Lab09-Recursively Enumerable Set(2)

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class

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1. Suppose A is an r.e. set. Prove the following statements.

   (a) Show that the sets $\bigcup\limits_{x\in A} W_x$ and $\bigcup\limits_{x\in A} E_x$ are both r.e.

   Proof:

   $y \in \bigcup\limits_{x\in A} W_x \Leftrightarrow \exists z(z \in A)(P_z(y)\downarrow)$

   So, the first set is r.e..

   $y \in \bigcup\limits_{x\in A} E_x \Leftrightarrow \exists z_1 \exists z_2(z_1 \in A)(P_{z_1}(z_2)\downarrow y)$

   So, the second set is r.e..

   (b) Show that $\bigcap\limits_{x\in A} W_x$ is not necessarily r.e. (*Hint*: $\forall t \in \mathbb{N}$ let $K_t = \{x : P_x(x)\downarrow \text{ in t steps}\}$. Show that for any t, K_t is recursive; moreover $K = \bigcup\limits_{t\in\mathbb{N}} K_t$ and $\overline{K} = \bigcap\limits_{t\in\mathbb{N}} \overline{K}_t$.)

   Proof:

   $\forall t \in \mathbb{N}$ let $K_t = \{x : P_x(x)\downarrow \text{ in t steps}\}$, the characteristic function of K_t is:

$$c_{K_t} = \begin{cases} 1 & , P_x(x)\downarrow \text{ in t steps} \\ 0 & , \text{otherwise} \end{cases}$$

   So, c_{K_t} is computable, thus K_t and $\overline{K_t}$ are recursive.

   Moreover, $K = \bigcup\limits_{t\in\mathbb{N}} K_t$ and $\overline{K} = \bigcap\limits_{t\in\mathbb{N}} \overline{K}_t$.

   Since $\overline{K} = \bigcap\limits_{t\in\mathbb{N}} \overline{K}_t$ is not r.e.. Let $A^* = \mathbb{N}$, $\bigcap\limits_{t\in A^*} W_x$ is not r.e. So the set is not necessarily r.e..

2. Prove that $A \subseteq \mathbb{N}^n$ is r.e. iff $A = \varnothing$ or there is a total computable function $f : \mathbb{N} \to \mathbb{N}^n$ such that $A = Ran(\boldsymbol{f})$. (A *computable function* $\boldsymbol{f}$ from $\mathbb{N}$ to $\mathbb{N}^n$ is an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$ where each f_i is a unary computable function and $\boldsymbol{f}(x) = (f_1(x), \ldots, f_n(x))$.)

   Proof:

   First, we proof the $\Rightarrow$:

   If $A = \varnothing$, then completed.

   If $A \neq \varnothing$, then $\exists a \in A$. For P to be the program to compute the partial characteristic function of A. We have:

$$f(x) = \begin{cases} ((x)_1, (x)_2, ...(x)_n) & , P((x)_1, (x)_2, ...(x)_n)\downarrow \text{in } (x)_0 \text{ steps} \\ a & , \text{otherwise} \end{cases}$$

   So, f is computable, thus $A = Ran(f)$

   Then, we proof the $\Leftarrow$:

   If $A = \varnothing$, then completed.

If $A \neq \varnothing$, then we assume that there is a total and computable function f that $A = Ran(f)$.

Then, we have:$(x_1, x_2, ...x_n) \in A \Leftrightarrow \exists y(f(y) = (x_1, x_2, ...x_n))$.

Because of **Graph Theorem**, the right part is partially decidable. So,the set A is r.e..

3. Suppose that f is a total computable function, A is a recursive set and B is an r.e.set. Show that $f^{-1}(A)$ is recursive and that $f(A)$, $f(B)$ and $f^{-1}(B)$ are r.e. but not necessarily recursive. What extra information about these sets can be obtained if f is a bijection?

   Proof:

   $x \in f^{-1}(A) \Leftrightarrow f(x) \in A$

   So, $f^{-1}(A)$ is recursive.

   $x \in f(A) \Leftrightarrow \exists y(y \in A)(f(y) = x)$

   So, $f(A)$ is r.e..

   $x \in f(B) \Leftrightarrow \exists y(y \in B)(f(y) = x)$

   So, $f(B)$ is r.e..

   $x \in f^{-1}(B) \Leftrightarrow f(x) \in B$

   So, $f^{-1}(B)$ is r.e..

   If f is a bijection, $f(A)$ is recursive.

4. A set D is the difference of r.e. sets ($d.r.e.$) iff $D = A - B$ where A, B are both $r.e.$.

   (a) Show that the set of all $d.r.e.$ sets is closed under the formation of intersection.

   Proof:

   We assume thatA_1, A_2, B_1, B_2 are all r.e. and $D_1 = A_1 - B_1$ and $D_2 = A_2 - B_2$.

   So, there are computable functions f,g that $f(A_1, A_2, B_1, B_2)$ and $g(A_1, A_2, B_1, B_2)$ are both r.e..

   So $D_1 \cap D_2 = f(A_1, A_2, B_1, B_2) - g(A_1, A_2, B_1, B_2)$. So the set of all d.r.e. sets is closed under the formation of intersection.

   (b) Show that if $C_n = \{x \mid |W_x| = n\}$, then C_n is $d.r.e.$ for all $n \geq 0$.

   Proof:

   We assume that U is the set of all computable functions, and $C'_n = \{x \mid |W_x| \neq n\}$.

   Since $|W_x| = \sum \mathbf{1}\ (y \in W_x)$, so C'_n is r.e..

   So, $C_n = U - C'_n$ is d.r.e..