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Identifiability of EIV Dynamic Systems with Non-Stationary Data

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Abstract: This paper presents novel results related to the identifiability of EIV dynamic systems based on exploiting properties of non-stationary data. We analyze single-input single-output systems using second order properties. Our results show that, it is possible to establish identifiability of EIV systems under mild conditions when the data is non-stationary.

1. INTRODUCTION

Error in variables (EIV) systems or measurement error models are systems where the input and output measurements are contaminated with noise [Cheng and Van Ness, 1999, Fuller, 1987]. A recent survey of this area is given in Söderström [2007]. The two main streams of research in this area are:

Identifiability: In many cases the system (or the parameters that define it) cannot be determined uniquely even when the probability density function for the signals is exactly known. In this case we say that the system is not identifiable. In the *System Identification* literature there exist many results which establish conditions for identifiability. For example, it is well known that systems operating under feedback can be made identifiable by using switching controllers (see e.g. Ljung et al. [1974], Söderström et al. [1976], Sin and Goodwin [1980]). In the case of EIV systems, it is possible to establish identifiability results from high order moments (see Reiersøl [1950], Anderson and Deistler [1984], Deistler [1986]). Recent research in this area has focused on the problem of retrieving dynamic system models from given second order properties of the input and output signals [Maravall, 1979, Söderström, 1980, Solo, 1986, Anderson et al., 1987, Nowak, 1992, Agüero and Goodwin, 2008].

Estimation algorithms: Once identifiability is established for a particular EIV application, estimation algorithms can be developed. Research in this area has focused on bias compensation, Frisch schemes and related algorithms [Beghelli et al., 1990, Zheng, 2002, Söderström et al., 2002, Ekman et al., 2006, Mahata, 2007]. Recently, Maximum Likelihood approaches have also been developed in both time and frequency domains [Pintelon and Schoukens, 2007, Diversi and Soverini, 2007]. One of the main difficulties in this area is that it is difficult to find the system structure and consequently it is usually assumed to be known.

It is interesting to note that, in general, identifiability is established by developing a construction procedure to retrieve the system from signal spectra. In this case, it is also possible to develop estimation algorithms by using spectra estimates obtained from the data instead of the true signal spectra.

In this paper we analyze the Identifiability of EIV dynamic systems when the data is non-stationary. The use of non-stationary data to develop an estimation algorithm for EIV systems has a long history going back to the work of Wald [1940] for the static case. Recently an estimation algorithm has been developed in [Markovsky et al., 2006] for a particular class of dynamic EIV systems. In [Markovsky et al., 2006] it is assumed that i) the input and output noise are white, and ii) that the noise free input, $u_o(t)$, exhibits two different types of behavior. This topic is also briefly discussed in [Söderström, 2007].

In the current paper, we will analyze the identifiability of a general class of EIV dynamic systems when the data is non-stationary. In particular, we assume that the non-stationary signals are due to changes in the system structure. This paper has been motivated by an application to Transient Electromagnetic Mineral Exploration explained in detail in a companion paper [Lau et al., 2008].

The layout of the remainder of the paper is as follows: In Section 2 we describe the system of interest. In Section 3 we list assumptions. In Section 4 Identifiability in the case of stationary data is reviewed. Identifiability in the case of non-stationary data is presented in Section 5. Finally some comments and conclusions are presented in Section 6.

2. SYSTEM DESCRIPTION

2.1 Errors in variables

The type of system considered is shown in Figure 1, and is of the form

$$u(t) = u_0(t) + \eta_1(t), \quad y(t) = y_0(t) + \eta_2(t)$$

where $G_o(z)$ is a linear dynamic system, $u(t)$ and $y(t)$ are the measured input and output respectively, $\eta_1(t)$ and $\eta_2(t)$ are the input and output noise.

2.2 Input-Output spectra

We will utilize the joint input-output spectrum:

$$\Phi(e^{j\omega}) = \begin{bmatrix} \Phi_y(e^{j\omega}) & \Phi_{yu}(e^{j\omega}) \\ \Phi_{yu}(e^{-j\omega}) & \Phi_u(e^{j\omega}) \end{bmatrix} \tag{1}$$

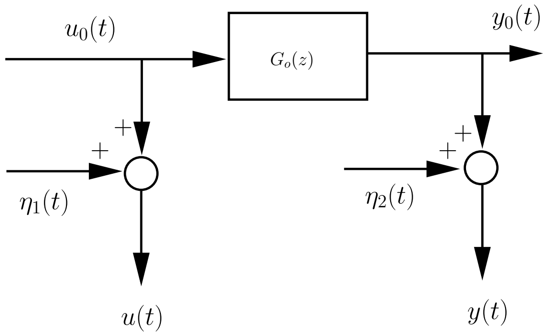


Fig. 1. Errors in variables system.

