AN ANALYSIS OF DIFFERENTIAL GEOMETRIC AND DIFFERENTIAL ALGEBRAIC METHOD FOR DISTURBANCE DECOUPLING OF NONLINEAR SYSTEMS

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Abstract: In the past few years two methods of nonlinear control theory which have become increasingly popular are differential geometry and differential algebra. In order to achieve higher performance and accuracy in practice, nonlinear controllers – based on these methods – may be applied in new CACSD-packages. This paper deals with the disturbance decoupling problem (DDP) which means that any undesired input no longer effects the output. Both methods are presented and an analysis shows the characteristic advantages and disadvantages of each method. The differential geometric and differential algebraic computations for the DDP-controller are demonstrated using an example system. A simulation study demonstrates the disturbance decouplability. *Copyright©2000 IFAC*

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1. INTRODUCTION

In practice physical dynamical systems can often be approximated by nonlinear differential equations. To achieve higher performance and accuracy, it is expedient to apply *nonlinear controllers*. Especially the task of decoupling undesired inputs of a system such that the outputs are no longer affected by these inputs may be allocated of high importance. This task is referred to as the *disturbance decoupling problem (DDP)*.

Based on nonlinear mathematical *system modelling*, there exist two methods that have become more popular in the past few years – *differential geometry* (Isidori, 1995; Schwarz, 1999) and *differential algebra* (Ritt, 1950; Fliess, 1987; Delaleau and Fliess, 1994). This paper investigates the condition of disturbance decouplability for both methods as well as the computation of the DDP-controller law. An analysis yields advantages and disadvantages of differential geometric and differential algebraic disturbance decoupling which are characteristic for both methods.

The computation of a nonlinear DDP-controller may be used in a *CACSD-package*. To demonstrate the mode of operation, a simulation study is presented for an example system. The numerical software MATLAB® and the CAS software MAPLE® are the main packages used for the simulation and computation study. Additionally, the NSAS-package (Lemmen *et al.*, 1995) is applied to solve the DDP with the differential geometric method.

2. SOLVING THE DDP WITH DIFFERENTIAL GEOMETRY

The differential geometric method according to Isidori (1995) is applicable for *analytical input-affine systems* of the form

$$\Sigma_{\mathrm{ALS}} \begin{array}{ll} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{p}(\boldsymbol{x})d, & \boldsymbol{x} \in \mathbb{R}^n, \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}), & d \in \mathbb{R}, \quad \boldsymbol{u}, \boldsymbol{y} \in \mathbb{R}^m, \end{array} \quad (1)$$

with $\boldsymbol{G}(\boldsymbol{x}) := [\boldsymbol{g}_1(\boldsymbol{x}), \ldots, \boldsymbol{g}_m(\boldsymbol{x})]^{\mathrm{T}}$.

The relative degree $\boldsymbol{r}$ of an undisturbed system is defined as (Isidori, 1995):

(i) $$\mathrm{L}_{\boldsymbol{g}_j} \mathrm{L}_{\boldsymbol{f}}^k h_i(\boldsymbol{x}) = 0$$

for all $1 \leq j \leq m$, for all $0 < k < r_i - 1$, for all $1 \leq i \leq m$, and for all $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{x}_0$,

(ii) the decoupling matrix

$$\boldsymbol{A}(\boldsymbol{x}) := \begin{bmatrix} \mathrm{L}_{\boldsymbol{g}_1}\mathrm{L}_{\boldsymbol{f}}^{r_1-1} h_1(\boldsymbol{x}) & \cdots & \mathrm{L}_{\boldsymbol{g}_m}\mathrm{L}_{\boldsymbol{f}}^{r_1-1} h_1(\boldsymbol{x}) \\ \mathrm{L}_{\boldsymbol{g}_1}\mathrm{L}_{\boldsymbol{f}}^{r_2-1} h_2(\boldsymbol{x}) & \cdots & \mathrm{L}_{\boldsymbol{g}_m}\mathrm{L}_{\boldsymbol{f}}^{r_2-1} h_2(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \mathrm{L}_{\boldsymbol{g}_1}\mathrm{L}_{\boldsymbol{f}}^{r_m-1} h_m(\boldsymbol{x}) & \cdots & \mathrm{L}_{\boldsymbol{g}_m}\mathrm{L}_{\boldsymbol{f}}^{r_m-1} h_m(\boldsymbol{x}) \end{bmatrix}$$

is non-singular at $\boldsymbol{x} = \boldsymbol{x}_0$.

