Lab11-Reducibility

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class

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1. Recall that $A \otimes B = \{\pi(a,b) \mid a \in A, b \in B\}$. Prove the following statements.

   (a) For any sets A, B, if $B \neq \varnothing$ then $A \leq_m A \otimes B$.

   Proof:

   Because $B \neq \varnothing$, so we have the total computable function $f(x) = \pi(x,b)$ where $b \in B$. So we have $x \in A$ iff $f(x) \in A \otimes B$. Therefore, we have $A \leq_m A \otimes B$.

   (b) $A \equiv_m A \otimes \mathbb{N}$ for any set A,

   Proof:

   According to (a), we have $A \leq_m A \otimes N$. Then, we have the function $f(x) = \pi_1(x)$, and we have $x \in A \otimes N$ iff $f(x) \in A$. Thus we have $A \otimes N \leq_m A$.
   Therefore, $A \equiv_m A \otimes N$.

   (c) $A \equiv_m A \otimes B$ if $A \neq \mathbb{N}$ and B is a non-empty recursive set.

   Proof:

   According to (a), we have $A \leq_m A \otimes B$. Then we define a total computable function:

   $$f(x) = \begin{cases} \pi_1(x) & , \pi_2 \notin B \\ k & , \text{otherwise} \end{cases}, k \notin A$$

   So, we have $x \in A \otimes N$ iff $f(x) \in A$.Thus we have $A \otimes N \leq_m A$.
   Therefore, $A \equiv_m A \otimes B$.

2. Let $\mathbf{a}$, $\mathbf{b}$ be m-degrees.

   (a) Show that the least upper bound of $\mathbf{a}$, $\mathbf{b}$ is uniquely determined; denote this by $\mathbf{a} \cup \mathbf{b}$;

   Proof:

   According to the theorem, any pair of m-degrees $\mathbf{a}$, $\mathbf{b}$ have a least upper bound. If $\mathbf{c}$, $\mathbf{d}$ are both least upper bounds, then $\mathbf{c} \leq_m \mathbf{d}$ and $\mathbf{d} \leq_m \mathbf{c}$, so $\mathbf{c} = \mathbf{d}$.

   (b) Show that if $\mathbf{a} \leq_m \mathbf{b}$ then $\mathbf{a} \cup \mathbf{b} = \mathbf{b}$;

   Proof:

   According to the definition, $\mathbf{b} \leq_m \mathbf{a} \cup \mathbf{b}$. So $\mathbf{b}$ is an upper bound of $\mathbf{a}$ and $\mathbf{b}$. So, $\mathbf{a} \cup \mathbf{b} \leq_m \mathbf{b}$. So, $\mathbf{c} = \mathbf{d}$.

   (c) Show that if $\mathbf{a}$,$\mathbf{b}$ are r.e., then so is $\mathbf{a} \cup \mathbf{b}$;

   Proof:

   Pick $A \in \mathbf{a}$, $B \in \mathbf{b}$, and because $\mathbf{a}$ and $\mathbf{b}$ are both r.e.. So, A and B are both r.e.. So, their direct sum is still r.e.. By the proof of Theorem 2.8 in the textbook, $\mathbf{a} \cup \mathbf{b}$ is r.e..

   (d) Let $A \in \mathbf{a}$ and let $\mathbf{a}^*$ denote $d_m(\overline{A})$. (Check that $\mathbf{a}^*$ is independent of the choice of $A \in \mathbf{a}$.) Show that $(\mathbf{a} \cup \mathbf{a}^*)^* = \mathbf{a} \cup \mathbf{a}^*$.

