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Gradient Based Discrete-Time Modeling and Control of Hamiltonian Systems

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Abstract: A gradient based discrete-time model of continuous Hamiltonian systems with input is proposed and a procedure is given to construct the discrete-time model. The model validation for both separable and non-separable case is done considering the energy relation, additionally stabilizability condition is given and the model is also tested especially for the well-known non-separable Hamiltonian systems by simulations. After then, the discrete-time counterpart of PBC technique is developed for n-degree of freedom mechanical systems using this proposed discrete-time model. The discrete-time control rules which correspond to potential energy shaping and damping assignment are designed directly using the discrete time model of the desired system and the discrete time model of the open loop systems. To illustrate the effectiveness of the proposed method, two non-separable examples are investigated and the simulation results are given.

1. INTRODUCTION

The port-controlled Hamiltonian (PCH) approach has been introduced not only for modeling of physical systems but also for control of a wide class of nonlinear systems. There is large number of publications on this subject(Van der Schaft (1996), Ortega et al. (2002), Ortega and Garcia-Canseco (2004)). PCH approach has been considered mostly for nonlinear systems especially in systems where electrical and mechanical sub-systems have to be considered together. Furthermore, the passivity-based control (PBC) is a powerful design technique for stabilizing nonlinear systems and especially set point regulation problem for both Euler-Lagrange systems and PCH systems.

In continuous-time context, the PBC design is completed in two-step; first the energy shaping control rule $u_{es}(t)$ is designed to assign the desired energy function as the total energy of the system, second, the damping injection control rule $u_{di}(t)$ is designed to achieve asymptotic stability at desired equilibrium point, which is an isolated and strict minimum of the desired energy function. One can find the detail of design methodology in (Van der Schaft (1996), Ortega et al. (1998)) and references therein.

On the other hand, technological advancements in digital processors, the widespread use of computer controlled systems in engineering practice are in need of a theory to analyse and design sampled-data systems and techniques to obtain discrete-time model of non-linear systems crucially. A framework on the issue of the stabilization of sampled-data non linear systems using their approximate discrete time models can be found in (Nesic and Teel (2004), Nesic et al. (1999)).

In control literature, to the best of our knowledge, there is limited number of works utilizing the discrete-time models of Hamiltonian systems for control applications as (Laila and Astolfi (2005), Laila and Astolfi (2006a), Laila and Astolfi (2006b)). In these works, in order to obtain the discrete-time model, Euler method is used and as mentioned by the authors, “the Euler model is not Hamiltonian conserving, but better preserves the Hamiltonian structure of the plant”. In mathematics literature, the discrete Hamiltonian systems are considered for different purposes. In some of these works, discrete Hamiltonian systems are considered under the section of “symplectic difference systems” or “discrete symplectic systems” and a lot of analysis is carried out for these systems (Kratz (2003), Hilscher and Zeidan (2003), Shi (2006), Bohner (1996)). Other works on this subject deal with numerical computation of Hamiltonian dynamics and focuses on the integration methods. The survey paper (McLachlan et al. (1999)) summarizes the already existing integration methods, thoroughly and in Gonzalez and Simo (1996), the integration methods for Hamiltonian systems can be found.

In this study, mainly, a gradient based discrete-time model of continuous Hamiltonian systems with input is proposed and a procedure is given to construct a discrete-time model. Moreover, model validation is done considering the energy relation, additionally stabilizability condition and the model is also tested especially for well-known Hamiltonian systems by simulations. After then, the discrete-time counterpart of PBC technique is developed for n-degree of freedom mechanical system using this proposed discrete-time model. The discrete-time control rules $u_{es}(k)$ and $u_{di}(k)$ which correspond potential energy shaping and damping injection respectively are designed directly using the discrete time model of the desired system and the discrete time model of the open loop systems. To illustrate the effectiveness of the proposed method, two non-linear examples are investigated and the simulation results are given.

