 2 of (Tsachouridis, 2000) is restated. The new theorem 2 includes in its synthesis the scaling and the homogeneous projective transformation, of sections 2 and 3.

Theorem 2 (probability-1 polynomial homotopy); Let

$$F\left(\bar{x}\right) := 10^{e_o} * \left\{\left[\left(B_1^T \otimes A_1 + B_2^T \otimes A_2\right)L_X + \left(E_1^T \otimes C_1\right)\left(I_p \otimes \left(D_1\left(10^{u_X} * \bar{X}\right)\right)\right) + \left(E_2^T \otimes C_2\right)\left(I_p \otimes \left(D_2\left(10^{u_X} * \bar{X}\right)\right)\right)\right] \times \left(10^{u_X} * \bar{x}\right) + vec\left(GL_X^2\right)\right\} = 0 \quad (9)$$

Where, $x = vec\left(\bar{X}\right)$ and u_X, e_o and L_X evaluated according to sections 2 and 3. Also, let $H: C^m \times [0,1] \to C^m$, $m = np$, be a homotopy mapping as in definition 1 of (Tsachouridis, 2000), with easy problem $F_0\left(\bar{x}\right) = A * \bar{x} * \bar{x} - L_X^2 B$, where, $A, B \in C^m - \{0\}$, and with the homotopy equation $H\left(\bar{x}, \varepsilon\right) := \varepsilon F\left(\bar{x}\right) - \gamma(1-\varepsilon)F_0\left(\bar{x}\right)$, $\gamma \in C - \{0\}$.

Also let $H_a: C^m \times [0,1] \times C^m \to C^m$ be a homotopy mapping as in definition 2 of (Tsachouridis, 2000) with homotopy equation $H_\alpha\left(\bar{x}, \varepsilon, a\right) := \varepsilon F\left(\bar{x}\right) - \gamma(1-\varepsilon)F_0\left(\bar{x}, a\right)$, where, $F_0\left(\bar{x}, a\right) = \bar{x} * \bar{x} - a$. Furthermore, suppose that the conditions below hold.

(a) The mapping H is complex analytic.

(b) The mapping H_α is transversal to zero.

Then almost always, the set

$$H_\alpha^{-1}(0) := \left\{\left(\bar{x}, \varepsilon, a\right) \in C^m \times [0,1) \times C^m \;\middle|\; H\left(\bar{x}, \varepsilon, a\right) = 0\right\},$$

contains in general 2^m unique smooth curves connecting the 2^m zero solutions of $F_0\left(\bar{x}, a\right) = 0$, with the 2^m solutions of $F\left(\bar{x}\right) = 0$ in 1-1 relationship and with the problems (i)-(vi) of (Tsachouridis, 2000) not applying. Furthermore for the homotopy mapping H, for almost every $A, B \in C^m - \{0\}$, there are 2^m unique smooth curves in general connecting the 2^m solutions of $F_0\left(\bar{x}\right) = 0$, with the 2^m solutions of $F\left(\bar{x}\right) = 0$ in 1-1 relationship and with problems (i)-(vi) not applying.

Note that in theorem 2 of the present paper the sufficient condition of the set $H_\alpha^{-1}(0)$ being bounded in (Tsachouridis, 2000) is relaxed.

Hence, all the homotopy paths can be viewed as trajectories of the initial value problem (13) of (Tsachouridis, 2000) with $x = \bar{x}$, emanating from $\varepsilon = 0$ and terminating at $\varepsilon = 1$. Note that under theorem 2 above, none of the homotopy paths diverges to infinity and the solutions to the original equation (1) are obtained according to remark 3.1.

5. NUMERICAL EXAMPLES

The numerical example of subsection 4.2 of (Tsachouridis, 2000) is considered, and all solutions were computed via an algorithm based on theorem 2 of the present paper. For computing all finite solutions X_1, X_2, X_3 and X_4 the algorithm was initialized exactly as in (Tsachouridis, 2000). Also, from a pseudo-random generator it was chosen $A_a := [0.3528 \quad 0.8131 \quad 0.0098 \quad 0.1388]$, $a_{np+1} = 0.2027$. For all possible initial conditions, all solutions have been calculated, including those at infinity. In the last case L_X was found to tend to zero. Here the results presented consider the finite only solutions. This is in order to see the effects of scaling and the homogeneous projective transformation on the homotopy paths and on the overall computational process. The complete set of solutions (including those at infinity) will be reported in another paper. For each solution, six different options for the algorithm were implemented. All these computations are summarized in table 1. In this table S, P indicate scaling and homogeneous projective transformation. In addition V indicates that there is the additional constraint (see section 2) in scaling problem of the minimization of element variation. N indicates that the specific option after it is not take place. For example SVNP, indicates a case with scaling including minimization of element variation and with no homogeneous projective transformation, etc.

