THE USE OF SYMBOLIC ALGEBRA IN CONTROL ENGINEERING

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Abstract: Several tools for linear model manipulation, model-order reduction, analysis, and the design/synthesis of control systems, developed using Mathematica, are presented. New control objects have been introduced in the framework of Mathematica's Control System Professional package, with several new associated transformations and model format manipulations. A quasi-generalised singular perturbation model-order reduction method has been implemented, and some interesting observations on appropriate pole assignment algorithms for use with system models containing symbols are reported. Tools for the analysis of systems with parametric uncertainty, and some recent results on the systematic design of PID controllers are considered. *Copyright © 2000 IFAC.*

Keywords: Symbolic algebra, model-order reduction, pole assignment, uncertain systems, PID controller design.

1 Introduction

Mathematica (1991), like other similar computing environments, in addition to its kernel system, is enhanced by various domain-specific packages, such as the "Control System Professional" (1997). This latter package provides the classical single-input single-output control system analysis and design tools, the established Kalman tests for controllability and observability, the controllability and observability Gramians, minimal realisation algorithms, pole assignment, optimal control system design, system interconnection facilities, system time-response plots, and many other tools.

In recent years, the author has developed several packages using Mathematica, initially to provide CAD tools to support the teaching of control theory, but also for use in research and in solving real industrial control problems. In the following, some new facilities for linear system model manipulation, system analysis, model-order reduction, pole assignment for uncertain systems, the analysis of both multivariable systems and nonlinear SISO systems with structured parametric uncertainty, and the systematic design of PID controllers will be considered. The Linear Models Package and the System Analysis Package have both already been made fully compatible with Mathematica's Control System Professional.

2 Model Manipulation

The Linear Models Package provides the various transformations needed to automatically manipulate linear system models between any of the following standard forms; state space, transfer-function (matrix), left or right matrix-fraction form, and Rosenbrock's (1970) system matrix in state-space or polynomial form. The format for StateSpace objects is in the form used by the H_∞ robust control system design method. With the transformations to Left/RightMatrixFraction forms, also now widely used in the H_∞ design method, the resulting system models can also be readily reduced to least-order, or coprime, form.

Using the facilities currently available in the Control System Professional (CSP), a user can enter a system description either as a TransferFunction or StateSpace object, as shown below:

```
tf = TransferFunction[s, {{1/(s+1), 2/(s+3)}}]

ss = StateSpace[{{0, 1},{-3, -4}},{{0},{1}},
                {{3, 1},{2, 2}}]
```

These can then be manipulated into the alternative model form by simply applying the appropriate object wrapper around the form concerned; e.g.

```
sstf = StateSpace[tf]
tfss =TransferFunction[s, ss]
```

The resulting model data held in these objects can also be inspected by the CSP command //ReviewForm. However, these model formats have now been improved so that entering the state space object **ss** or transfer function object **tf** automatically results in the more compact and pleasing output formats, shown below

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -3 & -4 & 1 \\ \hline 3 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right)^{S} \qquad \left(\begin{array}{c} \frac{1}{s+1} \\ \frac{2}{s+3} \end{array}\right)^{T}$$

These new model formats, which are fully interactively editable, have been supplemented by Rosenbrock's system matrix description for systems in state space or polynomial form and matrix-fraction descriptions; e.g.

ps = SystemMatrix[s, ss]

results in $\left(\begin{array}{cc|c} s & -1 & 0 \\ 3 & s+4 & 1 \\ \hline -1 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right)^{M}$

rmf = RightMatrixFraction[tf]

results in $\left(\begin{array}{c} s+3 \\ 2(s+1) \end{array}\right)((s+1)(s+3))^{-1}|^{T}$

SystemMatrix[tf, TargetForm->RightFraction]

results in $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & (s+1)(s+3) & 1 \\ \hline 0 & -s-3 & 0 \\ 0 & -2(s+1) & 0 \end{array}\right)^{M}$

3 System Analysis

The System Analysis Package, contains a range of well-established analysis tools. In addition to the established state-space model controllability and observability tests, provided in the CSP facility, the user can now also request the controllability or observability of a system described by a polynomial system matrix model, by simply entering Controllable[ps] or Observable[ps], where ps is the name of the SystemMatrix object concerned.