   Proof:

   We assume that $\mathbf{b} = \mathbf{a} \cup \mathbf{a}^*$.
   We first claim that $\mathbf{b}^*$ is an upper bound for $\mathbf{a}$ and $\mathbf{a}^*$, so $\mathbf{b} \leq_m \mathbf{b}^*$.
   Then, for $X \in \mathbf{b}^*, \overline{X} \in \mathbf{b}$ by definition. So for $Y \in \mathbf{a}, \overline{Y} \in \mathbf{a}^* \leq_m \mathbf{b}$, so $\overline{Y} \leq_m \overline{X}$ and $Y \leq_m X$. So, $\mathbf{a} \leq_m \mathbf{b}^*, \mathbf{a}^* \leq_m \mathbf{b}^*$, so $\mathbf{b} \leq_m \mathbf{b}^*$.
   Because $\mathbf{b}^*$ is well-defined, so $\mathbf{b}^* \leq_m (\mathbf{b}^*)^* = \mathbf{b}$, so $\mathbf{b} = \mathbf{b}^*$.

3. Show that the following sets all belong to the same m-degree:

   (a) $\{x \mid \phi_x = 0\}$,

   (b) $\{x \mid \phi_x \text{ is total and constant}\}$,

   (c) $\{x \mid W_x \text{ is infinite}\}$.

   Proof:

   First, we proof $(a) \leq_m (b)$.

   According the s-m-n theorem, there exists a total computable function k such that:

$$\forall x, \phi_{k(x)}(y) = y \sum_{z=0}^{y} \phi_x(z)$$

   If $\phi_x = 0$, then $\phi_{k(x)} = 0$, so $\phi_{k(x)}$ is total and constant.

   If $\phi_x \neq 0$, then either ϕ_x is total or it is not. If ϕ_x is not total, then neither is $\phi_{k(x)}$. If ϕ_x is total, choose y such that $\phi_x(y) > 0$. Then:

$$\phi_{k(x)}(y+1) = (y+1) \sum_{z=0}^{y+1} \phi_x(z) \geq (y+1) \sum_{z=0}^{y} \phi_x(z) \geq \phi_{k(x)}(y) + \phi_x(y) > \phi_{k(x)}(y)$$

   So, $\phi_{k(x)}$ is not constant, so $(a) \leq_m (b)$.

   Then, we proof $(b) \leq_m (c)$.

   According the s-m-n theorem there exists a total computable function s such that:

$$\forall x, \phi_{s(x)}(0) = \phi_x(0), \phi_{s(x)}(y+1) = \begin{cases} \phi_x(y+1) & , \phi_{s(x)}(y) \text{ is defined and } \phi_x(y+1) = \phi_{s(x)}(y) \\ \uparrow & , \text{otherwise} \end{cases}$$

   If ϕ_x is total and constant, then $\phi_{s(x)} = \phi_x$, so $W_{s(x)} = \mathbb{N}$ is infinite.

   If ϕ_x is not total, then there exists y such that $\phi_x(y)$ is undefined; by construction then, $\phi_{s(x)}(z)$ is undefined for all $z \geq y$, so $W_{s(x)}$ is finite.

   If ϕ_x is total but not constant, then there exists a least $y > 0$ such that $\phi_x(y) \neq \phi_x(0)$; by construction again, $\phi_{s(x)}(z)$ is undefined for all $z \geq y$, so $W_{s(x)}$ is finite.

   So, $(b) \leq_m (c)$.

   Finally, we proof $(c) \leq_m (a)$.

   According to the s-m-n theorem there exists a total computable function u such that:

$$\forall x, \phi_{u(x)} = \mathbf{0}(\mu t(\bigvee_{i=0}^{t} H_1(x, y+i, t)))$$

   If W_x is infinite, then for any y there exists $z \geq y$ such that $\phi_x(z)$ is defined, so $\phi_{u(x)}(y) = 0$. Since y was arbitrary, $\phi_{u(x)} = 0$.

   If W_x is finite, there exists some y such that for all $z \geq y$, $\phi_{u(x)}(y)$ is undefined and $\phi_{u(x)} \neq 0$.

   So, $(c) \leq_m (a)$.

   Since $(a) \leq_m (b) \leq_m (c) \leq_m (a)$, so these sets all belong to the same m-degree.