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CADE30001

Advanced structures and materials

Lecture 4: Energy based methods in fracture mechanics

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Course Content

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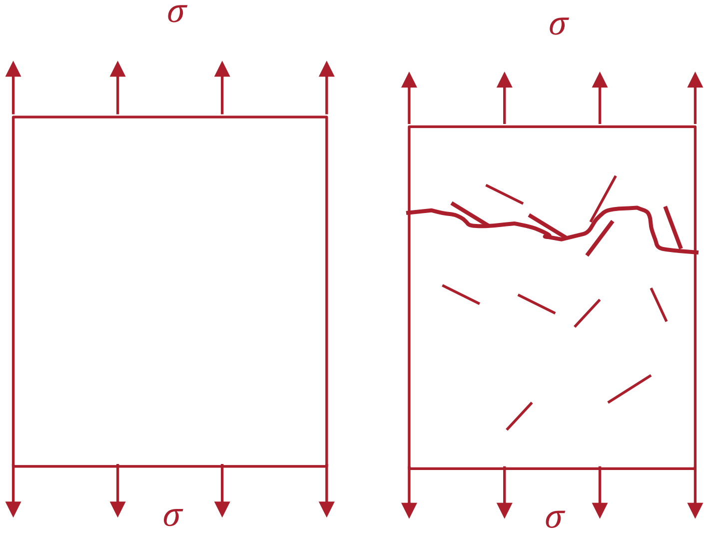
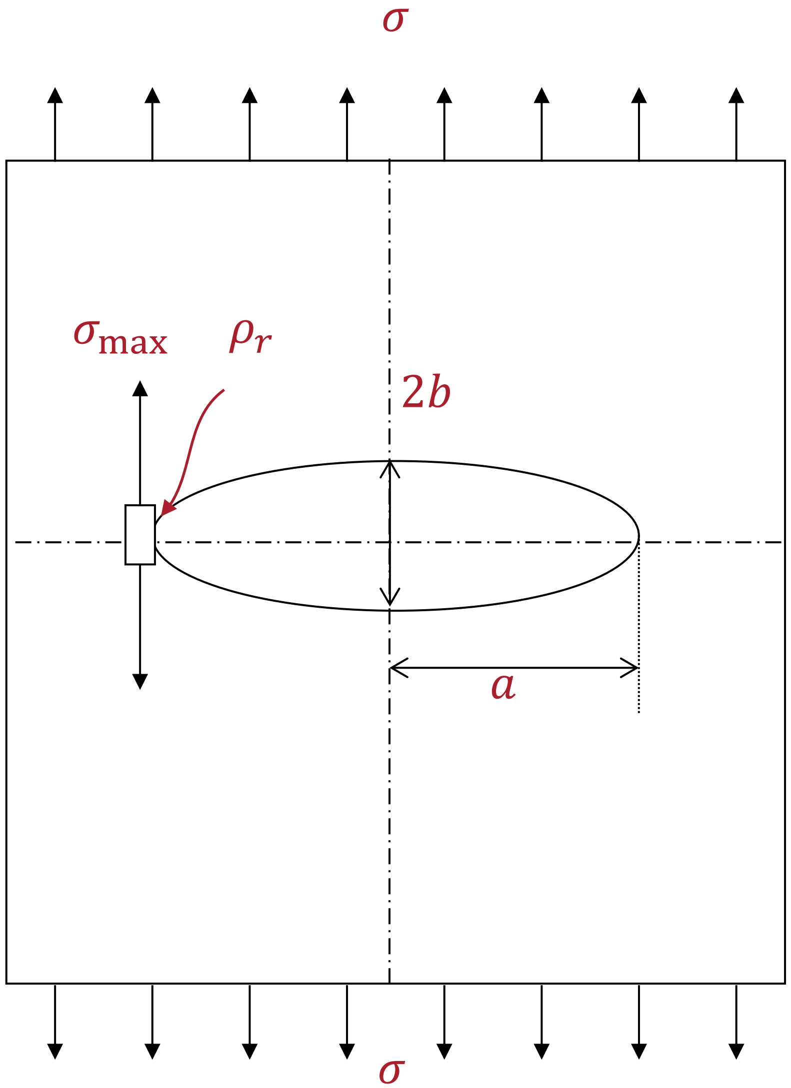
Previously...



(a) (b)



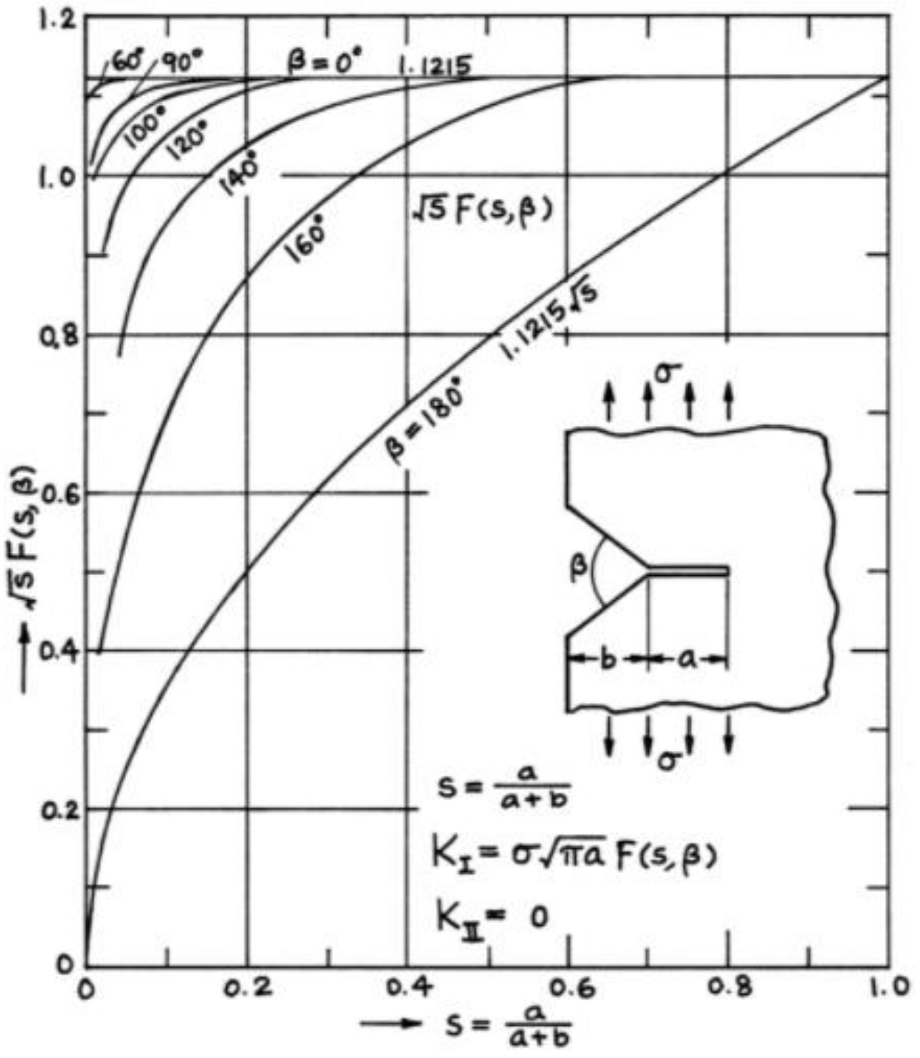
$$\sigma_{max} = \sigma\left(1 + 2\sqrt{\frac{a}{\rho_r}}\right) \qquad \rho_r = \frac{b^2}{a}$$

