Gödel's Incompleteness Theorem*

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Outline

Gödel's Incompleteness Theorem, 1931.

Logic, arithmetic, and recursion theory.

Formalization of Arithmetic

Definition. The formalization of arithmetic is specifying an adequate formal logical language L and making statements of ordinary arithmetic of the natural numbers (First order logic with equality).

Functional symbols (alphabet): $0, 1, +, \times, =$.

Logical notions: $\neg$ (not); $\wedge$ (and); $\vee$ (or); $\rightarrow$ (implies); $\forall$ (for all); $\exists$ (exists).

Variables: $x, y, z, \ldots$.

Other symbols: brackets (and), $\neq$, etc.

Statements (Formulas)

Definition. The statements (formulas) of L are the meaningful finite sequences of symbols from the alphabet of L.

Example 1. $\exists \mathsf{y}(\mathsf{y} \times (1+1) = \mathsf{x})$ is the formal counterpart of the informal statement 'x is even'.

Example 2. $\exists \mathsf{y}(\mathsf{y} \times 2 = 5)$ is the formal expression of the false informal statement '5 is even'.

Example 3. $\exists \mathsf{z}(\neg(\mathsf{z} = 0) \wedge (\mathsf{y} + \mathsf{z} = \mathsf{x}))$ asserts '$\mathsf{x} > \mathsf{y}$'.

Example 4.
$(\mathsf{x} \neq 0) \wedge (\mathsf{x} \neq 1) \wedge \forall \mathsf{y} \forall \mathsf{z}.(\mathsf{x} = \mathsf{y} \times \mathsf{z} \to (\mathsf{y} = 1 \vee \mathsf{z} = 1))$ asserts 'x is a prime'.

The standard model $\mathbb{N}$: the ordinary arithmetic.

Formal Arithmetic

Define:

$\mathscr{S}$ be the set of all possible meaningful statements.

$\mathscr{T}$ be the set of all statements that are true in the ordinary arithmetic on $\mathbb{N}$.

$\mathscr{F}$ be the set of all statements that are false in the ordinary arithmetic on $\mathbb{N}$.

Notice. *true* and *false* are model theoretical concepts.

$$\text{Thus } \mathscr{T} \cap \mathscr{F} = \varnothing \text{ and } \mathscr{T} \cup \mathscr{F} = \mathscr{S}.$$

Discovery

Consider two questions:

1. Is $\mathscr{T}$ recursive or recursively enumerable?

2. Is there a recursive subset of $\mathscr{T}$ (a set of axioms) from which all other statements in $\mathscr{T}$ can be derived?

Standard Coding

It is straightforward to assign a Gödel number to every member of $\mathscr{S}$ in a uniform manner.

$$\mathscr{S} = \{\theta_0, \theta_1, \theta_2, \ldots\}.$$

We can use it to code any set of statements $\mathscr{X}$ by the set of number $\mathbf{X} = \{n \mid \theta_n \in \mathscr{X}\}$.

$$\text{We say that } \mathscr{X} \text{ is } \left\{ \begin{array}{l} \textit{recursive} \\ \textit{r.e.} \\ \textit{productive} \\ \textit{creative} \\ \textit{etc.} \end{array} \right\} \text{ if } \mathbf{X} \text{ is } \left\{ \begin{array}{l} \text{recursive} \\ \text{r.e.} \\ \text{productive} \\ \text{creative} \\ \text{etc.} \end{array} \right\}$$

Gödel's Lemma

Lemma. Suppose $M(x_1, \ldots, x_n)$ is a decidable predicate. Then it is possible to construct a statement $\sigma(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ that is a formal counterpart of $M(x_1, \ldots, x_n)$ in the following sense: for all $a_1, \ldots, a_n \in \mathbb{N}$, $M(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ holds iff $\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n) \in \mathscr{T}$.

Applying Gödel's Lemma

$x \in K$ iff $\exists y.R(x, y)$ for some decidable predicate $R(x, y)$.

Let $\sigma_R(\mathsf{x}, \mathsf{y})$ be a particular formal counterpart of $R(x, y)$.

Then $\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ is a formal counterpart of $n \in K$, and $\neg\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ is a formal counterpart of $n \notin K$.