where

$$\Phi_y(e^{j\omega}) = G_o(e^{j\omega})\Phi_0(e^{j\omega})G_o(e^{-j\omega}) + \Phi_2(e^{j\omega}) \quad (2)$$
$$\Phi_{yu}(e^{j\omega}) = G_o(e^{j\omega})\Phi_0(e^{j\omega}) \quad (3)$$
$$\Phi_u(e^{j\omega}) = \Phi_0(e^{j\omega}) + \Phi_1(e^{j\omega}) \quad (4)$$

and $\Phi_0(e^{j\omega})$, $\Phi_1(e^{j\omega})$, and $\Phi_2(e^{j\omega})$ are the spectrum of $u_0(t)$, $\eta_1(t)$, and $\eta_2(t)$ respectively.

We can then state the identifiability problem from second order properties as being equivalent to retrieving the unknown system, $G_o(e^{j\omega})$ from the input-output signal spectrum, $\Phi(e^{j\omega})$. The latter quantity is assumed to be available, and it is sometimes called the *standard data* [Anderson, 1985].

Note that a rational spectrum $\Phi(e^{j\omega})$ is usually written as a function of a complex variable z, i.e. $\Phi(e^{j\omega}) = \Phi(z)|_{z=e^{j\omega}}$. In addition, using the theorem of analytic continuation (see e.g. [Churchill and Brown, 1990, page 323]) we have that equations (2) to (4) hold for $z = e^{j\omega}$, if and only if, they hold for any z in the complex plane [1]. Thus, in the sequel we indistinguishable use equations (2) to (4) written in terms of $e^{j\omega}$, or in terms of the complex variable [2] z.

2.3 Non-stationary data

Most of the results in System Identification have been developed under the assumption that the data is stationary. However, this assumption is always considered as an approximation [Priestley, 1981]. Indeed, there has been substantial interest in non-stationary systems in the last twenty years (see e.g. Maddala and Kim [1998]).

Non-stationary behavior can be due to an unstable system generating the data or to changes in the model structure. Recent research on the latter case has focused on Markov Switching Models where the structure changes in an abrupt manner. The main difficulty here is that in some applications the system structure usually changes gradually (see e.g. Giordani et al. [2007] and the references therein).

In a recent paper regarding EIV dynamic systems, Markovsky et al. [2006] have assumed that the system input, $u_0(t)$, is non-stationary in the sense that it changes its *behavior*. An algorithm to cluster the data is used, and then an estimation algorithm is proposed. In Söderström [2007] it was pointed out that when it is possible to cluster the data from these two different behavior, the problem can be understood as data coming from two different experiments.

Here, we assume that the different signals in the EIV system are generated as follows:

$$y_0(t) = G_o(z)u_0(t), \quad u_0(t) = L_0(z)\bar{u}_0(t)$$
$$\eta_1(t) = L_1(z)\bar{\eta}_1(t), \quad \eta_2(t) = L_2(z)\bar{\eta}_2(t)$$

Here, the complex variable z should be understood as the forward shift operator. $G_o(z)$, $L_0(z)$, $L_1(z)$, $L_2(z)$ are linear transfer functions.

We analyze an EIV dynamic system where the data shows two different *behaviors*. These two different behaviors originate from changes in (one or more of) the transfer functions which generate data. We call these behaviors scenario $\{i\}$, $i = 1, 2$ and write:

$$y\{i\}(t) = G_o\{i\}u_o\{i\}(t) + \eta_2\{i\}(t)$$
$$u\{i\}(t) = u_o\{i\}(t) + \eta_1\{i\}(t) \quad (5)$$

where the index $\{i\}$ represents the corresponding scenario with a different system structure generating the data.

We also assume that the data can be clustered to separate the data coming from these two scenarios.

In order to simplify the notation, in the sequel we omit the variable t in the equations.

In [Priestley, 1981, chapter 11] it was pointed out that whenever it is possible to cluster the data (to know when the system commutes from one model to the other) the signals can be considered as stationary with two different covariance functions. In the current paper we follow this approach. Indeed, in Section 5 we analyze the identifiability of EIV dynamic system with non-stationary data by using a similar approach to the one developed in [Agüero and Goodwin, 2008] for the stationary case.

Cases of interest: It can be seen that there are different possibilities for the change in the system structure. In fact here are 4 variables that can change: G_o, Φ_0, Φ_1, Φ_2. Thus, the non-stationary behavior can be explained by 2^4 cases. Each case imposes additional constraints, which might lead to identifiability of the process.

3. ASSUMPTIONS

It is well known that EIV dynamic systems are identifiable from second order properties when a-priori knowledge is used (see e.g [Agüero and Goodwin, 2008]). This a-priori knowledge is usually stated as extra assumptions.

