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The servomechanism problem for unknown SISO positive LTI systems via tuning regulators and clamping

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Abstract: In this paper we consider the servomechanism problem for SISO positive LTI systems. In particular, we solve the robust servomechanism problem of nonnegative constant reference signals for stable SISO positive unknown LTI systems with constant nonnegative unmeasurable disturbances under strictly nonnegative control inputs, using a clamping tuning regulator.

Keywords: positive linear systems; tuning regulator; servomechanism problem; positive control inputs; switching control

1. INTRODUCTION

In this paper we consider the tracking problem of nonnegative constant reference signals for unknown stable SISO positive linear systems with nonnegative constant unmeasurable disturbances under strictly nonnegative switching control inputs. In practice, the knowledge of the system's model is commonly unknown, particularly for industrial systems, and therefore control design for unknown systems, which we adopt in this paper, is very advantageous.

A positive linear system is nothing else but a linear system with the constraint that state, output and/or input variables are nonnegative for all time. A special class of positive systems that appears quite frequently in the literature is the class of compartmental systems. In recent years, the interest in positive systems and their counterparts, compartmental systems, has grown considerably and various general results have been presented in the literature; we direct the interested reader to Roszak and Davison (July, 2007) for numerous citations.

The problem of the servomechanism problem for positive systems has had limited consideration; Roszak and Davison (September, 2007) solves a subclass of the servomechanism problem under measurable disturbances with feedforward compensators and tuning regulators; Roszak and Davison (July, 2007) takes into account the tracking/regulation problem for SISO positive LTI systems with *almost*-positivity [1]. Other related papers include Haddad et al. (2003), Leenheer and Aeyels (2001) and Kaczorek (1998). The first considers the tracking problem for positive linear systems with no disturbances under a special class of input matrices. The second, solves the problem of stabilization to a strictly positive equilibrium for SISO positive LTI known systems using techniques of positive observers presented in Van den Hof (1998). The third, provides quadratic programming techniques for the stabilization of known positive LTI systems. In this paper, we take into account unmeasurable disturbances under strictly nonnegative control inputs, using clamping control.

The paper is organized as follows. Preliminaries are given first, where the terminology, positive systems and compartmental systems, tuning regulators, and singular perturbation theory are discussed. All assumptions on the system plant treated in the paper and the Problem Statement are described in Section 3. Section 4 provides the main results of the paper, while all concluding remarks complete the paper.

2. BACKGROUND AND PRELIMINARIES

2.1 Terminology

Let the set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, the set $\mathbb{R}_+^n := \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_+,\ \forall i = 1, ..., n\}$. If exclusion of 0 from the sets will be necessary, then we will denote the sets in the standard way $\mathbb{R}_+^n \setminus \{0\}$. The set of eigenvalues of a matrix $\mathcal{A}$ will be denoted as $\sigma(\mathcal{A})$. The ij^{th} entry of a matrix $\mathcal{A}$ will be denoted as a_{ij}. A *nonnegative* matrix $\mathcal{A}$ has all of its entries greater or equal to 0, $a_{ij} \in \mathbb{R}_+$. A *Metzler* matrix $\mathcal{A}$ is a matrix for which all off-diagonal elements of $\mathcal{A}$ are nonnegative, i.e. $a_{ij} \in \mathbb{R}_+$ for all $i \neq j$. A *compartmental* matrix $\mathcal{A}$ is a matrix that is Metzler, where the sum of the components within a column is less than or equal to zero, i.e. $\sum_{i=1}^{n} a_{ij} \leq 0$ for all $j = 1, 2, ..., n$.

2.2 Positive Linear Systems

In this section we give an overview of *positive linear systems* Luenberger (1979), Farina and Rinaldi (2000),

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[1] where the component of the state/output may exit the positive orthant by a small perturbation

Kaczorek (2002), and point out a key subset of positive systems known as *compartmental systems* Farina and Rinaldi (2000), Jacquez and Simon (1993).

We first define a positive linear system in the traditional sense Farina and Rinaldi (2000).

