Lab08-Recursively Enumerable Set

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class
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1. Let A, B be subsets of $\mathbb{N}$. Define sets $A \oplus B$ and $A \otimes B$ by

$$A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\},$$
$$A \otimes B = \{\pi(x,y) \mid x \in A \text{ and } y \in B\},$$

where π is the pairing function $\pi(x,y) = 2^x(2y+1) - 1$. Prove that

(a) $A \oplus B$ is recursive iff A and B are both recursive.

Proof:

First, we proof $\Rightarrow$:

If $A \oplus B$ is recursive, then the characteristic function $c_{A\oplus B}(x)$ is computable. Then $c_A(x) = c_{A\oplus B}(2x)$ and $c_B(x) = c_{A\oplus B}(2x+1)$ are both computable, so A and B are both recursive.

Then, we proof $\Leftarrow$:

If A and B are both recursive, so $c_A(x)$ and $c_B(x)$ are both computable.

So, $c_{A\oplus B}(x) = sg(c_A(qt(2,x))sg(div(2,x)) + c_B(qt(2,x-1))\bar{sg}(div(2,x)))$ is computable, thus $A \oplus B$ is recursive.

(b) If $A, B \neq \varnothing$, then $A \otimes B$ is recursive iff A and B are both recursive.

Proof:

First, we proof $\Rightarrow$:

If $A \otimes B$ is recursive, then $c_{A\otimes B}(x)$ is computable. Because $A, B \neq \varnothing$, so we assume $a \in A$, $b \in B$.

So, $c_A(x) = c_{A\otimes B}(\pi(x,b))$ and $c_B(x) = c_{A\otimes B}(\pi(a,x))$ are both computable, thus A,B is recursive.

Then, we proof $\Leftarrow$:

If A and B are both recursive, so $c_A(x)$ and $c_B(x)$ are both computable.

So, $c_{A\otimes B}(x) = c_A(\pi_1(x))c_B(\pi_2(x))$ is computable, thus $A \otimes B$ is recursive.

2. Which of the following sets are recursive? Which are r.e.? Which have r.e. complement? Prove your judgements.

(a) $\{x \mid P_m(x)\downarrow$ in t or fewer steps $\}$ $(m, t$ are fixed).

Solution:

It is recursive, r.e. and has r.e. complement.

The set's characteristic function is $H(m,x,t)$, it is computable. So, the set is recursive. The complement set of a recursive set is still recursive, so the set is r.e. complement.

(b) $\{x \mid x$ is a power of $2\}$;

Solution:

It is recursive, r.e. and has r.e. complement.

The set's characteristic function is: $c(x) = \begin{cases} 1 & ,x=0 \\ sg(x - (\mu z < x(2^z = x))) & ,x \neq 0 \end{cases}$

It's computable, so the set is recursive, r.e. and r.e. complement.

(c) $\{x \mid \phi_x \text{ is injective}\}$;

Solution:

It has r.e. complement.

The set is not r.e. Let $\mathscr{B} = \{f | f \in \mathscr{C}$ and f is injective $\}$. According to Rice theorem, ϕ_x is injective is not decidable. So it is not r.e..

Then the complement of the set is $A = \{x | \phi_x$ is not injective$\}$. A is r.e. because $x \in A$ iff $\exists y \exists z, y \neq z$ but $\phi_x(y) = \phi_x(z)$, which means ϕ_x is not injective is partially decidable.

(d) $\{x \mid y \in E_x\}$ (y is fixed);

Solution:

It's not recursive, it's r.e and it doesn't has r.e. complement.

According to the Kleene Normal Form Theorem. $x \in W_e \leftrightarrow \exists z.T(e, x, z)$, and $\phi_e(x) = (\mu z(T(e, x, z)))_1$. Then define $c(x) = sg(\mu z(T(x, (z)_1, (z)_2) \wedge ((z)_2)_1 = y) + 1)$. $c(x)$ is the partial characteristic function of $y \in E_x$. Therefore, the set is r.e.. By Rices Theorem, the set is not recursive. And then by complementation theorem, it does not have r.e. complement.

3. Prove following statements.

(a) Let $B \subseteq \mathbb{N}$ and $n > 1$; prove that B is r.e. then the predicate $M(x_1, \ldots, x_n)$ given by "$M(x_1, \ldots, x_n) \equiv 2^{x_1} 3^{x_2} \ldots p_n^{x_n} \in B$" is partially decidable.

Proof:

Because B is r.e. then its partial characteristic function χ_B is computable. So, the partial characteristic function of M is $\chi(x) = \chi(2^{x_1} 3^{x_2} ... p^{x_n})$ is computable. So, M is partially decidable.

(b) Prove that $A \subseteq \mathbb{N}^n$ is r.e. iff $\{2^{x_1} 3^{x_2} \ldots p_n^{x_n} \mid (x_1, \ldots, x_n) \in A\}$ is r.e..

Proof:

First, we proof $\Rightarrow$:

If $A \subseteq \mathbb{N}^n$ is r.e.. Then its partial characteristic function $\chi_A(\mathbf{x})$ is computable. So, the set's partial characteristic function $\chi_S(x) = c_A(2^{(x)_1} 3^{(x)_2} ... P_n^{(x)_n})$ is computable. So the set is r.e..

Then, we proof $\Leftarrow$:

If the set is r.e.. Then its partial characteristic function $c_S(x)$ is computable. So,$\chi_A(\mathbf{x})$ $= \chi_S(2^{x_1} 3^{x_2} ... P_n^{x_n})$ is computable, so $A \subseteq \mathbb{N}^n$ is r.e..