The terms $\mathrm{L}_{\boldsymbol{f}}^k h_i(\boldsymbol{x})$ are defined as the so called Lie derivatives of the order k of any real-valued function h_i along a vector field $\boldsymbol{f}$:

$$\mathrm{L}_{\boldsymbol{f}}^k h_i(\boldsymbol{x}) := \frac{\partial \mathrm{L}_{\boldsymbol{f}}^{k-1} h_i(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x}).$$

Respectively, the relative degree $\boldsymbol{r}_d$ of a disturbed system is defined as (Isidori, 1995):

(iii) $$\mathrm{L}_{\boldsymbol{p}} \mathrm{L}_{\boldsymbol{f}}^k h_i(\boldsymbol{x}) = 0$$

for all $0 < k < r_{d,i} - 1$, for all $1 \leq i \leq m$, and for all $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{x}_0$.

An Σ_{ALS} is disturbance decouplable if the condition

$$r_{d,i} \geq r_i, \quad (i = 1, \ldots, m), \quad (2)$$

with the relative degrees $\boldsymbol{r}_d = [r_{d,1}, \ldots, r_{d,m}]$ and $\boldsymbol{r} = [r_1, \ldots, r_m]$ holds. Subsequently, the disturbance decoupled substitute system can be determined by the static state feedback

$$\boldsymbol{u}(\boldsymbol{x}, d, \boldsymbol{v}) = \boldsymbol{a}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})\boldsymbol{v} + \boldsymbol{c}(\boldsymbol{x})d. \quad (3)$$

Thus, the matrix $\boldsymbol{B}(\boldsymbol{x})$ and the vectors $\boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{c}(\boldsymbol{x})$ are defined as

$$\boldsymbol{a}(\boldsymbol{x}) := -\boldsymbol{A}^{-1}(\boldsymbol{x}) \begin{bmatrix} \mathrm{L}_{\boldsymbol{f}}^{r_1} h_1(\boldsymbol{x}) \\ \mathrm{L}_{\boldsymbol{f}}^{r_2} h_2(\boldsymbol{x}) \\ \vdots \\ \mathrm{L}_{\boldsymbol{f}}^{r_m} h_m(\boldsymbol{x}) \end{bmatrix},$$

$$\boldsymbol{B}(\boldsymbol{x}) := \boldsymbol{A}^{-1}(\boldsymbol{x}),$$

$$\boldsymbol{c}(\boldsymbol{x}) := -\boldsymbol{A}^{-1}(\boldsymbol{x}) \begin{bmatrix} \mathrm{L}_{\boldsymbol{p}}\mathrm{L}_{\boldsymbol{f}}^{r_1-1} h_1(\boldsymbol{x}) \\ \mathrm{L}_{\boldsymbol{p}}\mathrm{L}_{\boldsymbol{f}}^{r_2-1} h_2(\boldsymbol{x}) \\ \vdots \\ \mathrm{L}_{\boldsymbol{p}}\mathrm{L}_{\boldsymbol{f}}^{r_m-1} h_m(\boldsymbol{x}) \end{bmatrix},$$

with $\boldsymbol{A}^{-1}(\boldsymbol{x})$ as the inverse decoupling matrix. If the condition $r_1 + r_2 + \ldots + r_m = n$ also holds, the linearized and disturbance decoupled substitute system is given by the transfer function matrix

$$H(s) = \begin{bmatrix} \frac{1}{s^{r_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{s^{r_2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{s^{r_m}} \end{bmatrix}$$

and behaves like an integrator chain of the order m.