Definition 1. A linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, and $D \in \mathbb{R}^{r\times m}$ is considered to be a *positive linear system* if for every nonnegative initial state and for every nonnegative input the state of the system and the output remain nonnegative.

Notice that Definition 1 states that the input to the system must be nonnegative, a restriction that we will abide to throughout this paper.

It turns out that Definition 1 has a very nice interpretation in terms of the matrix quadruple (A, B, C, D).

Theorem 2. (Farina and Rinaldi (2000)). A linear system (1) is positive if and only if the matrix A is a Metzler matrix, and B, C, and D are nonnegative matrices.

An interesting subset of positive systems is that of compartmental systems. The main mathematical distinction, for LTI systems, between a positive system and a compartmental system is that a positive system's A matrix is Metzler, while a compartmental system's A matrix is compartmental. The inclusion of compartmental systems is made because in general, compartmental systems are stable, a property of great significance throughout the paper. For a more complete study and interesting results on compartmental systems see Jacquez and Simon (1993) and references therein.

2.3 Tuning Regulators

In this section we describe a particular compensator, known as the tuning controller or tuning regulator, which solves the servomechanism tracking and regulation problem for *unknown* [2] stable linear systems under unknown constant unmeasurable disturbances. Such unknown systems often occur in industrial application problems. The results of this section can be found in their entirety and in their general form in Davison (1976) and Miller and Davison (1989).

Consider the plant

$$\begin{aligned} \dot{x} &= Ax + Bu + E\omega \\ y &= Cx + Du + F\omega \\ e &:= y_{ref} - y \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, the unmeasurable disturbance vector $\omega \in \mathbb{R}^{\Omega}$, and $y_{ref} \in \mathbb{R}^r$ is a desired tracking signal. Assume that the output y is measurable, that the matrix A is Hurwitz, $m = r$, and that the unmeasurable disturbance vector and tracking signals are constants. Then, the tuning regulator

$$\begin{aligned} \dot{\eta} &= \epsilon(y_{ref} - y) \\ u &= (D - CA^{-1}B)^{-1}\eta \end{aligned} \tag{3}$$

where $\epsilon \in (0, \epsilon^*]$, $\epsilon^* \in \mathbb{R}_+ \setminus \{0\}$, solves the robust servomechanism problem [3], i.e. (i) the closed loop system is stable, (ii) $e \to 0$ as $t \to \infty$ for all tracking signals and unmeasurable disturbances , and (iii), property (ii) occurs for all plant perturbations which maintain closed loop stability.

We summarize the above discussion by a Theorem for the case of SISO linear systems.

Theorem 3. (Davison (1976)). Consider the system (2), under the assumption that $y_{ref} \in \mathbb{R}$ and $\omega \in \mathbb{R}$ are constants. Then there exists an ϵ^* such that the tuning regulator (3) achieves robust tracking and regulation if and only if $rank(D - CA^{-1}B) = 1$.

We refer the interested reader to Davison (1976) for the procedure of experimentally obtaining the gain matrix $(D - CA^{-1}B)^{-1}$, for the case of unknown plant models (2). It is to be noted that if $rank(D - CA^{-1}B) = r$, then "on-line tuning" Davison (1976) is used to find an optimal value of ϵ in the controller (3).

2.4 Singular Perturbation

This section has been added for completeness and covers singular perturbation results needed in order to prove the main results of this paper. The following discussion has been taken from Khalil (2002), Chapter 11.

The standard singular perturbation model can be described as

$$\begin{aligned} \dot{q} &= f(t, q, z, \epsilon), \qquad q(t_0) = q_0 \\ \epsilon\dot{z} &= g(t, q, z, \epsilon), \qquad z(t_0) = z_0 \end{aligned} \tag{4}$$

where the functions f and g are continuously differentiable in their arguments $(t, q, z, \epsilon) \in [0, \infty) \times D_q \times D_z \times [0, \epsilon_0]$, with $D_q \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^s$ being open and connected sets. By setting $\epsilon = 0$, we obtain

$$0 = g(t, q, z, 0), \tag{5}$$

where we designate the real root [4] of (5) as

$$z = h(t, q). \tag{6}$$

To obtain a reduced model, we substitute (6) into (4) resulting in

$$\dot{q} = f(t, q, h(t, q), 0), \qquad q(t_0) = q_0. \tag{7}$$

Now denote the solution of (7) by $\overline{q}(t)$ and define

$$\overline{z}(t) = h(t, \overline{q}(t)),$$

which describes the behavior of z when $q = \overline{q}$.

