Signals, **Systems** and Control

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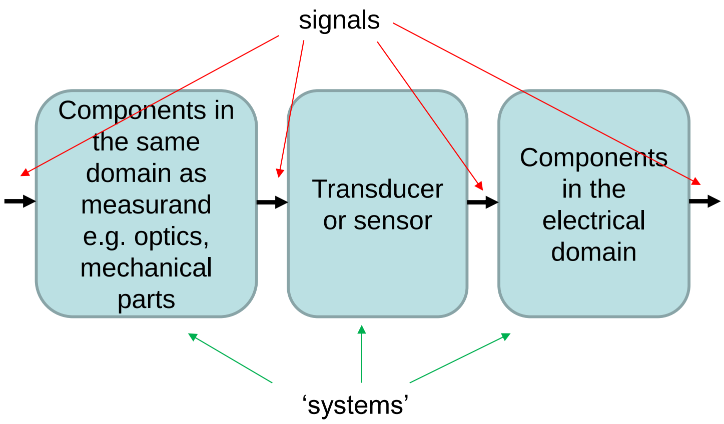
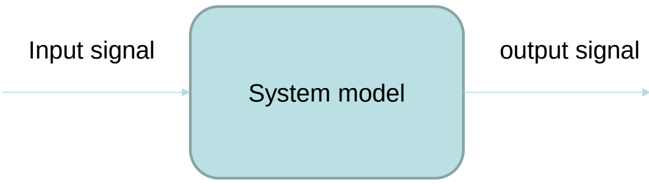
2.1.1 Modelling systems

So far, we have looked at modelling of signals and to complete our overall modelling framework we need to develop models of the components which act on the signals – the systems.



Often, we would like to have a model of our system such that it can be used to derive the output for a given input signal; in other situations, we might have input and output signals and want to understand the system caused the relation.



Although it might seem trivial to describe the output in terms of the input, for most systems it isn't quite as easy as you might first think. In the next few lectures we will look at modelling a class of systems known as Linear Time Invariant, or LTI, and see how the frequency domain can be a powerful tool in solving problems.

2.1.2 LTI Systems

The class of systems we are going to consider are 'Linear Time Invariant'. Perhaps the most significant properties of LTI systems are:

- **Linearity**

  If input $x_1(\mathrm{t})$ produces output $y_1(t)$, and input $x_2(\mathrm{t})$ produces output $y_2(t)$,

  then input $x_1(\mathrm{t}) + x_2(\mathrm{t})$ produces output $y_1(t) + y_2(t)$

  - This is called *SUPERPOSITION* and is a really useful property as it means you can break down you input signal into elements, consider the system response to those elements individually, and then sum the outputs.

- **Time invariance**

  If input $x_1(\mathrm{t})$ produces output $y_1(t)$, then the same input at a different time, $x_1(\mathrm{t} + \Delta t)$ produces the same output, at a corresponding time, $y_1(t + \Delta t)$

  - This is really saying that the system parameters do not change with time, just the input and output.

In addition, LTI systems have the additional properties:

- **Homogeneity**

  If input $x_1(\mathrm{t})$ gives output $y_1(t)$, then input $nx_1(\mathrm{t})$ will give output $ny_1(t)$

  In practice this means that the constant term in differential expressions has to be zero

- **Causal**

  As system is causal when the current output is a function of current or past inputs only

  i.e. the system has a memory of past inputs

  An acausal system is also a function of future inputs

- **Stability**

  For a Bounded Input, Bounded Output (BIBO)

2.1.3 Example LTI systems

To illustrate, let's think of some equations that might describe a system and if they pass the LTI test.

If y is the output, and x is the input, which equations could represent linear systems?