We next introduce the following mild assumptions (These assumptions were also used in the analysis in Agüero and Goodwin [2008]).

Assumption 1. $G_o(z)$, $L_0(z)$, $L_1(z)$ and $L_2(z)$ are rational of finite order.

Assumption 2. $G_o(z)$ contains no pole that is also a pole of $G_o(z^{-1})$.

Assumption 3. $G_o(z)$ contains no zero that is also a zero of $G_o(z^{-1})$.

[1] Note that the spectrum is analytic in a region excluding the poles.
[2] When we write $|L(z)|^2$ we mean $|L(z)|^2 = L(z)L(z^{-1}) \neq L(z)L(z^*)$ (where $*$ denotes complex-conjugate). However, when $|z| = 1$ both expressions are the same.

Assumption 4. There is no zero of $G_o(z)$ outside the unit circle that is also a pole of $L_0(z^{-1})$. There is no pole of $G_o(z)$ inside the unit circle that is a zero of $L_0(z)$.

Remark 5. Note that assumptions 2, 3 and 4 rule out various singular cases. Notice also that assumption 2 is satisfied when the process is either strictly causal or strictly anti-causal. ▽▽▽

We then, define the following class of systems

$$\bar{\mathcal{S}} = \{G(z) \in \mathcal{G} : G(z) \text{ satisfies assumptions 1 to 3}\} \quad (6)$$

where $\mathcal{G}$ is the set of linear systems defined as:

$$\mathcal{G} = \{L(z) : L(z) = \frac{1 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}, z \in \mathcal{A} \subset \mathbb{C}\} \quad (7)$$

where $\{a_i\} \in \mathbb{R}$, $\{b_i\} \in \mathbb{R}$, and $\mathcal{A}$ includes the unit circle so that the spectra are well defined. We define these transfer functions using the two-sided z-transform in order to deal with anti-causal systems $G_o(z)$ [Oppenheim and Schafer, 1989]. Specifically poles outside the unit circle are associated with anti-causal responses.

We say that a transfer function $G(z) \in \mathcal{G}$ does not have "symmetric" poles or zeros if $G(z) \in \bar{\mathcal{S}}$.

We next list assumptions which will be used in the analysis of identifiability of EIV systems in Section 5.

Assumption 6. The system belongs to $\bar{\mathcal{S}}$ for $i = 1, 2$.

Assumption 7. The process is the same for both scenarios: $G_o\{1\} = G_o\{2\} = G_o$.

Assumption 8. The input noise spectrum is the same for both scenarios: $\Phi_1\{1\} = \Phi_1\{2\} = \Phi_1$.

Assumption 9. The output noise spectrum is the same for both scenarios: $\Phi_2\{1\} = \Phi_2\{2\} = \Phi_2$.

Assumption 10. The input $\bar{u}_o$ has the same second order properties for every experiment: $\Phi_0\{1\} = \Phi_0\{2\} = \Phi_0$.

The assumptions listed above impose constraints on the problem in order to make the EIV system identifiable.

Notice that, in our analysis, we do not use all the assumptions at the same time.

4. IDENTIFIABILITY IN THE STATIONARY CASE

To set the scene for the non-stationary case, we first review identifiability of EIV system in the stationary case.

Identifiability of stationary EIV dynamic systems has been studied in [Anderson and Deistler, 1984, Anderson, 1985, Deistler, 1986] under very general conditions called the *standing assumptions* (i.e. signals with bounded spectra, stable and causal transfer functions, and independence of the input, and noise signals). Related results also appear in [Agüero and Goodwin, 2008].

The results for Identifiability from second order properties for EIV dynamic system can be summarized as follows

4.1 Bounds for G_o

In [Anderson and Deistler, 1984, Anderson, 1985, Deistler, 1986] it was found that for a given spectra (*the standard data*) $G_o(e^{j\omega})$ lies between the following bounds:

$$|G_{yu}(e^{j\omega})| \leq |G_o(e^{j\omega})| \leq |G_{uy}(e^{j\omega})| \quad (8)$$

where

$$G_{yu}(e^{j\omega}) = \frac{\Phi_{yu}(e^{j\omega})}{\Phi_u(e^{j\omega})}, \qquad G_{uy}(e^{j\omega}) = \frac{\Phi_y(e^{j\omega})}{\Phi_{uy}(e^{j\omega})} \quad (9)$$

If an estimate $\hat{G}(e^{j\omega})$, of $G_o(e^{j\omega})$ satisfies (8), then there exist corresponding values of $\Phi_0(e^{j\omega})$, $\Phi_1(e^{j\omega})$, and $\Phi_2(e^{j\omega})$ such that equations (2)-(4) also hold.