In order to present a very important result on singular perturbations, we need to perform a change of variables first $p = z - h(t, q)$, which shifts the state of z to the origin. In the new variables (q, p) with $\epsilon\frac{dp}{dt} = \frac{dp}{d\tau}$, hence $\frac{d\tau}{dt} = \frac{1}{\epsilon}$,

[2] by unknown we mean that there is no knowledge of (A, B, C, D)

[3] Davison (1976) does not assume that $m = r$ and that the tracking and unmeasurable disturbance signals are constant, but for the purpose of this paper this causes no loss of generality

[4] without loss of generality, we assume there is only one root

and using $\tau = 0$ as the initial value at $t = t_0$, we obtain, in the new time scale:

$$\begin{aligned} \dot{q} &= f(t,q,p+h(t,q),\epsilon), \qquad q(t_0) = q_0 \\ \frac{dp}{d\tau} &= g(t,q,p+h(t,q),\epsilon) - \epsilon\frac{\partial h}{\partial t} \\ &\quad -\epsilon\frac{\partial h}{\partial q} f(t,q,p+h(t,q),\epsilon), \\ &\quad p(t_0) = z_0 - h(t_0,q_0). \end{aligned} \tag{8}$$

By setting $\epsilon = 0$, the latter equation reduces to

$$\frac{dp}{d\tau} = g(t,q,p+h(t,q),0),\ p(t_0) = z_0 - h(t_0,q_0), \tag{9}$$

which is commonly referred to as the *boundary-layer model*.

We will also make use of the autonomous system

$$\frac{dp}{d\tau} = g(t_0,q_0,p+h(t_0,q_0),0),\ p(t_0) = z_0 - h(t_0,q_0) \tag{10}$$

which has an equilibrium at $p = 0$, and has been derived from (9) by setting $t = t_0$ and $q = q_0$. Define the solution of (10) as $\hat{p}(\tau)$.

The following theorem presents the singular perturbation result of interest in this paper.

Theorem 4. (Khalil (2002) pg.439). Consider the singular perturbation problem of (4). Assume that the following conditions are satisfied for all

$$[t, q, z - h(t,q), \epsilon] \in [0,\infty) \times D_q \times D_p \times [0,\epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_y \subset \mathbb{R}^s$, which contain their respective origins:

(1) On any compact subset of $D_x \times D_y$, the functions f, g, their first partial derivatives with respect to (q,z,ϵ), and the first partial derivative of g with respect to t are continuous and bounded, $h(t,q)$ and $[\partial g(t,q,z,0)/\partial z]$ have bounded first partial derivatives with respect to their arguments, and $[\partial f(t,q,h(t,q),0)/\partial q]$ is Lipschitz in q, uniformly in t;
(2) the origin is an exponentially stable equilibrium point of the reduced system (7);
(3) the origin is an exponentially stable equilibrium point of the boundary-layer model (9), uniformly in (t,q). Let $\mathcal{R}_p \subset D_p$ be the region of attraction of (10) and Γ_y be a compact subset of $\mathcal{R}_y$.

Then, for each compact set $\Gamma_q \subset \{W_2(x) \leq \xi c, 0 < \xi < 1\}$ there is a positive constant ϵ_1 such that for all $t_0 \geq 0$, $q_0 \in \Gamma_q$, $z_0 - h(t_0,q_0) \in \Gamma_p$, and $0 < \epsilon < \epsilon_1$, the singular perturbation problem has a unique solution $q(t,\epsilon)$, $z(t,\epsilon)$ on $[t_0,\infty)$, and

$$q(t,\epsilon) - \overline{q}(t,\epsilon) = O(\epsilon)$$
$$z(t,\epsilon) - h(t,\overline{q}(t)) - \hat{p}(\tau) = O(\epsilon)$$

We are now ready to introduce the main problem of the paper.

