CHAPTER

# Plastic Analysis of Beams and Frames

# 18

So far our analysis of the behaviour of structures has assumed that whether the structures are statically determinate or indeterminate the loads on them cause stresses which lie within the elastic limit. Design, based on this elastic behaviour, ensures that the greatest stress in a structure does not exceed the yield stress divided by an appropriate factor of safety.

An alternative approach is based on *plastic analysis* in which the loads required to cause the structure to *collapse* are calculated. The reasoning behind this method is that, in most steel structures, particularly redundant ones, the loads required to cause the structure to collapse are somewhat larger than the ones which cause yielding. Design, based on this method, calculates the loading required to cause complete collapse and then ensures that this load is greater than the applied loading; the ratio of collapse load to the maximum applied load is called the *load factor*. Generally, *plastic*, or *ultimate load* design, results in more economical structures.

In this chapter we shall investigate the mechanisms of plastic collapse and determine collapse loads for a variety of beams and frames.

## 18.1 Theorems of plastic analysis

Plastic analysis is governed by three fundamental theorems which are valid for elasto-plastic structures in which the displacements are small such that the geometry of the displaced structure does not affect the applied loading system.

### The uniqueness theorem

The following conditions must be satisfied simultaneously by a structure in its collapsed state:

The *equilibrium condition* states that the bending moments must be in equilibrium with the applied loads.

The *yield condition* states that the bending moment at any point in the structure must not exceed the plastic moment at that point.

The *mechanism condition* states that sufficient plastic hinges must have formed so that all, or part of, the structure is a mechanism.

### The lower bound, or safe, theorem

If a distribution of moments can be found which satisfies the above equilibrium and yield conditions the structure is either safe or just on the point of collapse.

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### The upper bound, or unsafe, theorem

If a loading is found which causes a collapse mechanism to form then the loading must be equal to or greater than the actual collapse load.

Generally, in plastic analysis, the upper bound theorem is used. Possible collapse mechanisms are formulated and the corresponding collapse loads calculated. From the upper bound theorem we know that all mechanisms must give a value of collapse load which is greater than or equal to the true collapse load so that the critical mechanism is the one giving the lowest load. It is possible that a mechanism, which would give a lower value of collapse load, has been missed. A check must therefore be carried out by applying the lower bound theorem.

## 18.2 Plastic analysis of beams

Generally plastic behaviour is complex and is governed by the form of the stress–strain curve in tension and compression of the material of the beam. Fortunately mild steel beams, which are used extensively in civil engineering construction, possess structural properties that lend themselves to a relatively simple analysis of plastic bending.

We have seen in Section 8.3, Fig. 8.8, that mild steel obeys Hooke's law up to a sharply defined yield stress and then undergoes large strains during yielding until strain hardening causes an increase in stress. For the purpose of plastic analysis we shall neglect the upper and lower yield points and idealize the stress–strain curve as shown in Fig. 18.1. We shall also neglect the effects of strain hardening, but since this provides an increase in strength of the steel it is on the safe side to do so. Finally we shall assume that both Young's modulus, $E$, and the yield stress, $\sigma_Y$, have the same values in tension and compression, and that plane sections remain plane after bending. The last assumption may be shown experimentally to be very nearly true.

### Plastic bending of beams having a singly symmetrical cross section

This is the most general case we shall discuss since the plastic bending of beams of arbitrary section is complex and is still being researched.

Consider the length of beam shown in Fig. 18.2(a) subjected to a positive bending moment, $M$, and possessing the singly symmetrical cross section shown in Fig. 18.2(b). If $M$ is sufficiently small the length of beam will bend elastically, producing at any section mm, the linear direct stress distribution of Fig. 18.2(c)

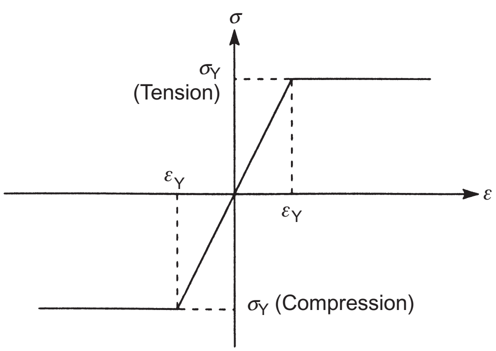


**FIGURE 18.1**

Idealized stress–strain curve for mild steel.

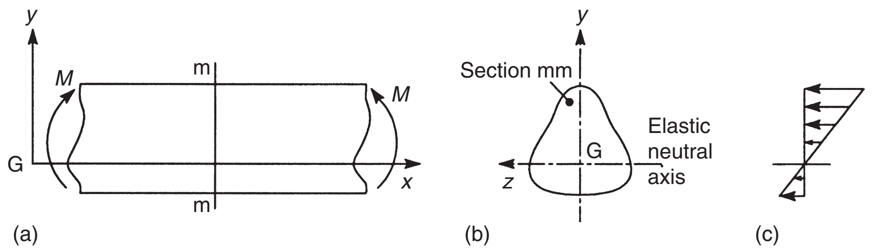


**FIGURE 18.2**

Direct stress due to bending in a singly symmetrical section beam.

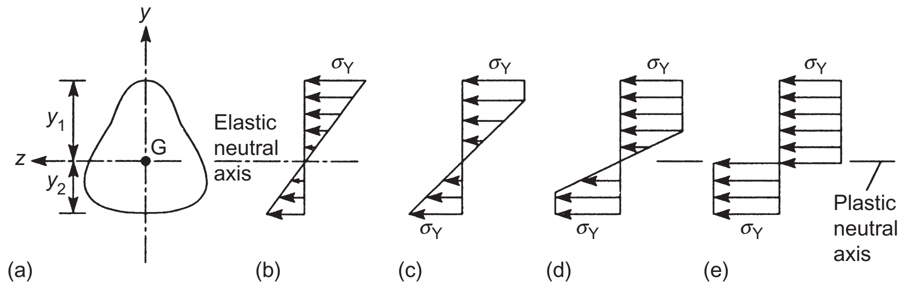


**FIGURE 18.3**

Yielding of a beam section due to bending.

where the stress, $\sigma$, at a distance $y$ from the neutral axis of the beam is given by Eq. (9.9). In this situation the *elastic neutral axis* of the beam section passes through the centroid of area of the section (Eq. (9.5)).

Suppose now that $M$ is increased. A stage will be reached where the maximum direct stress in the section, i.e. at the point furthest from the elastic neutral axis, is equal to the yield stress, $\sigma_Y$ (Fig. 18.3(b)). The corresponding value of $M$ is called the *yield moment,* $M_Y$, and is given by Eq. (9.9); thus

$$M_Y = \frac{\sigma_Y I}{y_1} \tag{18.1}$$

If the bending moment is further increased, the strain at the extremity $y_1$ of the section increases and exceeds the yield strain, $\varepsilon_Y$. However, due to plastic yielding the stress remains constant and equal to $\sigma_Y$ as shown in the idealized stress–strain curve of Fig. 18.1. At some further value of $M$ the stress at the lower extremity of the section also reaches the yield stress, $\sigma_Y$ (Fig. 18.3(c)). Subsequent increases in bending moment cause the regions of plasticity at the extremities of the beam section to extend inwards, producing a situation similar to that shown in Fig. 18.3(d); at this stage the central portion or 'core' of the beam section remains elastic while the outer portions are plastic. Finally, with further increases in bending moment the elastic core is reduced to a negligible size and the beam section is more or less completely plastic. Then, for all practical purposes the beam has reached its ultimate moment resisting capacity; the value of bending moment at this stage is known as the *plastic moment,* $M_P$, of the beam. The stress distribution corresponding to this moment may be idealized into two rectangular portions as shown in Fig. 18.3(e).

The problem now, therefore, is to determine the plastic moment, $M_P$. First, however, we must investigate the position of the neutral axis of the beam section when the latter is in its fully plastic state. One of the conditions used in establishing that the elastic neutral axis coincides with the centroid of a beam section was that stress is directly proportional to strain (Eq. (9.2)). It is clear that this is no longer the case for the stress distributions of Figs 18.3(c), (d) and (e). In Fig. 18.3(e) the beam section above the *plastic neutral axis* is subjected to a uniform compressive stress, $\sigma_Y$, while below the neutral axis the stress is tensile and also equal to $\sigma_Y$. Suppose that the area of the beam section below the plastic neutral axis is $A_2$, and that above, $A_1$ (Fig. 18.4(a)). Since $M_P$ is a pure bending moment the total direct load on the beam section must be zero. Thus from Fig. 18.4

$$\sigma_Y A_1 = \sigma_Y A_2$$

so that

$$A_1 = A_2 \tag{18.2}$$

Therefore if the total cross-sectional area of the beam section is $A$

$$A_1 = A_2 = \frac{A}{2} \tag{18.3}$$

and we see that the plastic neutral axis divides the beam section into two equal areas. Clearly for doubly symmetrical sections or for singly symmetrical sections in which the plane of the bending moment is perpendicular to the axis of symmetry, the elastic and plastic neutral axes coincide.

The plastic moment, $M_P$, can now be found by taking moments of the resultants of the tensile and compressive stresses about the neutral axis. These stress resultants act at the centroids $C_1$ and $C_2$ of the areas $A_1$ and $A_2$, respectively. Thus from Fig. 18.4

$$M_P = \sigma_Y A_1 \bar{y}_1 + \sigma_Y A_2 \bar{y}_2$$

or, using Eq. (18.3)

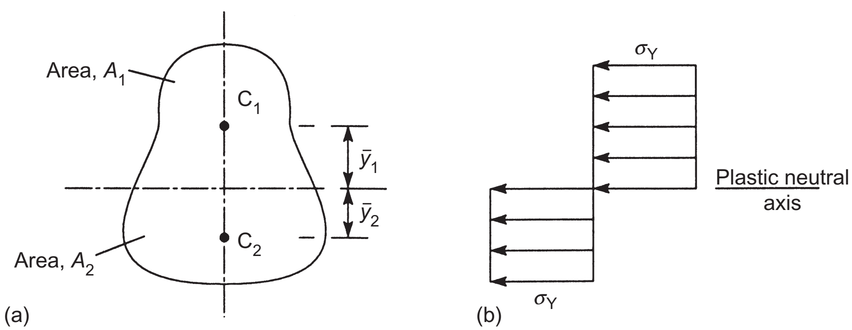


**FIGURE 18.4**

Position of the plastic neutral axis in a beam section.

$$M_P = \sigma_Y \frac{A}{2}(\bar{y}_1 + \bar{y}_2) \tag{18.4}$$

Equation (18.4) may be written in a similar form to Eq. (9.13); thus

$$M_P = \sigma_Y Z_P \tag{18.5}$$

where

$$Z_P = \frac{A(\bar{y}_1 + \bar{y}_2)}{2} \tag{18.6}$$

$Z_P$ is known as the *plastic modulus* of the cross section. Note that the elastic modulus, $Z_e$, has two values for a beam of singly symmetrical cross section (Eq. (9.12)) whereas the plastic modulus is single-valued.

## Shape factor

The ratio of the plastic moment of a beam to its yield moment is known as the *shape factor, f*. Thus

$$f = \frac{M_P}{M_Y} = \frac{\sigma_Y Z_P}{\sigma_Y Z_e} = \frac{Z_P}{Z_e} \tag{18.7}$$

where $Z_P$ is given by Eq. (18.6) and $Z_e$ is the minimum elastic section modulus, $I/y_1$. It can be seen from Eq. (18.7) that $f$ is solely a function of the geometry of the beam cross section.

