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Distributed Geometric Control of Wave Equation

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Abstract:
An approach for the geometric control of a one-dimensional non-autonomous linear wave equation is presented. The idea consists in reducing the wave equation to a set of first-order linear hyperbolic equations. Then based on geometric control concepts, a distributed control law that enforces stability and output tracking in the closed-loop system is designed. The presented control approach is applied to obtain a distributed control law that brings a stretched uniform string, modeled by a wave equation with Dirichlet boundary conditions, to rest in infinite time by considering the displacement of the middle point of the string as the controlled output. The controller performances have been evaluated in simulation.

Keywords: Distributed parameter systems, partial differential equation, wave equation, geometric control.

1. INTRODUCTION

The behavior of distributed parameter systems or infinite-dimensional systems is described by partial differential equations (PDEs). It occupies an important place in control and systems theory and constitutes an important research area, see e.g. Ray (1989); Omatu and Seinfeld (1989); Curtain and Zwart (1995); Lasiecka (1995); Christofides (2001a,b) for more information and references. Typically such systems are characterized by variables depending both on spatial location and time, and include the transport-reaction processes and wave equation problems.

In recent years, several control methods that directly take into account the distributed nature of the processes have been developed specifically for quasi-linear first-order hyperbolic systems (e.g. convection-reaction processes) and parabolic systems (e.g. diffusion-reaction processes). The book by Christofides (2001a) includes many results, applications and literature citations concerning transport-reaction processes. Some books are devoted to linear distributed parameter systems (Curtain and Zwart, 1995; Bensoussan et al., 2007)

However, second-order hyperbolic equations arising in wave propagation problems have been little studied with respect to control design, and most contributions deal with boundary control rather than distributed control. Control contributions include boundary control (Gunzburger and Nicolaides, 1989; Svobodny, 1992; Bastos, 1999), controllability studies (Vancostenoble, 2000; Curtain and Zwart, 1995; Bui, 2005), optimal control (Lagnese and Leugering, 2003) and the stabilization problem (Glowinski and He, 2003; Berrahmoune, 2004). To stabilize a system described by a one-dimensional linear wave equation in a bounded domain with appropriate boundary conditions, Morgül (1994) proposed a dynamic controller applied at the free end of the system. Chambolle and Santosa (2002) studied an initial boundary-value problem of a wave equation with time-dependent sound speed. The objective is to determine a sound-speed function which damps the vibration of the system, and by considering the case where the sound speed takes only two values, a simple control law is proposed. Alli and Singh (2004) addressed the problem of design of collocated and non collocated controllers for a uniform bar whose dynamics are described by the wave equation, without structural damping. The root-locus technique is used to control the non-collocated system by means of a time delay controller.

In this work, an approach, based on geometric control concepts, is proposed to design a distributed control law that ensures the stability and output tracking in closed-loop for a system described by a one-dimensional wave equation with a distributed input and an output defined as a function of state variables.

The proposed design approach consists in reducing the wave equation to a set of first-order linear hyperbolic equations. Then a general distributed control law has been derived for systems modeled by a wave equation, by exploiting notions of geometric control such as exposed in (Christofides, 2001a; Christofides and Daoutidis, 1996; Isidori, 1995). The approach is illustrated by an application concerning a transverse displacement of a uniform string with Dirichlet boundary conditions. The objective is to bring the string to rest in infinite time by imposing a desired displacement of the string middle point. Simulation results show that the proposed controller ensures a perfect

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tracking of the reference input. Finally, the implementation issue of the designed controller is discussed.

The remaining part of the paper is organized as follows: in section 2, the formulation of the distributed control problem of a wave equation is given. In section 3, the proposed design approach of the controller is discussed and followed by an illustrative application in Section 4. Section 5 is devoted to the conclusion.