The Smith form and Hermite form of a polynomial matrix and the McMillan form of a rational polynomial matrix have also been implemented. The invariant zeros of the Smith form can be used to indicate the existence of any possible input or output decoupling zeros present in a polynomial system matrix model form. The McMillan form can be used to determine the poles and zeros of a multivariable system, and hence to determine the existence of any right-half plane "transmission zeros". The position of these rhp-zeros in the diagonal terms of the McMillan form clearly indicates the point in any loop-closure procedure at which the non-minimum phase effect will manifest itself. The poles of the McMillan form also indicate the order of any minimal state space realisation of the transfer function matrix concerned. Here, a further minimal realisation algorithm (Patel and Munro, 1982), based on Rosenbrock's decoupling zeros theory has been implemented that carries out a minimum of numerical operations, compared with some other algorithms, in determining the desired minimal-order state space model.

An algorithm to determine coprime factorisations of a given transfer function matrix model has also been implemented. This can be used with an initial Left/RightMatrixFraction object that may not be in least-order form to detect and remove any Left/Right Matrix Common Factor in the resulting numerator and denominator matrices. Suppose that a system description has been entered as a SystemMatrix object in polynomial form;

e.g. **ps = SystemMatrix[t, u, v, w, s]**

$$\left(\begin{array}{c|c} T(s) & U(s) \\ \hline -V(s) & W(s) \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} s^2(s+1) & s^3+s^2-1 & 1-s^2(s+1) & 0 & 0 \\ s(s+2) & s^2+3s+2 & -s(s+2) & 0 & 1 \\ s(s+2) & s^2+3s+1 & 1-s(s+2) & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right)^{M}$$

Then, by entering **MatrixLeftGCD[s, t, u]**, the user can determine the Matrix Left Greatest Common Divisor, L(s), of the matrices T(s) and U(s), if any, yielding here

$$L(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & s^2(s+1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$T_r(s) = \begin{pmatrix} s^2(s+1) & s^3+s^2-1 & -(s^3+s^2-1) & 0 \\ s(s+2) & s^2+3s+2 & -s(s+2) & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and $$U_r(s)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad (3)$$

4 Model-order Reduction

Two of the best known methods for obtaining a reduced order model of a given high order state space system are the Balanced Truncation method by Moore (1981) and the Balanced Residualisation method by Samar, et al, (1995). It is also well known that a good frequency-domain fit with the original model is obtained at intermediate to high frequencies using the Balanced Truncation method, and at low to intermediate frequencies using the Balanced Residualisation method. However, a new model-order reduction method, known as the Quasi-Generalised Singular Perturbation method (Baki and Munro, 1998), which has been implemented and provides a compromise between these two approaches, is briefly described below.

Let [A, B, C, D] be an n-dimensional stable balanced linear time-invariant model for a system, and let the corresponding state space matrices be partitioned as

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} \quad B=\begin{bmatrix}B_1\\ B_2\end{bmatrix} \quad C=\begin{bmatrix}C_1 & C_2\end{bmatrix} \quad (4)$$

where A_{11} has dimensions r x r, then the corresponding transfer-function matrix model can be written as

$$G(s,s_0)=\tilde{C}(s_0)[sI_r-\tilde{A}(s_0)]^{-1}\tilde{B}(s_0)+\tilde{D}(s_0) \quad (5)$$

where
$$\begin{aligned}\tilde{A}(s_0) &= A_{11}+A_{12}\phi(s_0)A_{21}\\ \tilde{B}(s_0) &= B_1+A_{12}\phi(s_0)B_2\\ \tilde{C}(s_0) &= C_1+C_2\phi(s_0)A_{21}\\ \tilde{D}(s_0) &= D+C_2\phi(s_0)B_2\end{aligned} \quad (6)$$

and $$\phi(s_0)=(s_0I_{n-r}-A_{22})^{-1}. \quad (7)$$

Here, $\phi(s_0)$ is replaced with a new real valued matrix $\tilde{\phi}=-\mu A_{22}{}^{-1}$, where μ can be chosen to lie between 0 and 1. In this new formulation, $s_0=\infty$ corresponds to the Balanced Truncation case, and $s_0=0$ corresponds to the Balanced Residualisation case. Now, by varying μ between 0 and 1, the Quasi-GSP model-order reduction method provides a hybrid approach and allows the designer to select a compromise between the low and high frequency response fit of the reduced order model to suit the design purpose.

