University of BRISTOL

ADVANCED STRUCTURES & MATERIALS

Finite Element Analysis Principles – Lecture 3

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Recap

Assuming
$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{d}$$

we turned
$$\int_S \boldsymbol{p}^\mathsf{T} \delta\boldsymbol{u}\, dS + \int_V \boldsymbol{b}^\mathsf{T} \delta\boldsymbol{u}\, dV = \int_V \boldsymbol{\sigma}^\mathsf{T} \delta\boldsymbol{\varepsilon}\, dV$$

into
$$\boldsymbol{f} = \boldsymbol{K}\boldsymbol{d}$$

where
$$\boldsymbol{K} = \int_V \boldsymbol{B}^\mathsf{T}\boldsymbol{E}\boldsymbol{B}\, dV \qquad \boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T}\boldsymbol{p}\, dS + \int_V \boldsymbol{N}^\mathsf{T}\boldsymbol{b}\, dV$$

Properties of the Finite Element Representation

Properties of the Finite Element Representation

- **Nodal displacement vector $\boldsymbol{d}$:**
  - has $m \times n$ rows.
  - where n is the number of nodes of the element, and m is the number of degrees of freedom for each node.
  - in the most general case, each node has 6 degrees of freedom, making $\boldsymbol{d}$ a $6n \times 1$ vector.

- **Equivalent nodal forces $\boldsymbol{f}$:**
  - has $m \times n$ rows.
  - is calculated using $\boldsymbol{N}$, ensuring equivalency with the work done by the distributed forces.

Properties of the Finite Element Representation

- Stiffness matrix $\boldsymbol{K}$: Every element k_{ij} represents the force f_i due to a unit displacement d_j, when all other displacements are zero.
  - It has the following properties:
    - Square: It has dimensions $mn \times mn$.
    - Symmetric: Recall $(\boldsymbol{ABC})^\top = \boldsymbol{C}^\top\boldsymbol{B}^\top\boldsymbol{A}^\top$, and $\boldsymbol{E} = \boldsymbol{E}^\top$. Then $(\boldsymbol{B}^\top\boldsymbol{EB})^\top = \boldsymbol{B}^\top\boldsymbol{EB}$, which implies $\boldsymbol{K}^\top = \boldsymbol{K}$.
    - Singular: At least two rows of the matrix are linearly dependent, meaning $\det(\boldsymbol{K}) = 0$, preventing the inversion of $\boldsymbol{K}$. Not considering boundary conditions, the system has infinitely many solutions corresponding to rigid body motions.
    - Positive Definite: $\boldsymbol{f}^\top\boldsymbol{d} = \boldsymbol{d}^\top\boldsymbol{Kd} \geq 0$. This ensures that the deformation energy is positive.

Properties of the Finite Element Representation

Shape functions, $\boldsymbol{N}$, play a crucial role in approximating the displacement field within an element.

The **shape functions** should possess certain **properties** to ensure accurate and stable numerical solutions.

Shape Functions Requirements

Shape Functions Requirements

- Continuity:
  - Shape functions must be continuous within the element and across the boundaries of adjacent elements to ensure compatibility.
  - For most problems, shape functions are required to have C^0-continuity, meaning the displacement field is continuous across element boundaries but not necessarily its derivatives.
  - For problems like beam bending, where the solution involves second-order derivatives (curvature), C^1-continuity may be required.
- Completeness:
  - Shape functions must be able to represent all possible rigid body motions and constant strain states within the element. This ensures that as the element size tends to zero, the solution converges to the exact solution.
  - For 1D elements, a complete linear shape function must represent both constant and linear terms (e.g., $N(x) = a + bx$). For higher-order elements, higher-degree polynomials may be required.

Shape Functions Requirements

- **Interpolation**:
  - The shape functions must correctly interpolate nodal values.
  - This means at each node the corresponding shape function must equal 1 at that node and 0 at all other nodes.
  - Mathematically: $N_i(x_j) = \delta_{ij}$, where $N_i(x_j)$ is the shape function corresponding to node i evaluated at node j, and δ_{ij} is the Kronecker delta.
- **Partition of Unity**:
  - The sum of all shape functions must equal 1 at any point in the element.
  - This ensures that a uniform displacement field is accurately represented by a combination of nodal displacements.
  - Formally: $\sum N_i(x) = 1$

Shape Functions Requirements

By satisfying these properties, shape functions ensure the numerical solution obtained using finite elements is accurate, stable, and converges to the exact solution as the mesh is refined.

