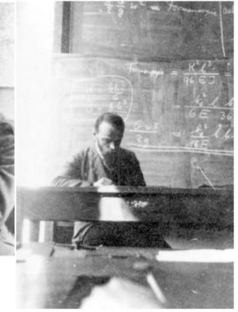
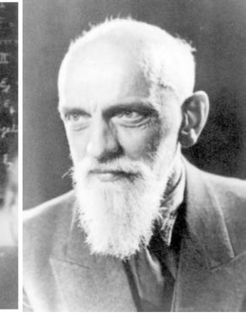
Aerodynamics and Numerical Simulation Methods

Flat-Plate Boundary Layer (Blasius solution)

University of BRISTOL

Blasius Solution



- For 2D, steady, incompressible flow along a flat plate, with no pressure gradients in the external flow, the boundary layer equations are reduced to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad\qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

Kinematic viscosity

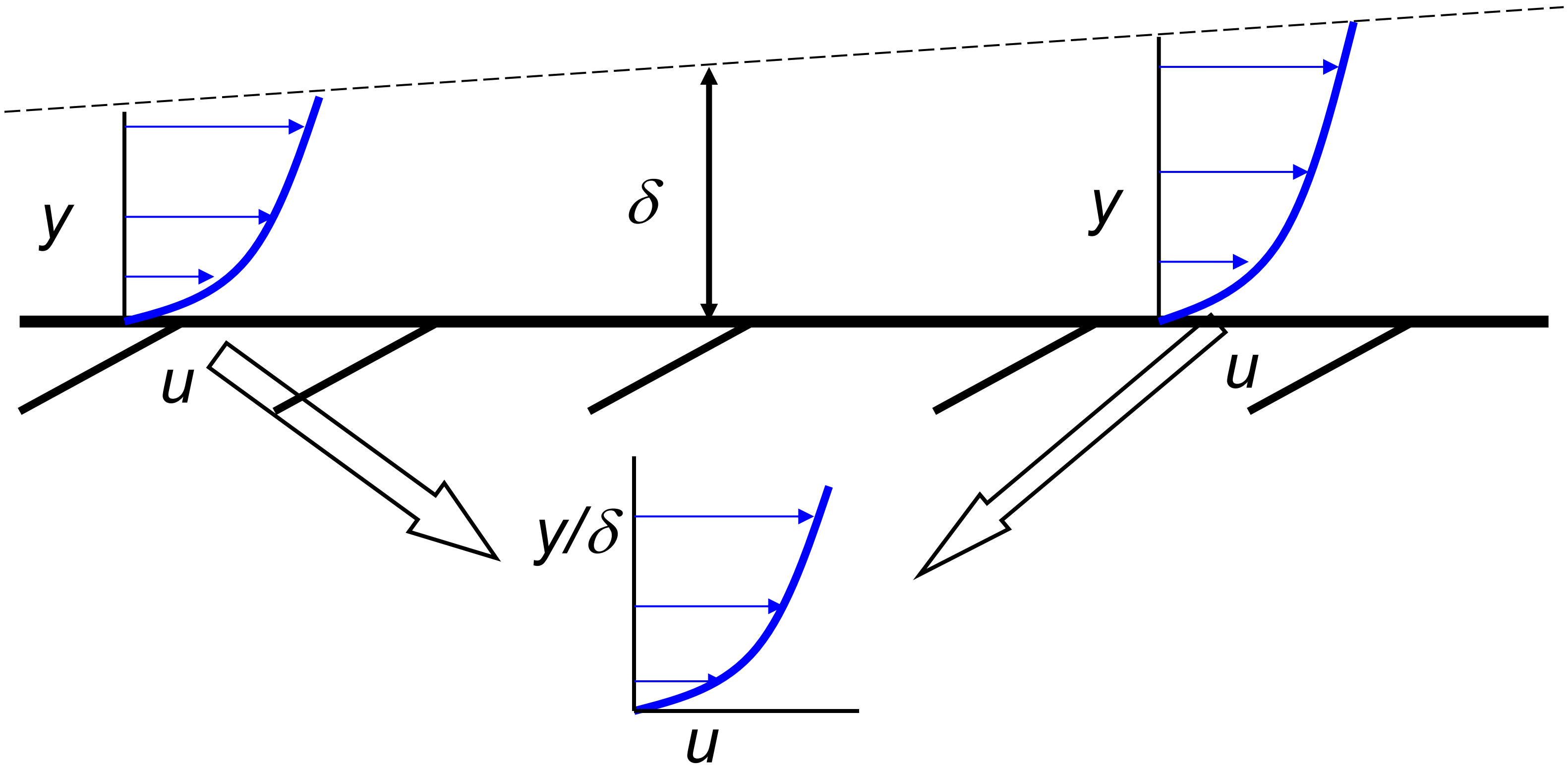
- With Boundary Conditions

$$u,v=0$$ at the wall, and

$$u=u_e,\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}=\ldots=0$$ at the boundary layer edge

- Blasius goal: Reduce PDE to ODE!

