AN APPLICATION OF SLIDING MODE CONTROL FOR TWO-WHEELED VEHICLE WITH NONHOLONOMIC CONSTRAINTS

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Abstract: This paper considers motion control of a riderless two-wheeled vehicle and proposes a systematic way to obtain a nonholonomic system's state equation from its Lagrange equation to design a control law. The proposed method is used to derive the state equation of a two-wheeled vehicle. Then, a sliding mode control method is applied to control the vehicle's motion. The controller's effectiveness has been clarified through computer simulations. *Copyright© 2000 IFAC*

Keywords: Nonholonomic Constraint, Nonlinear Control, Sliding Mode Control, Robust Control, Two-Wheeled Vehicle

1. INTRODUCTION

The wheel is one of the most important elements of machine systems. Analysis of dynamic systems with wheels is generally based on the assumption that wheels rotate without any slip. Therefore, wheel contact is constrained to have velocities parallel to the lines of intersection of wheel plane and ground plane. This condition is described by nonlinear differential equations that cannot be integrated. Such constraints are called nonholonomic constraints. Nonholonomic constraint conditions make it hard to analyze the system's dynamics and control. No systematic way has yet been developed to treat a nonholonomic system.

Some nonholonomic systems have recently been studied as control objects, for example, machine systems with several wheels, such as cars, bicycles and motorcycles, or which are operated in space or under water. Nonholonomic systems are very interesting from the viewpoint of theoretical automatic control, since their state equation is described as nonlinear, which is hard to control. Two-wheeled vehicles such as bicycles and motorcycles are particularly interesting systems as control objects, since they become nonholonomic systems under the assumption that their wheels rotate without any slip. Furthermore, they are unstable systems. Some studies have been carried out on control of two-wheeled vehicles. Sharp considered the model of a motorcycle with two frames. He described typical stability characteristics and how they depend on various parameters (R.S.Sharp, 1971). Getz proposed a method of reducing the nonlinear differential-algebraic equation of a two-wheeled vehicle's motion to a reduced state-space. A feedback control law was derived for the vehicle's model to stably track arbitrary smooth trajectories of roll angle and rear wheel velocity (Getz, 1994). Yavin considered deriving a pedaling moment and a directional moment for a bicycle when it moves between two differential points in a horizontal plane. A feedback control law was used to control its motion (Y.Yavin, 1998).

In these studies, the state equation of a two-wheeled vehicle have been described as a nonlinear form that can't be controlled using general

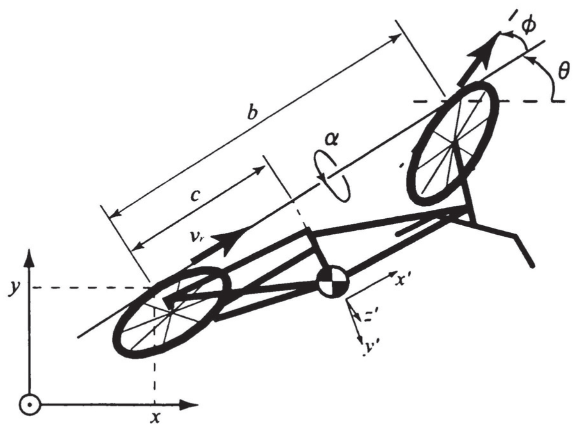

Fig. 1. Top view of two-wheeled vehicle model

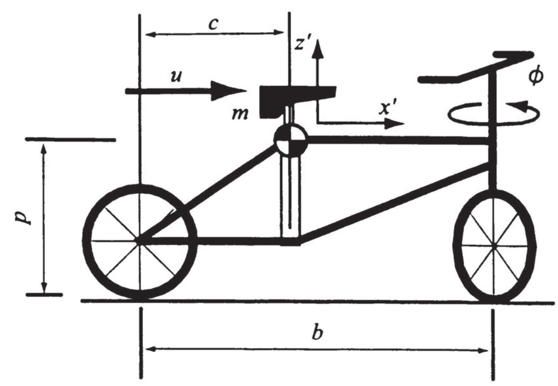

Fig. 2. Side view of two-wheeled vehicle model with roll angle $\alpha = 0$.

linear control theory. Thus, most of them were carried out using a feedback linearization method based on a state feedback method (Isidori, 1995). However, a feedback linearization control law is very sensitive to errors between models and actual systems.