**EXAMPLE 18.1**

Determine the yield moment, the plastic moment and the shape factor for a rectangular section beam of breadth $b$ and depth $d$.

The elastic and plastic neutral axes of a rectangular cross section coincide (Eq. (18.3)) and pass through the centroid of area of the section. Thus, from Eq. (18.1)

$$M_Y = \frac{\sigma_Y bd^3/12}{d/2} = \sigma_Y \frac{bd^2}{6} \tag{i}$$

and from Eq. (18.4)

$$M_P = \sigma_Y \frac{bd}{2}\left(\frac{d}{4} + \frac{d}{4}\right) = \sigma_Y \frac{bd^2}{4} \tag{ii}$$

Substituting for $M_P$ and $M_Y$ in Eq. (18.7) we obtain

$$f = \frac{M_P}{M_Y} = \frac{3}{2} \tag{iii}$$

Note that the plastic collapse of a rectangular section beam occurs at a bending moment that is 50% greater than the moment at initial yielding of the beam.

## EXAMPLE 18.2

Determine the shape factor for the I-section beam shown in Fig. 18.5(a).

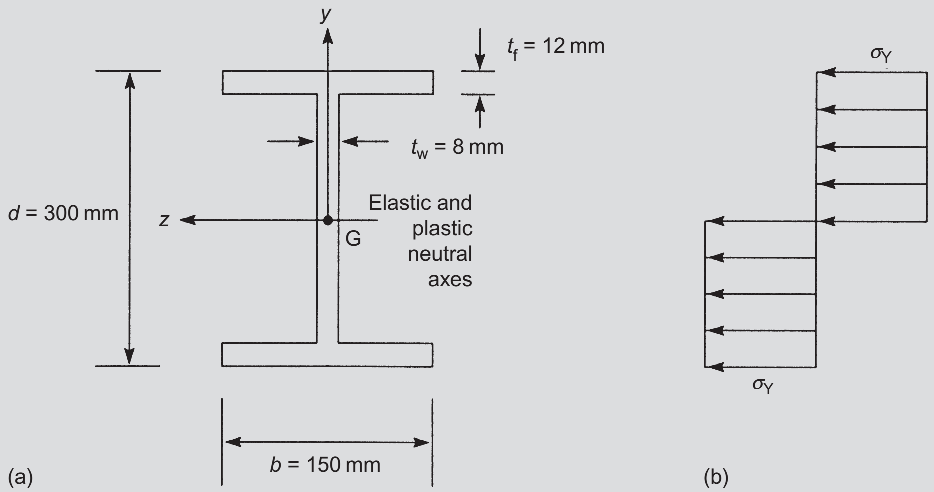


**FIGURE 18.5**

Beam section of Ex. 18.2.

Again, as in Ex. 18.1, the elastic and plastic neutral axes coincide with the centroid, G, of the section. In the fully plastic condition the stress distribution in the beam is that shown in Fig. 18.5(b). The total direct force in the upper flange is

$$\sigma_Y b t_f \,(\text{compression})$$

and its moment about G$z$ is

$$\sigma_Y b t_f\left(\frac{d}{2}-\frac{t_f}{2}\right) \equiv \frac{\sigma_Y b t_f}{2}(d-t_f) \qquad \text{(i)}$$

Similarly the total direct force in the web above G$z$ is

$$\sigma_Y t_w\left(\frac{d}{2}-t_f\right)(\text{compression})$$

and its moment about G$z$ is

$$\sigma_Y t_w\left(\frac{d}{2}-t_f\right)\frac{1}{2}\left(\frac{d}{2}-t_f\right) \equiv \frac{\sigma_Y t_w}{8}(d-2t_f)^2 \qquad \text{(ii)}$$

The lower half of the section is in tension and contributes the same moment about G$z$ so that the total plastic moment, $M_P$, of the complete section is given by

$$M_P = \sigma_Y\left[b t_f(d-t_f)+\frac{1}{4}t_w(d-2t_f)^2\right] \qquad \text{(iii)}$$

Comparing Eqs. (18.5) and (iii) we see that $Z_P$ is given by

$$Z_P = bt_f(d - t_f) + \frac{1}{4}t_w(d - 2t_f)^2 \qquad \text{(iv)}$$

Alternatively we could have obtained $Z_P$ from Eq. (18.6).

The second moment of area, $I$, of the section about the common neutral axis is

$$I = \frac{bd^3}{12} - \frac{(b - t_w)(d - 2t_f)^3}{12}$$

so that the elastic modulus $Z_e$ is given by

$$Z_e = \frac{I}{d/2} = \frac{2}{d}\left[\frac{bd^3}{12} - \frac{(b - t_w)(d - 2t_f)^3}{12}\right] \qquad \text{(v)}$$

Substituting the actual values of the dimensions of the section in Eqs (iv) and (v) we obtain

$$Z_P = 150 \times 12(300 - 12) + \frac{1}{4} \times 8(300 - 2 \times 12)^2 = 6.7 \times 10^5 \text{mm}^3$$

and

$$Z_e = \frac{2}{300}\left[\frac{150 \times 300^3}{12} - \frac{(150 - 8)(300 - 24)^3}{12}\right] = 5.9 \times 10^5 \text{mm}^3$$

Therefore from Eq. (18.7)

$$f = \frac{M_P}{M_Y} = \frac{Z_P}{Z_e} = \frac{6.7 \times 10^5}{5.9 \times 10^5} = 1.14$$

and we see that the fully plastic moment is only 14% greater than the moment at initial yielding.

## EXAMPLE 18.3

Determine the shape factor of the T-section shown in Fig. 18.6.

In this case the elastic and plastic neutral axes are not coincident. Suppose that the former is a depth $y_e$ from the upper surface of the flange and the latter a depth $y_P$. The elastic neutral axis passes through the centroid of the section, the location of which is found in the usual way. Hence, taking moments of areas about the upper surface of the flange

$$(150 \times 10 + 190 \times 7)y_e = 150 \times 10 \times 5 + 190 \times 7 \times 105$$

which gives

$$y_e = 52.0 \text{ mm}$$

The second moment of area of the section about the elastic neutral axis is then, using Eq. (9.38)

$$I = \frac{150 \times 52^3}{3} - \frac{143 \times 42^3}{3} + \frac{7 \times 148^3}{3} = 11.1 \times 10^6 \text{mm}^4$$

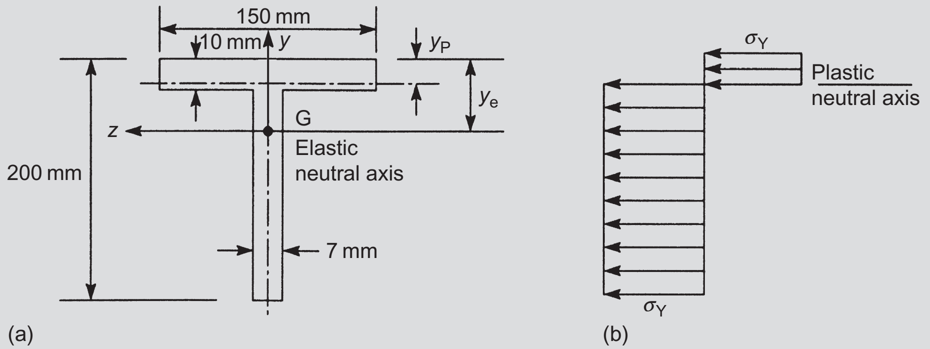


**FIGURE 18.6**

Beam section of Ex. 18.3.

Therefore

$$Z_e = \frac{11.1 \times 10^6}{148} = 75000\ \text{mm}^3$$

Note that we choose the least value for $Z_e$ since the stress will be a maximum at a point furthest from the elastic neutral axis.

The plastic neutral axis divides the section into equal areas (see Eq. (18.3)). Inspection of Fig. 18.6 shows that the flange area is greater than the web area so that the plastic neutral axis must lie within the flange. Hence

$$150y_P = 150(10 - y_P) + 190 \times 7$$

from which

$$y_P = 9.4\,\text{mm}$$

Equation (18.6) may be interpreted as the first moment, about the plastic neutral axis, of the area above the plastic neutral axis plus the first moment of the area below the plastic neutral axis. Hence

$$Z_P = 150 \times 9.4 \times 4.7 + 150 \times 0.6 \times 0.3 + 190 \times 7 \times 95.6 = 133\,800\,\text{mm}^3$$

The shape factor $f$ is, from Eq. (18.7)

$$f = \frac{M_P}{M_Y} = \frac{Z_P}{Z_e} = \frac{133800}{75000} = 1.78$$

## Moment–curvature relationships

From Eq. (9.8) we see that the curvature $k$ of a beam subjected to elastic bending is given by

$$k = \frac{1}{R} = \frac{M}{EI} \tag{18.8}$$

At yield, when $M$ is equal to the yield moment, $M_Y$

$$k_Y = \frac{M_Y}{EI} \tag{18.9}$$

The moment–curvature relationship for a beam in the linear elastic range may therefore be expressed in non-dimensional form by combining Eqs (18.8) and (18.9), i.e.

$$\frac{M}{M_Y} = \frac{k}{k_Y} \tag{18.10}$$

This relationship is represented by the linear portion of the moment–curvature diagram shown in Fig. 18.7. When the bending moment is greater than $M_Y$ part of the beam becomes fully plastic and the moment–curvature relationship is non-linear. As the plastic region in the beam section extends inwards towards the neutral axis the curve becomes flatter as rapid increases in curvature are produced by small increases in moment. Finally, the moment–curvature curve approaches the horizontal line $M = M_P$ as an asymptote when, theoretically, the curvature is infinite at the collapse load. From Eq. (18.7) we see that when $M = M_P$, the ratio $M/M_Y = f$, the shape factor. Clearly the equation of the non-linear portion of the moment–curvature diagram depends upon the particular cross section being considered.

Suppose a beam of rectangular cross section is subjected to a bending moment which produces fully plastic zones in the outer portions of the section (Fig. 18.8(a)); the depth of the elastic core is $d_e$. The total bending moment, $M$, corresponding to the stress distribution of Fig. 18.8(b) is given by

$$M = 2\sigma_Y b \frac{1}{2}(d - d_e)\frac{1}{2}\left(\frac{d}{2} + \frac{d_e}{2}\right) + 2\frac{\sigma_Y}{2} b \frac{d_e}{2}\frac{2}{3}\frac{d_e}{2}$$

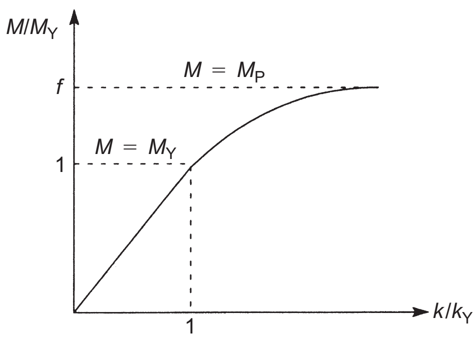


**FIGURE 18.7**

Moment–curvature diagram for a beam.

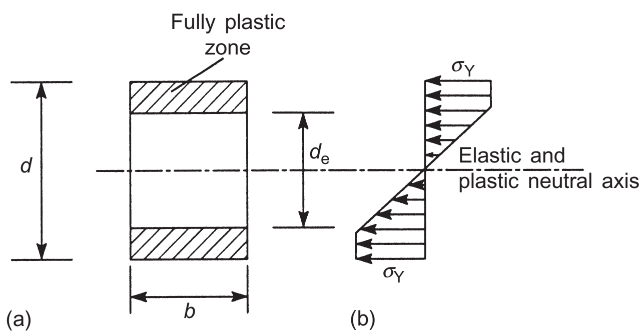


**FIGURE 18.8**

Plastic bending of a rectangular-section beam.

which simplifies to

$$M=\frac{\sigma_Y bd^2}{12}\left(3-\frac{d_e^2}{d^2}\right)=\frac{M_Y}{2}\left(3-\frac{d_e^2}{d^2}\right) \tag{18.11}$$

Note that when $d_e = d$, $M = M_Y$ and when $d_e = 0$, $M = 3M_Y/2 = M_P$ as derived in Ex. 18.1.

