Lab12-Incompleteness

CS363-Computability Theory, Xiaofeng Gao, Spring 2015

$*$ Please upload your assignment to FTP or submit a paper version on the next class.

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1. Write the formal counterpart of the following informal statements using the language L with alphabet $\{0, 1, +, \times, =\}$, logical notions $\{\neg, \vee, \wedge, \rightarrow, \forall, \exists\}$, variables, and other symbols.

   (a) x is a perfect square.

   (b) $x \bmod y = z$.

2. Show that $\mathscr{F}$ is productive.

3. Effectively Recursively Inseparable

   Let $(\mathscr{A}, \mathscr{D})$ be a consistent recursively axiomatized formal system for which the following two properties hold:

   (1) If $M(x_1, \cdots, x_n)$ is a decidable predicate, $\sigma(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ is the statement of L that is the formal counterpart of $M(x_1, \cdots, x_n)$, then for any $a_1, \cdots, a_n \in \mathbb{N}$,

      i. If $M(a_1, \cdots, a_n)$ holds, then $\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ is provable.

      ii. If $M(a_1, \cdots, a_n)$ does not hold, then $\neg\sigma(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ is provable.

   (2) For each natural number n,

      i. If $n \in K_0$ then $\mathsf{n} \in \mathsf{K}_0$ is provable.

      ii. If $n \in K_1$ then $\mathsf{n} \in \mathsf{K}_1$ is provable.

      iii. If $\mathsf{n} \in \mathsf{K}_1$ is provable, then $\mathsf{n} \notin \mathsf{K}_0$ is also provable.

   If we define $\begin{cases} Pr^{**} = \{n : \mathsf{n} \in \mathsf{K}_0 \text{ is provable}\}, \\ Ref^{**} = \{n : \mathsf{n} \in \mathsf{K}_0 \text{ is refutable}\} = \{n : \mathsf{n} \notin \mathsf{K}_0 \text{ is provable}\}, \end{cases}$ then answer the following two questions.

   (a) Show that Pr^{**} and Ref^{**} are effectively recursively inseparable.

   (b) Let $Pr = \{n : \theta_n \text{ is provable}\}$ and $Ref = \{n : \neg\theta_n \text{ is provable}\}$. Prove that Pr and Ref are effectively recursively inseparable. (*Hint.* Extend the idea of theorem 7-3.2 to pairs of effectively recursively inseparable sets.)