Also, no external gradients, so flow is *similar:*



i.e. the velocity profiles are identical everywhere in terms of a fraction of boundary layer thickness

- This is because there are no gradients in *x* direction – No way for changes to occur, apart from b.l. thickness.
- If we introduce a new **auxiliary variable** η which scales with boundary layer height, then *u* for constant η is constant.
- What choice for η? From dimensional analysis:

Kinematic viscosity

$$\boxed{\delta \propto \frac{x}{\mathrm{Re}_x^{\frac{1}{2}}}} \qquad \boxed{\mathrm{Re} = \frac{\rho u x}{\mu} = \frac{u x}{\nu}} \qquad \boxed{\frac{x}{\mathrm{Re}^{\frac{1}{2}}} = \frac{x \nu^{\frac{1}{2}}}{(ux)^{\frac{1}{2}}}}$$

- So try:

$$\eta = \frac{1}{2}\frac{y}{\delta} = \frac{1}{2}\frac{y}{\frac{x}{\mathrm{Re}^{\frac{1}{2}}}} = \frac{1}{2}\frac{y\,\mathrm{Re}^{\frac{1}{2}}}{x} = \frac{1}{2}\frac{y}{x}\left(\frac{u_e x}{\nu}\right)^{1/2} = \frac{1}{2}y\left(\frac{u_e}{x\nu}\right)^{1/2}$$

Kinematic viscosity

Source: Frank White page 467. Note that Frank White uses a slightly different η, you can use a different η too!

Also, as we have incompressible flow, we know that the stream function exists (from continuity), i.e.

$$u = \frac{\partial \varphi}{\partial y}, \qquad v = -\frac{\partial \varphi}{\partial x}$$

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \equiv 0$$

-Means we only need to find one variable, not two.

$$u = \frac{\partial \varphi}{\partial y}$$

With these two ideas (η, *φ*), we define a **second auxiliary variable**:

$$\varphi = (u_e x \nu)^{\frac{1}{2}}.F(\eta),$$

Kinematic viscosity

Where *F* is some function. Then:

$$\Rightarrow \frac{\partial \varphi}{\partial y} = (u_e x \nu)^{\frac{1}{2}}.F'(\eta)\frac{\partial \eta}{\partial y} + 0.F(\eta)$$

Also, from first auxiliary variable:

$$\eta = \frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} \Rightarrow \frac{\partial \eta}{\partial y} = \frac{1}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Kinematic viscosity

Hence:

$$u = (u_e x \nu)^{\frac{1}{2}} \frac{F'}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} = \frac{u_e F'}{2} \Rightarrow \boxed{\frac{u}{u_e} = \frac{F'}{2}}$$

Kinematic viscosity

$$v = -\frac{\partial \varphi}{\partial x}$$

Similarly,

$$\varphi = (u_e x \nu)^{\frac{1}{2}}.F(\eta)$$

Where *F* is some function. Then

$$\Rightarrow -\frac{\partial \varphi}{\partial x} = -(u_e x \nu)^{\frac{1}{2}}.F'(\eta)\frac{\partial \eta}{\partial x} - \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}.F(\eta)$$

Also, from first auxiliary variable:

$$\eta = \frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} \Rightarrow \frac{\partial \eta}{\partial x} = -\frac{1}{2x}\frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} = -\frac{\eta}{2x}$$

Hence:

$$v = (u_e x \nu)^{\frac{1}{2}}\frac{\eta}{2x}F' - \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}.F$$

$$\Rightarrow \boxed{v = \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}(F'\eta - F)}$$

Kinematic viscosity

Take a deep breath and let's recap

- We want to solve the boundary layer equations for a 2D, steady, incompressible flow along a flat plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

- We introduced 2 auxiliary variables

Kinematic viscosity

$$\eta = \frac{1}{2}\frac{y}{\delta} = \frac{1}{2}y\left(\frac{u_e}{x\nu}\right)^{1/2} \qquad \varphi = (u_e x\nu)^{\frac{1}{2}}.F(\eta)$$