The phase of $G_o(e^{j\omega})$ can be obtained from (3) since $\Phi_o(e^{j\omega})$ is a real function of ω. It is also assumed that $\Phi_o(e^{j\omega}) = 0$ only on a set of measure zero, and the phase of $G_o(e^{j\omega})$ can thus be obtained as a limit in this case. Assuming that the system is causal, it is possible to obtain the number of non-minimum phase zeros by using the *principle of the argument* (see e.g. Greenleaf [1972]).

Clearly, the bounds for G_o given in (8) also define bounds for Φ_0, Φ_1 and Φ_2. These bounds are given by:

$$\frac{|\Phi_{yu}(e^{j\omega})|^2}{\Phi_y(e^{j\omega})} \leq \Phi_0(e^{j\omega}) \leq \Phi_u(e^{j\omega}) \quad (10)$$

$$0 \leq \Phi_1(e^{j\omega}) \leq \Phi_u - \frac{|\Phi_{yu}|^2}{\Phi_y(e^{j\omega})} \quad (11)$$

$$0 \leq \Phi_2(e^{j\omega}) \leq \Phi_y - \frac{|\Phi_{yu}|^2}{\Phi_u(e^{j\omega})} \quad (12)$$

4.2 Identifiability up to a frequency independent parameter

In [Anderson and Deistler, 1984, Anderson, 1985, Deistler, 1986] it was established that: If the system is causal and minimum phase, then the magnitude of $G_o(e^{j\omega})$ can be retrieved from its phase up to a scalar frequency independent gain.

Recently, in [Agüero and Goodwin, 2008] the following related result was presented:

Theorem 11. Subject to assumptions 1 to 4, the equivalence class of compatible models is given by

$$\mathcal{P} = \left\{ \hat{G}(z) : \hat{G}(z) = \frac{\Phi_{yu}(z)}{\hat{\sigma}_0^2 |L_0(z)|^2} \right\} \quad (13)$$

where $\hat{\sigma}_0^2$ is any positive real number satisfying

$$\lambda_{min} \leq \hat{\sigma}_0^2 \leq \lambda_{max} \quad (14)$$

where

$$\lambda_{min} = \max_w \frac{|\Phi_{yu}(e^{j\omega})|^2}{\Phi_y(e^{j\omega})|L_0(e^{j\omega})|^2}, \lambda_{max} = \min_w \frac{\Phi_u(e^{j\omega})}{|L_0(e^{j\omega})|^2}$$

and where $L_0(z)$ is uniquely determined by taking those poles and zeros from $\Phi_{yu}(z)$ which are symmetrically placed with respect to the unit circle, i.e. were $(z - c)$ and $(1 - zc)$ are factors.

From this result we obtain tighter bounds (G_{min} and G_{max}) for G_o than the ones obtained earlier (see Figure 2).

4.3 Complete identifiability

From the result presented in Theorem 11, it is clear that the only impediment to complete identifiability [3] is

[3] Here and in the sequel, we use the term "complete identifiability" to the case when the system is identifiable, i.e. when it is possible to retrieve the phase and gain of G_o from second order properties of the input and output signals.

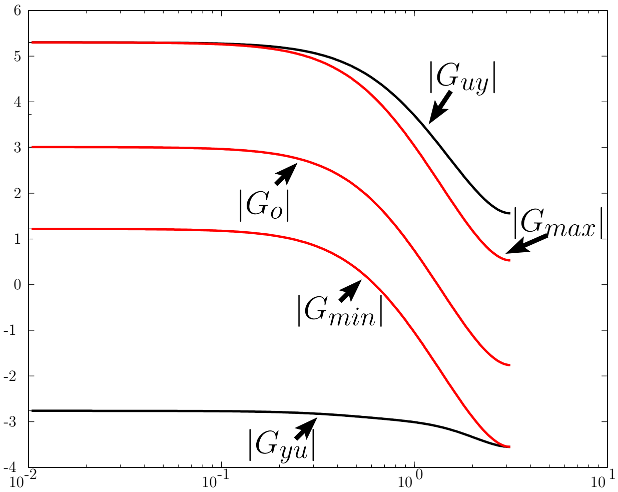


Fig. 2. Bounds for $|G_o(e^{j\omega})|$.

the knowledge of the gain of G_o. If extra information is available, then it may be possible to obtain σ_0^2. These results have been summarized in [Agüero and Goodwin, 2008], where extra knowledge regarding polynomial order, and existence of different poles in some of the transfer functions was used. These results have also been extended for a class of multivariable systems. We refer to [Agüero and Goodwin, 2008] for details.

In Section 5 we will see that the above results above also aid the understanding of identifiability results for EIV systems in the case of non-stationary data.