3. SOLVING THE DDP WITH DIFFERENTIAL ALGEBRA

Differential algebra is restricted to *rational systems* (Σ_{RS}). But in many cases the behaviour of nonlinear systems can be approximated by *analytical systems* Σ_{AS}:

$$\Sigma_{\mathrm{AS}} \begin{array}{ll} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}), & \boldsymbol{x} \in \mathbb{R}^n, \ \boldsymbol{u} \in \mathbb{R}^m, \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{d}), & \boldsymbol{d} \in \mathbb{R}^q, \ \boldsymbol{y} \in \mathbb{R}^p. \end{array}$$

Therefore, an additional *system transformation* is unavoidable to achieve a rational substitute system for the analytical system. An advanced algorithm (Senger, 1999; Bröcker and Polzer, 1999) based on differential algebra computes a DDP-controller for the rational substitute system. Hence, in the original system all non-rational partial functions $r_i(x_1, \ldots, x_n)$ are replaced with new state variables x_{n+i}. By establishing the time derivative(s) of the new state variables $\mathrm{d}x_{n+i}/\mathrm{d}t$, an extended system model is obtained, which is the rational substitute system. The system transformation can be achieved for all relevant non-rational functions of the original system, which can be expanded in a Taylor series. Such functions include e.g. trigonometrical, logarithmical, exponential, square root etc. functions (Polzer, 1998). To describe the dynamics of a nonlinear *rational system* with the disturbances $\boldsymbol{d}$, the state-space model

$$\Sigma_{\mathrm{RS}} \begin{array}{ll} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}), & \boldsymbol{x} \in \mathbb{R}^n, \ \boldsymbol{u} \in \mathbb{R}^m, \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{d}), & \boldsymbol{d} \in \mathbb{R}^q, \ \boldsymbol{y} \in \mathbb{R}^p \end{array}$$

can be specified with the meromorphic functions $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ and $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{d})$ which are quotients of polynomials. The model for the undisturbed system with $\boldsymbol{d} \equiv \boldsymbol{0}$ is defined as follows:

$$\Sigma_{\mathrm{RS}}^{\boldsymbol{d}\equiv\boldsymbol{0}} \begin{array}{ll} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})|_{\boldsymbol{d}\equiv\boldsymbol{0}}, & \boldsymbol{x} \in \mathbb{R}^n, \ \boldsymbol{u} \in \mathbb{R}^m, \\ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{d})|_{\boldsymbol{d}\equiv\boldsymbol{0}}, & \boldsymbol{d} \in \mathbb{R}^q, \ \boldsymbol{y} \in \mathbb{R}^p. \end{array}$$

In differential algebra, two parameters are significant for disturbance decouplability – the differential output rank ρ_d^* of the disturbed system and the differential output rank ρ^* of the undisturbed system. Both parameters are defined as *differential transcendence degrees* of a *differential field extension*:

$$\rho_d^* := \text{diff. trg } k\langle \boldsymbol{y}\rangle / k,$$

with $\rho_d^* \leq \min\{m+q, p\}$ (Fliess, 1987; Delaleau and Fliess, 1994). Where $k\langle \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{d}\rangle / k\langle \boldsymbol{u}, \boldsymbol{d}\rangle$ denotes the disturbed system.
Analogically, the differential output rank of the undisturbed system $k\langle \boldsymbol{y}, \boldsymbol{u}\rangle / k\langle \boldsymbol{u}\rangle$ is determined by

$$\rho^* := \text{diff. trg } k\langle \boldsymbol{y}\rangle / k,$$

with $\rho^* \leq \min\{m, p\}$.

The algorithm can be subdivided into four steps. The first step includes all possible methods of simplification. Subsequently, the time derivatives of the system output up to the system order n have to be specified in the second step. Verifying the differential algebraic condition of disturbance decouplability $\rho_d^* \leq \rho^*$ is checked in the third step. Hence, the control law is computed in the fourth and final step. The final step is an iterative process, which rewrites the time derivatives of the output $\tilde{\boldsymbol{y}}^{(l)}$ into the normal form $\boldsymbol{N}_l(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u})$. Solving this set of equations in normal form with respect to the inputs $\boldsymbol{u}$, yields the terms of inputs resp. the control variables held in the set $\mathbb{U}_l$. If the set of equations is uniquely solvable and the maximum number of possible inputs or step n has been reached, the algorithm terminates and the control law consists of the set terms.

