Gödel’s Incompleteness Theorem*

Xiaofeng Gao

Department of Computer Science and Engineering
Shanghai Jiao Tong University, P.R.China

CS363-Computability Theory

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Outline

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1. Formal Arithmetic

2. Gödel's Incompleteness Theorem
   - Simplified Gödel Theorem
   - Gödel's Incompleteness Theorem
   - Rosser's Refinement

3. Undecidability

Gödel’s Incompleteness Theorem, 1931.

Logic, arithmetic, and recursion theory.

Formalization of Arithmetic

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Other symbols: brackets (and), $\neq$, etc.

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The standard model $\mathbb{N}$: the ordinary arithmetic.

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Notice. *true* and *false* are model theoretical concepts.

$$\text{Thus } \mathscr{T} \cap \mathscr{F} = \varnothing \text{ and } \mathscr{T} \cup \mathscr{F} = \mathscr{S}.$$

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Standard Coding

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We can use it to code any set of statements $\mathscr{X}$ by the set of number $\mathbf{X} = \{n \mid \theta_n \in \mathscr{X}\}$.

We say that $\mathscr{X}$ is $\left\{\begin{array}{l} \textit{recursive} \\ \textit{r.e.} \\ \textit{productive} \\ \textit{creative} \\ \textit{etc.} \end{array}\right\}$ if $\mathbf{X}$ is $\left\{\begin{array}{l} \text{recursive} \\ \text{r.e.} \\ \text{productive} \\ \text{creative} \\ \text{etc.} \end{array}\right\}$

Gödel's Lemma

Lemma. Suppose $M(x_1, \ldots, x_n)$ is a decidable predicate. Then it is possible to construct a statement $\sigma(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ that is a formal counterpart of $M(x_1, \ldots, x_n)$ in the following sense: for all $a_1, \ldots, a_n \in \mathbb{N}$, $M(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ holds iff $\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n) \in \mathscr{T}$.

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Then $\exists \mathsf{y}.\sigma_R(\mathsf{n},\mathsf{y})$ is a formal counterpart of $n \in K$, and $\neg\exists \mathsf{y}.\sigma_R(\mathsf{n},\mathsf{y})$ is a formal counterpart of $n \notin K$.

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Write $\mathsf{n} \in \mathsf{K}$ for $\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ and $\mathsf{n} \notin \mathsf{K}$ for $\neg\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$.

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By previous lemma, $n \in K$ iff $\mathsf{n} \in \mathsf{K} \in \mathscr{T}$, and $n \notin K$ iff $\mathsf{n} \notin \mathsf{K} \in \mathscr{T}$.

Transform Lemma

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Proof. Given n we can effectively write down the statement $\neg\exists \mathsf{y}.\sigma_R(\mathsf{n},\mathsf{y})$, from which we can construct the index $g(n)$.

First Conclusion

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Proof. Let $\mathbb{T} = \{n \mid \theta_n \in \mathscr{T}\}$ and g be the function given in the previous lemma. We have

$$\begin{aligned} n \in \overline{K} &\Leftrightarrow n \notin K \\ &\Leftrightarrow \mathsf{n} \notin \mathsf{K} \in \mathscr{T} \\ &\Leftrightarrow g(n) \in \mathbb{T}. \end{aligned}$$

Since $\overline{K}$ is productive, so is $\mathbb{T}$.

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Formal System

A formal system $(\mathscr{A}, \mathscr{D})$ consists of a set $\mathscr{A} \subseteq \mathscr{S}$ (the axioms) and an explicit definition $\mathscr{D}$ of the notion of a formal proof of a statement in $\mathscr{S}$ from the axioms, satisfying the conditions:

- Proofs are finite objects.
- Provability is decidable if $\mathscr{A}$ is recursive.

'p is a proof of the statement σ from the axioms $\mathscr{A}$'.

Interpretation

Is there a simple-minded subset of $\mathscr{T}$ (a set of axioms) from which all other statements in $\mathscr{T}$ can be proved?

$\Longleftrightarrow$

Is there a formal system $(\mathscr{A}, \mathscr{D})$ for L such that

1. $\mathscr{A}$ is recursive, and
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2.1 Consistency: There is no statement σ such that both σ and $\neg\sigma$ are provable;

2.2 Completeness: For any statement σ, either σ is provable or $\neg\sigma$ is provable.

Judgements

A simplified version of Gödel's theorem shows that there is no formal system of arithmetic satisfying above conditions (1) and (2).

The full theorem of Gödel together with its improvement by Rosser shows that there is no formal system of arithmetic (of a certain minimal strength) satisfying conditions (1), (2.1) and (2.2).

Any consistent formal system of arithmetic having a recursive set of axioms is incomplete.

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So the set Pr of provable statements is r.e.