Write $\mathsf{n} \in \mathsf{K}$ for $\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ and $\mathsf{n} \notin \mathsf{K}$ for $\neg\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$.

By previous lemma, $n \in K$ iff $\mathsf{n} \in \mathsf{K} \in \mathscr{T}$, and $n \notin K$ iff $\mathsf{n} \notin \mathsf{K} \in \mathscr{T}$.

Transform Lemma

Lemma. There is a total computable function g such that for all n, $\theta_{g(n)}$ is $\mathsf{n} \notin \mathsf{K}$.

Proof. Given n we can effectively write down the statement $\neg\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$, from which we can construct the index $g(n)$.

First Conclusion

Theorem. $\mathscr{T}$ is productive.

Proof. Let $\mathbb{T} = \{n \mid \theta_n \in \mathscr{T}\}$ and g be the function given in the previous lemma. We have

$$\begin{aligned} n \in \overline{K} &\Leftrightarrow n \notin K \\ &\Leftrightarrow \mathsf{n} \notin \mathsf{K} \in \mathscr{T} \\ &\Leftrightarrow g(n) \in \mathbb{T}. \end{aligned}$$

Since $\overline{K}$ is productive, so is $\mathbb{T}$.

Formal System

A formal system $(\mathscr{A}, \mathscr{D})$ consists of a set $\mathscr{A} \subseteq \mathscr{S}$ (the axioms) and an explicit definition $\mathscr{D}$ of the notion of a formal proof of a statement in $\mathscr{S}$ from the axioms, satisfying the conditions:

- Proofs are finite objects.
- Provability is decidable if $\mathscr{A}$ is recursive.

'p is a proof of the statement σ from the axioms $\mathscr{A}$'.

Interpretation

Is there a simple-minded subset of $\mathscr{T}$ (a set of axioms) from which all other statements in $\mathscr{T}$ can be proved?

$\Longleftrightarrow$

Is there a formal system $(\mathscr{A}, \mathscr{D})$ for L such that

1. $\mathscr{A}$ is recursive, and
2. the provable statements are precisely those in $\mathscr{T}$?

The second condition relates provability to validity.

2.1 Consistency: There is no statement σ such that both σ and $\neg\sigma$ are provable;

2.2 Completeness: For any statement σ, either σ is provable or $\neg\sigma$ is provable.

Judgements

A simplified version of Gödel's theorem shows that there is no formal system of arithmetic satisfying above conditions (1) and (2).

The full theorem of Gödel together with its improvement by Rosser shows that there is no formal system of arithmetic (of a certain minimal strength) satisfying conditions (1), (2.1) and (2.2).

Any consistent formal system of arithmetic having a recursive set of axioms is incomplete.

Lemma to Establish Gödel Theorem

Lemma. In any recursively axiomatized formal system the set Pr of provable statements is r.e.

Proof. Since proofs are finite, they can be effectively numbered.

If $\mathscr{A}$ is a recursive set of axioms, then the following predicate $M(x,y)$ is decidable according to the definition of formal system:

'y is the number of a proof of θ_x from the axioms $\mathscr{A}$.'

It follows that θ_x is provable iff $\exists y.M(x,y)$ holds.

So the set Pr of provable statements is r.e.

Simplified Gödel Theorem

Theorem. Suppose that $(\mathscr{A}, \mathscr{D})$ is a recursively axiomatized formal system in which all provable statements are true. Then there is a statement σ that is true but not provable (and consequently $\neg\sigma$ is not provable either).

Proof. $\mathscr{P}$ is r.e. and $\mathscr{T}$ is productive. Since $\mathscr{P} \subseteq \mathscr{T}$, there is some $\sigma \in \mathscr{T} \setminus \mathscr{P}$.

Clearly $\neg\sigma$ is not provable, otherwise $\neg\sigma$ would be true.

Such σ can be found effectively from a specification of the formal system (yield an index for $\mathscr{P}$).

Refutable and Provable

A statement is refutable if its negation is provable.