3.1 Advanced algorithm

Step 1:

Simplify the rational system by factorization, expanding an expression, partial fraction, multidimensional division of polynomials and/or converting rational functions to a common denominator.

Step 2:

Compute the time derivatives of the system output up to the system order n: $\dot{\boldsymbol{y}}, \ddot{\boldsymbol{y}}, \ldots, \boldsymbol{y}^{(n)}$.

Step 3:

3.1: Calculate the *differential output rank* ρ_d^* of Σ_{RS} by computing the dimensions dim $\overline{\mathcal{E}}_k$ of the common (non-differential) vector spaces $\overline{\mathcal{E}}_k$ via determination of the rank of Jacobian matrices $\overline{\boldsymbol{J}}_k$ ($\overline{\boldsymbol{J}}_0 = \boldsymbol{0}$; $k = 1, \ldots, n$):

$$\dim \overline{\mathcal{E}}_k = n + \operatorname{rank} \overline{\boldsymbol{J}}_k,$$

$$\overline{\boldsymbol{J}}_k = \begin{bmatrix} \dfrac{\partial \dot{\boldsymbol{y}}}{\partial \bar{\boldsymbol{w}}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \bar{\boldsymbol{w}}} & \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \dot{\bar{\boldsymbol{w}}}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{0} \\ \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \bar{\boldsymbol{w}}} & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \dot{\bar{\boldsymbol{w}}}} & \cdots & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \bar{\boldsymbol{w}}^{(k-1)}} \end{bmatrix},$$

with $\bar{\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{d} \end{bmatrix} \Rightarrow \rho_d^* = \dim \overline{\mathcal{E}}_n - \dim \overline{\mathcal{E}}_{n-1}$.

3.2: Calculate the *differential output rank* ρ^* of $\Sigma_{\mathrm{RS}}^{d\equiv 0}$ by computing the dimensions dim $\mathcal{E}_k$ of the common (non-differential) vector spaces $\mathcal{E}_k$ via determination of the rank of Jacobian matrices $\boldsymbol{J}_k$ ($\boldsymbol{J}_0 = \boldsymbol{0}$; $k = 1, \ldots, n$) (see Di Benedetto *et al.*, 1989):

$$\dim \mathcal{E}_k = n + \operatorname{rank} \boldsymbol{J}_k,$$

$$\boldsymbol{J}_k = \begin{bmatrix} \dfrac{\partial \dot{\boldsymbol{y}}}{\partial \boldsymbol{u}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \boldsymbol{u}} & \dfrac{\partial \ddot{\boldsymbol{y}}}{\partial \dot{\boldsymbol{u}}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{0} \\ \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{u}} & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \dot{\boldsymbol{u}}} & \cdots & \dfrac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{u}^{(k-1)}} \end{bmatrix},$$

$$\Rightarrow \rho^* = \dim \mathcal{E}_n - \dim \mathcal{E}_{n-1}.$$

3.3: If the condition $\rho_d^* \leq \rho^*$ holds, the system is disturbance decouplable and the computation of the control law starts in step 4.

Step 4:

4.1: Rewrite the first time derivative of the output $\dot{\boldsymbol{y}}$ into the normal form

$$\boldsymbol{N}_1(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) = \dot{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) - \dot{\boldsymbol{y}} = \boldsymbol{0}.$$

Let $\mathbb{U}_0 := \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ be the initial set of all possible control variables. Therefore, the auxiliary set of control variables is defined by

$$\tilde{\mathbb{U}}_1 := \{\boldsymbol{u} \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m) | \boldsymbol{u} = \boldsymbol{r}_1(\boldsymbol{x}, \boldsymbol{d})\},$$

as the set of all roots of $\boldsymbol{u}$ for

$$\boldsymbol{N}_1(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) = \boldsymbol{0}.$$