Elements that meet the conditions of completeness and compatibility as defined are referred to as conforming elements: they exhibit a “bottom-up” convergence toward the true value of the total strain energy. In other words, coarser meshes are stiffer. Mesh refinement increasing compliance.

Element Formulation

Element Formulation

Picking element geometry, nodal d.o.f., and shape functions, the rest flows down.

Assuming $$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{d}$$

we turned $$\int_S \boldsymbol{p}^\mathsf{T} \delta \boldsymbol{u} \, dS + \int_V \boldsymbol{b}^\mathsf{T} \delta \boldsymbol{u} \, dV = \int_V \boldsymbol{\sigma}^\mathsf{T} \delta \boldsymbol{\varepsilon} \, dV$$

into $$\boldsymbol{f} = \boldsymbol{K}\boldsymbol{d}$$

where $$\boldsymbol{K} = \int_V \boldsymbol{B}^\mathsf{T} \boldsymbol{E} \boldsymbol{B} \, dV \qquad \boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T} \boldsymbol{p} \, dS + \int_V \boldsymbol{N}^\mathsf{T} \boldsymbol{b} \, dV$$

In some cases, this process can be skipped

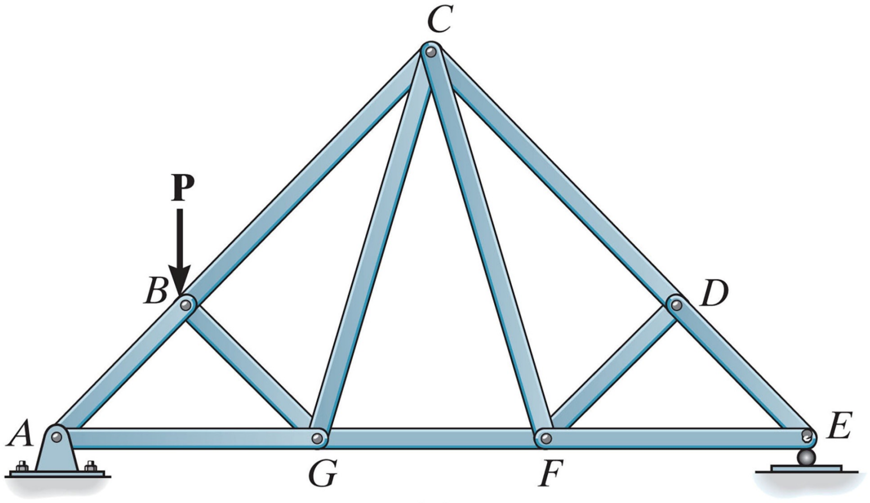
Direct Element Derivation

Structures with Discrete Joints: Direct Element Derivation

- In the case of a structure with discrete joints, the transition from the differential equations governing the mechanics of continuous elastic systems to a set of algebraic matrix equations can be more direct and based on simple mechanics.
- The differential equations for the elasticity of each structural element can initially be solved in terms of nodal displacements.
- The requirement to satisfy the boundary conditions of the various adjacent elements, which are essentially the conditions for proper assembly, leads directly to a set of algebraic equations, expressible in matrix form, used to solve the structure.
- These solutions must then be supplemented, taking into account the effects, in terms of deformations and stress characteristics, of possible loads acting in non-nodal positions.

1D Elasticity: Bar Elements

Truss Analysis – Assumptions



- Members are straight and 1D.
- Members of the truss are connected at their ends by frictionless pins or hinges.
- The truss is loaded and supported only at its joints.
- Forces in the members of the truss are purely axial.

Governing Equations of Linear Elasticity: Bar Elements

Ensuing kinematics

$$
\begin{aligned}
u_x &= u_x(x, y, z) = u_x(x) \\
u_y &= u_y(x, y, z) = 0 \\
u_z &= u_z(x, y, z) = 0
\end{aligned}
$$

Implying…

Governing Equations of Linear Elasticity: Bar Elements

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x},$$
$$\epsilon_{yy} = \frac{\partial u_y}{\partial y},$$
$$\epsilon_{zz} = \frac{\partial u_z}{\partial z},$$
$$\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x},$$
$$\epsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y},$$
$$\epsilon_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x},$$
$$\epsilon_{yy} = 0,$$
$$\epsilon_{zz} = 0,$$
$$\epsilon_{xy} = 0,$$
$$\epsilon_{yz} = 0,$$
$$\epsilon_{zx} = 0$$