5 Pole Assignment

The Pole Assignment Package provides various state and output-feedback pole assignment algorithms for systems with known and uncertain model parameters. However, it is interesting to note that pole assignment algorithms that are often considered undesirable when used with wholly numeric data, either because of numerical accuracy problems or time to execute, may not be as bad when used with systems containing symbols. Some recent tests by Soylemez and Munro (1998) have shown that the mapping algorithm developed by Young and Willems (1972), which is not considered as reliable or as fast as Ackermann's algorithm (1972), currently implemented in both MatLab and Mathematica's CSP, is in fact one of the best methods for use with symbolic data.

Given a state space system model $\dot{x}=Ax+Bu$, then using a feedback law $u=-Kx$, where K is expressed as the outer product of two vectors; $K=f\,m^t$; K is to be determined so that $A_c=A-BK$ has a desired set of eigenvalues $\{\gamma\}$. Here, the vector f is chosen such that the resulting single-input system with $b=Bf$ is completely controllable, and the pole assignment algorithm solves for the necessary row-vector m^t.

The mapping algorithm determines this row vector as

$$m^t=\left[\Phi^t\right]^{-1}X^{-1}\delta \quad (8)$$

where $$X=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ a_{n-1} & 1 & 0 & \cdots & 0\\ a_{n-2} & a_{n-1} & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_1 & a_2 & a_3 & \cdots & 1\end{bmatrix} \quad (9)$$

$$\delta=\left[\alpha_{n-1}-a_{n-1},\alpha_{n-2}-a_{n-2},\cdots\cdots,\alpha_0-a_0\right] \quad (10)$$

and Φ is the controllability matrix for the pair [A,b]. Here, the a_i and α_i are the coefficients of the open-loop and closed-loop system characteristic polynomials, respectively.

Figures 1 and 2 show the results of some recent tests on both the accuracy and speed of execution of several well established pole assignment algorithms; namely, those of Young and Willems, Ackermann, Munro, and Ogata (1987). In the top diagram of Figure 1, the y-axis shows the difference, in terms of digits before and after the decimal point, between the desired eigenvalues and the numerically computed values. When a symbolic environment is used numerical accuracy is no longer a problem since we can get exact results, and the property of interest becomes the speed of execution. It should be noted that two curves are shown for the mapping approach, since an automatic internal algorithm switch occurs with respect to the calculation of Det[sI_n-A] at n = 6. The full-line plot shows this effect and the dotted-line plot shows the result when this algorithm switch is suppressed.

6 Multivariable Systems

A Nyquist Array Design Package (Kontogiannis and Munro, 2000) has been implemented that provides Rosenbrock's Direct and Inverse Nyquist Array design methods for certain and uncertain multivariable systems. It not only provides the essential Nyquist array plots of the elements of the direct or inverse systems being considered, but also includes various other recent interaction measures and tests for diagonal dominance, and scaling procedures for multivariable systems. Both Limebeer's (1982) definition of 'generalised diagonal dominance', and Bryant and Yeung's (1996) definition of 'fundamental diagonal dominance', have also been implemented for uncertain multivariable systems.

For an uncertain multivariable system described by a square transfer-function matrix **G(s,q)**, where **q** is a vector of the uncertainties affecting the coefficients of the elements of G(s) in an affine linear manner, the definition of robust column diagonal dominance is briefly stated below:

$$\phi_{c,i} = \min_{q\in E(Q)} \left|g_{ii}(s,q)\right| - \sum_{\substack{j=1 \\ j\neq i}}^{m} \max_{q\in E(Q)} \left|g_{ji}(s,q)\right| \qquad (11)$$

where the $\phi_{c,\,i}$ give the worst-case dominance for each column.

Whilst it is known that this result is conservative, it can be improved by using a global optimisation method. Figure 3 shows the resulting Robust Direct Nyquist Array (RDNA) for the uncertain multivariable system described by

$$G(s,q) = \begin{bmatrix} \dfrac{s+4+q_1}{s^2+(6+q_2)s+5} & \dfrac{1}{5s+1+q_5} \\ \dfrac{s+1}{s^2+(10+q_3)s+(100+10q_4)} & \dfrac{2+q_6}{2s+1} \end{bmatrix} \qquad (12)$$

where $|q_1| \le 1$, $|q_2| \le 1$, $|q_3| \le 3$, $|q_4| \le 1$, $|q_5| \le 0.5$, $|q_6| \le 0.3$. In Figure 3, the 'X's are the nominal values obtained when **q = 0**.

