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H_∞ Control for Singularly Perturbed Bilinear Systems with Parameter Uncertainties Using Successive Galerkin Approximation

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Abstract: This paper presents a new algorithm for the closed-loop H_∞ composite control of singularly perturbed bilinear systems with time-varying parameter uncertainties and exogenous disturbance using the successive Galerkin approximation (SGA). The singularly perturbed bilinear system is decomposed into two subsystems of a slow-time scale and a fast-time scale via singular perturbation theory, and then two H_∞ control laws are obtained for each subsystem. H_∞ control theory guarantees robust closed-loop performance but the resulting problem is difficult to solve for bilinear systems. In order to overcome the difficulties inherent in the H_∞ control problem, the suitable robust H_∞ feedback control law can be constructed in term of the approximated solution to a Hamilton-Jacobi-Isaac equation using SGA. The composite control law consists of H_∞ control laws for each subsystem.

1. INTRODUCTION

The major importance of bilinear systems indeed lies in their applications to the real world systems, for example: the basic law of mass action, dynamics of heat exchanger with controlled flow, DC motor and induction motor drives, mechanical brake system [1, 2, 3]. These bilinear systems are linear in control and linear in state but not jointly linear in state and control. It is important to understand its real properties or to guarantee the global stability or to improve the performance by applying the various control techniques to bilinear system rather than its linearized system since the linearization of bilinear system loses its nature property [1, 2, 4, 5].

Singular perturbation theory has been a highly recognized and rapidly developing area of control system research in the last thirty years, and control methods to solve the singularly perturbed systems have received much attention by many researchers [2, 6, 7]. Recently, an excellent survey of the applications of the theory and control methods of singular perturbation and time scales and the importance features of the singularly perturbed systems have been reported in [8]. Also very efficient and high accurate optimal control methods for both continuous time and discrete time singularly perturbed linear systems are found in a recent book [9]. In the class of optimal control [2, 6, 7, 9, 10], design of the control law for the singularly perturbed system has ill-defined numerical problems. To avoid these problems, the full order system is decomposed into reduced slow and fast subsystems, and optimal control laws are designed for each subsystem. Thus, the near-optimal composite control law consists of two optimal sub-control laws [2, 6, 7, 9, 11].

Recently, robust control is issued and developed by many researchers for linear systems [12, 13, 14]. But in the class of bilinear and nonlinear systems, because conditions for the solvability of the robust H_∞ control design problem are hard, still there are a lot of problems to be developed. For bilinear and nonlinear systems, the H_∞ optimal control problem is reduced to the solution of the Hamilton-Jacobi-Isaac (HJI) equation, which is a nonlinear partial differential equation (PDE) [15, 16]. The solution of a nonlinear PDE is extremely difficult to solve and so researchers have looked for methods of approximating its solution. Specially, the practical method named successive Galerkin approximation (SGA) to improve a stabilizing feedback control were proposed in [15, 17]. The problem of improving the closed-loop performance of a stabilizing H_∞ control can be reduced to solving a first-order, linear PDE called the Generalized-Hamilton-Jacobi-Isaac (GHJI) equation [15]. An interesting fact is that when the process is iterated, the solution to the GHJI equation converges uniformly to the solution of the HJI equation which solves the H_∞ optimal control problem [15]. In addition, [17] shows how to find a uniform approximation to the HJI equation such that the approximate controls are still stable on a specified set using SGA. However, the SGA method has the difficulty that the complexity of computations increases according to order of system.

In this paper, we focus on the class of a H_∞ feedback control for singularly perturbed bilinear systems with time-varying parameter uncertainties and exogenous disturbance. In order to obtain the closed-loop H_∞ control law using the SGA method, one must compute n−dimensional integrals, and the number of computations increases according to n. Therefore, the full order system is decom-

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posed into the reduced order subsystems via singular perturbation theory, and we define GHJI equations for each subsystem, and then, we can obtain two sub-control laws for each subsystem through SGA method. In this case, n_1- and n_2-dimensional integrals are computed and the number of computations are decreased, where $n = n_1+n_2$. Thus, the near-optimal H_∞ composite control law consists of two optimal H_∞ sub-control laws.