Governing Equations of Linear Elasticity: Bar Elements

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right)$$
$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right)$$
$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right)$$
$$\epsilon_{xy} = 2\frac{1+\nu}{E}\sigma_{xy}$$
$$\epsilon_{yz} = 2\frac{1+\nu}{E}\sigma_{yz}$$
$$\epsilon_{zx} = 2\frac{1+\nu}{E}\sigma_{zx}$$

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E}$$
$$\epsilon_{yy} = -\frac{\nu}{E}\sigma_{xx}$$
$$\epsilon_{zz} = -\frac{\nu}{E}\sigma_{xx}$$
$$\epsilon_{xy} = 0$$
$$\epsilon_{yz} = 0$$
$$\epsilon_{zx} = 0$$

Governing Equations of Linear Elasticity: Bar Elements

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

$$\Rightarrow$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \cancel{b_x} = 0$$
$$0 = 0$$
$$0 = 0$$

Governing Equations of Linear Elasticity: Bar Elements

What shall we expect?

$$\frac{\partial \sigma_{xx}}{\partial x} = 0$$

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E}$$
$$\epsilon_{yy} = -\frac{\nu}{E}\sigma_{xx}$$
$$\epsilon_{zz} = -\frac{\nu}{E}\sigma_{xx}$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}$$

Constant σ_{xx} along x ➡ Constant ϵ_{xx} along x, ➡ Linear u_x along x,

$\epsilon_{yy}, \epsilon_{zz}$ dependent variables

2 nodes sufficient to capture the displacement field

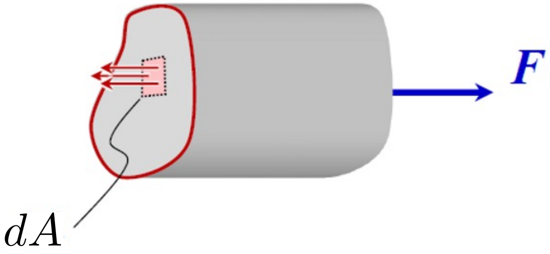
Bar Element Equations

Bar Element Equations

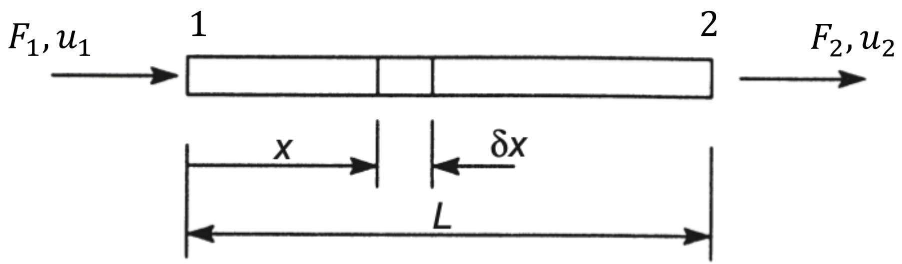
Noting that

$$F = \int_A \sigma_{xx} \, dA \quad \Rightarrow \quad \sigma_{xx} = \frac{F}{A}$$

we can then manipulate the governing equations.



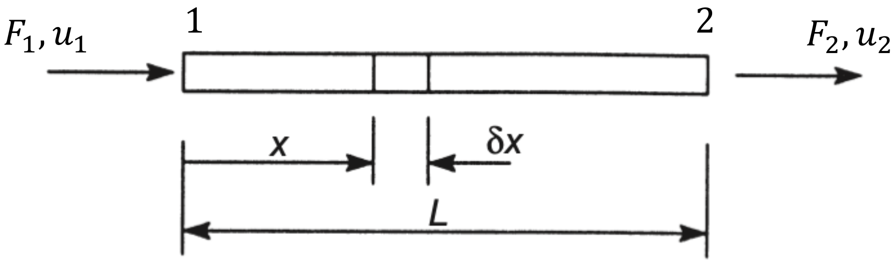
Bar Element Equations



$$\left.\begin{aligned} \epsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \epsilon_{xx} &= \frac{\sigma_{xx}}{E} \end{aligned}\right\} \quad du_x = \epsilon_{xx} dx = \frac{\sigma_{xx}}{E} dx = \frac{F}{EA} dx \quad \Rightarrow \quad \int_L du_x = \int_L \frac{F}{EA} dx$$