This paper considers motion control of a riderless two-wheeled vehicle. To succeed in this work, the authors proposed a systematic way to obtain a state equation of a special class of dynamic system including two-wheeled vehicles to design a control law. That is an extension of the formulation presented by Getz (H. Getz, 1993). Furthermore, a sliding mode control method is applied to the vehicle's model, which obtained from the proposed method. Sliding mode control is known as a nonlinear robust control method (E.Slotine and Li, 1991). The controller's effectiveness has been verified through computer simulations.

2. BRIEF DESCRIPTION OF TWO-WHEELED VEHICLE MODEL

This paper considers a simple model of a two-wheeled vehicle, as shown in Figures 1 and 2. The model's wheels are considered to have negligible inertial moments, mass, and width. They are also assumed to rotate without any slip. As shown in Figure 1, the vehicle is considered to have an inertial coordinate system with x and y axes on the ground plane and z axis perpendicular to the ground plane in the opposite direction to gravity. The coordinate system fixed in the vehicle's center of mass is constituted by x', y' and z' axes. The vehicle's yaw angle is given by θ and its roll angle is given by α. Yaw angle θ is assumed to be zero when a line passing through the front and rear-wheel contact points is in the x direction. Now ϕ and v_r indicate the steering angle and the velocity relative to the rear wheel, respectively. As shown in Figure 2, m indicates the vehicle's mass on its center of mass, c denotes the distance between the front- and rear-wheel contact points, p denotes the parallel distance between the rear-wheel contact point and the vehicle's center of mass, and b denotes the vertical distance between the ground and vehicle's center of mass. For simplicity, moments of inertia associated with the x', y' and z' axes are assumed to be identical, and are described by $J = jm$. The vehicle's model considered here has a force generator that supplies a driving force u, and it is assumed that the vehicle does not pitch.

The generalized coordinates of the two-wheeled vehicle are assumed to be $\mathbf{q} = [\alpha, \theta, x, y]^T$. From the assumption of no sideslip or skidding, the vehicle's model has the following nonholonomic constraint conditions:

$$-\dot{x}\sin\theta + \dot{y}\cos\theta = 0 \tag{1}$$

$$-\sin(\theta+\phi)\dot{x} + \cos(\theta+\phi)\dot{y} + b\dot{\theta}\cos\phi = 0 \tag{2}$$

These conditions will be described by matrix $\mathbf{A}(\mathbf{q}, \phi)$, which includes $\mathbf{q}$ and steering angle ϕ, as:

$$\mathbf{A}(\mathbf{q}, \phi)\dot{q} = 0 \tag{3}$$

In equation (3), matrix $\mathbf{A}(\mathbf{q}, \phi)$ is given as:

$$\mathbf{A}(\mathbf{q}, \phi) = \begin{bmatrix} 0 & 0 & -S_\theta & C_\theta \\ 0 & bC_\phi & -S_{\theta+\phi} & C_{\theta+\phi} \end{bmatrix} \tag{4}$$

where S_δ and C_δ are $\sin\delta$ and $\cos\delta$, respectively. The vehicle's model with nonholonomic constraints described by equation (1) and (2) is obtained using Lagrange equation as:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{E}(\mathbf{q})u + \mathbf{A}^T(\mathbf{q}, \phi)\lambda \tag{5}$$

where $\mathbf{M}(\mathbf{q})$ is a 4×4 inertia matrix, $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})$ is the centripetal and Coriolis force term, and $\mathbf{E}(\mathbf{q})$ is a 4×1 input transformation matrix for the driving force u. λ is called the Lagrange multiplier and it is an unknown vector function. Using λ, $\mathbf{A}^T(\mathbf{q}, \phi)\lambda$ implies nonholonomic constraint forces of the model, where $\mathbf{A}^T(\mathbf{q}, \phi)$ is the transposed matrix given in equation (4).

