ROBUST EIGENVALUE ASSIGNMENT IN DESCRIPTOR SYSTEMS VIA OUTPUT FEEDBACK

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Abstract — Based on a recently proposed parametric approach for eigenstructure assignment in descriptor linear systems via output feedback, the robust eigenvalue assignment problem in descriptor linear systems via output feedback is solved. The problem aims to assign a set of finite closed-loop eigenvalues which have minimum sensitivities with respect to perturbations in the closed-loop coefficient matrices, while at the same time, guarantee the closed-loop regularity. The approach optimizes the design parameters existing in the closed-loop eigenvectors to achieve the minimum eigenvalue sensitivities, and use the extra degree of freedom existing in the solution of the gain matrix to further minimize the magnitude of the gain matrix and enhance the robustness of the closed-loop regularity. The approach allows the finite closed-loop eigenvalues to be optimized within desired regions, and is demonstrated to be simple and effective.

Key words — Descriptor systems; output feedback, robust pole assignment, eigenvalue sensitivities, eigenstructure assignment.

1. INTRODUCTION

By making use of these degrees of freedom existing in eigenvalue assignment in multivariable linear systems, the closed-loop eigenvalues may be made to be as insensitive as possible to perturbations in the components of the closed-loop system coefficient matrices. This problem is known, in the literature, as robust pole assignment, and has been extensively studied by many authors for the case of conventional linear systems (e.g., Kautsky *et al.* 1985, Owens and O'Reilly 1989, Kautsky and Nickols 1990, Duan 1992, 1993, Lam and Yan 1995). However, for the case of descriptor systems, this problem has only been investigated by a few researchers (Kautsky and Nichols 1986, Kautsky *et al.* 1989, Syrmos and Lewis 1992 and Duan and Patton 1999). Kautsky and Nichols (1986) and Kautsky *et al.* (1989) extend their earlier well known techniques in Kautsky *et al.* (1985) developed for normal linear systems to the case of descriptor systems, and lay a special emphasis on the closed-loop regularity. Syrmos and Lewis (1992) develop a robustness theory for the generalized spectrum of descriptor linear systems, and present a compact theory for the robust eigenvalue assignment problem in descriptor linear systems using the concept of chordal metric. Different the above, Duan and Patton (1999) studied robust pole assignment in descriptor linear systems via proportional plus partial derivative state feedback. Due to the capacity of the derivative feedback, their work concentrates on the case that the closed-loop system possesses n (= the system order) finite closed-loop eigenvalues.

In this paper, robust pole assignment in multivariable descriptor linear systems via output feedback is investigated based on the eigenstructure assignment approach recently proposed in Duan (1999). Duan (1999) proposes parametric expressions for both the left and right closed-loop eigenvectors in terms of the closed-loop eigenvalues and two groups of parameter vectors $\{f_{ij}^k\}$ and $\{g_{ij}^k\}$. By using these parametric expressions of closed-loop eigenvectors and the perturbation theory of generalized eigenvalue problems proposed by Stewart (1975), closed-loop eigenvalues sensitivities in terms of these design parameters are obtained. These parameters existing in the closed-loop eigenvectors are then optimized to minimize the closed-loop eigenvalue sensitivities. When the closed-loop system resulted in by these parameters is sub-conventional (Duan 1999), there exists a unique corresponding output feedback gain matrix. However, when the resulted system is not sub-conventional, there also exist some extra degrees of freedom in the solution of the gain matrix, which is represented by the parameter matrix W_∞ or Z_∞. In this case, this parameter matrix is optimized to minimize the magnitude of the feedback gain and enhance the robustness of the closed-loop regularity.

As our earlier work in Duan (1992b, 1993b) and Duan and Patton (1999), a simple method is proposed for solving the robust pole assignment problem, which contains procedures in a sequential order, and no "going back" procedures are needed. Due to some advantages of the eigenstructure assignment approach used, the closed-loop finite eigenvalues may be easily included in the design parameters and are optimized within certain desired regions on the complex plane to improve the robustness, and the optimality of the solution to the whole robust pole assignment problem is solely dependent on the optimality of the solution to the eigenvalue sensitivity minimization problem converted.

