NUMERICAL SOLUTION OF A GENERAL QUADRATIC ALGEBRAIC MATRIX RICCATI EQUATION VIA PROBABILITY-1 HOMOTOPY ALGORITHMS [1]

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Abstract: The design of an algorithm for the numerical solution of a generalized quadratic algebraic matrix Riccati equation is presented. The approach is based on probability-1 homotopy methods. The algorithm is illustrated with numerical examples. *Copyright © 2000 IFAC*

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1. INTRODUCTION

The synthesis and implementation of an algorithm for the numerical solution of the general quadratic algebraic matrix Riccati equation below, is presented.

$$\begin{aligned} A_1XB_1 + A_2XB_2 + C_1XD_1XE_1 \\ + C_2XD_2XE_2 + G = 0 \end{aligned} \qquad (1)$$

Where,

$A_1, A_2, C_1, C_2 \in C^{n\times n}$, $B_1, B_2, E_1, E_2 \in C^{p\times p}$, $D_1, D_2 \in C^{p\times n}$, $G \in C^{n\times p}$, $0 \in C^{n\times p}$ are the constant matrix coefficients of the equation and $X \in C^{n\times p}$ is the equation unknown matrix. $C^{m\times l}$ and C^m denote the set of complex matrices with dimension $m\times l$ and $m\times 1$ respectively and m, n, l, p are non-zero physical numbers. In the sequel 0, denotes the zero matrix, vector or number of compatible dimensions.

Equation (1) appears in a wide range of engineering and control design problems in more special forms. For example, Algebraic Riccati and Lyapunov matrix equations for continuous time systems can be considered as special forms of the above equation. Also, equation (1) can be specialized to a second order polynomial equation for the solution of the quadratic eigenvalue problem, which is used for modeling oscillations in airplane wings.

At this point it has to be said that, the minimization of complex cost functions in post-modern control systems might result design equations as (1) above (i.e. equations more complex than standard Riccati equations). Therefore (1) might be useful for future complex designs. Hence it is apparent that the proposed method can unify the numerical solutions to a wide range of problems. Apart from its practical usefulness, equation (1) is mathematically attractive in the sense that standard existing methods for the numerical solution of general algebraic Riccati-type equations can not apply for its solution.

So far with respect to author's knowledge, there is no theory developed about the existence and geometry of solutions of (1). This is an open research problem. Despite the theoretical analysis of the equation, the present paper solves the problem of its numerical solution. The developed algorithm based on ideas arising from probability-1 homotopy methods, for the solution of algebraic systems of equations. A primitive homotopy algorithm, for a simpler quadratic equation than (1), was for the first time developed in (Tsachouridis and Postlethwaite, 1999). In (Tsachouridis and Postlethwaite, 1999), only real solutions are considered. The algorithm in the present paper is more robust and sophisticated and it can be used for equations with both complex and/or real coefficients. Also, in contrast to (Tsachouridis and Postlethwaite, 1999) the proposed algorithm is able to compute one or all the existing solutions real and/or complex. Computing all existing solutions discriminate the present method to be rigorous instead of being heuristic. Note that computing all existing solutions might be not practical for equations with large dimensions. This is because the number of

[1] Dedicated to the memory of my father Alexandros.

existing solutions in general is exponentially increasing with the dimension of the equation, according to the theorem of Bezout. In our case, there can be in general 2^{np} solutions for equation (1). Nevertheless, for equations with small dimensions and when computation time is not crucial, computing all existing solutions may be practically feasible. This will be illustrated with a numerical example in subsection 4.2 of the present paper.

The reason of using homotopy methods is that they possess good global convergence properties when compared with locally convergent methods (e.g. Newton, Newton-Gauss). More specifically, probability-1 homotopy methods guarantee global convergence from arbitrary initial conditions. Note that the selection of proper initial conditions is almost always a problem when dealing with other methods. Now, equation (1) can produce a non-convex problem (depending on its data) and therefore a wide range of convex numerical methods by default can not be used. In the last case convergence is laborious for any method in general, most of the time. This problem is not an issue in the present formulation since the proposed algorithm can be used for convex and non-convex problems under the guarantee of convergence with probability one. Generally a homotopy method for the solution of an algebraic equation, is to first solve an easy equation, similar to the original, and then to continuously deform the solution of the easy equation, to the solution of the original equation. For this purpose an appropriate homotopy map needs to be constructed. Based on this philosophy, theorems and propositions can be developed in order to guaranteed convergence and good numerical properties. At this point it should be stated, that such methods gain respect just during the last twenty years. The application of these methods exclusively for the numerical solution of (1) is novel. The core of mathematical tolls used for analysis and synthesis of homotopy methods, lie in the branches of differential topology and dynamical systems theory.

