EQUATIONS OF MOTION 1

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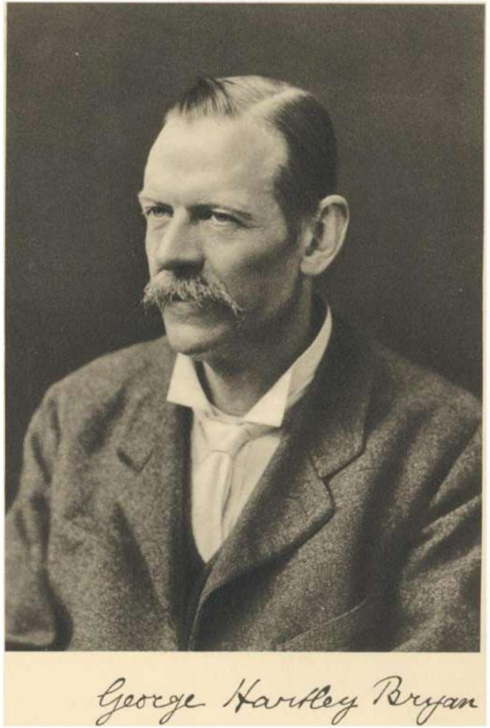
References

1. *Flight Dynamics Principles* – M V Cook, 1997, 2007 or 2013
2. David A. Caughey, M&AE 5070 “Introduction to Aircraft Stability and Control”, Sibley School of Mechanical & Aerospace Engineering, Cornell University, Ithaca, New York, 2011. Chapter 4 “Dynamical Equations for Flight Vehicles” https://courses.cit.cornell.edu/mae5070/DynamicEquations.pdf
3. *Aircraft Control and Simulation* – Stephens & Lewis, 2003
4. *Dynamics of Atmospheric Flight* – B. Etkin, 2012
5. (*Aircraft Handling Qualities* – Hodgkinson, 1999)

… and numerous other texts.

Refs 1 and 2 are used for the derivation in these notes.

Historical Background



- George Bryan addressed the problem of stability and control in the first decade of the 20th century.
- Bryan published 'Stability in Aviation' in 1911.
- The basis of his treatment, with very few changes, is still in everyday use.
- Bryan developed the general equations of motion of a rigid body with six degrees of freedom to describe the motion of the aircraft.

Why do we need a mathematical model?

- The equations of motion form a mathematical model, for a rigid body aircraft, in our case.
- The model provides a complete description of the response to controls (e.g. aileron, rudder and elevator) or disturbances, subject only to modelling limitations.
- Output can be measured in terms of displacement, velocity and acceleration.
- The mathematical model can be augmented to include equations which describe engine gyroscopic effects and control system dynamics.
- e.g. ASTRAEA II – Autonomous Air-to-Air Refuelling (AAAR).

ASTRAEA = Autonomous Systems Technology Related Airborne Evaluation & Assessment … *a UK industry-led consortium focusing on the technologies, systems, facilities, procedures and regulations to allow autonomous vehicles to operate safely and routinely in civil airspace over the UK.*

Why do we need a mathematical model?

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e.g. simulation

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Why do we need a mathematical model?