Define

$$\begin{aligned} Pr^* &= \{n \mid \mathsf{n} \in \mathsf{K} \text{ is provable}\}, \\ Ref^* &= \{n \mid \mathsf{n} \in \mathsf{K} \text{ is refutable}\} \\ &= \{n \mid \mathsf{n} \notin \mathsf{K} \text{ is provable}\} \\ &= \{n \mid \theta_{g(n)} \in Pr\}. \end{aligned}$$

Incompleteness

1. $Pr^* \subseteq K$ and $Ref^* \subseteq \overline{K}$ since provable statements are true by assumption.

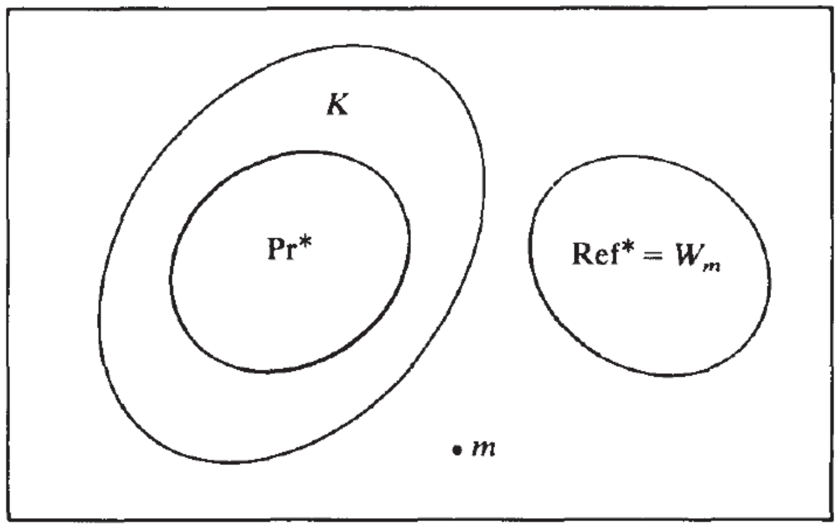
2. It follows from $n \in Ref^* \Leftrightarrow \theta_{g(n)} \in Pr$ that Ref^* is r.e.

3. So $Ref^* = W_m$ for some m.

4. By the productiveness of $\overline{K}$, we have $m \in \overline{K} \setminus Ref^*$, $m \notin K$, and $\mathsf{m} \notin \mathsf{K}$ is not provable.

5. Let σ be $\mathsf{m} \notin \mathsf{K}$. Clearly σ is true but not provable.

Illumination



Non-provability of $\neg\sigma$

$Pr^* \subseteq K$: for $m \notin K$, so $m \notin Pr^*$, i.e., $\mathsf{m} \in \mathsf{K}$ is not provable.

By the rules of formal proof $\neg\mathsf{m} \notin \mathsf{K}$ $(\neg\sigma)$ is not provable.

Illustration

σ is the formal counterpart of the statement $m \notin K$ $(m \notin W_m)$.

σ states that "I am true but not provable".

$$
\begin{array}{rcl}
\mathsf{m} \notin \mathsf{K} \in \mathscr{T} & \Leftrightarrow & m \notin W_m \\
 & \Leftrightarrow & m \notin K, \\
m \notin Ref^* & \Leftrightarrow & \mathsf{m} \notin \mathsf{K} \text{ is not provable} \\
 & \Leftrightarrow & \mathsf{m} \notin \mathsf{K} \notin Pr.
\end{array}
$$

Notation

The previous result is about the relationship between provability and validity.

Gödel's Incompleteness Theorem can be stated without any reference to any model.

Peano Arithmetic

First Order Peano Axioms:

PA1	$\forall x.(\mathsf{s}(x) \neq 0)$
PA2	$\forall xy.(\mathsf{s}(x) = \mathsf{s}(y) \Rightarrow x = y)$
PA3	$\forall x.(x = 0 \vee \exists y.\mathsf{s}(y) = x)$
PA4	$\forall x.(x < \mathsf{s}(x))$
PA5	$\forall xy.(x < y \Rightarrow \mathsf{s}(x) \leq y)$
PA6	$\forall xy.(\neg(x < y) \Leftrightarrow y \leq x)$
PA7	$\forall xy.((x < y) \wedge (y < z) \Rightarrow x < z)$

Important Fact to Devote Gödel's Proof

Lemma. (Gödel) Let $M(x_1, \ldots, x_n)$ be a decidable predicate, then there is a statement $\sigma(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ in Peano arithmetic that satisfies the following properties: for any $a_1, \cdots, a_n \in \mathbb{N}$,

(i) If $M(a_1, \ldots, a_n)$ holds, then $\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ is provable.