7 Nonlinear Systems

A Nonlinear Systems Package, which is currently under development, can be used for the prediction of limit cycles in nonlinear systems. This package can deal with the cases of systems with structured uncertainties in the linear part only, the nonlinearity only, and both the linear and nonlinear parts. This work has now been extended to cover the case of systems with multiple cascaded uncertain nonlinearities and linear parts, where the frequency band and amplitude band within which any limit cycle may exist can be determined.

For the system shown in Figure 4, where the uncertain linear part is described by

$$G(s,q) = \frac{[2,8]s + [97,103]}{s^4 + [7.4,8.6]s^3 + [31.4,32.6]s^2 + 50s + [59,60]} \qquad (13)$$

and the nonlinear part is a saturation element with an uncertain slope varying in the interval [1,4], a limit cycle exists if

$$\mathbf{1+N(A,p)G(s,q) = 0} \qquad (14)$$

Since, in general, the coefficients of this equation have a multilinear uncertainty structure, closed-loop stability can be checked by using the Zadeh and Desoer's Mapping Theorem (1963) to compute the associated value sets and perform a Zero Exclusion test. Since the describing function for the nonlinearity satisfies N(A,p) $\in$ [0,4], Figure 5 shows that the Zero Exclusion test is not satisfied for this system, and a limit cycle may exist.

8 PID Controller Design

Various researchers have recently been re-examining the design of PID controllers to meet certain absolute stability, and performance criteria. Munro and Soylemez (2000) have developed a very fast method of computing the limiting values of the proportional-gain, K_p, the integral-gain, K_i, and the derivative-gain, K_d, terms of a PID controller described by

$$K(s) = K_p + \frac{K_i}{s} + K_d\, s \qquad (15)$$

such that the resulting closed-loop system is Hurwitz stable, or has a desired relative stability, or damping ratio. Figure 6, shows the resulting K_p-K_i, space for the closed-loop Hurwitz stability of a system that has 2 right-half plane poles and 2 right half-plane zeros with transfer-function

$$g(s) = \frac{s^3+6s^2-2s+1}{s^5+3s^4+29s^3+15s^2-3s+60} \qquad (16)$$

For a system described by the transfer-function

$$g(s) = \frac{27}{(s+1)(s+3)^3} \qquad (17)$$

the limiting values of the three terms of a PID controller that guarantee that the resulting closed-loop system is Hurwitz stable are shown in Figure 7, which was obtained in 1 second on a 266 MHz PC running under Windows NT.

For the uncertain system, with interval uncertainty, described by Ho et al. (1998): namely,

$$g(s) = \frac{a_0 + a_1 s + s^2}{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4 + s^5} \quad (18)$$

where $a_0 \in [2, 4]$, $a_1 \in [-5, -4]$, $b_0 \in [-2, -1]$, $b_1 \in [8, 9]$, $b_2 \in [7, 9]$, $b_3 \in [5, 5]$, $b_4 \in [3, 4]$,

By gridding the proportional gain parameter Kp over the range, $K_p \in [1, 1.09]$, the resulting robust PID stabilizing parameter space obtained is shown in Fig. 8.

