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Delay-Dependent Regional Stability of a Class of Uncertain Nonlinear State-Delayed Systems

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Abstract: This paper proposes a convex approach to regional stability analysis of a class of nonlinear state-delayed systems subject to convex-bounded parameter uncertainty. Delay-dependent conditions are developed to ensure the system robust local stability and obtain an estimate of a domain of attraction of the origin inside a given polytopic region of the state-space. The proposed approach is based on a Lyapunov-Krasovskii functional with polynomial dependence on the system state and uncertain parameters and is formulated in terms of linear matrix inequalities. Numerical examples illustrate the potentials of the derived results.

1. INTRODUCTION

Time-delay and modeling uncertainty are frequently encountered in control problems of many dynamical systems and very often are the cause of instability and poor performance; see, e.g. Gu *et al.* [2003]. Over the last decade considerable attention has been devoted to the problem of robust stability analysis for linear state-delayed systems subject to parametric uncertainty; see, for instance, Xie and de Souza [1993], Li and de Souza [1997], de Souza and Li [1999], Moon *et al.* [2001], Fridman and Shaked [2003], Gu *et al.* [2003], Xia and Jia [2003], He *et al.* [2004], Xu and Lan [2005], and the references therein. In the context of nonlinear state-delayed systems, the problem of stability analysis is much more involved and a few approaches have been proposed in the literature, such as, the central manifold theory together with Taylor's expansion (Fridman [2003]), first-order approximation methods (Melchor-Aguillar and Niculescu [2007]) and the sum of squares approach for polynomial systems (Papachristodoulou [2004]). In spite of these developments, to the authors' knowledge, the theory of stability analysis of uncertain nonlinear systems with delayed states has not yet been fully developed.

This paper deals with the stability of a class of nonlinear state-delayed systems subject to convex-bounded parameter uncertainty. A delay-dependent linear matrix inequality (LMI) method of robust regional stability analysis is developed based on a parametric Lyapunov-Krasovskii functional with polynomial dependence on the system state and uncertain parameters. An estimate of a domain of attraction of the origin inside a given polytopic region of the state-space is also provided. The proposed method can handle the class of systems with rational functions of the state variables and uncertain parameters without singularities at the origin as well as some trigonometric nonlinearities. When specialized to uncertain linear state-delayed systems, the method of this paper provides a novel robust stability result based on a polynomially parameter-dependent Lyapunov-Krasovskii functional. For simplicity of presentation, the paper treats the case of a single and constant delay.

Notation. $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices, $\|\cdot\|$ is the Euclidean vector norm, 0_n and $0_{m\times n}$ are the $n\times n$ and $m\times n$ matrices of zeros, and I_n is the $n\times n$ identity matrix. For a real matrix S, S' denotes its transpose, $\mathrm{Her}\{S\}$ stands for $S+S'$ and $S>0$ means that S is symmetric and positive-definite. For two polytopes $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{A}\times\mathscr{B}$ denotes the meta-polytope obtained by the cartesian product and $\vartheta(\mathscr{A})$ is the set of all vertices of $\mathscr{A}$. The space of continuously differentiable functions $\phi:[-\tau,0]\to\mathbb{R}^n$ with finite norm $\|\phi\|_\tau=\sup_{-\tau\le t\le 0}\|\phi(t)\|$ is denoted by $\mathscr{C}_\tau^n$, $x_t\in\mathscr{C}_\tau^n$ is a segment of the function $x(\cdot)$ given by $x_t(\theta)=x(t+\theta)$, $\forall\,\theta\in[-\tau,0]$, and $x_t\in\mathscr{D}\subseteq\mathbb{R}^n$ means that $x(t+\theta)\in\mathscr{D}$, $\forall\,\theta\in[-\tau,0]$.

2. PROBLEM STATEMENT

Consider the nonlinear state-delayed system

$$\begin{cases}\dot{x}(t)=f(x(t),\tilde{x}(t),\delta), & \tilde{x}(t):=x(t-\tau),\\ x(t)=\phi(t),\ \forall\, t\in[-\tau,0]\end{cases}\tag{1}$$

where $x(t)\in\mathscr{X}\subseteq\mathbb{R}^n$ is the state vector, $\phi(\cdot)\in\mathscr{C}_\tau^n$ is the system initial function, $\tau\in\mathbb{R}_+$ is a constant time-delay, $\delta\in\mathbb{R}^{n_\delta}$ is a vector of uncertain constant parameters, and $f(x,\tilde{x},\delta)$ is a nonlinear vector function that satisfies the standard conditions for existence and uniqueness of a solution to (1) for all (x,δ) of interest (Hale and Lunel [1993]). The set $\mathscr{X}$ represents a given polytopic region of the state-space containing the origin in which local stability analysis will be performed. It is assumed that system (1) satisfies the following assumptions:

A1 The uncertain parameter vector $\delta:=[\,\delta_1\cdots\delta_{n_\delta}\,]'$ belongs to a given polytopic set $\Delta\subseteq\mathbb{R}^{n_\delta}$.

A2 The origin $x=0$ is an equilibrium point for all $\delta\in\Delta$.

Remark 1. For nonzero equilibrium points, x_e, that are independent of the parameter δ, Assumption **A2** is not restrictive as, without loss of generality, the state vector can be shifted to $\chi:=x-x_e$. As a result, in the new system coordinates the origin is an equilibrium point for all δ. When x_e depends on δ, the system in (1) can still be redefined in terms of χ with the additional algebraic constraint $f(x_e,x_e,\delta)=0$, and where $\chi=0$ is an equilibrium for all δ. In the latter case, the methodology proposed in the paper can be adapted to incorporate the underlying algebraic constraint.