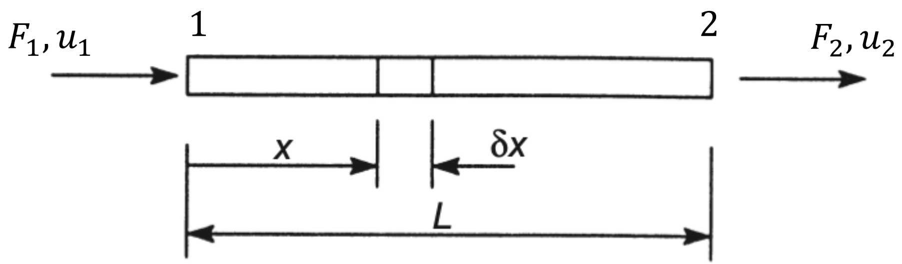
This allows us to relate forces and displacements at 1 and 2.

Bar Element Equations



$$\int_L du_x = \int_L \frac{F}{EA} dx \Rightarrow \int_0^x du_x = \frac{F}{EA} \int_0^x dx \Rightarrow u_x - u_1 = \frac{F}{EA} x \Rightarrow u_x = \frac{F}{EA} x + u_1$$

Bar Element Equations



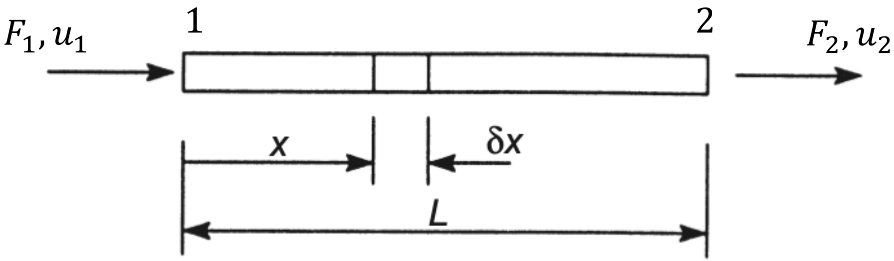
$$F = F_2 = -F_1$$

$$u_x = \frac{F}{EA}x + u_1 \quad \Rightarrow \quad u_2 = \frac{F_2}{EA}L + u_1$$

$$F_2 = \frac{EA}{L}(u_2 - u_1)$$

$$-F_1 = \frac{EA}{L}(u_2 - u_1)$$

Bar Element Equations



$$F_2 = \frac{EA}{L}(u_2 - u_1)$$

$$F_1 = \frac{EA}{L}(u_1 - u_2)$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

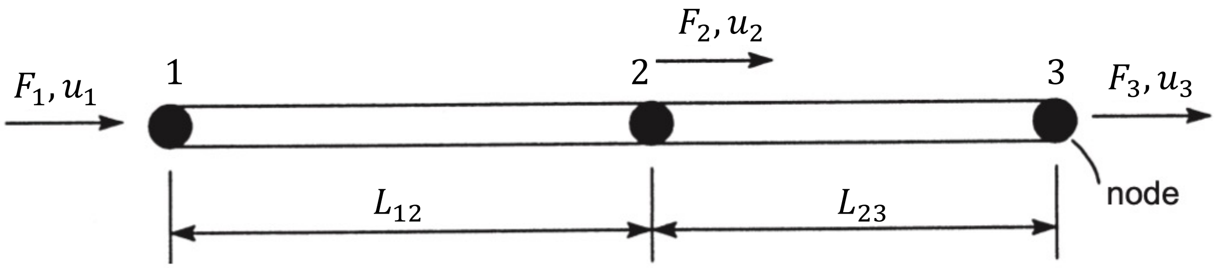
$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{u}$$

2 noded element
1 d.o.f. per node
2×2 stiffness matrix

Element Assembly

Let's consider an assembly of two elements

Suppose that we have two axially loaded members, 1-2 and 2-3, in line and connected at 2. The structure will be characterised by…



$\boldsymbol{F}$ – vector of 3 equivalent nodal forces

$\boldsymbol{u}$ – vector of 3 nodal displacements

$\boldsymbol{K}$ – 3 by 3 global stiffness matrix – from two 2 by 2 element matrices? How does this work?