2. CONTROL PROBLEM

2.1 Problem formulation

Non-autonomous systems modelled by the one-dimensional linear wave equation under the following state-space representation are studied

$$\frac{\partial^2 x(z,t)}{\partial t^2} = \gamma^2 \frac{\partial^2 x(z,t)}{\partial z^2} + u(z,t), \text{ for } 0 < z < L \tag{1}$$

subject to the boundary conditions

$$x(0,t) = x_0, \quad x(L,t) = x_L \tag{2}$$

and the initial conditions

$$x(z,0) = g_1(z), \quad \frac{\partial x(z,0)}{\partial t} = g_2(z) \tag{3}$$

with $\gamma > 0$ and g_1, g_2 given functions. The output variable $y(z,t)$ to be controlled by manipulating the input $u(z,t)$ is

$$y(t) = \mathcal{C}\big(h(x(z,t))\big) \tag{4}$$

where h is a linear function of the form $k\,x(z,t)$ and $\mathcal{C}(\,.\,)$ is a bounded linear operator depending on the desired performance specifications. In this work, this latter follows the form

$$y(t) = \mathcal{C}\big(k\,x(z,t)\big) = \int_0^L c(z)\,k\,x(z,t)\,dz \tag{5}$$

where $c(z)$ is a known smooth function of z and k is a scalar. As a specification of the control problem, let us assume that the distributed input $u(z,t)$ is given as follows

$$u(z,t) = b(z)\,v(t) \tag{6}$$

where $b(z)$ is a known smooth function of z.

The control problem consists in determining the control law $v(t)$, as a function of the variable $x(z,t)$, so as to achieve desired performances of the controlled variable $y(t)$. In order to design such a control law, we propose to reduce the wave equation (1) to a set of first-order linear hyperbolic equations, which allows to synthesize a state feedback controller that enforces output tracking and stability of the closed-loop system, following the control methodology proposed in (Christofides and Daoutidis, 1996; Christofides, 2001a).

Note that, in the formulation of the control problem, the two functions $b(z)$ and $c(z)$ must be specified. Thus, the following assumption ensures that, for systems of the form (1), the relative order σ of $y(t)$ with respect to $v(t)$ exists and is equal to 2.

Assumption. *The functions $b(z)$ and $c(z)$ are smooth and satisfy*

$$\int_0^L c(z)\,b(z)\,dz \neq 0 \tag{7}$$

In addition, the approximate controllability and observability of the system (1) depend on the shaping functions $b(z)$ and $c(z)$.

2.2 Controllability

This section gives the condition for approximate controllability of system (1). Since the considered system is an infinite-dimensional one, the controllability property is difficult to be proved, and it is often much simpler to prove the approximate controllability (Curtain and Zwart, 1995). According to Ray (1989), approximate controllability is sufficient for adequate design of a controller.

The controllability of system (1) has been discussed in detail by Curtain and Zwart (1995, page 165), and it is shown that the system is approximately controllable if and only if

$$\int_0^L b(z)\phi_n(z)\,dz \neq 0, \text{ for } n \geq 1 \tag{8}$$

where

$$\phi_n(z) = \sin(\frac{n\,\pi\,z}{L}) \tag{9}$$

are the eigenfunctions for the partial differential equation in (1).

3. CONTROLLER DESIGN

The proposed design approach is based on the reduction of the wave equation to a set of first-linear hyperbolic equations, then, by use of geometric control concepts, a general control law is derived for the formulated control problem.

As indicated above, in this section, it is assumed that the shaping functions $b(z)$ and $c(z)$ satisfy the conditions (7) and (8).

3.1 Reduction of the wave equation

To reduce the wave equation (1) to a set of first-order linear hyperbolic equations, it can be noticed that it suffices to introduce the new state vector and its auxiliary variables

$$\begin{aligned} X(z,t) &= \left[x(z,t), \frac{\partial x(z,t)}{\partial t}, \frac{\partial x(z,t)}{\partial z}\right]^T \\ &= \big[X_1(z,t), X_2(z,t), X_3(z,t)\big]^T \end{aligned} \tag{10}$$

Thus equation (1) can be reduced to the following set of first order linear hyperbolic equations (Wang, 2007)

$$\begin{aligned} \frac{\partial X_1(z,t)}{\partial t} &= X_2(z,t) \\ \frac{\partial X_3(z,t)}{\partial t} &= \frac{\partial X_2(z,t)}{\partial z} \\ \frac{\partial X_2(z,t)}{\partial t} &= \gamma^2 \frac{\partial X_3(z,t)}{\partial z} + b(z)\,v(t) \end{aligned} \tag{11}$$

with appropriate boundary and initial conditions, which are not required to specify the control design.