It is assumed that the system (1) can be described by the following differential-algebraic representation (DAR):

$$\begin{cases} \dot{x} = A_1(x,\tilde{x},\delta)x + A_2(x,\tilde{x},\delta)\tilde{x} + A_3(x,\tilde{x},\delta)\pi(x,\tilde{x},\delta) \\ 0 = \Omega_1(x,\tilde{x},\delta)x + \Omega_2(x,\tilde{x},\delta)\tilde{x} + \Omega_3(x,\tilde{x},\delta)\pi(x,\tilde{x},\delta) \end{cases} \quad (2)$$

where $\pi(x,\tilde{x},\delta) \in \mathbb{R}^{n_\pi}$ is an auxiliary nonlinear vector function of $(x, \tilde{x}, \delta)$ representing nonlinear terms in $f(x, \tilde{x}, \delta)$ and $A_1(\cdot), A_2(\cdot) \in \mathbb{R}^{n\times n}$, $A_3(\cdot) \in \mathbb{R}^{n\times n_\pi}$, $\Omega_1(\cdot), \Omega_2(\cdot) \in \mathbb{R}^{m\times n}$, and $\Omega_3(\cdot) \in \mathbb{R}^{m\times n_\pi}$ are affine matrix functions of $(x,\tilde{x},\delta)$. To simplify the notation, throughout the paper the arguments of $\pi(\cdot)$, $A_i(\cdot)$ and $\Omega(\cdot)$, $i = 1,2,3$ will often be omitted.

A broad class of nonlinear systems can be represented in the form (2), such as systems with rational nonlinearities, as well as some trigonometric nonlinearities. Indeed, it can be shown that (2) can model the class of systems where $f(\cdot)$ is a rational vector function of $(x(t),\tilde{x}(t),\delta)$ and the origin is an equilibrium point; see, e.g. Coutinho *et al.* [2006]. Note that the DAR of (2) does not involve any linearization of the system (1).

In order to guarantee that the DAR of (2) is well defined, the following assumption is adopted:

A3 The matrix function $\Omega_3(x,\tilde{x},\delta)$ has full column-rank for all $(x,\tilde{x},\delta)$ belonging to $\mathscr{X} \times \mathscr{X} \times \Delta$.

Example 2. To illustrate the DAR of (2), consider a scalar system with

$$f(x,\tilde{x},\delta) = a_1(1+\delta)x + a_2(1+\delta^2)\tilde{x} + bx\tilde{x} - \frac{c_0 x}{m(x)}$$

where $m(x) = c_1 + x + c_2x^2$.

The above system can be written in the DAR of (2) with:

$$A_1 = a_1(1+\delta),\ A_2 = a_2 + bx,\ A_3 = \begin{bmatrix} -a_2\delta & -c_0 & 0 \end{bmatrix},$$

$$\pi = \begin{bmatrix} -\delta\tilde{x} \\ \dfrac{x}{m(x)} \\ \dfrac{x^2}{m(x)} \end{bmatrix},\ \Omega_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},\ \Omega_2 = \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix},\ \Omega_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1+x & c_2x \\ 0 & -x & 1 \end{bmatrix}$$

Note that **A3** is equivalent to $m(x) \neq 0$, which is a regularity condition for $f(x,\tilde{x},\delta)$. □

In the sequel, we shall address the problem of robust regional stability of system (2). Specifically, the aim is to develop LMI based delay-dependent conditions for local asymptotic stability of the equilibrium point $x = 0$ of system (2) in $\mathscr{X}$ for all $\delta \in \Delta$, as well as to provide an estimate $\hat{\mathscr{D}}_a$ of the domain of attraction (DOA) of the origin, defined as the set

$$\mathscr{D}_a = \left\{ \phi \in \mathscr{C}_\tau^n : \lim_{t\to\infty} x(t) = 0,\ \forall \delta \in \Delta \right\}.$$

The DOA estimate $\hat{\mathscr{D}}_a$ proposed in this paper is defined by the following set:

$$\hat{\mathscr{D}}_a = \left\{ \phi \in \mathscr{C}_\tau^n : \alpha_m \|\phi\|_\tau^2 \le 1,\ \alpha_d \|\dot{\phi}\|_\tau^2 \le 1 \right\} \quad (3)$$

where α_m and α_d are positive scalars to be minimized.

We conclude this section by recalling an auxiliary result to be used in this paper.

Lemma 3. (Finsler's lemma). Given matrices $\Psi = \Psi' \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{m\times n}$, then

$$x'\Psi x > 0,\ \forall x \in \mathbb{R}^n : Hx = 0,\ x \neq 0$$

iff there exists a matrix $L \in \mathbb{R}^{n\times m}$ such that $\Psi + LH + H'L' > 0$.

3. PRELIMINARY RESULTS

This section presents basic results needed to derived an LMI method to the robust regional stability analysis. We shall introduce a parameter-dependent Lyapunov-Krasovskii functional candidate and develop some related properties. For notation simplicity, in the sequel the dependence on x, $\tilde{x}$ and δ of vectors and matrices will often be omitted.

This paper uses the following Lyapunov-Krasovskii functional:

$$V(x_t,\delta) = V_1(x,\delta) + V_2(x_t,\delta) + V_3(x_t,\delta) \quad (4)$$

where

$$V_1(x,\delta) = x'\mathscr{P}_1(x,\delta)x \quad (5)$$

$$V_2(x_t,\delta) = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{x}(\alpha)'\mathscr{P}_2(x(\alpha),\delta)\dot{x}(\alpha)d\alpha d\beta \quad (6)$$

$$V_3(x_t,\delta) = \int_{t-\tau}^{t} x(\alpha)'\mathscr{P}_3(x(\alpha),\delta)x(\alpha)d\alpha \quad (7)$$

with $\mathscr{P}_i(x,\delta)$, $i = 1,2,3$ being symmetric polynomial matrix functions of (x,δ) to be specified below, such that $V_1(x,\delta)$, $Z_2(x,\tilde{x},\delta) := \dot{x}'\mathscr{P}_2(x,\delta)\dot{x}$ and $Z_3(x,\tilde{x},\delta) := x'\mathscr{P}_3(x,\delta)x$ are positive polynomials over $\mathscr{X}\times\Delta$.

To obtain an LMI based stability condition, the following structure is adopted for the matrix $\mathscr{P}_i(x,\delta)$, $i = 1,2,3$:

$$\mathscr{P}_i(x,\delta) = \begin{bmatrix} \Theta_i(x,\delta) \\ I_n \end{bmatrix}' P_i \begin{bmatrix} \Theta_i(x,\delta) \\ I_n \end{bmatrix},\ i = 1,2,3 \quad (8)$$

where $P_i = P_i'$ is a constant matrix to be determined, $\Theta_i(x,\delta) \in \mathbb{R}^{n_i\times n}$ is a polynomial matrix function of (x,δ) of degree p_i having the structure as below and with $\Theta_3(\cdot) \equiv \Theta_1(\cdot)$ [1]:

$$\Theta_i(x,\delta) = \begin{bmatrix} \theta_{p_i}^{(i)}(x,\delta)\ldots\theta_2^{(i)}(x,\delta)\theta_1^{(i)}(x,\delta) \\ \vdots \\ \theta_2^{(i)}(x,\delta)\theta_1^{(i)}(x,\delta) \\ \theta_1^{(i)}(x,\delta) \end{bmatrix},\ i = 1,2 \quad (9)$$

where $\theta_k^{(i)}(x,\delta) \in \mathbb{R}^{r_{i,k}\times r_{i,(k-1)}}$, $k = 1,\ldots,p_i$, $i = 1,2$, are given linear matrix functions of (x,δ) with $r_{i,0} = n$.