Exam question

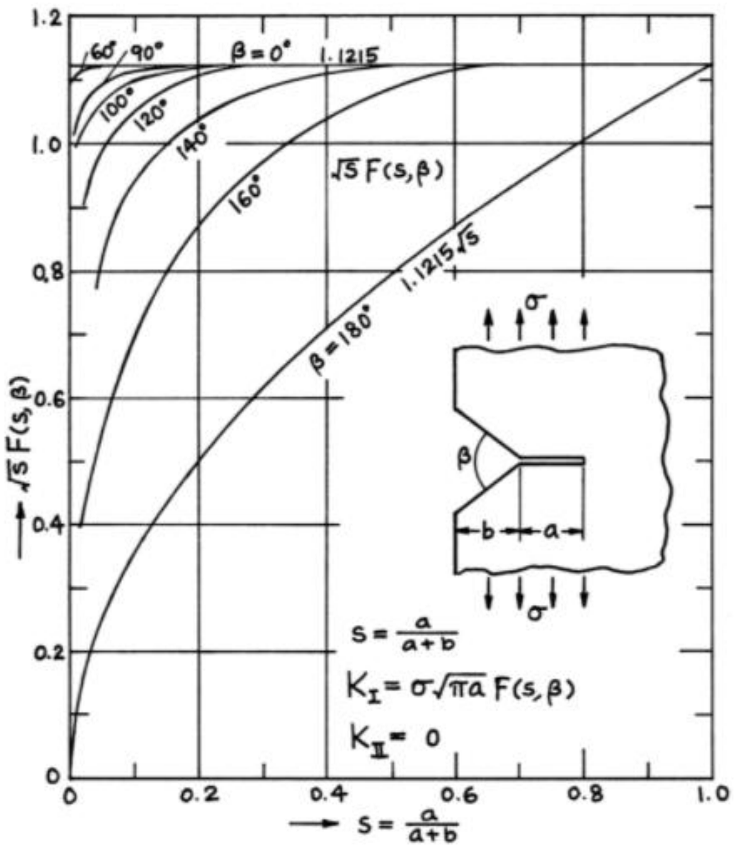
The figure shows a crack emanating from a V-notch under far-field tension. The associated expression of the stress intensity factor is given in the figure, together with the relevant geometrical factor F, which is illustrated in the chart.

The depth of the notch is given by b = 1 mm. The notch angle is β = 120°. Under a far field stress of 0.1 GPa, a crack 6 mm long experiences sudden propagation.

Calculate the fracture toughness of the material (in MPa $m^{1/2}$).



Solution



The depth of the notch is given by b = 1 mm. The notch angle is β = 120°. Under a far field stress of 0.1 GPa, a crack 6 mm long experiences sudden propagation.

The critical task in this question is to estimate the geometrical factor *F*. For the parameter *s* we observe that

$$s = \frac{a}{a+b} = \frac{6}{7} \cong 0.857$$

$$\sqrt{s}F = 1.1215$$

$$F = \frac{1.1215}{\sqrt{s}} = \frac{1.1215}{\sqrt{0.857}} \cong 1.2115$$

So, for the fracture toughness we have:

$$K_{IC} = F\sigma\sqrt{\pi a_C} = 1.2115 \cdot 0.1\text{GPa} \cdot \sqrt{\pi \cdot 6\text{mm}}$$

$$= 1.2115 \cdot 100\,\text{MPa} \cdot \sqrt{\pi \cdot 6 \cdot 10^{-3}\text{m}} \cong 16.6\text{MPa}\sqrt{\text{m}}$$

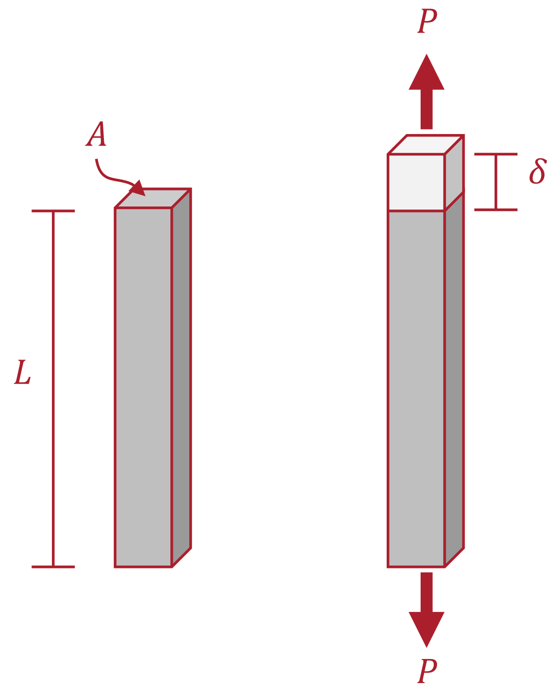
Griffith (1921) Theory

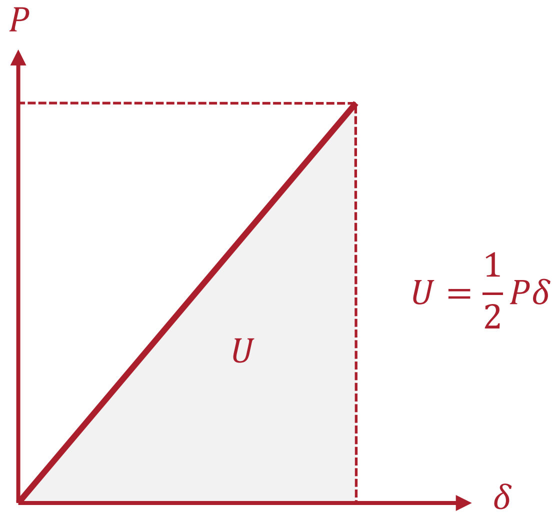
The ideas from Inglis paper were further developed in 1920 by a young engineer, Alan Arnold Griffith, who analysed the phenomena from an energy point of view

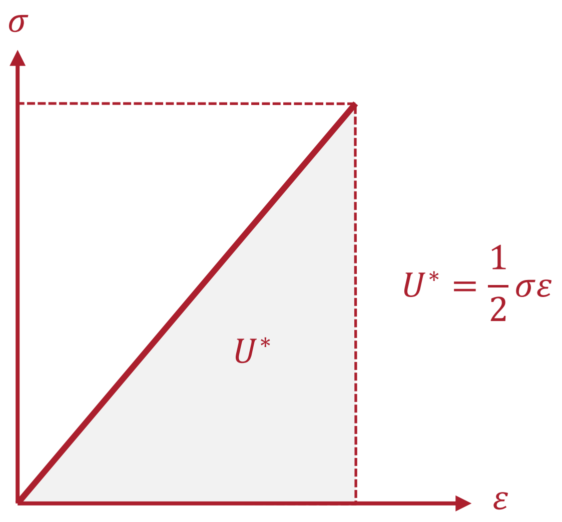
Griffith proposed that 'There is a simple energy balance consisting of the decrease in **potential energy** within the stressed body due to crack extension and this decrease is balanced by increase in **surface energy** due to increased crack surface'

Elastic strain Energy Density

- The relationship between the input energy U and the strain energy density U^* is:







$$\sigma = \frac{P}{A} \qquad \varepsilon = \frac{\delta}{L} \qquad E = \frac{\sigma}{\varepsilon} \qquad U = \frac{1}{2}P\delta = \frac{\sigma^2}{2E}V$$

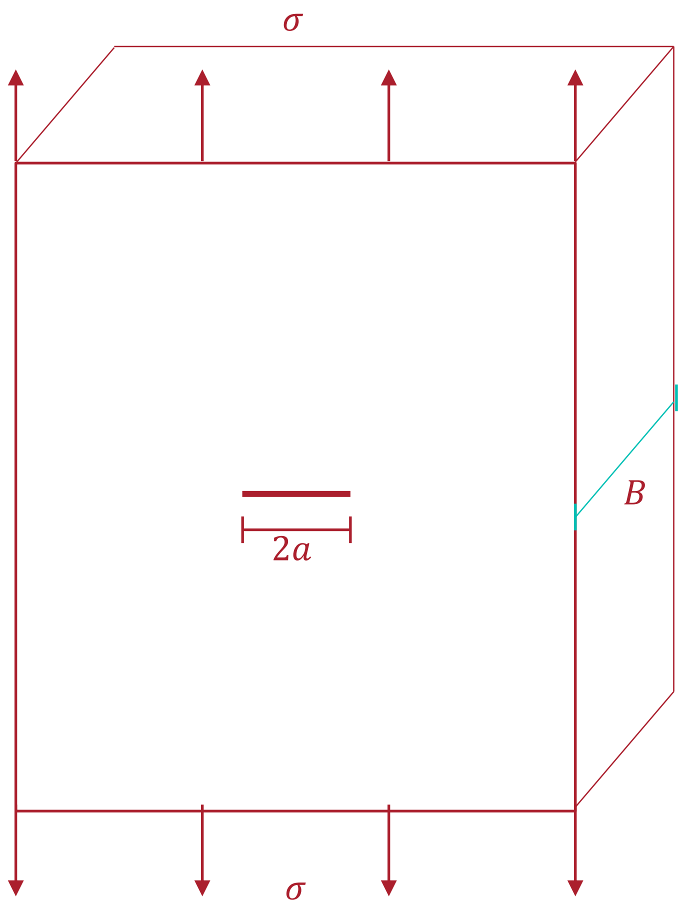
Griffith criterion for brittle fracture

Consider an infinitely wide plate with a thickness of B, loaded in tension. We determined the elastic strain energy to be:

$$U_0 = \frac{1}{2}\frac{\sigma^2}{E}V$$

Taking the same plate however with a crack. There will be a change in the strain energy for the same applied stress.