Also, $J = J_s(u_X, e_0)\big|_{(u_X, e_0)=(0,0)}$ and $J_s = J_s(u_X, e_0)\big|_{(u_X, e_0)\neq(0,0)}$. Furthermore, the homotopy paths for the cases listed in table 1 are shown in figures 1 and 2.

6. CONCLUSIONS

The matrix formulation and solution to the problems of homogeneous projective transformation and scaling of a generalized quadratic algebraic matrix Riccati equation was presented. The proposed method was applied to polynomial probability-1 homotopy algorithm of (Tsachouridis, 2000). From table 1 and figures 1 and 2, it is apparent that both the scaling and homogeneous projective transformation effects the computational process due to computational times accuracy and curvature of homotopy paths. In addition, the statements in section 1 are vindicated. The mathematical proofs of results and many other details concerning perturbation analysis, error estimates and how these are implemented in the software code will be presented in another paper.

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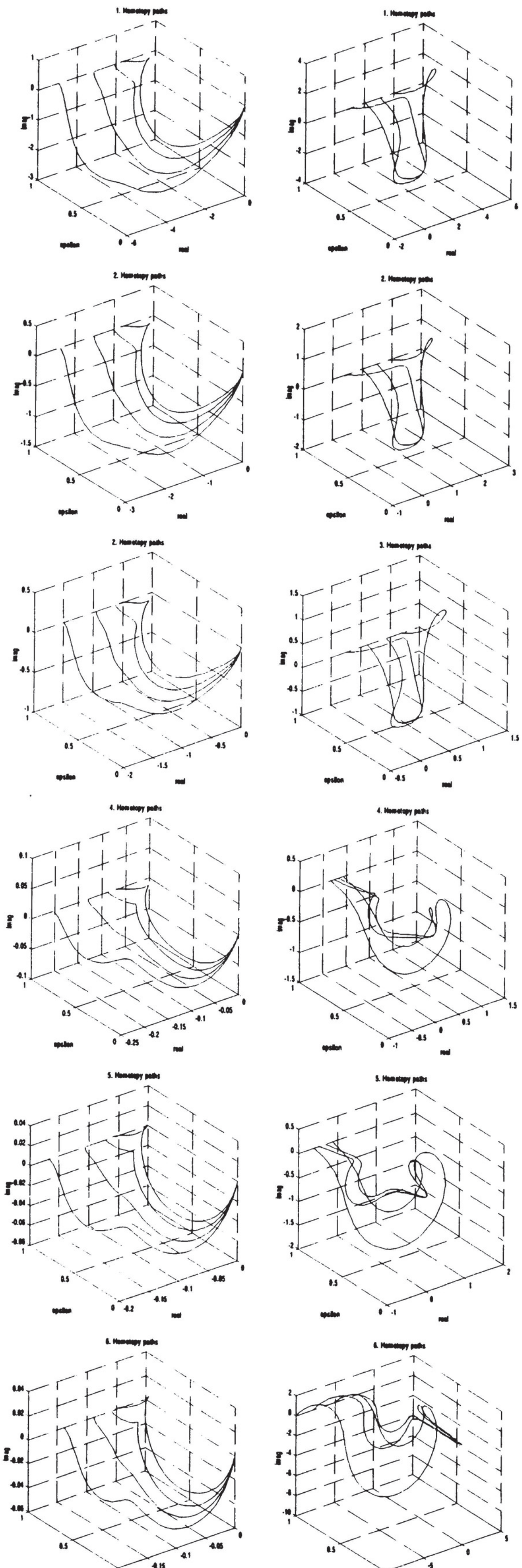
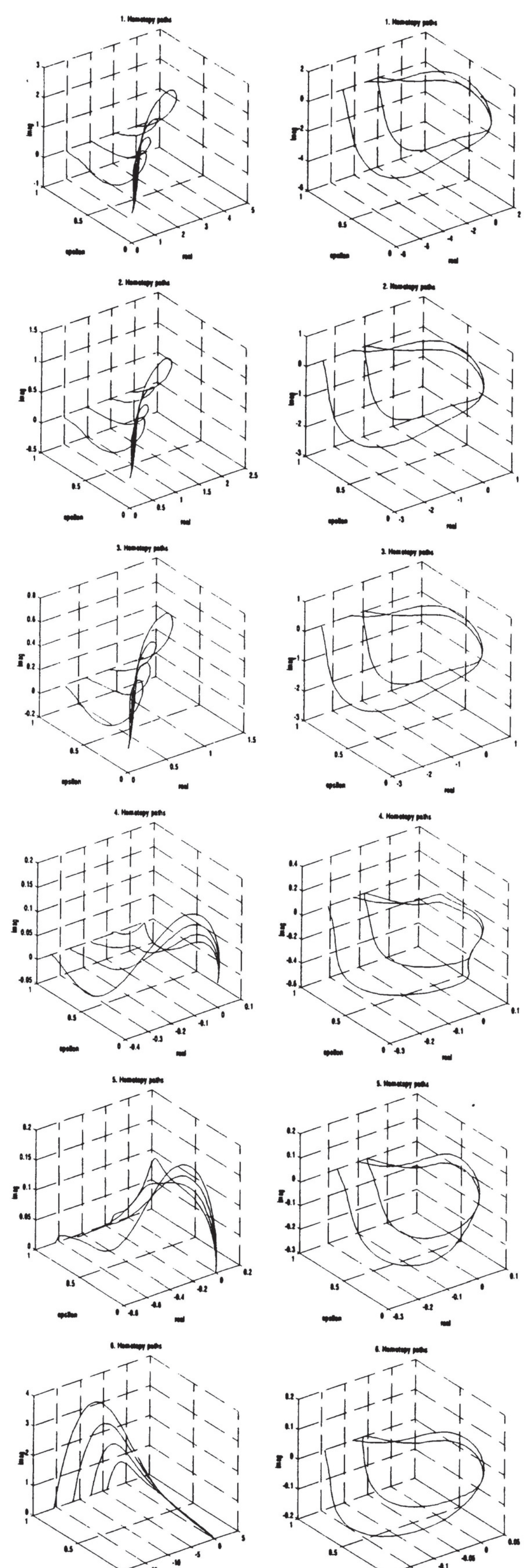
Table 1 Computation results for finite solutions.