9 Acknowledgements

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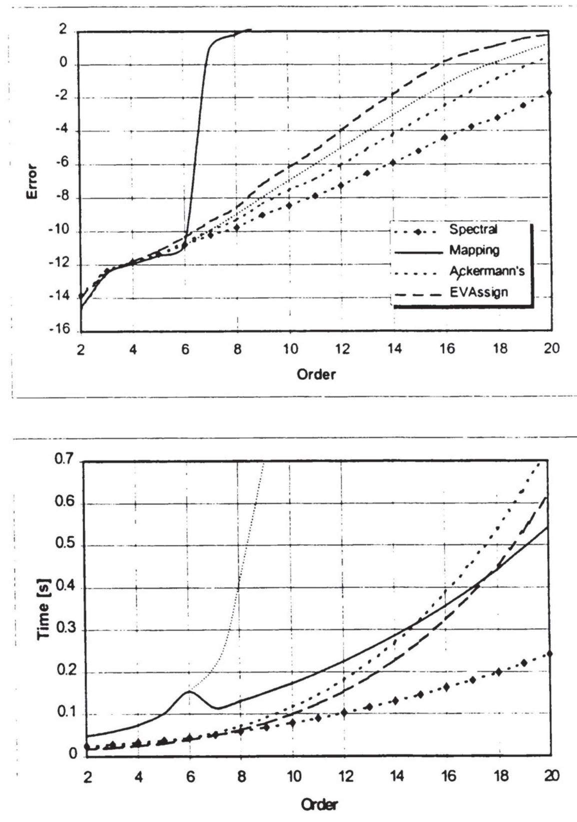

