Hints for Lab4-Various Sets

Exercise for CS363-Computability Theory, Xiaofeng Gao

1. Let A, B be subsets of $\mathbb{N}$. Define sets $A \oplus B$ and $A \otimes B$ by

$$A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\},$$
$$A \otimes B = \{\pi(x,y) : x \in A \text{ and } y \in B\},$$

where π is the pairing function $\pi(x,y) = 2^x(2y+1) - 1$ of Theorem 4-1.2. Prove that

(a) $A \oplus B$ is recursive iff A and B are both recursive.
"$\Leftarrow$" Consider the computability of $C_{A\oplus B}$ with the help of C_A and C_B.
"$\Rightarrow$" If $C_{A\oplus B}$ is computable, rewrite C_A and C_B by operations on $C_{A\oplus B}$.

(b) If $A, B \neq \varnothing$, then $A \otimes B$ is recursive iff A and B are both recursive.
Similar as the above question. Don't forget to prove the both directions.

(c) Suppose B is r.e. If A is creative, then so are $A \oplus B$ and $A \otimes B$ (provided $B \neq \varnothing$).

- For $A \oplus B$. We need to prove (1) $A \oplus B$ is r.e.; (2) $\overline{A \oplus B}$ is productive. (1) can be easily proved according to Q1(a). (2) Apply $\overline{A}$ is productive with Reduction Theorem of productive set such that $x \in \overline{A} \Leftrightarrow f(x) \in A \oplus B$.
- Similarly for $A \otimes B$. We need to prove (1) $A \otimes B$ is r.e.; (2) $\overline{A \otimes B}$ is productive. (1) can be easily proved according to Q1(b). (2) Apply $\overline{A}$ is productive with Reduction Theorem of productive set such that $x \in \overline{A} \Leftrightarrow f(x) \in A \otimes B$.

(d) If A is simple, then $A \otimes \mathbb{N}$ is r.e., but neither recursive, creative nor simple.

i. $A \otimes \mathbb{N}$ is r.e. Distinguish what is $A \otimes \mathbb{N}$ by pairing function $\pi_1(x)$. Then you can easily prove this property.

ii. $A \otimes \mathbb{N}$ is not recursive. Use the facts that A is not recursive and Q1(b); or prove by contradiction.

iii. $A \otimes \mathbb{N}$ is not creative. By contradiction if $\overline{A \otimes \mathbb{N}}$ is productive and $x \in \overline{A \otimes \mathbb{N}} \Leftrightarrow x \in \overline{A}$, then we can have A is creative, which violates the fact that A is simple.

iv. $A \otimes \mathbb{N}$ is not simple. $\overline{A \otimes \mathbb{N}} = \{x \mid \pi_1(x) \notin A\}$. Find an infinite r.e. subset in this set.

(e) If A, B are simple sets, then $A \oplus B$ is simple, $A \otimes B$ is not simple but $\overline{\overline{A} \otimes \overline{B}}$ is simple.

i. $A \oplus B$ is simple. (1) $A \oplus B$ is r.e. by Q1.(c). (2) $\overline{A \oplus B} = \overline{A} \oplus \overline{B}$. Prove it is infinite and use contradiction to prove it doesn't contain any r.e. subset.

ii. $A \otimes B$ is not simple. Prove $\overline{A \otimes B}$ has an infinite r.e. subset.

iii. $\overline{\overline{A} \otimes \overline{B}} = \overline{\{t \mid \pi_1(t) \in \overline{A} \wedge \pi_2(t) \in \overline{B}\}} = \{t \mid \pi_1(t) \in A \vee \pi_2(t) \in B\}$. Prove it is r.e.; $\overline{A} \otimes \overline{B}$ is infinite, and by contradiction $\nexists C \subseteq \overline{A} \otimes \overline{B}$, otherwise A or B will not be simple.