1. $y(t) = 5x(t)$

2. $y(t) = 5x(t) + 6$

3. $y(t) = x^2(t)$

4. $y(t) = \int x(t)\,dt$

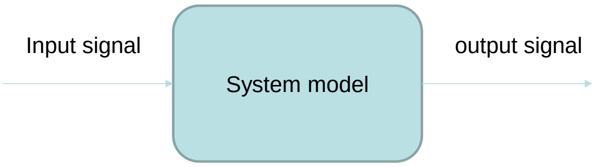
5. $y(t) = \frac{d(x(t))}{dt}$

6. $y(t) = \frac{d^2(x(t))}{dt^2} + \frac{d(x(t))}{dt} + \int x(t)\,dt$

$\boxed{\text{Answers: 1) Yes, 2) No, 3) No, 4) Yes, 5) Yes, 6) Yes}}$

2.1.4 Time domain Analysis

Thinking in the time domain

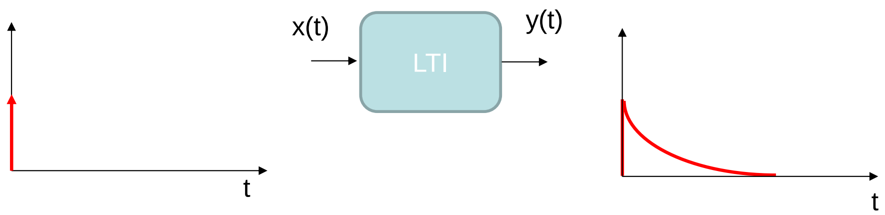


- If our system has no 'dynamics', i.e. it is just a gain or scaling factor, then the output only depends only on the current input. This case is simple to model.
- But we want to consider systems with *dynamic behaviour* – which is associated with systems which can store energy (so anything with a spring, mass, capacitor, inductor etc.). In these systems the output might respond slowly to the input and the action of a particular input extends beyond the instant at which it was applied. At any moment our output will be a function of present and past inputs.

We know that an LTI systems obeys superposition, so one approach to modelling a system is to 'add up' the contribution of inputs at various times.

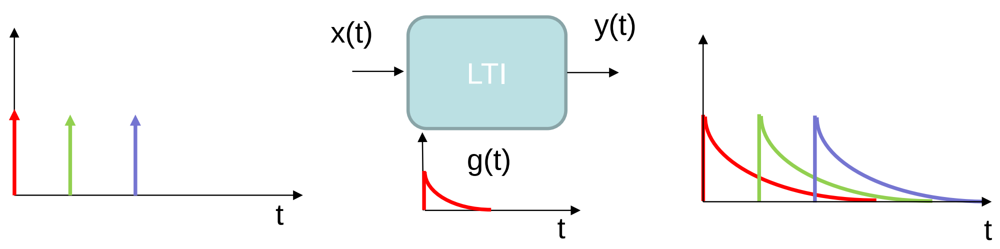
The impulse response of a system

The first step in time domain analysis is to understand how we approach the problem. We need to find some unitary signal in the time domain for which we can describe the response of the system. Then we describe our actual input signal as the sum of these unitary signals, and the output is the sum of all the individual responses. For example:

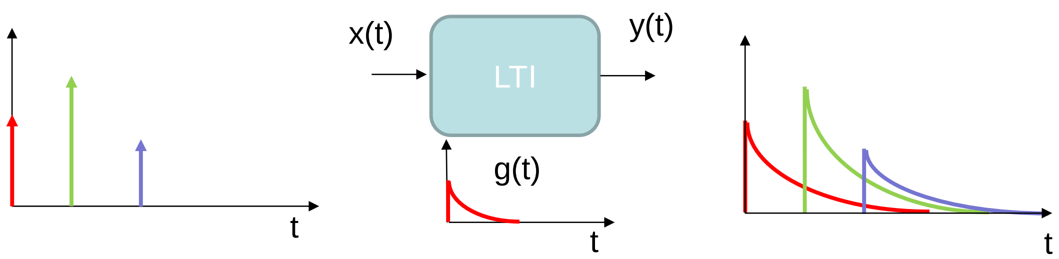


X(t) is the input to our system, y(t) is the output. If we make x(t) an impulse then the output, y(t) will be the ***impulse response*** of our system. This special case of output we will term g(t) – the system impulse response.