The mathematical presentation and establishment for above all, are comprehensively presented in the present paper. Considering the limited space and because of their extended length, proofs of theorems are omitted. Details of these and many other aspects, concerning the theoretical foundation and practical implementation of the present method, will be reported in another paper. Now two important numerical tolls, namely the scaling and the homogeneous projective transformations, which can be used together with the present algorithm, are reported in (Tsachouridis, 2000). This is because the issues in (Tsachouridis, 2000) have their own independent foundation and have too extended length to incorporate them in the present paper.

At this point it is only said that scaling assigns predetermined values to the coefficients and the unknown matrices of the original equation. The homogeneous projective transformation changes the structure of the original equation (1) in to a homogeneous equation. By doing this, possible solutions to infinity are avoided and therefore finite time calculations and smooth numerical operations take place. Scaling and the homogeneous projective transformation can be applied to (1) separately or in combination. Their effects on the numerical examples of the present paper are analyzed in (Tsachouridis, 2000).

The structure of this paper will be as follows. A comprehensive discussion of Homotopy methods is given in section 2. The formulation of the problem and its solution are given in section 3. The method is illustrated with numerical examples next in section 4. Finally, conclusions are given in section 5.

2. PROBABILITY-1 HOMOTOPY METHODS

This section provides a comprehensive description of probability-1 homotopy methods, which will used in this paper. The section considers the numerical solution to the nonlinear equation $F(\boldsymbol{x}) = \boldsymbol{0}$, $\boldsymbol{x} \in C^m$, under certain assumptions.

Definition 1; Consider the mappings $F: C^m \to C^m$ and $F_0: C^m \to C^m$. A continuous mapping $H: C^m \times [0,1] \to C^m$ is called a homotopy from F_0 to F if $H(\boldsymbol{x},0) = F_0(\boldsymbol{x})$ and $H(\boldsymbol{x},1) = F(\boldsymbol{x})$ for every $\boldsymbol{x} \in C^m$. F_0 is said to be homotopic to F and $H(\boldsymbol{x},\varepsilon) = \boldsymbol{0}$ is called the homotopy equation of the homotopy variable $\varepsilon \in [0,1]$.

Definition 2; The twice continuously differentiable mapping $H_a: C^m \times [0,1] \times C^m \to C^m$ is said to be transversal to zero if the Jacobian matrix $J_{H_\alpha(\boldsymbol{x},\varepsilon,\boldsymbol{a})}$ of $H_{\alpha(\boldsymbol{x},\varepsilon,\boldsymbol{a})}$ has full rank on the set

$$H_\alpha^{-1}(\boldsymbol{0}) := \left\{ (\boldsymbol{x},\varepsilon,\boldsymbol{a}) \in C^m \times [0,1) \times C^m \;\middle|\; H(\boldsymbol{x},\varepsilon,\boldsymbol{a}) = \boldsymbol{0} \right\}. \quad (2)$$

Note that along with definition (1), the mapping $H_a: C^m \times [0,1] \times C^m \to C^m$ in definition (2), is said to be a homotopy from $F_0: C^m \times C^m \to C^m$ to $F: C^m \to C^m$ if $H_\alpha(\boldsymbol{x},0,\boldsymbol{a}) = F_0(\boldsymbol{x},\boldsymbol{a})$ and $H_\alpha(\boldsymbol{x},1,\boldsymbol{a}) = F(\boldsymbol{x})$ for every $\boldsymbol{a} \in C^m$, $\boldsymbol{x} \in C^m$.

Similarly, F_0 is homotopic to F and $H_\alpha(x,\varepsilon,a) = \mathbf{0}$ is the homotopy equation of the homotopy variable $\varepsilon \in [0,1]$.

According to definition 1 a homotopy method for the solution of $F(x) = \mathbf{0}$, $x \in C^m$, is to initially solve an easy problem, say $F_0(x) = \mathbf{0}$, $x \in C^m$ and then to continuously deform the easy problem and its solution to the original problem and its solution $F(x) = \mathbf{0}$, $x \in C^m$. To do this a homotopy mapping H needs to be synthesized.