2. GRADIENT BASED DISCRETE MODEL OF HAMILTONIAN SYSTEMS

Consider the continuous-time Hamiltonian systems with dissipation and input

$$\dot{x} = [J(x) - R(x)] \nabla H(x) + G(x)u(t) \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the states, $u \in \mathbb{R}^m$ is the control input of the system and $J(x) = -J^T(x)$, $R(x) = R^T(x)$. The notation $\nabla_x H$ is used to denote the gradient vector of the scalar function of $H(x)$ with respect to x. Suppose the energy or the Hamiltonian function of the system is defined as,

$$H(x) = \frac{1}{2}\, x^T Z(x) x \tag{2}$$

As mentioned before, in this study, a gradient based discrete model will be constructed. To fulfill this, the discrete gradient definition given in Gonzalez and Simo (1996) and restated below will be considered.

Definition 1. Let $H(x)$ be a differentiable scalar function in $x \in \mathbb{R}^n$ then $\overline{\nabla} H(x_k, x_{k+1})$ is a *discrete gradient* of H if it is continuous and

$$\begin{aligned} \overline{\nabla}^T H(x_k, x_{k+1})\,[x_k - x_{k+1}] & \\ = H(x_{k+1}) - H(x_k) & \\ \overline{\nabla} H(x_k, x_k) = \nabla H(x) & \end{aligned} \tag{3}$$

The mean value theorem and the first condition of the discrete gradient imply that a satisfactory discrete gradient which leads us to obtain a discrete time model can be defined as follows.

Definition 2. Suppose a differentiable function in x given as $H(x) = \frac{1}{2}\, x^T Z(x) x$ and its gradient given in the form of $\nabla H(x) = Q(x)x$. The discrete gradient of a $H(x)$ is defined as,

$$\overline{\nabla} H(x) = \hat{Q}(x_{k+1}, x_k) \left[\frac{x_{k+1} + x_k}{2}\right] \tag{4}$$

where

$$\hat{Q}(x_{k+1}, x_k) = [Q(x_{k+1}) + Q(x_k)]\,/2 \tag{5}$$

Throughout the paper we will use this definition for discrete gradient, and to ease the notation, the following expression will be used,

$$\overline{\nabla} H(x) = \Phi(x_{k+1}, x_k)(x_{k+1} + x_k) \tag{6}$$

in which $\Phi(x_{k+1}, x_k) = \frac{1}{2}\, \hat{Q}(x_{k+1}, x_k)$.

It should be noted that the discrete gradient definition given here which is based on midpoint is slightly different than the one introduced by Gonzalez and Simo (1996).

If $Z(x)$ is a constant $n \times n$ matrix, then it can be easily shown that the discrete gradient given in Definition 2 satisfies exactly both of two conditions given in Definition 1. On the contrary, if $Z(x)$ is not a constant matrix, then it does not satisfy the first condition of Definition 1, precisely. A detailed analysis of the effect of this mismatching will be given later once the discrete-time model is constructed.

Consider the Hamiltonian system with dissipation and input, given in (1) and (2), using the Definition 2 for discrete gradient of $H(x)$, a gradient based discrete-time model of Hamiltonian system given in (1) can be constructed as follows,

$$x_{k+1} - x_k = T\,[J(x_k) - R(x_k)]\,\overline{\nabla} H(x) + TG(x_k)u(k) \tag{7}$$

-where T is sampling period- after the discrete gradient expression given in (6) is substituted in (7), the gradient based discrete-time model of Hamiltonian system is obtained as,

$$x_{k+1} = F(x_k)x_k + L(x_k)u(k) \tag{8}$$

where

$$\begin{aligned} F(x_k) &\hat{=} \{I - T\,[J(x_k) - R(x_k)]\,\Phi(x_{k+1}, x_k)\}^{-1} \\ &\qquad \{I + T\,[J(x_k) - R(x_k)]\,\Phi(x_{k+1}, x_k)\} \\ L(x_k) &\hat{=} T\{I - T\,[J(x_k) - R(x_k)]\,\Phi(x_{k+1}, x_k)\}^{-1} G(x_k) \end{aligned} \tag{9}$$

It should be noted that the model presented by (8) and (9) must not be regarded as an implicit or a non-causal model. In this study we use a simple approximation for x_{k+1} in calculation of $\Phi(x_{k+1}, x_k) \cong \Phi(\hat{x}_{k+1}, x_k)$ as follows, to avoid such misunderstandings,

$$x_{k+1} \cong \hat{x}_{k+1} = F(x_{k-1})x_k + L(x_{k-1})u(k-1) \tag{10}$$

This approximation might be explained as one-step ahead prediction.