2. H_∞ CONTROL FOR SINGULARLY PERTURBED BILINEAR SYSTEMS WITH PARAMETER UNCERTAINTIES

The infinite-time H_∞ control problem considers a class of singularly perturbed bilinear systems with parameter uncertainties and exogenous disturbances described by the following differential equations:

$$\dot{\alpha} = (A_1 + \Delta A)\alpha + A_2\beta + H\omega + (B_1 + \{\alpha M_1\})u \quad (1)$$

$$\epsilon\dot{\beta} = A_3\alpha + A_4\beta + (B_2 + \{\beta M_2\})u \quad (2)$$

$$z = \begin{bmatrix} \begin{bmatrix} C_1\alpha \\ C_2\beta \end{bmatrix} \\ Du \end{bmatrix} \quad (3)$$

$$\alpha(t_0) = \alpha^0, \qquad \beta(t_0) = \beta^0$$

with respect to the performance criterion:

$$J = \int_0^\infty \left(z^T z - \gamma^2\omega^T\omega\right) dt \quad (4)$$

where $\alpha \in R^{n_1}$, and $\beta \in R^{n_2}$ are states, $u \in R^m$ is a control input, $z \in R^q$ is a controlled output, $\omega \in R^p$ is a exogenous disturbance, A_1, A_2, A_3, A_4, B_1, B_2, M_1, M_2, C_1, C_2, D are constant matrices of appropriate dimensions, ϵ is a small positive parameter, and γ is a positive design parameter. The notation used for the bilinear term in (1-2) means $\{\alpha M_1\} = \sum_{j=1}^{n_1} \alpha_j M_{1j}$ and $\{\beta M_1\} = \sum_{j=1}^{n_2} \beta_j M_{2j}$, and we define that $\widetilde{B}_1 \equiv B_1 + \{\alpha M_1\}$ and $\widetilde{B}_2 \equiv B_2 + \{\beta M_2\}$. In addition, $\Delta A \in R^{n_1\times n_1}$ represents the uncertainty in the system and satisfy the following assumption.

Assumption 1:

$$\Delta A(t) = E_1 Q(t) E_2 \quad (5)$$

where E_1 and E_2 are known real constant matrices with appropriate dimensions and $Q(t)$ is an unknown matrix function with Lebesgue measurable elements such that $Q(t)^T Q(t) \le I$. $\diamondsuit$

For computational simplification, without loss of generality we assume that $D^T D = I$. Therefore, the performance criterion (4) can be written in the equivalent form:

$$J = \int_0^\infty \left(\alpha^T C_1^T C_1\alpha + \beta^T C_2^T C_2\beta + u^T u - \gamma^2\omega^T\omega\right) dt. \quad (6)$$

By the help of [7], we solve slow and fast optimal control problems and combine their solutions to form a composite control:

$$u_c = u_s^* + u_f^*. \quad (7)$$

Let us assume that the open-loop system (1-3) is a standard singularly perturbed system for every u_s, that is namely, the equation:

$$\beta_s = -A_4^{-1}\left(A_3\alpha_s + \widetilde{B}_2 u_s\right) \quad (8)$$

has a unique solution.

The slow time scale problem of order n_1 can be defined by eliminating β_f and u_f from (1-3) and (6) using (8). Then the resulting slow time scale problem becomes control of the following slow subsystem:

$$\dot{\alpha}_s = \left(A_0 + \Delta A\right)\alpha_s + H\omega + \widetilde{B}_s u_s \quad (9)$$
$$\alpha_s(t_0) = \alpha^0$$

with respect to the performance criterion:

$$J_s = \int_0^\infty \left(\alpha_s^T C_0\alpha_s + 2\alpha_s^T L_s u_s + u_s^T D_s u_s - \gamma^2\omega^T\omega\right) dt \quad (10)$$

where

$$\begin{aligned} A_0 &= A_1 - A_2A_4^{-1}A_3 \\ \widetilde{B}_s &= \widetilde{B}_1 - A_2A_4^{-1}\widetilde{B}_2 \\ C_0 &= C_1^T C_1 + A_3^T A_4^{-T} C_2^T C_2 A_4^{-1} A_3 \\ L_s &= A_3^T A_4^{-T} C_2^T C_2 A_4^{-1}\widetilde{B}_2 \\ D_s &= I + \widetilde{B}_2^T A_4^{-T} C_2^T C_2 A_4^{-1}\widetilde{B}_2. \end{aligned}$$

By the help of [13, 16], the approach adopted in this paper for solving the robust H_∞ control problem involves solving a parameter-dependent HJI equation associated with an H_∞ performance criterion and the uncertainty in the state function. Given the bilinear system (1-3), any desired γ, and some σ and δ, we can define the following HJI equation corresponding to the problem of quadratic stabilization:

$$\frac{\partial J_s^*}{\partial\alpha_s}^T A_s\alpha_s - \frac{1}{4}\frac{\partial J_s^*}{\partial\alpha_s}^T \Bigg(\widetilde{B}_s D_s^{-1}\widetilde{B}_s^T - \gamma^{-2}HH^T - 2\sigma E_1E_1^T\Bigg)\frac{\partial J_s^*}{\partial\alpha_s} + \alpha_s^T\left(\frac{1}{2\sigma}E_2^T E_2 + C_s + \delta I\right)\alpha_s = 0 \quad (11)$$

with the boundary condition:

$$J_s^*(0) = 0 \quad (12)$$

where

$$\begin{aligned} A_s &= A_0 - \widetilde{B}_s D_s^{-1} L_s^T \\ C_s &= C_0 - L_s D_s^{-1} L_s^T. \end{aligned}$$