5. IDENTIFIABILITY FOR NON-STATIONARY PROCESSES

With two scenarios, equations (2) to (4) give a total of 6 equations coming from the standard data (input, output, and cross spectrum for both scenarios) and 8 unknowns (G_o, Φ_0, Φ_1, Φ_2 for both scenarios). By assuming that some of the transfer functions are the same in both scenarios we add constraints which allow identifiability in some cases as shown below.

5.1 Bounds for G_o

Theorem 12. Under the *standing assumptions* we have that

(i) If assumption 7 holds ($G_o\{1\} = G_o\{2\}$), then G_o is bounded as follows:

$$\max_{i=1,2}\{|G_{yu}\{i\}\} \leq |G_o| \leq \min_{i=1,2}\{|G_{uy}\{i\}|\}$$

where G_{yu} and G_{uy} are defined in (9).

(ii) If assumption 8 holds ($\Phi_1\{1\} = \Phi_1\{2\}$) then,

$$\frac{\Phi_{yu}\{i\}}{\Phi_0^{max}} \leq G_o\{i\} \leq \frac{\Phi_{yu}\{i\}}{\Phi_0^{min}} \tag{15}$$

where Φ_0^{min} and Φ_0^{max} are defined as:

$$\Phi_0^{max}\{i\} = \Phi_u\{i\}$$

$$\Phi_0^{min}\{i\} = \Phi_u\{i\} - \min_{i=1,2}\{\Phi_u\{i\} - \frac{|\Phi_{yu}\{i\}|^2}{\Phi_y\{i\}}\}$$

(iii) If assumption 9 holds ($\Phi_2\{1\} = \Phi_2\{2\}$) then, G_o is bounded as in (15) where

$$\Phi_0^{max}\{i\} = \frac{|\Phi_{yu}\{i\}|^2}{\Phi_y\{i\} - \min_{i=1,2}\{\Phi_y\{i\} - \frac{|\Phi_{yu}\{i\}|^2}{\Phi_u\{i\}}\}}$$

$$\Phi_0^{min}\{i\} = \frac{|\Phi_{yu}\{i\}|^2}{\Phi_u\{i\}}$$

(iv) If assumption 10 holds ($\Phi_0\{1\} = \Phi_0\{2\}$) then, G_o is bounded as in (15) where

$$\Phi_0^{max} = \min_{i=1,2}\{\Phi_u\{i\}\}$$

$$\Phi_0^{min} = \max_{i=1,2}\{\frac{|\Phi_{yu}\{i\}|^2}{\Phi_y\{i\}}\}$$

Proof. Immediate from the inequalities in equations (8), (10), (11), and (12) $\square$.

We next analyze the case when we add two additional constraints and obtain complete identifiability.

5.2 Complete identifiability

The following Theorem establishes complete identifiability under different conditions.

Theorem 13. The EIV system described in (5) is identifiable if any of the following situations holds:

(i) Assumptions 7 and 8 hold, and $\Phi_0\{1\} \neq \Phi_0\{2\}$ (a.e.),
(ii) Assumptions 7 and 9 hold, and $\Phi_0\{1\} \neq \Phi_0\{2\}$ (a.e.),
(iii) Assumptions 9 and 10 hold, and $G_o\{1\} \neq G_o\{2\}$ (a.e.),
(iv) Assumptions 8, 9 hold, and $\tilde{\Phi}_u\tilde{\Phi}_y < 0$, $\forall\omega$, where $\tilde{\Phi}_u = \Phi_u\{1\} - \Phi_u\{2\}$ and $\tilde{\Phi}_y = \Phi_y\{1\} - \Phi_y\{2\}$.

In addition we have that

(v) if assumptions 8, 9 hold and $\tilde{\Phi}_u\tilde{\Phi}_y < 0$, $\forall\omega \in \Omega \subset (-\pi, \pi]$ then there is no a unique model. However, using the bounds in Theorem 12 it is possible to isolate a unique system.

Proof.

Our strategy will be to retrieve $G_o\{i\}$ from the standard data from both scenarios. Once $G_o\{i\}$ has been obtained $\Phi_0\{i\}$, $\Phi_1\{i\}$, $\Phi_2\{i\}$, $i = 1, 2$ can be retrieved as follows:

$$\hat{\Phi}_0\{i\} = \frac{\Phi_{yu}\{i\}}{G_o\{1\}}, \quad \hat{\Phi}_1\{i\} = \Phi_u\{i\} - \hat{\Phi}_0\{i\},$$

$$\hat{\Phi}_2\{i\} = \Phi_y\{i\} - |\hat{G}_o\{i\}|^2\hat{\Phi}_0\{i\}$$

(i) In this case we have that G_o and Φ_1 are the same for both scenarios. Then the input-output and input spectrum for the two scenarios are given by:

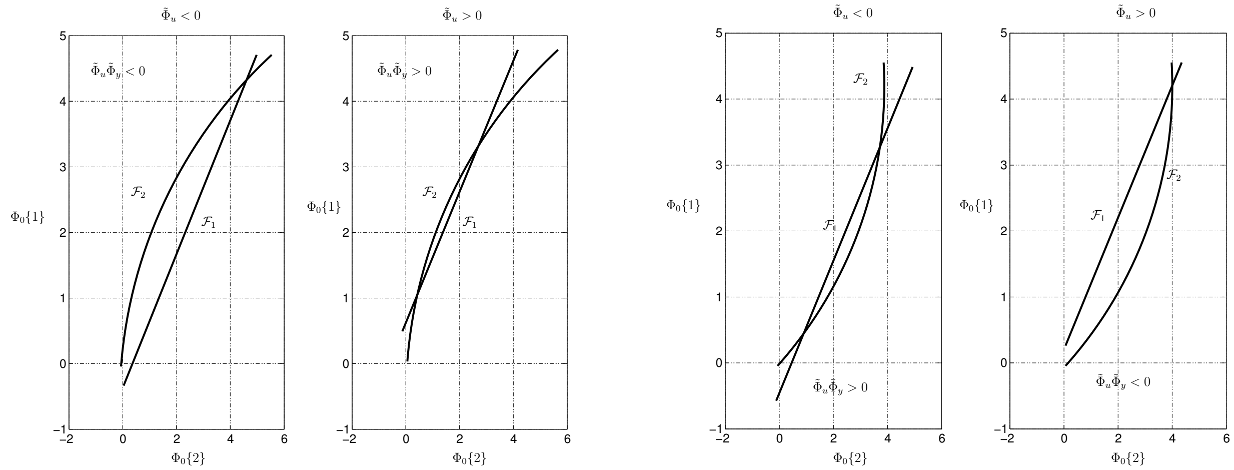
$$\Phi_{yu}\{i\} = G_o\Phi_0\{i\}, \quad \Phi_u\{i\} = \Phi_0\{i\} + \Phi_1$$

Thus, we have that G_o can be retrieved as follows

$$\hat{G}_o = \frac{\Phi_{yu}\{1\} - \Phi_{yu}\{2\}}{\Phi_u\{1\} - \Phi_u\{2\}} \tag{16}$$

Notice that we need $\Phi_u\{1\} \neq \Phi_u\{2\}$ (or equivalently $\Phi_0\{1\} \neq \Phi_0\{2\}$) almost everywhere (a.e.).

(ii) In this case we have that G_o and Φ_2 are the same for both scenarios. Then the cross (input, output) and output spectrum for the two scenarios are given by:



(a) Plot for $\tilde{\Phi}_y > 0$. Left: $\tilde{\Phi}_u < 0$, Right: $\tilde{\Phi}_u > 0$

(b) Plot for $\tilde{\Phi}_y < 0$. Left: $\tilde{\Phi}_u < 0$, Right: $\tilde{\Phi}_u > 0$.

Fig. 3. Plots of $\mathcal{F}_1$ and $\mathcal{F}_2$.

$$\Phi_y\{i\} = |G_o|^2\Phi_0\{i\} + \Phi_2, \quad \Phi_{yu}\{i\} = G_o\Phi_0\{i\}$$

Thus, G_o can be retrieved as follows:

$$G_o = \frac{\Phi_y\{1\} - \Phi_y\{2\}}{\Phi_{yu}\{1\}^* - \Phi_{yu}\{2\}^*} \tag{17}$$

Notice that we need $\Phi_{yu}\{1\} \neq \Phi_{yu}\{2\}$ (or equivalently $\Phi_0\{1\} \neq \Phi_0\{2\}$) a.e.

(iii) In this case we have that Φ_2 and Φ_0 are the same for both scenarios. Thus, we have that the spectra are given by:

$$\Phi_{yu}\{1\} = G_o\{1\}\Phi_0, \quad \Phi_{yu}\{2\} = G_o\{2\}\Phi_0$$

$$\tilde{\Phi}_y = \Phi_y\{1\} - \Phi_y\{2\} = |G_o\{1\}|^2\Phi_0 - |G_o\{2\}|^2\Phi_0$$

We have

$$G_o\{1\} = \rho G_o\{2\}, \quad \rho = \frac{\Phi_{yu}\{1\}}{\Phi_{yu}\{2\}}$$

$$\frac{\tilde{\Phi}_y}{\Phi_{yu}\{1\}} = \frac{|G_o\{1\}|^2 - |G_o\{2\}|^2}{G_o\{1\}} = G_o\{2\}^* \frac{|\rho|^2 - 1}{\rho}$$

Thus, we have that $G_o\{1\}$ and $G_o\{2\}$ can be obtained via:

$$G_o\{2\} = \frac{\tilde{\Phi}_y}{\Phi_{yu}\{1\}^*} \frac{\rho^*}{|\rho|^2 - 1}$$
$$G_o\{1\} = \rho G_o\{2\}$$

In this case we need that $|\rho| \neq 1$ almost everywhere, which is satisfied whenever $|G_o\{1\}| \neq |G_o\{2\}|$.