Remark 1. Suppose the system given in (1) with $R(x) = 0$, $u(t) = 0$ and $J(x) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Then it can be easily shown that the gradient based discrete model of this system constructed by (8) and (9) defines "*a symplectic difference system*" (Hilscher and Zeidan (2003)) with energy conserving property. □

2.1 Model Validation Based on Both Energy Relation and Stabilizability Property

Consider the Hamiltonian system given in (1) and (2), when $R(x) > 0$, $u(t) = 0$, and $H(x)$ has a local(global) strict minimum at $x = x^*$, then this system has a local(global) asymptotically stable equilibrium at point x^* (Van der Schaft (1996)), and the following inequality holds

$$\dot{H}(t) = \nabla^T H(x)\,[J(x) - R(x)]\,\nabla H(x) < 0 \tag{11}$$

On the other hand, the analogy between continuous and discrete cases would give rise to a similar energy relation as the one in (11) for discrete case using (7) as following,

$$\overline{\nabla}^T H\,[x_{k+1} - x_k] = T\overline{\nabla}^T H\,[J(x_k) - R(x_k)]\,\overline{\nabla} H \tag{12}$$

Using the relation $\nabla H(x) = Q(x)x = [Z(x) + S(x)]\,x$, and after some algebraic manipulations, the following energy relation is obtained for the proposed discrete model,

$$\begin{aligned} \frac{H(x_{k+1}) - H(x_k)}{T} = \overline{\nabla}^T H\,[J(x_k) - R(x_k)]\,\overline{\nabla} H & \\ + \epsilon(x_{k+1}, x_k) & \end{aligned} \tag{13}$$

This relation implies that the discrete model creates an extra energy or extra dissipation according to the sign of

$\epsilon(x_{k+1}, x_k) \in \mathbb{R}$, Obviously, for $T \to 0$ this extra term tends zero, i.e. $\epsilon(x_{k+1}, x_k) \to 0$. As a consequence of the above analysis the following remark can be given on the stabilizability property.

Remark 2. When the gradient based discrete model proposed here is used to design a control rule to stabilize the sampled-data Hamiltonian systems, the extra term created by discrete model does not effect stabilizability condition of continuous system, if $\epsilon(x_{k+1}, x_k) < 0$, namely the discrete model creates an extra energy. On the other hand, if $\epsilon(x_{k+1}, x_k) > 0$ namely, the discrete model creates an extra dissipation, the control rule should be designed considering this fact, especially when slow sampling is used. As it can be obviously followed from (13), the stabilizability condition of continuous Hamiltonian system by discrete-time control remains same, i.e. asymptotic stability can be achieved by adding an extra dissipation. □

2.2 Examples and Model Validation by Simulation

In this section we present two examples for modeling of Hamiltonian systems and investigated the models by simulations.

Example 1. Van der Pol oscillator given by

$$\dot{x} = [J - R(x)]\nabla H, \quad H(x) = \frac{1}{2}\, x^T I_2 x$$
$$R(x) = \begin{bmatrix} 0 & 0 \\ 0 & -\mu(1 - x_1^2) \end{bmatrix}$$

Using the procedure given in the previous section, the discrete model of Van der Pol circuit is obtained as follows,

$$x_{k+1} = \left[I - \frac{T}{2}\left(J - R(x_k)\right)\right]^{-1} \left[I + \frac{T}{2}\left(J - R(x_k)\right)\right] x_k$$

Figure 1 and 2 show the phase portraits and time responses of the discrete and continuous dynamics respectively, for $\mu = 1$ and T = 0.01s.

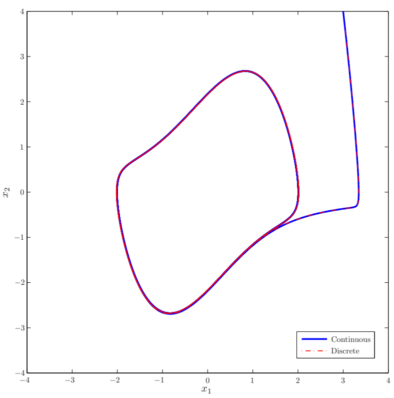

Fig. 1. Phase Portraits of Discrete and Continuous Dynamics of Van der Pol Oscillator