3. TRANSFORMATION FROM LAGRANGE EQUATION TO STATE EQUATIONS

To begin with, consider the general form of nonholonomic systems. The generalized coordinates of the system are assumed to be $\mathbf{q} = [q_1, q_2, \ldots, q_N]^T$. Nonholonomic constraint conditions for the system are described as:

$$\sum_{j=1}^{N} a_{ij}\dot{q}_j = 0, \quad i = 1, \ldots, M \tag{6}$$

These M equations are independent of each other. Then, it is known that the system's model is given from the Lagrange equation as (Moon, 1998):

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{E}(\mathbf{q})\mathbf{u} + \mathbf{A}^T(\mathbf{q})\lambda \tag{7}$$

where $\mathbf{A}(\mathbf{q})$ is an $M \times N$ matrix composed of a_{ij} in equation (6). Now, the Lagrange multiplier λ is an unknown vector function. When transforming from such a Lagrange equation to state equations in order to design the control law, the problem is how to treat λ.

Comparing equation (7) with two-wheeled vehicle's equation (5), it is found that matrix $\mathbf{A}$ in equation (5), which expresses constraint forces, includes one of the inputs ϕ of the two-wheeled vehicle. This means that the two-wheeled vehicle has a special input that can only affect the vehicle's behavior through the nonholonomic constraint term $\mathbf{A}^T\lambda$. It is therefore very difficult to obtain its state equation. This paper proposes a systematic way of obtaining the state equation of such special class systems from their Lagrange equation.

Consider a dynamic system with N-dimensional generalized coordinates subject to M nonholonomic independent constraints that are in the form:

$$\mathbf{A}(\mathbf{q}, \xi)\dot{\mathbf{q}} = 0 \tag{8}$$

where $\mathbf{A}(\mathbf{q}, \xi)$ is an $M \times N$ full-rank matrix and ξ is a continuous function that includes a special input like ϕ in equation (5). It is assumed that the system has P such inputs. The dynamic system's model with such nonholonomic constraint conditions is obtained from the Lagrange equation as:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{E}(\mathbf{q})\mathbf{u} + \mathbf{A}^T(\mathbf{q}, \xi)\lambda \tag{9}$$

Now it can be found $M \times N$ linearly independent vectors of the form:

$$\mathbf{A}(\mathbf{q}, \xi)s(\mathbf{q}, \xi) = 0 \tag{10}$$

Let $\mathbf{S}(\mathbf{q}, \xi) = [s_1(\mathbf{q}, \xi), \ldots, s_{N-M}(\mathbf{q}, \xi)]$ be an $N \times (N - M)$ full-rank matrix made up of these vectors such that:

$$\mathbf{A}(\mathbf{q}, \xi)\mathbf{S}(\mathbf{q}, \xi) = 0 \tag{11}$$

Equations (8) and (11) imply the existence of an $(N - M)$-dimensional velocity vector $\mathbf{z} = [z_1, \ldots, z_{N-M}]^T$ such that:

$$\dot{\mathbf{q}} = \mathbf{S}(\mathbf{q}, \xi)\mathbf{z} \tag{12}$$

Differentiating equation (12), it becomes:

$$\ddot{\mathbf{q}} = \mathbf{S}\dot{\mathbf{z}} + \dot{\mathbf{S}}\mathbf{z} \tag{13}$$

where $\dot{\mathbf{S}}\mathbf{z}$ is represented in the form:

$$\dot{\mathbf{S}}\mathbf{z} = \mathbf{K}(\mathbf{q}, \mathbf{z}, \xi)\dot{\mathbf{q}} + \mathbf{L}(\mathbf{q}, \mathbf{z}, \xi)\dot{\xi} \tag{14}$$