2. PROBLEM FORMULATION

Consider the following time-invariant linear descriptor system

$$\begin{cases} E\delta x = Ax + Bu \\ y = Cx \end{cases} \tag{2.1}$$

where δ denotes the differential operator d/dt for continuous-time systems, or the one-step forward operator q (defined by $qx(k) = x(k+1)$) for discrete-time systems; $x \in R^n$, $u \in R^r$, $y \in R^m$ are, respectively, the descriptor-variable vector, the input vector and the output vector; $A, E \in R^{n \times n}$, $B \in R^{n \times r}, C \in R^{m \times r}$ and $D \in R^{n \times p}$ are known matrices with $\text{rank}(E) = n_0 \leq n$, $\text{rank}(B) = r$, $\text{rank}(C) = m$, and they satisfy the following controllability and observability assumption

$$A1: \text{rank}\left([sE - A \quad B]\right) = \text{rank}\left(\begin{bmatrix} sE - A \\ C \end{bmatrix}\right) = n, \quad \forall s \in C$$

When the following output feedback control law

$$u = Ky, \quad K \in R^{r \times m} \tag{2.2}$$

is applied to the system (2.1), the closed-loop system is obtained in the following form

$$E\delta x = A_c x, \quad A_c = A + BKC \tag{2.3}$$

Refer to the fact that non-defective matrix pair $[A_c E_c]$ possesses relative eigenvalues which are less sensitive to the matrix parameter perturbations (Kautsky *et al.* 1989), it is restricted here the closed-loop finite eigenvalues to be a set of n_0 distinct, but self-conjugate complex numbers. The robust pole assignment problem for system (2.1) via the output feedback control law (2.2) can then be stated as follows.

Problem RPA: *Given system* (2.1) *satisfying Assumption A1, and a series of regions* Ω_i, $i = 1,2,\cdots,n_0$, *on the complex plane, seek an output feedback controller in the form of* (2.2), *such that the following requirements are met:*

1. *The closed-loop system* (2.3) *is regular, and has* n_0 *number distinct finite relative eigenvalues.*
2. *The finite closed-loop eigenvalues* $s_i, i = 1,2,\cdots,n_0$, *satisfy the location conditions* $s_i \in \Omega_i$ $i = 1, 2,\cdots,n_0$, *and are as insensitive as possible to parameter perturbations in the closed-loop systems matrices E and* A_c.

3. PRELIMINARIES

3.1 *An Eigenstructure Assignment Result*

Under Assumption A1, there exist a pair of right coprime polynomial matrices $N(s) \in R^{n \times r}$ and $D(s) \in R^{r \times r}$ and a pair of right coprime polynomial matrices $H(s) \in R^{n \times m}$ and $L(s) \in R^{m \times m}$ satisfying

$$(A - sE)N(s) + BD(s) = 0 \tag{3.1}$$

and

$$(A - sE)^T H(s) + C^T L(s) = 0 \tag{3.2}$$

Let the infinite eigenvalue of the closed-loop system be denoted by s_∞. Then s_∞ is a multiple eigenvalue with both geometric and algebraic multiplicities being equal to $n - n_0$. Therefore, there are $n - n_0$ left and right eigenvectors associated with s_∞. Denote the left and the right eigenvector matrices of the closed-loop system (2.3) by $T_\infty \in C^{(n-n_0) \times n}$ and $V_\infty \in C^{n \times (n-n_0)}$, then by definition,

$$EV_\infty = 0, \quad \text{rank}(V_\infty) = n - n_0 \tag{3.3}$$

and

$$T_\infty^T E = 0, \quad \text{rank}(T_\infty) = n - n_0 \tag{3.4}$$

Following the main result in Duan (1999) the following result can be obtained which gives the general form for all the output feedback controllers which meet the first condition in Problem RPA.