The curvature of the beam at the section shown may be found using Eq. (9.2) and applying it to a point on the outer edge of the elastic core. Thus

$$\sigma_Y = E\frac{d_e}{2R}$$

or

$$k=\frac{1}{R}=\frac{2\sigma_Y}{Ed_e} \tag{18.12}$$

The curvature of the beam at yield is obtained from Eq. (18.9), i.e.

$$k_Y=\frac{M_Y}{EI}=\frac{2\sigma_Y}{Ed} \tag{18.13}$$

Combining Eqs (18.12) and (18.13) we obtain

$$\frac{k}{k_Y}=\frac{d}{d_e} \tag{18.14}$$

Substituting for $d_e/d$ in Eq. (18.11) from Eq. (18.14) we have

$$M=\frac{M_Y}{2}\left(3-\frac{k_Y^2}{k^2}\right)$$

so that

$$\frac{k}{k_Y}=\frac{1}{\sqrt{3-2M/M_Y}} \tag{18.15}$$

Equation (18.15) gives the moment–curvature relationship for a rectangular section beam for $M_Y \le M \le M_P$, i.e. for the non-linear portion of the moment–curvature diagram of Fig. 18.7 for the particular case of a rectangular section beam. Corresponding relationships for beams of different section are found in a similar manner.

We have seen that for bending moments in the range $M_Y \le M \le M_P$ a beam section comprises fully plastic regions and a central elastic core. Thus yielding occurs in the plastic regions with no increase in stress whereas in the elastic core increases in deformation are accompanied by increases in stress. The deformation of the beam is therefore controlled by the elastic core, a state sometimes termed *contained plastic flow*. As $M$ approaches $M_P$ the moment–curvature diagram is asymptotic to the line $M = M_P$ so that large increases in deformation occur without any increase in moment, a condition known as *unrestricted plastic flow.*

In Eq. (iii) of Ex. 18.1 we have seen that, for a rectangular section beam, the ratio of the plastic moment to the yield moment is 1.5:1, that is

$$M_Y=\frac{2}{3}M_P$$

Then, substituting for $M_Y$ in Eq. (18.15) and rearranging we obtain

$$M = \left[1 - \frac{1}{3}\left(\frac{k_Y}{k}\right)^2\right] M_P$$

For a range of values of $k_Y/k$ we can obtain the applied moment in terms of the plastic moment as shown in Table 18.1.

Therefore we see that when the applied moment is approaching 99% of the plastic moment the beam curvature is only five times greater than that at the onset of yield.

**Table 18.1**

| | | | | | |
|---|---|---|---|---|---|
| $k/k_Y$ | 1 | 2 | 3 | 4 | 5 |
| $M/M_P$ | 0.667 | 0.917 | 0.963 | 0.979 | 0.987 |

## Plastic hinges

The presence of unrestricted plastic flow at a section of a beam leads us to the concept of the formation of *plastic hinges* in beams and other structures.

Consider the simply supported beam shown in Fig. 18.9(a); the beam carries a concentrated load, $W$, at mid-span. The bending moment diagram (Fig. 18.9(b)) is triangular in shape with a maximum moment equal to $WL/4$. If $W$ is increased in value until $WL/4 = M_P$, the mid-span section of the beam will be fully plastic with regions of plasticity extending towards the supports as the bending moment decreases; no plasticity occurs in beam sections for which the bending moment is less than $M_Y$. Clearly, unrestricted plastic flow now occurs at the mid-span section where large increases in deformation take place with no increase in

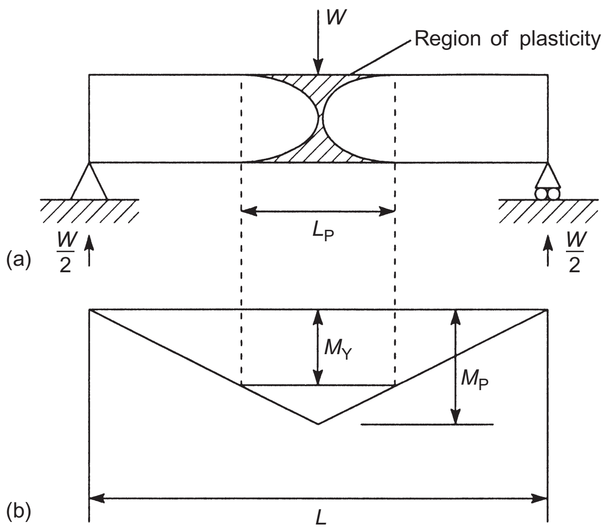


**FIGURE 18.9**

Formation of a plastic hinge in a simply supported beam.

load. The beam therefore behaves as two rigid beams connected by a *plastic hinge* which allows them to rotate relative to each other. The value of $W$ given by $W = 4M_P/L$ is the *collapse load* for the beam.

The length, $L_P$, of the plastic region of the beam may be found using the fact that at each section bounding the region the bending moment is equal to $M_Y$. Thus

$$M_Y = \frac{W}{2}\left(\frac{L - L_P}{2}\right)$$

Substituting for $W(= 4M_p/L)$ we obtain

$$M_Y = \frac{M_P}{L}(L - L_P)$$

from which

$$L_P = L\left(1 - \frac{M_Y}{M_P}\right)$$

or, from Eq. (18.7)

$$L_P = L\left(1 - \frac{1}{f}\right) \tag{18.16}$$

For a rectangular section beam $f = 1.5$ (see Ex. 18.1), giving $L_P = L/3$. For the I-section beam of Ex. 18.2, $f = 1.14$ and $L_P = 0.12L$ so that the plastic region in this case is much smaller than that of a rectangular section beam; this is generally true for I-section beams.

It is clear from the above that plastic hinges form at sections of maximum bending moment.

## Plastic analysis of beams

We can now use the concept of plastic hinges to determine the collapse or ultimate load of beams in terms of their individual yield moment, $M_P$, which may be found for a particular beam section using Eq. (18.5).

For the case of the simply supported beam of Fig. 18.9 we have seen that the formation of a single plastic hinge is sufficient to produce failure; this is true for all statically determinate systems. Having located the position of the plastic hinge, at which the moment is equal to $M_P$, the collapse load is found from simple statics. Thus for the beam of Fig. 18.9, taking moments about the mid-span section, we have

$$\frac{W_U}{2}\frac{L}{2} = M_P$$

or

$$W_U = \frac{4M_P}{L} \text{(as deduced before)}$$

where $W_U$ is the ultimate value of the load $W$.

### EXAMPLE 18.4

Determine the ultimate load for a simply supported, rectangular section beam, breadth $b$, depth $d$, having a span $L$ and subjected to a uniformly distributed load of intensity $w$.

The maximum bending moment occurs at mid-span and is equal to $wL^2/8$ (see Section 3.4). The plastic hinge therefore forms at mid-span when this bending moment is equal to $M_P$, the corresponding ultimate load intensity being $w_U$. Thus

$$\frac{w_U L^2}{8} = M_P \tag{i}$$

From Ex. 18.1, Eq. (ii)

$$M_P = \sigma_Y \frac{bd^2}{4}$$

so that

$$w_U = \frac{8M_P}{L^2} = \frac{2\sigma_Y bd^2}{L^2}$$

where $\sigma_Y$ is the yield stress of the material of the beam.

## EXAMPLE 18.5

The simply supported beam ABC shown in Fig. 18.10(a) has a cantilever overhang and supports loads of $4W$ and $W$. Determine the value of $W$ at collapse in terms of the plastic moment, $M_P$, of the beam.

The bending moment diagram for the beam is constructed using the method of Section 3.4 and is shown in Fig. 18.10(b). Clearly as $W$ is increased a plastic hinge will form first at D, the point of application of the $4W$ load. Thus, at collapse

$$\frac{3}{4} W_U L = M_P$$

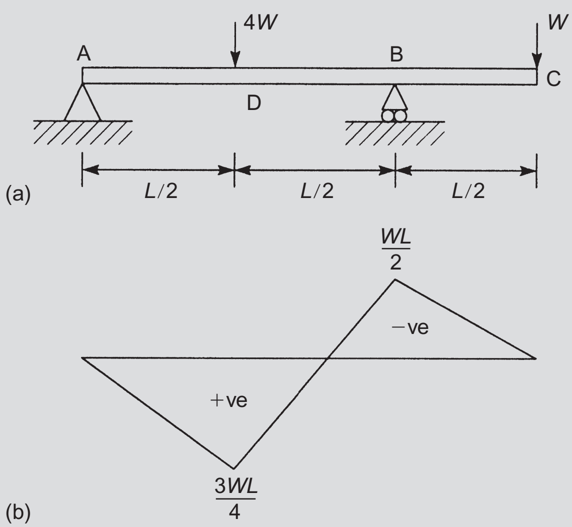


**FIGURE 18.10**

Beam of Ex. 18.5.

so that

$$W_U = \frac{4M_P}{3L}$$

where $W_U$ is the value of $W$ that causes collapse.

The formation of a plastic hinge in a statically determinate beam produces large, increasing deformations which ultimately result in failure with no increase in load. In this condition the beam behaves as a mechanism with different lengths of beam rotating relative to each other about the plastic hinge. The terms *failure mechanism* or *collapse mechanism* are often used to describe this state.

In a statically indeterminate system the formation of a single plastic hinge does not necessarily mean collapse. Consider the propped cantilever shown in Fig. 18.11(a). The bending moment diagram may be drawn after the reaction at C has been determined by any suitable method of analysis of statically indeterminate beams (see Chapter 16) and is shown in Fig. 18.11(b).

As the value of $W$ is increased a plastic hinge will form first at A where the bending moment is greatest. However, this does not mean that the beam will collapse. Instead it behaves as a statically determinate beam with a point load at B and a moment $M_P$ at A. Further increases in $W$ eventually result in the formation of a second plastic hinge at B (Fig. 18.11(c)) when the bending moment at B reaches the value $M_P$. The beam now behaves as a mechanism and failure occurs with no further increase in load. The bending moment diagram for the beam is now as shown in Fig. 18.11(d) with values of bending moment of $-M_P$ at A and $M_P$ at B. Comparing the bending moment diagram at collapse with that corresponding to the elastic deformation of the beam (Fig. 18.11(b)) we see that a redistribution of bending moment has occurred. This is generally the case in statically indeterminate systems whereas in statically determinate systems the bending moment diagrams in the elastic range and at collapse have identical shapes (see Figs. 18.9(b) and 18.10(b)). In the beam of Fig. 18.11 the elastic bending moment diagram has a maximum at A. After the formation of the plastic hinge at A the bending moment remains constant while the bending moment at B increases until the second plastic hinge forms. Thus this redistribution of moments tends to increase the ultimate strength of statically indeterminate structures since failure at one section leads to other portions of the structure supporting additional load.