The elements of matrix $\mathbf{K}$ and $\mathbf{L}$ are described as:

$$K_{ik} = \sum_{j=1}^{N-M} \frac{\partial S_{ij}(\mathbf{q}, \xi)}{\partial q_k} z_j \tag{15}$$

$$L_{il} = \sum_{j=1}^{N-M} \frac{\partial S_{ij}(\mathbf{q}, \xi)}{\partial \xi_l} z_j \tag{16}$$

where $i, k = 1, \ldots, N$ and $l = 1, \ldots, P$, respectively. From equations (12) and (14), equation (13) can be written as:

$$\ddot{\mathbf{q}} = \mathbf{S}\dot{\mathbf{z}} + \mathbf{KSz} + \mathbf{L}\dot{\xi} \tag{17}$$

A method of transforming equation (9) to a state equation is proposed by using these results. It becomes $\mathbf{AS} = \mathbf{S}^T\mathbf{A}^T = 0$ from equation (11). Therefore, left-multiplying by $\mathbf{S}^T$ in equation (9) to eliminate the unknown function λ and substituting equation (17) gives:

$$\mathbf{S}^T\mathbf{M}(\mathbf{S}\dot{\mathbf{z}} + \mathbf{KSz} + \mathbf{L}\dot{\xi}) - \mathbf{S}^T\mathbf{V} = \mathbf{S}^T\mathbf{Eu} \tag{18}$$

Now $\mathbf{S}^T\mathbf{MS}$ is assumed to have its inverse matrix. Then left-multiplying by $(\mathbf{S}^T\mathbf{MS})^{-1}$ in equation (18) and solving this for $\dot{\mathbf{z}}$ results in the following state equation:

$$\dot{\mathbf{z}} = \mathbf{F} + \mathbf{BU} \tag{19}$$

where

$$\mathbf{F} = (\mathbf{S}^T\mathbf{MS})^{-1}\mathbf{S}^T(\mathbf{V} - \mathbf{MKSz}) \tag{20}$$

$$\mathbf{B} = \begin{bmatrix} -(\mathbf{S}^T\mathbf{MS})^{-1}\mathbf{S}^T\mathbf{ML} \\ (\mathbf{S}^T\mathbf{MS})^{-1}\mathbf{S}^T\mathbf{E} \end{bmatrix}^T \tag{21}$$

$$\mathbf{U} = \begin{bmatrix} \dot{\xi} \\ \mathbf{u} \end{bmatrix} \tag{22}$$

Using that transformation, the unknown function λ is eliminated from the Lagrange equation, and special inputs, which can only have an effect through the nonholonomic constraint term $\mathbf{A}^T\lambda$, appear in the state equation in the form of $\dot{\xi}$. In equation (19), matrices $\mathbf{F}$ and $\mathbf{B}$ are described as nonlinear matrices.

According to nonlinear control theory, choosing a control input $\mathbf{U}$ of a state equation such as:

$$\mathbf{U} = \mathbf{B}^{-1}(\mathbf{F} + \mathbf{V}) \tag{23}$$

and substituting equation (23) into (19) gives:

$$\dot{\mathbf{z}} = \mathbf{V} \tag{24}$$

Thus, the stabilization problem for the nonlinear system has been reduced to a stabilization problem for a controllable linear system (K.Khalil, 1996). Therefore, state equation (19) will be controllable if matrix $\mathbf{B}$ in equation (21) has its inverse matrix $\mathbf{B}^{-1}$.