(iv) In this case we have that Φ_1 and Φ_2 are the same for both scenarios. To treat this case we will use a slightly different argument.

$$\tilde{\Phi}_u = \Phi_0\{1\} - \Phi_0\{2\}, \quad \tilde{\Phi}_y = \frac{|\Phi_{yu}\{1\}|^2}{\Phi_0\{1\}} - \frac{|\Phi_{yu}\{2\}|^2}{\Phi_0\{2\}}$$

We have that $\Phi_0\{1\}$ and $\Phi_0\{2\}$ are in the following graphs:

$$\mathcal{F}_1: \quad \Phi_0\{1\} = \Phi_0\{2\} + \delta \tag{18}$$

$$\mathcal{F}_2: \quad \Phi_0\{1\} = \frac{\alpha\Phi_0\{2\}}{\gamma\Phi_0\{2\} + \beta} \tag{19}$$

Combining the above equations we obtain the following second order equation for $\Phi_0\{1\}$:

$$\gamma\Phi_0\{1\}^2 + ((\beta - \alpha) - \delta\gamma)\Phi_0\{1\} + \delta\alpha = 0 \tag{20}$$

where

$$\alpha = |\Phi_{yu}\{1\}|^2, \quad \beta = |\Phi_{yu}\{2\}|^2, \quad \gamma = \tilde{\Phi}_y, \quad \delta = \tilde{\Phi}_u$$

Notice that α, β, γ, δ can be calculated with the available data.

We then have that there exist a solution [4] iff

$$\Delta \geq 0 \Leftrightarrow [(\beta - \alpha) - \delta\gamma]^2 \geq 4\alpha\gamma\delta \tag{21}$$

Now by assumption $\tilde{\Phi}_u\tilde{\Phi}_y = \gamma\delta < 0$. We see that this condition is sufficient to obtain only one solution (see Figure 3).

We finally analyze a case which presents some potential ambiguity.

(v) Here, we have the same conditions as in (iv) save that $\tilde{\Phi}_u\tilde{\Phi}_y = \gamma\delta \geq 0$. Here, the quadratic equation has potentially two solutions. We can obtain further insight as follows:

(1) We observe that the second derivative of the curve $\mathcal{F}_2$ is given by:

$$\frac{\partial^2\Phi_0\{1\}}{\partial\Phi_0\{2\}^2} = -\frac{2\alpha\beta\gamma}{(\gamma\Phi_0\{2\} + \beta)^3} = -\frac{\beta\gamma\Phi_0\{1\}^3}{\alpha^2\Phi_0\{2\}^3} \tag{22}$$

Thus ,if $\gamma > 0$ then the function is concave since $\Phi_0\{i\} > 0$. If $\gamma < 0$ then function is convex since $\Phi_0\{i\} > 0$.

(2) Next, we examine the first derivative of the curve $\mathcal{F}_2$ at the origin, namely:

$$\left.\frac{\partial\Phi_0\{1\}}{\partial\Phi_0\{2\}}\right|_{\Phi_0\{2\}=0} = \frac{\alpha}{\beta} \tag{23}$$

We see that the first derivative is greater than one when $\gamma > 0$ and $\delta > 0$, and less than one for $\gamma < 0$ and $\delta < 0$. We can see in Figure 3 that there exist two solutions for this case. Thus, there is a potential ambiguity. However,

[4] We assume that this holds since, by assumption, the true system satisfies the model.

using the bounds found in Theorem 12 it is possible to isolate a unique system. Notice that it is possible to check this condition with the available data. □

Remark 14. Using assumption 6 and an extra assumption such as $\tilde{\Phi}_u \neq 0$, or $\tilde{\Phi}_y \neq 0$, it is possible to establish identifiability using a similar procedure to the one in the proof of Theorem 14 in [Agüero and Goodwin, 2008]. ▽▽▽

6. COMMENTS AND CONCLUSIONS

In this paper we have analyzed the identifiability of EIV dynamic systems when the data is non-stationary. The non-stationarity feature is described via structural change in one or more of the transfer functions. We have analyzed the problem by considering that the data is collected from two different scenarios. We have found bounds for the system when only one of the transfer functions is the same in both scenarios. We have also given conditions under which the system is identifiable.

REFERENCES

J. C. Agüero and G. C. Goodwin. Identifiability of errors in variables dynamic systems. *Automatica*, 44(2):371–382, 2008.