Based on the above philosophy when using homotopy methods, some problems rising during the deformation of the easy problem to the original. More specific, the curves μ of zeros of $H(x,\varepsilon) = \mathbf{0}$, starting from $(x_0,0)$ (where, $H(x,\varepsilon) = F(x_0) = \mathbf{0}$),

(i) may have turning points,
(ii) may bifurcate,
(iii) may overlap each other,
(iv) may not exist for some ε ,
(v) may tend to infinity or spiraling endlessly for some $\varepsilon \in (0,1)$,
(vi) may form closed loops for some $\varepsilon \in (0,1)$.

The above problems (i)-(vi) can also be observed if we start tracing curves μ backwards i.e. starting from $(x_1,0)$ (where, $H(x,1) = F(x_1) = \mathbf{0}$).

Because of the above problems, the main consideration when constructing homotopy mappings H is to ensure almost always (i.e. with probability 1), that every solution x_0 of an easy problem $F_0(x) = \mathbf{0}$ is connected with a solution x_1 of the original problem via a smooth continuous curve μ (homotopy path). Such mappings are known as probability-1 homotopy mappings.

In order to include to the study the characteristic of global convergence from almost every solution x_0 of easy problems $F_0(x) = \mathbf{0}$ with a specific structure, the homotopy mapping H_α in definition 2 is used very often. The difference between the two mappings, although they very often designed identical in structure, is that the specific easy problem $F_0(x) = \mathbf{0}$ in the homotopy H is parameterized with the variable $a \in C^m$ as the easy problem $F_0(x,a) = \mathbf{0}$ in the homotopy H_α. Very often, the above parameterization is done in such a way that the structure of the easy problem $F_0(x) = \mathbf{0}$ is preserved. An immediate result of this structure preservation is the same structure for H and H_α, plus the fact that effectively only the solutions x_0 are change as a function of $a \in C^m$ i.e. $x_0 = f(a)$. Hence, the homotopy H can be viewed as a special case of the homotopy H_α for a specific $a \in C^m$. Thus, ensuring that H_α is a probability-1 homotopy we actually cover almost all possible homotopies H in which, each easy problem solution x_0 correspond to a specific value of $\alpha \in C^m$, i.e. $F_0(x) \equiv F_0(x,a)$. Therefore global convergence characteristics can be guaranteed for a family of easy problems with the same structure. Now note that the set $H_\alpha^{-1}(\mathbf{0})$ in equation (2) defines all possible homotopy curves (paths) μ with respect to all possible easy problem solutions.

The synthesis of probability-1 homotopy mappings can be done via theorems called probability-1 homotopy theorems. For the numerical solution of $F(x) = \mathbf{0}$, $x \in C^m$, two such theorems are stated next.

Theorem 1 (probability-1 fixed point homotopy); Let $H: C^m \times [0,1] \to C^m$ be a homotopy mapping as in definition 1 with easy problem

$$F_0(x) = x - x_0 \tag{3}$$

and with the homotopy equation

$$H(x,\varepsilon) := \varepsilon F(x) - \gamma(1-\varepsilon) F_0(x) \tag{4}$$

where, $\gamma \in C - \{0\}$.

Also let $H_a: C^m \times [0,1] \times C^m \to C^m$ be a homotopy mapping as in definition 2 with homotopy equation

$$H_\alpha(x,\varepsilon,a) := \varepsilon F(x) - \gamma(1-\varepsilon) F_0(x,a) \tag{5}$$

where,

$$F_0(x,a) = x - a\ . \tag{6}$$

Furthermore suppose that the conditions below hold.
(a) The mapping H is complex analytic.
(b) The mapping H_α is transversal to zero.
(c) The set $H_\alpha^{-1}(\mathbf{0})$ is bounded.

Then almost always, $H_\alpha^{-1}(\mathbf{0})$ contains a unique smooth curve which connects the zero solution a , of $F_0(x,a) = \mathbf{0}$, with one solution of $F(x) = \mathbf{0}$ and with the problems (i)-(vi) (stated previously) not applying. Furthermore for the homotopy mapping

H, for almost every $x_0 \in C^m$, there is a unique smooth curve which connects the zero solution x_0, of $F_0(x)=0$, with one solution of $F(x)=0$ and with problems (i)-(vi) not applying.