Lemma 3.1: *Let Assumption A1 be satisfied, and* T_∞ *and* V_∞ *be the infinite left and right closed-loop eigenvector matrices given in* (3.3) *and* (3.4), *respectively. Then*

1) *All the output feedback controllers in the form of (2.2) for the descriptor linear system (2.1), which satisfy the first condition in Problem RPA can be parameterized by*

$$K = \left(WV^T + W_\infty V_\infty^T\right)C^T\left[C(VV^T + V_\infty V_\infty^T)C^T\right]^{-1} \quad (3.5)$$

or

$$K = \left[B^T(TT^T + T_\infty T_\infty^T)B)\right]^{-1}B^T\left(TZ^T + T_\infty Z_\infty^T\right) \quad (3.6)$$

with the matrices T, V, W and Z given by

$$V = [N(s_1)f_1 \quad N(s_2)f_2 \quad \cdots \quad N(s_{n_0})f_{n_0}] \quad (3.7a)$$

$$W = [D(s_1)f_1 \quad D(s_2)f_2 \quad \cdots \quad D(s_{n_0})f_{n_0}] \quad (3.7b)$$

$$T = [H(s_1)g_1 \quad H(s_2)g_2 \quad \cdots \quad H(s_{n_0})g_{n_0}] \quad (3.8a)$$

$$Z = [L(s_1)g_1 \quad L(s_2)g_2 \quad \cdots \quad L(s_{n_0})g_{n_0}] \quad (3.8b)$$

where W_∞, Z_∞, f_i *and* g_i, $i = 1,2,\cdots,n_0$, *are the design parameters satisfying the following constraints:*

C1: $f_i = \bar{f}_l$, $g_i = \bar{g}_l$ if $s_i = \bar{s}_l$

C2: $g_i^T H^T(s_i)EN(s_j)f_j = \delta_{ij}$, $i,j = 1,2,\cdots,n_0$

C3: $\begin{cases} g_i^T H^T(s_i)Bw_j^\infty = g_i^T L^T(s_i)Cv_j^\infty \\ (t_j^\infty)^T BD(s_i)f_i = (z_j^\infty)^T CN(s_i)f_i \\ i = 1,2,\cdots,n_0, \; j = 1,2,\cdots,n-n_0 \end{cases}$

C4: $(t_k^\infty)^T Bw_j^\infty = (z_k^\infty)^T Cv_j^\infty$, $j,k = 1,2,\cdots,n-n_0$

C5: $\det(T_\infty^T AV_\infty + T_\infty^T BW_\infty) \neq 0$ or

$\det(T_\infty^T AV_\infty + Z_\infty^T CV_\infty) \neq 0$

where δ_{ij} *represents the Kronecker function, and* t_i^∞ and v_i^∞ *are the columns of matrices* T_∞ and V_∞, *respectively.*

2) *The matrices T and V given above are a pair of normalized left and right finite eigenvector matrices for the closed-loop system, that is, they satisfy*

$$T^T A_c = \Lambda T^T E, \quad A_c V = EV\Lambda \quad (3.9)$$

and

$$T^T A_c V = \Lambda, \quad T^T EV = I_{n_0} \quad (3.10)$$

with

$$A_c = A + BKC, \quad \Lambda = \mathrm{diag}[s_1 \quad s_2 \quad \cdots \quad s_{n_0}] \quad (3.11)$$

3.2. *Closed-loop Eigenvalue Sensitivities*

To derive the closed-loop eigenvalue sensitivity measures, the following perturbation result of generalized eigenvalue problem of matrix pairs is needed.