In view of (8), the positivity of $V_1(x,\delta)$ and $Z_i(x,\tilde{x},\delta)$, $i = 2,3$, can be expressed as

$$V_1 = \xi_1(x,\delta)'P_1\xi_1(x,\delta) > 0, \forall (x,\delta) \in \mathscr{X}\times\Delta, x \neq 0 \quad (10)$$

$$Z_2 = \xi_2(x,\tilde{x},\delta)'P_2\xi_2(x,\tilde{x},\delta) > 0, \forall (x,\tilde{x},\delta) \in \mathscr{X}\times\mathscr{X}\times\Delta,\ (x,\tilde{x}) \neq 0 \quad (11)$$

$$Z_3 = \xi_1(x,\delta)'P_3\xi_1(x,\delta) > 0, \forall (x,\delta) \in \mathscr{X}\times\Delta, x \neq 0 \quad (12)$$

where $\xi_i \in \mathbb{R}^{n_{\xi_i}}$, $n_{\xi_i} = n_i + n$, $n_i = r_{i,p_i} + \cdots + r_{i,2} + r_{i,1}$, $i = 1,2$, are given by

$$\xi_1(x,\delta) = \begin{bmatrix} \Theta_1(x,\delta) \\ I_n \end{bmatrix} x,\quad \xi_2(x,\tilde{x},\delta) = \begin{bmatrix} \Theta_2(x,\delta) \\ I_n \end{bmatrix} \dot{x}. \quad (13)$$

In the light of (10), the time-derivative of $V_1(x,\delta)$ along the trajectory of (2) is as follows:

$$\dot{V}_1 = 2\xi_1(x,\delta)'P_1\dot{\xi}_1(x,\delta),\ \dot{\xi}_1(x,\delta) = \frac{\partial\xi_1(x,\delta)}{\partial x}\dot{x}. \quad (14)$$

As $\xi_1(\cdot)$ is a polynomial vector function of (x,δ), the elements of $\partial\xi_1(x,\delta)/\partial x$ are polynomial functions of (x,δ). Hence,

[1] The condition $\Theta_3(\cdot) \equiv \Theta_1(\cdot)$ does not imply any loss of generality

there exists a decomposition of $\dot{\xi}_1(x,\delta)$ of the following form (Coutinho *et al.* [2006]):

$$\begin{cases} \dot{\xi}_1(x,\delta) = E_1(x,\delta)\dot{x} + E_2(x,\delta)\rho \\ 0 = \Upsilon_1(x,\delta)\dot{x} + \Upsilon_2(x,\delta)\rho \end{cases} \tag{15}$$

where $\rho := \rho(x,\dot{x},\delta) \in \mathbb{R}^{n_\rho}$ is a polynomial function of (x,δ) and linear in $\dot{x}$, and $E_1(\cdot) \in \mathbb{R}^{n_{\xi_1}\times n}$, $E_2(\cdot) \in \mathbb{R}^{n_{\xi_1}\times n_\rho}$, $\Upsilon_1(\cdot) \in \mathbb{R}^{q\times n}$ and $\Upsilon_2(\cdot) \in \mathbb{R}^{q\times n_\rho}$ are affine matrix functions of (x,δ).

Similarly to **A3**, the following assumption is adopted to assure that the decomposition in (15) is well-posed.

A4 $\Upsilon_2(x,\delta)$ has full column-rank for all $(x,\delta) \in (\mathscr{X}\times\Delta)$.

In view of (11)-(15), the time-derivative of $V(x_t,\delta)$ along the trajectory of the system (2) is given by

$$\begin{aligned} \dot{V} &= 2\xi_1' P_1 [E_1(x,\delta)\dot{x} + E_2(x,\delta)\rho] + \xi_1' P_3 \xi_1 - \tilde{\xi}_1' P_3 \tilde{\xi}_1 \\ &\quad + \tau\xi_2(t)' P_2 \xi_2(t) - \int_{t-\tau}^{t} \xi_2(\alpha)' P_2 \xi_2(\alpha) d\alpha \\ &= \frac{1}{\tau}\int_{t-\tau}^{t} \zeta(t,\alpha)'\Phi(x,\delta)\zeta(t,\alpha) d\alpha \end{aligned} \tag{16}$$

where $\tilde{\xi}_1 := \xi_1(\tilde{x}(t),\delta)$ and

$$\zeta(t,\alpha) = [\,\xi_1(t)'\ \tilde{\xi}_1(t)'\ \xi_2(t)'\ \xi_2(\alpha)'\ \pi(t)'\ \rho(t)'\,]' \tag{17}$$

$$\Phi(x,\delta) = \begin{bmatrix} P_3 & \star & \star & \star & \star & \star \\ 0 & -P_3 & \star & \star & \star & \star \\ N_2' E_1(x,\delta)' P_1 & 0 & \tau P_2 & \star & \star & \star \\ 0 & 0 & 0 & -\tau P_2 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \star \\ E_2(x,\delta)' P_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

where the symbol $\star$ stands for blocks that can be induced by symmetry, and $\xi_i(t)$, $\pi(t)$ and $\rho(t)$ respectively denote the vector functions $\xi_i(\cdot)$, $\pi(\cdot)$ and $\rho(\cdot)$ with their arguments at time t, and N_2 is a constant matrix such that $N_2\xi_2 = \dot{x}$, namely

$$N_2 = [\,0_{n\times n_2}\ \ I_n\,].$$

Observe that the partitions of $\zeta(\cdot)$ as defined in (17) satisfy the following equality constraints:

$$A_1(x,\tilde{x},\delta)x + A_2(x,\tilde{x},\delta)\tilde{x} - \dot{x}(t) + A_3(x,\tilde{x},\delta)\pi = 0 \tag{19}$$