(ii) If $M(a_1, \ldots, a_n)$ does not hold, then $\neg\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ is provable.

Corollary

Corollary. For any natural number n, if $n \in K$ then $\mathsf{n} \in \mathsf{K}$ is provable in Peano arithmetic.

Proof. Suppose $n \in K$. Then there is a natural number m such that $R(n, m)$ holds. By the above lemma $\sigma_R(\mathsf{n}, \mathsf{m})$ is provable. So $\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ is provable. Hence $\mathsf{n} \in \mathsf{K}$ is provable.

ω-consistent

A formal system is ω-consistent if there is no statement $\tau(\mathsf{y})$ such that all of the following are provable.

$$\exists \mathsf{y}.\tau(\mathsf{y}), \neg\tau(\mathsf{0}), \neg\tau(\mathsf{1}), \neg\tau(\mathsf{2}), \ldots$$

Another Lemma

Lemma. Suppose that Peano arithmetic is ω-consistent. Then for any natural number n, if $\mathsf{n} \in \mathsf{K}$ is provable then $n \in K$.

Proof. Suppose $n \notin K$. Then for each $m \in \mathbb{N}$ one has that $R(n, m)$ does not hold. So $\neg\sigma_R(\mathsf{n}, \mathsf{m})$ is provable for each m, which contradicts to the provability of $\exists \mathsf{y}.\sigma_R(\mathsf{n}, \mathsf{y})$ by ω-consistency.

Gödel's Incompleteness Theorem

Theorem (Gödel's Incompleteness Theorem, 1931).
There is a statement σ of Peano arithmetic such that
(i) If Peano arithmetic is consistent, then σ is not provable;
(ii) If Peano arithmetic is ω-consistent, then $\neg\sigma$ is not provable

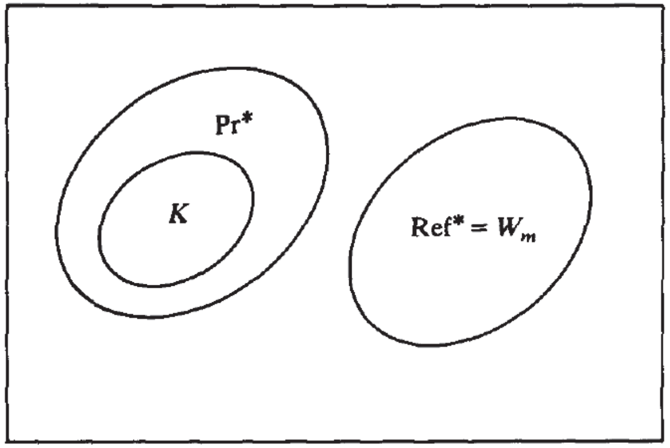
Proof. (i) Recall that

$$\begin{aligned} Pr^* &= \{n \mid \mathsf{n} \in \mathsf{K} \text{ is provable}\}, \\ Ref^* &= \{n \mid \mathsf{n} \in \mathsf{K} \text{ is refutable}\} \end{aligned}$$

By previous corollary, $K \subseteq Pr^*$. By consistency $Pr^* \cap Ref^* = \emptyset$. So $Ref^* \subseteq \overline{K}$. Since Ref^* is r.e., $Ref^* = W_m$ for some m.