Having located the positions of the plastic hinges and using the fact that the moment at these hinges is $M_P$, we may determine the ultimate load, $W_U$, by statics. Therefore taking moments about A we have

$$M_P = W_U \frac{L}{2} - R_C L \tag{18.17}$$

where $R_C$ is the vertical reaction at the support C. Now considering the equilibrium of the length BC we obtain

$$R_C \frac{L}{2} = M_P \tag{18.18}$$

Eliminating $R_C$ from Eqs (18.17) and (18.18) gives

$$W_U = \frac{6M_P}{L} \tag{18.19}$$

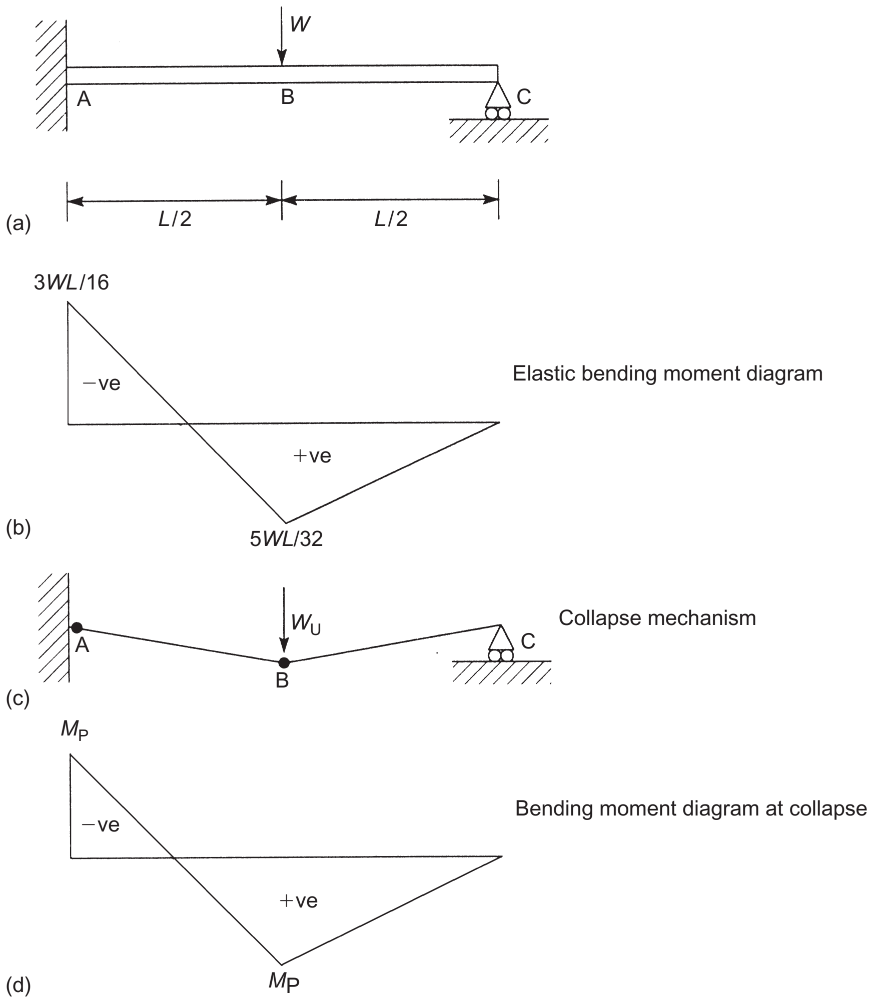


**FIGURE 18.11**

Plastic hinges in a propped cantilever.

Note that in this particular problem it is unnecessary to determine the elastic bending moment diagram to solve for the ultimate load which is obtained using statics alone. This is a convenient feature of plastic analysis and leads to a much simpler solution of statically indeterminate structures than an elastic analysis. Furthermore, the magnitude of the ultimate load is not affected by structural imperfections such as a sinking support, whereas the same kind of imperfection would have an appreciable effect on the elastic behaviour of a structure. Note also that the principle of superposition (Section 3.7), which is based on the linearly elastic behaviour of a structure, does not hold for plastic analysis. In fact the plastic behaviour of a structure depends upon the order in which the loads are applied as well as their final values. We therefore assume in plastic analysis that all loads are applied simultaneously and that the ratio of the loads remains constant during loading.

An alternative and powerful method of analysis uses the principle of virtual work (see Section 15.2), which states that for a structure that is in equilibrium and which is given a small virtual displacement, the sum of the work done by the internal forces is equal to the work done by the external forces.

Consider the propped cantilever of Fig. 18.11(a); its collapse mechanism is shown in Fig. 18.11(c). At the instant of collapse the cantilever is in equilibrium with plastic hinges at A and B where the moments are each $M_P$ as shown in Fig. 18.11(d). Suppose that AB is given a small rotation, $\theta$. From geometry, BC also rotates through an angle $\theta$ as shown in Fig. 18.12; the vertical displacement of B is then $\theta L/2$. The external forces on the cantilever which do work during the virtual displacement are comprised solely of $W_U$ since the vertical reactions at A and C are not displaced. The internal forces which do work consist of the plastic moments, $M_P$, at A and B and which resist rotation. Hence

$$W_U\theta\frac{L}{2} = (M_P)_A\theta + (M_P)_B 2\theta (\text{see Section 15.1})$$

from which $W_U = 6M_P/L$ as before.

We have seen that the plastic hinges form at beam sections where the bending moment diagram attains a peak value. It follows that for beams carrying a series of point loads, plastic hinges are located at the load positions. However, in some instances several collapse mechanisms are possible, each giving different values of ultimate load. For example, if the propped cantilever of Fig. 18.11(a) supports two point loads as shown in Fig. 18.13(a), three possible collapse mechanisms are possible (Fig. 18.13(b–d)). Each possible collapse mechanism should be analysed and the lowest ultimate load selected.

The beams we have considered so far have carried concentrated loads only so that the positions of the plastic hinges, and therefore the form of the collapse mechanisms, are easily determined. This is not the case when distributed loads are involved.

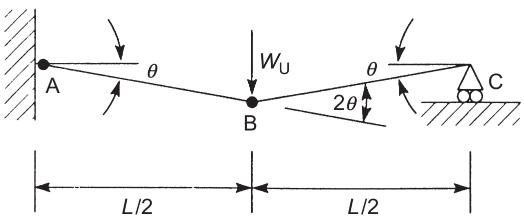


**FIGURE 18.12**

Virtual displacements in propped cantilever of Fig. 18.11.

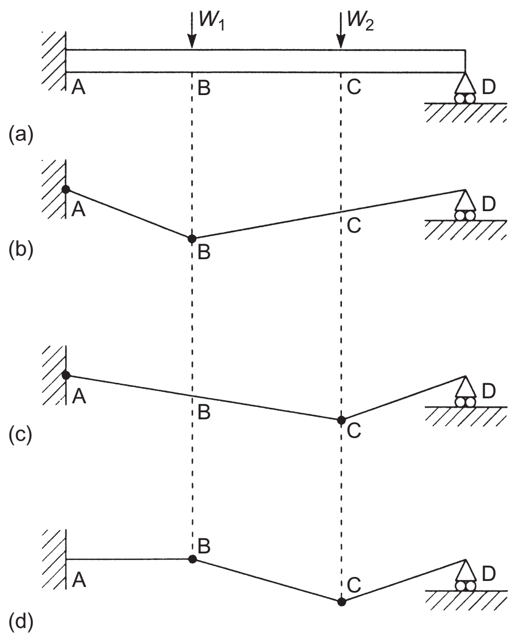


**FIGURE 18.13**

Possible collapse mechanisms in a propped cantilever supporting two concentrated loads.

## EXAMPLE 18.6

The propped cantilever AB shown in Fig. 18.14(a) carries a uniformly distributed load of intensity $w$. If the plastic moment of the cantilever is $M_P$ calculate the minimum value of $w$ required to cause collapse.

Peak values of bending moment occur at A and at some point between A and B so that plastic hinges will form at A and at a point C a distance $x$, say, from A; the collapse mechanism is then as shown in Fig. 18.14(b) where the rotations of AC and CB are $\theta$ and $\phi$ respectively. Then, the vertical deflection of C is given by

$$\delta = \theta x = \phi(L - x) \tag{i}$$

so that

$$\phi = \theta \frac{x}{L - x} \tag{ii}$$

The total load on AC is $wx$ and its centroid (at $x/2$ from A) will be displaced a vertical distance $\delta/2$. The total load on CB is $w(L-x)$ and its centroid will suffer the same vertical displacement $\delta/2$. Then, from the principle of virtual work

$$wx\frac{\delta}{2} + w(L-x)\frac{\delta}{2} = M_P\theta + M_P(\theta + \phi)$$

Note that the beam at B is free to rotate so that there is no plastic hinge at B. Substituting for $\delta$ from Eq. (i) and $\phi$ from Eq. (ii) we obtain

$$wL\frac{\theta x}{2} = M_P\theta + M_P\left(\theta + \theta\frac{x}{L-x}\right)$$

or

$$wL\frac{\theta x}{2} = M_P\theta\left(2 + \frac{x}{L-x}\right)$$

Rearranging

$$w = \frac{2M_P}{Lx}\left(\frac{2L-x}{L-x}\right) \tag{iii}$$

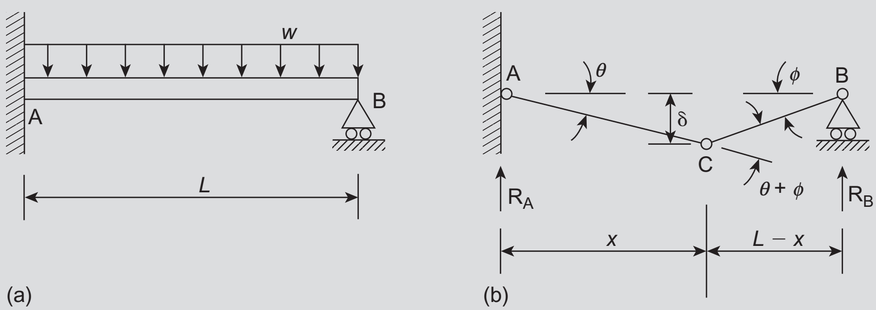


**FIGURE 18.14**

Collapse mechanism for a propped cantilever.

For a minimum value of $w$, $(\mathrm{d}w/\mathrm{d}x) = 0$. Then

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{2M_\mathrm{P}}{L}\left[\frac{-x(L-x)-(2L-x)(L-2x)}{x^2(L-x)^2}\right] = 0$$

which reduces to

$$x^2 - 4Lx + 2L^2 = 0$$

Solving gives

$$x = 0.586L(\text{the positive root is ignored})$$

Then substituting for $x$ in Eq. (iii)

$$w(\text{at collapse}) = \frac{11.66M_\mathrm{P}}{L^2}$$

We can now use the lower bound theorem to check that we have obtained the critical mechanism and thereby the critical load. The internal moment at A at collapse is hogging and equal to $M_\mathrm{P}$. Then, taking moments about A

$$R_\mathrm{B}L - w\frac{L^2}{2} = -M_\mathrm{P}$$

which gives

$$R_\mathrm{B} = \frac{4.83M_\mathrm{P}}{L}$$

Similarly, taking moments about B gives

$$R_\mathrm{A} = \frac{6.83M_\mathrm{P}}{L}$$

Summation of $R_\mathrm{A}$ and $R_\mathrm{B}$ gives $11.66M_\mathrm{P}/L = wL$ so that vertical equilibrium is satisfied. Further, considering moments of forces to the right of C about C we have

$$M_\mathrm{C} = R_\mathrm{B}(0.414L) - w\frac{0.414L^2}{2}$$

Substituting for $R_\mathrm{B}$ and $w$ from the above gives $M_\mathrm{C} = M_\mathrm{P}$. The same result is obtained by considering moments about C of forces to the left of C. The load therefore satisfies both vertical and moment equilibrium.

