Lab10-Various Sets

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

$*$ Please upload your assignment to FTP or submit a paper version on the next class
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1. Prove the following statements.

   (a) If B is r.e. and $A \cap B$ is productive, then A is productive.

   Proof:

   First, we claim there exists a total computable function k such that $\forall x, W_{k(x)} = W_x \cap B$. Since B is r.e., by the s-m-n theorem there exists k that:

   $$\phi_{k(x)}(y) = \begin{cases} \phi_x(y) & , y \in B \\ \text{undefined} & , \text{otherwise} \end{cases}$$

   Let f be a productive function for $A \cap B$. We claim $g = f \circ k$ is productive for A. Since g is total computable, and given $W_x \subseteq A, W_{k(x)} \subseteq A \cap B$, so:

   $$g(x) = f(k(x)) \in A \cap BW_{k(x)} = A \cap BW_x \cap B \subseteq AW_x$$

   So A is productive.

   (b) If C is creative and A is an r.e. set such that $A \cap C = \varnothing$, then $C \cup A$ is creative.

   Proof:

   Since A and C are r.e., $A \cup C$ is r.e.
   We claim that $\overline{A \cup C} = \overline{A} \cap \overline{C}$ is productive.
   Then, we fix a total computable function k such that $\forall x, W_{k(x)} = W_x \cup A$, and let f be a productive function for $\overline{C}$.
   We claim that $g = f \circ k$ is productive for $\overline{A} \cap \overline{C}$. g is total computable, and given $W_x \subseteq \overline{A} \cap \overline{C}$, $W_{k(x)} \subseteq \overline{C}$, so:

   $$g(x) = f(k(x)) \in \overline{C} - W_{k(x)} = \overline{C} - W_x \cup A \subseteq \overline{A} \cap \overline{C} - W_x$$

   Thus $\overline{A} \cap \overline{C}$ is productive, and so $A \cup C$ is creative.

2. Let $\mathscr{B}$ be a set of unary computable functions, and suppose that $g \in \mathscr{B}$ is such that for all finite $\theta \subseteq g$, $\theta \notin \mathscr{B}$. Prove that the set $\{x \mid \phi_x \in \mathscr{B}\}$ is productive.

   Proof:

   According to the s-m-n theorem, there exists k such that:

   $$\forall x, \phi_{k(x)}(y) = \begin{cases} g(y) & , \neg H_1(x, x, y) \\ \text{undefined} & , \text{otherwise} \end{cases}$$

   If $x \in \overline{K}$, then $\phi_{k(x)} = g \in \mathscr{B}$, and if $x \notin K$, then $\phi_{k(x)} \subseteq g$ is finite, hence $\phi_{k(x)} \notin B$.

   So, the set $\{x \mid \phi_x \in \mathscr{B}\}$ is productive.

3. If $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$, $A \otimes B = \{\pi(x, y) \mid x \in A \text{ and } y \in B\}$, prove the following statements.

(a) Suppose B is r.e. If A is creative, then so are $A \oplus B$ and $A \otimes B$ (provided $B \neq \varnothing$).

Proof:

It is obvious that $\{2x|x \in A\}$ is creative and $\{2x+1|x \in B\}$ is r.e..

Then, from exercise 1.b),$A \oplus B$ is creative.

Because $B \neq \varnothing$, we can find $b \in B$. Then, $x \in \overline{A}$ iff $\pi(x,b) \in A \otimes B$, so, $\overline{A \otimes B}$ is productive. So we can easily prove that $A \otimes B$ is r.e..

In all, $A \otimes B$ is creative.

(b) If B is recursive, then the implications in (a) reverse.

Proof:

We define $f(x) = \mu z((2z \geq x) \wedge (2z \in \overline{A \oplus B}))$. $f(x)$ is total and computable, so we have $x \in \overline{A \oplus B}$ iff $f(x) \in \overline{A}$. According to the reduction theorem, $\overline{A}$ is productive means that A is creative.

We define $f(x) = \pi_1(\mu z((z \geq x) \wedge (\pi_2(z) \in B)))$. $f(x)$ is total and computable since B is recursive, so, $x \in \overline{A \otimes B}$ iff $f(x) \in \overline{A}$. So, A is creative.

(c) If A, B are simple sets, prove that $A \otimes B$ is not simple but that $\overline{\overline{A} \otimes \overline{B}}$ is simple.

Proof:

i. $A \otimes B$ is not simple.
We have $x \in \overline{A \otimes B} \Leftrightarrow \pi_1(x) \notin A$ or $pi_2(x) \notin B$.
$\overline{B}$ is infinite, so we can find some $b \notin B$. Then we have $\pi(N,b) \subseteq A \otimes B$.
Because $\pi(N,b)$ is r.e., so that $A \otimes B$ is not simple.

ii. $\overline{\overline{A} \otimes \overline{B}}$ is simple.
$\overline{\overline{A} \otimes \overline{B}} = \{\pi(x,y)|x \in A \text{ or } y \in B\}$ is r.e., and $\overline{A} \otimes \overline{B}$ is infinite.
We suppose that there is an r.e. set $C \subseteq \overline{A} \otimes \overline{B}$.
Then, because $x \in \overline{A} \otimes \overline{B} \Leftrightarrow \pi_1(x) \in \overline{A}$ and $\pi_2(x) \in \overline{B}$. We have that $\pi_1(C) \subseteq \overline{A}$ and $\pi_2(C) \subseteq \overline{B}$ and $\pi_1(C)$ and $\pi_2(C)$ are both r.e., which contradicts that A and B are simple.
So, $\overline{\overline{A} \otimes \overline{B}}$ is simple.