Definition 3.1 [Stewart (1975)]: *Let* $M, N \in R^{n\times n}$, λ *be a simple finite relative eigenvalue of the matrix pair [M N]. A pair of right and left eigenvectors x and y of the matrix pair [M N] associated with eigenvalue* λ *are said to be a normalized pair if*

$$y^T N x = \lambda, \quad y^T M x = 1 \quad (3.12)$$

Proposition 3.1 [Stewart (1975)]: *Let* $M, N \in R^{n\times n}$, λ *be a simple finite relative eigenvalue of the matrix pair [M N]. Then the sensitivity of* λ *to perturbations in the components of M and N depends upon the following condition number*

$$c(\lambda) = \frac{\|y\|_2 \|x\|_2}{[1 + |\lambda|^2]^{1/2}} \quad (3.13)$$

where x and y are a pair of normalized right and left eigenvectors of the matrix pair [M N] associated with eigenvalue λ.

Based on Lemma 3.1 and the above proposition, the following result can be easily proven (proof omitted).

Lemma 3.2: *Let Assumption A1 be satisfied, and the matrices K, T and V be given by Lemma 3.1. Then the closed-loop system (2.3) takes* s_i, $i = 1,2,\cdots,n_0$, *as the set of closed-loop finite eigenvalues, and the sensitivity of* s_i, $i = 1,2,\cdots,n_0$, *to perturbations in the components of matrices E and* A_c *depends on the following condition numbers*

$$c_i = \frac{\|t_i\|_2 \cdot \|v_i\|_2}{[1 + |s_i|^2]^{1/2}} = \frac{\|H(s_i)g_i\|_2 \cdot \|N(s_i)f_i\|_2}{[1 + |s_i|^2]^{1/2}} \quad (3.14)$$

$$i = 1,2,\cdots,n_0$$

4. ROBUST POLE ASSIGNMENT

It follows from Lemma 3.1 that the design freedom existing in the closed-loop eigenstructure assignment consists of the following three parts:

- The closed-loop eigenvalues s_i, $i = 1,2,\cdots,n_0$.
- The group of vectors f_i, g_i, $i = 1,2,\cdots,n_0$.
- The parameter matrices W_∞ and Z_∞.

These parameters are required to satisfy Constraints C1~C5. In many cases, these parameters exist and are generally not unique. The Proposition 1 in Duan (1999) has stated a sufficient condition for existence of these parameters based on the pole assignment result proposed by Goodwin and Fletcher (1995). To solve Problem RPA, in the following proper choices of these three parts of design parameters are sought to meet the two requirements in our Problem RPA stated in Section 2.

4.1. *Optimizing Parameters* f_i, g_i *and* s_i, $i = 1,2,\cdots,n_0$

It follows from Lemma 3.1 that the feedback gain matrix given by (3.5) or (3.6), with the matrices V, W and T, Z given by (3.7) and (3.8), and the design parameters $\{f_i\}$ and $\{g_i\}$, W_∞ and Z_∞ satisfying Constraints C1~C5, meet the first requirement in Problem RPA. Note that the eigenvalue sensitivity measures given in Lemma 3.2 have relations only with the parameters f_i, g_i and s_i, $i = 1,2,\cdots,n_0$, to further meet the second requirement in Problem RPA, a natural idea is to minimize the closed-loop eigenvalue sensitivity measures c_i, $i = 1,2,\cdots,n_0$, defined in Lemma 3.2 by optimizing the parameters f_i, g_i and s_i, $i = 1,2,\cdots,n_0$.