The bending moment at any distance $x_1$, say, from B is given by

$$M = R_\mathrm{B}x_1 - w\frac{x_1^2}{2}$$

Then

$$\frac{\mathrm{d}M}{\mathrm{d}x_1} = R_\mathrm{B} - wx_1 = 0$$

so that a maximum occurs when $x_1 = R_\mathrm{B}/w$. Substituting for $R_\mathrm{B}$, $x_1$ and $w$ in the expression for $M$ gives $M = M_\mathrm{P}$ so that the yield criterion is satisfied. We conclude, therefore, that the mechanism of Fig. 18.14 (b) is the critical mechanism.

## Plastic design of beams

It is now clear that the essential difference between the plastic and elastic methods of design is that the former produces a structure having a more or less uniform factor of safety against collapse of all its components, whereas the latter produces a uniform factor of safety against yielding. The former method in fact gives an indication of the true factor of safety against collapse of the structure which may occur at loads only marginally greater than the yield load, depending on the cross sections used. For example, a rectangular section mild steel beam has an ultimate strength 50% greater than its yield strength (see Ex. 18.1), whereas for an I-section beam the margin is in the range 10–20% (see Ex. 18.2). It is also clear that each method of design will produce a different section for a given structural component. This distinction may be more readily understood by referring to the redistribution of bending moment produced by the plastic collapse of a statically indeterminate beam.

Two approaches to the plastic design of beams are indicated by the previous analysis. The most direct method would calculate the working loads, determine the required strength of the beam by the application of a suitable load factor, obtain by a suitable analysis the required plastic moment in terms of the ultimate load and finally, knowing the yield stress of the material of the beam, determine the required plastic section modulus. An appropriate beam section is then selected from a handbook of structural sections. The alternative method would assume a beam section, calculate the plastic moment of the section and hence the ultimate load for the beam. This value of ultimate load is then compared with the working loads to determine the actual load factor, which would then be checked against the prescribed value.

### EXAMPLE 18.7

The propped cantilever of Fig. 18.11(a) is 10 m long and is required to carry a load of 100 kN at mid-span. If the yield stress of mild steel is 300 N/mm$^2$, suggest a suitable section using a load factor against failure of 1.5.

The required ultimate load of the beam is $1.5\times 100 = 150$ kN. Then from Eq. (18.19) the required plastic moment $M_P$ is given by

$$M_P = \frac{150\times 10}{6} = 250\,\text{kNm}$$

From Eq. (18.5) the minimum plastic modulus of the beam section is

$$Z_P = \frac{250\times 10^6}{300} = 833333\ \text{mm}^3$$

Referring to an appropriate handbook we see that a Universal Beam, 406 mm$\times$140 mm $\times$46 kg/m, has a plastic modulus of 886.3 cm$^3$. This section therefore possesses the required ultimate strength and includes a margin to allow for its self-weight. Note that unless some allowance has been made for self-weight in the estimate of the working loads the design should be rechecked to include this effect.

## Effect of axial load on plastic moment

We shall investigate the effect of axial load on plastic moment with particular reference to an I-section beam, one of the most common structural shapes, which is subjected to a positive bending moment and a compressive axial load, $P$, Fig. 18.15(a)).

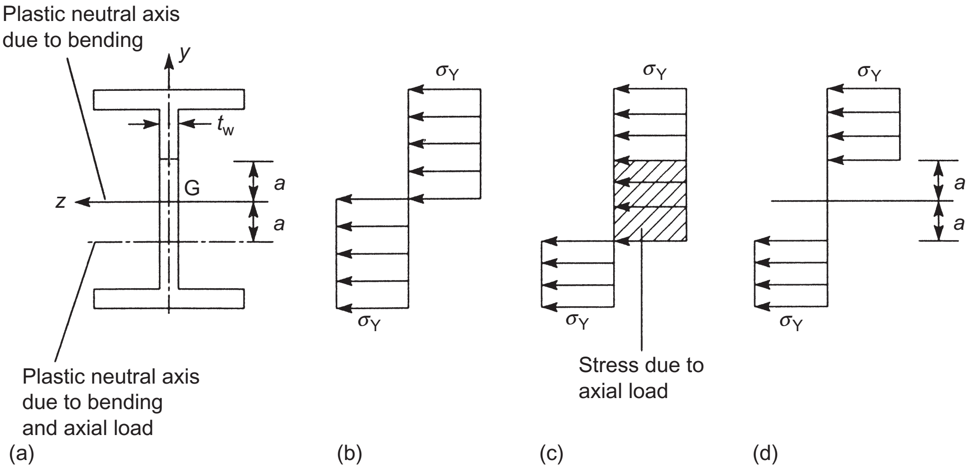


**FIGURE 18.15**

Combined bending and axial compression.

If the beam section were subjected to its plastic moment only, the stress distribution shown in Fig. 18.15 (b) would result. However, the presence of the axial load causes additional stresses which cannot, obviously, be greater than $\sigma_Y$. Thus the region of the beam section supporting compressive stresses is increased in area while the region subjected to tensile stresses is decreased in area. Clearly some of the compressive stresses are due to bending and some due to axial load so that the modified stress distribution is as shown in Fig. 18.15(c).

Since the beam section is doubly symmetrical it is reasonable to assume that the area supporting the compressive stress due to bending is equal to the area supporting the tensile stress due to bending, both areas being symmetrically arranged about the original plastic neutral axis. Thus from Fig. 18.15(d) the reduced plastic moment, $M_{P,R}$, is given by

$$M_{P,R} = \sigma_Y(Z_P - Z_a) \tag{18.20}$$

where $Z_a$ is the plastic section modulus for the area on which the axial load is assumed to act. From Eq. (18.6)

$$Z_a = \frac{2at_w}{2}\left(\frac{a}{2}+\frac{a}{2}\right) = a^2 t_w$$

also

$$P = 2at_w\sigma_Y$$

so that

$$a = \frac{P}{2t_w\sigma_Y}$$

Substituting for $Z_a$, in Eq. (18.20) and then for $a$, we obtain

$$M_{P,R} = \sigma_Y\left(Z_P - \frac{P^2}{4t_w\sigma_Y^2}\right) \tag{18.21}$$

Let $\sigma_a$ be the mean axial stress due to $P$ taken over the complete area, $A$, of the beam section. Then

$$P = \sigma_a A$$

Substituting for $P$ in Eq. (18.21)

$$M_{P,R} = \sigma_Y \left( Z_P - \frac{A^2}{4t_w} \frac{\sigma_a^2}{\sigma_Y^2} \right) \tag{18.22}$$

Thus the reduced plastic section modulus may be expressed in the form

$$Z_{P,R} = Z_P - Kn^2 \tag{18.23}$$

where $K$ is a constant that depends upon the geometry of the beam section and $n$ is the ratio of the mean axial stress to the yield stress of the material of the beam.

Equations (18.22) and (18.23) are applicable as long as the neutral axis lies in the web of the beam section. In the rare case when this is not so, reference should be made to advanced texts on structural steel design. In addition the design of beams carrying compressive loads is influenced by considerations of local and overall instability, as we shall see in Chapter 21.

**EXAMPLE 18.8**

If the propped cantilever of Ex. 18.7 is subjected to an axial load of 150 kN in addition to the 100 kN load at mid-span determine whether or not the selected Universal Beam is still adequate.

From Steel Tables the cross sectional area of the beam is 58.9 cm$^2$ and its web thickness is 6.9 mm. The mean axial stress is then

$$\sigma_a = \frac{150 \times 10^3}{58.9 \times 10^2} = 25.5\,\mathrm{N/mm^2}$$

Then, from Eqs. (18.22) and (18.23) the reduced plastic section modulus is given by

$$Z_{P,R} = 886.3 - \frac{(58.9)^2}{4 \times (6.9/10)} \times \frac{25.5^2}{300^2}$$

which gives

$$Z_{P,R} = 877.2\,\mathrm{cm^3}$$

The required plastic modulus of the beam section is 833.3 cm$^3$ so that the beam section is still adequate.

## 18.3 Plastic analysis of frames

The plastic analysis of frames is carried out in a very similar manner to that for beams in that possible collapse mechanisms are identified and the principle of virtual work used to determine the collapse loads. A complication does arise, however, in that frames, even though two-dimensional, can possess collapse mechanisms which involve both *beam* and *sway* mechanisms since, as we saw in Section 16.10 in the moment distribution analysis of portal frames, sway is produced by any asymmetry of the loading or frame. Initially we shall illustrate the method by a comparatively simple example.

## EXAMPLE 18.9

Determine the value of the load $W$ required to cause collapse of the frame shown in Fig. 18.16(a) if the plastic moment of all members of the frame is 200 kNm. Calculate also the support reactions at collapse.

We note that the frame and loading are unsymmetrical so that sway occurs. The bending moment diagram for the frame takes the form shown in Fig. 18.16(b) so that there are three possible collapse mechanisms as shown in Fig. 18.17.

In Fig. 18.17(a) the horizontal member BCD has collapsed with plastic hinges forming at B, C and D; this is termed a *beam mechanism.* In Fig. 18.17(b) the frame has swayed with hinges forming at A, B, D and E; this, for obvious reasons, is called a *sway mechanism.* Fig. 18.17(c) shows a *combined mechanism* which incorporates both the beam and sway mechanisms. However, in this case, the moments at B due to

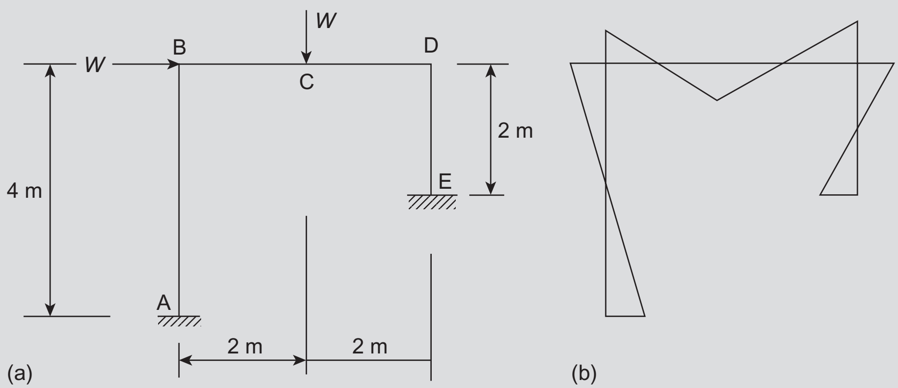


**FIGURE 18.16**

Portal frame of Ex. 18.9.

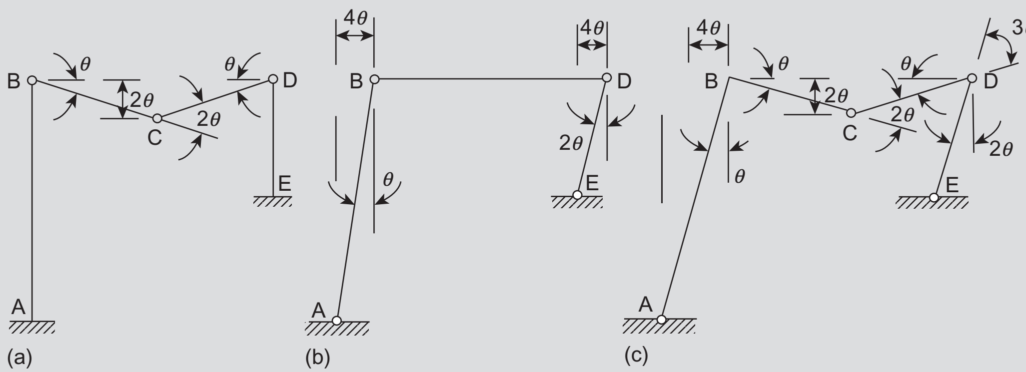


**FIGURE 18.17**

Collapse mechanisms for the frame of Ex. 18.9.

the vertical load at C and the horizontal load at B oppose each other so that the moment at B will be the smallest of the five peak moments and plastic hinges will form at the other locations. We say, therefore, that there is a *hinge cancellation* at B; the angle ABC then remains a right angle. We shall now examine each mechanism in turn to determine the value of $W$ required to cause collapse. We shall designate the plastic moment of the frame as $M_P$.