Simplified Gödel Theorem

Theorem. Suppose that $(\mathscr{A}, \mathscr{D})$ is a recursively axiomatized formal system in which all provable statements are true. Then there is a statement σ that is true but not provable (and consequently $\neg\sigma$ is not provable either).

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Proof. $\mathscr{P}$ is r.e. and $\mathscr{T}$ is productive. Since $\mathscr{P} \subseteq \mathscr{T}$, there is some $\sigma \in \mathscr{T} \setminus \mathscr{P}$.

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Clearly $\neg\sigma$ is not provable, otherwise $\neg\sigma$ would be true.

Such σ can be found effectively from a specification of the formal system (yield an index for $\mathscr{P}$).

Refutable and Provable

A statement is refutable if its negation is provable.

Define

$$
\begin{aligned}
Pr^* &= \{n \mid \mathsf{n} \in \mathsf{K} \text{ is provable}\}, \\
Ref^* &= \{n \mid \mathsf{n} \in \mathsf{K} \text{ is refutable}\} \\
&= \{n \mid \mathsf{n} \notin \mathsf{K} \text{ is provable}\} \\
&= \{n \mid \theta_{g(n)} \in Pr\}.
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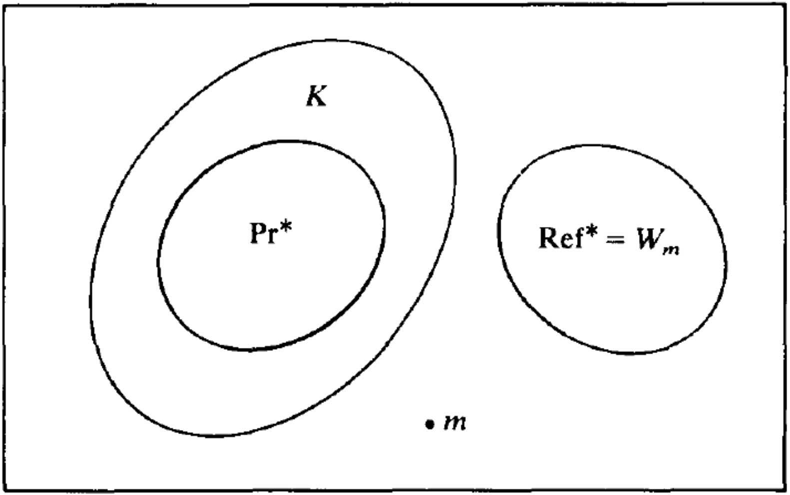
5. Let σ be $\mathsf{m} \notin \mathsf{K}$. Clearly σ is true but not provable.

Illumination



Non-provability of $\neg\sigma$

$Pr^* \subseteq K$: for $m \notin K$, so $m \notin Pr^*$, i.e., $\mathsf{m} \in \mathsf{K}$ is not provable.

By the rules of formal proof $\neg\mathsf{m} \notin \mathsf{K}$ $(\neg\sigma)$ is not provable.

Illustration

σ is the formal counterpart of the statement $m \notin K$ ($m \notin W_m$).

σ states that "I am true but not provable".

$$\begin{aligned} \text{m} \notin \text{K} \in \mathscr{T} &\Leftrightarrow m \notin W_m \\ &\Leftrightarrow m \notin K, \\ m \notin Ref^* &\Leftrightarrow \text{m} \notin \text{K is not provable} \\ &\Leftrightarrow \text{m} \notin \text{K} \notin Pr. \end{aligned}$$

Notation

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Gödel's Incompleteness Theorem can be stated without any reference to any model.

Peano Arithmetic

First Order Peano Axioms:

PA1	$\forall x.(\mathsf{s}(x) \neq 0)$
PA2	$\forall xy.(\mathsf{s}(x) = \mathsf{s}(y) \Rightarrow x = y)$
PA3	$\forall x.(x = 0 \vee \exists y.\mathsf{s}(y) = x)$
PA4	$\forall x.(x < \mathsf{s}(x))$
PA5	$\forall xy.(x < y \Rightarrow \mathsf{s}(x) \leq y)$
PA6	$\forall xy.(\neg(x < y) \Leftrightarrow y \leq x)$
PA7	$\forall xy.((x < y) \wedge (y < z) \Rightarrow x < z)$

Important Fact to Devote Gödel's Proof

Lemma. (Gödel) Let $M(x_1, \ldots, x_n)$ be a decidable predicate, then there is a statement $\sigma(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ in Peano arithmetic that satisfies the following properties: for any $a_1, \cdots, a_n \in \mathbb{N}$,

(i) If $M(a_1, \ldots, a_n)$ holds, then $\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ is provable.

(ii) If $M(a_1, \ldots, a_n)$ does not hold, then $\neg\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ is provable.