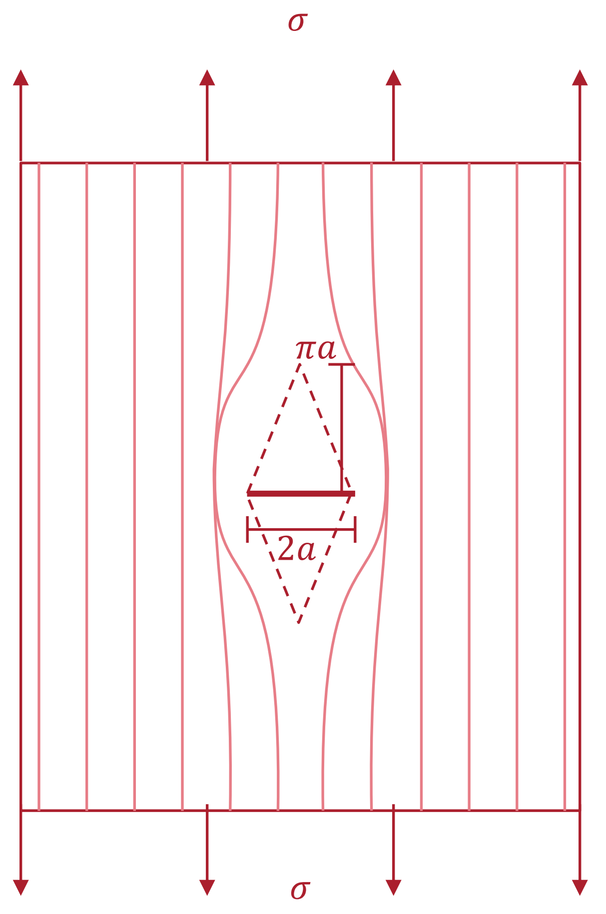
Using the Inglis solution, Griffith was able to estimate this energy change by working out the volume of the material no longer being stressed.



Griffith criterion for brittle fracture

By visualising this as triangular region with a height of πa we can estimate the strain energy of the unstressed region to be:

$$U_{\text{unstressed}} = \frac{1}{2}\frac{\sigma^2}{E}2\pi a^2 B \qquad = \frac{\sigma^2 \pi a^2 B}{E}$$



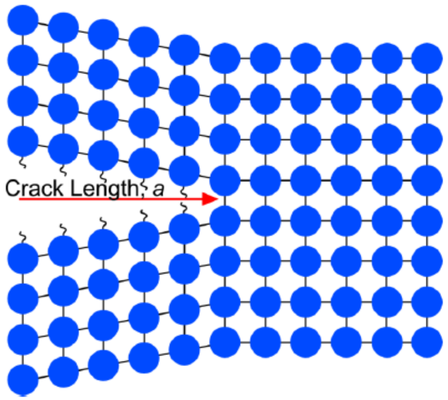
Therefore the strain energy for the plate with a crack is:

$$U_{\text{cracked}} - U_0 = -\frac{\sigma^2 \pi a^2 B}{E}$$

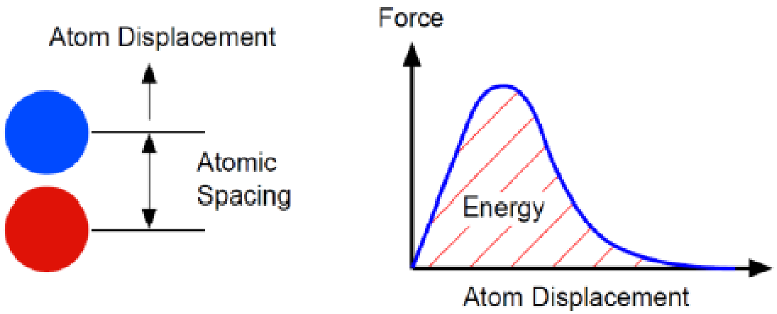
Energy balance requires that the change is strain energy must be transferred somewhere.

Atomic view of fracture

- The bond strength is supplied by the attractive forces between atoms.
- A material fractures when sufficient stress and work are applied at the atomic level to break the bonds that hold atoms together.
- A tensile force is required to increase the separation distance from the equilibrium value, this force must exceed the cohesive force to sever the bond.



$$E_{bond} = 2\gamma_s a B$$



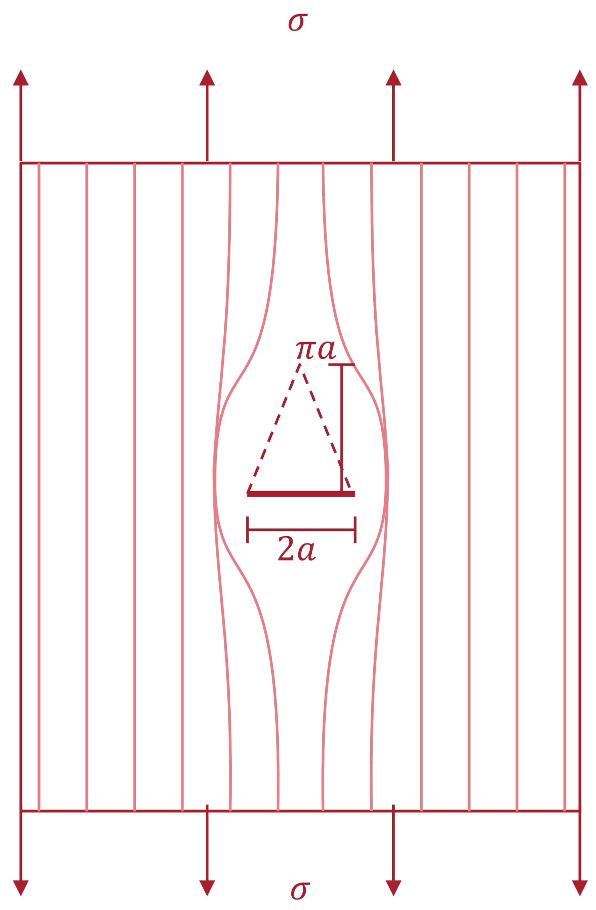
γs represents the energy required to break atomic bonds per unit surface area created by the crack

Griffith criterion for brittle fracture

- The surface energy S associated with a crack of length $2a$ is thus:

$$U_S = 2\gamma A = 4\gamma a B$$

  - where the energy needed to create a surface is γ.
- This energy is in effect absorbed by the material.
- Strain energy for the plate with a crack is equivalent to the energy needed in forming the fracture surface.