### Beam mechanism

Suppose that BC is given a small rotation $\theta$. Since CD = CB then CD also rotates through the angle $\theta$ and the relative angle between CD and the extension of BC is $2\theta$. Then, from the principle of virtual work

$$W2\theta = M_P\theta + M_P 2\theta + M_P\theta \qquad \text{(i)}$$

which gives

$$W = 2M_P$$

In the virtual work equation $2\theta$ is the vertical distance through which $W$ moves and the first, second and third terms on the right hand side represent the internal work done by the plastic moments at B, C and D respectively.

### Sway mechanism

The vertical member AB is given a small rotation $\theta$, ED then rotates through $2\theta$. Again, from the principle of virtual work

$$W4\theta = M_P\theta + M_P\theta + M_P 2\theta + M_P 2\theta \qquad \text{(ii)}$$

i.e.

$$W = \frac{3}{2}M_P$$

### Combined mechanism

Since, now, there is no plastic hinge at B there is no plastic moment at B. Then, the principle of virtual work gives

$$W4\theta + W2\theta = M_P\theta + M_P 2\theta + M_P 3\theta + M_P 2\theta \qquad \text{(iii)}$$

from which

$$W = \frac{4}{3}M_P$$

We could have obtained Eq. (iii) directly by adding Eqs (i) and (ii) and anticipating the hinge cancellation at B. Eq. (i) would then be written

$$W2\theta = \{M_P\theta\} + M_P 2\theta + M_P\theta \qquad \text{(iv)}$$

where the term in curly brackets is the internal work done by the plastic moment at B. Similarly Eq. (ii) would be written

$$W4\theta = M_P\theta + \{M_P\theta\} + M_P 2\theta + M_P 2\theta \qquad \text{(v)}$$

Adding Eqs (iv) and (v) and dropping the term in curly brackets gives

$$W6\theta = 8M_P\theta$$

as before.

From Eqs (i), (ii) and (iii) we see that the critical mechanism is the combined mechanism and the lowest value of $W$ is $4M_P/3$ so that

$$W = \frac{4 \times 200}{3}$$

i.e.

$$W = 266.7\,\text{kN}$$

Figure 18.18 shows the support reactions corresponding to the collapse mode. The internal moment at D is $M_P$ (D is a plastic hinge) so that, taking moments about D for the forces acting on the member ED

$$R_{E,H} \times 2 = M_P = 200\,\text{kNm}$$

so that

$$R_{E,H} = 100\,\text{kN}$$

Resolving horizontally

$$R_{A,H} + 266.7 - 100 = 0$$

from which

$$R_{A,H} = -166.7\,\text{kN} \quad \text{(to the left)}$$

Taking moments about A

$$R_{E,V} \times 4 + R_{E,H} \times 2 - 266.7 \times 2 - 266.7 \times 4 = 0$$

which gives

$$R_{E,V} = 350.1\,\text{kN}$$

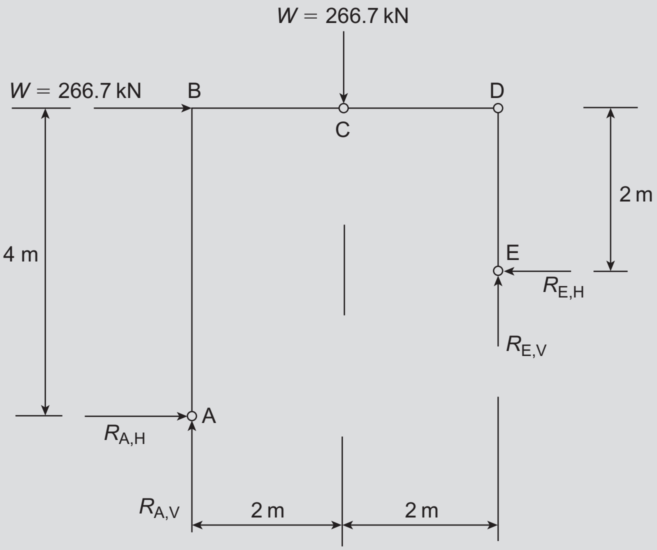


**FIGURE 18.18**

Support reactions at collapse in the frame of Ex. 18.9.

Finally, resolving vertically

$$R_{A,V} + R_{E,V} - 266.7 = 0$$

i.e.

$$R_{A,V} = -83.4\,\text{kN (downwards)}$$

In the portal frame of Ex. 18.9 each member has the same plastic moment $M_P$. In cases where the members have different plastic moments a slightly different approach is necessary.

## EXAMPLE 18.10

In the portal frame of Ex. 18.9 the plastic moment of the member BCD is $2M_P$. Calculate the critical value of the load $W$.

Since the vertical members are the weaker members plastic hinges will form at B in AB and at D in ED as shown, for all three possible collapse mechanisms, in Fig. 18.19. This has implications for the virtual work equation because in Fig. 18.19(a) the plastic moment at B and D is $M_P$ while that at C is $2M_P$. The virtual work equation then becomes

$$W2\theta = M_P\theta + 2M_P 2\theta + M_P\theta$$

which gives

$$W = 3M_P$$

For the sway mechanism

$$W4\theta = M_P\theta + M_P\theta + M_P 2\theta + M_P 2\theta$$

so that

$$W = \frac{3}{2}M_P$$

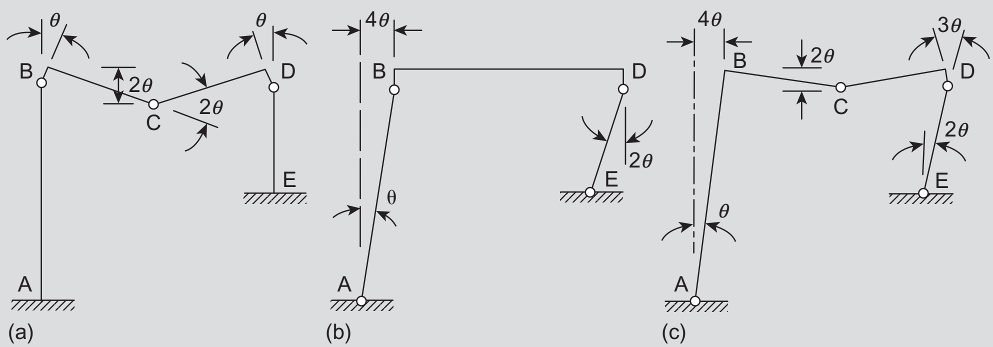


**FIGURE 18.19**

Collapse mechanisms for the frame of Ex. 18.10.

and for the combined mechanism

$$W4\theta + W2\theta = M_P\theta + 2M_P2\theta + M_P3\theta + M_P2\theta$$

from which

$$W = \frac{5}{3}M_P$$

Here we see that the minimum value of $W$ which would cause collapse is $3M_P/2$ and that the sway mechanism is the critical mechanism.

We shall now examine a portal frame having a pitched roof in which the determination of displacements is more complicated.

## EXAMPLE 18.11

The portal frame shown in Fig. 18.20(a) has members which have the same plastic moment $M_P$. Determine the minimum value of the load $W$ required to cause collapse if the collapse mechanism is that shown in Fig. 18.20(b).

In Exs 18.9 and 18.10 the displacements of the joints of the frame were relatively simple to determine since all the members were perpendicular to each other. For a pitched roof frame the calculation is more difficult; one method is to use the concept of *instantaneous centres*.

In Fig. 18.21 the member BC is given a *small* rotation $\theta$. Since $\theta$ is small C can be assumed to move at right angles to BC to C′. Similarly the member DE rotates about E so that D moves horizontally to D′. Further, since C moves at right angles to BC and D moves at right angles to DE it follows that CD rotates about the instantaneous centre, I, which is the point of intersection of BC and ED produced; the lines IC and ID then rotate through the same angle $\phi$.

From the triangles BCC′ and ICC′

$$CC' = BC\theta = IC\phi$$

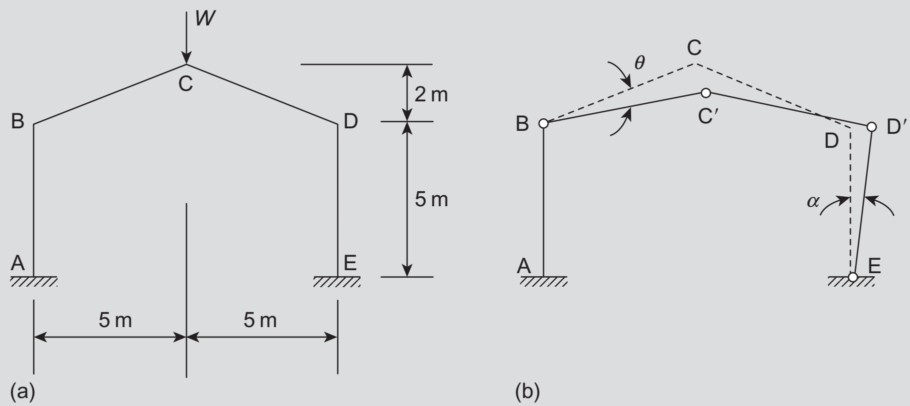


**FIGURE 18.20**

Collapse mechanism for the frame of Ex. 18.11.

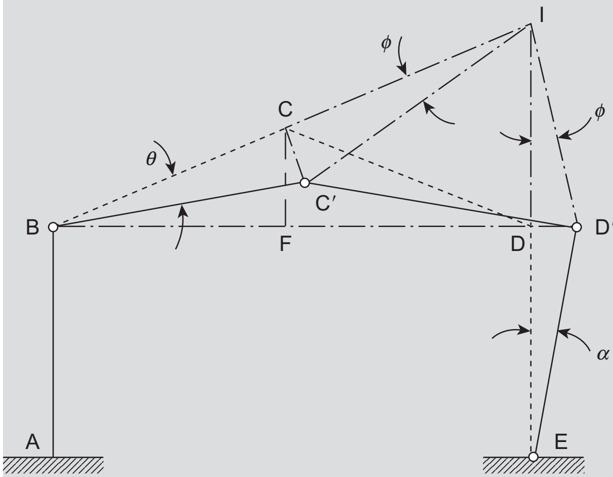


**FIGURE 18.21**

Method of instantaneous centres for the frame of Ex. 18.11.

so that

$$\phi = \frac{BC}{IC}\theta \tag{i}$$

From the triangles EDD′ and IDD′

$$DD' = ED\alpha = ID\phi$$

Therefore

$$\alpha = \frac{ID}{ED}\phi = \frac{ID}{ED}\frac{BC}{IC}\theta \tag{ii}$$

Now we drop a perpendicular from C to meet the horizontal through B and D at F. Then, from the similar triangles BCF and BID

$$\frac{BC}{CI} = \frac{BF}{FD} = \frac{5}{5} = 1$$

so that BC = CI and, from Eq. (i), $\phi = \theta$. Also

$$\frac{CF}{ID} = \frac{BF}{BD} = \frac{5}{10} = \frac{1}{2}$$

from which ID = 2CF = 4 m. Then, from Eq. (ii)

$$\alpha = \frac{4}{5}\theta$$

Finally, the vertical displacement of C to C′ is BF$\theta$ (=5$\theta$).