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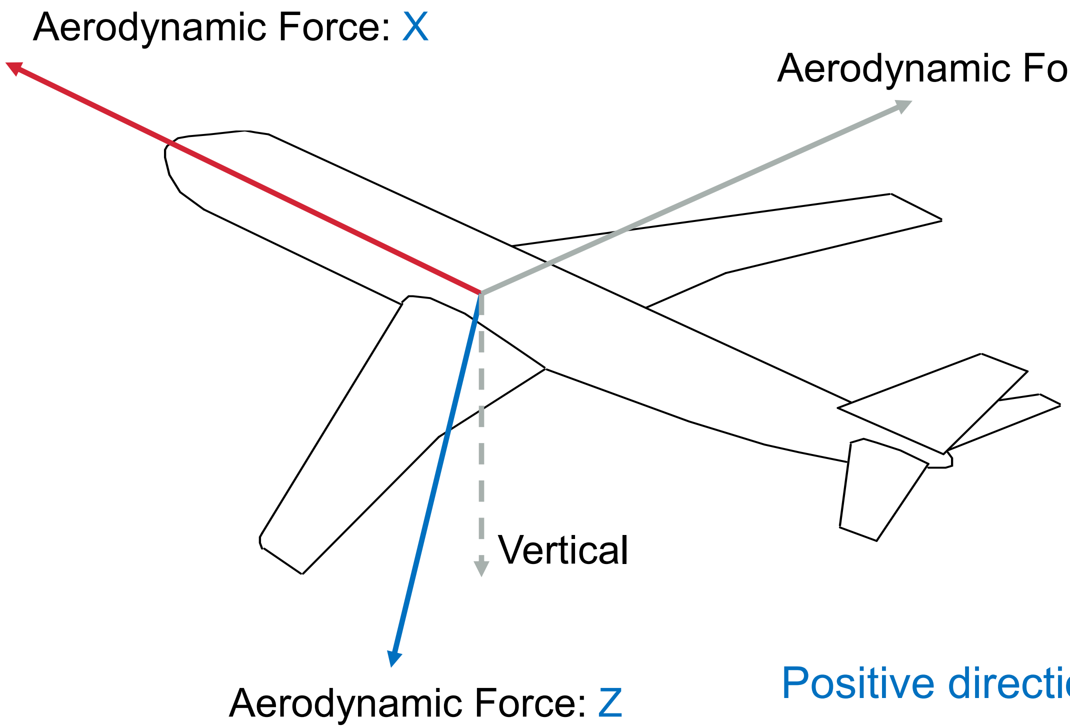
- Simulation
  - e.g. pilot training
  - UAS simulation
  - step & piloted response studies
  - air accident analysis
- Stability analysis
  - linear and nonlinear
- Handling qualities
- Aircraft design & development
- Control system design
- Flight clearance

Introduction to the Equations of Motion

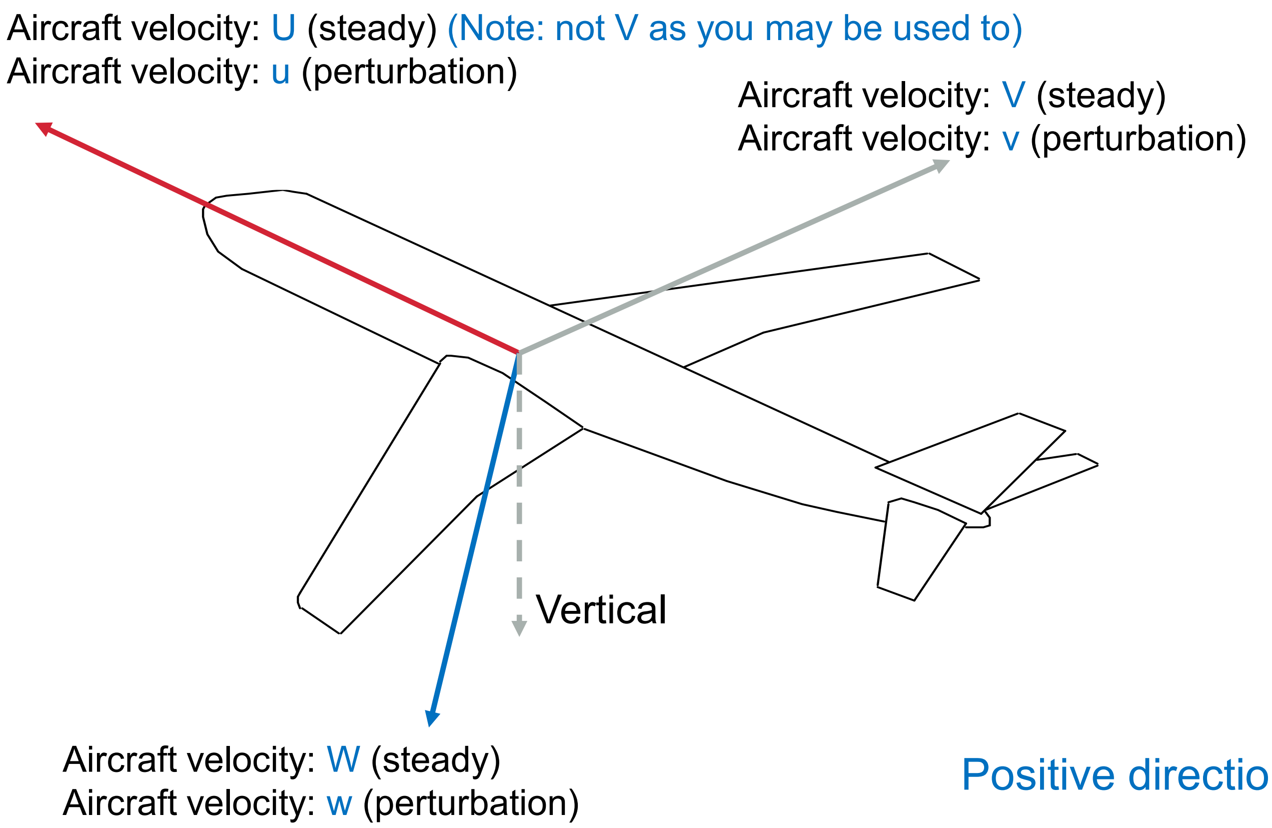
- You are NOT required to memorise a full derivation of the equations.
- You DO need to understand their structure and the terms involved.
- The derivation of the equations is mainly from Cook 2013 and partly from Caughey 2011.
- The general nonlinear equations are given first and simplified versions presented later.
- You DO *need to learn the notation from previous lectures!*