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Proof. Suppose $n \in K$. Then there is a natural number m such that $R(n, m)$ holds. By the above lemma $\sigma_R(\mathsf{n}, \mathsf{m})$ is provable. So $\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ is provable. Hence $\mathsf{n} \in \mathsf{K}$ is provable.

ω-consistent

A formal system is ω-consistent if there is no statement $\tau(\mathsf{y})$ such that all of the following are provable.

$$\exists \mathsf{y}.\tau(\mathsf{y}), \neg\tau(\mathsf{0}), \neg\tau(\mathsf{1}), \neg\tau(\mathsf{2}), \ldots$$

Another Lemma

Lemma. Suppose that Peano arithmetic is ω-consistent. Then for any natural number n, if $\mathsf{n} \in \mathsf{K}$ is provable then $n \in K$.

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Proof. Suppose $n \notin K$. Then for each $m \in \mathbb{N}$ one has that $R(n, m)$ does not hold. So $\neg\sigma_R(\mathsf{n}, \mathsf{m})$ is provable for each m, which contradicts to the provability of $\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ by ω-consistency.

Gödel's Incompleteness Theorem

Theorem (Gödel's Incompleteness Theorem, 1931).
There is a statement σ of Peano arithmetic such that
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By previous corollary, $K \subseteq Pr^*$. By consistency $Pr^* \cap Ref^* = \emptyset$. So $Ref^* \subseteq \overline{K}$. Since Ref^* is r.e., $Ref^* = W_m$ for some m.

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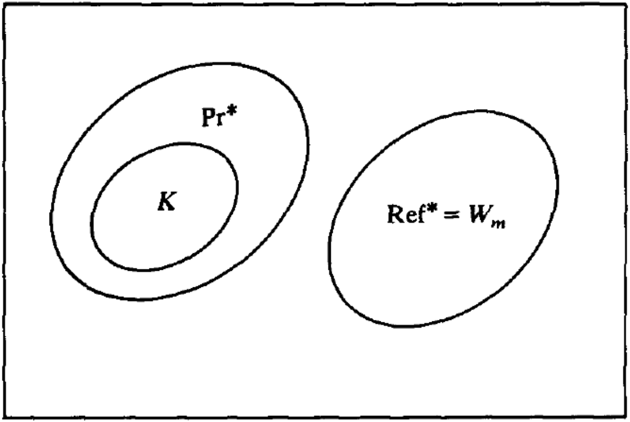
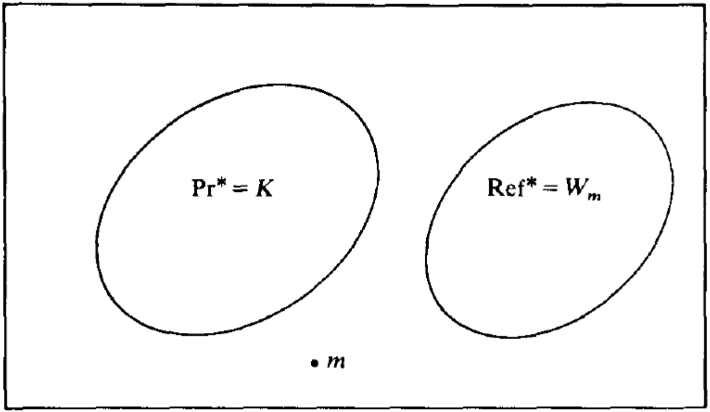
By the productiveness of $\overline{K}$, $m \in \overline{K} \setminus Ref^*$.
In particular $m \notin Ref^*$ means that $\mathsf{m} \notin \mathsf{K}$ is not provable. Let σ be $\mathsf{m} \notin \mathsf{K}$.

Illumination



Proof (2)

(ii) $Pr^* = K$ by ω-consistency. Now $m \notin K$ implies $m \notin Pr^*$ implies that $\mathsf{m} \in \mathsf{K}$ is not provable. Hence $\neg\sigma$ is not provable.



Rosser's Improvement

Rosser's contribution was to remove the ω-consistency condition.

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Let $K_0 = \{x \mid \phi_x(x) = 0\}$ and $K_1 = \{x \mid \phi_x(x) = 1\}$.

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Both K_0 and K_1 are r.e.

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Two disjoint sets A, B are recursively inseparable if there is no recursive set C such that $A \subseteq C$ and $B \subseteq \overline{C}$.

Two disjoint sets A, B are recursively inseparable iff whenever $A \subseteq W_a, B \subseteq W_b$ and $W_a \cap W_b = \emptyset$ then there is a number $x \notin W_a \cup W_b$.

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Fact. K_0 and K_1 are recursively inseparable.

Proof. Suppose such C existed. Let m be an index of the characteristic function of C. A contradiction can be easily derived.