$$\frac{1}{\tau}\int_{t-\tau}^{t} [x(t) - \tilde{x}(t) - \tau\dot{x}(\alpha)]\, d\alpha = 0 \tag{20}$$

$$\Omega_1(x,\tilde{x},\delta)x + \Omega_2(x,\tilde{x},\delta)\tilde{x} + \Omega_3(x,\tilde{x},\delta)\pi = 0 \tag{21}$$

$$\Upsilon_1(x,\delta)\dot{x} + \Upsilon_2(x,\delta)\rho = 0 \tag{22}$$

$$\mathscr{N}_{\bar{\xi}_i}(x,\delta)\xi_i = 0,\ i = 1,2 \tag{23}$$

where

$$\mathscr{N}_{\bar{\xi}_i}(x,\delta) = \begin{bmatrix} I & -\theta_{p_i}^{(i)}(x,\delta) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & I & -\theta_2^{(i)}(x,\delta) & 0 \\ 0 & \cdots & 0 & I & -\theta_1^{(i)}(x,\delta) \end{bmatrix} \tag{24}$$

4. ROBUST REGIONAL STABILITY ANALYSIS

This section develops an LMI approach to robust regional stability analysis of the system (2).

4.1 Local Stability

Basically, the system (2) is locally stable if $V > 0$ and $\dot{V} < 0$ for all $(x,\tilde{x},\delta) \in \mathscr{X}\times\mathscr{X}\times\Delta$. From Section 3, these inequalities are of the form

$$v(\hat{x},\delta)'\Lambda(\hat{x},\delta)v(\hat{x},\delta) > 0,\ \forall(\hat{x},\delta) \in (\hat{\mathscr{X}}\times\Delta) \tag{25}$$

where $\hat{x} \in \hat{\mathscr{X}}$ is either x or $[x'\ \ \tilde{x}']'$ depending on the context[2], $\Lambda(\hat{x},\delta) \in \mathbb{R}^{n_v\times n_v}$ depends affinely on $(\hat{x},\delta)$ and $v(\hat{x},\delta) \in \mathbb{R}^{n_v}$ is a nonlinear vector function of $(\hat{x},\delta)$. Notice that the above inequality could be tested via the LMIs

$$\Lambda(\hat{x},\delta) > 0,\ \forall(\hat{x},\delta) \in \vartheta(\hat{\mathscr{X}}\times\Delta).$$

However, the above is conservative because: (a) v is not an arbitrary vector in $\mathbb{R}^{n_v}$; (b) v and Λ are coupled.

A way to reduce the above conservatism is to use Finsler's lemma together with a *linear annihilator* of $\hat{x}$, namely a matrix function $\mathscr{N}(\hat{x})$ linear in $\hat{x}$ and such that $\mathscr{N}(\hat{x})\hat{x} = 0$. More specifically, if $\mathscr{N}_v(\hat{x},\delta)$ is a full row-rank matrix with affine dependence on $(\hat{x},\delta)$ that comprises $\mathscr{N}(\hat{x})$ and is such that

$$\mathscr{N}_v(\hat{x},\delta)v(\hat{x},\delta) = 0,\ \forall(\hat{x},\delta) \in (\hat{\mathscr{X}}\times\Delta)$$

then by Lemma 3, (25) holds if

$$\Lambda(\hat{x},\delta) + \mathrm{Her}\{L\mathscr{N}_v(\hat{x},\delta)\} > 0,\ \forall(\hat{x},\delta) \in \vartheta(\hat{\mathscr{X}}\times\Delta)$$

where L is a scaling matrix to be determined.

Note that the matrix representation of a linear annihilator $\mathscr{N}(\hat{x})$ of $\hat{x} = [\hat{x}_1\ \cdots\ \hat{x}_{\hat{n}}]'$ is not unique. A natural choice of $\mathscr{N}(\hat{x}) \in \mathbb{R}^{(\hat{n}-1)\times\hat{n}}$ is

$$\mathscr{N}(\hat{x}) = \begin{bmatrix} \hat{x}_2 & -\hat{x}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \hat{x}_3 & -\hat{x}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & \hat{x}_{\hat{n}} & -\hat{x}_{\hat{n}-1} \end{bmatrix}.$$

Example 4. The illustrate the potential of linear annihilators in reducing the conservatism of testing state-dependent LMIs, let the following globally stable delay-free bilinear system:

$$\dot{x} = A(x)x,\quad A(x) = \begin{bmatrix} -(1+x_2) & 0.5(x_1+x_2) \\ 0.5(x_1+x_2) & -(1+x_1) \end{bmatrix}$$

where $x = [x_1\ \ x_2]' \in \mathscr{X}$ and $\mathscr{X} = \{x \in \mathbb{R}^2 : |x_i| \le \sigma,\ i = 1,2\}$. The above system is locally asymptotically stable in $\mathscr{X}$ if there exists a matrix $P > 0$ such that

$$x'[A(x)'P + PA(x)]x < 0,\ \forall x \in \mathscr{X}. \tag{26}$$

The stability condition (26) could be checked via the following set of state-dependent LMIs:

$$P > 0,\quad A(x)'P + PA(x) < 0,\ \forall x \in \vartheta(\mathscr{X}).$$

The largest σ such that the latter LMIs are feasible is 0.5, demonstrating that state-dependent LMIs can be very conservative. On the other hand, applying Finsler's lemma together with the linear annihilator $\mathscr{N}(x) = [x_2\ \ -x_1]$, (26) holds if there exist matrices P and L satisfying the LMIs

$$P > 0,\quad \mathrm{Her}\{PA(x) + L\mathscr{N}(x)\} < 0,\ \forall x \in \vartheta(\mathscr{X})$$

which are feasible for σ arbitrary large, i.e. for all $x \in \mathbb{R}^2$. □

The above procedure will be applied to obtain a solution to the robust stability problem in terms of state- and parameter-dependent LMIs. To this end, let the following matrices:

[2] The domain $\hat{\mathscr{X}}$ for $\hat{x}$ is defined similarly.