Example 2. Consider the double pendulum given by,

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = J\nabla H, \; H(q,p) = \frac{1}{2}\, p^T M^{-1}(q)p + V(q)$$

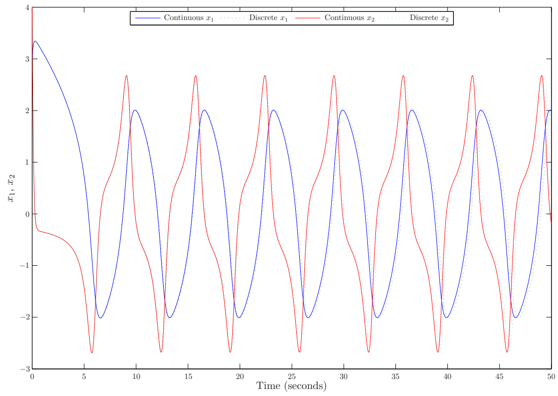

Fig. 2. Time Response of Discrete and Continuous Dynamics of Van der Pol Oscillator

where

$$M = \begin{bmatrix} l_1^2(m_1 + m_2) & m_2 l_1 l_2 cos(q_1 - q_2) \\ m_2 l_1 l_2 cos(q_1 - q_2) & l_2^2 m_2 \end{bmatrix}$$
$$V(q) = -m_2 g l_2 cos(q_2) + (m_1 + m_2) g l_1 cos(q_1)$$

After the discrete model of the system is obtained using the procedure given in the previous section, the simulation results are obtained for parameter values $m_1 = m_2 = 1kg$, $l_1 = 0.2m$, $l_2 = 0.2m$, $g = 9.81ms^{-2}$, $T = 0.005s$, for $x_0 = [0.5 \; 0.3 \; 0 \; 0]^T$ and they are presented in Figure 3 as phase portraits of Discrete and Continuous Dynamics of the double pendulum system. It should be noted that the series expansions of trigonometric functions have been used in the calculation of discrete gradient of $V(q)$ to avoid the discontinuity.

3. DISCRETE-TIME CONTROL OF HAMILTONIAN SYSTEMS VIA ENERGY SHAPING AND DAMPING INJECTION

In the previous section we dealt with a more general structure while obtaining the discrete-time model of a Hamiltonian system here we will focus on the following continuous-time port-controlled Hamiltonian systems

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u(t) \qquad (14)$$
$$y(t) = B^T(q)\nabla_p H$$

where $q \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ are generalized coordinates of the system, $u \in \mathbb{R}^m$ is control input of the system and $B(q) \in \mathbb{R}^{n\times m}$ is the input matrix. The notation $\nabla_{(\cdot)}H$ is used to denote the gradient vector of the scalar function of $H(q,p)$ with respect to $(\cdot)$.The energy or the Hamiltonian function of the system is defined as,

$$H(q,p) = K(q,p) + V(q) = \frac{1}{2} p^T M^{-1}(q)p + V(q) \qquad (15)$$

namely, the Hamiltonian function is the sum of kinetic and potential energy. In this relation, the matrix $M(q)$ is a symmetric and positive generalized inertia matrix, and if $M(q) = M \in \mathbb{R}^{n\times n}$, then the system is called "*separable* Hamiltonian system" whereas if $m = n$ and $rankB(q) = n$, the system is full-actuated. In the sequel, we assume that the system is full-actuated. In order to obtain discrete time