Theorem 2 (probability-1 polynomial homotopy); Let $H: C^m \times [0,1] \to C^m$ be a homotopy mapping as in definition 1 with easy problem

$$F_0(x) = A * x * x - B \tag{7}$$

where, $A, B \in C^m - \{0\}$, and with the homotopy equation

$$H(x,\varepsilon) := \varepsilon F(x) - \gamma(1-\varepsilon)F_0(x) \tag{8}$$

where, $\gamma \in C - \{0\}$ and $*$ denote the Hadamard matrix product operator.

Also let $H_a: C^m \times [0,1] \times C^m \to C^m$ be a homotopy mapping as in definition 2 with homotopy equation

$$H_\alpha(x,\varepsilon,a) := \varepsilon F(x) - \gamma(1-\varepsilon)F_0(x,a) \tag{9}$$

where,

$$F_0(x,a) = x * x - a. \tag{10}$$

Furthermore suppose that the conditions below hold.

(a) $F(x)=0$ is a second order polynomial system.

(b) The mapping H is complex analytic.

(c) The mapping H_α is transversal to zero.

(d) The set $H_\alpha^{-1}(0)$ is bounded.

Then almost always, $H_\alpha^{-1}(0)$ contains in general 2^m unique smooth curves connecting the 2^m zero solutions of $F_0(x,a)=0$, with the 2^m solutions of $F(x)=0$ in 1-1 relationship and with the problems (i)-(vi) (stated previously) not applying. Furthermore for the homotopy mapping H, for almost every $A, B \in C^m - \{0\}$, there are 2^m unique smooth curves in general connecting the 2^m solutions of $F_0(x)=0$, with the 2^m solutions of $F(x)=0$ in 1-1 relationship and with problems (i)-(vi) not applying.

It is apparent that theorem 1 can be used in cases which only one solution of $F(x)=0$ is required. Also in theorem 1, the homotopy mapping H can be considered as the specific case of the homotopy mapping H_α with $a = x_0$. Now, theorem 2 is able to provide all the set of solutions when $F(x)=0$ is a second order polynomial system. In theorem 2, the homotopy mapping H can be considered as the specific case of the homotopy mapping H_α with $a = B \div A$ ($\div$ denote the element by element division operator of B and A).

Now, the major task in a probability-1 homotopy algorithm is to track the curves μ, which lead to the solutions. One method of doing this continuously is to view the homotopic deformation process as a dynamic flow. To do this, recall that $H(x,\varepsilon)$ is continuous and twice differentiable and suppose that for every $\varepsilon \in [0,1]$, there is a continuous differentiable with respect to ε solution $x = x(\varepsilon)$ of $H(x,\varepsilon)=0$. Then $H(x(\varepsilon),\varepsilon)=0$ is continuously differentiable with respect to ε on $[0,1]$. Now since it is required $H(x,\varepsilon)=0$, it follows that

$$\frac{d}{d\varepsilon}\left[H(x(\varepsilon),\varepsilon)\right] = J_{H(x(\varepsilon),\varepsilon)}\begin{bmatrix} 1 \\ \dfrac{d(x(\varepsilon))}{d\varepsilon}\end{bmatrix} = 0 \tag{11}$$

where,

$$J_{H(x(\varepsilon),\varepsilon)} := \begin{bmatrix} \dfrac{\partial[H(x(\varepsilon),\varepsilon)]}{\partial\varepsilon} & \dfrac{\partial H[(x(\varepsilon),\varepsilon)]}{\partial(x(\varepsilon)^{\mathrm{T}})}\end{bmatrix} \tag{12}$$

is the Jacobian matrix of $H(x(\varepsilon),\varepsilon)$.

From (11), (12) and from H being complex analytic the dynamical system below is well defined.

$$\begin{cases} \dfrac{d(x(\varepsilon))}{d\varepsilon} = -\left(\dfrac{\partial[H(x(\varepsilon),\varepsilon)]}{\partial(x(\varepsilon)^{\mathrm{T}})}\right)^{-1}\dfrac{\partial[H(x(\varepsilon),\varepsilon)]}{\partial\varepsilon} \\ x(0) \in C^m, \quad F_0(x(0)) = 0 \end{cases} \tag{13}$$

Hence, desirable homotopy paths μ can be viewed as trajectories of the initial value problem (13), emanating from $\varepsilon = 0$ and terminating at $\varepsilon = 1$.

The application of the above all for the numerical solution of equation (1) follows in next section.