$$\mathcal{N}_{\xi_1}(x,\delta) = \left[\tilde{\mathcal{N}}_{\xi_1}(x,\delta)' \quad (\mathcal{N}(x)N_1)'\right]' \tag{27}$$

$$\tilde{\mathcal{N}}_{\xi_1}(\tilde{x},\delta) = \mathcal{N}_{\xi_1}(\tilde{x},\delta) \tag{28}$$

$$\mathcal{N}_{\xi_2}(x,\delta) = \tilde{\mathcal{N}}_{\xi_2}(x,\delta), \quad \mathcal{N}_{\xi_3}(x,\delta) = \mathcal{N}_{\xi_1}(x,\delta) \tag{29}$$

$$\mathcal{N}_{\zeta}(x,\tilde{x},\delta) = \begin{bmatrix} A_1N_1 & A_2N_1 & -N_2 & 0_n & A_3 & 0 \\ N_1 & -N_1 & 0 & -\tau N_2 & 0 & 0 \\ \Omega_1N_1 & \Omega_2N_1 & 0 & 0 & \Omega_3 & 0 \\ 0 & 0 & \Upsilon_1N_2 & 0 & 0 & \Upsilon_2 \\ \mathcal{N}_{\xi_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{N}}_{\xi_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{N}_{\xi_2} & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

where $A_i := A_i(x,\tilde{x},\delta)$, $\Omega_i := \Omega_i(x,\tilde{x},\delta)$, $\mathcal{N}_{\xi_i} := \mathcal{N}_{\xi_i}(x,\delta)$, $i = 1,2,3$, $\Upsilon_i := \Upsilon_i(x,\delta)$, $i=1,2$, $\tilde{\mathcal{N}}_{\xi_1} := \tilde{\mathcal{N}}_{\xi_1}(\tilde{x},\delta)$, $\mathcal{N}(x)$ and $\mathcal{N}(\tilde{x})$ are linear annihilators of x and $\tilde{x}$, respectively, and N_1 is a constant matrix such that $N_1\xi_1 = x$, namely $N_1 = \left[0_{n\times n_1} \quad I_n\right]$.

Note that in view of (19)-(23), we have that

$$\mathcal{N}_{\xi_i}\xi_i = 0,\, i=1,2, \;\; \tilde{\mathcal{N}}_{\xi_1}\tilde{\xi}_1 = 0, \;\; \frac{1}{\tau}\int_{t-\tau}^{t} \mathcal{N}_{\zeta}\zeta(t,\alpha)d\alpha = 0. \tag{31}$$

Theorem 5. Consider system (1) and its DAR as in (2) satisfying **A1**-**A3**. Let $\Theta_i(x,\delta)$, $i=1,2$ be given polynomial matrix functions of (x,δ) as in (9) and consider the nonlinear decomposition of $\xi_1(x,\delta)$ as defined in (15) and satisfying **A4**. Suppose that there exist matrices L_i, $P_i = P_i'$, $i = 1,2,3$ and M satisfying the following LMIs:

$$P_i + \mathrm{Her}\left\{L_i\mathcal{N}_{\xi_i}(x,\delta)\right\} > 0, \forall (x,\delta) \in \vartheta(\mathcal{X}\times\Delta), i=1,2,3 \tag{32}$$

$$\Phi(x,\delta) + \mathrm{Her}\left\{M\mathcal{N}_{\zeta}(x,\tilde{x},\delta)\right\} < 0, \quad \forall (x,\tilde{x},\delta) \in \vartheta(\mathcal{X}\times\mathcal{X}\times\Delta) \tag{33}$$

Then, the equilibrium point $x = 0$ of system (2) is locally asymptotically stable in $\mathcal{X}$ for all $\delta \in \Delta$.

Proof. First, if the LMIs of (32) and (33) are feasible, then, by convexity, they are also satisfied for all $(x,\tilde{x},\delta) \in (\mathcal{X}\times\mathcal{X}\times\Delta)$.

As $\mathcal{N}_{\xi_i}\xi_i = 0$, $i = 1,2,3$, where $\xi_3 = \xi_1$, then pre- and post-multiplying (32) by ξ_i' and ξ_i, respectively, leads to

$$V_1(x,\delta) > 0, \; \forall (x,\delta) \in (\mathcal{X}\times\Delta), \;\; x \neq 0 \tag{34}$$

$$V_i(x_t,\delta) > 0, \forall (x_t,\delta) \in (\mathcal{X}\times\Delta), \; x_t \neq 0, \;\; i=2,3. \tag{35}$$

In addition, since (x,δ) belongs to a polytope, and taking into account (4)-(7), (34) and (35), there exist positive scalars ε_1 and ε_2 such that

$$\varepsilon_1\|\phi(0)\|^2 \le V(\phi,\delta) \le \varepsilon_2\|\phi\|_{\tau}^2, \; \forall (\phi,\delta) \in (\mathcal{X}\times\Delta). \tag{36}$$

Next, as the inequality (33) is strict and taking into account (31), there exists a positive and sufficiently small scalar ε_3 such that the following holds for all $(x_t,\delta) \in (\mathcal{X}\times\Delta)$:

$$\frac{1}{\tau}\int_{t-\tau}^{t} \zeta(t,\alpha)'\Phi(x,\delta)\zeta(t,\alpha)d\alpha + \varepsilon_3\|x\|^2 \le 0.$$

Hence, considering (16), the latter inequality implies that

$$\dot{V}(\phi,\delta) \le -\varepsilon_3\|\phi(0)\|^2, \; \forall (\phi,\delta) \in (\mathcal{X}\times\Delta). \tag{37}$$

Finally, from (36) and (37) it follows that $V(x_t,\delta)$ as defined by (4)-(9) is a Lyapunov-Krasovskii functional for system (2) inside the domain $(\mathcal{X}\times\Delta)$. ∇∇∇

Note that Theorem 5 also ensures the robust local asymptotic stability of the system (2) for any time-delay smaller than τ.

Remark 6. A quadratic Lyapunov-Krasovskii functional, i.e. a functional which is quadratic in (x,x_t) and independent of δ, can be represented as in (4)-(8) by simply setting $\Theta_i(x,\delta) \equiv 0$, $i = 1,2,3$, or equivalently, zeroing the first n_i rows and columns of the matrix P_i in (8). Therefore, Theorem 5 can be also applied to solve quadratic stability analysis problems. □

Example 7. Let the example in Papachristodoulou [2004]:

$$\begin{cases} \dot{x}_1 = -(x_1 + 10/3)(x_1 + x_2) \\ \dot{x}_2 = -10x_2 + 10\tilde{x}_2 + 35\tilde{x}_1 + 3\tilde{x}_1\tilde{x}_2 \end{cases} \tag{38}$$

where $x = [x_1 \;\; x_2]' \in \mathcal{X} \subset \mathbb{R}^2$.