Body Axes Notation and Sign Conventions



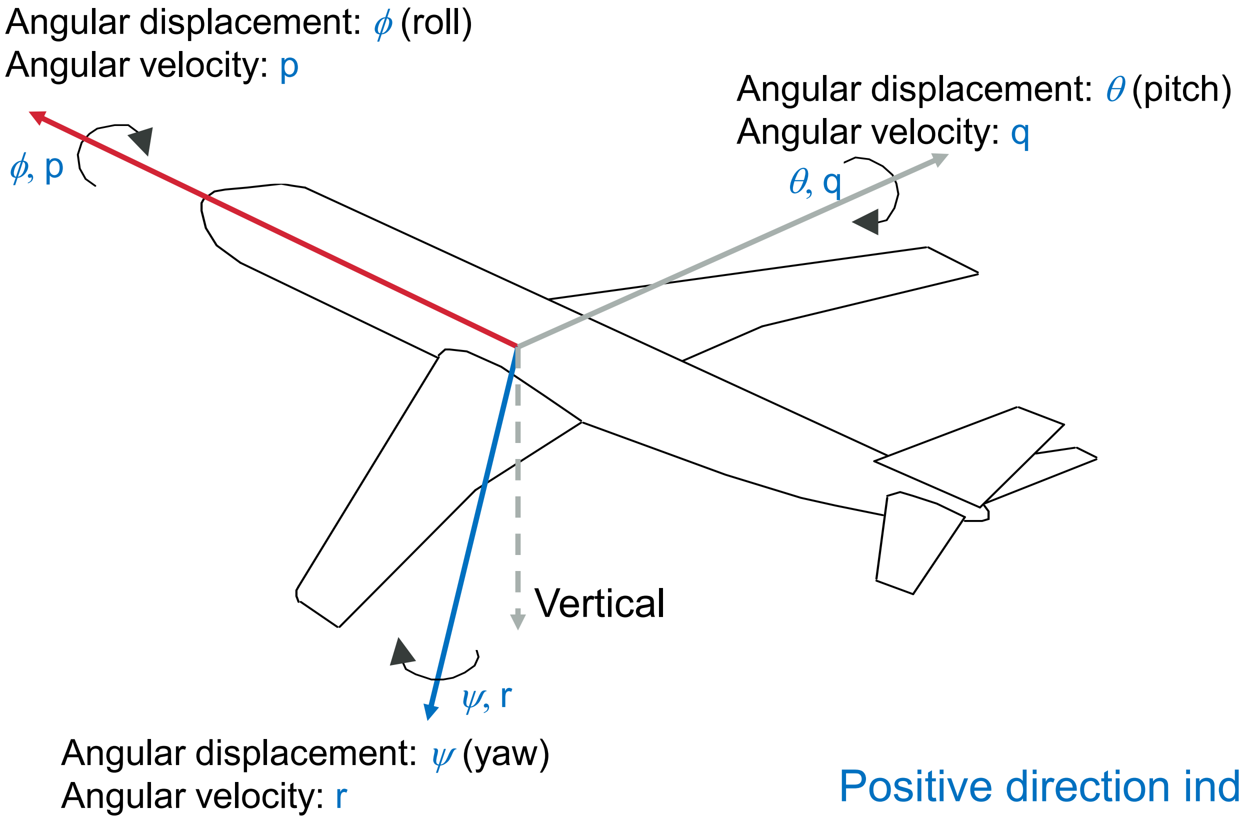
Positive direction indicated

Body Axes Notation and Sign Conventions



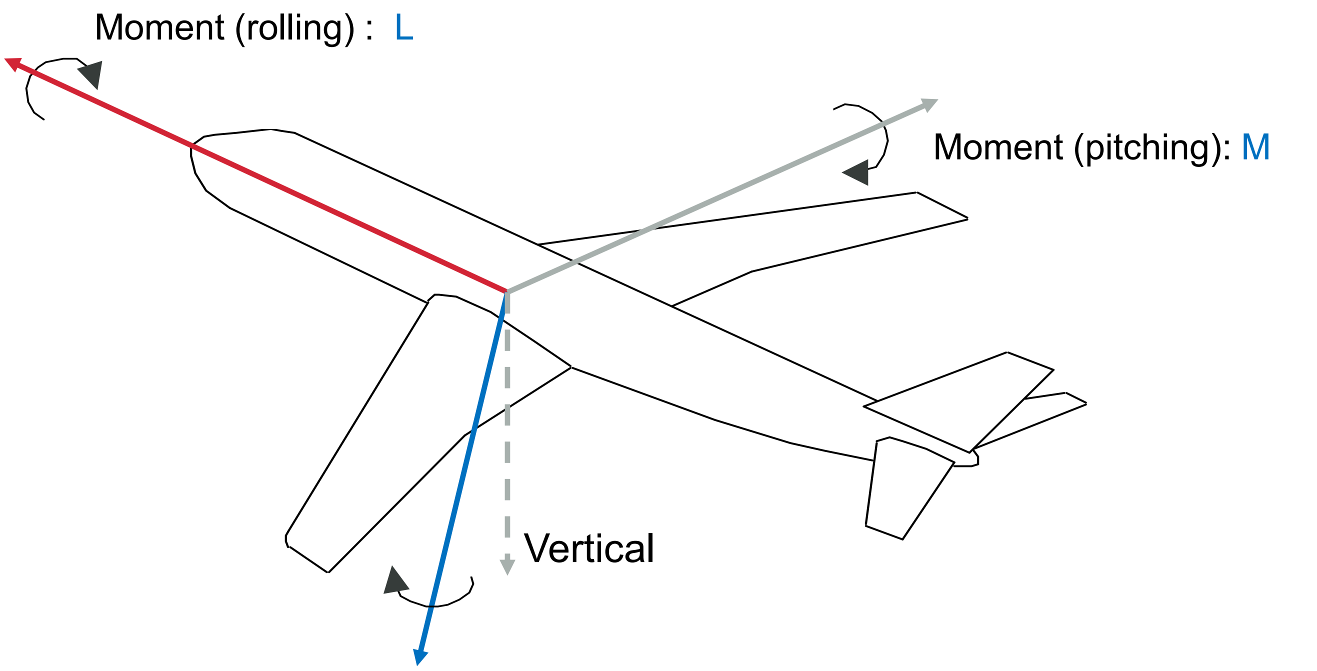
Positive direction indicated

Body Axes Notation and Sign Conventions



Positive direction indicated

Body Axes Notation and Sign Conventions



Positive direction indicated

Derivation of the Equations of Motion

Generalised Newton's Second Law applied to rigid body:

rate of change of linear momentum of a rigid body in a particular direction	=	sum of the components of the external forces acting on the body in that direction

and:

rate of change of angular momentum of a rigid body about a particular fixed axis	=	sum of the moments of the external forces acting on the body about that axis

where motions are relative to an **inertial** frame.