Here, $\boldsymbol{r}_1(\boldsymbol{x}, \boldsymbol{d})$ represents the solution of the set of equations. The set of all possible control variables is further restricted to

$$\mathbb{U}_1 := \begin{cases} \mathbb{U}_0 \cap \tilde{\mathbb{U}}_1 & \text{, for } \tilde{\mathbb{U}}_1 \neq \emptyset \\ \mathbb{U}_0 & \text{, for } \tilde{\mathbb{U}}_1 = \emptyset. \end{cases}$$

4.2: If the number of determined control variables of $\mathbb{U}_1$ does not conform with the number of system inputs $\boldsymbol{u}$, the remaining terms for $\boldsymbol{u}$ are computed in the next step k.

Step k ($k \geq 5$): Let $l := k - 3$.

k.1: Case distinction:

a) If $2 \leq l \leq n$, then no further computation of $\boldsymbol{y}^{(l)}$ is needed (c. f. step 2).

b) If $l > n$, then compute $\boldsymbol{y}^{(l)}$.

k.2: Rewrite the time derivative of the output

$$\tilde{\boldsymbol{y}}^{(l)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{y}^{(l-1)}\big|_{\boldsymbol{u}\in\mathbb{U}_l}\right)$$

into the normal form

$$\boldsymbol{N}_l(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u}) = \tilde{\boldsymbol{y}}^{(l)} - \boldsymbol{y}^{(l)} = \boldsymbol{0} \quad .$$

The auxiliary set of control variables for the roots of $\boldsymbol{N}_l(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u})$ w.r.t. $\boldsymbol{u}$ is defined as:

$$\tilde{\mathbb{U}}_l := \{\boldsymbol{u} \in \mathrm{C}^\infty(\mathbb{R},\mathbb{R}^m) | \boldsymbol{u} = \boldsymbol{r}_l(\boldsymbol{x},\boldsymbol{d})\}.$$

The set $\mathbb{U}_l$ is given by:

$$\mathbb{U}_l := \begin{cases} \mathbb{U}_{l-1} \cap \tilde{\mathbb{U}}_l & , \text{ for } \tilde{\mathbb{U}}_l \neq \emptyset \\ \mathbb{U}_{l-1} & , \text{ for } \tilde{\mathbb{U}}_l = \emptyset. \end{cases}$$

k.3: If the number of terms for $\boldsymbol{u}$ in $\mathbb{U}_l$ equals the number of inputs m or if step n has been reached, then stop. Furthermore, the set of equations $\boldsymbol{N}_l(\boldsymbol{x},\boldsymbol{d},\boldsymbol{u})$ has to be uniquely solvable. If this does not hold, another iteration step – step k.1 – follows.

The control variables that decouple the system from disturbances are given by all $\boldsymbol{u} \in \mathbb{U}_l$. The highest time derivatives of the output $y_i^{(j)}$ ($i = 1,\ldots,p$ and $j = 1,\ldots,l$) which occur in $\boldsymbol{u}$, have to be substituted with the new inputs v_i. If the algorithm yields $\rho_d^* > \rho^*$, the system is not disturbance decouplable using static state feedback.

4. SYNOPSIS OF AN EXAMPLE SYSTEM

Consider the Σ_{ALS} with inputs u_1, u_2, disturbance d and outputs y_1, y_2 given as:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} x_1 + e^{x_2}u_1 + x_2 d \\ \sqrt{x_1}u_1 + x_2u_2 + x_1^2 d \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (4)$$

4.1 Differential geometric method

In order to determine the DDP-controller law with the geometric method, the state-space model (4) may be rewritten to the form (1) with the vectors

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad g_1(\boldsymbol{x}) = \begin{bmatrix} e^{x_2} \\ \sqrt{x_1} \end{bmatrix},$$

$$g_2(\boldsymbol{x}) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \quad \boldsymbol{p}(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ x_1^2 \end{bmatrix}.$$