Define the objective

$$J = J(s_i, f_i, g_i, i = 1,2,\cdots,n_0) = \sum_{i=1}^{n_0} \tau_i c_i^2 \qquad (4.1)$$

with c_i, $i = 1,2,\cdots,n_0$, being the closed-loop eigenvalue measures defined by (3.14), and τ_i, $i = 1,2,\cdots,n$, being a group of positive scalars representing the weighting factors. Then the parameters f_i, g_i and s_i $i = 1,2,\cdots,n_0$ can be sought by the following optimization problem:

$$\begin{aligned} &\text{minimize } J(s_i, f_i, g_i, i = 1,2,\cdots,n_0) \\ &s.t \quad s_i \in \Omega_i, i = 1,2,\cdots,n_0 \\ &\qquad \text{Constraints C1 and C2} \end{aligned} \qquad (4.2)$$

4.2. *Optimizing Parameter W_∞ or Z_∞*

The third part of parameter W_∞ or Z_∞ is associated with the solution of the feedback gain K. It follows from the Theorem 4 in Duan (1999) that, for arbitrary f_i, g_i and s_i, $i = 1,2,\cdots,n_0$ satisfying Constraints C1 and C2, there always exist W_∞ or Z_∞ satisfying Constraints C3 and C4. In this subsection it is assumed that, with the parameters f_i, g_i and s_i, $i = 1,2,\cdots,n_0$ obtained by solving the optimization problem (6.6), the matrix W_∞ or Z_∞ satisfying Constraints C3 and C4 are not unique. In this case, a good use of this extra degree of freedom should be made.

There are two criteria to select the parameter W_∞ or Z_∞ satisfying Constraints C3~C5. One is to make the magnitude of the feedback gain matrix K given by (3.5) or (3.6) to be small. The other is to make the solution to the inverse of the matrix

$$\Sigma_\infty = T_\infty^T A V_\infty + T_\infty^T B W_\infty$$

or

$$\Sigma_\infty = T_\infty^T A V_\infty + Z_\infty^T C V_\infty$$

to be well-conditioned. This is because the non-singularity of this matrix determines the closed-loop regularity (Duan 1999) and making the inverse of this matrix well-conditioned enhances the robustness of the closed-loop regularity against system parameter perturbations. Combining these two aspects, yields the following objective

$$J_\infty = J_\infty(W_\infty \text{or} Z_\infty) = \beta\|K\|_F + \gamma\|\Sigma_\infty\|_F\|\Sigma_\infty^{-1}\|_F \qquad (4.3)$$

where K is given by (3.5) or (3.6), β and γ are two positive scalars which represent the weighting factors. Then the parameter W_∞ or Z_∞ can be found by solving the following optimization problem

$$\begin{cases} \text{minimize } J_\infty(W_\infty \text{or} Z_\infty) \\ \quad s.t \quad \text{Constraints C3} \sim \text{C5} \end{cases} \qquad (4.4)$$

4.3. *Closed-loop Regularity*

Closed-loop regularity is actually guaranteed by Constraint C5 (Duan 1999). When the parametric matrix W_∞ or Z_∞ satisfying Constraints C3 and C4 are not unique, solving the optimization problem (4.4) produces a parameter matrix W_∞ or Z_∞ satisfying Constraint C5. Note that Constraint C5 is actually the closed-loop regularity condition, the final gain matrix determined by the parameters f_i, g_i and s_i, $i = 1,2,\cdots,n_0$ and W_∞ or Z_∞ naturally meets the closed-loop regularity requirement. However, when the parameter matrix W_∞ or Z_∞ satisfying Constraints C3 and C4 is unique, there is obviously no need to optimize this parameter matrix. Such a case is defined in Duan (1999) as the sub-conventional case because, as in the case of conventional linear systems, the feedback gain is now totally determined by the closed-loop eigenvectors (or the design parameters f_i, g_i and s_i, $i = 1,2,\cdots,n_0$). Thus in this case, the closed-loop regularity is also completely determined by the parameters f_i, g_i and s_i, $i = 1,2,\cdots,n_0$.