Moreover, from robust H_∞ control theory [12], [?], it is well known that if $J_s^*(\alpha_s)$ is a unique positive-definite solution of (11). Then, the H_∞ control of the slow time scale problem is given by

$$u_s^* = -D_s^{-1}\left(L_s^T + \frac{1}{2}\widetilde{B}_s^T P_s\right)\alpha_s = G_s\alpha_s \quad (13)$$

where $P_s\alpha_s = \partial J_s^*/\partial\alpha_s$.

The fast time scale problem of order n_2 is defined by freezing the slow variable α_s and shifting the equilibrium of the fast subsystem to the origin.

$$\epsilon\dot{\beta}_f = A_4\beta_f + \widetilde{B}_2 u_f \tag{14}$$
$$\beta_f(t_0) = \beta^0 + A_4^{-1}\{A_3(\alpha^0) + \widetilde{B}_2 u_s(t_0)\}$$

where $\beta_f = \beta - \beta_s$. The performance criterion of the fast time scale problem is given by

$$J_f = \int_0^\infty \big(\beta_f^T C_2^T C_2 \beta_f + u_f^T u_f\big)dt. \tag{15}$$

If $J_f^*(\beta_f)$ is a unique positive-definite solution of the HJI equation:

$$\frac{\partial J_f^*}{\partial \beta_f}^T A_4\beta_f - \frac{1}{4}\frac{\partial J_f^*}{\partial \beta_f}^T \widetilde{B}_2\widetilde{B}_2^T \frac{\partial J_f^*}{\partial \beta_f} + \beta_f^T C_2^T C_2\beta_f = 0 \tag{16}$$

with the boundary condition

$$J_f^*(0) = 0 \tag{17}$$

then the H_∞ control of the fast time scale problem is given by

$$u_f^* = -\frac{1}{2}\widetilde{B}_2^T P_f\beta_f = G_f\beta_f \tag{18}$$

where $P_f\beta_f = \partial J_f^*/\partial\beta_f$.

A realizable composite control requires that the system states α_s and β_f be expressed in terms of the actual system states α and β. Specifically, this can be achieved by replacing α_s by α and β_f by $\beta - \beta_s$ so that

$$u_c = G_s\alpha + G_f\big[\beta + A_4^{-1}(A_3\alpha + \widetilde{B}_2 G_s\alpha)\big]. \tag{19}$$

3. H_∞ COMPOSITE CONTROL USING SUCCESSIVE GALERKIN APPROXIMATION

In order to obtain the H_∞ composite control law u_c, we need to find the solutions, $\partial J_s^*/\partial\alpha_s$ and $\partial J_f^*/\partial\beta_f$. In this section, we present the new algorithm to obtain approximation of these solutions using SGA.

3.1 GHJI equations to H_∞ composite control

Assumption 2
Ω_s and Ω_f are compact sets of R^{n_1} and R^{n_2}, respectively. Slow and fast time scale states are bounded on Ω_s and Ω_f, respectively. ♢

Under Assumption and by the help of [11, 17], we can define the GHJI equations for singular perturbed bilinear systems.

Definition 1
If initial control laws, $u_s^{(0)} : R^m \times \Omega_s \longrightarrow R^m$ and $u_f^{(0)} : R^m \times \Omega_f \longrightarrow R^m$, are admissible and functions, $J_s^{(i)} : R^{n_1} \times \Omega_s \longrightarrow R^{n_1}$ and $J_f^{(i)} : R^{n_2} \times \Omega_f \longrightarrow R^{n_2}$, satisfy the following Generalized-Hamilton-Jacobi-Isaacs equations, written by $GHJI(J_s^{(i)}, u_s^{(i)}) = 0$, namely

$$\begin{aligned}
&\frac{1}{4}\frac{\partial J_s^{(i-1)}}{\partial\alpha_s}^T\Big(\widetilde{B}_s D_s^{-1}\widetilde{B}_s^T - \gamma^{-2}HH^T - 2\sigma E_1E_1^T\Big)\frac{\partial J_s^{(i-1)}}{\partial\alpha_s} \\
&-\frac{1}{2}\frac{\partial J_s^{(i)}}{\partial\alpha_s}^T\Big(\widetilde{B}_s D_s^{-1}\widetilde{B}_s^T - \gamma^{-2}HH^T - 2\sigma E_1E_1^T\Big)\frac{\partial J_s^{(i-1)}}{\partial\alpha_s} \\
&+\frac{\partial J_s^{(i)}}{\partial\alpha_s}^T A_s\alpha_s + \alpha_s^T\big(\frac{1}{2\sigma}E_2^T E_2 + C_s + \delta I\big)\alpha_s = 0
\end{aligned} \tag{20}$$