Derivation of the Equations of Motion

These are commonly interpreted as:

Force = mass × inertial acceleration $\boxed{ma_i = \sum F_i}$

Moment = moment of inertia × angular accel. $\boxed{I_j \alpha_j = \sum M_j}$

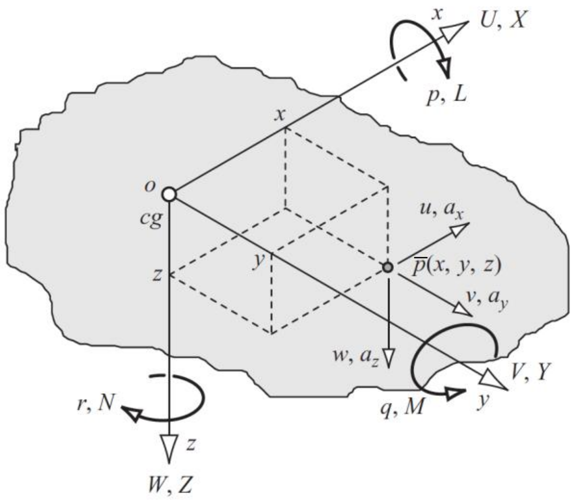
where:
m = mass of the body
a_i = acceleration in direction i
F_i = the set of external forces in direction i

I_j = moment of inertia of the body about chosen axis j
α_j = angular acceleration about axis j
M_j = the set of external moments about axis j.

Components of inertial acceleration

We start by working towards defining the inertial accelerations arising from the components of external force acting on the aircraft.

Consider an orthogonal set of body axes with **origin fixed at the CG** of the rigid body shown in the figure.



Motion referred to generalized body axes (Cook)

$\bar{p}$ is an arbitrary point within the body axes at position (x, y, z).
Local components of *linear* velocity and acceleration at $\bar{p}$ relative to body axes are (u, v, w) and (a_x, a_y, a_z).
Components of the *angular* velocity of $\bar{p}$ rel. to body axes are (p, q, r).

The body axes themselves are in motion relative to an external inertial or earth frame, with linear velocities (U, V, W).

Components of inertial acceleration

To determine the acceleration of point $\bar{p}$, we first need to derive the contributions to its velocity from the point's linear velocity components (u, v, w) in each body axis direction as well as the effects of the rotation rates (p, q, r) about each axis.

The contribution of p, q, r to the velocity arises through the Coriolis equation. You may recall this in vector form as $\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{r}}$ where $\vec{\boldsymbol{\Omega}}$ is the rotation vector (with body axis components p, q, r) and $\vec{\boldsymbol{r}}$ is the position vector of point $\bar{p}$ relative to origin o (with components x, y, z in body axes).

In these notes, we do not derive the equations using vector notation, although many textbooks do, e.g. Etkin 2012.

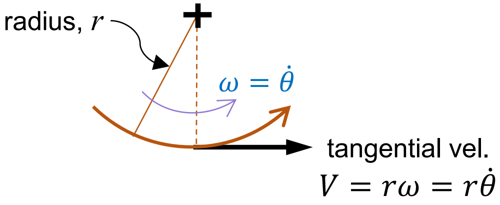
Components of inertial acceleration

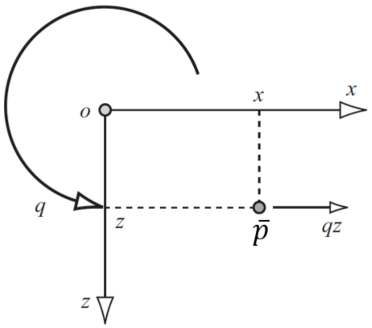
Velocity components at $\bar{p}(x, y, z)$ relative to origin o are:

$$
\begin{aligned}
u &= \dot{x} - ry + qz \\
v &= \dot{y} - pz + rx \qquad\qquad (1) \\
w &= \dot{z} - qx + py
\end{aligned}
$$

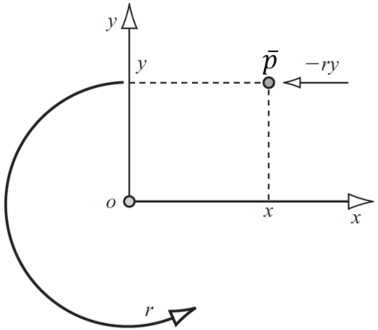
We see that in addition to the linear terms in each equation, there are two additional terms due to rotary motion – the *tangential velocity* components.

Reminder:
circular motion:





Looking into axes system along y axis



Looking into axes system along z axis

Tangential velocity terms due to rotary motion (Cook)

Components of inertial acceleration

Since the body (the aircraft) is assumed to be rigid, $\dot{x} = \dot{y} = \dot{z} = 0$.

