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A ROOT LOCUS CONCEPT FOR 2-DIMENSIONAL SYSTEMS

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An increasingly important area of control system design focuses on systems modelled with two independent frequency variables. The paper describes a root locus-like tool to aid in the analysis and design of such systems.

Key Words: Compensators, root-locus, stability tests, 2-dimensional systems, computer-aided control system design.

1. INTRODUCTION

Classical control theory makes use of a wide range of graphical design tools which allow engineers to assess the performance of a system across a range of parameter variations. Such tools include the root locus approach which demonstrates the way in which pole positions change in a closed loop system as loop gain is varied and hence enables the designer to estimate dynamic response at different loop gains and, in particular, to assess closed loop stability. The same tool is used for dynamic compensation via a set of well understood design rules, which indicate how the shape of the root locus changes with the addition of extra poles and zeros in the forward path. A similar approach would clearly be attractive for 2-dimensional systems (systems with two independent frequency variables), but the obvious analogues involve four dimensions (real and imaginary values for each variable) and this does not lead readily to an easily visualised approach. The work presented here demonstrates how the notion of root maps, used by multi-dimensional filter theorists for filter stability tests, can be extended into the closed loop to provide a root locus-like tool. Results are presented in the discrete domain, but these also map to the continuous domain (Seekings, 2000).

In Section 2, the notion of root maps is introduced and it is shown how the 2-dimensional root map can be extended to provide 3-dimensional visualisations which allow stability in performance variation with closed loop gain to be assessed directly. Section 3 demonstrates the use of dynamic compensation with the tool.

2. ROOT MAPS FOR 2-DIMENSIONAL SYSTEMS

By analogy with the 1-dimensional case, 2-dimensional systems can be described in the frequency domain by a rational transfer function $G(z_1^{-1}, z_2^{-2})$ in two, independent frequency variables. The numerator polynomial $nG(z_1, z_2)$ of such a transfer function defines the **zero set** of the transfer function, the denominator $dG(z_1,z_2)$ the **pole set.** The **zero set** and the **pole set** in the 2-dimensional case are defined as being **infinite continua**. A **'Zero'** is a particular point in the zero set. A '**pole**' is a particular point in the pole set. Note however, that z_i^{-1} is used to indicate a unit delay. The stability test used for this work first transforms the system to positive powers so that the region of stability is inside the unit circle. The **rootmap** for z_1 is the set of loci traced out by the roots of $dG(z_1, a)$ as 'a' traces out the unit circle ($a = e^{j\theta 2}$, $0 \le \theta_1 \le 2\pi$). The rootmap for z_2 is the set of loci traced out by the roots of $dG(a, z_2)$ as 'a' traces out the unit circle ($a = e^{j\theta 1}$, $0 \le \theta_2 \le 2\pi$). If both sets of loci are inside their respective unit circles then the system is stable (Dudgeon and Mersereau, 1994).

Now consider a closed loop system with open loop transfer function $G(z_1, z_2)$, forward path dynamic compensation $H(z_1, z_2)$ and feedback gain K.

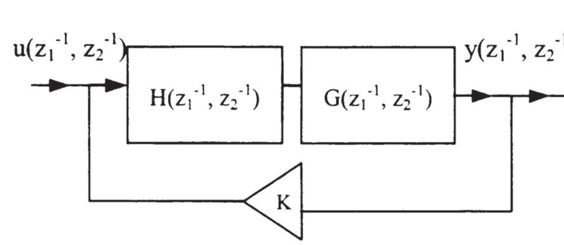

Fig. 1 Closed loop gain feedback control system.

Consider first the case with gain feedback only (ie no dynamic compensation). The closed loop transfer function is given by:

$$\frac{y(z_1^{-1}, z_2^{-1})}{u(z_1^{-1}, z_2^{-1})} = \frac{G(z_1^{-1}, z_2^{-1})}{1 + KG(z_1^{-1}, z_2^{-1})} \tag{1}$$

The rootmap now also depends on the gain, K, and K becomes the third dimension on each of a pair of 3d plots, this is termed the 3-d rootmap. The region of stability is now the unit cylinder centred on the K axis (Seekings *et al*, 1998).