3. PROBLEM STATEMENT

In this section, we provide the details of the plant, all accompanying assumptions made on the plant, and the problem of interest.

Throughout this paper we consider the following LTI SISO plant:

$$\begin{aligned} \dot{x} &= Ax + bu + e_\omega \omega \\ y &= cx + du + f\omega \\ e &:= y_{ref} - y \end{aligned} \tag{11}$$

where A is an $n \times n$ Metzler stable matrix, $b \in \mathbb{R}^n_+$, $c \in \mathbb{R}^{1\times n}_+$, $d \in \mathbb{R}_+$, $e_\omega\omega \in \Omega_1 \subset \mathbb{R}^n_+$, $f\omega \in \Omega_2 \subset \mathbb{R}_+$, $y_{ref} \in Y_{ref} \subset \mathbb{R}_+$.

Next, we provide an important assumption which will be commonly used in the sequel. The assumption is needed in order to ensure that the steady state values of the closed loop system be nonnegative, under the choice of the reference signals and the unmeasurable disturbances of the plant. If this assumption was not true, then clearly we cannot attempt to satisfy any sort of nonnegativity of the states.

Assumption 1. Given (11) assume that the existence condition $rank(d - cA^{-1}b) = 1$ holds and that the sets Ω_1, Ω_2, and Y_{ref} are chosen such that the steady state values of the plant's states and input are nonnegative, i.e. for all tracking and disturbance signals in question, it is assumed that the steady-state of the system (11) is given by

$$\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = -\begin{bmatrix} A & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} e_\omega & 0 \\ f & -1 \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix} \tag{12}$$

and has the property that $x_{ss} \in \mathbb{R}^n_+$ and $u_{ss} \in \mathbb{R}_+$. It is to be noted that the indicated inverse exists if and only if $rank(d - cA^{-1}b) = 1$, which is the same condition as that of Theorem 3.

Notice that because $y_{ref} \in \mathbb{R}_+$, then $y_{ss} = cx_{ss} + du_{ss} + f\omega \in \mathbb{R}_+$.

Before we present the problem of interest, we would like to point out that one can easily check if the existence condition $rank(d - cA^{-1}b) = 1$ holds true. We provide a very simple algorithm to do this. The following algorithm is a modification of the result given in Davison (1976).

Algorithm 1. It is assumed that the output of the system is measurable and the input is excitable the disturbance set to zero, i.e. $\omega = 0$.

(1) Apply an input $u = \overline{u}$ to (11), with $\overline{u}$ having a nonzero steady-state value.
(2) Measure the corresponding steady-state value of the output $y = \overline{y}$.
(3) If $\overline{y} \neq 0$, then the existence condition holds true.

With the above plant and assumption given, we outline the main problem of interest.

Problem 5. Consider the plant (11), with initial condition $x_0 \in \mathbb{R}^n_+$, under Assumption 1. Find a nonnegative controller u that

(a) guarantees closed loop stability;
(b) ensures the plant (11) is nonnegative, i.e. the states x and the output y are nonnegative for all time; and
(c) ensures tracking of the reference signals, i.e. $e = y - y_{ref} \to 0$, as $t \to \infty$, $\forall y_{ref} \in Y_{ref}$ and $\forall \omega \in \Omega$. In addition,
(d) assume that a controller has been found so that conditions (a), (b), (c) are satisfied; then for all pertur-

bations of the nominal plant modal which maintain properties (a) and (b), it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. property (c) still holds.

We present two more assumptions that will be used throughout the paper.

Assumption 2. The control input will be of the form:

$$\begin{aligned} \dot{\eta} &= \epsilon(y_{ref} - y), \quad \eta_0 = 0 \\ u &= k\eta \end{aligned}, \tag{13}$$

where

$$k = \begin{cases} 0 & \eta \leq 0 \\ 1 & \eta > 0 \end{cases}$$

The above control law is called a clamping controller; it incorporates the tuning regulator and saturation at zero.

