Lab09-Simple Set

CS363-Computability Theory, Xiaofeng Gao, Spring 2015

∗ Please upload your assignment to FTP or submit a paper version on the next class.
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1. Suppose f is a total injective computable function such that $Ran(f)$ is not recursive ((a) showed that such functions abound). Show that $A = \{x : \exists y(y > x \wedge f(y) < f(x))\}$ is simple. (*Hint*. To see that $\overline{A}$ is infinite, assume the contrary and show that there would then be a sequence of numbers $y_0 < y_1 < y_2 < \ldots$ such that $f(y_0) > f(y_1) > f(y_2) > \ldots$ To see that $\overline{A}$ does not contain an infinite r.e. set B, suppose to the contrary that $B \subseteq \overline{A}$. Then show that the problem $z \in Ran(f)$ is decidable as follows. Given z, find $n \in B$ such that $f(n) > z$; now use the fact that $n \notin A$ to devise a finite procedure for testing whether $z \in Ran(f)$.)

2. If $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$, $A \otimes B = \{\pi(x, y) : x \in A \text{ and } y \in B\}$, prove the following statements.

   (a) If A is simple, then $A \otimes \mathbb{N}$ is r.e., but neither recursive, creative nor simple.

   (b) If A, B are simple sets, then $A \oplus B$ is simple, $A \otimes B$ is not simple but $\overline{\overline{A} \otimes \overline{B}}$ is simple.

3. Recursively Inseparable and Effectively Recursively Inseparable

   Disjoint sets A, B are said to be *recursively inseparable* if there is no recursive set C such that $A \subseteq C$ and $B \subseteq \overline{C}$. Furthermore, A and B are said to be *effectively recursively inseparable* if there is a total computable function f such that whenever $A \subseteq W_a$, $B \subseteq W_b$ and $W_a \cap W_b = \varnothing$ then $f(a, b) \notin W_a \cup W_b$ (see the right figure). Note: Recursive inseparability for a pair of disjoint sets corresponds to non-recursiveness for a single set; pair of recursively inseparable sets that are also r.e. correspond to r.e. sets that are not recursive.

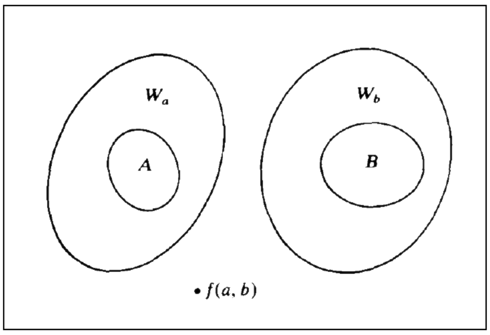

Fig. Effectively Recursively Inseparable Sets

   (a) Show that two disjoint sets A, B are recursively inseparable iff whenever $A \subseteq W_a$, $B \subseteq W_b$ and $W_a \cap W_b = \varnothing$, there is a number $x \notin W_a \cup W_b$.

   (b) Suppose A, B are effectively recursively inseparable. Prove that if A, B are both r.e. then they are both creative. (*Note*. Extending the idea of effectiveness to a pair of recursively inseparable sets in this way parallels the step from a nonrecursive set to a set having productive complement; the counterpart to a single creative set is then a pair of effectively recursively separable sets that are both r.e.)

   (c) Let $K_0 = \{x : \phi_x(x) = 0\}$ and $K_1 = \{x : \phi_x(x) = 1\}$. Show that K_0 and K_1 are r.e. (in particular neither K_0 nor K_1 is recursive), and that they are both recursively inseparable and effectively recursively inseparable. (*Hint*. For recursively inseparable, suppose that there is such a set C and let m be an index for its characteristic function; consider whether or not $m \in C$. For effectively recursively inseparable, find a total computable function f such that if $W_a \cap W_b = \varnothing$, then $\phi_{f(a,b)}(x) = \begin{cases} 1 & \text{if } x \in W_a, \\ 2 & \text{if } x \in W_b, \\ \text{undefined} & \text{otherwise.} \end{cases}$)