To aid the designer points on the map are colour coded. Blue points are inside the unit cylinder, Magenta points are outside the unit cylinder. To ensure stability in the closed loop a value of gain(K) is chosen for which all loci are inside the unit circle on both rootmaps i.e. All points are coloured blue.

As an example of this consider the unstable first order system :

$$G(z_1^{-1}, z_2^{-1}) = \frac{1}{1 - 0.26z_1^{-1} - 0.72z_2^{-1} - 1.1z_1^{-1}z_2^{-1}} \tag{2}$$

Plotting the Rootmap for this shows that it is unstable, according to the definition in Dudgeon and Mersereau (1994), and plotting the impulse response confirms this

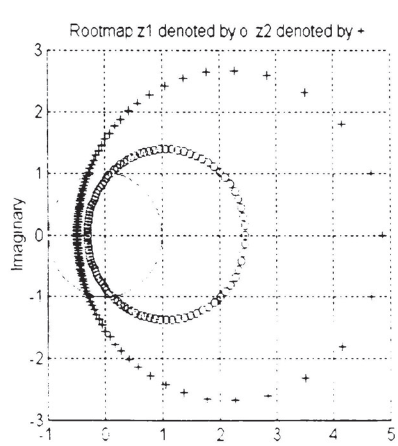

Fig. 2 Rootmap for Eq(2)

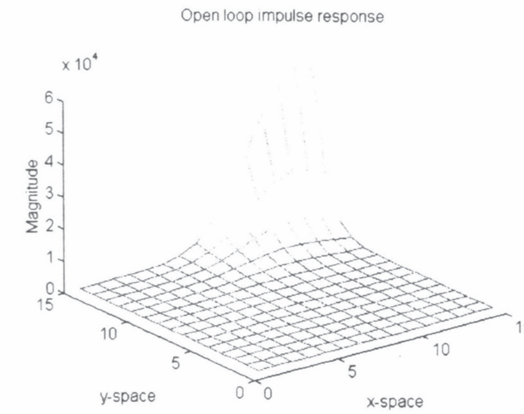

Fig 3. Impulse response of Eq(2)

Generating the 3d rootmaps w.r.t. z_1 and w.r.t. z_2 (Figs 4a, 4b):

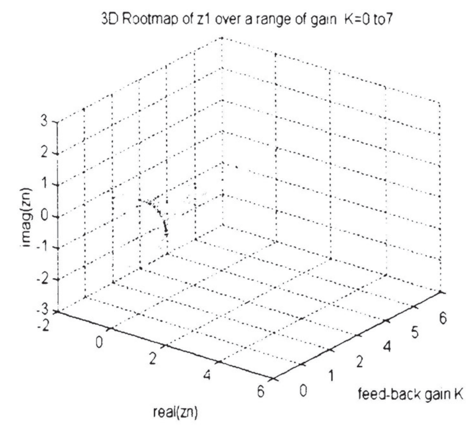

Fig. 4a 3-d rootmap of Eq(2), w.r.t. z_1.

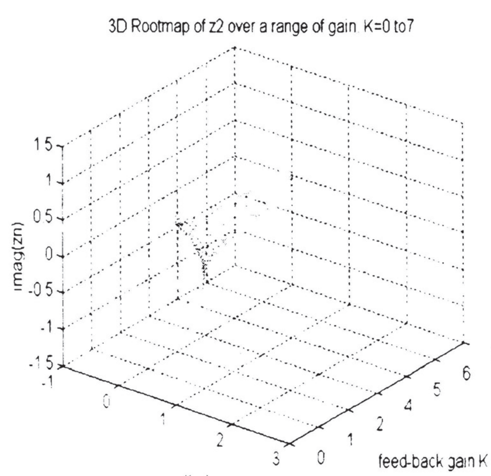

Fig. 4b 3-d rootmap of Eq(2), w.r.t. z_2.