Energy balance

Let's define the Griffith energy balance:

The potential energy of the system is the sum of the strain energy U, plus the surface energy of the crack S occur when

$$Potential\ energy\ E = U + S$$

The maximum of this total energy will occur when

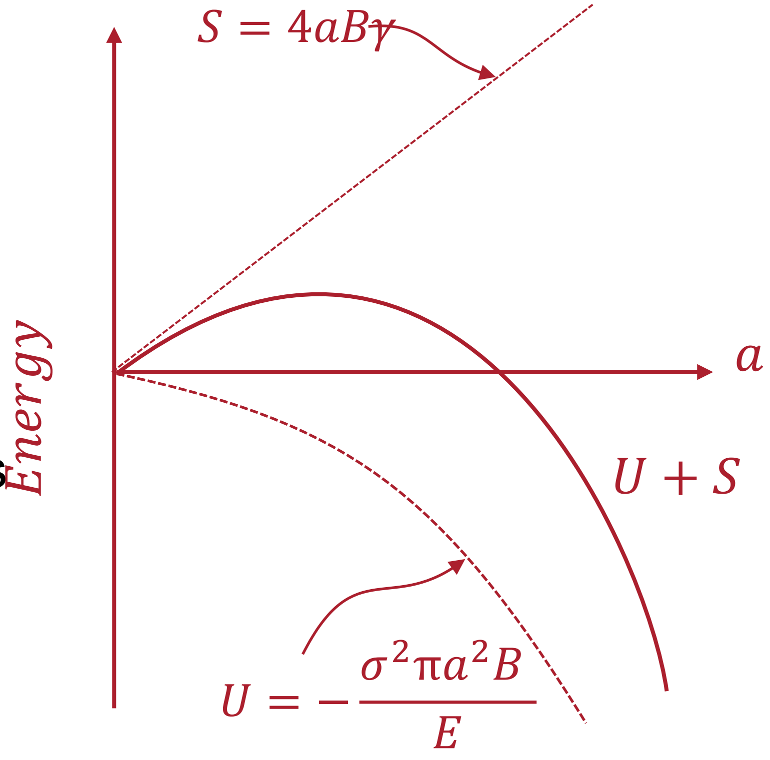
$$\frac{dE}{da} = 0$$

Where change of strain energy release per unit crack area is

$$\frac{dU}{da} = \frac{d}{da}\left(-\frac{\sigma^2 \pi a^2 B}{E}\right) = -\frac{2\sigma^2 \pi a B}{E}$$

and surface energy per area of crack area

$$\frac{dS}{da} = 4\gamma B$$



Critical crack length

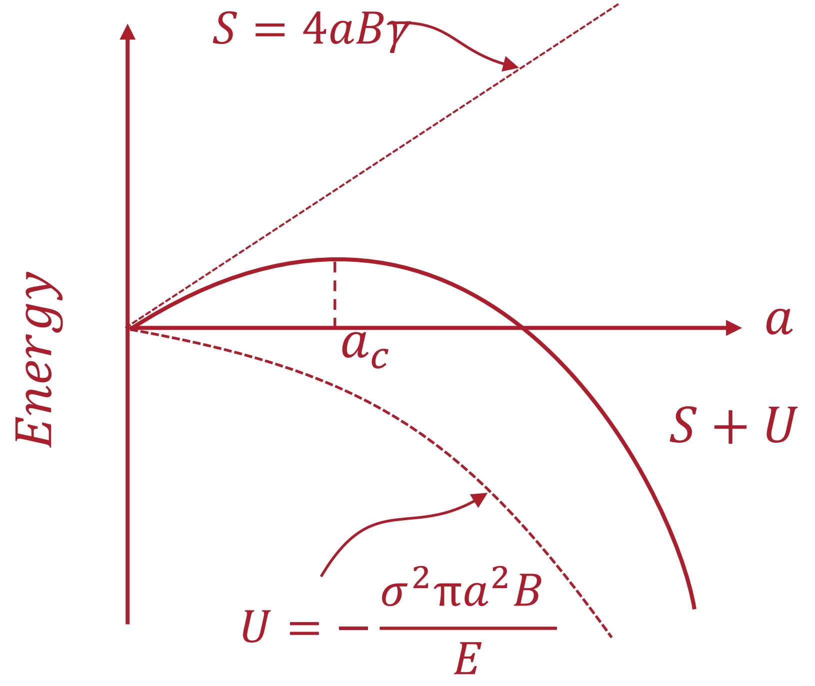
$$\frac{d(U+S)}{da} = 0 \Rightarrow -\frac{2\sigma^2 \pi a B}{E} + 4\gamma B = 0$$

$$\frac{2\sigma^2 \pi a B}{E} = 4\gamma B$$

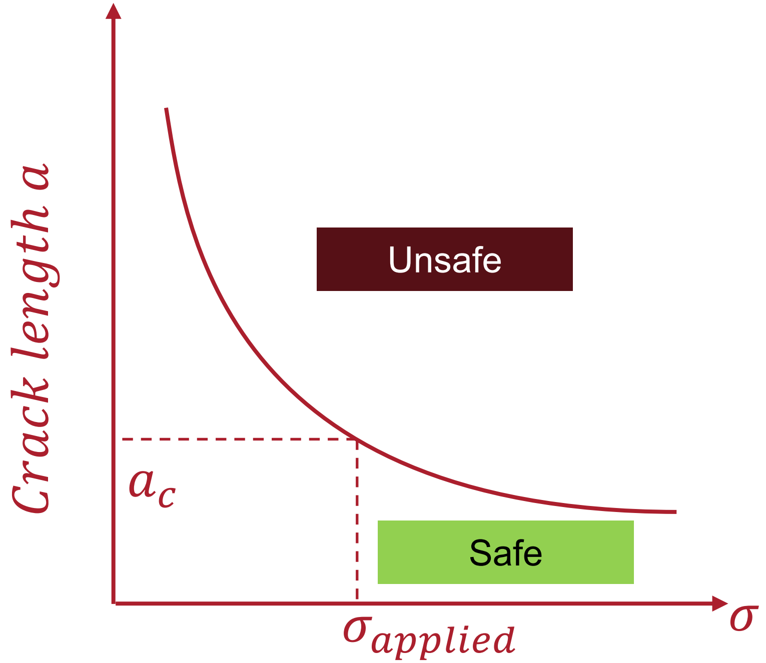
$$a_c = \frac{2E\gamma}{\pi\sigma^2}$$

Cracks with length a < a_c , the crack will not grow

Cracks with length a > a_c , the crack will grow

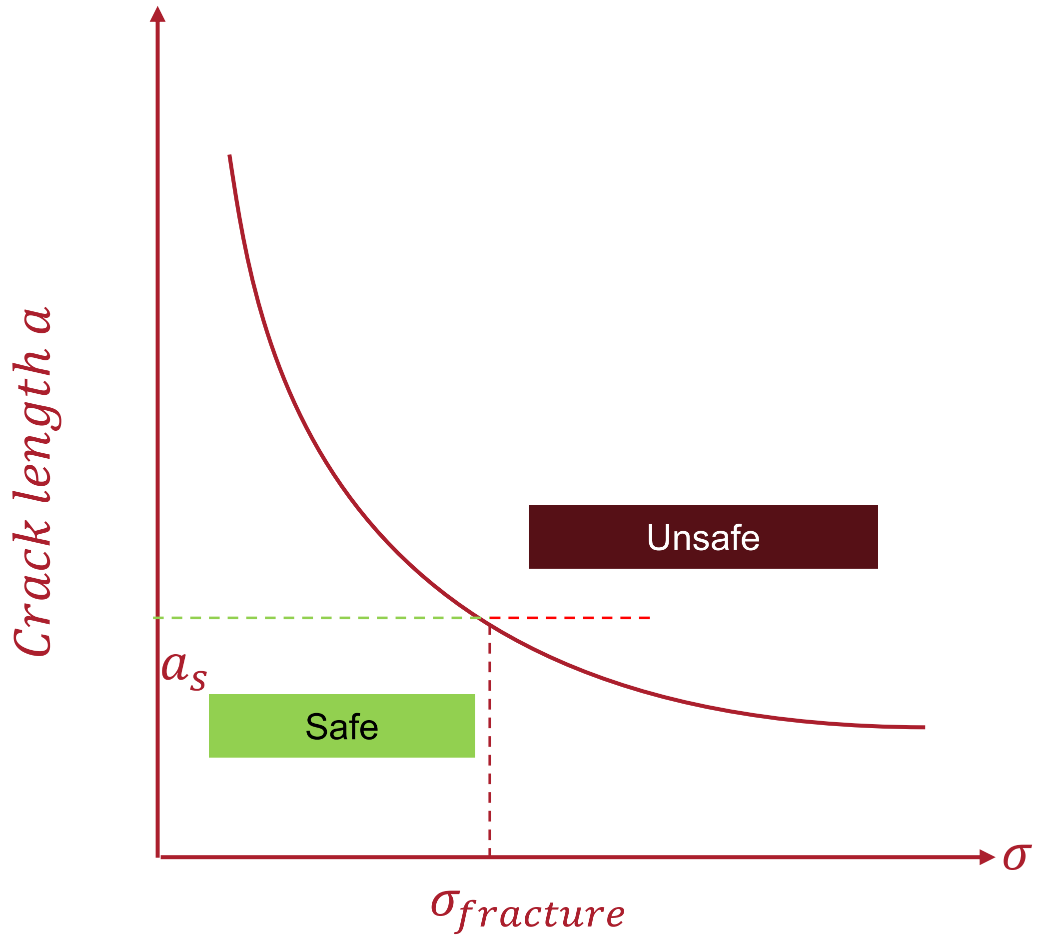


a_c is the critical crack length when two forms of energies are balanced.