Computing the relative degrees $\boldsymbol{r}$ and $\boldsymbol{r}_d$ yields $\boldsymbol{r} = \boldsymbol{r}_d = [1,1]$. Condition (2) holds and the system can be decoupled from the disturbance d. With equation (3) the control law $\boldsymbol{u}(\boldsymbol{x},d,\boldsymbol{v}) = [u_1,u_2]^{\mathrm{T}}$ is determined:

$$u_1 = \frac{v_1 - x_1 - x_2 d}{e^{x_2}},$$

$$u_2 = \frac{-\sqrt{x_1}v_1 + v_2e^{x_2} + x_1^{3/2} + d\sqrt{x_1}x_2 - dx_1^2e^{x_2}}{e^{x_2}x_2}.$$

The substitute system can be specified as

$$\dot{\boldsymbol{x}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4.2 Differential algebraic method

Determining the non-rational partial functions $x_3 := r_1(x_2) = e^{x_2}$, $x_4 := r_2(x_1) = \sqrt{x_1}$ and its time derivatives

$$\dot{x}_3 := \frac{\mathrm{d}}{\mathrm{d}t}r_1(x_2) = \dot{x}_2e^{x_2} = \dot{x}_2x_3,$$

$$\dot{x}_4 := \frac{\mathrm{d}}{\mathrm{d}t}r_2(x_1) = \frac{\dot{x}_1}{2\sqrt{x_1}} = \frac{\dot{x}_1}{2x_4}$$

leads to the rational substitute system Σ_{RS}

$$\dot{\boldsymbol{x}} = \begin{bmatrix} x_3u_1 + x_1 + x_2d \\ x_4u_1 + x_2u_2 + x_1^2d \\ x_3x_4u_1 + x_3x_2u_2 + x_3x_1^2d \\ \dfrac{x_3u_1 + x_1 + x_2d}{2x_4} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The advanced algorithm can now be used to investigate if the rational substitute system is disturbance decouplable and if so, then to compute the DDP-controller.

Step 1:

No simplification is needed.

Step 2:

Compute the time derivatives of the system output up to the system order $n = 4$:

$$\dot{\boldsymbol{y}} = \begin{bmatrix} x_3u_1 + x_1 + x_2d \\ x_4u_1 + x_2u_2 + x_1^2d \end{bmatrix}.$$

The higher time derivatives $\ddot{\boldsymbol{y}}$, $\boldsymbol{y}^{(3)}$, $\boldsymbol{y}^{(4)}$ are too complex for detailled illustration, but have to be computed.

Step 3:

3.1: With the Jacobian matrices for $\bar{\boldsymbol{w}} = [u_1\ u_2\ d]^{\mathrm{T}}$

$$\bar{\boldsymbol{J}}_1 := \frac{\partial\dot{\boldsymbol{y}}}{\partial\bar{\boldsymbol{w}}} = \begin{bmatrix} x_3 & 0 & x_2 \\ x_4 & x_2 & x_1^2 \end{bmatrix}$$

and $\overline{J}_2$, $\overline{J}_3$, $\overline{J}_4$, the differential output rank ρ_d^* of Σ_{RS} can be calculated as:

$$\begin{aligned} &\dim \overline{\mathcal{E}}_2 = n + \operatorname{rank} \overline{J}_2 = 4 + 4 = 8 , \\ &\dim \overline{\mathcal{E}}_3 = n + \operatorname{rank} \overline{J}_3 = 4 + 6 = 10 , \\ &\Rightarrow \rho_d^* = \dim \overline{\mathcal{E}}_3 - \dim \overline{\mathcal{E}}_2 = 10 - 8 = 2. \end{aligned}$$