B. D. O. Anderson. Identification of scalar errors-in-variables models with dynamics. *Automatica*, 21(6):709–716, 1985.

B. D. O. Anderson and M. Deistler. Identifiability in dynamic errors-in-variables model. *Journal of Time Series Analysis*, 5(1):001–013, 1984.

B. D. O. Anderson, M Green, and M. Deistler. Errors-in-variables identification: Linear gaussian stationary results. In *10th IFAC world congress*, pages 294–299, Munich, Germany, 1987.

S. Beghelli, R. P. Guidorzi, and U. Soverini. The frisch scheme in dynamic system identification. *Automatica*, 26(1):171–176, 1990.

C. L. Cheng and J. W. Van Ness. *Kendall's library of statistics: Statistical regression with measurement error*, volume 6. Edward Arnold, 1999.

R. V. Churchill and J. W. Brown. *Complex variables and applications.* McGraw Hill Book Company, 5th edition, 1990.

M. Deistler. Linear dynamic errors-in-variables models. *Essays in time series and allied processes. Edited by J. Gani and M. B. Priestley. Journal of Applied Probability*, 23 A:23–39, 1986.

R. Diversi, R.and Guidorzi and U. Soverini. Maximum likelihood identification of noisy input-output models. *Automatica*, 43(3):464–472, 2007.

M. Ekman, M. Hong, and T. Söderström. Identification of linear system with errors in variables using separable nonlinear least-squares. In *14th IFAC symposium on system identification*, Newcastle, Australia, 2006.

W. A. Fuller. *Measurement error models.* Wiley, 1987.

P. Giordani, R. Kohn, and D. van Dijk. A unified approach to nonlinearity, structural change, and outliers. *Journal of Econometrics*, 137(1):112–133, 2007.

F. P. Greenleaf. *Introduction to complex variables.* W. B. Saunders company, 1972.

K. Lau, J. H. Braslavsky, J. C. Agüero, and G. C. Goodwin. Application of non-stationary EIV methods to transient electromagnetic mineral exploration. In *IFAC World Congress*, 2008.

L. Ljung, I. Gustavsson, and T. Söderström. Identification of linear, multivariable systems operating under linear feedback control. *Automatic Control, IEEE Transactions on*, 19(6):836–840, 1974.

G. S. Maddala and I. M. Kim. *Unit roots, cointegration and structural change.* Cambridge University Press, 1998.

K. Mahata. An improved bias-compensation approach for errors-in-variables model identification. *Automatica*, 43 (8):1339–1354, 2007.

A. Maravall. *Identification in dynamic shock-error models.* Lecture notes in economics and mathematical systems 165, Springer-Verlag, Berlin, 1979.

I. Markovsky, A. Kukush, and S. Van Huffel. On errors-in-variables estimation with unknown noise variance. In *14th IFAC symposium on system identification*, Newcastle, Australia, 2006.

E. Nowak. Identifiability in multivariable dynamic linear error-in-variables models. *Journal of the American Statistical Association*, 87(419):714–723, 1992.

A. V. Oppenheim and R. W. Schafer. *Discrete-time signal processing.* Prentice Hall, 1989.

R. Pintelon and J. Schoukens. Frequency domain maximum likelihood estimation of linear dynamic errors-in-variables models. *Automatica*, 43(4):621–630, 2007.

M. B. Priestley. *Spectral analysis and time series.* Elsevier Academic Press, 1981.

O. Reiersøl. Identifiability of a linear relation between variables which are subject to errors. *Econometrica*, 18 (4):375–389, 1950.

K. S. Sin and G. C. Goodwin. Checkable conditions for identifiability of linear systems operating in closed loop. *Automatic Control, IEEE Transactions on*, 25(4):722–729, 1980.

T. Söderström. Spectral decomposition with application to identification. In F. Archetti and M. Cugiani, editors, *Numerical techniques for stochastic systems*, pages 59–79. North-Holland, Amsterdam, 1980.

T. Söderström. Errors-in-variables methods in system identification. *Automatica*, 43(6):939–958, 2007.

T. Söderström, L. Ljung, and I. Gustavsson. Identifiability conditions for linear multivariable systems operating under feedback. *Automatic Control, IEEE Transactions on*, 21(6):837–840, 1976.

T. Söderström, U. Soverini, and K. Mahata. Perspectives on errors-in-variables estimation for dynamic systems. *Signal Processing*, 82:1039–1154, 2002.

V. Solo. Identifiability of time series models with errors in variables. *Journal of Applied Probability*, 23 A:63–71, 1986.

A. Wald. The fitting of straight lines if both variables are subject to error. *The Annals of Mathematical Statistics, Series B*, 11(3):284–300, 1940.

W. X. Zheng. A bias correction method for identification of linear dynamic Errors-in-Variables models. *IEEE Transactions on Automatic Control*, 47(7):1142–1147, 2002.