4. STATE EQUATION OF TWO-WHEELED VEHICLE MODEL

Now let's apply the proposed method to Lagrange equation (5) to obtain the state equation of the two-wheeled vehicle model. The order of the generalized coordinates is $N = 4$. Since 2 nonholonomic conditions exist in the system, $\mathbf{z}$ becomes a 2-order vector. The special input of the model that can only affect the vehicle's behavior through the nonholonomic constraints is ϕ, which indicates $P = 1$. The matrix $\mathbf{S}$ that satisfies equation (11) is obtained as:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi & \cos\theta & \sin\theta \end{bmatrix}^T \tag{25}$$

where $\xi = \tan\phi/b$. Therefore, the velocity vector $\mathbf{z}$ that satisfies equation (12) can be found as:

$$\mathbf{z} = \begin{bmatrix} \dot{\alpha} \\ v_r \end{bmatrix} \tag{26}$$

Now calculating matrix $\mathbf{K}$ in equation (15) and matrix $\mathbf{L}$ in equation (16), respectively, produce:

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -v_r S_\theta & 0 & 0 \\ 0 & v_r C_\theta & 0 & 0 \end{bmatrix} \tag{27}$$

$$\mathbf{L} = \begin{bmatrix} 0 & v_r & 0 & 0 \end{bmatrix}^T \tag{28}$$

Therefore, the state equation of the two-wheeled vehicle model is obtained from equation (19) in the form:

$$\begin{bmatrix} \ddot{\alpha} \\ \dot{v}_r \end{bmatrix} = \mathbf{F}(\mathbf{q}, \mathbf{z}, \xi) + \mathbf{B}(\mathbf{q}, \mathbf{z}, \xi) \begin{bmatrix} \dot{\xi} \\ u \end{bmatrix} \tag{29}$$

where $\mathbf{B}(\mathbf{q}, \mathbf{z}, \xi)$ is invertible for all $-\pi/2 < \alpha < \pi/2$ and $v_r > 0$. Thus, this equation is controllable in that domain.

5. SLIDING MODE CONTROL LAW

A sliding mode control method is applied to the state equation (29) of the two-wheeled vehicle model to control its motion. The control objective is that the vehicle's state variables α, $\dot{\alpha}$ and v_r will track its desired states α_d, $\dot{\alpha}_d$ and v_{rd}, respectively. Now $\alpha_e = \alpha - \alpha_d$ and $v_{re} = v_r - v_{rd}$ define the tracking error of the state variable α and v_r. In this paper, the assumed initial values of each states are $\alpha(0) = -\pi/4$, $\dot{\alpha}(0) = 0$ and $v_r(0) = 2$, and the desired states are $\alpha_d = \pi/8$, $\dot{\alpha}_d = 0$ and $v_{rd} = 4$, respectively.

The concept of sliding mode control is that the system trajectory is tracked to a certain surface in phase plane, called the sliding surface, by using a switching control law and makes the system stable. To achieve the motion control, the sliding surfaces for each state $\sigma_1 = 0$ and $\sigma_2 = 0$ are introduced, where:

$$\sigma_1 = \dot{\alpha}_e + \gamma\alpha_e \tag{30}$$

$$\sigma_2 = v_{re} \tag{31}$$

The dynamics after the state trajectory reaches the sliding surface $\sigma_i = 0$ can be written as $\dot{\sigma}_i = 0$. Then, it is found from equation (31) that v_r converges to its desired value v_{rd} when state trajectory reaches $\sigma_2 = 0$. Furthermore, from equation (30), $-\gamma$ becomes eigenvalues of the states α and $\dot{\alpha}$. Therefore, α is asymptotically stable if $\gamma > 0$. Now γ is chosen as $\gamma = 2$.

Since the implementation of the associated control switching is necessarily imperfect, this leads to chattering. Now, choosing the following sliding control law to achieve tracking to the desired state while avoiding chattering:

$$\begin{bmatrix} \dot{\xi} \\ u \end{bmatrix} = \mathbf{B}^{-1} \left\{ \begin{bmatrix} \ddot{\alpha}_d + \gamma\dot{\alpha}_e \\ \dot{v}_{rd} \end{bmatrix} - \mathbf{F} - \begin{bmatrix} k_1 \operatorname{sat}(\sigma_1/\Phi_1) \\ k_2 \operatorname{sat}(\sigma_2/\Phi_2) \end{bmatrix} \right\} \tag{32}$$

The stable condition of the sliding mode control is described by:

$$\frac{1}{2}\frac{d}{dt}\sigma_i^2 \leq -\eta_i|\sigma_i|, \quad i = 1, 2 \tag{33}$$

where $\eta_i > 0$.

