Lab10-Various Sets

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

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1. Prove the following statements.

   (a) If B is r.e. and $A \cap B$ is productive, then A is productive.

   (b) If C is creative and A is an r.e. set such that $A \cap C = \varnothing$, then $C \cup A$ is creative.

Solution. (a) We can define

$$f(x,y) = \begin{cases} 1 & \text{if } y \in W_x \wedge y \in B \\ undefined & \text{otherwise.} \end{cases}$$

According to the partial decidability of '$y \in W_x$' and '$x \in B$', $f(x,y)$ is computable. The s-m-n theorem provides a total computable function $k(x)$ such that $f(x,y) \simeq \phi_{k(x)}(y)$, and $W_{k(x)} = W_x \cap B$. For whenever $W_x \subseteq A$, we have

$$\begin{aligned} W_x \subseteq A &\Rightarrow W_{k(x)} \subseteq A \cap B \Rightarrow g(k(x)) \in A \cap B \backslash W_{k(x)} \\ &\Rightarrow g(k(x)) \in A \cap B \backslash (W_x \cap B) = (A \backslash W_x) \cap B \\ &\Rightarrow g(k(x)) \in A \backslash W_x \end{aligned}$$

where g is the productive function of $A \cap B$. Hence, $g(k(x))$ is the productive function of A. Therefore, A is productive.

(b) For both A and C are r.e., $A \cup C$ is r.e. according to Theorem 7-2.13. We can define

$$f(x,y) = \begin{cases} 1 & \text{if } y \in W_x \vee y \in A \\ \uparrow & \text{otherwise.} \end{cases}$$

According to the partial decidability of '$y \in W_x$' and '$x \in A$', $f(x,y)$ is computable. The s-m-n theorem provides a total computable function $k(x)$ such that $f(x,y) \simeq \phi_{k(x)}(y)$.

$$W_{k(x)} = W_x \cup A$$

For whenever $W_x \subseteq \overline{C} \cap \overline{A}$, we have

$$\begin{aligned} W_x \subseteq \overline{C} \cap \overline{A} &\Rightarrow W_x \subseteq \overline{C} \Rightarrow W_x \cup A \subseteq \overline{C} \cup A \Rightarrow W_{k(x)} \subseteq \overline{C} \\ &\Rightarrow g(k(x)) \in \overline{C} \backslash (A \cup W_x) \Rightarrow g(k(x)) \in \overline{C} \cap \overline{A} \backslash W_x \end{aligned}$$

where g is the productive function of $\overline{C}$. Hence, $g(k(x))$ is the productive function of $\overline{C} \cap \overline{A}$. Therefore, $C \cup A$ is creative.

□

2. Let $\mathscr{B}$ be a set of unary computable functions, and suppose that $g \in \mathscr{B}$ is such that for all finite $\theta \subseteq g$, $\theta \notin \mathscr{B}$. Prove that the set $\{x \mid \phi_x \in \mathscr{B}\}$ is productive.

Solution. Let $B = \{x \mid \phi_x \in \mathscr{B}\}$. Define $f(x, y)$ as

$$f(x,y) = \begin{cases} g(y) & \text{if } \neg H_n(x,x,y), \\ \text{undefined} & \text{otherwise,} \end{cases}$$

According to s-m-n theorem, there is a total computable function $k(x)$, such that $\phi_{k(x)}(y) \simeq f(x, y)$. Then

$$x \in W_x \Leftrightarrow \phi_{k(x)} \text{ is finite } \Leftrightarrow k(x) \notin B$$
$$x \notin W_x \Leftrightarrow \phi_{k(x)} = g \Leftrightarrow k(x) \in B$$

Therefore, $x \in \overline{K} \Leftrightarrow k(x) \in B$. Since $\overline{K}$ is productive, B is productive. □

3. If $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$, $A \otimes B = \{\pi(x, y) \mid x \in A \text{ and } y \in B\}$, prove the following statements.

   (a) Suppose B is r.e. If A is creative, then so are $A \oplus B$ and $A \otimes B$ (provided $B \neq \varnothing$).

   (b) If B is recursive, then the implications in (a) reverse.

   (c) If A, B are simple sets, prove that $A \otimes B$ is not simple but that $\overline{\overline{A} \otimes \overline{B}}$ is simple.