It follows from the Theorem 7 in Duan (1999) that the sub-conventional case occurs if and only if

$$(CV)(CV)^- = I \quad \text{or} \quad (T^T B)^-(T^T B) = I \qquad (4.5)$$

where M^- represents the inner inverse of the matrix M. When (4.5) is met, the gain matrix is given by

$$K = W(CV)^- \text{ or } K = (T^T B)^- Z^T \qquad (4.6)$$

and Constraint C5 is given by

$$\mathbf{C^*5:} \begin{cases} \det\{T_\infty^T (A + BW(CV)^- C)V_\infty\} \neq 0 \\ \text{or} \\ \det\{T_\infty^T (A + B(T^T B)^- Z^T C)V_\infty\} \neq 0 \end{cases}$$

It follows from the above that, after solving the optimization problem (4.2), the sub-conventional condition (4.5) need to be checked. If this condition is not met, solve the optimization problem (4.4). If the sub-conventional condition (4.5) is met, check Constraint C^*5. If this Constraint is met, calculate the gain matrix K, otherwise seek parameters f_i, g_i and s_i, $i = 1,2,\cdots,n_0$ again by solving the following optimization problem

$$\begin{aligned} &\text{minimize } J(s_i, f_i, g_i, i = 1,2,\cdots,n_0) \\ &s.t \begin{cases} s_i \in \Omega_i, i = 1,2,\cdots,n_0 \\ \text{Constraints C1 and C2} \\ \text{Constraints } C^*5 \end{cases} \end{aligned} \qquad (4.7)$$

5. AN ILLUSTRATIVE EXAMPLE

Consider a system in the form of (2.1) with the following coefficient matrices (Duan 1999):

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix}1 & 0 & 0\\ 1 & -1 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, C=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

It is easy to verify that with this example system Assumption A1 holds. Moreover, it can be obtained that the left and right infinite eigenvector matrices T_∞ and V_∞ defined by (3.3) and (3.4) are

$$V_\infty = T_\infty = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^{\mathrm{T}}$$

5.1. *Eigenstructure Assignment*

Restrict the closed-loop eigenvalues s_i, $i=1,2,3$ to be distinct and real, and denote

$$f_i = f_{i1}^1 = \begin{bmatrix}\alpha_{i1}\\ \alpha_{i2}\\ \alpha_{i3}\end{bmatrix},\quad g_i = g_{i1}^1 = \begin{bmatrix}\beta_{i1}\\ \beta_{i2}\end{bmatrix},\quad i=1,2,3$$

and

$$W_\infty = w_1^\infty = \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix},\quad Z_\infty = z_1^\infty = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} \tag{5.1}$$

then the matrices T, V, W and Z are given as follows

$$T=\begin{bmatrix}\beta_{11} & \beta_{21} & \beta_{31}\\ s_1\beta_{11} & s_2\beta_{21} & s_3\beta_{31}\\ \beta_{12} & \beta_{22} & \beta_{32}\\ s_1\beta_{12}-\beta_{11} & s_2\beta_{22}-\beta_{21} & s_3\beta_{32}-\beta_{31}\end{bmatrix} \tag{5.2}$$

$$Z=\begin{bmatrix}s_1^2\beta_{11}-\beta_{12} & s_2^2\beta_{21}-\beta_{22} & s_3^2\beta_{31}-\beta_{32}\\ \beta_{12} & -\beta_{22} & -\beta_{32}\end{bmatrix} \tag{5.3}$$

and

$$V=\begin{bmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\\ \alpha_{12} & \alpha_{22} & \alpha_{32}\\ \alpha_{13} & \alpha_{23} & \alpha_{33}\\ \varphi_1 & \varphi_2 & \varphi_3\end{bmatrix} \tag{5.4}$$

$$W=\begin{bmatrix}s_1\alpha_{11}-\alpha_{13} & s_2\alpha_{21}-\alpha_{23} & s_3\alpha_{31}-\alpha_{33}\\ \psi_1 & \psi_2 & \psi_3\\ -\alpha_{13} & -\alpha_{23} & -\alpha_{33}\end{bmatrix} \tag{5.5}$$

with

$$\varphi_i = (-\alpha_{i1}+\alpha_{i2}+\alpha_{i3})s_i - \alpha_{i1} - \alpha_{i2} + 3\alpha_{i3} \tag{5.6}$$