with boundary condition:

$$J_s^{(i)}(0) = 0, \tag{21}$$

here i-th slow control law is given by

$$u_s^{(i)} = -\frac{1}{2}D_s^{-1}\left(L_s^T\alpha_s + \frac{1}{2}\widetilde{B}_s^T\frac{\partial J_s^{(i-1)}}{\partial\alpha_s}\right), \tag{22}$$

and $GHJI(J_f^{(i)}, u_f^{(i)}) = 0$, namely

$$\begin{aligned}
&\frac{\partial J_f^{(i)}}{\partial\beta_f}^T A_4\beta_f + \frac{1}{4}\frac{\partial J_f^{(i-1)}}{\partial\beta_f}^T \widetilde{B}_2\widetilde{B}_2^T\frac{\partial J_f^{(i-1)}}{\partial\beta_f} \\
&-\frac{1}{2}\frac{\partial J_f^{(i)}}{\partial\beta_f}^T \widetilde{B}_2\widetilde{B}_2^T\frac{\partial J_f^{(i-1)}}{\partial\beta_f} + \beta_f^T C_2^T C_2\beta_f = 0
\end{aligned} \tag{23}$$

with boundary condition:

$$J_f^{(i)}(0) = 0, \tag{24}$$

here i-th fast control law is given by

$$u_f^{(i)} = -\frac{1}{2}\widetilde{B}_2^T\frac{\partial J_f^{(i-1)}}{\partial\beta_f}, \tag{25}$$

and i is iteration number.

3.2 Galerkin projections of the GHJB equations

In this section, we use Galerkin's projection method to derive approximate solutions to the GHJI equations on the compact set, Ω. We find an approximate solution $J_N^{(i)}$ to the equation $GHJB(J^{(i)}, u^{(i)}) = 0$ by letting

$$J_N^{(i)}(x) = \sum_{j=1}^N c_j^{(i)}\phi_j(x) \tag{26}$$

where the coefficients c_j are constant in the infinite-time case. Substituting this expression into the GHJI equation results in an approximation error:

$$error = GHJI\Big(\sum_{j=1}^N c_j^{(i)}\phi_j, u^{(i)}\Big). \tag{27}$$

The coefficients c_j are determined by setting the projection of the error (27) on the finite basis, $\{\phi_j\}_1^N$, to zero for all states, $x \in \Omega$,

$$\langle GHJB\Big(\sum_{j=1}^N c_j^{(i)}\phi_j, u^{(i)}\Big), \phi_n\rangle_\Omega = 0, \quad n = 1, \cdots, N \tag{28}$$

Then (28) becomes N equations with N unknown constants.

To represent (28) by the matrix equations, we define

$$\Phi_N(x) \equiv (\phi_1(x), \cdots, \phi_N(x))^T \tag{29}$$

and let $\nabla\Phi_N$ be the Jacobian Φ_N. If $\eta : R^N \longrightarrow R^N$ is a vectoer valued function, then we define the notation

$$\langle \eta, \Phi_N \rangle_\Omega \equiv \begin{bmatrix} \langle \eta_1, \phi_1 \rangle_\Omega & \cdots & \langle \eta_N, \phi_1 \rangle_\Omega \\ \vdots & \ddots & \vdots \\ \langle \eta_1, \phi_N \rangle_\Omega & \cdots & \langle \eta_N, \phi_N \rangle_\Omega \end{bmatrix}$$

where the inner product is defined as

$$\langle f, g \rangle_\Omega \equiv \int_\Omega f(x)g(x)dx, \tag{30}$$

and then

$$J_N \equiv \mathbf{c}_N^T \Phi_N \tag{31}$$

where $\mathbf{c}_N \equiv (c_1, c_2, \cdots, c_N)^T$.