2. (a) Let $B \subseteq \mathbb{N}$ and $n > 1$; prove if B is recursive (or r.e.) then the predicate $M(x_1, \ldots, x_n)$ given by "$M(x_1, \ldots, x_n) \equiv 2^{x_1}3^{x_2} \ldots p_n^{x_n} \in B$" is decidable (or partially decidable).
$C_M = C_B(2^{x_1}3^{x_2} \ldots p_n^{x_n}) \quad \chi_M = \chi_B(2^{x_1}3^{x_2} \ldots p_n^{x_n})$

(b) Prove that $A \subseteq \mathbb{N}^n$ is recursive (or r.e.) iff $\{2^{x_1}3^{x_2} \ldots p_n^{x_n} : (x_1, \ldots, x_n) \in A\}$ is recursive (or r.e., respectively).
$C_M((\mathbf{x})_1, \cdots, (\mathbf{x})_n) \Leftrightarrow C_A(\mathbf{x}) \quad \chi_M((\mathbf{x})_1, \cdots, (\mathbf{x})_n) \Leftrightarrow \chi_A(\mathbf{x})$

(c) Prove that $A \subseteq \mathbb{N}^n$ is r.e. iff $A = \varnothing$ or there is a total computable function $\boldsymbol{f} : \mathbb{N} \to \mathbb{N}^n$ such that $A = Ran(\boldsymbol{f})$. (A *computable function* $\boldsymbol{f}$ from $\mathbb{N}$ to $\mathbb{N}^n$ is an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$ where each f_i is a unary computable function and $\boldsymbol{f}(x) = (f_1(x), \ldots, f_n(x))$.)

Mimic the proof of Listing Theorem with Coding technique.

3. Which of the following sets are recursive? Which are r.e.? Which are productive? Which are creative? Prove your judgements.

(a) $\{x : x \in E_x\}$,

(b) $\{x : x \text{ is a perfect square}\}$,

(c) $\{x : \phi_x \text{ is not injective}\}$,

(d) $\{x : \phi_x \text{ is not surjective}\}$,

(e) $\{x : \phi_x(x) = f(x)\}$, where f is any total computable function.

You can (1) consider the C or χ function, or (2) consider reduction, or (3) try theorems (like Quasi-Rice Theorem) ; and (3) consider the complement of each set.

4. Suppose A is an r.e. set. Prove the following statements.

(a) Show that the sets $\bigcup_{x \in A} W_x$ and $\bigcup_{x \in A} E_x$ are both r.e.

Describe each form as a combination of quantifier + partial decidable predicates.

(b) Show that $\bigcap_{x \in A} W_x$ is not necessarily r.e. (*Hint*: $\forall t \in \mathbb{N}$ let $K_t = \{x : P_x(x) \downarrow \text{ in t steps}\}$. Show that for any t, K_t is recursive; moreover $K = \bigcup_{t \in \mathbb{N}} K_t$ and $\overline{K} = \bigcap_{t \in \mathbb{N}} \overline{K}_t$.)

$\bigcap_{x \in A} W_x$ is not necessarily r.e. means there exists special r.e. set A^*, such that $\bigcap_{x \in A^*} W_x$ is not r.e. Firstly, prove all the descriptions from the Hint. Next, since K_t is recursive, $\overline{K_t}$ is also recursive, but $\overline{K} = \bigcap_{t \in \mathbb{N}} \overline{K}_t$ is not r.e. Thus let $A^* = \mathbb{N}$.

5. Suppose that f is a total computable function, A is a recursive set and B is an r.e.set. Show that $f^{-1}(A)$ is recursive and that $f(A)$, $f(B)$ and $f^{-1}(B)$ are r,e, but not necessarily recursive. What extra information about these sets can be obtained if f is a bijection?

Describe $f(A)$, $f(B)$, $f^{-1}(A)$, $f^{-1}(B)$ by quantifier + predicates and then use reduction. If f is bijection, then you may have stronger conclusion for some of these sets.

6. Prove Rice's theorem (Theorem 6-1.7) from Rice-Shapiro theorem (Theorem 7-2.16). (*Hint.* Suppose that '$\phi_x \in \mathscr{B}$' is decidable; then both $\mathscr{B}$ and $\mathscr{C}_1 \backslash \mathscr{R}$ satisfy the conditions of Rice-Shapiro: consider the cases $f_\varnothing \in \mathscr{B}$ and $f_\varnothing \notin \mathscr{B}$.)

Just follow the hint.

7. Let $\mathscr{B}$ be a set of unary computable functions, and suppose that $g \in \mathscr{B}$ is such that for all finite $\theta \subseteq g$, $\theta \notin \mathscr{B}$. Prove that the set $\{x : \phi_x \in \mathscr{B}\}$ is productive. (*Hint.* Follow the first part of the proof of the Rice-Shapiro theorem.)

Define appropriate function then reduce from $\overline{K}$.

8. Prove the following statements.

(a) If B is r.e. and $A \cap B$ is productive, then A is productive.

Construct a f with $Dom(f) = K \cap B$, then construct the productive function of A with the help of f + s-m-n theorem

(b) If C is creative and A is an r.e. set such that $A \cap C = \varnothing$, then $C \cup A$ is creative.