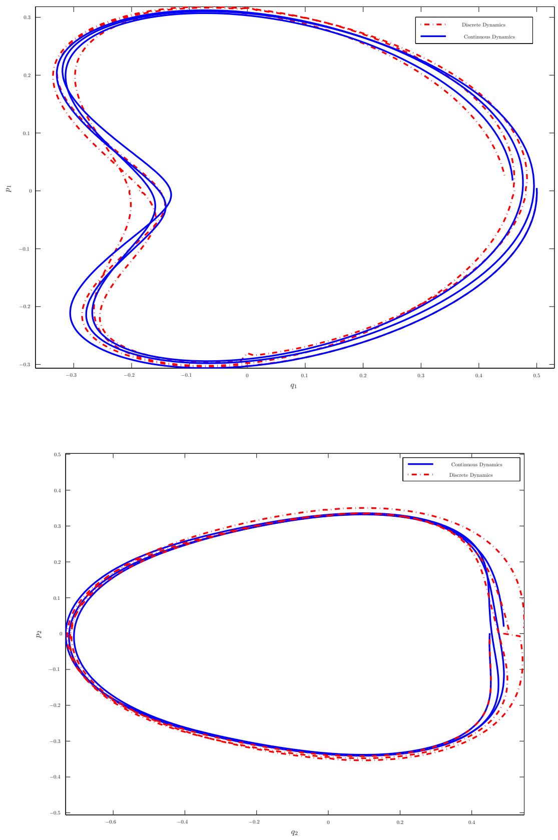

Fig. 3. Discrete and Continuous Time Responses of Double Pendulum System

model of the system, first we have to construct discrete gradient of $H(q,p)$ in terms of $\nabla H(q,p)$, namely

$$\begin{aligned}\nabla H(q,p) &= \begin{bmatrix} V_{gr}(q) & S(q,p) \\ 0 & M^{-1}(q) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \\ &= Q(q,p) \begin{bmatrix} q \\ p \end{bmatrix} \\ S(q,p) &= \begin{bmatrix} p^T \dfrac{\partial M^{-1}(q)}{\partial q_1} \\ p^T \dfrac{\partial M^{-1}(q)}{\partial q_2} \\ \vdots \\ p^T \dfrac{\partial M^{-1}(q)}{\partial q_n} \end{bmatrix}\end{aligned} \tag{16}$$

in which the matrix $V_{gr}(q)$ is described as,

$$\nabla V(q) = V_{gr}(q) q \tag{17}$$

then the discrete gradient expression is obtained as,

$$\overline{\nabla} H(q,p) = \Phi_d(k+1,k) \begin{bmatrix} q_{k+1} + q_k \\ p_{k+1} + p_k \end{bmatrix} \tag{18}$$

where

$$\begin{aligned}\Phi(k+1,k) &= \Phi(\hat{q}_{k+1}, \hat{p}_{k+1}, q_k, p_k) \\ &= \frac{1}{4}\left(Q(q_k,p_k) + Q(\hat{q}_{k+1}, \hat{p}_{k+1})\right) \\ &= \begin{bmatrix} \Phi_{11}(k+1,k) & \Phi_{12}(k+1,k) \\ 0 & \Phi_{22}(k+1,k) \end{bmatrix}\end{aligned} \tag{19}$$

Then, the more versatile relations are obtained as follows,

$$\begin{aligned}\overline{\nabla}_q H &= \overline{\nabla}_q K + \overline{\nabla}_q V(q) \\ &= \left[\, \Phi_{11}(k+1,k) \;\; \Phi_{12}(k+1,k) \,\right] \begin{bmatrix} q_{k+1} + q_k \\ p_{k+1} + p_k \end{bmatrix} \\ \overline{\nabla}_p H &= \overline{\nabla}_p K = \Phi_{22}(k+1,k)\,(p_{k+1} + p_k)\end{aligned} \tag{20}$$

To obtain the gradient based discrete model of the system given in (14), the following equation should be written,

$$\begin{aligned}\begin{bmatrix} q_{k+1} \\ p_{k+1} \end{bmatrix} - \begin{bmatrix} q_k \\ p_k \end{bmatrix} &= T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \overline{\nabla}_q H \\ \overline{\nabla}_p H \end{bmatrix} \\ &\quad + T \begin{bmatrix} 0 \\ B(q_k) \end{bmatrix} u(k) \\ &= T \begin{bmatrix} \overline{\nabla}_p K \\ -\overline{\nabla}_q K - \overline{\nabla}_q V \end{bmatrix} + T \begin{bmatrix} 0 \\ B(q_k) \end{bmatrix} u(k)\end{aligned} \tag{21}$$

and substituting (20) in (21) the following relation is obtained,

$$\begin{aligned}&\begin{bmatrix} q_{k+1} \\ p_{k+1} \end{bmatrix} - \begin{bmatrix} q_k \\ p_k \end{bmatrix} = T \begin{bmatrix} 0 \\ B(q_k) \end{bmatrix} u(k) \\ &+ T \begin{bmatrix} 0 & \Phi_{22}(k+1,k) \\ -\Phi_{11}(k+1,k) & -\Phi_{12}(k+1,k) \end{bmatrix} \begin{bmatrix} q_{k+1} + q_k \\ p_{k+1} + p_k \end{bmatrix}\end{aligned} \tag{22}$$

