Lab06-Universal Program

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

$*$ Please upload your assignment to FTP or submit a paper version on the next class

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1. (a) Show that there is a decidable predicate $Q(x, y, z)$ such that

   i. $y \in E_x$ if and only if $\exists z.Q(x, y, z)$

   ii. if $y \in E_x$ and $Q(x, y, z)$, then $\phi_x((z)_1) = y$.

   Solution:

   Let $Q(x, y, z) = S(x, (z)_1, y, (z)_2)$, and $z = 2^{(z)_1}3^{(z)_2}$. Assume that $p = (z)_1, q = (z)_2$, $\phi_x(p) \downarrow y$ in q steps.

   If $y \in E_x$, there exist p, q satify $z = 2^p3^q$, so there exist z that makes $Q(x, y, z)$ true, which in turn can make sure $y \in E_x$.

   Then if $y \in E_x$ and there exists such z so that $Q(x, y, z)$ holds, then according to the definition that $z = 2^p3^q$, we have $\phi_x((z)_1) = \phi_x(p) = y$.

   (b) Deduce that there is a computable function $g(x, y)$ such that

   i. $g(x, y)$ is defined if and only if $y \in E_x$.

   ii. if $y \in E_x$, then $g(x, y) \in W_x$ and $\phi_x(g(x, y)) = y$; i.e. $g(x, y) \in \phi_x^{-1}(\{y\})$.

   Solution:

   $g(x, y) = \mu z(Q(x, y, z))_1$

   If $g(x, y)$ is defined, then exist $z, Q(x, y, z)$ is true, which implies $y \in E_x$. According to the conclusion before, the other direction is also clear.

   Then if $y \in E_x$, then there exist $z, Q(x, y, z)$ which means $g(x, y) = (z)_1W_x$ and $\phi_x(g(x, y)) = y$.

   (c) Deduce that if f is a computable injective function (not necessarily total or surjective) then f^{-1} is computable. (cf. exercise 2-5.4(1)).

   Solution:

   By the s-m-n theory there is a total computable function k such that $g(x, y) = \phi_{k(x)}(y)$, then we have:

   $W_{k(x)} = E_x$

   $E_{k(x)} \subseteq W_x$ and if $y \in E_x$, then $\phi_x(\phi_{k(x)}(y)) = y$

   Hence, if ϕ_x is injective, then $\phi_{k(x)} = \phi^{-1}$ and $E_{k(x)} = W_x$, so the f^{-1} is computable.

2. (cf. example 3-7.1(b)) Suppose that f and g are unary computable functions; assuming that T_1 has been formally proved to be decidable, prove formally that the function $h(x)$ defined by $h(x) = \begin{cases} 1 & \text{if } x \in Dom(f) \text{ or } x \in Dom(g), \\ \uparrow & \text{otherwise,} \end{cases}$ is computable.

   Solution:

   Let $f = \phi_m, g = \phi_n$. Then $h(x) = \mathbf{1}(\mu t(H_1(m, x, t) or H_1(n, x, t)))$.

3. Show that there is a total computable function $k(e_1, e_2)$ such that $\phi_{k(e_1,e_2)}(x)$ is the characteristic function for predicate "$M_1(x)$ and $M_2(x)$", where M_1 and M_2 are both decidable predicate and $\phi_{e_1} = c_{M_1}$, $\phi_{e_2} = c_{M_2}$.

   Solution:

$f(e_1, e_2, x) = sg(\phi_{e_1}(x) + \phi_{e_2}(x)1) = sg(\psi(e_1, x) + \psi(e_2, x) - 1)$.

Obiviously, $f(e_1, e_2, x)$ is the characteristic function for predicate $M_1(x)$ and $M_2(x)$.

According to the s-m-n theorem, there exists a total computable function $k(e_1, e_2)$ such that $\phi_{k(e1,e2)}(x) = f(e_1, e_2, x)$.

4. Show that there is a total computable function $s(x, y)$ such that for all x, y, $E_{s(x,y)} = W_x \cup E_y$.

   Solution:

   Let $f(x, y, z) = \begin{cases} \pi_1(z) - 1 & , \pi_1(z) - 1 \in W_x, \pi_2(z) = 0 \\ p & , \pi_1(z) = 0, \phi_y(\pi_2(z) - 1) \downarrow p \\ undefined & , otherwise \end{cases}$

   According to the above definition, for all $x, y, E_{s(x,y)} = W_x \cup E_y$.

   According Churchs Thesis, $f(x, y, z)$ is a computable function.

   According to s-m-n Theorem, there is a total computable function $s(x, y)$ such that $\phi_{s(x,y)}(z) = f(x, y, z)$.

5. Suppose that $f(x)$ is computable; show that there is a total computable function $k(x)$ such that for all x, $W_{k(x)} = f^{-1}(W_x)$.

   Solution:

   $\psi(x, y) = \begin{cases} 1 & , f(y) \in W_x \\ undefined & , otherwise \end{cases}.$

   According to the s-m-n theorem, there is a total computable function $k(x)$ that $\phi_{k(x)}(y) = f(x, y)$. So the domain of $\phi_{k(x)}(y)$ is $f^{-1}(W_x)$.