For a given applied stress, what is the critical crack length

Critical stress for given crack size



For a given crack length, what is the fracture stress

$$-\frac{2\sigma^2\pi a\mathrm{B}}{E} + 4\gamma B = 0$$

$$\boxed{\boldsymbol{\sigma_f = \sqrt{\frac{2E\gamma}{\pi a}}}}$$

Griffith criterion for brittle fracture

Strain energy release rate G – change of strain energy per unit crack area

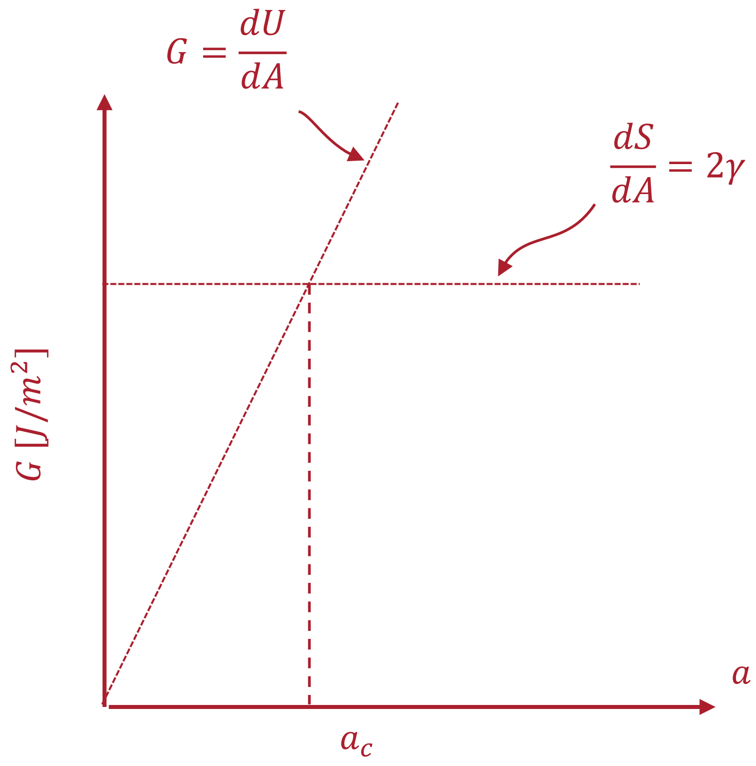
$$\mathcal{G} = \left|\frac{dU}{dA}\right|$$

When the material is stressed, unstable fracture will occur only if the strain energy release rate becomes more than the atomic bond surface energy

$$\mathcal{G}_c = 2\gamma$$

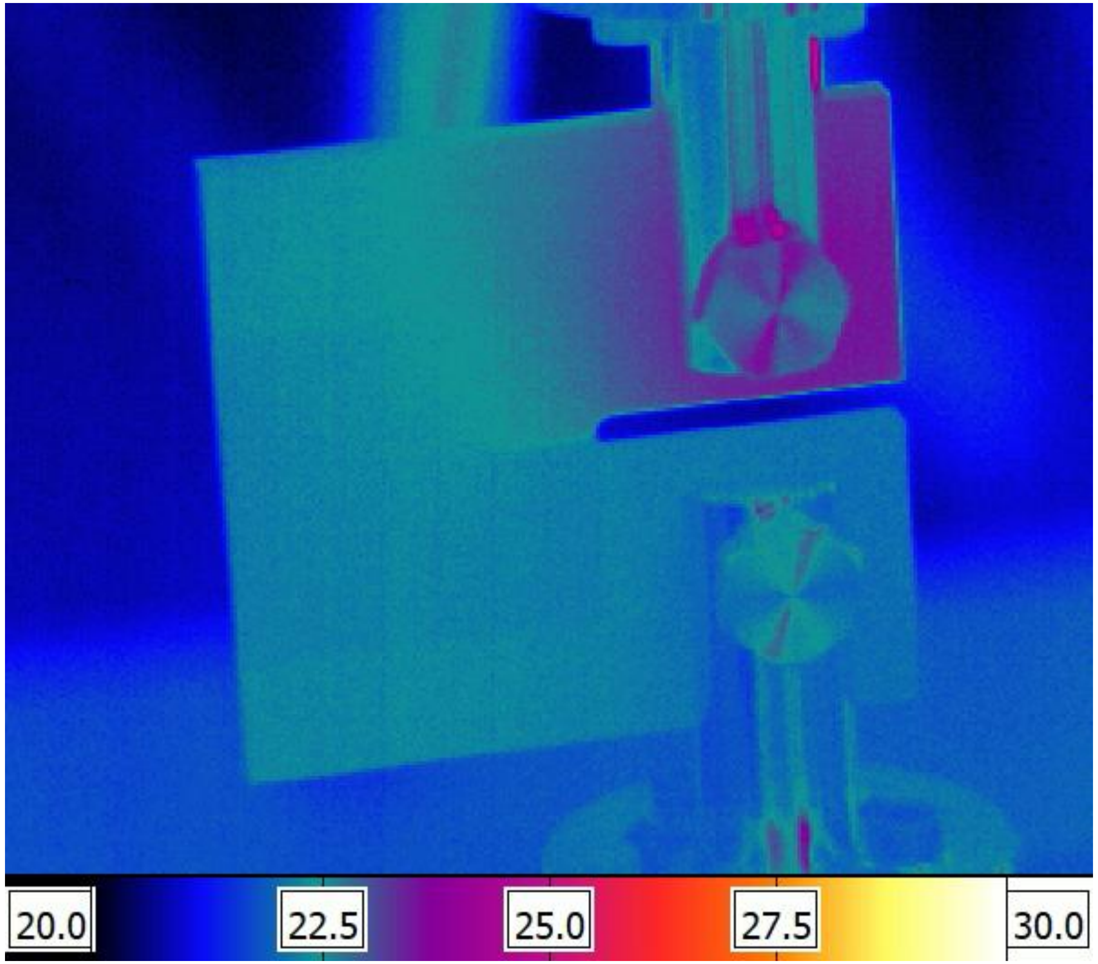
We can see that this corresponds to a critical crack length at which the material is expected to fracture unstably