Finally, the discrete time model of the system given in (14) is obtained using (8), (9) and (19) as follows,

$$\begin{bmatrix} q_{k+1} \\ p_{k+1} \end{bmatrix} = F(q_k,p_k) \begin{bmatrix} q_k \\ p_k \end{bmatrix} + L(q_k,p_k) u(k) \tag{23}$$

where

$$\begin{aligned}F(q_k,p_k) &\mathrel{\hat{=}} \left[I - TJ\Phi(k+1,k)\right]^{-1} \left[I + TJ\Phi(k+1,k)\right] \\ L(q_k,p_k) &\mathrel{\hat{=}} T\left[I - TJ\Phi(k+1,k)\right]^{-1} \begin{bmatrix} 0 \\ G(q_k) \end{bmatrix}\end{aligned} \tag{24}$$

In order to stabilize the system (14) at the point q^*, one good candidate for the closed loop energy function is the following,

$$\begin{aligned}H_d(q,p) &= K(q,p) + V_d(q) \\ &= \frac{1}{2}\left(p^T M^{-1}(q) p + (q - q^*)^T K_p (q - q^*)\right)\end{aligned} \tag{25}$$

where $K_p = K_p^T > 0$, and so the desired Hamiltonian system can be written as,

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{26}$$

to construct the discrete gradient of $H_d(q,p)$ in terms of $\nabla H_d(q,p)$, namely,

$$\begin{aligned}\nabla H_d(q,p) &= \begin{bmatrix} K_p & S_d(q,p) \\ 0 & M^{-1}(q) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \\ &= Q_d(q,p) \begin{bmatrix} q \\ p \end{bmatrix} \\ S_d(q,p) &= \begin{bmatrix} p^T \dfrac{\partial M^{-1}(q)}{\partial q_1} \\ p^T \dfrac{\partial M^{-1}(q)}{\partial q_2} \\ \vdots \\ p^T \dfrac{\partial M^{-1}(q)}{\partial q_n} \end{bmatrix} = S(q,p)\end{aligned} \tag{27}$$

then the discrete gradient expression is obtained as

$$\overline{\nabla} H_d(q,p) = \Phi(k+1,k) \begin{bmatrix} q_{k+1} + q_k \\ p_{k+1} + p_k \end{bmatrix} \tag{28}$$

where

$$\begin{aligned} \Phi_d(k+1,k) &= \Phi_d(\hat{q}_{k+1}, \hat{p}_{k+1}, q_k, p_k) \\ &= \frac{1}{4}\left(Q_d(q_k,p_k) + Q_d(\hat{q}_{k+1},\hat{p}_{k+1})\right) \\ &= \begin{bmatrix} \Phi_{d11}(k+1,k) & \Phi_{d12}(k+1,k) \\ 0 & \Phi_{d22}(k+1,k) \end{bmatrix} \end{aligned} \tag{29}$$

When one compares the desired energy function given in (25) and the energy function given in (15), the only difference between two expressions is the change in the potential energy. Therefore the following relations are obtained for the discrete model of the desired system,

$$\begin{aligned} \overline{\nabla}_q H_d &= \overline{\nabla}_q K + \overline{\nabla}_q V_d(q) \\ &= \left[\, \Phi_{d11}(k+1,k) \;\; \Phi_{12}(k+1,k) \,\right] \begin{bmatrix} q_{k+1} + q_k \\ p_{k+1} + p_k \end{bmatrix} \\ \overline{\nabla}_p H_d &= \overline{\nabla}_p K = \Phi_{22}(k+1,k)\,(p_{k+1} + p_k) \end{aligned} \tag{30}$$