Rosser's Improvement

Let decidable predicates $R_0(x, y), R_1(x, y)$ be such that

- $n \in K_0$ iff there is y such that $R_0(n, y)$.
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Clearly $K_0 \cap K_1 = \emptyset$. So we also have

- $n \in K_0$ iff there is y such that $R_0(n,y)$ and, for all $z \leq y$, $R_1(n,z)$ does not hold.

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Let $\sigma_{R_0}(\mathsf{x},\mathsf{y}), \sigma_{R_1}(\mathsf{x},\mathsf{y})$ be the counterparts of $R_0(x,y), R_1(x,y)$ in Peano arithmetic.

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Rosser's trick was to use the following statement

$$\exists \mathsf{y}.(\sigma_{R_0}(\mathsf{n},\mathsf{y}) \wedge \forall \mathsf{z} \leq \mathsf{y}.\neg\sigma_{R_1}(\mathsf{n},\mathsf{z}))$$

for the formal counterpart of $n \in K_0$, denoted by $\mathsf{n} \in \mathsf{K}_0$. Similarly we write $\mathsf{n} \in \mathsf{K}_1$ for

$$\exists \mathsf{y}.(\sigma_{R_1}(\mathsf{n},\mathsf{y}) \wedge \forall \mathsf{z} \leq \mathsf{y}.\neg\sigma_{R_0}(\mathsf{n},\mathsf{z})).$$

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Lemma. For each natural number n, the following are valid in Peano arithmetic.

1. If $n \in K_0$ then $\mathsf{n} \in \mathsf{K}_0$ is provable.
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Proof. Suppose $\mathsf{n} \in \mathsf{K}_1$ is provable. Then some m exists such that

$$\sigma_{R_1}(\mathsf{n}, \mathsf{m}) \wedge \forall \mathsf{z} \leq \mathsf{m}.\neg\sigma_{R_0}(\mathsf{n}, \mathsf{z})$$

is provable. In other words, $\sigma_{R_1}(\mathsf{n}, \mathsf{m}), \neg\sigma_{R_0}(\mathsf{n}, \mathsf{0}), \ldots, \neg\sigma_{R_0}(\mathsf{n}, \mathsf{m})$ are provable. It follows that

$$\forall \mathsf{y}.(\neg\sigma_{R_0}(\mathsf{n}, \mathsf{y}) \vee \exists \mathsf{z} \leq \mathsf{y}.\sigma_{R_1}(\mathsf{n}, \mathsf{z}))$$

is provable.

Rosser's Improvement

Theorem. (Gödel-Rosser Incompleteness Theorem)
There is a statement τ such that if Peano arithmetic is consistent, then neither τ nor $\neg\tau$ is provable.

Rosser's Improvement

Proof. Define

$$\begin{aligned} Pr^{**} &= \{n \mid \mathsf{n} \in \mathsf{K}_0 \text{ is provable}\}, \\ Ref^{**} &= \{n \mid \mathsf{n} \in \mathsf{K}_0 \text{ is refutable}\} = \{n \mid \mathsf{n} \notin \mathsf{K}_0 \text{ is provable}\}. \end{aligned}$$

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So we may let τ be $p \in \mathsf{K}_0$.

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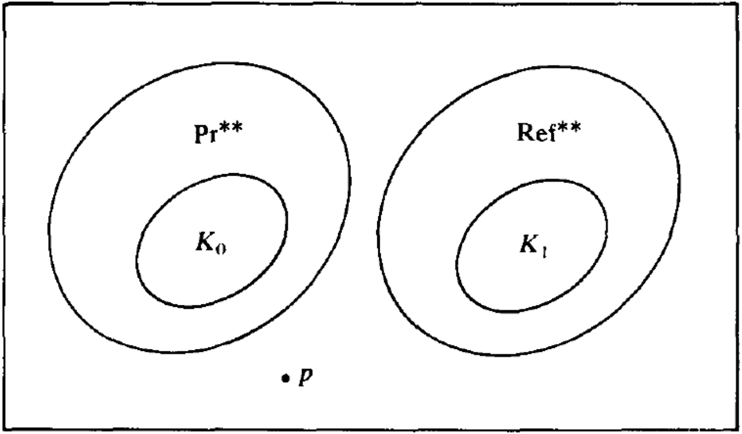

Outline

1. Formal Arithmetic

2. Gödel's Incompleteness Theorem
   - Simplified Gödel Theorem
   - Gödel's Incompleteness Theorem
   - Rosser's Refinement

3. Undecidability

Undecidability

When considering sets of statements the terms decidable and undecidable are often used to mean recursive and non-recursive.

Undecidability

Theorem. Suppose that $(\mathscr{A}, \mathscr{D})$ is an ω-consistent formal system of arithmetic in which all decidable predicates are representative. Then the set of provable statements is creative.

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So Pr is creative.

Undecidability

Corollary. If Peano arithmetic is ω-consistent, then the provable statements form a creative set.

Provability is creative.

Truth (validity) is productive.