Assumption 3. A necessary result for Assumption 1 to hold is that

$$y_{ref} - f\omega \geq 0;$$

thus, without loss of generality, we can assume that $f = 0$.

3.1 Breakdown of Assumption 1

Assumption 1 provides a nice algebraic expression for x_{ss} and u_{ss}; however, equation (12) is only useful if we know the value of ω and the matrices e_ω and f. It would be of great interest to actually know how large or small the disturbances can be in order for Assumption 1 to hold, i.e. what are the feasible sets $E\omega \subset \Omega_1 \subset \mathbb{R}^n_+$, $F\omega \subset \Omega_2 \subset \mathbb{R}_+$ and $Y_{ref} \subset \mathbb{R}_+$, where $e_\omega\omega \in E\omega$, $f\omega \in F\omega$ and $y_{ref} \in Y_{ref}$, such that (12) holds. In this subsection, we consider the latter problem of finding the feasible sets $E\omega$, $F\omega$, and Y_{ref}.

Solution to Assumption 1

First, recall where (12) comes from:

$$\dot{x} = 0 = Ax_{ss} + bu_{ss} + e_\omega\omega \tag{14}$$

$$\dot{\eta} = \epsilon(y_{ref} - y) = 0 = cx_{ss} + du_{ss} + f\omega - y_{ref} \tag{15}$$

Taking equation (14) and isolating it for x_{ss} we get:

$$x_{ss} = -A^{-1}bu_{ss} - A^{-1}e_\omega\omega. \tag{16}$$

Now, substituting equation (16) into equation (15) and isolating for u_{ss} we obtain:

$$u_{ss} = \frac{cA^{-1}e_\omega\omega - f\omega + y_{ref}}{d - cA^{-1}b}. \tag{17}$$

From above and the fact that we need $u_{ss} \in \mathbb{R}_+$, we obtain the equation

$$\begin{aligned} \frac{cA^{-1}e_\omega\omega - f\omega + y_{ref}}{d - cA^{-1}b} &\geq 0 \\ cA^{-1}e_\omega\omega - f\omega + y_{ref} &\geq 0; \end{aligned} \tag{18}$$

however, $K^{-1} = d - cA^{-1}b > 0$ by Roszak and Davison (July, 2007), resulting in:

$$\begin{aligned} (e_\omega\omega, f\omega, y_{ref}) \in S := \{(\xi_1, \xi_2, \xi_3) \mid \\ cA^{-1}\xi_1 - \xi_2 + \xi_3 \geq 0,\ \xi_1 \geq 0, \xi_2 \geq 0, \xi_3 \geq 0\} \end{aligned} \tag{19}$$

In the case of unmeasurable disturbances, we can see from (18) that if the disturbances are small in comparison to the tracking signal, i.e. $y_{ref} \geq (f - cA^{-1}e_\omega)\omega$, then the assumption will hold true. However, if (18) does not hold, Problem 5 is unsolvable.

Note that if the system matrices are known, then one can use (19) directly to find S.

Remark 6. Notice that if $u_{ss} \geq 0$, then

$$x_{ss} = -A^{-1}bu_{ss} - A^{-1}e_\omega\omega \geq 0$$

as all matrices and vectors are nonnegative, i.e. $-A^{-1}$ exists and is nonnegative Luenberger (1979) and b along with $e_\omega\omega$ are nonnegative by assumption.

4. MAIN RESULTS

In this section, we present the main results of the paper. First, we present a theorem that assumes a strictly positive steady state value for the input, then we present a corollary where the steady state of the input is allowed to be nonnegative.

Theorem 7. Consider system (11) under Assumption 1 and Assumption 2. Further assume that $x_0 \in \mathbb{R}^n_+$ and $u_{ss} > 0$. Then there exists an ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ the controller (13) solves Problem 5.