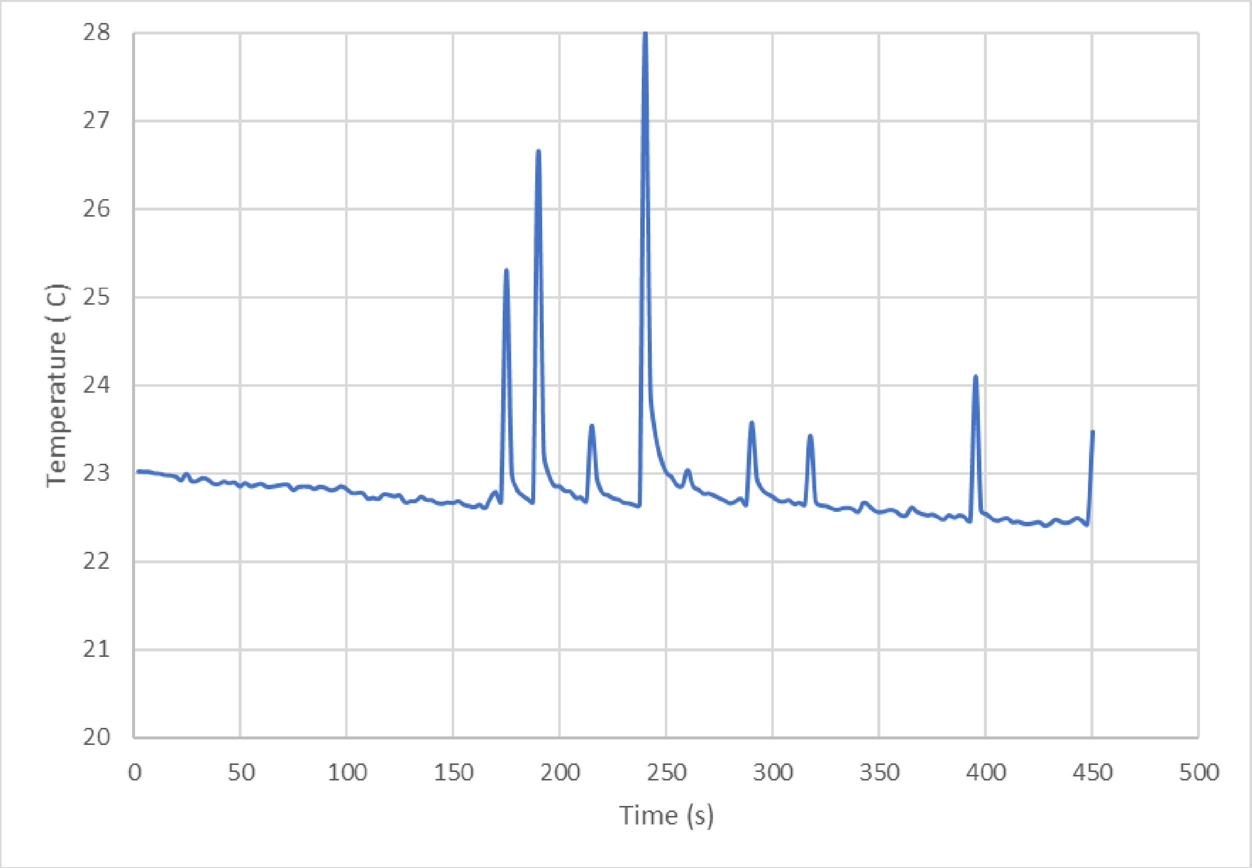
Re-writing the equation we get $$\boxed{\boldsymbol{\sigma_f = \sqrt{\frac{\mathcal{G}_c E}{\pi a}}}}$$



The energy balance is broken once a_c is reached.

Thermal imaging





Example

Use the Griffith's criterion to estimate the critical failure stress for a given crack length in a glass pane when a 10 mm crack is present.

- The modulus of glass is E = 70 GPa
- The critical energy release rate G_c = 7 J/m^2.

$$\boldsymbol{\sigma_f = \sqrt{\frac{\mathcal{G}_c E}{\pi a}} = \ = \sqrt{\frac{7 * 70e9}{\pi * 10e-3}}} = 3.95\text{E+06} \boldsymbol{\frac{N}{m^2} = 3.95\ MPa}$$

- It is well known that glass panes are brittle, and are especially susceptible to shattering when cracks are present.
- This is a very low failure stress. It can be easily exceeded when bending loads are imposed on a pane of glass. For comparison, the yield strength of aircraft-grade aluminium (e.g., Al 7075-T73) is approximately 400 MPa

Exercise

In the previous problem of a glass pane, what is the critical crack length if the applied stress is 10 MPa.

- The modulus of glass is E = 70 GPa
- The critical energy release rate G_c = 7 J/m^2.

$$a_c = \frac{\mathcal{G}_c \mathrm{E}}{\pi \sigma^2}$$

- Answer: 1.56 mm

Limitations

- This procedure helped define a structural strength to two inherent material properties, modulus and surface energy of the atomic bonds
- However, this procedure was found to agree well only to highly brittle materials, like **glass.**
- However, metals are not ideally brittle and normally fail with certain amounts of plastic deformation, the fracture stress is increased due to blunting of the crack tip.
- This is because Griffith assumes an **infinitely sharp crack** where the stresses at the tip are very high
- When applied to **ductile materials** this procedure severely underestimated its fracture strength

Key slide!

Irwin (1957)

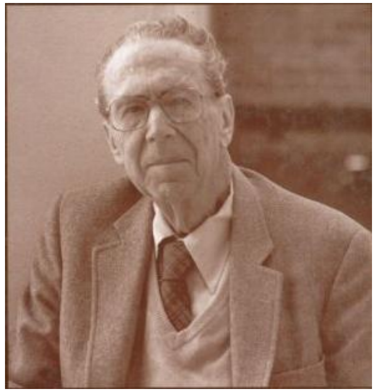
Griffith Critical Energy Release Rate, Gc

$$\mathcal{G}_c = 2\gamma \qquad \boxed{\sigma_f = \sqrt{\frac{E\mathcal{G}_c}{\pi a}}} \qquad \boxed{a_c = \frac{E\mathcal{G}_c}{\pi\sigma^2}}$$

This expression describes the relationship between three important parameters involved in the fracture process:

- the stress level σ_f
- and the size, a, of the flaw
- the material, as evidenced in the critical strain energy release rate G_C

In 1957, a professor from the Leigh University, George Rankine Irwin, showed that Griffith's relation should include the work done in the plastic region, i.e. the crack will propagate if the strain energy is bigger than the total energy necessary (work done to create new crack surfaces and the work done in plastic region).



Irwin Theory

Key slide!

Irwin suggested that a crack will propagate if the strain energy release rate G is bigger than the critical work necessary to create new crack surfaces.

$$G = \frac{dU}{dA} > G_C$$

G_C is known as the fracture energy of a material.

Irwin simply argued that definition of $\frac{\mathrm{d}S}{dA} = 2\gamma$ (the energy required to break bonds) requires an extra plastic energy, γ_P

$$G_C = 2[\gamma_S + \gamma_P]$$

Application in design

The modified Griffith criterion can then be written as

$$\sigma_S = \sqrt{\frac{EG_C}{\pi a}}$$

$$G_c = \frac{(\sigma_s\sqrt{\pi a})^2}{E}$$

Then for a given fracture energy G_C of a material the safe level of stress σ_S can be determined. The structure would then be sized so as to keep the working stress comfortably below this critical value.

In a design situation, one might choose a value of a based on the smallest crack that could be easily detected.

Fracture toughness vs. Fracture energy

Key slide!

As you can see stress intensity factor K can be related to the strain energy release rate G

$$K = \sigma\sqrt{\pi a} \qquad G = \frac{(\sigma\sqrt{\pi a})^2}{E} \qquad G = \frac{K^2}{E}$$

For critical strain energy release rate G_C (fracture energy) [J/m^2], we have the relationship to the critical stress intensity factor K_C (fracture toughness) [Pa m$^{1/2}$]