The equation of virtual work is then

$$W5\theta = M_P\theta + M_P(\theta + \alpha) + M_P(\phi + \alpha) + M_P\alpha$$

Substituting for $\phi$ and $\alpha$ in terms of $\theta$ from the above gives

$$W = 1.12M_P$$

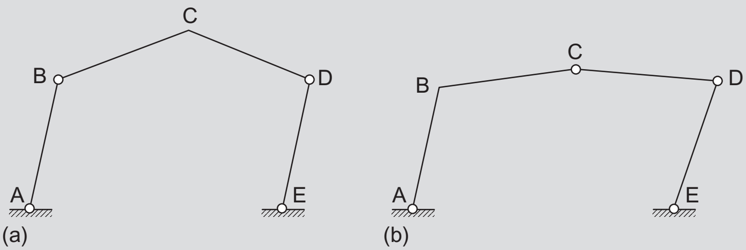


**FIGURE 18.22**

Possible collapse mechanisms for the frame of Ex. 18.11 with sway.

The failure mechanism shown in Fig. 18.20(b) does not involve sway. If, however, a horizontal load were applied at B, say, then sway would occur and other possible failure mechanisms would have to be investigated; two such mechanisms are shown in Fig. 18.22. Note that in Fig. 18.22(a) there is a hinge cancellation at C and in Fig. 18.22(b) there is a hinge cancellation at B. In determining the collapse loads of such frames the method of instantaneous centres still applies.

Loads applied to frames are not always concentrated and may be distributed along one or more members. To illustrate the method of analysis of such frames we shall consider the relatively simple frame of Ex. 18.12.

## EXAMPLE 18.12

The portal frame shown in Fig. 18.23(a) carries a uniformly distributed load of total value $4W$ across the horizontal member BD in addition to a horizontal concentrated load of $W$ at B. If the plastic moment of the member BD is $2M_P$ while that of the vertical columns is $M_P$ determine the critical value of the load $W$.

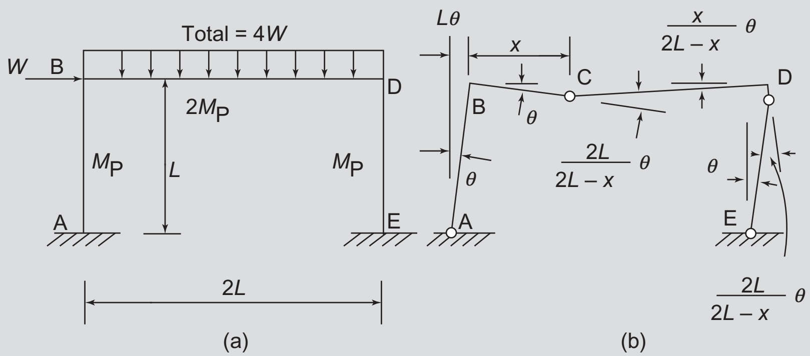


**FIGURE 18.23**

Frame carrying a uniformly distributed load.

Since the columns are the weaker members plastic hinges will form at B in AB and at D in DE. However, as in Ex. 18.9, we can assume a hinge cancellation at B so that the collapse mechanism is that shown in Fig. 18.23(b). There will be a further hinge at C in BD where the bending moment is a maximum, a distance $x$, say, from B. The problem then is to determine the value of $x$.

Referring to Fig. 18.23(b) the principle of virtual work gives

$$M_P\theta + M_P\theta + M_P\left(\frac{2L}{2L-x}\right)\theta + 2M_P\left(\frac{2L}{2L-x}\right)\theta = WL\theta + \frac{4W}{2L}x.\frac{x}{2}\theta + \frac{4W}{2L}(2L-x)\left(\frac{2L-x}{2}\right)\left(\frac{x}{2L-x}\right)\theta$$

which simplifies to

$$M_P = W\frac{(2L^2+3Lx-2x^2)}{10L-2x} \qquad \text{(i)}$$

This will be a maximum when $dM_P/dx = 0$. Then, differentiating the above and equating to zero gives

$$4x^2 - 40Lx + 34L^2 = 0$$

the solution of which is

$$x = 0.94L$$

Substituting in Eq. (i) gives

$$M_P = 0.376WL$$

If it is assumed that the hinge in BD is midway along its length then the virtual work equation is

$$M_P\theta + M_P\theta + M_P2\theta + 2M_P2\theta = WL\theta + 2\times\frac{4W}{2L}\frac{L\theta}{2}$$

from which

$$M_P = 0.375WL$$

which differs from the accurately calculated value by 0.27%.

## PROBLEMS

**P.18.1** Determine the plastic moment and shape factor of a beam of solid circular cross section having a radius $r$ and yield stress $\sigma_Y$.

*Ans.* $M_P = 1.33\sigma_Y r^3$, $f = 1.69$.

**P.18.2** Determine the plastic moment and shape factor for a thin-walled box girder whose cross section has a breadth $b$, depth $d$ and a constant wall thickness $t$. Calculate $f$ for $b = 200$ mm, $d = 300$ mm.

*Ans.* $M_P = \sigma_Y td(2b + d)/2$, $f = 1.17$.

**P.18.3** A beam having the cross section shown in Fig. P.18.3 is fabricated from mild steel which has a yield stress of 300 $N/mm^2$. Determine the plastic moment of the section and its shape factor.

*Ans.* 256.5 kNm, 1.52.

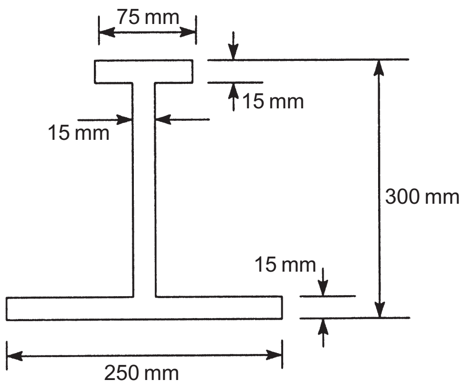


**FIGURE P.18.3**

**P.18.4** A cantilever beam of length 6 m has an additional support at a distance of 2 m from its free end as shown in Fig. P.18.4. Determine the minimum value of $W$ at which collapse occurs if the section of the beam is identical to that of Fig. P.18.3. State clearly the form of the collapse mechanism corresponding to this ultimate load.

*Ans.* 128.3 kN, plastic hinge at C.

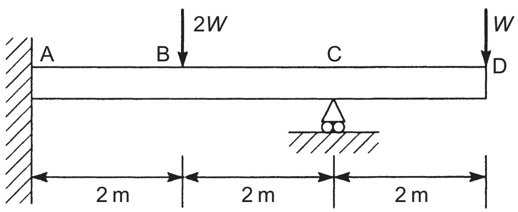


**FIGURE P.18.4**

**P.18.5** A beam of length $L$ is rigidly built-in at each end and carries a uniformly distributed load of intensity $w$ along its complete span. Determine the ultimate strength of the beam in terms of the plastic moment, $M_P$, of its cross section.

*Ans.* $16M_P/L^2$.

**P.18.6** A simply supported beam has a cantilever overhang and supports loads as shown in Fig. P.18.6. Determine the collapse load of the beam, stating the position of the corresponding plastic hinge.

*Ans.* $2M_P/L$, plastic hinge at D.

**P.18.7** Determine the ultimate strength of the propped cantilever shown in Fig. P.18.7 and specify the corresponding collapse mechanism.

*Ans.* $W = 4M_P/L$, plastic hinges at A and C.

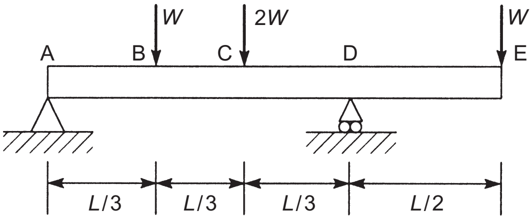


FIGURE P.18.6

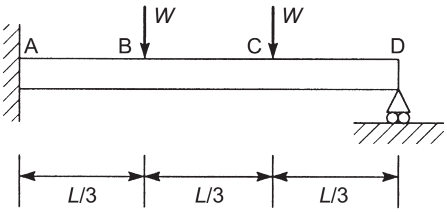


FIGURE P.18.7

**P.18.8** The working loads, $W$, on the propped cantilever of Fig. P.18.7 are each 150 kN and its span is 6 m. If the yield stress of mild steel is 300 N/mm$^2$, suggest a suitable section for the beam using a load factor of 1.75 against collapse.

*Ans.* Universal Beam, 406 mm×152 mm×67 kg/m.

**P.18.9** If the propped cantilever of Fig. P.18.7 is subjected to an axial load of 200 kN in addition to the two concentrated loads of 150 kN determine whether or not the beam section chosen in P.18.8 remains satisfactory.

*Ans*. Marginally satisfactory with no allowance for self-weight therefore use a UB 406×152×74 kg/m.

**P.18.10** The members of a steel portal frame have the relative plastic moments shown in Fig. P.18.10. Calculate the required value of $M_P$ for the ultimate loads shown.

Ans. 36.2 kNm.

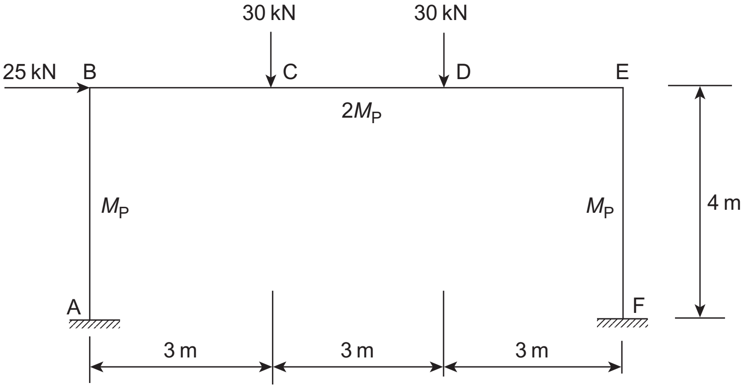


FIGURE P.18.10

**P.18.11** The frame shown in Fig. P.18.11 is pinned to its foundation and has relative plastic moments of resistance as shown. If $M_P$ has the value 108 kNm calculate the value of $W$ that will just cause the frame to collapse.

*Ans.* 60 kN.

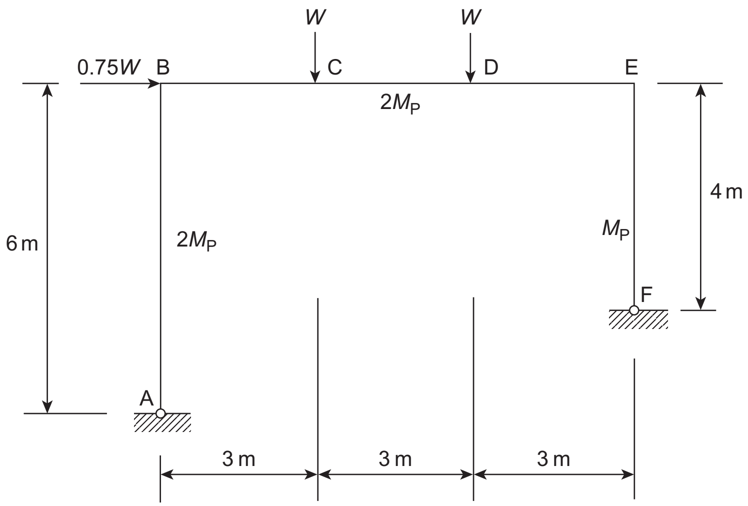


**FIGURE P.18.11**

**P.18.12** Fig. P.18.12 shows a portal frame which is pinned to its foundation and which carries vertical and horizontal loads as shown. If the relative values of the plastic moments of resistance are those given determine the relationship between the load $W$ and the plastic moment parameter $M_P$. Calculate also the foundation reactions at collapse.