It can be seen from Figs 4a and 4b that values of gain, K ≥~1.5 will give a stable closed loop system. (All blue points). Looking at the case K=1.5 in more detail:

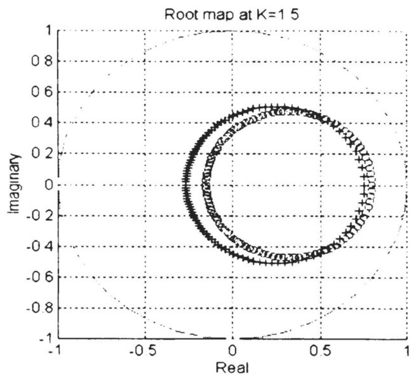

Fig. 5 Rootmap for closed loop system at K=1.5

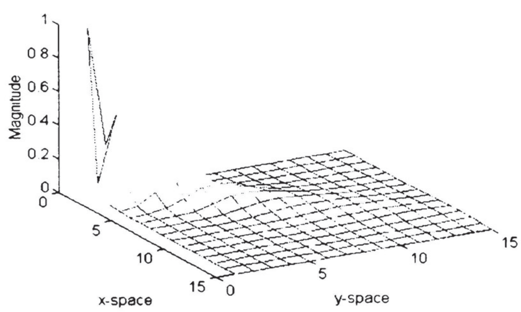

Fig. 6 Impulse response for closed loop system at K=1.5

The rootmap at K=1.5 Fig(5) is inside the unit circle, and therefore stable. This is confirmed by looking at the impulse response Fig(6).

3. DYNAMIC COMPENSATION

In this section the use of the 3d rootmap for the design of dynamic compensators is examined.

The forward path of the system now contains poles and zeros from the compensator in addition to those in the original system so that the closed loop transfer function is given by:

$$\frac{y\,(z_1^{-1}, z_2^{-1})}{u\,(z_1^{-1}, z_2^{-1})} = \frac{G\,(z_1^{-1}, z_2^{-1})\,H\,(z_1^{-1}, z_2^{-1})}{1 + KG\,(z_1^{-1}, z_2^{-1})\,H\,(z_1^{-1}, z_2^{-1})}$$

Such pre-compensator poles can be used to repel the rootmap away from those poles, and likewise pre-compensator zeros can be used as attractors. This is analogous to the 1-d case.

Take the example of a system:

$$G(z_1^{-1}, z_2^{-1}) = \frac{1+05z_1^{-1}-023z_2^{-1}+04z_1^{-1}z_2^{-1}+05z_1^{-2}z_2^{-2}}{1+01z_1^{-1}+01z_1^{-2}-01z_2^{-1}+02z_1^{-1}z_2^{-1}-01z_1^{-2}z_2^{-1}-01z_2^{-2}+01z_1^{-1}z_2^{-2}-z_1^{-2}z_2^{-2}}$$

(3)

This has a non-minimum phase numerator and an unstable denominator. Using gain feedback there is only a small range (K≈2 to 3) of gain for which the system is stable, Figs(7a) & (7b).

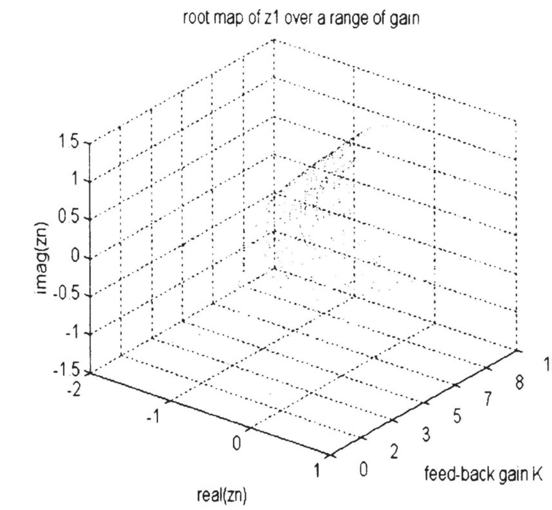

Fig. 7a 3-d rootmap (w.r.t. z_1) of Eq(3).