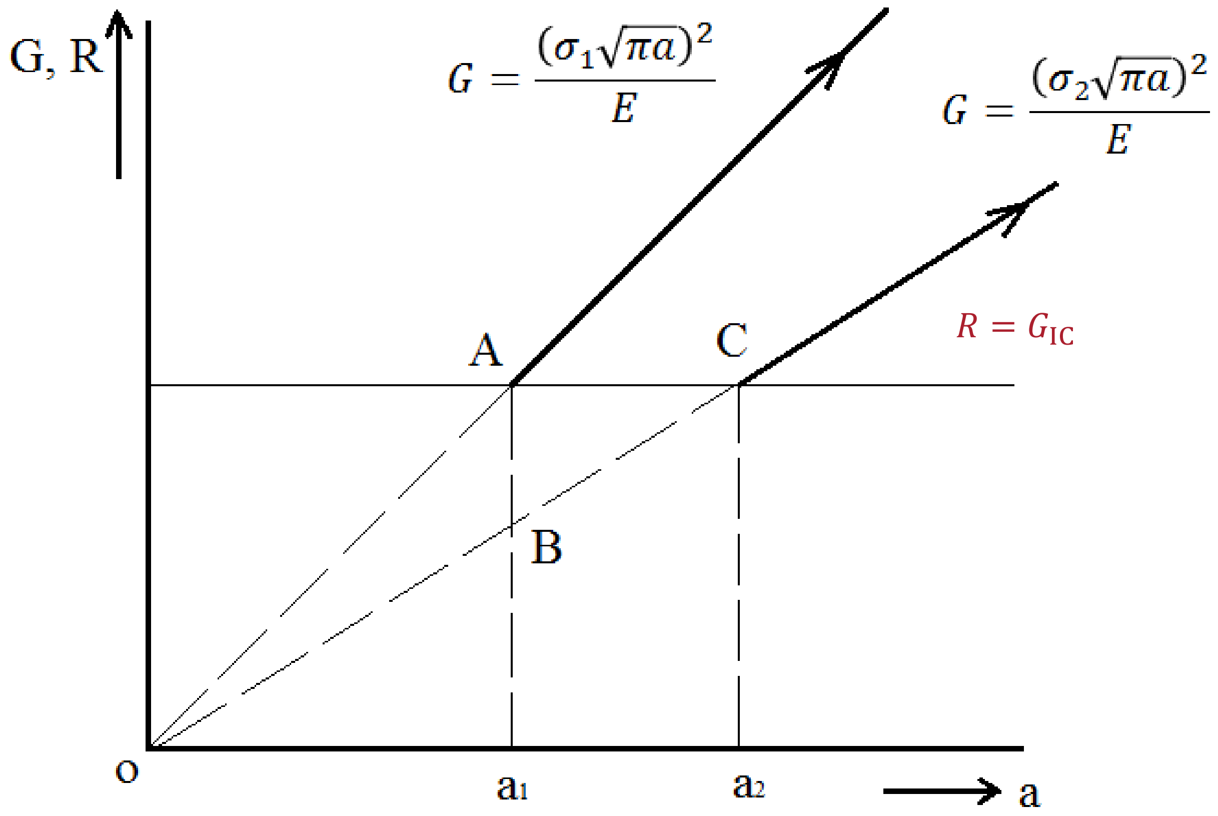
In Plane Stress

$$G_C = \frac{K_C^2}{E}$$

In Plane Strain

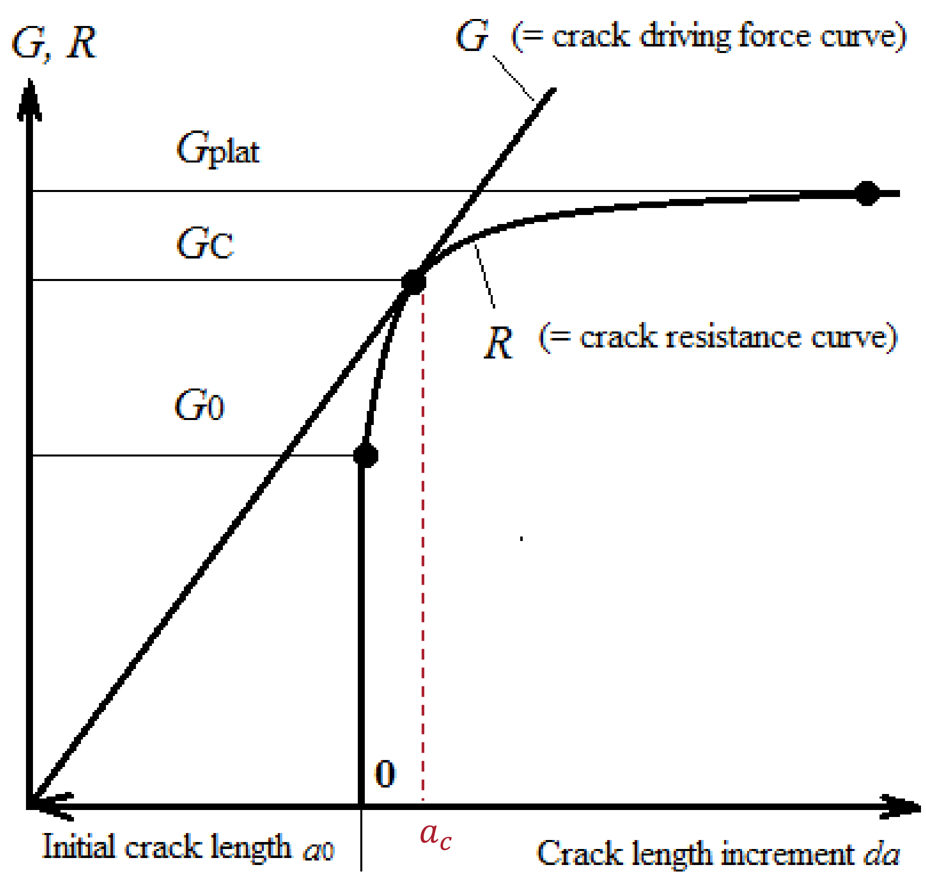
$$G_C = \frac{(1-\nu^2)K_C^2}{E}$$

The Energy Principle



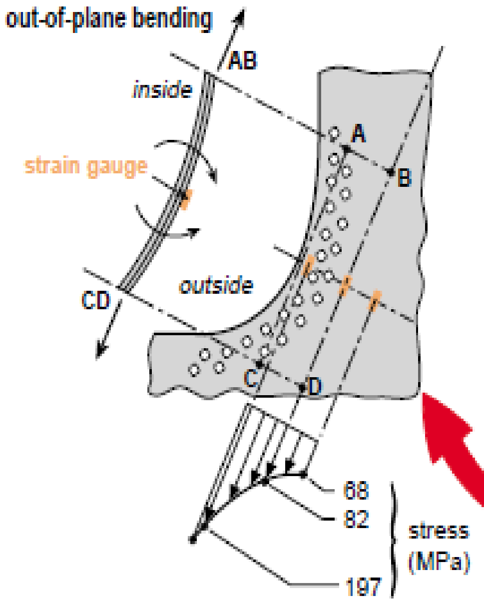
Materials with a constant G_C: Energy balanced at a_1 and a_2 for two different crack driving force curves

R-curve

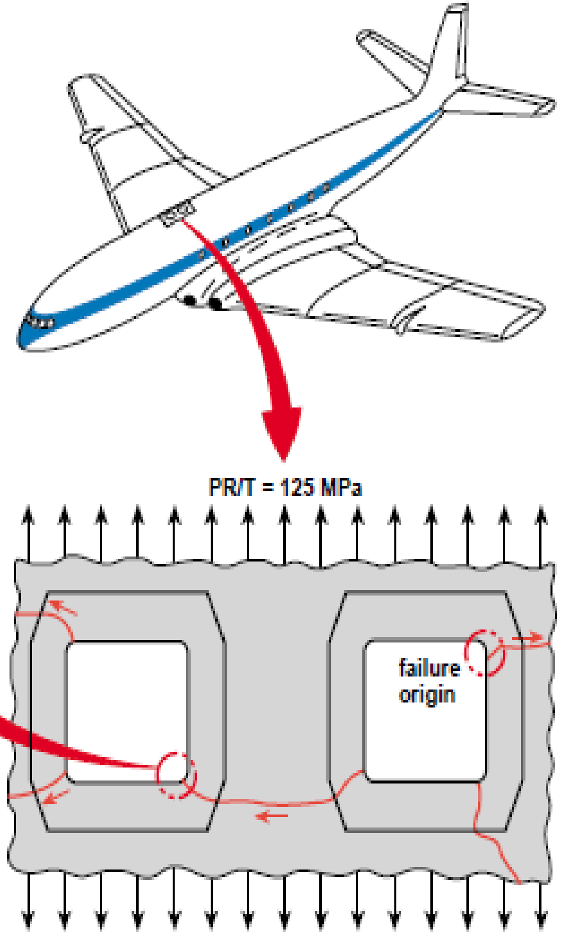


Materials with a R-curve: Energy balanced at a_C for one crack driving force curve