	1. NSNP	2. SNP	3. SVNP	4. NSP	5. SP	6. SVP
u_X	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0.3172 & 0.3172\\0.3172 & 0.3172\end{bmatrix}$	$\begin{bmatrix}0.4982 & 0.5278\\0.5243 & 0.5179\end{bmatrix}$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0.3172 & 0.3172\\0.3172 & 0.3172\end{bmatrix}$	$\begin{bmatrix}0.4982 & 0.5278\\0.5243 & 0.5179\end{bmatrix}$
e_0	$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}-5.8680\\-6.0358\\-5.9519\\-6.1118\end{bmatrix}$	$\begin{bmatrix}-1.0455\\-1.0176\\-1.0032\\-1.0360\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}-5.8680\\-6.0358\\-5.9519\\-6.1118\end{bmatrix}$	$\begin{bmatrix}-1.0455\\-1.0176\\-1.0032\\-1.0360\end{bmatrix}$
J	5.5581	5.5581	5.5581	5.5581	5.5581	5.5581
J_s	-	2.7634	6.2147	-	2.7634	6.2147
L_{X_1}	1	1	1	0.0414	0.0705	0.0930
$E_{rX_1} \times 10^{-9}$	9.2341	11.3730	12.4520	47.416	28.570	95.973
CPU_{X_1} (sec)	7.29	6.45	6.45	5.48	5.07	4.86
L_{X_2}	1	1	1	-0.0844	-0.3179	-8.3692
$E_{rX_2} \times 10^{-9}$	37.9300	34.314	46.551	67.671	38.4770	1.0896
CPU_{X_2} (sec)	7.08	6.37	5.45	7.22	7.56	8.02
L_{X_3}	1	1	1	-0.08564	-0.3258	-12.7550
$E_{rX_3} \times 10^{-9}$	1.5309	1.0690	1.1648	22.9980	2.8814	1.3199
CPU_{X_3} (sec)	6.99	6.33	5.63	5.62	5.36	5.82
L_{X_4}	1	1	1	0.0417	0.0709	0.0934
$E_{rX_4} \times 10^{-9}$	18.0230	18.2650	18.2650	235.620	495.160	608.390
CPU_{X_4} (sec)	7.02	6.33	6.45	5.78	5.06	4.68



X_1 X_2

Fig. 1. Homotopy paths $X_1(\varepsilon)$, $X_2(\varepsilon)$, $\varepsilon \in [0,1]$.



X_3 X_4

Fig. 2. Homotopy paths $X_3(\varepsilon)$, $X_4(\varepsilon)$, $\varepsilon \in [0,1]$.