Since equations (11) are linear, the control law $v(t)$ takes the following form

$$v(t) = \mathcal{Q}\, X(z,t) + q\, v^{ext}(t) \tag{12}$$

which will be expressed, for implementation purpose, as a function of the state variable $x(z,t)$, of the original system (1), using relations (10). So the control law (12) can be written as

$$v(t) = \mathcal{S}\, x(z,t) + s\, v^{ext}(t) \tag{13}$$

Note that $\mathcal{Q}$ is a vector of linear operators while $\mathcal{S}$ is a linear operator, q and s are invertible functionals, and $v^{ext}(t)$ is the reference or external input.

To solve the formulated control problem, the concept of relative order is introduced.

3.2 Relative order

According to relation (6), it is clear that the relative order between the output $y(t)$ and the input $u(t)$ is the same as between the output $y(t)$ and the input $v(t)$. Thus, to determine the relative order, one can consider the successive derivatives of the output (4) with respect to time, which yields

$$\begin{aligned}
y(t) &= \mathcal{C}\big(kx(z,t)\big)\\
\frac{dy(t)}{dt} &= \frac{d}{dt}\mathcal{C}\big(kx(z,t)\big) = \mathcal{C}\left(k\frac{dx(z,t)}{dt}\right) = \mathcal{C}\big(kX_2(z,t)\big)\\
\frac{dy^2(t)}{dt^2} &= \frac{d}{dt}\mathcal{C}\big(kX_2(z,t)\big) = \mathcal{C}\left(k\frac{\partial X_2(z,t)}{\partial t}\right)\\
&= \mathcal{C}\left(k\gamma^2\frac{\partial X_3(z,t)}{\partial z}\right) + \mathcal{C}\big(kb(z)\big)\, v(t)
\end{aligned} \tag{14}$$

Now, since

$$\mathcal{C}\big(kb(z)\big) = k\mathcal{C}\big(b(z)\big) = k\int_0^L c(z)\, b(z)\, dz \neq 0 \tag{15}$$

the relative order of $y(t)$ with respect to $v(t)$ is equal to 2.

In the same way, by substituting the control law (13) into the system (11), the closed-loop system is obtained, and by differentiating the output $y(t)$ with respect to time, it can be demonstrated that the control law (13) preserves the relative order σ. This means that the relative order of the output $y(t)$ with respect to the external input $v^{ext}(t)$ in the closed-loop system is also equal to 2, which suggests requesting the following input-output response of the closed-loop system

$$\frac{d^2y(t)}{dt^2} + c_1\frac{dy(t)}{dt} + c_0\, y(t) = v^{ext}(t) \tag{16}$$

where the adjustable controller parameters c_0 and c_1 are chosen to guarantee the input-output stability and enforce the desired performance specifications for the closed-loop system (Christofides and Daoutidis, 1996; Christofides, 2001a).

3.3 State feedback control

As demonstrated above, a second-order input-output response is requested in the closed loop system. Thus from (14) and (16), it is easy to show that the control law $v(t)$, under the form (12), that enforces this response is

$$\begin{aligned}
v(t) = &\frac{1}{k\,\mathcal{C}\big(b(z)\big)} \times \Big\{v^{ext}(t) - c_0\, k\,\mathcal{C}\big(x(z,t)\big)\\
&- c_1\, k\,\mathcal{C}\Big(X_2(z,t)\Big) - k\,\gamma^2\,\mathcal{C}\Big(\frac{\partial X_3(z,t)}{\partial z}\Big)\Big\}
\end{aligned} \tag{17}$$

which can be written under the form (13) as follows

$$\begin{aligned}
v(t) = &\frac{1}{k\,\mathcal{C}\big(b(z)\big)} \times \Big\{v^{ext}(t) - c_0\, k\,\mathcal{C}\Big(x(z,t)\Big)\\
&- c_1\, k\,\mathcal{C}\Big(\frac{\partial x(z,t)}{\partial t}\Big) - k\,\gamma^2\,\mathcal{C}\Big(\frac{\partial^2 x(z,t)}{\partial z^2}\Big)\Big\}
\end{aligned} \tag{18}$$

Thus, for on line implementation, the estimation of the state $x(z,t)$ is required, and since the controller is of distributed nature, the calculation of the control requires algebraic manipulation, differentiation and integration with respect to space.