$$\psi_i = (s_i+1)\alpha_{i1} - s_i\alpha_{i2} - 3\alpha_{i3} \tag{5.7}$$

Since all the finite closed-loop eigenvalues are real, the parameters α_{ij}'s and β_{ij}'s can be restricted to be real. Therefore, Constraint C1 holds automatically, while Constraints C2~C5 give the following set of equations:

$$\xi_3 = \eta_2 \tag{5.8}$$

$$\xi_3 = \eta_2 \neq 0 \tag{5.9}$$

$$\alpha_{i2}\eta_1 + \varphi_i\eta_2 = -\alpha_{i3}, i=1,2,3 \tag{5.10}$$

$$(\alpha_{j1}+s_i\alpha_{j2})\beta_{i1} + \alpha_{j3}\beta_{i2} = \delta_{ij}\ \ i,j=1,2,3 \tag{5.11}$$

$$\beta_{i1}(s_i+1)\xi_1 + (\beta_{i2}-s_i\beta_{i1})\xi_2 = \rho_i, i=1,2,3 \tag{5.12}$$

where

$$\rho_i = -[(2s_i-1)\beta_{i1} + s_i\beta_{i2}]\eta_2 - \beta_{i2}, i=1,2,3 \tag{5.13}$$

For a thorough treatment of this set of constraints, refer to the subsections 4.1 and 4.2 in Duan (1999).

5.2. *Robust Pole Assignment*

This subsection considers robust pole assignment in this example system based on the above general result of eigenstructure assignment. The main task in solving the robust pole assignment problem is to solve the optimization problem (4.2).

Obviously, the condition numbers defined by (3.14) are

$$c_i = \frac{\left\{(1+s_i^2)\beta_{i1}^2 + \beta_{i2}^2 + (s_i\beta_{i2}-\beta_{i1})^2\right\}^{1/2}\left\{\alpha_{i1}^2+\alpha_{i2}^2+\alpha_{i3}^2+\varphi_i^2\right\}^{1/2}}{\left\{1+|s_i|^2\right\}^{1/2}} \tag{5.14}$$

where φ_i, $i=1,2,3$, are given by (5.6). It follows from the Fact 7.1 in Duan (1999) that Constraint C2, that is, condition (5.11), is satisfied if and only if

$$\Delta_1 = \det\begin{bmatrix}\beta_{11} & s_1\beta_{11} & \beta_{12}\\ \beta_{21} & s_2\beta_{21} & \beta_{22}\\ \beta_{31} & s_3\beta_{31} & \beta_{32}\end{bmatrix} \neq 0 \tag{5.15}$$

and in this case, there holds

$$\begin{bmatrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\\ \alpha_{12} & \alpha_{22} & \alpha_{32}\\ \alpha_{13} & \alpha_{23} & \alpha_{33}\end{bmatrix} = \begin{bmatrix}\beta_{11} & s_1\beta_{11} & \beta_{12}\\ \beta_{21} & s_2\beta_{21} & \beta_{22}\\ \beta_{31} & s_3\beta_{31} & \beta_{32}\end{bmatrix}^{-1} \tag{5.16}$$

Through substituting (5.16) into (5.14), these condition numbers can be finally arranged into the form represented only by the part of parameters β_{ij}'s. In this case, Constraint C2 in the optimization problem (4.2) becomes condition (5.15). Further note that the closed-loop eigenvalues are restricted to be real, Constraint C1 holds automatically. Therefore, the optimization problem (4.2) is finally turned into the following

$$\begin{cases}\text{minimise } \sum_{i=1}^{3}\tau_i c_i^2\\ \quad s.t.\quad s_i \in \Omega_i, i=1,2,3\\ \qquad\quad \text{Inequality (5.15)}\end{cases} \tag{5.17}$$

where the condition numbers $c_i, i=1,2,3$ are given by (5.14) and (5.16).