Given an initial control $u_s^{(0)}$, we compute an approximation to its cost $J_{sN}^{(0)} = \mathbf{c}_{sN}^{T(0)} \Phi_{sN}$ where $\mathbf{c}_{sN}^{(0)}$ is the solution of Galerkin approximation of GHJB equation (20), i.e.,

$$\mathbf{a}_s^{(0)} \mathbf{c}_{sN}^{(0)} + \mathbf{b}_s^{(0)} = 0 \tag{32}$$

where

$$\begin{aligned} \mathbf{a}_s^{(0)} &= \langle \nabla\Phi_{sN} A_0 \alpha_s, \Phi_{sN} \rangle_{\Omega_s} + \langle \nabla\Phi_{sN} \widetilde{B}_s u_s^{(0)}, \Phi_{sN} \rangle_{\Omega_s} \\ \mathbf{b}_s^{(0)} &= \langle \alpha_s^T C_0 \alpha_s, \Phi_{sN} \rangle_{\Omega_s} \\ &\quad + \langle 2\alpha_s^T L_s u_s^{(0)} + u_s^{(0)^T} D_s u_s^{(0)}, \Phi_{sN} \rangle_{\Omega_s}. \end{aligned}$$

Here we can compute the updated control law that is based on the approximated solution, $J_{sN}^{(i-1)}$.

$$u_s^{(i)} = -D_s^{-1} \left(L_s^T \alpha_s - \frac{1}{2} \widetilde{B}_s^T \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)} \right) \tag{33}$$

Then we can obtain the approximation:

$$J_{sN}^{(i)} = \mathbf{c}_{sN}^{T(i)} \Phi_{sN} \tag{34}$$

here $\mathbf{c}_{sN}^{(i)}$ is the solution of

$$\mathbf{a}_s^{(i)} \mathbf{c}_{sN}^{(i)} + \mathbf{b}_s^{(i)} = 0 \tag{35}$$

where

$$\begin{aligned} \mathbf{a}_s^{(i)} &= \langle \nabla\Phi_{sN} A_s \alpha_s, \Phi_{sN} \rangle_{\Omega_s} - \frac{1}{2} \langle \nabla\Phi_{sN} (\widetilde{B}_s D_s^{-1} \widetilde{B}_s^T \\ &\quad - \gamma^{-2} H H^T - 2\sigma E_1 E_1^T) \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN} \rangle_{\Omega_s} \\ \mathbf{b}_s^{(i)} &= \langle \alpha^T \big(C_s + \frac{1}{2\sigma} E_2^T E_2 + \delta I \big) \alpha_s, \Phi_{sN} \rangle_{\Omega_s} \\ &\quad + \frac{1}{4} \langle \mathbf{c}_{sN}^{T(i-1)} \nabla\Phi_{sN} (\widetilde{B}_s D_s^{-1} \widetilde{B}_s^T \\ &\quad - \gamma^{-2} H H^T - 2\sigma E_1 E_1^T) \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN} \rangle_{\Omega_s} \end{aligned}$$

and i is iteration number.

Similarly, given an initial control $u_f^{(0)}$, we can compute an approximation to the cost $J_{fN}^{(i)} = \mathbf{c}_{fN}^{T(i)} \Phi_{fN}$ where $\mathbf{c}_{fN}^{(i)}$ is the solution of Galerkin approximation of GHJB equation for the fast-time case. The following lemma states the existence of unique solutions, $\mathbf{c}_{sN}^{(i)}$ and $\mathbf{c}_{fN}^{(i)}$ of Galerkin approximation.

Lemma 1
Suppose that $\{\phi_{sj}\}_1^N$ and $\{\phi_{fj}\}_1^N$ are linearly independent respectively, then $\mathbf{a}_s$ and $\mathbf{a}_f$ are invertible. Furthermore, existence of the unique solutions is guaranteed.
The proof of this lemma can be drawn from (Randal W. Beard, 1995, [17]).

3.3 The new algorithm to H_∞ composite control

The following algorithm presents that the H_∞ composite control can be designed by two closed-loop control laws of fast- and slow-subsystem using the SGA method for singularly perturbed bilinear systems.

Algorithm

Initial Step

Compute

$$\begin{aligned} \mathbf{a}_s^{(0)} &= \langle \nabla\Phi_{sN} A_0 \alpha_s, \Phi_{sN} \rangle_{\Omega_s} + \langle \nabla\Phi_{sN} \widetilde{B}_s u_s^{(0)}, \Phi_{sN} \rangle_{\Omega_s} \\ \mathbf{b}_s^{(0)} &= \langle \alpha_s^T C_0 \alpha_s, \Phi_{sN} \rangle_{\Omega_s} \\ &\quad + \langle 2\alpha_s^T L_s u_s^{(0)} + u_s^{(0)^T} D_s u_s^{(0)}, \Phi_{sN} \rangle_{\Omega_s} \\ \mathbf{a}_f^{(0)} &= \langle \nabla\Phi_{fN} A_4 \beta_f, \Phi_{fN} \rangle_{\Omega_f} + \langle \nabla\Phi_{fN} \widetilde{B}_2 u_f^{(0)}, \Phi_{fN} \rangle_{\Omega_f} \\ \mathbf{b}_f^{(0)} &= \langle \beta_f^T C_2^T C_2 \beta_f, \Phi_{fN} \rangle_{\Omega_f} + \langle u_f^{(0)^T} u_f^{(0)}, \Phi_{fN} \rangle_{\Omega_f}. \end{aligned}$$