Now let's consider if we apply a train of pulses to our system:



The output will be the combined contribution of all the impulse responses. There is something important to tease out here – the first 'mind-bending' bit, and that is the input and output signal and the impulse response are all functions of time, but we don't want these all to share a common time axis variable (t). The impulse response needs a different argument that allows us to look backward in time to see the contribution of past input; this variable still exists on the time axis; From the point of view of the system the current time is always zero.



It is (hopefully) only a short stretch to suggest that if the magnitude of the input impulses change, then so do the output components – remember this is an LTI system!

The questions that some of you will now be asking may be along the lines – 1) 'the input signal has a bounded magnitude at any moment but the impulse (or Dirac delta function) has an infinite amplitude. Here you are implying that the input signal can be represented by a set of variable-amplitude impulses – how is this possible?' and related 2) 'the impulse function is infinite in magnitude – how do we apply this to a real system to determine the impulse response in the first place?'

We will answer the second question later, but for the first there are two options:

1. Take it on trust that a signal can be represented in this way – you don't *need* to understand it for this course

Or

2. you can read the following section!

2.1.5 The sifting property of the impulse function

The impulse function has a particular property called the *sifting* (or sometimes *sampling*) property defined as;

$$x(t) = \int x(\tau)\delta(t-\tau)d\tau$$

Where x is our signal and δ is the impulse function

You should note that 't' and 'τ' are both variables operating along the same axis. Mathematically this is relatively easy to think about, but if you want to give it physical context then 't' is our time now, as we would observe it, and 'τ' is our sliding time – looking backwards or forwards from where we are now.

To work out what is going on you need to remember that:

- The impulse (δ) function has a value that is zero everywhere except at 0, hence the integral is zero everywhere except when t=τ

- The impulse function has the property that its integral is 1. i.e. $\int_{-\infty}^{\infty} \delta(t)dt = 1$

- Hence when t=τ the integral returns the value of the signal at 't'

You might be thinking this looks quite pointless: why express the signal as the integral of weighted impulse functions when we could just directly substitute 't' into 'x(t)'? – after all we need to know the function 'x' to solve the integral anyway. But remember, all we wanted to show was that a signal *can* be represented as the sum of impulse functions, not for it to be any use *per-se*. (but note that is it useful for other things!).

Thus, using the definition of the sifting property we have shown that a signal can be represented by series of weighted (scaled) impulse functions.

The other interesting aspect is that the sifting property integral is very similar to the convolution integral we will look at next……

2.1.6 Convolution

Now that we have shown that our input signal is a series of impulses, and we know the response of our system to an impulse, we should be able to work out the output for any input – but be warned the concept we must use – convolution – can be hard to understand at first.

The integral describing convolution of ‘x’ (the input) with ‘g’ (the impulse response) is:

$$y(t) = \int x(\tau)g(t-\tau)d\tau$$

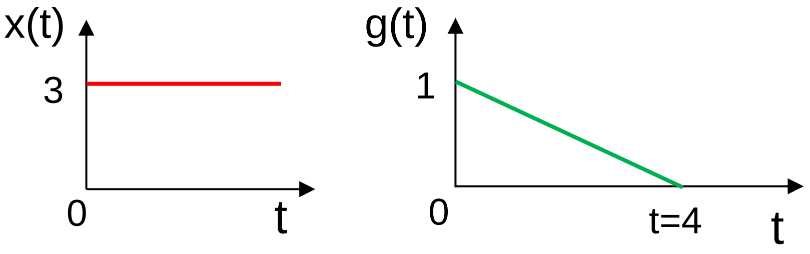
This is sometimes written as *(xconv.g)(t)*

Pay close attention to the arguments of the functions. Notice that the output (y) is a function of time and integral is over the variable τ (this is the opposite of the cross-correlation integrals we looked at in lecture 1.5). And notice because the argument of the impulse response is (t-τ) and t is fixed (the integral is over τ) this has the effect of ‘reversing’ the impulse response.

