Lab11-Solution

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

1. Recall that $A \otimes B = \{\pi(a,b) \mid a \in A, b \in B\}$. Prove the following statements.

   (a) For any sets A, B, if $B \neq \varnothing$ then $A \leq_m A \otimes B$.

   (b) $A \equiv_m A \otimes \mathbb{N}$ for any set A,

   (c) $A \equiv_m A \otimes B$ if $A \neq \mathbb{N}$ and B is a non-empty recursive set.

   Solution.

   (a) Let $b_0 \in B$, then $a \in A \Leftrightarrow \pi(a, b_0) \in A \otimes B$, so $A \leq_m A \otimes B$.

   (b)
   - '$A \leq_m A \otimes \mathbb{N}$' has been proved in 1a;
   - $x \in A \otimes \mathbb{N} \iff \pi_1(x) \in A$. Thus $A \otimes \mathbb{N} \leq_m A$.

   (c) Since '$A \leq_m A \otimes B$' has been proved in 1a, we only need to show that $A \otimes B \leq_m A$. To prove this, let $a_0 \notin A$, then define the following function

   $$f(x) = \begin{cases} \pi_1(x) & \pi_2(x) \in B \\ a_0 & \text{otherwise} \end{cases}$$

   $$x \in A \otimes B \Leftrightarrow f(x) \in A$$

   Thus $A \otimes B \leq_m A$.

   □

2. Let $\mathbf{a}$, $\mathbf{b}$ be m-degrees.

   (a) Show that the least upper bound of $\mathbf{a}$, $\mathbf{b}$ is uniquely determined; denote this by $\mathbf{a} \cup \mathbf{b}$;

   (b) Show that if $\mathbf{a} \leq_m \mathbf{b}$ then $\mathbf{a} \cup \mathbf{b} = \mathbf{b}$;

   (c) Show that if $\mathbf{a}$,$\mathbf{b}$ are r.e., then so is $\mathbf{a} \cup \mathbf{b}$;

   (d) Let $A \in \mathbf{a}$ and let $\mathbf{a}^*$ denote $d_m(\overline{A})$. (Check that $\mathbf{a}^*$ is independent of the choice of $A \in \mathbf{a}$.) Show that $(\mathbf{a} \cup \mathbf{a}^*)^* = \mathbf{a} \cup \mathbf{a}^*$.

   Solution.

   (a) Any pair of m-degrees $\mathbf{a}, \mathbf{b}$ have a least upper bound, denoted by $\mathbf{c}$. Suppose there is another m-degree $\mathbf{d}$, which is also a least upper bound of $\mathbf{a}, \mathbf{b}$. Then we have $\mathbf{c} \leq_m \mathbf{d}$ and $\mathbf{d} \leq_m \mathbf{c}$ by the properties of least upper bound, i.e. $\mathbf{c} \equiv_m \mathbf{d}$. Therefore, from the definition of m-degree, we have $\mathbf{c} = \mathbf{d}$.

   (b) Apparently we have $\mathbf{a} \leq_m \mathbf{b}$ and $\mathbf{b} \leq_m \mathbf{b}$. Furthermore, for any upper bound $\mathbf{c}$ of $\mathbf{a}, \mathbf{b}$, we have $\mathbf{b} \leq_m \mathbf{c}$. Then from the definition of the least upper bound, $\mathbf{b}$ is a least upper bound of $\mathbf{a}, \mathbf{b}$.

   (c) Since $\mathbf{a}$,$\mathbf{b}$ are r.e., then we have $\mathbf{a} \leq_m K$ and $\mathbf{b} \leq_m K$, indicating that K is an upper bound of $\mathbf{a}$,$\mathbf{b}$. Hence by definition, $\mathbf{a} \cup \mathbf{b} \leq_m K$. Therefore $\mathbf{a} \cup \mathbf{b}$ is r.e.

