Lab08-Recursively Enumerable Set

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class

∗ If there is any problem, please contact: steinsgate@sjtu.edu.cn

∗ Name:________ StudentId: ________ Email: ________

1. Let A, B be subsets of $\mathbb{N}$. Define sets $A \oplus B$ and $A \otimes B$ by

$$A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\},$$
$$A \otimes B = \{\pi(x,y) \mid x \in A \text{ and } y \in B\},$$

where π is the pairing function $\pi(x,y) = 2^x(2y+1) - 1$. Prove that

(a) $A \oplus B$ is recursive iff A and B are both recursive.

(b) If $A, B \neq \varnothing$, then $A \otimes B$ is recursive iff A and B are both recursive.

Solution.

(a) Let C_A and C_B be characteristic function of set A and B respectively.

$\Leftarrow$: If A and B are both recursive, then C_A and C_B are computable. Consider the characteristic function of $A \oplus B$: $f(x) = \begin{cases} C_A(x/2) & \text{if } x \text{ is even,} \\ C_B((x-1)/2) & \text{if } x \text{ is odd.} \end{cases}$

$f(x)$ is computable, therefore $A \oplus B$ is recursive.

$\Rightarrow$: Let f denote the characteristic function of $A \oplus B$. Then we can build the characteristic function of A and B with the help of f, namely $\begin{cases} C_A(x) = f(2x), \\ C_B(x) = f(2x+1). \end{cases}$ Obviously, both C_A and C_B are computable, making A and B are recursive.

(b) $\Rightarrow$: $\begin{cases} x \in A \Leftrightarrow z = \pi(x,y) \in A \otimes B \text{ (randomly pick } y \text{ from } B) \\ y \in B \Leftrightarrow z = \pi(x,y) \in A \otimes B \text{ (randomly pick } x \text{ from } A) \end{cases}$

π is computable, hence A and B are both recursive if $A \otimes B$ is recursive.

(Some students may point out that it is not easy to pick an element from $A(B)$. However, we don't care how to find an exact element from $A(B)$. Instead, we only need to show there indeed exists a computable characteristic function.)

$\Leftarrow$: $z \in A \otimes B \Leftrightarrow \pi_1(z) \in A \wedge \pi_2(z) \in B$.

let $x = \pi_1(z)$, $y = \pi_2(z)$, $z \in A \otimes B \Leftrightarrow x \in A \wedge y \in B$. For both π_1 and π_2 are computable functions, we conclude that $A \otimes B$ is recursive if A and B are both recursive.

□

2. Which of the following sets are recursive? Which are r.e.? Which have r.e. complement? Prove your judgements.

(a) $\{x \mid P_m(x)\downarrow \text{ in } t \text{ or fewer steps }\}$ (m, t are fixed).

(b) $\{x \mid x \text{ is a power of } 2\}$;

(c) $\{x \mid \phi_x \text{ is injective}\}$;

(d) $\{x \mid y \in E_x\}$ (y is fixed);

Solution.

(a) We have the characteristic function $C(x) = \begin{cases} 1, & \text{if } H_1(m,x,t) \text{ holds} \\ 0, & \text{if } H_1(m,x,t) \text{ not holds} \end{cases}$. Since H is primitive recursive, the characteristic function C is computable. Thus the set is recursive which means it is also and has r.e. complement.

(b) x is a power of 2 $\Leftrightarrow \exists y \leq x(2^y = x)$. Obviously, the predicate is decidable. Thus the set is recursive which means it is r.e. and has r.e. complement.

(c) According to the **Rice's Theorem**, the set is not recursive. On the other hand, ϕ_x is not infective $\Leftrightarrow \exists z_1 \exists z_2 (z_1 \neq z_2 \wedge \phi_x(z_1) = \phi_x(z_2))$. Since the right predicate is partially decidable, the original set should have a r.e. complement. Thus the set is not r.e..

(d) Similarly, according to the **Rice's Theorem**, the set is not recursive. The partial characteristic function $C_\chi(x) = \begin{cases} 1, & \text{if } \exists z(\phi_x(z) = y) \\ \uparrow, & \text{otherwise} \end{cases}$ is computable, therefore the set is r.e.. Thus it should have no r.e. complement.

□

3. Prove following statements.

(a) Let $B \subseteq \mathbb{N}$ and $n > 1$; prove that B is r.e. then the predicate $M(x_1, \ldots, x_n)$ given by "$M(x_1, \ldots, x_n) \equiv 2^{x_1}3^{x_2} \ldots p_n^{x_n} \in B$" is partially decidable.

Solution. B is r.e., $f(x) = \begin{cases} 1, & \text{if } x \in B, \\ \uparrow, & \text{if } x \notin B. \end{cases}$ is the partial characteristic function of B. Since B is r.e., $f(x)$ is computable.

The partial characteristic function of M is $g(x) = \begin{cases} 1, & \text{if } f(2^{x_1}3^{x_2} \ldots p_n^{x_n}) = 1, \\ \uparrow, & \text{otherwise.} \end{cases}$

Since power function and f are computable, $g(x)$ is computable. Thus the predicate $M(x_1, \ldots, x_n)$ given by "$M(x_1, \ldots, x_n) \equiv 2^{x_1}3^{x_2} \ldots p_n^{x_n} \in B$" is partially decidable. □

(b) Prove that $A \subseteq \mathbb{N}^n$ is r.e. iff $\{2^{x_1}3^{x_2} \ldots p_n^{x_n} \mid (x_1, \ldots, x_n) \in A\}$ is r.e..

Solution. Define $B \subseteq \mathbb{N}$ to be $\{2^{x_1}3^{x_2} \ldots p_n^{x_n} : (x_1, \ldots, x_n) \in A\}$.

$\Rightarrow$: $A \subseteq \mathbb{N}^n$ is r.e.. $f(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } (x_1, \ldots, x_n) \in A, \\ \uparrow, & \text{if } (x_1, \ldots, x_n) \notin A. \end{cases}$ is the partial characteristic function of A. Since A is r.e., $f(x_1, \ldots, x_n)$ is computable.

The partial characteristic function of B is $g(x) = \begin{cases} 1, & \text{if } f((x)_1, (x)_2 \ldots (x)_n) = 1, \\ \uparrow & \text{otherwise.} \end{cases}$

Since $(x)_y$ is computable, by substitution, $f((x)_1, (x)_2 \ldots (x)_n)$ is computable, hence $g(x)$ is computable. Thus, B is r.e.

$\Leftarrow$: $g(x) = \begin{cases} 1, & \text{if } x \in B, \\ \uparrow, & \text{if } x \notin B. \end{cases}$ is the partial characteristic function of B. Since B is r.e., $g(x)$ is computable.

The characteristic function of A is $f(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } g(2^{x_1}3^{x_2} \ldots p_n^{x_n}) = 1, \\ \uparrow, & \text{otherwise.} \end{cases}$ Since the computation of power is computable, by substitution, $g(2^{x_1}3^{x_2} \ldots p_n^{x_n})$ is computable, hence f is computable. Thus, A is r.e.. □