Lab09-Recursively Enumerable Set(2)

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class

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1. Suppose A is an r.e. set. Prove the following statements.

   (a) Show that the sets $\bigcup_{x\in A} W_x$ and $\bigcup_{x\in A} E_x$ are both r.e.

   (b) Show that $\bigcap_{x\in A} W_x$ is not necessarily r.e..

   Solution.

   (a) We know that :

   $$y \in \bigcup_{x\in A} W_x \Leftrightarrow \exists x(x \in A \wedge y \in W_x)$$

   Since "$x \in A \wedge y \in W_x$" is partial decidable, the right part should also be partial decidable. Hence the set $\bigcup_{x\in A} W_x$ is r.e..

   Similarly, we have:

   $$y \in \bigcup_{x\in A} E_x \Leftrightarrow \exists x(x \in A \wedge y \in E_x)$$

   Thus the set $\bigcup_{x\in A} E_x$ is also r.e..

   (b) Let $K_t = \{x : P_x(x)\downarrow \text{ in t steps}\}$. Obviously, K_t is recursive. Then we define a function:

   $$f(t,x) = \begin{cases} \uparrow, & \text{if } P_x(x)\downarrow \text{in t steps },\\ 1, & \text{otherwise }. \end{cases} \tag{0.1}$$

   Based on s-m-n theorem, there exists a total computable function $m(t)$ that $W_{m(t)} = \overline{K_t}$. Additionally, according to the **List Theorem** $range(m)$ is an r.e. set. Then we have $\bigcap_{x\in range(m)} W_x = \bigcap_{m(t)\in range(m)} W_{m(t)} = \bigcap_{t\in\mathbb{N}} W_{m(t)} = \bigcap_{t\in\mathbb{N}} \overline{K_t} = \overline{K}$ which is not r.e., hence we find a counterexample that $\bigcap_{x\in A} W_x$ is not r.e..

   □

2. Prove that $A \subseteq \mathbb{N}^n$ is r.e. iff $A = \varnothing$ or there is a total computable function $f : \mathbb{N} \to \mathbb{N}^n$ such that $A = Ran(\boldsymbol{f})$. (A *computable function* $\boldsymbol{f}$ from $\mathbb{N}$ to $\mathbb{N}^n$ is an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$ where each f_i is a unary computable function and $\boldsymbol{f}(x) = (f_1(x), \ldots, f_n(x))$.)

   Solution. In the homework last week, we have proved that:

   $$A \subseteq \mathbb{N}^n \text{ is r.e.} \quad\Leftrightarrow\quad B = \{2^{x_1}3^{x_2}\ldots p_n^{x_n} : (x_1,\ldots,x_n) \in A\} \text{ is r.e.}$$

   By Listing Theorem,

   $$\begin{aligned} B \text{ is r.e.} \quad &\Leftrightarrow \quad \text{either } B = \emptyset \text{ or } B \text{ is the range of a unary total computable function.}\\ &\Leftrightarrow \quad B = \emptyset \text{ or there exists a total computable function } g,\ B = Ran(g)\\ &\Leftrightarrow \quad A = \emptyset \text{ or } A = Ran(\boldsymbol{f}) \end{aligned}$$

where $\boldsymbol{f} = ((g)_1, (g)_2, \dots, (g)_n)$ and it is a total computable function. Therefore, $A \subseteq \mathbb{N}^n$ is r.e. iff $A = \varnothing$ or there is a total computable function $\boldsymbol{f} : \mathbb{N} \to \mathbb{N}^n$ such that $A = Ran(\boldsymbol{f})$.

□

3. Suppose that f is a total computable function, A is a recursive set and B is an r.e.set. Show that $f^{-1}(A)$ is recursive and that $f(A)$, $f(B)$ and $f^{-1}(B)$ are r.e. but not necessarily recursive. What extra information about these sets can be obtained if f is a bijection?

   Solution. We have:

$$\begin{aligned} x \in f(A) &\Leftrightarrow \exists y(y \in A \land x = f(y)) \\ x \in f^{-1}(A) &\Leftrightarrow f(x) \in A \end{aligned}$$

   Since $f(x)$ is a total computable function, it is obvious that $f(A)$ is r.e. and $f^{-1}(A)$ is recursive. Similarly, we can get that $f(B)$ is r.e. and $f^{-1}(B)$ is also r.e..

   According to the **Equivalence Theorem**, for any r.e. set A, there exists a total computable function whose range is exactly A. Thus there exists a total computable function g_1 whose range is K. Additionally, we can define another total computable funtion $g_2(x) = x$. Then let $A = \mathbb{N}, B = K$, we can see that $g_1(A) = K$ and $g_2(B) = K, g_2^{-1}(B) = K$ are all not recursive.

   If f is a bijection, then f^{-1} is also a total computable function. Therefore:

$$x \in f(A) \Leftrightarrow f^{-1}(x) \in A$$

   Thus $f(A)$ is recursive.

□

4. A set D is the difference of r.e. sets (*d.r.e.*) iff $D = A - B$ where A, B are both *r.e.*.

   (a) Show that the set of all *d.r.e.* sets is closed under the formation of intersection.

   (b) Show that if $C_n = \{x \mid |W_x| = n\}$, then C_n is *d.r.e.* for all $n \geq 0$.

   Solution.

   (a) Assume any two *d.r.e.* sets $D_1 = A_1 - B_1, D_2 = A_2 - B_2$ where A_1, A_2, B_1, B_2 are all r.e. sets.

$$\begin{aligned} D_1 \cap D_2 &= (A_1 - B_1) \cap (A_2 - B_2) \\ &= (A_1 \cap \overline{B_1}) \cap (A_2 \cap \overline{B_2}) \\ &= (A_1 \cap A_2) \cap (\overline{B_1} \cap \overline{B_2}) \\ &= (A_1 \cap A_2) - \overline{(\overline{B_1} \cap \overline{B_2})} \\ &= (A_1 \cap A_2) - (B_1 \cup B_2) \end{aligned}$$

   Thus $D_1 \cap D_2$ is also an r.e. set.

   (b) Let $T_n = \{x \mid |W_x| \geq n\}$. Since "$x \in T_n$" $\Leftrightarrow$ "$\exists x_1 \exists x_2 \dots \exists x_n (x_i \in W_x$ for any $i \leq n \land x_i \neq x_j$ for any $i \neq j$)", T_n is an r.e. set.

   Since $C_n = T_n - T_{n+1}$, according to the definition, C_n is a *d.r.e* set.

□