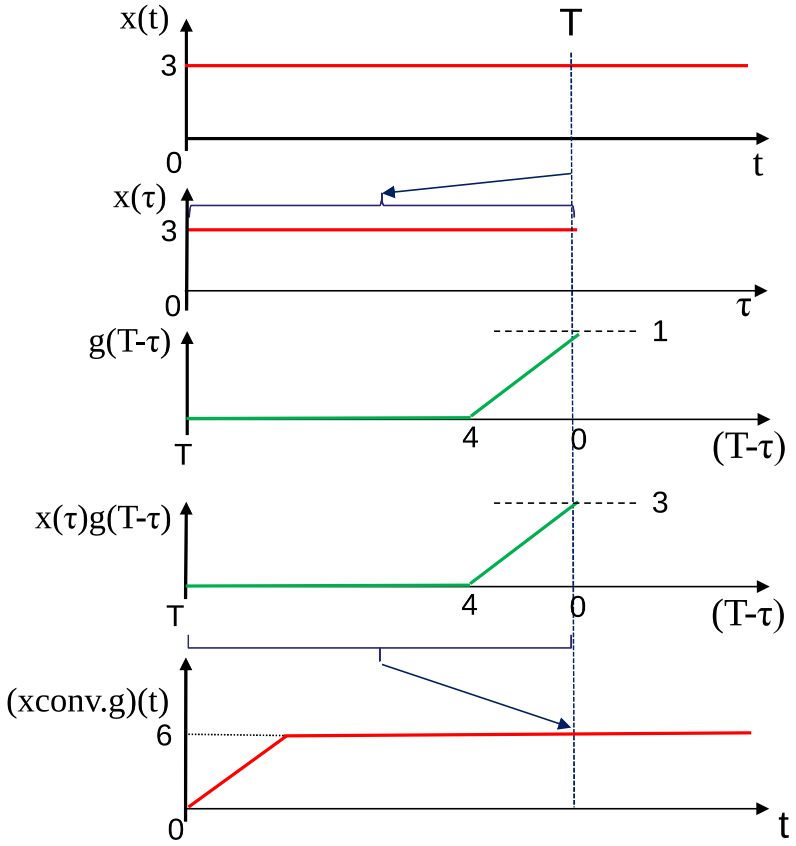
In words this integral is saying ‘the value at the instant in time now is the combined sum of the contribution from the current input and all the contributions from past inputs’.

Example

Let us consider a system with an input signal that starts at t = 0 and is a constant ‘3’. The impulse response of our system is given by $g(t) = 1 - {}^{t}/_{4}$, for 0<t<4



We can work through the convolution integral and work out the output. In the example below you can see the input signal at the top and the system output at the bottom; the steps in between are illustrating how the value of the output for a particular value of time, T, is calculated. The section of the convolution integral shown at each step is highlighted in red.



1) This is the input signal; we are going to look at just one instant in time, T.

2) $y(T) = \int x(\tau)g(T-\tau)d\tau$

3) $y(T) = \int x(\tau)g(T-\tau)d\tau$

4) $y(T) = \int x(\tau)g(T-\tau)d\tau$

5) $y(T) = \int x(\tau)g(T-\tau)d\tau$

An important property to realise when trying to understand convolution is that the impulse response is reversed in time because of how the argument is expressed. Some examples you will see, for example in online resources, use symmetrical functions which mask this.

If you are still struggling with the concept then this common analogy might help:

Sometimes convolution is described a 'windowing function' where the input signal is 'viewed' through the moving 'window' of the impulse response. If you want to push the analogy, think of sitting on a train, facing backwards with a window adjacent to you. The scenery you pass is the input signal, the window somewhat like the system response. Out of your window you will see the scenery you are passing right now plus, into the distance, scenery that you past a few seconds ago. The convolution is the aggregate picture.