Consider the DAR of (2) with $\pi(x,\tilde{x}) \equiv 0$,

$$A_1(x) = \begin{bmatrix} -(10/3 + x_1) & -(10/3 + x_1) \\ 0 & -10 \end{bmatrix},$$

$$A_2(\tilde{x}) = \begin{bmatrix} 0 & 0 \\ (35 + 1.5\tilde{x}_2) & (10 + 1.5\tilde{x}_1) \end{bmatrix}$$

and the remaining matrices are set to be zero.

Considering a quadratic Lyapunov-Krasovskii functional and the same polytope $\mathcal{X}$ as adopted in Papachristodoulou [2004], Theorem 5 assures that the equilibrium $x = 0$ of system (38) is locally asymptotically stable for any $\tau \in [0, \;\; 0.04]$, which is the same result obtained in Papachristodoulou [2004]. Next, applying Theorem 5 with matrices $\Theta_i(x)$, $i = 1,2,3$ that are linear in x, the maximum achievable value of τ is $\tau = 0.0536$, which is close to the maximum admissible value $\tau = 0.0541$ for local stability of the equilibrium $x = 0$ and larger than the value $\tau = 0.053$ obtained in Papachristodoulou [2004] with a Lyapunov-Krasovskii functional of similar complexity.

4.2 Uncertain Linear State-Delayed Systems

In the sequel, Theorem 5 is specialized to the robust asymptotic stability analysis of linear state-delayed systems subject to rational convex-bounded parameter uncertainty. To this end, consider the following uncertain linear system

$$\begin{cases} \dot{x}(t) = A(\delta)x(t) + A_d(\delta)\tilde{x}(t) \\ x(t) = \phi(t), \;\; \forall t \in [-\tau, 0] \end{cases} \tag{39}$$

where $A(\delta)$ and $A_d(\delta)$ are rational matrix functions of the uncertain parameter vector $\delta \in \mathbb{R}^{n\delta}$ and the remaining variables are as in system (1). In this case, the matrices $A_i(x,\tilde{x},\delta)$ and $\Omega_i(x,\tilde{x},\delta)$ in the DAR of (2) are affine functions of δ only and will be denoted by $A_i(\delta)$ and $\Omega_i(\delta)$, respectively.

Let the Lyapunov-Krasovskii functional matrix $\Theta_i(x,\delta)$ of (8) be chosen as a polynomial matrix of δ, i.e. $\Theta_i(\delta)$. Hence, the matrices in the decomposition of $\xi_1(x,\delta)$ of (15) become affine functions of δ, namely $E_i(\delta)$ and $\Upsilon_i(\delta)$, $i = 1,2$. Moreover, the matrices $\tilde{\mathcal{N}}_{\xi_i}(x,\delta)$, $i = 1,2$ in (24) are now affine functions of δ only, which will be denoted by $\tilde{\mathcal{N}}_{\xi_i}(\delta)$.

In the above setting, Theorem 5 leads to a novel robust asymptotic stability condition for system (39) based on a parametric Lyapunov-Krasovskii functional with polynomial dependence on the uncertain parameter δ as below. To this end, introduce the following matrix:

$$\bar{\mathcal{N}_\zeta}(\delta) = \begin{bmatrix} A_1(\delta)N_1 & A_2(\delta)N_1 & -N_2 & 0_n & A_3(\delta) & 0 \\ N_1 & -N_1 & 0_n & -\tau N_2 & 0 & 0 \\ \Omega_1(\delta)N_1 & \Omega_2(\delta)N_1 & 0 & 0 & \Omega_3(\delta) & 0 \\ 0 & 0 & \Upsilon_1(\delta)N_2 & 0 & 0 & \Upsilon_2(\delta) \\ \bar{\mathcal{N}_{\xi_1}}(\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\mathcal{N}_{\xi_1}}(\delta) & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\mathcal{N}_{\xi_2}}(\delta) & 0 & 0 & 0 \end{bmatrix}$$

and let $\bar{\Phi}(\delta)$ denote the matrix $\Phi(x,\delta)$ of (18) with $E_1(x,\delta)$ and $E_2(x,\delta)$ replaced by $E_1(\delta)$ and $E_2(\delta)$, respectively. Then, the next corollary follows straightforwardly from Theorem 5

Corollary 8. Consider system (39) and its DAR of (2) satisfying **A1** and **A3**. Let $\Theta_i(\delta)$, $i=1,2$ be given polynomial matrix functions of δ defined as before. Suppose that there exist matrices L_i, $P_i = P_i'$, $i=1,2,3$ and M satisfying the LMIs:

$$P_i + \mathrm{Her}\{L_i \bar{\mathcal{N}_{\xi_i}}(\delta)\} > 0,\ \forall \delta \in \vartheta(\Delta),\ i=1,2,3 \quad (40)$$

$$\bar{\Phi}(\delta) + \mathrm{Her}\{M \bar{\mathcal{N}_\zeta}(\delta)\} < 0,\ \forall \delta \in \vartheta(\Delta) \quad (41)$$

where $\bar{\mathcal{N}_{\xi_3}}(\delta) \equiv \bar{\mathcal{N}_{\xi_1}}(\delta)$. Then, system (39) is globally asymptotically stable for all $\delta \in \Delta$. □

Consider next the case of linear state-delayed systems with affine polytopic uncertainty, namely when the matrices $A(\delta)$ and $A_d(\delta)$ of system (39) are affine functions of δ. In this situation, (39) is already in the DAR form of (2) with

$$\begin{cases} \pi = 0,\ A_1(x,\tilde{x},\delta) = A(\delta),\ A_2(x,\tilde{x},\delta) = A_d(\delta), \\ A_3(x,\tilde{x},\delta) \equiv 0,\ \Omega_1 = 0,\ \Omega_2 = 0,\ \Omega_3 = 0. \end{cases} \quad (42)$$

In this setting, let the Lyapunov matrix $\mathscr{P}_i(x,\delta)$ of (8) be chosen as an affine matrix function of δ, i.e. $\mathscr{P}_i(x,\delta) = P_i(\delta) = P_{i,0} + \sum_{k=1}^{n_\theta} \delta_k P_{i,k}$, and introduce the following notations:

$$\hat{\Phi}(\delta) = \begin{bmatrix} P_3(\delta) & \star & \star & \star \\ 0 & -P_3(\delta) & \star & \star \\ P_1(\delta) & 0 & \tau P_2(\delta) & \star \\ 0 & 0 & 0 & -\tau P_2(\delta) \end{bmatrix},$$

$$\hat{\mathcal{N}_\zeta}(\delta) = \begin{bmatrix} A(\delta) & A_d(\delta) & -I_n & 0_n \\ I_n & -I_n & 0_n & -\tau I_n \end{bmatrix}.$$

Then, Theorem 5 specializes to the following result.