Let's consider an assembly of two elements

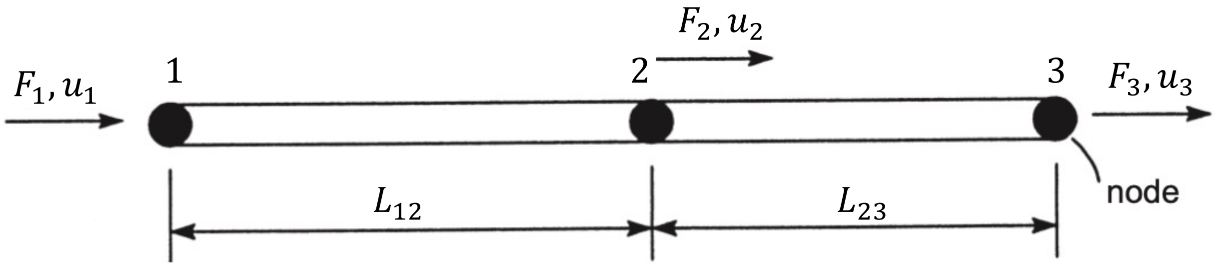
The global governing equations will then take the form

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

How do we assemble the global stiffness matrix?

Let's start filling it with zeros.

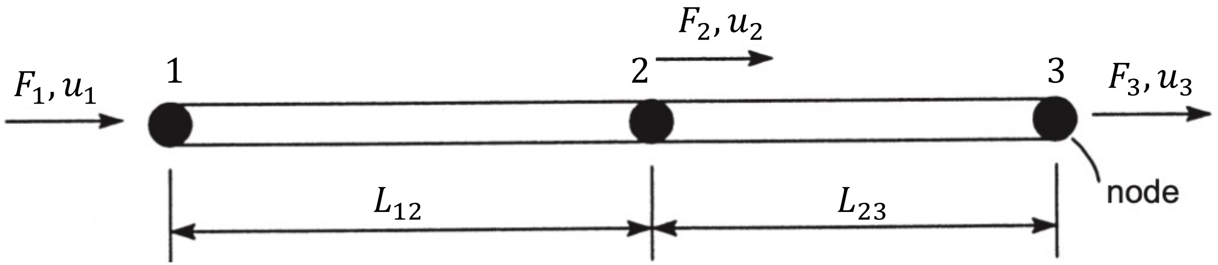
Let's consider an assembly of two elements



Recall displacements must be continuous and forces balanced. Hence,

$$u_2^{12} = u_2^{23} = u_2 \qquad\qquad F_2 = F_2^{12} + F_2^{23}$$

Let's consider an assembly of two elements



It follows that

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} + F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^{12} \\ u_2^{12} = u_2^{23} \\ u_3^{23} \end{bmatrix}$$

Let's consider an assembly of two elements

Considering the matrix equation for each element

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix} \qquad \begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$\Downarrow$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \frac{EA}{L_{12}} u_2^{12} + \frac{EA}{L_{23}} u_2^{23} - \frac{EA}{L_{23}} u_3$$

$$\Downarrow$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) u_2 - \frac{EA}{L_{23}} u_3$$

Let's consider an assembly of two elements

So, we have established that

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix} \qquad \begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) u_2 - \frac{EA}{L_{23}} u_3$$

Let's put it all together…

Let's consider an assembly of two elements

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix} \qquad \begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) u_2 - \frac{EA}{L_{23}} u_3$$

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} + F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^{12} \\ u_2^{12} = u_2^{23} \\ u_3^{23} \end{bmatrix}$$

Let's consider an assembly of two elements

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix} \qquad \begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) u_2 - \frac{EA}{L_{23}} u_3$$

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} + F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^{12} \\ u_2^{12} = u_2^{23} \\ u_3^{23} \end{bmatrix}$$

Let's consider an assembly of two elements

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix} \qquad \begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) u_2 - \frac{EA}{L_{23}} u_3$$

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} + F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} & 0 \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^{12} \\ u_2^{12} = u_2^{23} \\ u_3^{23} \end{bmatrix}$$

Let’s consider an assembly of two elements

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix} \qquad \begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) u_2 - \frac{EA}{L_{23}} u_3$$

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} + F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} & 0 \\ -\frac{EA}{L_{12}} & \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) & -\frac{EA}{L_{23}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^{12} \\ u_2^{12} = u_2^{23} \\ u_3^{23} \end{bmatrix}$$

Let's consider an assembly of two elements

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix} \qquad \begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$F_2 = F_2^{12} + F_2^{23} = -\frac{EA}{L_{12}} u_1 + \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) u_2 - \frac{EA}{L_{23}} u_3$$