4. APPLICATION EXAMPLE

4.1 System description and modeling

Consider the displacement applied to a uniform string of length L in the coordinate system shown in Figure 1. The extremities of a string are fixed at the z-axis. The displacement x, from the z-axis, is a function of time t and of the spatial coordinate z.

ρ being the mass density of the string expressed in units of mass per unit length is assumed to be constant. α is the tension in the string. When the string is plucked, it is assumed that the tension remains constant throughout the string which corresponds to a hypothesis of small displacements. The force of gravity is also assumed to be much weaker than the tension ($\rho\, g\, L \ll \alpha$) so that it does not influence the motion of the string notably and therefore can be neglected.

It is assumed that a distributed force $u(z,t) = b'(z)\, v(t)$, with units of force per unit length is applied to the string with $b'(z)$ following the quadratic Bezier function

$$b'(z) = \rho\, z\,(L - z) \tag{19}$$

so that, at the boundaries, the force is not active, i.e. $u(0,t) = 0$ and $u(L,t) = 0$.

Initially, the string is at rest, i.e. $u(x,0) = 0$, therefore $x(z,0) = 0$ for $0 \leq z \leq L$. Thus, the motion of the string (Curtain and Zwart, 1995) is described by

$$\rho\,\frac{\partial^2 x(z,t)}{\partial t^2} = \alpha\,\frac{\partial^2 x(z,t)}{\partial z^2} + u(z,t) \tag{20}$$

As the string is fixed at its extremities, this leads to Dirichlet boundary conditions

$$x(0,t) = 0, \quad x(L,t) = 0 \tag{21}$$

The following initial conditions apply

$$x(z,0) = 0, \quad \frac{\partial x(z,0)}{\partial t} = 0 \tag{22}$$

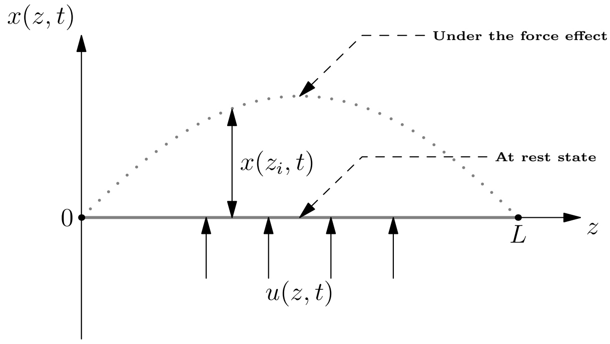

Fig. 1. Coordinate system for a string that will undergo vertical displacements

4.2 Control law

Let us consider that the desired objective is to find the control law $v(t)$, i.e. a force, that allows to bring the string to rest in infinite time by imposing a desired displacement $x^d(t)$ at position $z = L/2$. In this case, the controlled output takes the form

$$y(t) = \int_0^L \delta(z - L/2)\, x(z,t)\, dz = x(\frac{L}{2}, t) \tag{23}$$

which means that $k = 1$ and $c(z) = \delta(z - L/2)$ where δ is the Dirac distribution.

Now, by setting $\gamma^2 = \alpha/\rho$, according to the derived control law (18), taking into account a reference the control law results

$$\begin{aligned} v(t) &= \frac{1}{b(L/2)} \times \Big\{ v^{ext}(t) - c_0\, x(L/2,t) \\ &\quad - c_1 \frac{\partial x(z,t)}{\partial t}\Big|_{z=L/2} - \frac{\alpha}{\rho}\frac{\partial^2 x(z,t)}{\partial z^2}\Big|_{z=L/2} \Big\} \\ &= \frac{1}{b(L/2)} \times \Big\{ v^{ext}(t) - c_0\, y(t) \\ &\quad - c_1 \frac{\partial y(t)}{\partial t} - \frac{\alpha}{\rho}\frac{\partial^2 x(z,t)}{\partial z^2}\Big|_{z=L/2} \Big\} \end{aligned} \tag{24}$$

Note that for the formulated problem, the chosen functions $b(z)$ and $c(z)$ satisfy the Eqs. (7) and (8) corresponding to the existence of the relative order and the controllability conditions, respectively. However, to take into account the reference trajectory denoted as "ref", the previous control law has been modified as

$$\begin{aligned} v(t) = \frac{1}{b(L/2)} &\times \{ v^{ext}(t) - c_0\,(y(t) - y^{ref}(t)) \\ &- c_1 \left(\frac{\partial y(t)}{\partial t} - \left(\frac{\partial y(t)}{\partial t}\right)^{ref} \right) - \frac{\alpha}{\rho}\, \frac{\partial^2 x(z,t)}{\partial z^2}\Big|_{z=L/2}^{ref} \Big\} \end{aligned} \tag{25}$$