*Ans.* $W = 0.3P$. Horizontal: $0.44W$ at A, $0.56W$ at G. Vertical: $0.89W$ at A, $2.11W$ at G.

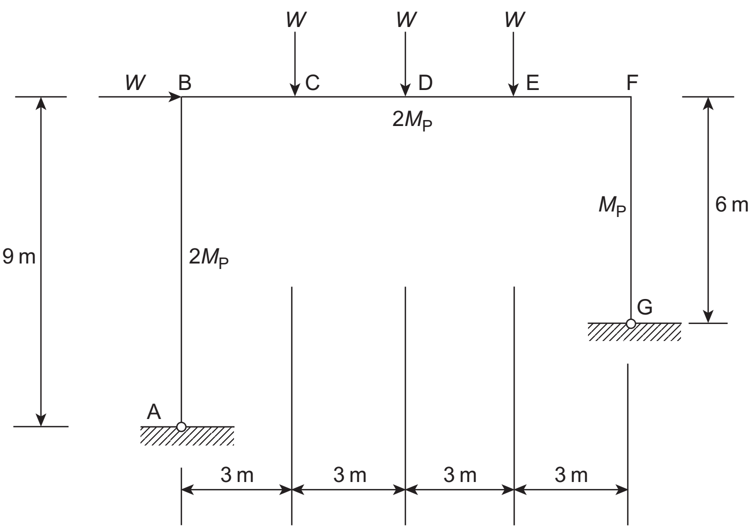


**FIGURE P.18.12**

**P.18.13** The steel frame shown in Fig. P.18.13 collapses under the loading shown. Calculate the value of the plastic moment parameter $M_P$ if the relative plastic moments of resistance of the members are as shown. Calculate also the support reactions at collapse.

*Ans.* $M_P = 56$ kNm. Vertical: 32 kN at A, 48 kN at D. Horizontal: 13.3 kN at A, 33.3 kN at D.

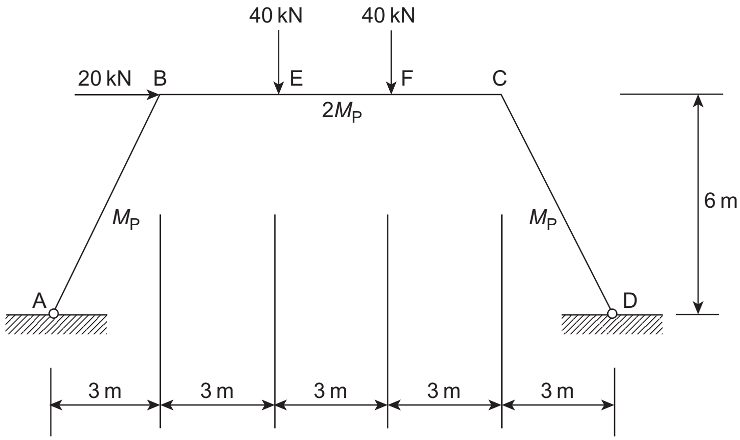


**FIGURE P.18.13**

**P.18.14** The pitched roof portal frame shown in Fig. P.18.14 has columns with a plastic moment of resistance equal to $M_P$ and rafters which have a plastic moment of resistance equal to $1.3M_P$. Calculate the smallest value of $M_P$ that can be used so that the frame will not collapse under the given loading.

*Ans.* $M_P = 24$ kNm.

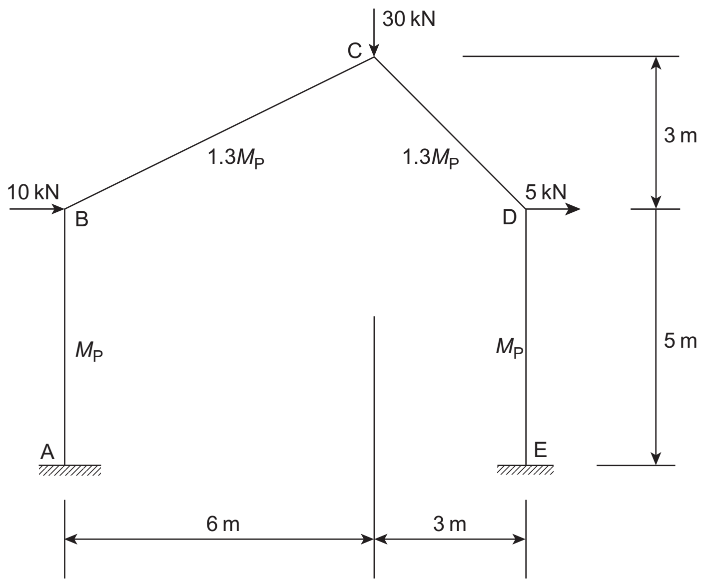


**FIGURE P.18.14**

**P.18.15** The frame shown in Fig. P.18.15 is pinned to the foundation at D and to a wall at A. The plastic moment of resistance of the column CD is 200 kNm while that of the rafters AB and BC is 240 kNm. For the loading shown calculate the value of $M$ at which collapse will take place.

*Ans.* $M = 106.3$ kN.

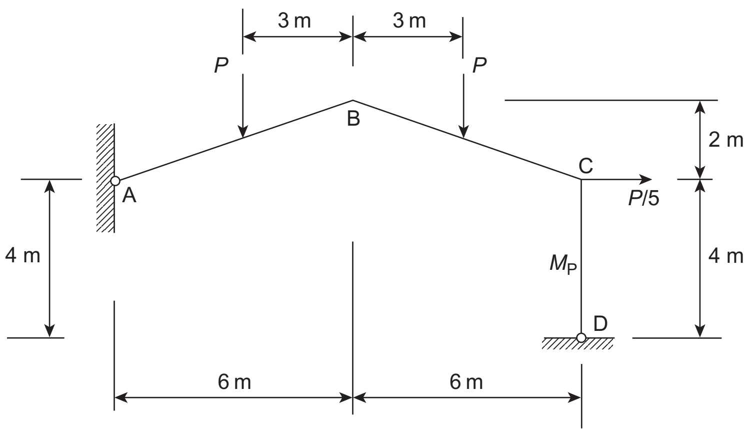


**FIGURE P.18.15**

**P.18.16** The steel portal frame shown in Fig. P.18.16 is pinned to its foundations at A and E and the plastic moment of resistance of all the members of the frame is the same. If the frame is on the point of collapse under the loading shown calculate the actual plastic moment of resistance. Sketch the bending moment diagram of the frame at the onset of collapse giving the value of the bending moment at each joint.

*Ans.* $M_P = 47.83$ kNm. $M_B = 27.15$ kNm. $M_C = 21.57$ kNm, $M_D = 47.83$ kNm.

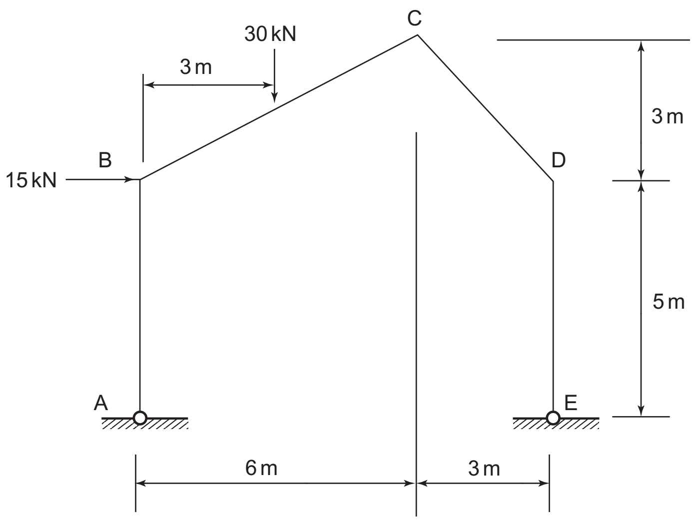


**FIGURE P.18.16**

**P.18.17** If the vertical load $W$ applied to the frame of Fig. 18.16(a) is increased to $4W$, is uniformly distributed along the member BD and the horizontal concentrated load $W$ remains in position at B determine the value of $W$ required to cause collapse. Assume that the plastic hinge in BD still occurs at C, the mid-point of BD.

*Ans.* $M_P = W$ (both the sway and combined mechanisms produce the same value).

**P.18.18** Repeat P.18.17 but consider the more accurate positioning of the plastic hinge in BD. Comment on the result obtained.

*Ans.* $M_P = 1.125W$ (combined mechanism). See Solutions Manual for comment.

**P.18.19** The steel frame shown in Fig. P.18.19 has members with different values of plastic moment as shown. The forces $P$ and $W$ may act independently or together, the load factor $\lambda$ applying to both forces. Initially, calculate the value of $M_P$, the plastic moment parameter, that will just prevent plastic collapse of the frame when the vertical forces $W$, *alone*, are applied and the load factor is 1.5.

Also show that the maximum value of $M$ that can be applied without plastic collapse occurring is $0.604W$ when both $P$ and $W$ are applied together with a load factor of 1.2 and where $M_P = 0.5WL$. Finally, calculate the support reactions.

*Ans.* $M_P = 0.5WL$, $R_{H,A} = 0.225W$, $R_{V,A} = 1.05W$, $R_{H,G} = 0.5W$, $R_{V,G} = 1.35W$.

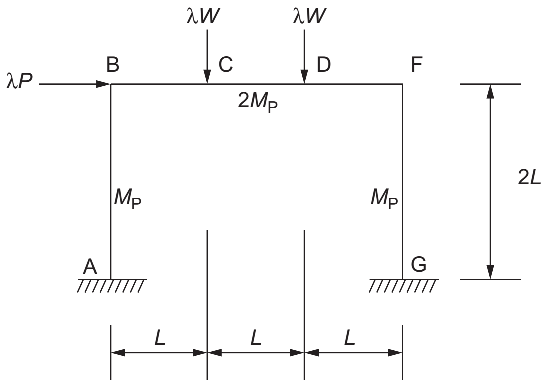


FIGURE P.18.19

**P.18.20** The steel portal frame shown in Fig. P.18.20 is subjected to the loading shown. Also shown are the plastic moments of resistance of each member. Determine the minimum value of $P$ in terms of $M_P$ at which plastic collapse will occur.

*Ans.* $P$ (min.) $= 1.39M_P$.

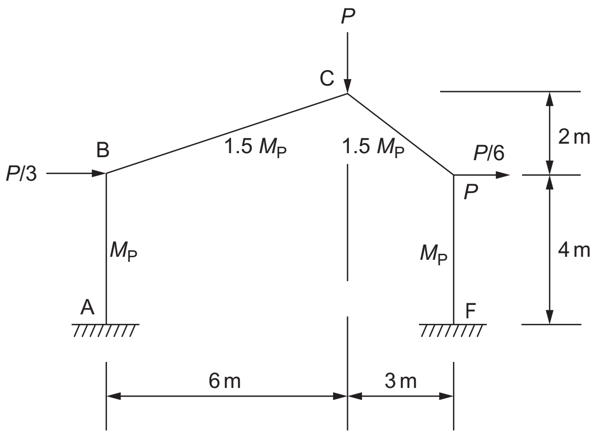


FIGURE P.18.20