3.2: With the Jacobian matrices for $\boldsymbol{u} = [u_1\ u_2]^{\mathrm{T}}$

$$J_1 := \frac{\partial \dot{\boldsymbol{y}}}{\partial \boldsymbol{u}} = \begin{bmatrix} x_3 & 0 \\ x_4 & x_2 \end{bmatrix}$$

and J_2, J_3, J_4, the differential output rank ρ^* of $\Sigma_{\mathrm{RS}}^{d\equiv 0}$ can be calculated as:

$$\begin{aligned} &\dim \mathcal{E}_3 = n + \operatorname{rang} J_3 = 4 + 6 = 10 , \\ &\dim \mathcal{E}_4 = n + \operatorname{rang} J_4 = 4 + 8 = 12 , \\ &\Rightarrow \rho^* = \dim \mathcal{E}_4 - \dim \mathcal{E}_3 = 12 - 10 = 2. \end{aligned}$$

3.3: Consequently, the condition $\rho_d^* \leq \rho^*$ holds and the control law for the disturbance decoupling can be worked out by computation as follows in step 4.

Step 4:

4.1: The normal form $\boldsymbol{N}_1(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) = \boldsymbol{0}$ is given by

$$\boldsymbol{N}_1(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}) = \begin{bmatrix} x_3 u_1 + x_1 + x_2 d - \dot{y}_1 \\ x_4 u_1 + x_2 u_2 + x_1^2 d - \dot{y}_2 \end{bmatrix} .$$

The solution of this set of equations w.r.t. the inputs $\boldsymbol{u}$ is given by

$$\mathbb{U}_1 = \left\{ \boldsymbol{u} \middle| u_1 = \frac{-x_1 - x_2 d + \dot{y}_1}{x_3}, \right.$$
$$\left. u_2 = \frac{x_4 x_1 + x_4 x_2 d - x_4 \dot{y}_1 - x_3 x_1^2 d + \dot{y}_2 x_3}{x_3 x_2} \right\},$$

since $\mathbb{U}_0 := \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m)$,

$$\tilde{\mathbb{U}}_1 := \{\boldsymbol{u} \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^m) | \boldsymbol{u} = \boldsymbol{r}_1(\boldsymbol{x}, \boldsymbol{d})\}$$

and hence, $\mathbb{U}_1 := \mathbb{U}_0 \cap \tilde{\mathbb{U}}_1$.

4.2: The number of controls in $\mathbb{U}_1$ complies exactly with the number of inputs $m = 2$ and furthermore, the set of equations is uniquely solvable. Thus, the algorithm terminates.

Substituting the time derivatives of the outputs $\dot{y}_1$, $\dot{y}_2$ with the new inputs v_1, v_2 yields the control law

$$\boldsymbol{u}(\boldsymbol{x}, d, \boldsymbol{v}) = \begin{bmatrix} \dfrac{-x_1 - x_2 d + v_1}{x_3} \\ \dfrac{x_4(x_1 + x_2 d - v_1) + x_3(v_2 - x_1^2 d)}{x_2 x_3} \end{bmatrix},$$

$$\forall\ (x_2 \neq 0) \wedge (x_3 \neq 0). \tag{5}$$

The disturbance decoupled substitute system can be determined to

$$\dot{\boldsymbol{x}} = \begin{bmatrix} v_1 \\ v_2 \\ v_2 x_3 \\ \dfrac{v_2}{2x_4} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

4.3 Simulation study

The disturbed analytical system (4) and the computed DDP-controller (5) are implemented in MATLAB®/SIMULINK®. As initial state variables $x_{1,0} = 1$ and $x_{2,0} = 1$ are allocated. Also as new inputs $v_1 := 5 \cdot 1(t)$, $v_2 := 2 \cdot 1(t)$ of the DDP-controller with

$$1(t) := \begin{cases} 1, & \forall t \geq 0 \\ 0, & \forall t < 0 \end{cases}$$

are chosen. For any constant disturbance $d = c \cdot 1(t)$, $c \in \{0; 5; 10; 50\}$ the system outputs y_1, y_2 are disturbance decoupled (see figure 1). The desired input-output behaviour consists of two integrators for each system output. The mode of operation of the control law (5) is demonstrated in figure 2.

