Lab05-Numbering Programs

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class

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1. Show that there is a total computable function k such that for each n,

   (a) $k(n)$ is an index of the function $\lfloor \sqrt[n]{x} \rfloor$.
   Solution:
   Let $f(n,x) = \lfloor \sqrt[n]{x} \rfloor$
   So, $f(n,x) = \mu z \leq ((z+1)^n > x)$
   So, $f(n,x)$ is computable. So, according to the s-m-n theory, there is a total computable function $k(n)$ such that for any fixed n, $f(n,x) = \phi_{k(n)}(x)$

   (b) $W_{k(n)}^{(m)} = \{(y_1, \ldots, y_m) : y_1 + y_2 + \ldots + y_m = n\}$ $(m \geq 1)$.
   Solution:
   Let $f(n, y_1, ...y_m) = \mu z(n - (y_1 + ...y_m) + z = 0)$
   So, $f(n, y_1, ...y_n)$ is computable. So, according to teh s-m-n theory, there is a total computable function $k(n)$ such that for any fixed n, $f(n, y_1, ...y_m) = \phi_{k(n)}^m(y_1, ...y_m)$, and $W_{k(n)}^m = \{(y_1, ...y_m) | y_1 + ...y_m = n\}$

   (c) $E_{k(n)} = W_n$.
   Solution:
   For each n, We assume the last register used by P_n is R_n.
   So, $T(1, R_n + 1), P_n, T(R_n + 1, 1)$ has the range as same as the domain of P_n.
   Let $k(n) = \gamma(T(1, R_n + 1), P_n, T(R_n + 1, 1))$, $k(n)$ is total and computable.

2. (a) Find P_{1028}. Distinguish what are $\phi_{1028}(x)$ and $\phi_{1028}^{(n)}(x_1, \cdots, x_n)$ and their corresponding $W_{1028}(x)$, $E_{1028}(x)$ and $W_{1028}^{(n)}(x)$, $E_{1028}^{(n)}(x)$;
   Solution:
   1)

$$1028 = 2^0 + 2^2 + 2^{10} - 1$$
$$\beta(I_1) = 0, \beta(I_2) = 1 + 1, \beta(I_3) = 7 + 1 + 2$$
$$I_1 : Z(1), I_2 : S(1), I_3 : J(2, 1, 1)$$

   So, $P_{1028} = Z(1); S(1); J(2, 1, 1)$.
   2)

$$\phi_{1028}(x) = 1$$
$$W_{1028}(x) = \mathbb{N}$$
$$E_{1028}(x) = \{1\}$$

   3)

$$\phi_{1028}^{(n)}(x_1, ...x_n) = \begin{cases} 1 & , x_2 \neq 1 \\ undefined & , x_2 = 1 \end{cases}$$

$$W_{1028}^{(n)}(x) = \mathbb{N} \times (\mathbb{N} - \{1\}) \times \mathbb{N}^{n-2}$$
$$E_{1028}^{(n)}(x) = \{1\}$$

(b) Let P be the program J(1,2,4), Z(1), S(1). Calculate $\gamma(P)$.
Solution:

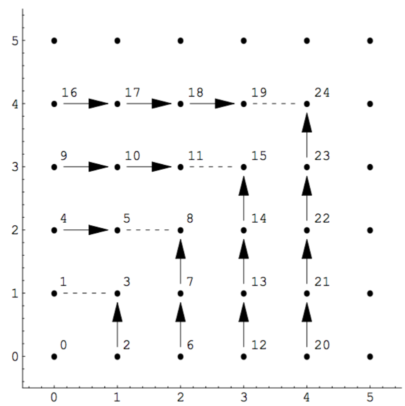
$$\beta(J(1,2,4)) = 4 * 27 + 3 = 111$$
$$\beta(Z(1)) = 0$$
$$\beta(S(1)) = 1$$
$$\gamma(P) = 2^{111} + 2^{112} + 2^{114} - 1$$

3. (a) (Cantor) Show that the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable.
Proof:
If the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is denumerable, then we assume $f_1, f_2, ...f_n$ is an enumeration of functions from $\mathbb{N}$ to $\mathbb{N}$, so we define that $g(n) = f_n(n) + 1$, for each $n, g \neq f_n$, however, g is also a function from $\mathbb{N}$ to $\mathbb{N}$.
So, there's contradiction, the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable.

(b) Show that the set of all non-computable total functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable.
Proof:
If the set of all non-computable total functions from $\mathbb{N}$ to $\mathbb{N}$ is denumerable. Since we know that the set of all computable total functions from $\mathbb{N}$ to $\mathbb{N}$ is denumerable, the set of all functions from $\mathbb{N}$ to $\mathbb{N}$, however,the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable which has been already proved.
So, there's contradiction, the set of all non-cmoputable total functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable.

4. Alternative Selection of π

The π function where $\pi(x, y) = 2^x(2y + 1) - 1$ can enumerate linearly all pairs of natural numbers $(x, y) \in \mathbb{N} \times \mathbb{N}$. However, it does not generate a trace in the first quadrant of the plane. Correspondingly, instead of applying this π function, we can define an alternative bijection π', such that $\pi' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and it grows horizontally and vertically according to the right figure. Thus we have:
$\pi'(0, 0) = 0$, $\pi'(0, 1) = 1$, $\pi'(1, 0) = 2$,
$\pi'(1, 1) = 3$, $\pi'(0, 2) = 4$, $\pi'(1, 2) = 5$,
$\pi'(2, 0) = 6$, $\pi'(2, 1) = 7$, $\pi'(2, 2) = 8$, etc.



Now please develop a mathematical formula for π', (like the notation of original π), and prove the correctness of your design.

Solution:

We can see that if the $x = 0, \pi'(0, y) = 0, 1, 4, 9, ..., y = 0, 1, 2, 3...$

If the $y = 0, \pi'(x, 0) = 0, 2, 6, 12..., x = 0, 1, 2, 3...$

We define that $\pi'(x, y) = \begin{cases} x + y^2, & x < y \\ x(x + 1) + y, & x \geq y \end{cases}$

Proof:

First, we prove the function is a injective function, we assume two different pair (x_1, y_1) and (x_2, y_2).

We assume that $x_1 < y_1, x_2 < y_2$, so $\pi'(x_1, y_1) = x_1 + y_1^2, \pi'(x_2, y_2) = x_2 + y_2^2$, if $\pi'(x_1, y_1) = \pi'(x_2, y_2)$, we get $x_1 - x_2 = (y_2 + y_1)(y_2 - y_1)$

If $x_1 = x_2$, $y_1 = y_2$, so we assume $x_1 > x_2$, then $y_2 - y_1$ must equal to at least 1, and $y_2 + y_1 > x_1 + x_2 > x_1 - x_2$, so $x_1 = x_2, y_1 = y_2$

Similarily, we can prove the equlation is hold in other three conditions, so the function is injective.

Next, we prove the function is a surjective function, we can see that $\forall z \in \mathbb{N}$,we can find $n \in \mathbb{N}$ that $n \leq z \leq (n+1)^2$

We assume $m = z - n^2$, if $m < n$, then (m, n) is the corresponding pair (x, y), if $m \geq n$, then $(n, m - n)$ is the corresponding pair.

So, we've proved $\forall z \in \mathbb{N}$,there is a corresponding (x, y), so the function is a surjective function.

So, our design is coorect.