In equation (32), the stable condition (33) is satisfied if k_1 and k_2 are chosen above η_1 and η_2, respectively. When k_i is chosen as $k_i = \eta_i$, k_i can be found from the time the state trajectory reaches the sliding surface from its initial values. In this paper, it is assumed that the state trajectory reaches the sliding surface after 0.5 sec, resulting in $k_1 = 3\pi/4$ and $k_2 = 4$.

6. TRACKING RESPONSES TO MOTION CONTROL

This section describes computer simulations carried out to verify the performance of the proposed controller. Table 1 shows the physical parameters of the two-wheeled vehicle model used in these simulations. The assumed sampling time for the simulations is 0.01 sec.

Table 1. Physical parameters of two-wheeled vehicle model

Symbol	Unit	Value
b	m	1.0
c	m	0.5
p	m	0.5
m	kg	20.0
j	m^2	0.1
g	m/s^2	9.8

Figure 3 shows the tracking responses of the roll angle α and the velocity of the rear-wheel v_r, to be applied to the proposed sliding mode control law and general feedback linearization law based on the linear state feedback control method. In the sliding mode control law (32), Φ_i is chosen as $\Phi_1 = 0.1$ and $\Phi_2 = 0.2$ from previous simulation results. It is thus found that the response of sliding mode control is very good. Furthermore, the response of sliding mode control converges faster than that of feedback linearization control. Figure 4 shows the trajectory of the rear-wheel contact on the ground plane obtained from the sliding control law. Figure 4 shows that sliding mode control achieves a stable trajectory for the two-wheeled vehicle.

Next, to verify the robustness of these controllers, simulations were carried out with a mass perturbation of the model. As a result, the tracking response of the roll angle α for each controller was

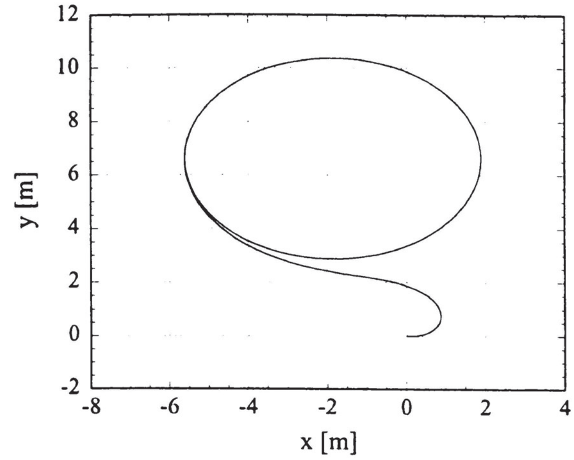

Fig. 4. Trajectory of the rear-wheel contact in the ground plane.

obtained as shown in figure 5. From figure 5, with feedback linearization control, a 27% increase of the mass causes α to reach $\pi/2$ rad after about 2.3 sec. That is, it indicates that the vehicle model falls down. However, sliding mode control maintains almost the same response as no perturbation of the mass with a 27% increase. Furthermore, it produces a stable response with a 56% increase in the mass. Similarly, the velocity v_r of the rear-wheel also indicates a stable response.