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} + F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} & 0 \\ -\frac{EA}{L_{12}} & \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}} \right) & -\frac{EA}{L_{23}} \\ 0 & -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^{12} \\ u_2^{12} = u_2^{23} \\ u_3^{23} \end{bmatrix}$$

Let's consider an assembly of two elements

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} \\ -\frac{EA}{L_{12}} & \frac{EA}{L_{12}} \end{bmatrix} \end{array} \begin{bmatrix} u_1^{12} \\ u_2^{12} \end{bmatrix}$$

$$\begin{bmatrix} F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} \frac{EA}{L_{23}} & -\frac{EA}{L_{23}} \\ -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \end{array} \begin{bmatrix} u_2^{23} \\ u_3^{23} \end{bmatrix}$$

$$\begin{bmatrix} F_1^{12} \\ F_2^{12} + F_2^{23} \\ F_3^{23} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{EA}{L_{12}} & -\frac{EA}{L_{12}} & 0 \\ -\frac{EA}{L_{12}} & \left(\frac{EA}{L_{12}} + \frac{EA}{L_{23}}\right) & -\frac{EA}{L_{23}} \\ 0 & -\frac{EA}{L_{23}} & \frac{EA}{L_{23}} \end{bmatrix} \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^{12} \\ u_2^{12} = u_2^{23} \\ u_3^{23} \end{bmatrix}$$

Global Matrix Assembly

The formation of the stiffness matrix for a complete structure is carried out as follows:

- Terms of the form k_{ii} on the main diagonal consist of the sum of the stiffnesses of all the structural elements meeting at node i.
- Off-diagonal terms of the form k_{ij} consist of the sum of the stiffnesses of all the elements connecting node i to node j.

From local to global coordinates

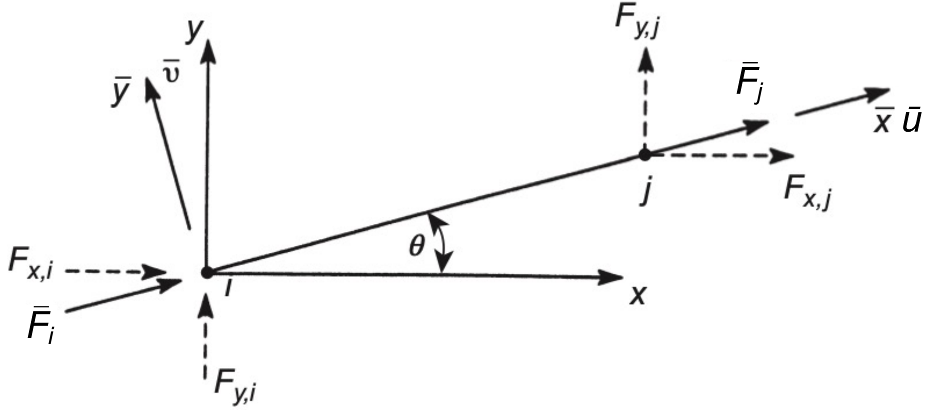
From local to global coordinates

- In a practical situation a bar element could be part of a truss comprising **members** set at various angles to one another.
- To assemble a stiffness matrix for a complete structure, we need to refer axial forces and displacements to a common, or global, axis system.

From local to global coordinates

Consider the member in figure.

- It is inclined at an angle θ to a global axis system denoted by xy.
- It connects node i to node j.
- It has *local* axes $\bar{x}$, $\bar{y}$.
- Nodal forces and displacements referred to local axes are written as $\bar{F}$, $\bar{u}$.