Proof. We first concentrate on showing that tracking of y_{ref} occurs. This is broken down into two steps:

(1) First, we will show that if initially $\eta_0 = 0$ and $\dot{\eta}(0) \leq 0$, then there exists a time $t_1 > 0$ such that $\eta(t_1) = 0$ and $\dot{\eta}(t_1) > 0$, i.e. in (13) $k \neq 0$ for all time.
(2) Second, we will show that if there exists a time t_2 such that $\eta(t_2) > 0$, then there exists an ϵ^* such that for all time $t \geq t_2$ and all $\epsilon \in (0, \epsilon^*]$ the controller (13) maintains nonnegativity of the states, outputs, and the input and solves Problem 5.

Thus, let us show (1). By contradiction, assume there does not exist a time t_1 such that $\eta(t_1) = 0$ and $\dot{\eta}(t_1) > 0$, i.e. $k = 0$ for all time. Therefore, the closed loop system becomes

$$\begin{aligned} \dot{x} &= Ax + e_\omega\omega \\ \dot{\eta} &= \epsilon(y_{ref} - cx - f\omega) \end{aligned}$$

and since A is stable

$$x \to -A^{-1}e_\omega\omega = \overline{x}_{ss}, \quad t \to \infty.$$

Recall,

$$\begin{aligned} 0 &= Ax_{ss} + bu_{ss} + e_\omega\omega \\ -e_\omega\omega &= Ax_{ss} + bu_{ss} \\ -A^{-1}e_\omega\omega &= x_{ss} + A^{-1}bu_{ss} \\ \overline{x}_{ss} &= x_{ss} + A^{-1}bu_{ss}, \end{aligned}$$

i.e. if $k = 0$ for all time $t > 0$, then the system tends toward $\overline{x}_{ss}$ as $t \to \infty$, but this implies that

$$
\begin{aligned}
\dot{\eta} &= \epsilon(y_{ref} - c\overline{x}_{ss} - d(0) - f\omega) \\
&= \epsilon(y_{ref} - c(x_{ss} + A^{-1}bu_{ss}) - d(0) - f\omega) \\
&= \epsilon(y_{ref} - cx_{ss} - du_{ss} - f\omega) + \epsilon(d - cA^{-1}b)\frac{K}{K}u_{ss}) \\
&= 0 + \frac{\epsilon}{K}u_{ss} > 0;
\end{aligned}
$$

(recall $K = (d - cA^{-1}b)^{-1} > 0$ by Roszak and Davison (July, 2007)) therefore, by continuity there exists a time t_1 such that $\dot{\eta}(t_1) > 0$, $\eta(t_1) = 0$, and hence there exists a t_2 such that $\eta(t_2) > 0$, a contradiction to the assumption made that $\eta \leq 0$ for all time.

Next, let us show (2); we proceed to illustrate that if for some time $t_2 \geq 0$, $\eta(t_2) > 0$, then there exists an ϵ^* such that for all time $t \geq t_2$ and all $\epsilon \in (0, \epsilon^*]$ the controller (13) maintains nonnegativity of the states, outputs, and the input and thus solves Problem 5. In order to prove the above, we use the results of singular perturbation. The closed loop system with the tuning regulator for $\eta > 0$ is of the form:

$$
\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & b \\ -\epsilon c & -\epsilon d \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} e_\omega & 0 \\ -\epsilon f & \epsilon 1 \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix}. \quad (20)
$$

First, let us show that the equilibrium of the closed loop system is independent of ϵ. This is easily seen by noticing

$$
\dot{\eta} = \epsilon(-cx - d\eta - f\omega + y_{ref}) = 0
$$

$$
\text{implies that} \quad -cx - d\eta - f\omega + y_{ref} = 0.
$$

Now, since the equilibrium (x_{ss}, η_{ss}) is independent of ϵ and invariant, we can transform the system as needed, i.e. let $z = x - x_{ss}$ and $q = \eta - \eta_{ss}$. in (20), resulting in the new system

$$
\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\epsilon d & -\epsilon c \\ b & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}. \quad (21)
$$