3. NUMERICAL SOLUTION OF (1) VIA PROBABILITY-1 HOMOTOPY METHODS

Consider the problem of finding the numerical solution of equation (1). This equation can be written in vector format as shown below.

$$
\begin{aligned}
F(x) := \left(B_1^T \otimes A_1 + B_2^T \otimes A_2\right)x \\
+\left\{\left(E_1^T \otimes C_1\right)\left[I_p \otimes (XD_1)\right]\right. \\
\left.+\left(E_2^T \otimes C_2\right)\left[I_p \otimes (XD_2)\right]\right\}x \\
+\overline{G} = 0
\end{aligned}
\tag{14}
$$

Where,

$$x := vec(X), \tag{15}$$

$$\overline{G} := vec(G), \tag{16}$$

$vec: C^{n\times p} \to C^{np}$, $vec(\cdot)$ is the column-wise vector operator of a matrix, $\otimes$ is the Kronecker product matrix operator, and I_p is the identity $p \times p$ matrix.

It is apparent from (1) and (14) that the numerical solution X of (1) can be recovered from the numerical solution x of (14) by equation (17) below.

$$X = vec^{-1}(x). \tag{17}$$

Where, $vec^{-1}: C^{np} \to C^{n\times p}$, $vec^{-1}(\cdot)$ is the inverse $vec(\cdot)$ vector operator. Therefore, the problem now is to solve (14). Hence, equation (14) can be viewed as the nonlinear problem $F(x) = 0$, $x \in C^m$ of section 2, with $m = np$.

It has been proved by the author (the proof will be presented in another paper) that, equation (14) satisfies all sufficient conditions of theorems 1, 2 in section 2 apart from condition (c) ($H_\alpha^{-1}(0)$ is bounded). With this condition not satisfied and if there are solutions at infinity, the algorithms will try to trace them via homotopy paths of infinite length. Hence infinite time of computation is required, which is practically impossible. However, this problem is solved using the homogeneous projective transformation technique in (Tsachouridis, 2000). Hence the probability-1 characteristics of both algorithms proposed via theorems 1 and 2, is always satisfied. For the numerical examples of present paper, although condition (c) is not satisfied, the algorithms are able to compute finite solutions. For the solutions at infinity the algorithms fail in terms of infinite computational time. Now, as far it is concern the implementation of the algorithms, in both cases of theorems 1, 2 the initial value problem (13) was used. The solutions are then recovered from (17), as $X = vec^{-1}(x)\big|_{\varepsilon=1}$.

3.1 Specialization to Real Arithmetic Probability-1 Homotopy Methods.

The analysis and synthesis of the probability-1 algorithms assumes complex arithmetic, and that the theorems 1 and 2 in section 2 hold for complex arithmetic only. Nevertheless the algorithms can be restricted to real arithmetic as well for particular problems (having some or all of their solutions real) for which the conditions (a)-(c) of theorems 1,2 in section 2 hold for real arithmetic. Note that, form the computational point of view in most computer systems and programming languages real arithmetic is usually faster than the complex. This is for example the case where equation (1) is specialized to continuous time algebraic matrix Riccati and Lyapunov equations, having some real solutions. For these matrix equations, the author has proved that, real solutions of specific kind (e.g. positive or negative define/semi-definite) can be computed using real arithmetic in theorem 1. Particularly for the continuous time algebraic matrix Riccati and Lyapunov equations having positive or negative definite solutions, it has been proved that with an easy problem solution $x_0 = vec(X_0)$ any positive or negative define matrix X_0 and with $\gamma = +1$ (for positive definite solution) or $\gamma = -1$ (for negative definite solution), a homotopy path consisting with only positive or only negative define solutions $X(\varepsilon)$, $\varepsilon \in [0,1]$, is defined. Hence for the above cases, when only one real solution of a special kind is required, it is preferable of using an algorithm based on theorem 1 with real arithmetic. The proof of the above are too long, and will be reported in a more detailed paper.

4. NUMERICAL EXAMPLES

In this section the methods presented in this paper are illustrated with two numerical examples. The first example in subsection 4.1 illustrates an algorithm based on theorem 1. The second example in subsection 4.2 illustrates an algorithm based on theorem 2. In both cases for the initial value problem (13), a modified Runge-Kutta 4.5 algorithm with error tolerance 10^{-8}, developed by the author, were used. The algorithms were implemented via m-files, developed by the author, using MATLAB 5.3 for UNIX operating system in a two parallel Pentium III PC at 550 MHz.