   (d) Let $A \in \mathbf{a}$ and $B \in \mathbf{a} \cup \mathbf{a}^*$, and $\overline{A} \in \mathbf{a}^*$. Then we have $A \leq_m B$ and $\overline{A} \leq_m B$. By the theorem that "$A \leq_m B$ iff. $\overline{A} \leq_m \overline{B}$". We have $\overline{A} \leq_m \overline{B}$ and $A \leq_m \overline{B}$. Thus $d_m(\overline{B})$ is a upper bound of $\mathbf{a} \cup \mathbf{a}^*$. From the definition of least upper bound, we could get $B \leq_m \overline{B}$. Then we apply the theorem again, we get $\overline{B} \leq_m B$. Hence $B \equiv_m \overline{B}$, i.e. $(\mathbf{a} \cup \mathbf{a}^*)^* = \mathbf{a} \cup \mathbf{a}^*$.

$\square$

3. Show that the following sets all belong to the same m-degree:
   (a) $\{x \mid \phi_x = 0\}$,
   (b) $\{x \mid \phi_x \text{ is total and constant}\}$,
   (c) $\{x \mid W_x \text{ is infinite}\}$.

   Solution.

   - $(a) \Rightarrow (b)$

     We construct the function given as
     $$f(x,y) = \begin{cases} c & \text{if } \phi_x(y) = 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$
     Since $f(x,y)$ is obviously computable, there is a total computable function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.
     Consequently, $\phi_x = 0 \Leftrightarrow \phi_{k(x)} = c$.
     And therefore, $\{x \mid \phi_x = 0\} \leq_m \{x \mid \phi_x \text{ is total and constant}\}$.
   - $(b) \Rightarrow (c)$

     We construct the function given as
     $$f(x,y) = \begin{cases} 1 & \text{if } \phi_x(0)\downarrow \wedge H(x,z,\phi_x(0)) \text{ for any } 0 \leq z \leq y \\ \text{undefined} & \text{otherwise} \end{cases}$$
     Since $f(x,y)$ is obviously computable, there is a total computable function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.
     Hence, it is clear that ϕ_x is total and constant$\Rightarrow W_{k(x)}$ is infinite.
     And if ϕ_x is not total or not constant, then there is a z such that $\phi_x(z)\uparrow$ or $\phi_x(z) \neq \phi_x(0)$ and therefore $\forall t \in W_{k(x)}$, $t < z$. Thus, $W_{k(x)}$ is not infinite.
     Therefore, ϕ_x is total and constant $\Leftrightarrow W_{k(x)}$ is infinite.
     And $\{x \mid \phi_x \text{ is total and constant }\} \leq_m \{x \mid W_x \text{ is infinite}\}$.
   - $(c) \Rightarrow (a)$

     Consider the function shown below:
     $$f(x,y) = \begin{cases} 0 & \text{if } \exists z > y(\phi_x(z)\downarrow) \\ \text{undefined} & \text{otherwise} \end{cases}$$
     Obviously, $f(x,y)$ is computable.
     Then, there is a total computable function $k(x)$ such that $\phi_{k(x)} \simeq f(x,y)$.
     And it is obvious that W_x is infinite $\Rightarrow \phi_{k(x)} = 0$.
     On the other hand, if W_x is finite, then there is a z such that $\forall t \in W_x$, $t \leq z$ and therefore $\phi_{k(x)}(z)$ is undefined. Thus, $\phi_{k(x)} \neq 0$.
     Hence, W_x is infinite $\Leftrightarrow \phi_{k(x)} = 0$.
     Therefore, $\{x \mid W_x \text{ is infinite}\} \leq_m \{x \mid \phi_x = 0\}$.
     And since $\leq_m$ is transitive, we can conclude that
     $$\{x \mid \phi_x = 0\} \equiv_m \{x \mid \phi_x \text{ is total and constant}\} \equiv_m \{x \mid W_x \text{ is infinite}\}.$$

$\square$