Figure 1: Comparison of Dyadic Methods under Numeric Considerations

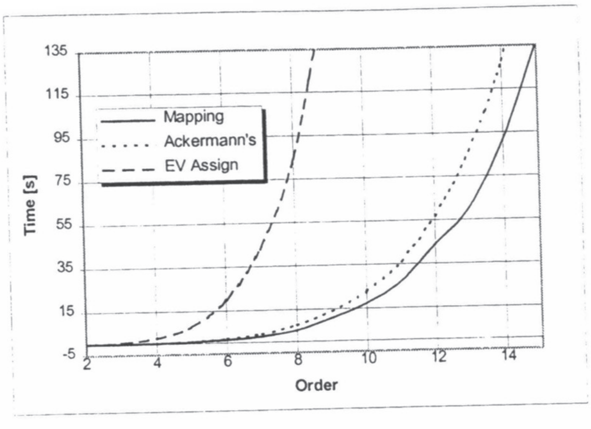

Figure 2: Comparison of Dyadic Methods under Symbolic Considerations

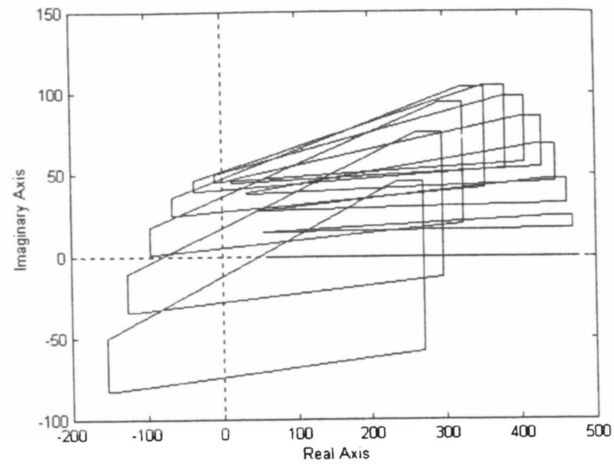

Figure 5: Value sets for the uncertain system characteristic polynomial

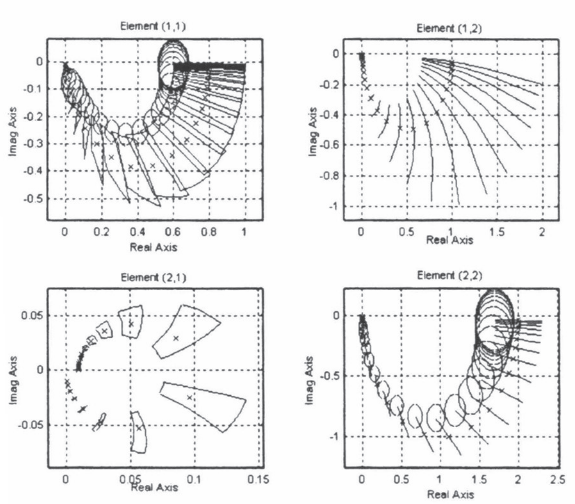

Figure 3: Nyquist Array of uncertain system

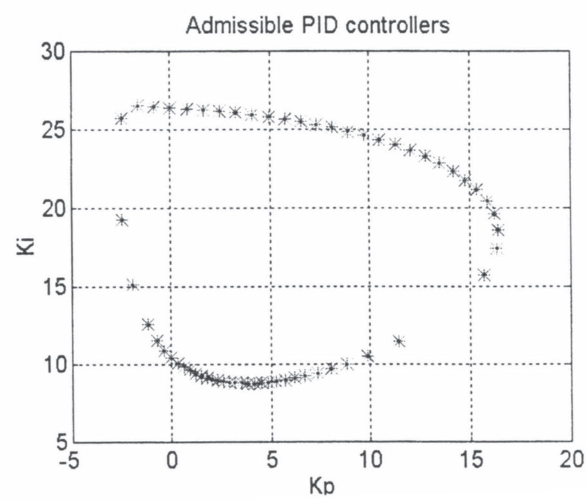

Figure 6: The admissible PI compensator space

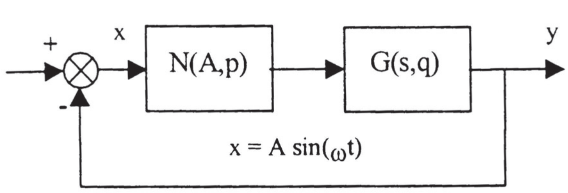

Figure 4: Uncertain nonlinear system

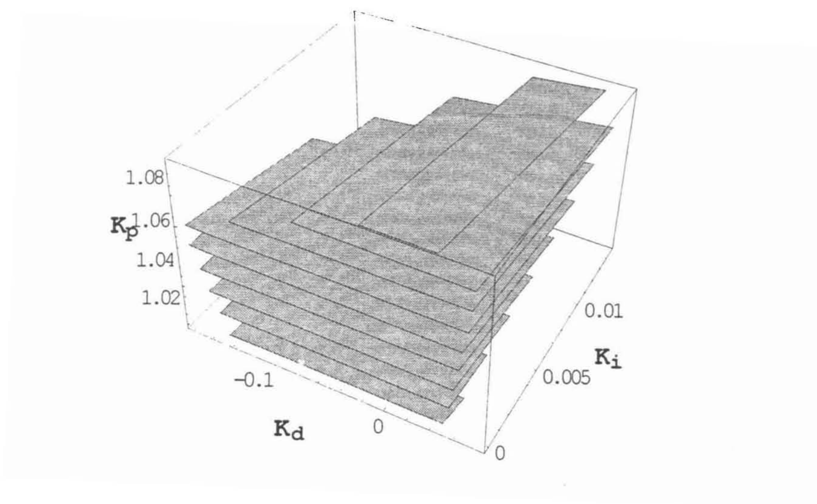

Figure 8: Robust stabilizing PID space

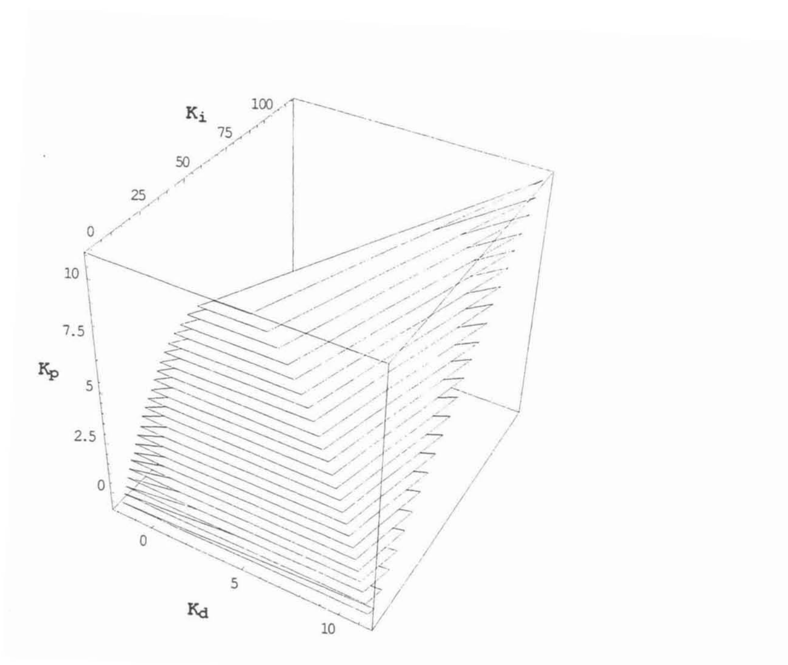

Figure 7: Stabilizing PID compensators