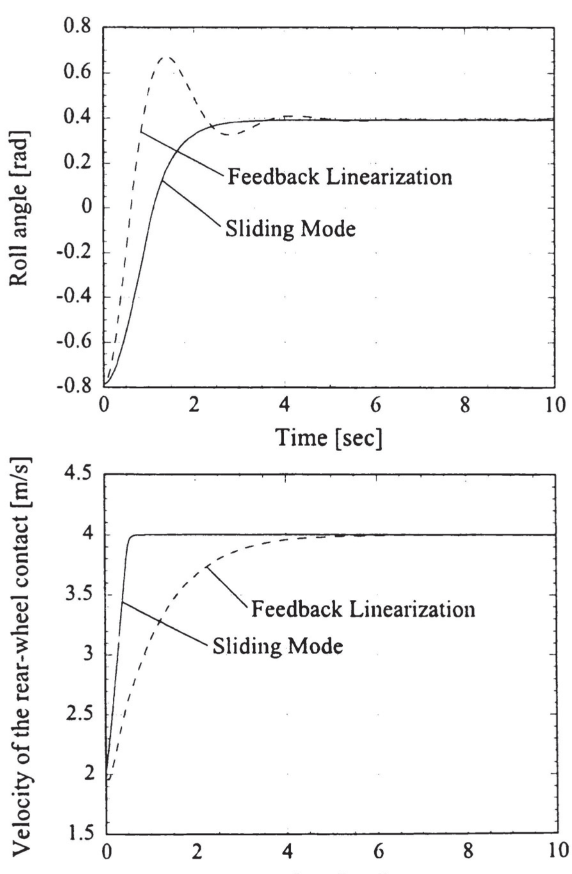

Fig. 3. Tracking response of the roll-angle α and velocity to the rear-wheel v_r.

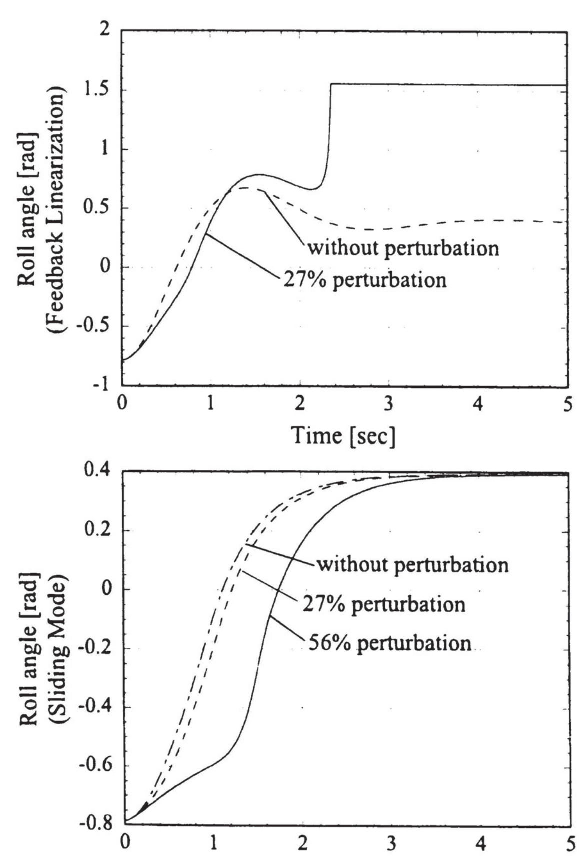

Fig. 5. Tracking response of the roll-angle α with perturbation of the vehicle's mass

These simulations have clarified that the sliding mode controller is effective in controlling the motion of two-wheeled vehicles with nonholonomic constraints, because of its high control activity against tracking and its robustness.

7. CONCLUSION

This paper has described a method for controlling the motion of a riderless two-wheeled vehicle with nonholonomic constraints. It has also proposed a systematic way of obtaining a nonholonomic system's state equation from its Lagrange equation. A sliding mode control method was applied to the state equation of the vehicle's model obtained from the proposed method.

The sliding mode controller's performance was verified by computer simulations. The following results were obtained:

(1) The sliding mode control method produces very fast convergence to the desired value and good response.
(2) The sliding mode control method is robust against parameter perturbations of the model.

Therefore, the sliding mode control method is much more effective in controlling the two-wheeled vehicle motion than the feedback linearization method.

There are many nonholomic systems that are similar to the two-wheeled vehicle dealt with in this paper. A car is one such system, and there are many requirements for its control. The method proposed here would be very effective for such nonholonomic systems.

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