4.1 Continuous Time Algebraic Riccati Equation.

Consider the specialization of equation (1) in to the continuous time algebraic matrix Riccati equation with data, $D_1 = -U_4$, $G = U_4$, $C_2 = D_2 = E_2 = 0$, $B_1 = A_2 = C_1 = E_1 = I_4$, $B_2 = A_1^T$ and

$$A_1 = \begin{bmatrix} -1e-20 & -1 & 0 & 0 \\ 1 & -1e-20 & 0 & 0 \\ 0 & 0 & 1e-20 & -1 \\ 0 & 0 & 1 & 1e-20 \end{bmatrix}.$$

Where, $1e+k = 10^k$ and U_4 is the 4×4 unit matrix. For the above data, the positive definite solution is $X = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

The Hamiltonian of the above equation, see (Benner and Byers, 1998), has a conjugate pair of eigenvalues very close to the imaginary axis. Because of these 'bad' data, all the state-of-the-art numerical routines for solving Riccati equations in MATLAB 5.3 and other routines such as the sign function method fail to compute the positive definite solution above. On the contrary using a probability-1 algorithm based on theorem 1 in section 2 with real arithmetic ($\gamma = 1$), the above positive definite solution was successfully obtained. The algorithm was initialized with an easy problem solution $x_0 = vec(X_0)$ a positive definite matrix $X_0 = 1e-08 \times I_4$. The CPU time was 1.8 sec. The computed solution X_c is

$$X_c = \begin{bmatrix} 0.5000 & 0.0000 & 0.5000 & 0.0000 \\ 0.0000 & 0.5000 & 0.0000 & 0.5000 \\ 0.5000 & 0.0000 & 0.5000 & 0.0000 \\ 0.0000 & 0.5000 & 0.0000 & 0.5000 \end{bmatrix}.$$

The Frobenious forward $E_f := \|X - X_c\|_F \|X_c\|_F^{-1}$ and backward errors $E_b := \|X - X_c\|_F \|X\|_F^{-1}$ are $E_f = E_b = 3.7813e\text{-}08$.

4.2 Second Order Matrix Polynomial equation.

Consider the specialization of equation (1) in to the second order matrix polynomial equation with data,

$G = -\begin{bmatrix} 8 & 12 \\ 18 & 26 \end{bmatrix}$, $A_1 = A_2 = B_1 = B_2 = C_1 = I_2$, $C_2 = D_1 = D_2 = E_1 = E_2 = I_2$. The above equation, see (Davis, 1983), has the four real solvents $X_1 = -\begin{bmatrix} 1.8056 & 2.0889 \\ 3.1334 & 4.9390 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0.8056 & 2.0889 \\ 3.1334 & 3.9390 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $X_4 = -\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$, plus 12 solutions at infinity. The finite solutions were obtained using a probability-1 algorithm based on theorem 2 in section 2 with complex arithmetic ($\gamma = 1 + 1i$). The easy problem was set with $A = B = [1\ 1\ 1\ 1]^T$. Its solution is $x(0) = \pm[v\ v\ v\ v]^T$, $v = 2.0276e-5$. From 16 various easy problem solutions the ones that lead via homotopic paths to the finite solutions X_1, X_2, X_3, X_4, are $-[v\ v\ v\ v]^T$, $[v\ -v\ -v\ v]^T$, $[v\ v\ v\ v]^T$ and $[-v\ v\ v\ -v]^T$ respectively. Each one of the rest 12 easy problem solutions led to an infinite length homotopic path. In this cases the algorithm was stopped after 5 min. In (Tsachouridis, 2000), the solutions at infinity are obtained in finite time computation without any problem. The CPU times and residual Frobenious norm errors over the computed solutions X_c norm $E_r := \frac{\|residual\|_F}{\|X_c\|_F}$ are shown in table 1 below.

Table 1 CPU times and Errors.

	X_1	X_2	X_3	X_4
CPU sec	7.29	7.08	6.99	7.02
$E_r \times 10^{-9}$	9.234	37.930	1.5390	18.023

CONCLUSIONS

A robust probability-1 homotopy algorithm for the numerical solution of a generalized quadratic algebraic matrix Riccati equation was presented. The mathematical proofs of results and many other details concerning the perturbation analysis and error estimates of (1) and how these are used in the software code will be presented in another paper. Also, the superiority of the presented algorithm over other state-of-the-art algorithms will be analyzed and stated in a mathematical basis.

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