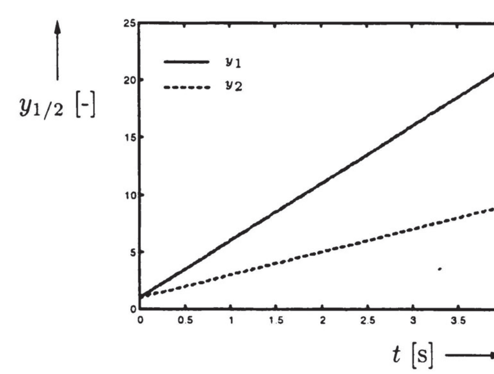

Fig. 1. System outputs y_1 and y_2 with DDP-controller

4.4 Analysis of both methods

For this analytical input-affine system both methods lead to the same DDP controller. Also the linearized substitute system is equal for the first n states. However, the differential algebraic method leads to an extended system model. Computing the DDP-controller differential geometrically can be solved with the NSAS-package automatically whereas with the differential algebraic method an additional system transformation is needed. The differential geometric method according to Isidori (1995) computes the control law much easier than the differential algebraic method but is restricted to Σ_{ALS} with the same number of inputs and outputs. The differential algebraic algorithm can

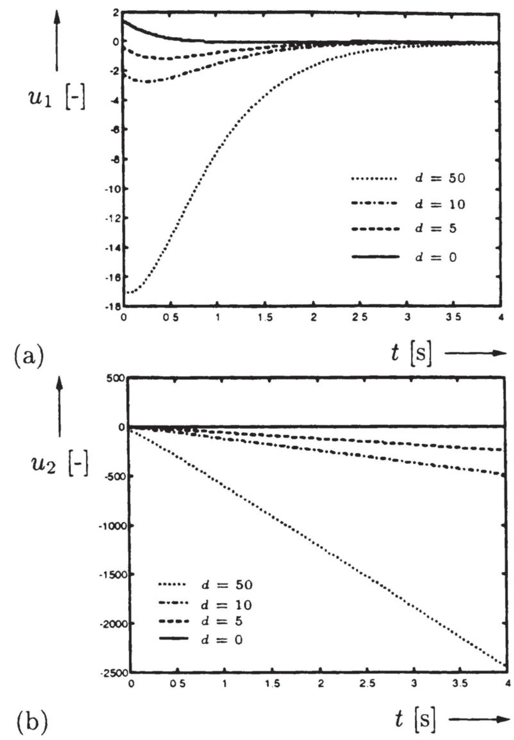

Fig. 2. Control variables with DDP-controller

compute the DDP controller also for non-affine analytical systems with different numbers of inputs and outputs. The main advantages and disadvantages of both methods are summarized in table (1).

Table 1 Advantages ($\oplus$) and disadvantages ($\ominus$) of both nonlinear control methods

Differential geometric method	*Differential algebraic method*
$\ominus$ solvable for Σ_{ALS}	$\oplus$ solvable for Σ_{AS}
$\oplus$ computation of the DDP-controller automatically	$\ominus$ computation of an additional system transformation
$\ominus$ modelling of one disturbance ($q = 1$)	$\oplus$ modelling of multiple disturbances
$\ominus$ $m = p$, $m :=$ inputs, $p :=$ outputs	$\oplus$ $m \neq p$, $m :=$ inputs, $p :=$ outputs
$\oplus$ linearized substitute system	$\ominus$ nonlinear substitute system

5. CONCLUSION

In this paper, two nonlinear methods - differential geometry and differential algebra - are presented for solving the disturbance decoupling problem. Both methods have characteristic advantages and disadvantages. The anaylysis shows that a new advanced algorithm based on differential algebra can solve the DDP for an extended class of nonlinear systems (Σ_{RS}) with an additional system transformation ($\Sigma_{AS} \rightarrow \Sigma_{RS}$). The algorithm is independent of the number of disturbances, inputs or outputs. The future goal is to implement this algorithm in a CACSD-package in order to compute the DDP-controller law automatically. An example system demonstrates how the DDP-controller is determined by differential geometric and differential algebraic methodology. A simulation study shows the mode of operation of the DDP-controller as well as the fact that the system is now disturbance decoupled.

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