2.1.7 Comparison of Convolution and Cross - Correlation

Both correlation and correlation use two variables operating on the time axis, we are using ‘t’ for our normal time, and ‘τ’ for our ‘sliding’, or ‘offset’ into past/future time.

$$(f corr. g)(\tau) = \int f\,(t) g(t + \tau) dt$$

Correlation is a function of offset, ‘τ’, but the integral is over time, ‘t’. It is the sum of the product of the functions over time, ‘t’, for various offsets, ‘τ’. The integral is solved for one value of offset, ‘τ’,

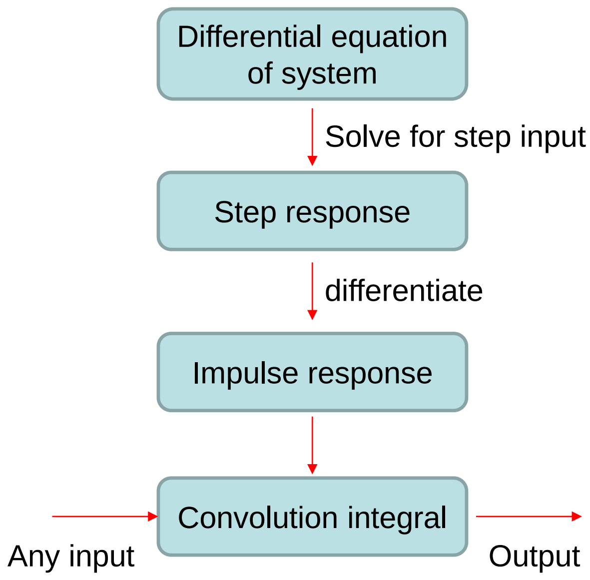
$$(f conv. g)(t) = \int f(\tau) g(t - \tau) d\tau$$

Convolution is a function of ‘t’, but the integral is over ‘τ’. The way the argument to ‘g’ is expressed ‘reverses’ the response. It is the sum of the product of the functions for over past times ‘τ’, at a particular time, ‘t’. The integral is solved for one value of time, ‘t’

2.1.8 Solving systems in the time domain

We have outlined some the steps needed to analyse systems in the time domain, but we have not shown the important step of how to derive the impulse response.

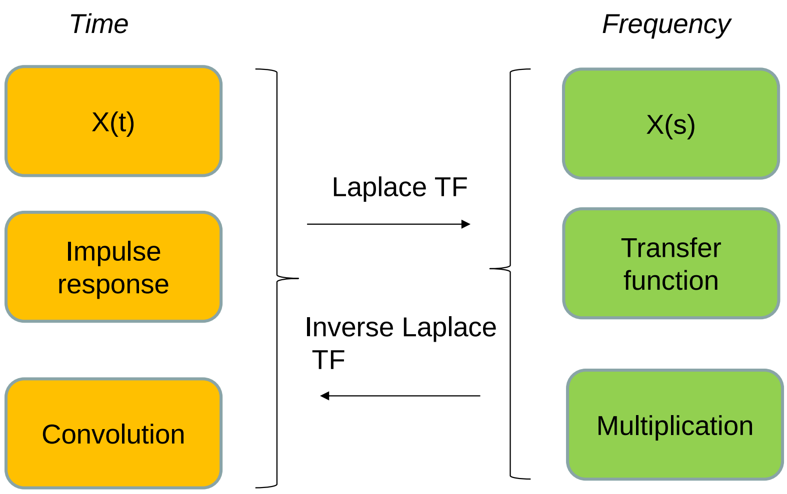
In practice we have to start with the differential equations that describe the system. We can solve these for a step input. The step response can them be differentiated to give the impulse response. Once we have the impulse response we can then use convolution to solve for any input.



2.1.9 Solving in the frequency domain

If solving our LTI systems in the time domain sounds complicated, then you would be right! The whole purpose of this lecture (alongside introducing the process convolution) was to illustrate just how difficult it can be to calculate the output of even a simple system.

The solution to this is to solve in the frequency domain - In the frequency domain impulse functions become transfer functions and convolution become multiplication



2.1.10 Test yourself

1) What are the key characteristics of an LTI system?
2) What is the impulse response of a system?
3) What is the name of mathematical process used to calculate the response of a system in the time domain?
4) In this process why do we need two independent variables operating on the time axis?
5) Starting with the differential equations describing a system, what are the steps required to be able to determine the output for any input?
6) When comparing solving in the time domain with the frequency domain, what is the frequency equivalent of the impulse response? And the time domain equivalence of multiplication in the frequency domain?