Next, let us scale the derivatives (i.e. scaling of time) by $\epsilon dt = d\tau$ resulting in the transformed system

$$
\begin{bmatrix} \overset{\odot}{q} \\ \epsilon \overset{\odot}{z} \end{bmatrix} = \begin{bmatrix} -d & -c \\ b & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \quad (22)
$$

with $\epsilon\overset{\odot}{q} = \dot{q}$ and $\epsilon\overset{\odot}{z} = \dot{z}$. We have now transformed our model into that of the singular perturbation model (4). In order to use Theorem 4, we must show that all assumptions of Theorem 4 hold true. However, as (22) is linear and time invariant and we are only interested in η it suffices to show that the reduced model yields exponential stability; all other assumptions clearly hold. By setting $\epsilon = 0$ we obtain $z = h(q) = -A^{-1}bq$, as A is Hurwitz, $h(q)$ exists and is unique. Next by substituting $h(q)$ into $\overset{\odot}{q}$ we obtain the reduced model:

$$
\overset{\odot}{q} = -dq + cA^{-1}bq = -(d - cA^{-1}b)\frac{K}{K}q = -\frac{q}{K},
$$

implying that $\overset{\odot}{q} = -\frac{q}{K} \Rightarrow \frac{\dot{q}}{\epsilon} = -\frac{q}{K} \Rightarrow \dot{q} = -\frac{\epsilon q}{K}$ clearly exponentially stable. Thus, by Theorem 4 we have:

$$
q - \overline{q} = O(\epsilon) \text{ or } \eta - \overline{\eta} = O(\epsilon) \quad \forall t \geq t_2;
$$

where

$$
\begin{aligned}
\overline{\eta} &= \overline{q} + \eta_{ss} \\
&= \eta_{ss} + e^{-\epsilon\frac{t}{K}}(\eta(t_2) - \eta_{ss})
\end{aligned}
$$

and since $\eta(t_2) > 0$, then for all time $t \geq t_2$, there exists an ϵ^* such that $\eta > 0$ for all $\epsilon \in (0, \eta^*]$ and $t \geq t_2$ since $\overline{\eta}$ is monotonically approaching η_{ss}. Thus, $y \to y_{ref}$ as $t \to \infty$ if $u_{ss} > 0$.

Finally, nonnegativity holds since $u \geq 0$ for all time, and the fact that all other conditions of Problem 5 hold are also satisfied by Roszak and Davison (July, 2007); we omit the details.

We note that the above proof uses the method of contradiction to prove the Theorem; we clearly do not need to let $\epsilon \to 0$ to obtain the needed clamping controller. We point the interested reader to Davison (1976) where "on-line tuning" is used to find the ideal ϵ^*; although our clamping controller is different, the procedure is the same.

Our next result presents a Corollary encapsulating the case when $u_{ss} \geq 0$, under the assumption that $x_0 = \overline{x}_{ss} = -A^{-1}e_\omega\omega$. Intuitively this means that the control law (13) with $k = 0$ is used until all the natural dynamics die off.

Corollary 8. Consider system (11) under Assumption 1. Further assume that $x_0 = \overline{x}_{ss} = -A^{-1}e_\omega\omega$ and $u_{ss} \geq 0$. Then there exists an ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ the controller (13) solves Problem 5.

Proof. The result for the case $u_{ss} > 0$ follows from the proof of Theorem 7; thus, let us concentrate on $u_{ss} = 0$. However, since $u_{ss} = 0$, then $x_{ss} = -A^{-1}e_\omega\omega = x_0$ and $y_{ss} = -cA^{-1}e_\omega\omega + f\omega = cx_0 + f\omega = y_{ref}$, completing the result.

Lastly, the paper considers unmeasurable disturbances, therefore, next we will introduce an algorithm that uses controller (13) to solve the servomechanism problem if the property of the steady-state existence conditions (Assumption 1 are satisfied. Otherwise, if the steady-state existence conditions are not satisfied, then the contoller (13) automatically shuts itself off so that in finite time the control input is equal to zero, and remains at 0 for all time. We note to the reader that this is the best which any controller can do, given the limited information which we have.