The control law (26) can be seen as the addition of three terms

$$v(t) = \frac{1}{b(L/2)}(v^{ext}(t) + v_{pole}(t) + v_{steady}(t)) \tag{26}$$

where v_{pole} stands for the pole placement term based on c_0 and c_1, while v_{steady} concerns the second order spatial derivative. Clearly, the nature of v_{steady} is different from the other terms as it is related to the concavity of the string. If the reference y^{ref} is known, the steady value of this second order spatial derivative can be calculated a priori as the solution of the following stationary equation resulting from the dynamic model (20) of the string

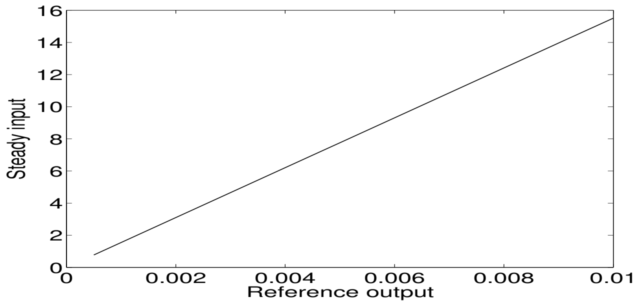

Fig. 2. Steady input with respect to the reference output

$$0 = \alpha \frac{\partial^2 x(z,t)}{\partial z^2} + b'(L/2)v \tag{27}$$

This equation is solved iteratively until convergence of $x(L/2)$ towards y^{ref}, which yields the numerical value of $v = v_{steady}$. Fig 2 shows that the steady input v_{steady} depends linearly on the reference output.

The control law (26) allows the application of the linear control theory to the resulting linear input-output system to handle uncertainty and unmodeled dynamics, by adopting the control strategy given by Fig. 3. In our study, a PI controller is used for this purpose, thus the external input $v^{ext}(t)$ is defined by means of a PI controller (Kravaris and Kantor, 1990) as following

$$v^{ext}(t) = K_c \left[(y^d(t) - y(t)) + \frac{1}{\tau_I}\int_0^t (y^d(\xi) - y(\xi))\, d\xi \right] \tag{28}$$

where K_c, τ_I and $y^d(t)$ are respectively the proportional gain, the integral time constant and the desired output.

4.3 Evaluation of controller performance

In this section, the capabilities of the controller to track the reference input in presence of model uncertainties are addressed. In the control law (26), the term v_{steady} plays the most important role. The other terms stand for noise and model uncertainty. To evaluate the controller robustness, the value of α considered for the calculation of v_{steady} has been chosen equal to $0.8 \times \alpha$ (i.e. an error of 20%), whereas α is used for the model of the string.

The spatial partial derivative of the string model is approximated by the finite difference

$$\frac{\partial^2 x(z,t)}{\partial z^2} = \frac{x(z + \Delta z, t) - 2\, x(z,t) + x(z - \Delta z, t)}{(\Delta z)^2} \tag{29}$$

for $z \in [\Delta z, L - \Delta z]$ where $\Delta z = L/N$ is the distance between two adjacent discretization points, with N number of discretization points taken as $N = 21$. For $z = 0$ and $z = L$, specific non centered finite differences are used.

The control law has been implemented under discrete form and the sampling period is equal to $T_s = 0.25$s. The values used for system and controller parameters are given in Table 1 and Table 2. The controller parameters are determined based on the pole placement technique. The string undergoes severe sustained oscillations if it is submitted to any sudden change step input. For this reason, the set point and the input v_{steady} were filtered by a second order filter with a time constant equal to 2 s and damping factor equal to 1.2. The parameter c_1 was chosen small because of the large sensitivity of the string which tends to oscillate to step variations, so that the derivative undergoes large variations which could be also avoided by adequate filtering.