Find $\mathbf{c}_{sN}^{(0)}$ and $\mathbf{c}_{fN}^{(0)}$ satisfying the following linear equations:

$$\mathbf{a}_s^{(0)} \mathbf{c}_{sN}^{(0)} + \mathbf{b}_s^{(0)} = 0, \qquad \mathbf{a}_f^{(0)} \mathbf{c}_{fN}^{(0)} + \mathbf{b}_f^{(0)} = 0.$$

Set $i = 1$.

Iterative Step

Improved controllers are given by

$$\begin{aligned} u_s^{(i)} &= -D_s^{-1} \left(L_s^T - \frac{1}{2} \widetilde{B}_s^T P_s^{(i)} \right) \alpha_s = G_s^{(i)} \alpha_s, \\ u_f^{(i)} &= -\frac{1}{2} \widetilde{B}_2^T P_f^{(i)} \beta_f = G_f^{(i)} \beta_f. \end{aligned}$$

where $P_s^{(i)} \alpha_s = \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}$ and $P_f^{(i)} \beta_f = \nabla\Phi_{fN}^T \mathbf{c}_{fN}^{(i-1)}$.

Compute

$$\begin{aligned} \mathbf{a}_s^{(i)} &= \langle \nabla\Phi_{sN} A_s \alpha_s, \Phi_{sN} \rangle_{\Omega_s} - \frac{1}{2} \langle \nabla\Phi_{sN} (\widetilde{B}_s D_s^{-1} \widetilde{B}_s^T \\ &\quad - \gamma^{-2} H H^T - 2\sigma E_1 E_1^T) \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN} \rangle_{\Omega_s} \\ \mathbf{b}_s^{(i)} &= \langle \alpha^T \big(C_s + \frac{1}{2\sigma} E_2^T E_2 + \delta I \big) \alpha_s, \Phi_{sN} \rangle_{\Omega_s} \\ &\quad + \frac{1}{4} \langle \mathbf{c}_{sN}^{T(i-1)} \nabla\Phi_{sN} (\widetilde{B}_s D_s^{-1} \widetilde{B}_s^T \\ &\quad - \gamma^{-2} H H^T - 2\sigma E_1 E_1^T) \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN} \rangle_{\Omega_s} \\ \mathbf{a}_f^{(i)} &= \langle \nabla\Phi_{fN} A_4 \beta_f, \Phi_{fN} \rangle_{\Omega_f} \\ &\quad - \frac{1}{2} \langle \nabla\Phi_{fN} \widetilde{B}_2 \widetilde{B}_2^T \nabla\Phi_{fN}^T \mathbf{c}_{fN}^{(i-1)}, \Phi_{fN} \rangle_{\Omega_f} \\ \mathbf{b}_f^{(i)} &= \langle \beta_f^T C_2^T C_2 \beta_f, \Phi_{fN} \rangle_{\Omega_f} \end{aligned}$$

$$+\frac{1}{4}\langle \mathbf{c}_{fN}^{T(i-1)}\nabla\Phi_{fN}\widetilde{B}_2\widetilde{B}_2^T\nabla\Phi_{fN}^T\mathbf{c}_{fN}^{(i-1)},\Phi_{fN}\rangle_{\Omega_f}.$$

Find $\mathbf{c}_{sN}^{(i)}$ and $\mathbf{c}_{fN}^{(i)}$ satisfying the following linear equations:

$$\mathbf{a}_s^{(i)}\mathbf{c}_{sN}^{(i)}+\mathbf{b}_s^{(i)}=0, \qquad \mathbf{a}_f^{(i)}\mathbf{c}_{fN}^{(i)}+\mathbf{b}_f^{(i)}=0.$$

Set $i=i+1$.