Therefore, the velocity equations reduce to:

$$\left.\begin{aligned} u &= qz - ry \\ v &= rx - pz \\ w &= py - qx \end{aligned}\right\} \qquad (2)$$

Without loss of generality, we can consider the inertial frame to be instantaneously coincident with the body axes.

We can then obtain the absolute velocity components (u', v', w') of the point $\bar{p}(x, y, z)$ by adding the velocity components (U, V, W) of the origin/CG to the local velocity components (u, v, w) from eqn (2):

Components of inertial acceleration

eqn (2):

$$u = qz - ry$$
$$v = rx - pz$$
$$w = py - qx$$

$$\begin{aligned} u' &= \dot{x}' = U + u = U - ry + qz \\ v' &= \dot{y}' = V + v = V - pz + rx \\ w' &= \dot{z}' = W + w = W - qx + py \end{aligned} \qquad (3)$$

where x', y', z' are the displacements of point $\bar{p}$ in the inertial axes and u', v', w' are the velocity components in the inertial axes.

We can use these velocities to obtain the accelerations in inertial axes, as required for applying Newton's 2nd Law.

Components of inertial acceleration

The corresponding accelerations in the inertial frame are therefore:

$$\begin{aligned}
\dot{u}' &= \ddot{x}' = \frac{d}{dt}(U - ry + qz) = \dot{U} - \dot{r}y - r\dot{y} + \dot{q}z + q\dot{z} \\
\dot{v}' &= \ddot{y}' = \frac{d}{dt}(V - pz + rx) = \dot{V} - \dot{p}z - p\dot{z} + \dot{r}x + r\dot{x} \\
\dot{w}' &= \ddot{z}' = \frac{d}{dt}(W - qx + py) = \dot{W} - \dot{q}x - q\dot{x} + \dot{p}y + p\dot{y}
\end{aligned} \tag{4}$$

Given that the inertial frame is considered to be instantaneously coincident with the body axes, $x' = x$, $y' = y$ and $z' = z$ and $\dot{x} = \dot{x}'$, $\dot{y} = \dot{y}'$, $\dot{z} = \dot{z}'$.

Substituting the expressions for $\dot{x}'$, $\dot{y}'$ and $\dot{z}'$ from (3) for $\dot{x}$, $\dot{y}$ and $\dot{z}$ in eqn (4), the expressions for acceleration components in the inertial frame become:

$$\begin{aligned}
\ddot{x}' &= \dot{U} - \dot{r}y - r(V - pz + rx) + \dot{q}z + q(W - qx + py) = a'_x \\
\ddot{y}' &= \dot{V} - \dot{p}z - p(W - qx + py) + \dot{r}x + r(U - ry + qz) = a'_y \\
\ddot{z}' &= \dot{W} - \dot{q}x - q(U - ry + qz) + \dot{p}y + p(V - pz + rx) = a'_z
\end{aligned} \tag{5}$$

Components of inertial acceleration

Alternatively, accelerations can be derived using vector notation, resulting in the equation (for a rigid body):

$$\vec{\boldsymbol{a}}_{total} = \vec{\boldsymbol{a}}_{rel} + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{\Omega}} \times \left(\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{r}}\right)$$

where $\vec{\boldsymbol{a}}_{rel}$ is the linear acceleration of the body axes relative to inertial (earth) axes and $\vec{\boldsymbol{\alpha}}$ is the angular acceleration of the body about its origin (CG), i.e. $\dot{\vec{\boldsymbol{\Omega}}}$.

The components of $\vec{\boldsymbol{a}}_{total}$ in the inertial axes x, y, z directions would be the same as those in eqn (5).

Generalised force equations

Now, applying Newton's 2nd Law to obtain the equations of motion, we consider an incremental mass δm at each point $\bar{p}(x, y, z)$ in the rigid body.

Thus, the incremental components of force acting on the mass will be given by $(\delta m.a'_x,\ \ \delta m.a'_y,\ \ \delta m.a'_z)$.