The following equation -which will be used the gradient based model of the desired system - can be written as,

$$\begin{aligned} \begin{bmatrix} q_{k+1} \\ p_{k+1} \end{bmatrix} - \begin{bmatrix} q_k \\ p_k \end{bmatrix} &= T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \overline{\nabla}_q H_d \\ \overline{\nabla}_p H_d \end{bmatrix} \\ &= T \begin{bmatrix} \overline{\nabla}_p K \\ -\overline{\nabla}_q K - \overline{\nabla}_q V \end{bmatrix} \end{aligned} \tag{31}$$

and by substituting (30) in (31) the following relation is obtained,

$$\begin{aligned} &\begin{bmatrix} q_{k+1} \\ p_{k+1} \end{bmatrix} - \begin{bmatrix} q_k \\ p_k \end{bmatrix} = T \begin{bmatrix} \overline{\nabla}_p K \\ -\overline{\nabla}_q K - \overline{\nabla}_q V_d(q_k) \end{bmatrix} \\ &= T \begin{bmatrix} 0 & \Phi_{22}(k+1,k) \\ -\Phi_{d11}(k+1,k) & -\Phi_{12}(k+1,k) \end{bmatrix} \begin{bmatrix} q_{k+1} + q_k \\ p_{k+1} + p_k \end{bmatrix} \end{aligned} \tag{32}$$

Finally, the discrete time model of the desired system given by (26) is obtained using (8), (9) and (29) as follows,

$$\begin{bmatrix} q_{k+1} \\ p_{k+1} \end{bmatrix} = F_d(q_k,p_k) \begin{bmatrix} q_k \\ p_k \end{bmatrix} \tag{33}$$

where

$$F_d(q_k,p_k) \widehat{=} \left[I - TJ\Phi_d(k+1,k)\right]^{-1} \left[I + TJ\Phi_d(k+1,k)\right]$$

If the right hand side of (22) is equated to the right hand side of (32), the discrete time controller responsible for energy shaping of the system (14) is obtained as follows in terms of discrete gradients,

$$u_{es}(k) = -B^{-1}(q_k)\left(\overline{\nabla}_q V_d - \overline{\nabla}_q V\right) \tag{34}$$

The discrete gradients in (34) are obtained as follows,

$$\begin{aligned} \overline{\nabla}_q V &= \Phi_V(q_{k+1},q_k)(q_{k+1} + q_k) \\ \overline{\nabla}_q V_d &= K_p\left(\frac{1}{2}(q_{k+1} + q_k) - q^*\right) \end{aligned} \tag{35}$$

in which

$$\begin{aligned} \Phi_V(q_{k+1},q_k) &= \frac{1}{4}\left(V_{gr}(q_{k+1}) + V_{gr}(q_k)\right) \\ \nabla_q V(q) &= V_{gr}(q)q \end{aligned} \tag{36}$$

In these relations, the term q_{k+1} can be calculated in two different ways; one approach would be to use the expression in (10), the other way is to use the output variable. It should be noted that there were no difference between the two approaches, in simulation results We prefer to use the output variable for the calculation of q_{k+1}, since its formulation is easily tractable. In discrete time setting, the output equation given in (14) can be written as follows,

$$y(k) = B^T(q_k)\overline{\nabla}_p H = B^T(q_k)\frac{q_{k+1} - q_k}{T} \tag{37}$$

so q_{k+1} can be obtained in term of output as,

$$q_{k+1} = TB^{-T}(q_k)y(k) + q_k \tag{38}$$

Finally, the discrete time controller responsible for the energy shaping of the system (14) is obtained with (36),

$$\begin{aligned} u_{es}(k) &= -B^T\left[T\left(\frac{K_p}{2} - \Phi_V(q_{k+1},q_k)\right)B^{-T}(q_k)y(k)\right. \\ &\quad \left. + 2\left(\frac{K_p}{2} - \Phi_V(q_{k+1},q_k)\right)q_k - K_p q^*\right] \end{aligned} \tag{39}$$

It is obviously seen that the system obtained when applying $u_{es}(k)$ to the system (21) gives the discrete model (23) of desired Hamiltonian system (14) with the Hamiltonian function (25). Consequently, the analysis and so Remark 2 given in Section 2.1 is valid. Moreover, in order to drive the state of the system to the equilibrium point, the damping injection controller $u_{di}(k)$ must be constructed. When the desired system is taken as follows,

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & R_d \end{bmatrix} \right) \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{40}$$

with $R_d = B(q)K_v B^T(q)$. It is easily realized that the damping injection control rule $u_{di}(k)$ can be found using the output variable $y(k)$ as follows,

$$u_{di}(k) = -K_v B^T(q_k)\overline{\nabla}_p H = -K_v y(k) \tag{41}$$

Therefore, the discrete-time control rules $u_{es}(k)$ and $u_{di}(k)$, which correspond potential energy shaping and damping injection respectively have been designed directly using the discrete time model of the desired system and the discrete time model of the open loop systems.