In the simulation run, two set point steps have been specified at $t = 100$ s and $t = 200$ s corresponding respectively to $y^d(t) = 8$ mm and $y^d(t) = 4$ mm. The output (Fig. 4) converges slowly towards the desired reference trajectory in spite of the parameter uncertainty, whereas the control moves (Fig. 5) are physically acceptable. In addition, the spatial profiles of the displacement $x(z, t)$ at $t = 100$ s and $t = 200$ s (Fig. 6) are realistic and correspond to the profile defined by the choice of the shaping function $b(z)$. In Fig 7, it appears clearly that the steady term of control law (26) takes the larger part in the manipulated input, but that the other actions are necessary to obtain exactly the desired trajectory. The output labelled as "steady" obtained with only the steady input displays a permanent deviation. A faster convergence towards the reference trajectory could be obtained by decreasing the value of the parameter β of Table 2, corresponding to an increase of the role of the PID with respect to the state feedback law. A three-dimensional plot of the profiles (Fig. 8) during the tracking of the reference trajectories confirms the good behavior of the controller. In the case where no parameter uncertainty is considered, the output follows very closely the reference trajectory. This case was not illustrated in any figure as it presents less interest than the robustness study.

Symbol	Value	Designation
ρ	0.5	String mass density $[\mathrm{kg\,.\,m^{-1}}]$
α	20	String tension $[\mathrm{kg\,.\,m\,.\,s^{-2}}]$
L	1	String length $[\mathrm{m}]$

Table 1. System parameters

	State feedback law		PI Controller	
Parameter	c_0	c_1	K_c	τ_I
Value	$\beta\, 2.15\omega_0^2$	$0.01\,\omega_0$	$(1-\beta)\, 2.15\omega_0^2$	$0.5\, K_c/\omega_0^3$

Table 2. Controller tuning parameters (with $\omega_0 = 1.1\sqrt{v_{steady}}$ and $\beta = 0.8$)

5. CONCLUSION

In this paper, the geometric control of a one-dimensional non autonomous linear wave equation is investigated, and a general distributed control law, which enforces stability and output tracking is derived. The main idea consists in reducing the wave equation to a set of first-order linear equations, then the design problem is solved based on the concept of relative order. It is shown that under certain assumptions made concerning the control problem formulation, a second-order input/output response results in closed loop, and desired dynamics can be achieved by tuning the controller parameters. Note that the proposed approach can be easily applied to a quasilinear wave equation.

The effectiveness of the proposed approach is illustrated by an application concerning the control of the displacement of a string. In the final control law, a steady term playing the larger role can be calculated a priori from the reference trajectory. The robustness is demonstrated by imposing a parameter uncertainty. In these conditions, the controller still behaves correctly and ensures a satisfactory tracking.