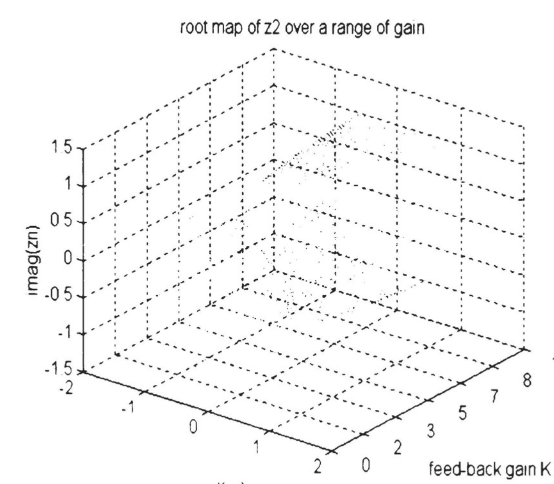

Fig. 7b 3-d rootmap (w.r.t. z_2) of Eq(3).

Now suppose compensation is added in the form of 5 1-dimensional separable zeros in each dimension, i.e.:

$$nH = \prod \left(z_1^{-1} - 0.7e^{in}\right)\left(z_2^{-1} - 0.7e^{in}\right)$$

(4)

where $n = \frac{2}{5}k\pi$ for k=0, 1, 2,.....,5 and nH is the numerator of the forward compensator H.

The compensator zeros act as a ring of attractors in each closed loop rootmap.

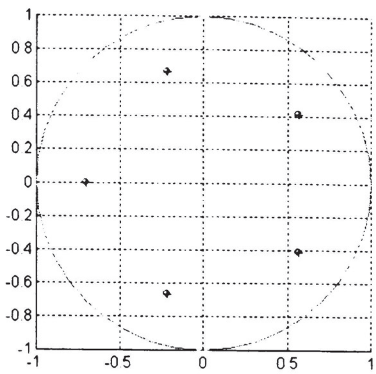

Fig. 8 Rootmap of a separable function in the form of Eq(4)

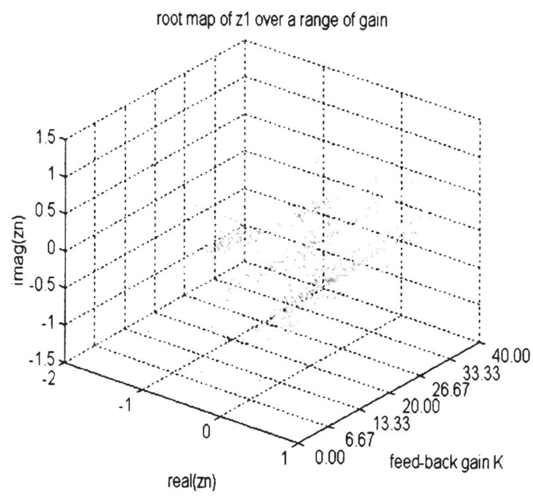

Fig. 9a 3-d rootmap (w.r.t. z_1) of closed loop system with separable compensator zeros.