Theorem 9. Consider system (39) with $A(\delta)$ and $A_d(\delta)$ affine in δ satisfying **A1**. Suppose there exist matrices $P_{i,k} = P_{i,k}'$, $k = 0,\dots,n_i$, $i=1,2,3$ and M satisfying the LMIs:

$$P_i(\delta) > 0,\ i=1,2,3,\ \forall \delta \in \vartheta(\Delta) \quad (43)$$

$$\hat{\Phi}(\delta) + \mathrm{Her}\{M \hat{\mathcal{N}_\zeta}(\delta)\} < 0,\ \forall \delta \in \vartheta(\Delta) \quad (44)$$

Then, system (39) with the matrices $A(\delta)$ and $A_d(\delta)$ affine in δ is asymptotically stable for all $\delta \in \Delta$.

Example 10. Let the following uncertain linear state-delayed system of Example 4.A from Fridman and Shaked [2003]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 12\delta - 0.12 \\ 1 & -0.465 - \delta \end{bmatrix} x(t) + \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix} x(t-\tau)$$

where $\delta \in \Delta = [-0.035\ \ 0.035]$ is an uncertain parameter.

We aim to find the maximum time delay such that the above system is asymptotically stable for all $\delta \in \Delta$. Applying Theorem 9, a maximum $\tau = 0.863$ is obtained, which is the same result of Fridman and Shaked [2003] and He *et al.* [2004]. On the other hand, using a Lyapunov-Krasovskii functional as in (4) with $\Theta_1(\delta) = \mathscr{P}_2(\delta) = \mathscr{P}_3(\delta) = \delta I_2$ and applying Corollary 8 we obtain a maximum time-delay of $\tau = 0.876$, illustrating that more complex Lyapunov-Krasovskii functionals can lead to less conservative results for uncertain systems at the cost of extra computations.

4.3 Domain of Attraction Estimate

Under the assumption that Theorem 5 holds, the region

$$\mathscr{D} := \{x_t \in \mathscr{C}_\tau^n : V(x_t,\delta) \le \kappa,\ \delta \in \Delta\} \quad (45)$$

where $V(\cdot)$ is defined in (4) and $\kappa \in \mathbb{R}_+$, and subject to $\mathscr{D} \subset \mathscr{X}$ is a positively invariant set. However, the condition $\mathscr{D} \subset \mathscr{X}$ is of difficult computation because $V(x_t,\delta)$ depends not only on $x(t), t \ge 0$, but also on the function x_t. The approach adopted in this paper to overcome this problem is to determine a bounding set $\mathscr{D}_1 \subset \mathscr{X}$ such that $\mathscr{D} \subseteq \mathscr{D}_1$ and the condition $\mathscr{D}_1 \subset \mathscr{X}$ can be numerically tested.

Since $V_1(x,\delta) \le V(x_t,\delta), \forall (x_t,\delta)$, the following set

$$\mathscr{D}_1 = \{x \in \mathscr{X} : V_1(x,\delta) \le \kappa,\ \delta \in \Delta\} \quad (46)$$

is such that $\mathscr{D} \subseteq \mathscr{D}_1$. Moreover, the condition $\mathscr{D}_1 \subset \mathscr{X}$ turns out to be an LMI problem. To see this, first the polytopic region $\mathscr{X}$ is recast as a set of linear inequalities [3] (Boyd *et al.* [1994]):

$$\mathscr{X} = \{x \in \mathbb{R}^n : c_k' x \le 1,\ i = 1,\dots,n_e\} \quad (47)$$

where c_k are given constant vectors defining the n_e edges of $\mathscr{X}$. Hence, $\mathscr{D}_1 \subset \mathscr{X}$ can be described by the condition:

$$2 - 2c_k' x \ge 0,\ \forall\, x \in \mathbb{R}^n : x' \mathscr{P}_1(x,\delta) x - \kappa \le 0,\ \forall \delta \in \Delta \quad (48)$$

Since $x' \mathscr{P}_1(x,\delta) x = \xi_1' P_1 \xi_1$, applying the $\mathscr{S}$-procedure (Boyd *et al.* [1994]) leads to the following conditions for (48) to hold:

$$\xi_{1a}' \Pi_k \xi_{1a} \ge 0,\ k = 1,\dots,n_e \quad (49)$$

where

$$\xi_{1a} = \begin{bmatrix} 1 \\ \xi_1 \end{bmatrix},\ \Pi_k = \begin{bmatrix} 2\varsigma - \kappa & -\varsigma c_k' N_1 \\ -\varsigma N_1' c_k & P_1 \end{bmatrix} \quad (50)$$

where $\varsigma > 0$ is a free scalar introduced by the $\mathscr{S}$-procedure.

In addition to the above conditions, in order for the set $\hat{\mathscr{D}}_a$ of (3) to be a DOA estimate it is required that $V(\phi,\delta) \le \kappa, \forall \delta \in \Delta$. Considering (3) and subject to the constraints

$$\dot{x}' \mathscr{P}_2(x,\delta) \dot{x} \le \alpha_d \dot{x}' \dot{x},\quad x' \mathscr{P}_3(x,\delta) x \le \alpha_m x' x,\ \text{ over } (\mathscr{X} \times \Delta) \quad (51)$$

it can be readily verified that $V(\phi,\delta) \le \kappa, \forall \delta \in \Delta$ if the condition as below holds:

$$x' \mathscr{P}_1(x,\delta) x - b(\kappa,\tau) \le 0, \forall\, x \in \mathbb{R}^n : \alpha_m x' x \le 1, \forall \delta \in \Delta \quad (52)$$

where $b(\kappa,\tau) = \kappa - \tau - 0.5\tau^2$. By applying the $\mathscr{S}$-procedure we get the following condition for (52) to be satisfied:

$$\xi_{1a}' \Xi \xi_{1a} \ge 0 \quad (53)$$

where ξ_{1a} is as in (50) and

$$\Xi = \begin{bmatrix} b(\kappa,\tau) - 1 & 0 \\ 0 & \alpha_m N_1' N_1 - P_1 \end{bmatrix}. \quad (54)$$

Applying Lemma 3 to (49), (51) and (53), we get the result:

[3] Note that $\mathscr{X}$ can be equivalently defined by its vertices.