Final Step

The H_∞ composite control law is

$$u_c=G_s\alpha+G_f\left[\beta+A_4^{-1}(A_3\alpha+\widetilde{B}_2G_s\alpha)\right] \qquad \diamondsuit$$

4. A NUMERICAL EXAMPLE

Consider a fourth-order example representing the physical model of induction motor drives (Figalli et al., 1984, [3]) with parameter uncertainties and exogenous disturbance. The states variables are given by $\alpha=[x_1\ x_2]^T=[\phi_{ds}\ \phi_{qs}]^T$ and $\beta=[x_3\ x_4]^T=[i_{ds}\ i_{qs}]^T$, and the control variable is given by $u=[u_1\ u_2\ u_3]^T=[v_{ds}\ v_{qs}\ w_s]^T$, where ϕ_{ds} and ϕ_{qs} are projections of the stator flux, i_{ds} and i_{qs} are projections of stator current, v_{ds} and v_{qs} are projections of the supply voltages, and w_s is a slip angular frequency. The problem matrices of a bilinear system represented in (1-3) have the following values:

$$A_1=\begin{bmatrix}0 & 321.57\\ -321.57 & 0\end{bmatrix},\ A_2=\begin{bmatrix}-0.312 & 0\\ 0 & -312\end{bmatrix},$$
$$A_3=\begin{bmatrix}98.87 & 27059\\ -27059 & 98.87\end{bmatrix},\ A_2=\begin{bmatrix}-44.93 & 2.57\\ -2.57 & -44.93\end{bmatrix},$$
$$B_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -7.3\end{bmatrix},\ B_2=\begin{bmatrix}87.3 & 0 & 87.8\\ 0 & 87.3 & -53\end{bmatrix},$$
$$M_{11}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix},\ M_{12}=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix},$$
$$M_{21}=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ M_{22}=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix},$$
$$H=[1\ 1\ 0\ 0]^T,\ C_1=I_2,\ C_2=I_2,\ D=I_3.$$

In this paper, we choose that $\epsilon=0.01$. We assume that the parameter uncertain matrices, $E_1=[1\ 1\ 0\ 0]^T$, $E_2=[21\ 18\ 0\ 0]$ and $Q(t)=\sin(16\pi t)$, and the exogenous disturbance, $\omega=0.5\sin(74\pi t)-0.4\cos(83\pi t)$. The simulation results are presented with initial states, $[\alpha(t_0)\ \beta(t_0)]=[-0.07\ 0.04\ 15\ 47]$, in the figures 1-4, where every upper plots are the state trajectories obtained from full-order SGA method and lower plots are the state trajectories obtained from proposed algorithm. In the figure 5, a dashed line, (−−), is the trajectory of performance criterion obtained from full-order SGA method and a sold line, (—-), is the trajectory of performance criterion obtained from proposed algorithm. The figures show that the proposed algorithm is more robust than the full-order SGA method, because errors of the full-order SGA method are bigger than those of the proposed algorithm. In this simulation, for the full-order SGA method, 7-dimensional basis and computed 4-dimensional integrals for $7\times(1+7+49)=399$ times. But, in the proposed algorithm, we can use only 3-dimensional basis and computed 2-dimensional integrals for $3\times(1+3+9)=39$ times for each subsystem, respectively. Therefore, we can say that the computational complexity is greatly reduced without loss of performance.