Algorithm 2.

(1) Check the existence condition $rank(d - cA^{-1}b) = 1$ by Algorithm 1.
    (a) If Algorithm 1 returns $\overline{y} = 0$, then there does not exist a solution to the servomechanism problem.
    (b) Otherwise, go to Step 2.
(2) Apply the clamping controller (13) to the unknown plant, by using "on-line tuning" Davison (1976).
    (a) If the clamping controller remains at zero for $t \in [t_+, \infty)$, where $t_+ \geq 0$, and no tracking/regulation occurs, then the servomechanism problem is not solvable under any control law.
    (b) Otherwise, the clamping controller (13) solves Problem 5.

5. EXAMPLE

In this section, we illustrate the results presented in this paper via an example.

Example 9. The following plant, which is a stable compartmental system, has been taken from Farina and Rinaldi (2000) pg.105. Consider the reservoirs network of Figure 1, with u as the input flow of water and ω an input flow disturbance. The system is of dimension 6, as we assume the pump dynamics can be neglected. As pointed out in Farina and Rinaldi (2000), the dynamics of each reservoir can be captured by a single differential equation: $\dot{x}_i = -\alpha_i x_i + v$, $\alpha_i > 0$, $i = 1, ..., 6$, where x_i represents the depth of the water in each reservoir.

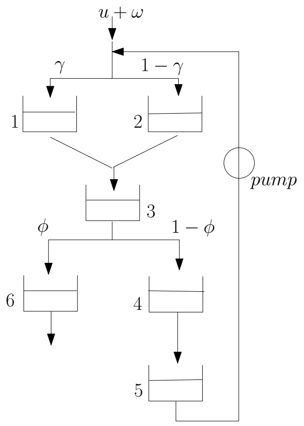

Fig. 1. System set up for Example 9.

Consider the case where $\gamma = 0.5$, $\phi = 0.9$, $\alpha_1 = 2$, $\alpha_2 = 1.7$, $\alpha_3 = 1.5$, $\alpha_4 = 1$, $\alpha_5 = 2$, and $\alpha_6 = 2$. This results in the following system:

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1.7 & 0 & 0 & 0 & 0 \\ 2 & 1.7 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1.35 & 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \omega$$

$$y = [0\ 0\ 0\ 0\ 0\ 1]x.$$

The existence condition $rank(d - cA^{-1}b) = 1$ holds using Algorithm 1. Assume now that we would like to track the reference input $y_{ref} = 1$, subject to the disturbance $\omega = 0.5$. For simulation purposes we assume $x_0 = [2\ 4\ 1\ 0.5\ 0.5\ 2]$. In this case using Algorithm 2, the application of controller (13) with $\epsilon = 0.5$, solves the tracking problem. Note that the condition (19) holds in this case for the problem, although this information was not used in order to implement the controller (13). Figure 2 illustrates both the output y and the input u. The plots of the states x are omitted, however, it is easy to deduce that they are nonnegative as $u \geq 0$.

6. CONCLUSION

In this paper, we have used a switching controller, known as the clamping controller, in order to solve the servomechanism problem for stable unknown SISO positive LTI systems with nonnegative control inputs for the case of constant tracking signals and constant unmeasurable disturbance signals. We point out that the control law can be implemented without any knowledge of the system's model.

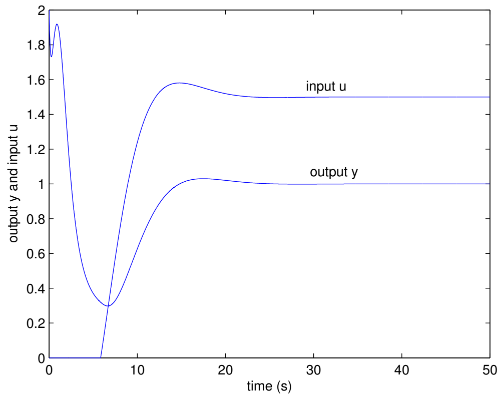

Fig. 2. Output and input response for Example 9.

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