Theorem 11. Consider system (1) and its DAR of (2) satisfying **A1-A3**. Let $\Theta_i(x,\delta)$, $i=1,2$ be given polynomial matrix functions of (x,δ) as in (9) and consider the nonlinear decomposition of $\dot{\xi}_1(x,\delta)$ defined in (15) and satisfying **A4**. Let $\alpha_d>0$ be a given scalar defining the admissible initial function time-derivative in the set $\hat{\mathscr{D}}_a$ of (3). Suppose there exist matrices $M, R, S, L_i, P_i=P_i'$, $i=1,2,3$ and K_i, $i=1,\ldots,n_e$, and positive scalars κ, ς and α_m solving the convex optimization problem:

minimize α_m, subject to (32), (33) and

$$\Pi_k+\mathrm{Her}\{K_k\mathscr{N}_{\xi_{1a}}\}>0,\ \forall(x,\delta)\in\vartheta(\mathscr{X}\times\Delta),\quad k=1,\ldots,n_e \tag{55}$$

$$\Xi+\mathrm{Her}\{R\mathscr{N}_{\xi_{1a}}\}>0,\ \forall(x,\delta)\in\vartheta(\mathscr{X}\times\Delta) \tag{56}$$

$$\alpha_d N_2'N_2-P_2+\mathrm{Her}\{R\mathscr{N}_{\xi_2}\}\geq 0,\forall(x,\delta)\in\vartheta(\mathscr{X}\times\Delta) \tag{57}$$

$$\alpha_m N_1'N_1-P_3+\mathrm{Her}\{S\mathscr{N}_{\xi_1}\}\geq 0,\forall(x,\delta)\in\vartheta(\mathscr{X}\times\Delta) \tag{58}$$

where

$$\mathscr{N}_{\xi_{1a}}(x,\delta)=\begin{bmatrix}0 & \mathscr{N}_{\xi_1}(x,\delta)\\ x & -N_1\end{bmatrix}. \tag{59}$$

Then, $\mathscr{D}_1\subset\mathscr{X}$ and for any initial function $\phi\in\hat{\mathscr{D}}_a$ the system trajectory $x(t)$ lies within $\mathscr{D}_1$, $\forall t\geq 0$, and $x(t)\to 0$ as $t\to\infty$ for all $\delta\in\Delta$, where $\mathscr{D}_1$ is as in (46). □

Theorem 11 gives a DOA estimate $\hat{\mathscr{D}}_a$ as in (3) with a region $\mathscr{A}=\{\phi\in\mathscr{C}_\tau^n:\alpha_m\|\phi\|_\tau^2\leq 1\}$ of maximum size for a given region $\mathscr{R}=\{\dot{\phi}\in\mathscr{C}_\tau^n:\alpha_d\|\dot{\phi}\|_\tau^2\leq 1\}$. Note that the size of the region $\mathscr{R}$ can be also maximized by finding the minimum value of the scalar $\alpha_d>0$ such that the convex optimization problem of Theorem 11 is solvable. The optimal α_d can be readily obtained by performing a line search on α_d.

Example 12. Consider the Example 5.4 of Melchor-Aguilar and Niculescu [2007], namely

$$\begin{cases}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_1(t)-2x_2(t)-2x_2(t-1)+0.5x_2^3(t-1)\end{cases} \tag{60}$$

Let the DAR of (2) for system (60) with $\pi=x_2^2(t-1)$ and

$$A_1=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix},\ A_2=\begin{bmatrix}0 & 0\\ 0 & -2\end{bmatrix},\ A_3=\begin{bmatrix}0\\ 0.5x_2(t-1)\end{bmatrix},$$
$$\Omega_1=\begin{bmatrix}0 & 0\end{bmatrix},\ \Omega_2=\begin{bmatrix}0 & x_2(t-1)\end{bmatrix},\ \Omega_3=-1.$$

Note that the nonlinear term $x_2^3(t-1)$ turns the system (60) unstable for system initial functions ϕ of large size $\|\phi\|_1$. Thus, the equilibrium $x=0$ of system (60) is not globally asymptotically stable. To analyze the regional stability of (60), we consider the following parameterized polytope

$$\mathscr{X}(\sigma)=\left\{[x_1\ \ x_2]'\in\mathbb{R}^2:|x_i|\leq\sigma,\ i=1,2\right\} \tag{61}$$

where σ is as large as possible.

As the system (60) is linear with respect to $x(t)$, we apply Theorem 11 with a quadratic Lyapunov-Krasovskii functional (see Remark 6). The minimum values of α_m and α_d such that the LMIs of Theorem 11 are feasible with a maximum $\sigma=2.8$ are $\alpha_m=0.64$ and $\alpha_d=10^{-8}$, resulting in the DOA estimate:

$$\hat{\mathscr{D}}_a=\left\{\phi\in\mathscr{C}_1^2:\|\phi\|_1\leq 1.25,\quad\|\dot{\phi}\|_1\leq 10^4\right\}$$

and the bounding set $\mathscr{D}_1=\{x(t)\in\mathbb{R}^2:\|x(t)\|\leq 1.77,\forall t\geq 0\}$ for $x(t)$. Note that, from a practical point of view, the restriction on $\|\dot{\phi}\|_1$ in $\hat{\mathscr{D}}_a$ is insignificant.

In contrast, the procedure of Melchor-Aguilar and Niculescu [2007] to determine a DOA estimate with a computationally tractable size calculation gives the following DOA estimate $\{\phi\in\mathscr{C}_1^2:\|\phi\|_1<0.0075\}$, which is much smaller than the domain for $\|\phi\|_1$ in $\hat{\mathscr{D}}_a$. Note that Melchor-Aguilar and Niculescu [2007] uses a quadratic Lyapunov-Krasovskii functional which is more complex than the one used with Theorem 11. □

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