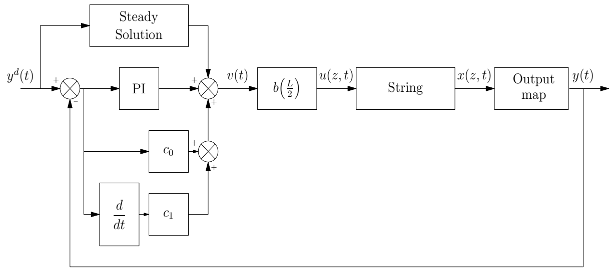

Fig. 3. Control strategy

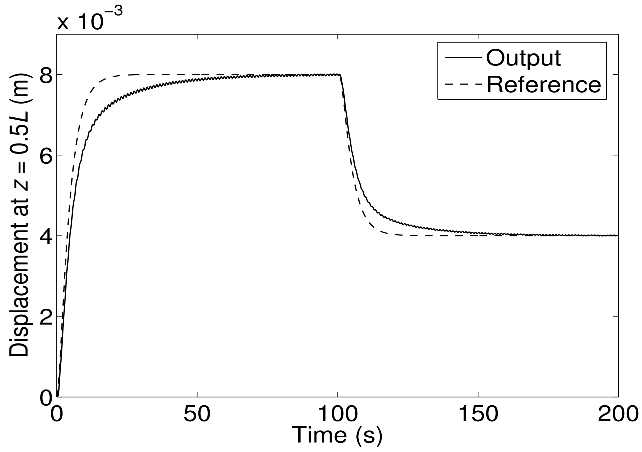

Fig. 4. Set point change: Controlled output

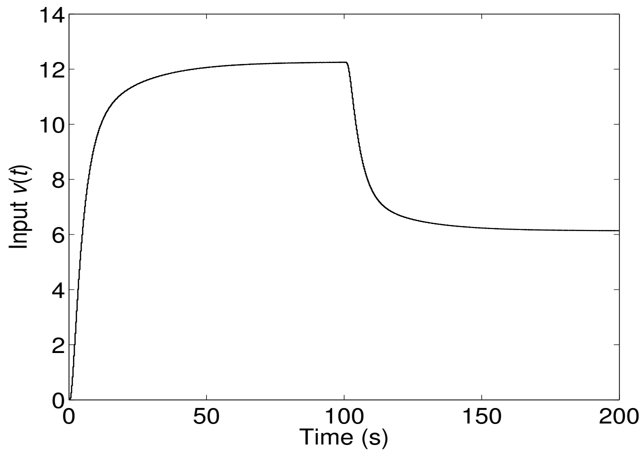

Fig. 5. Set point change: Manipulated input

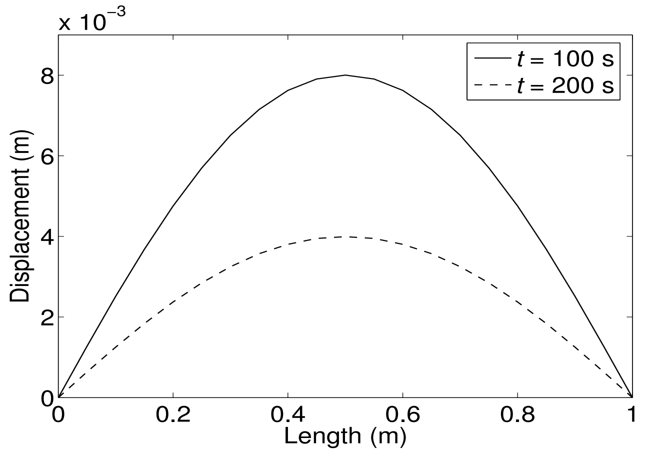

Fig. 6. Set point change: Profiles of the respective displacements at $t = 100\,\mathrm{s}$ and at $t = 200\,\mathrm{s}$.

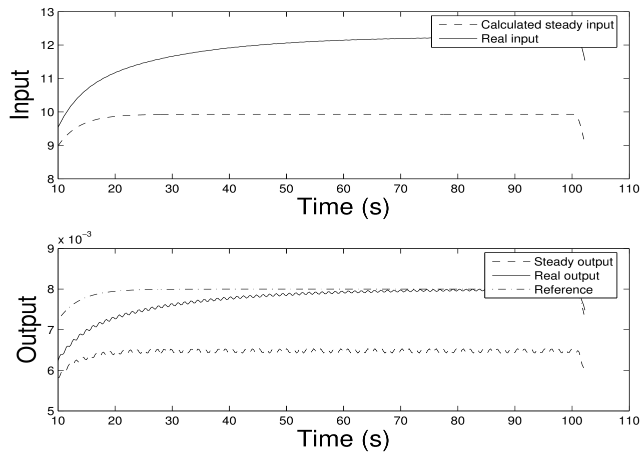

Fig. 7. Set point change: Comparison between the actual manipulated input and the "steady" input (top), Comparison between the actual output and the "steady" output with reference (bottom)

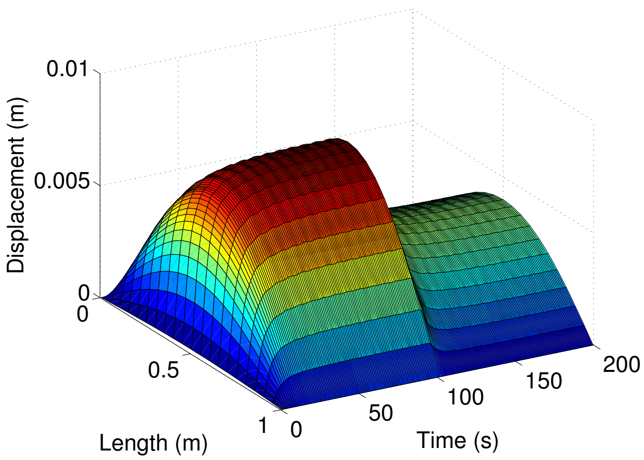

Fig. 8. Set point change: Profile of evolution of the string displacement

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