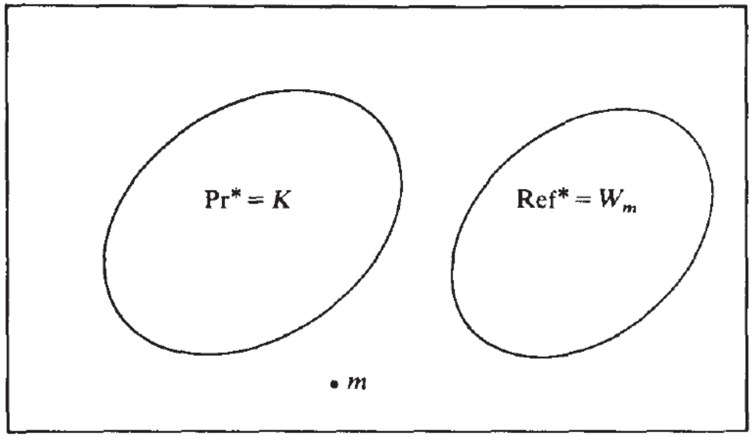
By the productiveness of $\overline{K}$, $m \in \overline{K} \setminus Ref^*$.
In particular $m \notin Ref^*$ means that $\mathsf{m} \notin \mathsf{K}$ is not provable. Let σ be $\mathsf{m} \notin \mathsf{K}$.

Illumination



Proof (2)

(ii) $Pr^* = K$ by ω-consistency. Now $m \notin K$ implies $m \notin Pr^*$ implies that $\mathsf{m} \in \mathsf{K}$ is not provable. Hence $\neg\sigma$ is not provable.



Rosser's Improvement

Rosser's contribution was to remove the ω-consistency condition.

Rosser's Improvement

Let $K_0 = \{x \mid \phi_x(x) = 0\}$ and $K_1 = \{x \mid \phi_x(x) = 1\}$.

Both K_0 and K_1 are r.e.

Rosser's Improvement

Two disjoint sets A, B are recursively inseparable if there is no recursive set C such that $A \subseteq C$ and $B \subseteq \overline{C}$.

Two disjoint sets A, B are recursively inseparable iff whenever $A \subseteq W_a$, $B \subseteq W_b$ and $W_a \cap W_b = \emptyset$ then there is a number $x \notin W_a \cup W_b$.

Fact. K_0 and K_1 are recursively inseparable.

Proof. Suppose such C existed. Let m be an index of the characteristic function of C. A contradiction can be easily derived.

Rosser's Improvement

Let decidable predicates $R_0(x, y), R_1(x, y)$ be such that

- $n \in K_0$ iff there is y such that $R_0(n, y)$.
- $n \in K_1$ iff there is y such that $R_1(n, y)$.

Clearly $K_0 \cap K_1 = \emptyset$. So we also have

- $n \in K_0$ iff there is y such that $R_0(n, y)$ and, for all $z \leq y$, $R_1(n, z)$ does not hold.

Rosser's Improvement

Let $\sigma_{R_0}(\mathsf{x}, \mathsf{y}), \sigma_{R_1}(\mathsf{x}, \mathsf{y})$ be the counterparts of $R_0(x, y), R_1(x, y)$ in Peano arithmetic.

Rosser's trick was to use the following statement

$$\exists \mathsf{y}.(\sigma_{R_0}(\mathsf{n}, \mathsf{y}) \wedge \forall \mathsf{z} \leq \mathsf{y}.\neg\sigma_{R_1}(\mathsf{n}, \mathsf{z}))$$

for the formal counterpart of $n \in K_0$, denoted by $\mathsf{n} \in \mathsf{K}_0$. Similarly we write $\mathsf{n} \in \mathsf{K}_1$ for

$$\exists \mathsf{y}.(\sigma_{R_1}(\mathsf{n}, \mathsf{y}) \wedge \forall \mathsf{z} \leq \mathsf{y}.\neg\sigma_{R_0}(\mathsf{n}, \mathsf{z})).$$

Rosser's Improvement

Lemma. For each natural number n, the following are valid in Peano arithmetic.

1. If $n \in K_0$ then $\mathsf{n} \in \mathsf{K}_0$ is provable.
2. If $n \in K_1$ then $\mathsf{n} \in \mathsf{K}_1$ is provable.
3. If $\mathsf{n} \in \mathsf{K}_1$ is provable, then $\mathsf{n} \notin \mathsf{K}_0$ is also provable.