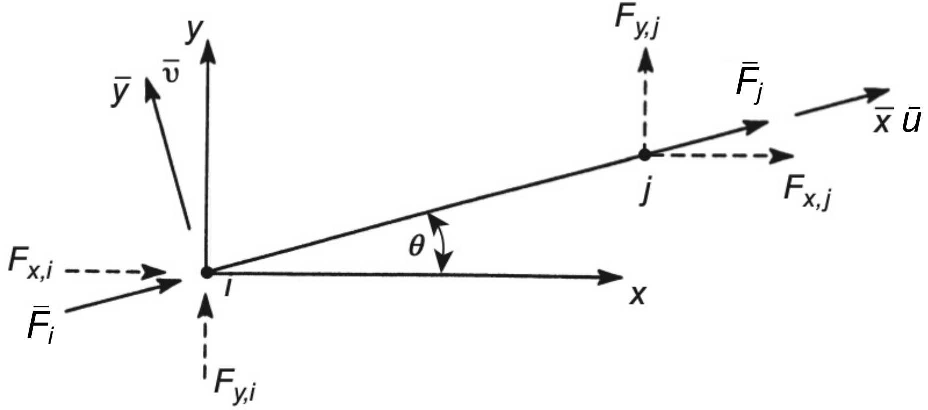
From local to global coordinates

by comparison with

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

we see that

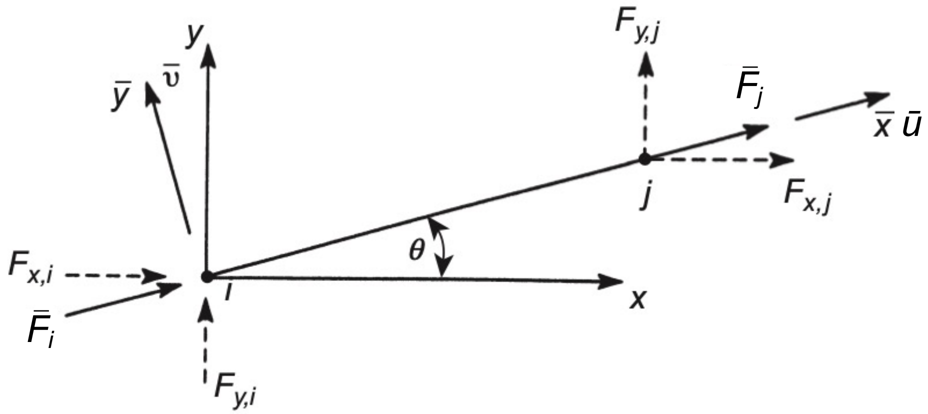
$$\begin{bmatrix} \bar{F}_i \\ \bar{F}_j \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \bar{u}_i \\ \bar{u}_j \end{bmatrix}$$



From local to global coordinates

Also,

$$F_{x,i} = \bar{F}_i \cos\theta \qquad F_{x,j} = \bar{F}_j \cos\theta$$
$$F_{y,i} = \bar{F}_i \sin\theta \qquad F_{y,j} = \bar{F}_j \sin\theta$$



which, in matrix form, becomes

$$\begin{bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} \bar{F}_i \\ \bar{F}_j \end{bmatrix}$$

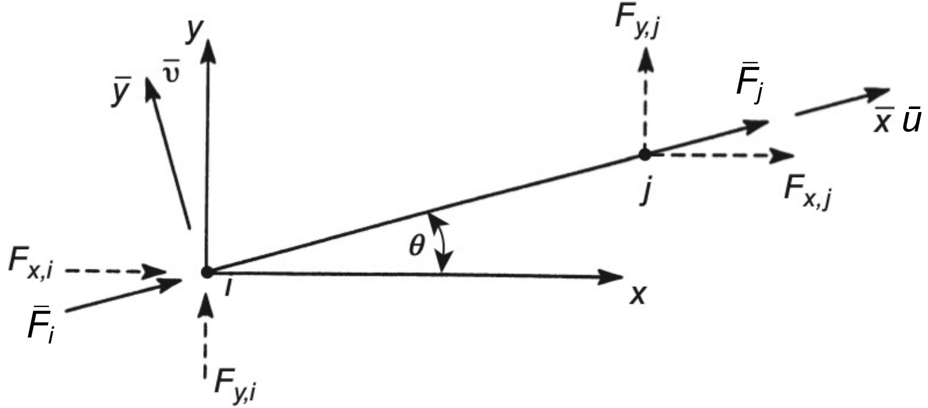
From local to global coordinates

Similarly,

$$\bar{F}_i = F_{x,i}\cos\theta + F_{y,i}\sin\theta$$
$$\bar{F}_j = F_{x,j}\cos\theta + F_{y,j}\sin\theta$$

which, in matrix form, becomes

$$\begin{bmatrix}\bar{F}_i\\ \bar{F}_j\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0 & 0\\ 0 & 0 & \cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}F_{x,i}\\ F_{y,i}\\ F_{x,j}\\ F_{y,j}\end{bmatrix}$$