For the body as a whole, we substitute the expressions for components of inertial acceleration from eqn (5) and then sum these increments of mass $\times$ accel. over the body in each component direction: these will equal the total body-axis force components, X, Y and Z in the x, y and z directions respectively:

$$\sum \delta m.a'_x = X \qquad \sum \delta m.a'_y = Y \qquad \sum \delta m.a'_z = Z \tag{6}$$

Generalised force equations

Substituting the expressions for inertial accelerations in eqn (5) into eqn (6) we get:

$$X = \sum \delta m \left(\dot{U} - \dot{r}\cancel{y} - r(V - p\cancel{z} + r\cancel{x}) + \dot{q}\cancel{z} + q(W - q\cancel{x} + p\cancel{y})\right)$$

$$Y = \sum \delta m\left(V - \dot{p}\cancel{z} - p(W - q\cancel{x} + p\cancel{y}) + \dot{r}\cancel{x} + r(U - r\cancel{y} + q\cancel{z})\right) \tag{7}$$

$$Z = \sum \delta m \left(\dot{W} - \dot{q}\cancel{x} - q(U - r\cancel{y} + q\cancel{z}) + \dot{p}\cancel{y} + p(V - p\cancel{z} + r\cancel{x})\right)$$

From the definition of centre of gravity (CG), $\sum \delta m.x = \sum \delta m.y = \sum \delta m.z = 0$.

Hence, all the terms in eqn (7) containing an x, y or z are zero when summed over the body.

Generalised force equations

This yields the three translational equations of motion:

$$\boxed{\begin{aligned} m(\dot{U} - rV + qW) &= X \\ m(\dot{V} - pW + rU) &= Y \\ m(\dot{W} - qU + pV) &= Z \end{aligned}} \qquad (8)$$

m is the total mass of the body (aircraft).

Note that eqn (8) describes motion of the CG (extra terms are needed if the origin is not at the CG).

Generalised moment equations

Consider the incremental *moments* produced by forces acting on the incremental mass δm at point $\bar{p}(x, y, z)$ in the rigid body.

Summing these over the body yields the moment equations.

For example, the total rolling moment, L, about the ox axis is:

$$\sum \delta m\left(y.a'_z - z.a'_y\right) = L \tag{9}$$

Substituting the expressions for a'_y and a'_z from eqn (5) and recalling that $\sum \delta m.x = \sum \delta m.y = \sum \delta m.z = 0$, eqn (9) can be written as:

$$\begin{pmatrix} \dot{p}\sum \delta m\left(y^2 + z^2\right) + qr\sum \delta m\left(y^2 - z^2\right) + \left(r^2 - q^2\right)\sum \delta m.yz - \\ \left(pq + \dot{r}\right)\sum \delta m.xz + \left(pr - \dot{q}\right)\sum \delta m.xy \end{pmatrix} = L \tag{10}$$

Generalised moment equations

The terms in the summation signs in eqn (10) are moments of inertia.
We define:

$$
\begin{aligned}
I_{xx} &= \sum \delta m\left(y^2 + z^2\right) \qquad & I_{xy} &= \sum \delta m.xy \\
I_{yy} &= \sum \delta m\left(x^2 + z^2\right) \qquad & I_{xz} &= \sum \delta m.xz \\
I_{zz} &= \sum \delta m\left(x^2 + y^2\right) \qquad & I_{yz} &= \sum \delta m.yz
\end{aligned}
\tag{11}
$$

Eqn (10) can therefore be written as:

$$
I_{xx}\dot{p} - \left(I_{yy} - I_{zz}\right)qr + I_{xy}(pr - \dot{q}) - I_{xz}(pq + \dot{r}) + I_{yz}(r^2 - q^2) = L \tag{12}
$$

Generalised moment equations

Similarly, the total moments M and N about the oy and the oz axes are:

$$\sum \delta m(z.a_x' - x.a_z') = M \qquad \sum \delta m\left(x.a_y' - y.a_x'\right) = N \tag{13}$$

Substituting a_x', a_y' and a_z' from eqn (5) into (13), recalling again that $\sum \delta m.x = \sum \delta m.y = \sum \delta m.z = 0$ and using the inertia definitions in eqn (11), we get:

$$I_{yy}\dot{q} + (I_{xx} - I_{zz})pr + I_{yz}(pq - \dot{r}) + I_{xz}(p^2 - r^2) - I_{xy}(qr + \dot{p}) = M \tag{14}$$

$$I_{zz}\dot{r} - \left(I_{xx} - I_{yy}\right)pq - I_{yz}(pr + \dot{q}) + I_{xz}(qr - \dot{p}) + I_{xy}(q^2 - p^2) = N \tag{15}$$

Generalised moment equations

These equations are the generalized rigid body moment equations for the rotational motion about the orthogonal axes through the CG, since the axes origin is at the CG.