4. EXAMPLES

In order to make a comparison with the results given in Laila and Astolfi (2006b), first we considered the control of a cart and pendulum system (Bloch et al. (1997)) with

$$M = \begin{bmatrix} ml^2 & mlcos(q_1) \\ mlcos(q_1) & M_s + m \end{bmatrix}, \; G = \begin{bmatrix} 0_{2\times 2} \\ I_2 \end{bmatrix}$$
$$V(q) = mglcos(q_1)$$

and the cart mass $M_s = 0.14kg$, the pendulum bob mass $m = 0.44kg$, the pendulum length $l = 0.215m$ and $g = 9.81ms^{-2}$, $q_1 = \theta$, the pendulum angle from its upright position and $q_2 = s$ is the cart position. The control objective is to stabilize the continuous system at origin and also to swing up of the pendulum as oppose to the problem considered in Laila and Astolfi (2006b). They have assumed that the swinging up of the pendulum has been achieved by a separate control but they have considered the underactuated case. Figure 4 illustrates the simulation results for values

$$[\, q_{10}\ q_{20}\ p_{10}\ p_{20}\,] = [\, 3.14\ -1.5\ 0.1\ 0.1\,]\,,\ R_d = 25 I_2$$

with $K_p = 75 I_2$ and $T = 0.01s$. The control input is constructed by summing $u_{es}(k)$ and $u_{di}(k)$ which are obtained using the relations given in (39) and (41).

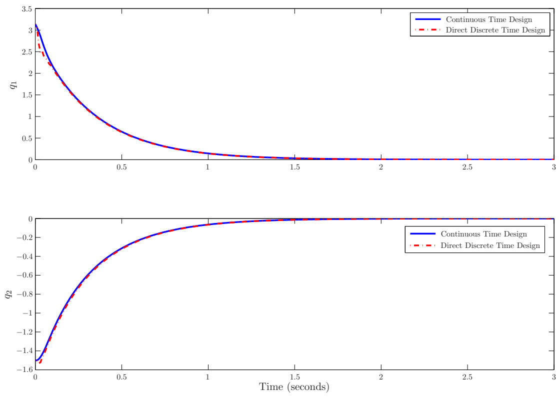

Fig. 4. Discrete and Continuous Time Responses of Cart and Pendulum System

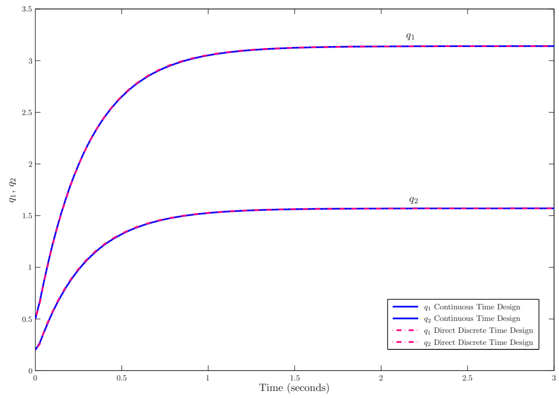

Fig. 5. Discrete and Continuous Time Responses of Double Pendulum System

In order to investigate the effectiveness of the proposed method on a more complicated system, we also considered the control of double pendulum -described in Section 2.2- at a desired position $[\, q_1^*\ q_2^*\ p_1^*\ p_2^*\,] = [\, 3.14\ 1.57\ 0\ 0\,]$. The simulation results presented in Figure 5 for $T = 0.025s$, $K_p = 50 I_2$, $R_d = 15 I_2$ and $[\, q_{10}\ q_{20}\ p_{10}\ p_{20}\,] = [\, 0.5\ 0.2\ 0.1\ 0.1\,]$.

It must be noticed that the emulation of the continuous controllers have destabilized the system in both two examples. These results reveal that the discrete time controller design method proposed in this study yields a good performance for sampled data Hamiltonian systems.

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