In the following, two different cases are considered.

5.2.1. $\Omega_i = \{-i\}, i=1,2,3$.

In this case the closed-loop eigenvalues are pre-assigned to -1, -2 and -3, and are not optimized within any fields on the complex plane by minimizing the condition numbers. Therefore, the above optimization problem has only 6 parameters β_{ij}, $i=1,2,3; j=1,2$. By using the MATLAB command *constr*, the following solution (truncated to 7 figures) to the optimization problem (5.17) is obtained:

$$\begin{bmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\\ \beta_{31} & \beta_{32}\end{bmatrix} = \begin{bmatrix}-0.2365024 & 0.0113553\\ -0.0547700 & 0.1490114\\ 0.6877025 & 0.0849793\end{bmatrix}$$

With this group of parameters, the following robust pole assignment solution (truncated to 7 figures) for this system is derived:

Solution 1:

$$K = (T^T B)^{-1} Z^T = \begin{bmatrix} -4.5998628 & -0.9960638 \\ -3.9379684 & -2.9943938 \\ -1.6249757 & -0.9819272 \end{bmatrix}$$

With this solution, the closed-loop finite eigenvalues are given by

$s_1 = -0.9999998,\ s_1 = -2.0000000,\ s_1 = -3.0000005$

Further, note $T_\infty^T (A + BKC) V_\infty = -0.9819272 \neq 0$, the closed-loop system is regular.

5.2.2. $\Omega_1 = [-1 - 0.4], \Omega_2 = [-3 - 1.5], \Omega_3 = [-5 - 3.5]$

In this case the closed-loop eigenvalues are included as the optimizing parameters, and now the optimization problem (5.17) has 9 parameters to be optimized. Again by using the MATLAB command *constr*, the following solution (truncated to 7 figures) to the optimization problem (5.17) is obtained:

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} = \begin{bmatrix} 3.2642800 & 1.7657230 \\ -3.1715436 & 4.0664588 \\ -2.9145538 & -1.7866234 \end{bmatrix}$$

$$s_1 = -0.4,\ s_2 = -2.0007953,\ s_3 = -5$$

With this group of parameters, the following robust pole assignment solution (truncated to 7 figures) for this system is derived:

Solution 2:

$$K = (T^T B)^{-1} Z^T = \begin{bmatrix} -3.2806791 & 0.2904322 \\ -0.2009044 & -5.8268859 \\ -0.8810492 & -2.3643821 \end{bmatrix}$$

With this solution, the closed-loop finite eigenvalues are calculated exactly the same as given in the sought parameters above when truncated by 7 figures. Further, note $T_\infty^T (A + BKC) V_\infty = -2.3643821 \neq 0$, the closed-loop system is regular.

The condition numbers and the norm of the above robust Solutions 1 and 2 are listed in Table 5.1.

Table 5.1 Condition numbers and magnitudes

Solution	c_1	c_2	c_3	$\|K\|$
1	2.11263	1.23063	1.32079	6.92911
2	1.57834	0.88162	0.48421	6.30989

6. CONCLUDING REMARKS

This paper proposes, based on a recently proposed eigenvalue assignment approach, a simple and effective algorithm for eigenstructure assignment with minimum sensitivities in descriptor linear systems via output feedback. The eigenvalue condition numbers for matrix pairs are used to measure the eigenvalue sensitivities with respect to the closed-loop parameter perturbations. Explicit parametric forms for these eigenvalue sensitivities are established in terms of the design parameters. The method first minimizes the closed-loop eigenvalue sensitivities by optimizing the first two parts of the design parameters, and then use the third part of design parameters (if exists) to minimize the magnitude of the gain matrix and enhance the robustness of the closed-loop regularity. An example is presented to illustrate the effect of the proposed approach.

ACKNOWLEDGEMENTS

This work was supported in part by the Chinese Outstanding Youth Science Foundation under Grant No.69504002.

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