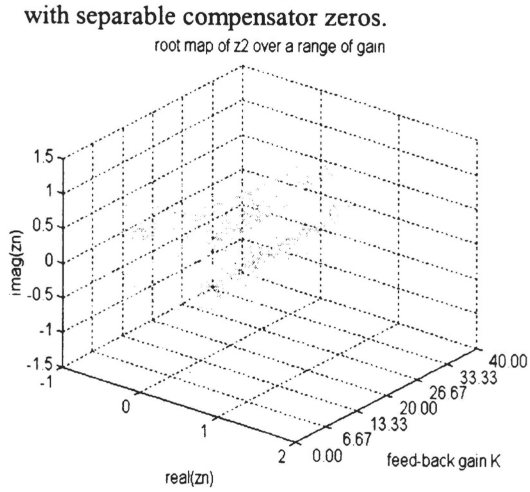

Fig. 9b 3-d rootmap (w.r.t. z_2) of closed loop system with separable compensator zeros.

Figs 9a, 9b show that the compensated system has a stability range from K = ~6 to K = ~35. However, the controller is 10th order!

A better alternative is to examine non-separable 2-dimensional compensators that have similar rootmaps. A simple function with a predictable rootmap is:

$$nH = 1+az_1^{-1}z_2^{-1} \tag{5}$$

Whose rootmaps form a circle of radius 'a'.

Thus, by adding a compensator:

$$(nH(z_1^{-1}, z_2^{-1}) = 1+0.7z_1^{-1}z_2^{-1}) \tag{6}$$

to the open loop system, the closed loop rootmaps of Figs 10a, 10b are obtained. Here the range of stability is from K=~2 to ~35. This far exceeds that using gain feedback, and is about the same as that obtained by using separable one dimensional zeros, except here the controller is only 2nd order.

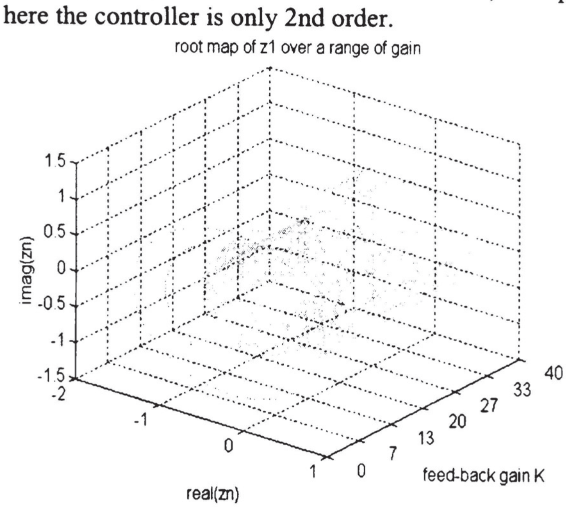

Fig. 10a 3-d rootmap (w.r.t. z_1) of closed loop system with non-separable compensator zeros.

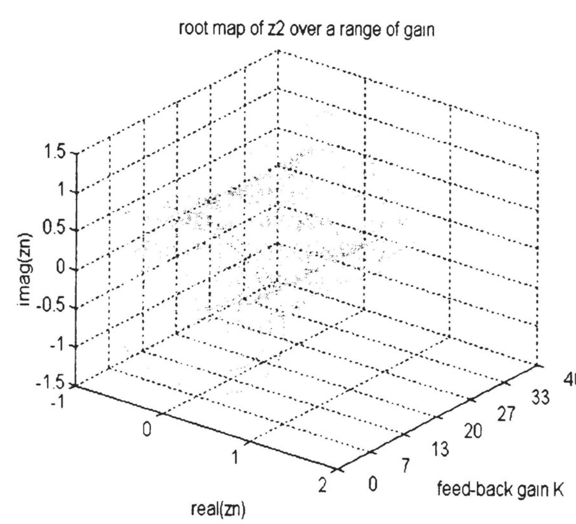

Fig. 10b 3-d rootmap (w.r.t. z_2) of closed loop system with non-separable compensator zeros.

4. CONCLUSIONS

It has been demonstrated that the notion of root maps, used to assess the stability of 2-dimensional filters, can be extended to offer a root locus-like tool for 2-dimensional system design. Ongoing work is focused on techniques for design of robust, low order compensators. It can also be shown that a similar approach can be taken with continuous systems.

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