Solution. (a) i. Since A is creative, $\overline{A}$ is productive.

$$\begin{cases} x \in A \oplus B \Leftrightarrow \dfrac{x}{2} \in A, & \text{If } x \text{ is an even number,} \\ x \in A \oplus B \Leftrightarrow \dfrac{x-1}{2} \in B, & \text{If } x \text{ is an odd number.} \end{cases}$$

Since A and B are both r.e., $A \oplus B$ is also r.e.

$$x \in \overline{A} \Leftrightarrow x \notin A \Leftrightarrow 2x \notin A \oplus B \Leftrightarrow 2x \in \overline{A \oplus B}$$

Since $\overline{A}$ is productive, $\overline{A \oplus B}$ is productive according to Theorem 7-3.2. Therefore, $A \oplus B$ is creative.

ii. By the definition of $A \otimes B$, $z \in A \otimes B \Leftrightarrow \pi_1(z) \in A \wedge \pi_2(z) \in B$. Since both A and B are r.e., $A \otimes B$ is also r.e.

For $B \neq \varnothing$, randomly pick y from B, such that

$$x \in \overline{A} \Leftrightarrow x \notin A \Leftrightarrow \pi(x, y) \notin A \otimes B \Leftrightarrow \pi(x, y) \in \overline{A \otimes B}.$$

Since $\overline{A}$ is productive, $\overline{A \otimes B}$ is productive according to Theorem 7-3.2. Therefore, $A \otimes B$ is creative.

(b) i. If $A \oplus B$ is creative:

$$x \in A \Leftrightarrow 2x \in A \oplus B$$

$A \oplus B$ is r.e., therefore A is r.e..
Suppose g is the productive function of $\overline{A \oplus B}$. Define

$$f(x,y) = \begin{cases} 1 & \text{if } y \text{ is even and } y/2 \in W_X \\ 1 & \text{if } y \text{ is odd and } (y-1)/2 \notin B \\ \text{undefined} & \text{otherwise,} \end{cases}$$

There exists a total computable function $k(x)$ such that $\phi_{k(x)} \simeq f(x, y)$ by the s-m-n theorem. For any $W_x \subseteq \overline{A}$, we have $W_{k(x)} \subseteq \overline{A \oplus B}$ and $g(k(x)) \in \overline{A \oplus B} \setminus W_{k(x)} = \{x \mid x \text{ is even and } x/2 \in \overline{A} \text{ and } x/2 \notin W_x\}$. Therefore $g(k(x))/2 \in \overline{A} \setminus W_x$, which is a productive function for $\overline{A}$.
Now we can conclude A is creative.

ii. If $A \otimes B$ is creative:
Choose an arbitrary number $y_0 \in B$, we have

$$x \in A \Leftrightarrow \pi(x, y_0) \in A \otimes B$$

$A \otimes B$ is r.e., therfore A is r.e.
Choose an arbitrary number $y_1 \notin B$, define

$$f(x, y) = \begin{cases} 1 & \text{if } \pi_1(y) \in W_x \text{ or } \pi_2(y) \notin B \\ \text{undefined} & \text{otherwise,} \end{cases}$$

There exists a total computable function $k(x)$ such that $\phi_{k(x)} \simeq f(x, y)$ by the s-m-n theorem. For any $W_x \subseteq \overline{A}$, we have $W_{k(x)} \subseteq \overline{A \otimes B}$ and $g(k(x)) \in \overline{A \otimes B} \setminus W_{k(x)} = \overline{A \otimes B} \setminus \overline{\overline{W_x} \otimes B}$. $\pi_1(g(k(x))) \in \overline{A} \setminus W_x$, which is a productive function for $\overline{A}$.
Now we can conclude A is creative.

(c) i. Since $\overline{A}$ is infinite, there is an $a \notin A$.
Since both $\{a\}$ and B are r.e., we have $\{a\} \otimes B \subset \overline{A \otimes B}$ is r.e. On the other hand, since B is not recursive, it must be infinite. Thus $|\{a\} \otimes B| = |B|$ is also infinite. Therefore, $\overline{A \otimes B}$ has an infinite r.e. subset, indicating it is not simple.

ii. By definition, $\overline{\overline{A} \otimes \overline{B}} = \{\pi(x, y) | x \in A \vee y \in B\}$, then $x \in \overline{\overline{A} \otimes \overline{B}} \Leftrightarrow \pi_1(x) \in A \vee \pi_2(x) \in B$, which is partially decidable since both A and B are r.e.
Therefore $\overline{\overline{A} \otimes \overline{B}}$ is r.e. And since $\overline{A}$ and $\overline{B}$ are both infinite, $\overline{A} \otimes \overline{B}$ is infinite. Furthermore, if there is a r.e. set $C \subseteq \overline{A} \otimes \overline{B}$, then $\pi_1(C) \subseteq \overline{A}$ and $\pi_2(C) \subseteq \overline{B}$, where $|\pi_1(C)| * |\pi_2(C)| = |C|$. Thus either $\pi_1(C)$ and $\pi_2(C)$ is an infinite set which leads to a contradiction.
Hence $\overline{A} \otimes \overline{B}$ contains no r.e. subset. Therefore, $\overline{\overline{A} \otimes \overline{B}}$ is simple.

□