(1) $C \cup A$ is r.e. (2) $\overline{C} \cap \overline{A}$ is productive by a construction of f with $Dom(f) = C \cup A$.

(c) Every productive set contains an infinite recursive subset.

r. subsubset $\subseteq$ r.e. subset $\subseteq$ productive set

9. (a) (Cf. Theorem 7-2.14) Let A be an infinite r.e. set. Show that A can be enumerated without repetitions by a total computable function.

Listing Theorem + μ operator

(b) Suppose f is a total injective computable function such that $Ran(f)$ is not recursive ((a) showed that such functions abound). Show that $A = \{x : \exists y(y > x \wedge f(y) < f(x))\}$ is simple. (*Hint.* To see that $\overline{A}$ is infinite, assume the contrary and show that there would then be a sequence of numbers $y_0 < y_1 < y_2 < \ldots$ such that $f(y_0) > f(y_1) > f(y_2) > \ldots$ To see that $\overline{A}$ does not contain an infinite r.e. set B, suppose to the contrary that $B \subseteq \overline{A}$. Then show that the problem $z \in Ran(f)$ is decidable as follows. Given z, find $n \in B$ such that $f(n) > z$; now use the fact that $n \notin A$ to devise a finite procedure for testing whether $z \in Ran(f)$.)

Just follow the hint. Notice that $f(y_0)$ is finite.

10. Recursively Inseparable and Effectively Recursively Inseparable

Disjoint sets A, B are said to be *recursively inseparable* if there is no recursive set C such that $A \subseteq C$ and $B \subseteq \overline{C}$. Furthermore, A and B are said to be *effectively recursively inseparable* if there is a total computable function f such that whenever $A \subseteq W_a$, $B \subseteq W_b$ and $W_a \cap W_b = \varnothing$ then $f(a,b) \notin W_a \cup W_b$ (see the right figure). Note: Recursive inseparability for a pair of disjoint sets corresponds to non-recursiveness for a single set; pair of recursively inseparable sets that are also r.e. correspond to r.e. sets that are not recursive.

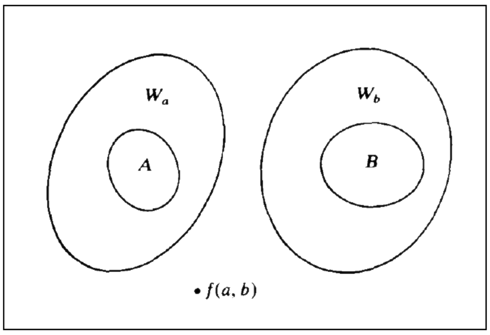

Fig. Effectively Recursively Inseparable Sets

(a) Show that two disjoint sets A, B are recursively inseparable iff whenever $A \subseteq W_a$, $B \subseteq W_b$ and $W_a \cap W_b = \varnothing$, there is a number $x \notin W_a \cup W_b$.

Use contradiction, assume $\forall x$, $x \in W_a \cup W_b$, then $W_a \cup W_b = \mathbb{N}$.

(b) Suppose A, B are effectively recursively inseparable. Prove that if A, B are both r.e. then they are both creative. (*Note.* Extending the idea of effectiveness to a pair of recursively inseparable sets in this way parallels the step from a nonrecursive set to a set having productive complement; the counterpart to a single creative set is then a pair of effectively recursively separable sets that are both r.e.)

For symmetry, you only need to prove $\overline{A}$ is productive. Construct computable function g with $Dom(g) = K \cup B$, then create the productive function from f and g.

(c) Let $K_0 = \{x : \phi_x(x) = 0\}$ and $K_1 = \{x : \phi_x(x) = 1\}$. Show that K_0 and K_1 are r.e. (in particular neither K_0 nor K_1 is recursive), and that they are both recursively inseparable and effectively recursively inseparable. (*Hint.* For recursively inseparable, suppose that there is such a set C and let m be an index for its characteristic function; consider whether or not $m \in C$. For effectively recursively inseparable, find a total computable function f such that if $W_a \cap W_b = \varnothing$, then $\phi_{f(a,b)}(x) = \begin{cases} 1 & \text{if } x \in W_a, \\ 2 & \text{if } x \in W_b, \\ \text{undefined} & \text{otherwise.} \end{cases}$)

Prove by contradiction for recursively inseparable. Then follow the hint for effectively recursively inseparable.