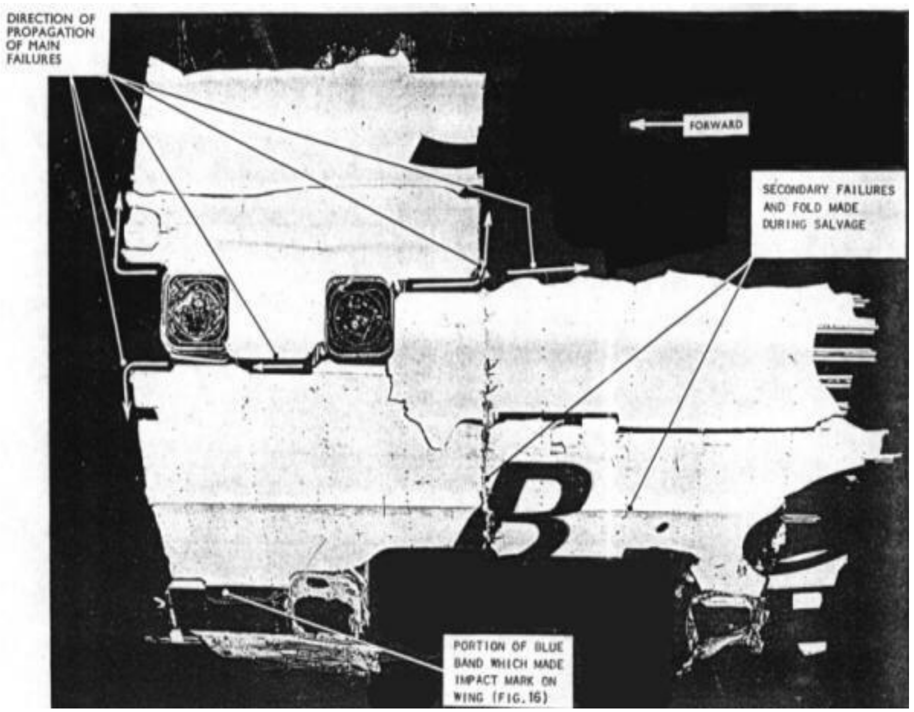
Example: DeHavilland Comet Failure



Stress distribution at 56.9 kPa cabin pressure
and 1.3 g inertia loading



automatic direction finding (ADF) windows



Example of Comet Failure

Key slide!

Remember: The story of the DeHavilland Comet aircraft of the early 1950's, in which at least two aircraft disintegrated in flight, provides a tragic but fascinating insight into the importance of fracture theory.

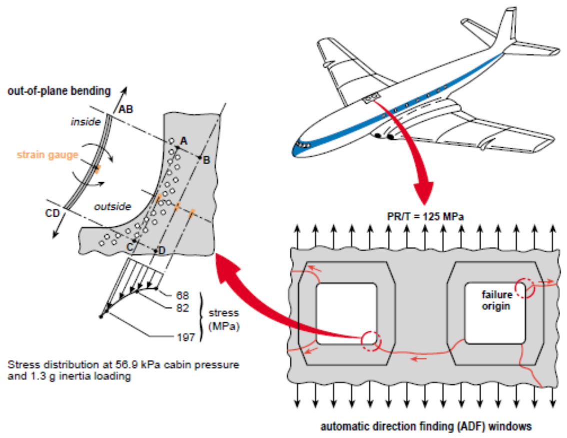
The Comet aircraft had a aluminium fuselage, with G_C ≈ 300 in-psi. The hoop stress due to relative cabin pressurization was 20,000 psi, and at that stress the length of crack that will propagate catastrophically is:

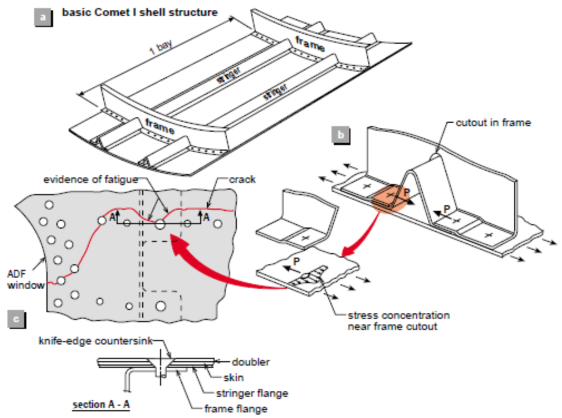
$$G_c = \frac{(\sigma_s\sqrt{\pi a})^2}{E} \quad \Rightarrow \quad a = \frac{G_C\,E}{\pi\,\sigma_s^{\,2}} = \frac{(300)(11\times10^6)}{\pi(20\times10^3)^2} = 2.62\ \text{inches} \cong 66.5\ \text{mm}$$

A crack would presumably be detected in routine inspection long before it could grow to this length!

Example of Comet Failure

But in the case of the Comet, the cracks were propagating from rivet holes near the cabin windows. When the crack reached the window, the size of the window opening was effectively added to the crack length, plus the absence of any crack-arrest design features ultimately lead to catastrophic disaster.





Countersunk rivet heads + stress concentrations @ windows + absence of crack arrest design =
DISASTER!

Exam question

A large thin plate contains a crack 2.4 mm long in total. When loaded in tension normal to the crack line, the plate fails at a stress of 188 MPa. Calculate the fracture energy (in kJ/m2) for the plate material, knowing that its Young's modulus is 70 GPa and its Poisson's ratio is 0.3.

$$K_{IC} = \sigma\sqrt{\pi a_C}, \quad G_{IC} = \frac{K_{IC}^2}{E} \; \square \; G_{IC} = \frac{\pi\sigma^2 a_C}{E}$$

A large thin plate contains a crack 2.4 mm long in total. When loaded in tension normal to the crack line, the plate fails at a stress of 188 MPa. Calculate the fracture energy (in kJ/m2) for the plate material, knowing that its Young's modulus is 70 GPa and its Poisson's ratio is 0.3.

From the formula of the stress intensity factor and the relation between the energy release rate and the stress intensity factor <u>in plane stress</u>, we find

$$K_{IC} = \sigma\sqrt{\pi a_C}, \quad G_{IC} = \frac{K_{IC}^2}{E} \; \square \; G_{IC} = \frac{\pi\sigma^2 a_C}{E}$$

$$G_{Ic} = \frac{\pi \, * 188^2 * 2.4}{70 \, * 10^3} = 3.807 \; kJ/m^2$$

Pro tip: watch your units

Summary

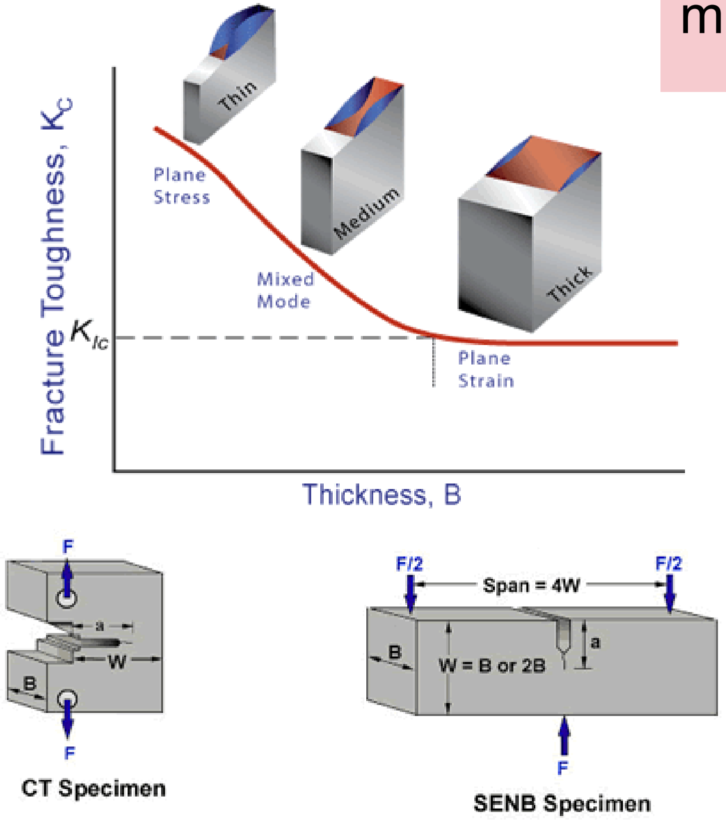
- Griffith
  - concept of an energy balance, the relationship between stress, crack length atomic bonds strength
- Irwin
  - The plastic deformation ahead of crack contributes to toughness
- In a design situation, one might choose a value of crack length based on the smallest crack that could be easily detected.
- R-curve for crack resistance

$$G = \frac{(\sigma\sqrt{\pi a})^2}{E}$$

$$G_c = \frac{(\sigma_s\sqrt{\pi a})^2}{E}$$

$$G_C = \frac{K_C^2}{E}$$

Next: Measurement of Fracture Toughness



Objective: Understand the influence of geometry and loading method on fracture toughness

- It is important to understand the role of material thickness and application of loading direction in establishing fracture toughness
- This can be accomplished via understanding of
  - Plane Strain and Plane Stress
  - Crack tip state of tension
  - Isotropic vs Anisotropic Toughness