From local to global coordinates

Setting

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} = \boldsymbol{T}$$

Transformation Matrix

$$\begin{bmatrix} \bar{F}_i \\ \bar{F}_j \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{bmatrix}$$

$$\bar{\boldsymbol{F}} = \boldsymbol{T}\boldsymbol{F}$$

$$\begin{bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} \bar{F}_i \\ \bar{F}_j \end{bmatrix}$$

$$\boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\bar{\boldsymbol{F}}$$

From local to global coordinates

Displacements transform in the same way

$$\bar{\boldsymbol{F}} = \boldsymbol{T}\boldsymbol{F} \qquad \Rightarrow \qquad \bar{\boldsymbol{u}} = \boldsymbol{T}\boldsymbol{u}$$

$$\boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\bar{\boldsymbol{F}} \qquad \qquad \boldsymbol{u} = \boldsymbol{T}^{\mathsf{T}}\bar{\boldsymbol{u}}$$

Since,

$$\left.\begin{array}{c} \begin{bmatrix} \bar{F}_i \\ \bar{F}_j \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} \bar{u}_i \\ \bar{u}_j \end{bmatrix} \\ \bar{\boldsymbol{F}} = \bar{\boldsymbol{K}}\bar{\boldsymbol{u}} \end{array}\right\} \quad \boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\bar{\boldsymbol{F}} \Rightarrow \boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\bar{\boldsymbol{K}}\bar{\boldsymbol{u}} \Rightarrow \boldsymbol{F} = \boldsymbol{T}^{\mathsf{T}}\bar{\boldsymbol{K}}\boldsymbol{T}\boldsymbol{u}$$

From local to global coordinates

In conclusion, the nodal forces vector, $\boldsymbol{F}$, referred to the global axes is related to the corresponding nodal displacements, $\boldsymbol{u}$, by

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{u}$$

in which the element stiffness matrix referred to global coordinates is

$$\boldsymbol{K} = \boldsymbol{T}^{\mathsf{T}}\bar{\boldsymbol{K}}\boldsymbol{T}$$

From local to global coordinates

For completeness,

$$\boldsymbol{K} = \boldsymbol{T}^\mathsf{T} \bar{\boldsymbol{K}} \boldsymbol{T} = \frac{AE}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Solution of the System of Equations

On the Issue of the Singularity of $\boldsymbol{K}$

Having derived equations of the form

$$\boldsymbol{f} = \boldsymbol{K}\boldsymbol{u}$$

the issue of the singularity of $\boldsymbol{K}$ must be addressed.

Otherwise, we can’t compute

$$\boldsymbol{u} = \boldsymbol{K}^{-1}\boldsymbol{f}$$

The solution is immediate!

Partitioning of the Global Stiffness Matrix

- By applying the boundary conditions, the nodal displacement vector can be split into two sub-vectors, denoted by $\boldsymbol{u}_u$ and $\boldsymbol{u}_k$, which respectively indicate the unknown and known displacements.
- Where kinematic conditions are imposed, the mechanical conditions will be unknown.
- Consequently, at the constrained nodes related to $\boldsymbol{u}_k$, unknown forces (reaction forces) will act.
- This results in the force vector being divided into two sub-vectors, called $\boldsymbol{f}_k$ and $\boldsymbol{f}_u$, which respectively contain known and unknown quantities.

Partitioning of the Global Stiffness Matrix

It follows that $\boldsymbol{K}$ and the solving equations can be rearranged as

$$\begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{uk} \\ \boldsymbol{K}_{ku} & \boldsymbol{K}_{kk} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_u \\ \boldsymbol{u}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_k \\ \boldsymbol{f}_u \end{bmatrix}$$

In rearranging $\boldsymbol{K}$, it is essential to remember that shifts in the rows of $\boldsymbol{u}$ and $\boldsymbol{f}$ correspond to movements of the columns and rows of $\boldsymbol{K}$, respectively.

Partitioning of the Global Stiffness Matrix

Following the rearrangement, the system

$$\begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{uk} \\ \boldsymbol{K}_{ku} & \boldsymbol{K}_{kk} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_u \\ \boldsymbol{u}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_k \\ \boldsymbol{f}_u \end{bmatrix}$$

can be solved to yield

$$\boldsymbol{u}_u = \boldsymbol{K}_{uu}^{-1} \left(\boldsymbol{f}_k - \boldsymbol{K}_{uk} \boldsymbol{u}_k \right)$$

$$\boldsymbol{f}_u = \boldsymbol{K}_{ku} \boldsymbol{K}_{uu}^{-1} \left(\boldsymbol{f}_k - \boldsymbol{K}_{uk} \boldsymbol{u}_k \right) + \boldsymbol{K}_{kk} \boldsymbol{u}_k$$