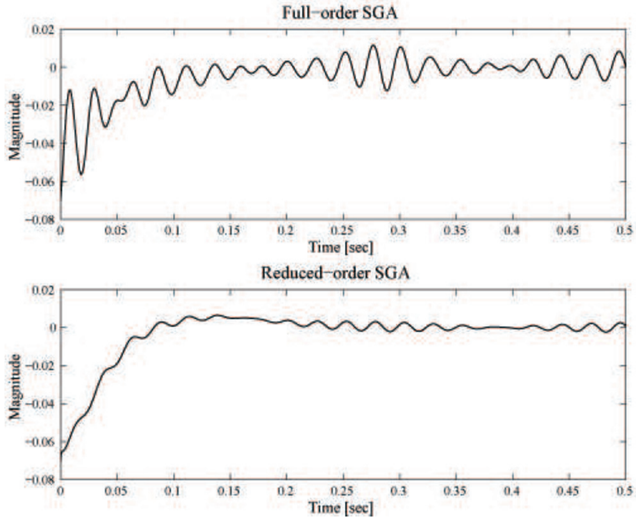

Fig. 1. Trajectories of x_1

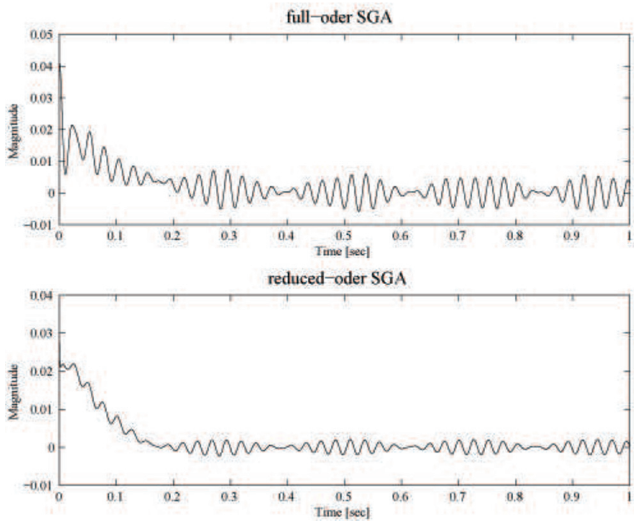

Fig. 2. Trajectories of x_2

5. CONCLUSIONS

In order to overcome the difficulties inherent in the robust H_∞ control problem for singularly perturbed bilinear systems with time-varying parameter uncertainties and exogenous disturbance, we have presented the closed-loop H_∞ composite control scheme and a new algorithm using the SGA. In this paper, the suitable robust H_∞ composite control law has been designed by H_∞ control laws of slow and fast subsystems. Each control law has been constructed in term of approximated solution to each Hamilton-Jacobi-Isaac equation. The advantages of proposed algorithm are as follows: (i) all of the computations can be performed off-line, (ii) the resulting control law is in feedback form, (iii) the algorithm guarantees uniformly the H_∞ performance, (iv) computational complexities can be greatly reduced. Through the presented simulation results, it should be noted that the proposed algorithm are more effective than the full order SGA method.

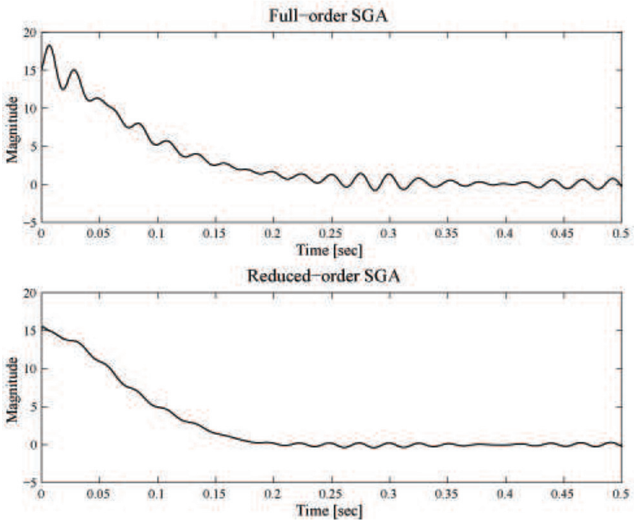

Fig. 3. Trajectories of x_3

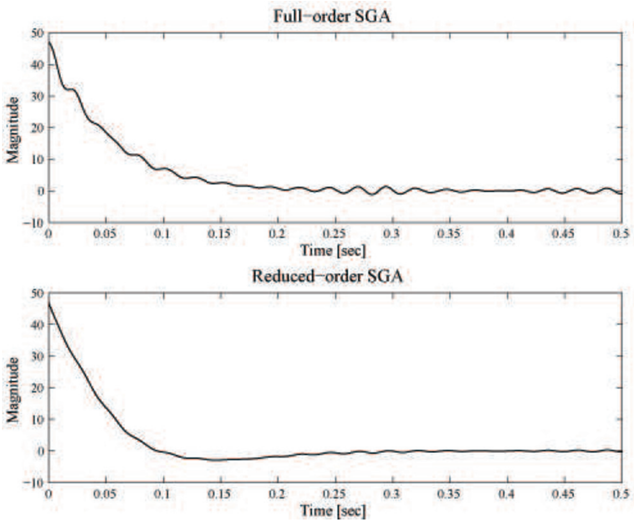

Fig. 4. Trajectories of x_4

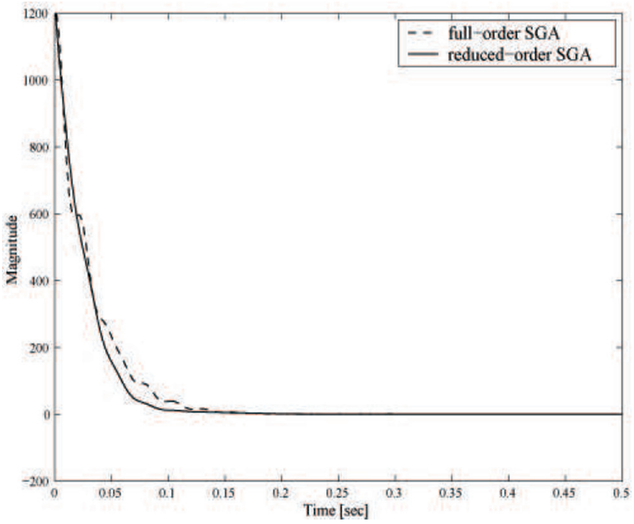

Fig. 5. Trajectories of Performance Criterion

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