In conventional fixed-wing aircraft, *x-z* is a plane of symmetry in terms of the mass distribution so that $\boxed{I_{xy} = I_{yz} = 0}$.

The moment equations (12), (14) and (15) then simplify to:

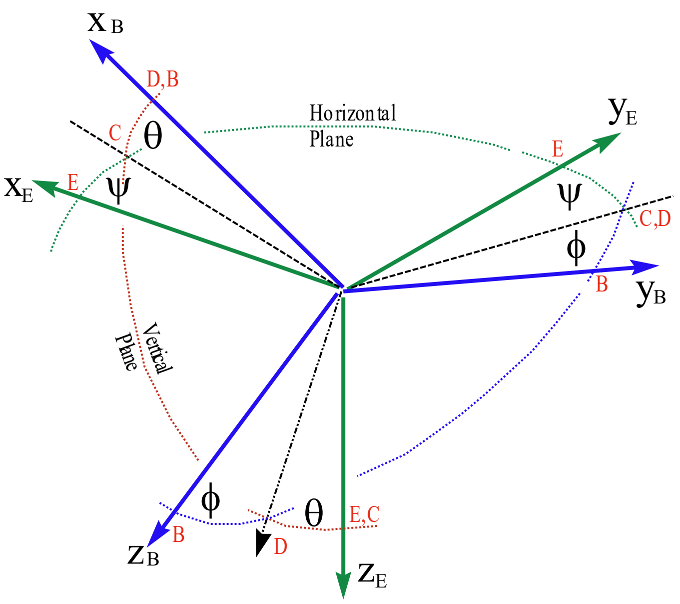
$$\boxed{\begin{aligned} I_{xx}\dot{p} - \left(I_{yy} - I_{zz}\right)qr - I_{xz}(pq + \dot{r}) &= L \\ I_{yy}\dot{q} + \left(I_{xx} - I_{zz}\right)pr + I_{xz}(p^2 - r^2) &= M \\ I_{zz}\dot{r} - \left(I_{xx} - I_{yy}\right)pq + I_{xz}(qr - \dot{p}) &= N \end{aligned}} \quad (16)$$

Kinematic equations for rotation

Recall the definition of the Euler angles in Session 2, where the aircraft body axes were related to the inertial axes via a sequence of rotations about non-orthogonal axes.

To go from inertial to body axes:

1. Rotate through ψ about z_E to intermediate frame F_C
2. Rotate through θ about y_c to intermediate frame F_D
3. Rotate through ϕ about x_B to body axes frame F_B.



Kinematic equations for rotation

We can find the relationship between body axes rates (p,q,r) and attitude (Euler angle) rates $(\dot{\phi},\dot{\theta},\dot{\psi})$ by carrying out the necessary trigonometric transformations according to the sequences referred to in the previous slide.

This yields:

$$\boxed{\begin{aligned}\dot{\phi} &= p + \tan\theta\left(q\sin\phi + r\cos\phi\right)\\ \dot{\theta} &= q\cos\phi - r\sin\phi\\ \dot{\psi} &= \left(q\sin\phi + r\cos\phi\right)/\cos\theta\end{aligned}} \qquad (17)$$

Kinematic equations for rotation

We can also use the Euler angles to transform between body axes velocities (U, V, W) and inertial axes velocities $(\dot{x}_E, \dot{y}_E, \dot{z}_E)$.

These form the so-called navigation equations (not derived here):

$$\boxed{\begin{aligned}
\dot{x}_E &= U\cos\theta\cos\psi + V(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) + \\
&\quad W(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \\
\dot{y}_E &= U\cos\theta\sin\psi + V(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) + \\
&\quad W(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \\
\dot{z}_E &= -U\sin\theta + V\sin\phi\cos\theta + W\cos\phi\cos\theta
\end{aligned}} \tag{18}$$

Kinematic equations for rotation

Equations (8), (16), (17) and (18) form the 12 equations of motion for general atmospheric 6 degree-of-freedom flight.

In the next session we will look in more detail at:

1. the force and moment terms on the right-hand sides of equations (8) and (16);
2. alternative forms of the equations of motion;
3. some features of the equations; and
4. how to implement them in, for example, a flight simulation.

Next Session
Equations of Motion 2

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EQUATIONS OF MOTION 1

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