Proof. Suppose $\mathsf{n} \in \mathsf{K}_1$ is provable. Then some m exists such that

$$\sigma_{R_1}(\mathsf{n}, \mathsf{m}) \wedge \forall \mathsf{z} \leq \mathsf{m}. \neg\sigma_{R_0}(\mathsf{n}, \mathsf{z})$$

is provable. In other words, $\sigma_{R_1}(\mathsf{n}, \mathsf{m}), \neg\sigma_{R_0}(\mathsf{n}, 0), \ldots, \neg\sigma_{R_0}(\mathsf{n}, \mathsf{m})$ are provable. It follows that

$$\forall \mathsf{y}. (\neg\sigma_{R_0}(\mathsf{n}, \mathsf{y}) \vee \exists \mathsf{z} \leq \mathsf{y}. \sigma_{R_1}(\mathsf{n}, \mathsf{z}))$$

is provable.

Rosser's Improvement

Theorem. (Gödel-Rosser Incompleteness Theorem)
There is a statement τ such that if Peano arithmetic is consistent, then neither τ nor $\neg\tau$ is provable.

Rosser's Improvement

Proof. Define

$$\begin{aligned} Pr^{**} &= \{n \mid \mathsf{n} \in \mathsf{K}_0 \text{ is provable}\}, \\ Ref^{**} &= \{n \mid \mathsf{n} \in \mathsf{K}_0 \text{ is refutable}\} = \{n \mid \mathsf{n} \notin \mathsf{K}_0 \text{ is provable}\}. \end{aligned}$$

Consistency of PA means that $Pr^{**} \cap Ref^{**} = \emptyset$.

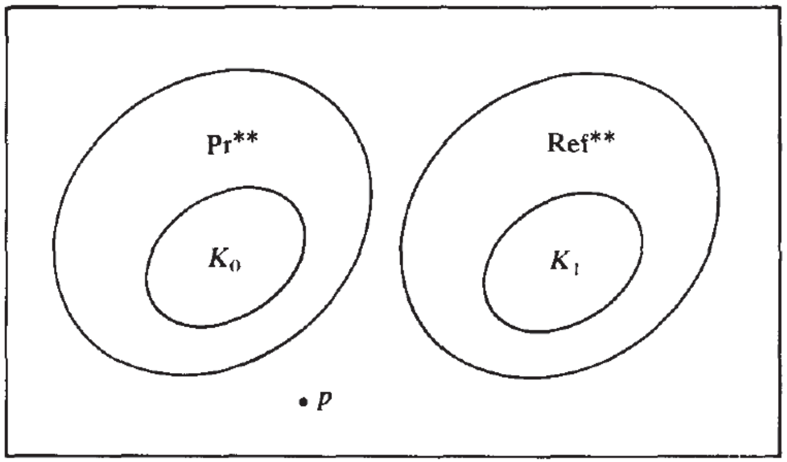
By the previous lemma, $K_0 \subseteq Pr^{**}$ and $K_1 \subseteq Ref^{**}$ (since $\mathsf{n} \in \mathsf{K}_1$ implies $\mathsf{n} \notin \mathsf{K}_0$ is provable).

Since Pr^{**} and Ref^{**} are both r.e., the recursive inseparability of K_0 and K_1 means that there is some $p \notin Pr^{**} \cup Ref^{**}$.

Now $p \notin Pr^{*}$ means that $\mathsf{p} \in \mathsf{K}_0$ is not provable; and $p \notin Ref^{*}$ means that $\mathsf{p} \notin \mathsf{K}_0$ is not provable.

So we may let τ be $p \in \mathsf{K}_0$.

Illumination



Undecidability

When considering sets of statements the terms decidable and undecidable are often used to mean recursive and non-recursive.

Undecidability

Theorem. Suppose that $(\mathscr{A}, \mathscr{D})$ is an ω-consistent formal system of arithmetic in which all decidable predicates are representative. Then the set of provable statements is creative.

Proof. By the proof of Gödel's Incompleteness Theorem,
$K = Pr^* = \{n \mid \mathsf{n} \in \mathsf{K} \text{ is provable}\}$.

Now let $Pr = \{n \mid \theta_n \text{ is provable}\}$.

We can find a computable function h such that $\mathsf{n} \in \mathsf{K}$ is $\theta_{h(n)}$.

$n \in K$ iff $n \in Pr^*$ iff $h(n) \in Pr$.

So Pr is creative.

Undecidability

Corollary. If Peano arithmetic is ω-consistent, then the provable statements form a creative set.

Provability is creative.

Truth (validity) is productive.