- We solved for u and v using the stream functions

$$u = \frac{\partial \varphi}{\partial y},\ v = -\frac{\partial \varphi}{\partial x} \Rightarrow \boxed{\frac{u}{u_e} = \frac{F'}{2}} \quad \boxed{v = \frac{1}{2}\left(\frac{u_e\nu}{x}\right)^{\frac{1}{2}}(F'\eta - F)}$$

- We still need to solve for derivatives of u and v…

  In the same way, we can differentiate u w.r.t, x, y & y again (try it yourself – this is examinable!):

$$\frac{\partial u}{\partial x} = -\frac{u_e}{2} F'' \frac{y}{4x}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} \qquad \frac{\partial u}{\partial y} = u_e \frac{F''}{4}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_e^2 F'''}{8\nu x}$$

Kinematic viscosity

We can then insert these into the x momentum equation:

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}}$$

First term of x-momentum equation:

$$u = \frac{u_e F'}{2}, \qquad \frac{\partial u}{\partial x} = -\frac{u_e}{2} F'' \frac{y}{4x}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Hence

$$u\frac{\partial u}{\partial x} = -\frac{u_e F'}{2}\frac{u_e F''}{2}\frac{y}{4x}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

$$= -\frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}\left(\frac{u_e^2 F' F''}{8x}\right)$$

$$= -\left(\frac{\eta u_e^2}{8x}\right)F' F''$$

Kinematic viscosity

Second term of x-momentum equation:

$$v = \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}(F'\eta - F) \qquad \frac{\partial u}{\partial y} = u_e \frac{F''}{4}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Kinematic viscosity

Hence

$$v\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}(F'\eta - F)u_e \frac{F''}{4}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

$$= \frac{u_e^2 \eta}{8x} F'F'' - \frac{u_e^2}{8x} FF''$$

$$= \left(\frac{u_e^2 F''}{8x}\right)(F'\eta - F)$$

RHS of x-momentum equation:

$$\nu \frac{\partial^2 u}{\partial y^2} = \nu \frac{u_e^2 F'''}{8\nu x} = \frac{u_e^2 F'''}{8x}$$

Kinematic viscosity

All the terms derived thus far have a common factor $u_e^2/8x$, so we divide by this, and add up the terms, giving:

Kinematic viscosity

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow -\eta F' F'' + \eta F' F'' - FF'' = F'''$$

i.e.

$$FF'' + F''' = 0$$

$$\boxed{FF'' + F''' = 0}$$

- i.e. we want a function that when multiplied by its second derivative w.r.t. η, is the negative of its third derivative.
- The form of this ODE can change depending on your initial choice of the definition of η.
- This ODE can be solved numerically and is quite simple on today's computers. Can even be done by hand.
- The first to do so was Blasius (a student of Prandtl's), hence it is named after him. It may be considered *exact* for incompressible, steady, 2D flow over a flat plate

Quick refresher on solving ODEs…

Example 1

Solve: $\frac{dx}{dt} = 5x - 3$

$$\frac{dx}{5x-3} = dt$$

$$\int \frac{dx}{5x-3} = \int dt$$

$$x(t) = Ce^{5t} + \frac{3}{5}$$

Example 2

Solve: $\frac{dy}{dx} = 7\mathrm{y}^2\mathrm{x}^3$ with initial condition $\mathrm{y}(2) = 3$

$$\int y^{-2} dy = \int 7x^3 dx$$

$$y(x) = -\frac{1}{\frac{7}{4}x^4 + C}$$

Solve for C using I.C.

$$y(x) = -\frac{1}{\frac{7}{4}x^4 - \frac{85}{3}}$$

How to solve?

$$\boxed{FF''+F'''=0}$$

- We know $F(\eta)$ and $\eta(x,y)=\frac{y}{2\delta}$
- Boundary conditions are key to solving the ODE
- At the wall, u=v=0 (no-slip, no-flux)
  - $F'(\eta=0)=0$ (u equation)
  - $F(\eta=0)=0$ (v equation)
- At edge of boundary layer, $u=u_e$
  - $F'(\eta\to\infty)=2$
- This is still very difficult to solve analytically… Blasius did it using an analytical series solution technique
- We have computers now – let's solve it numerically

$$\boxed{\frac{u}{u_e}=\frac{F'}{2}}$$

$$\boxed{v=\frac{1}{2}\left(\frac{u_e\nu}{x}\right)^{\frac{1}{2}}(F'\eta-F)}$$

Kinematic viscosity

Shooting method $\boxed{FF''+F'''=0}$

- Cast the third order ODE as three coupled first order ODE:

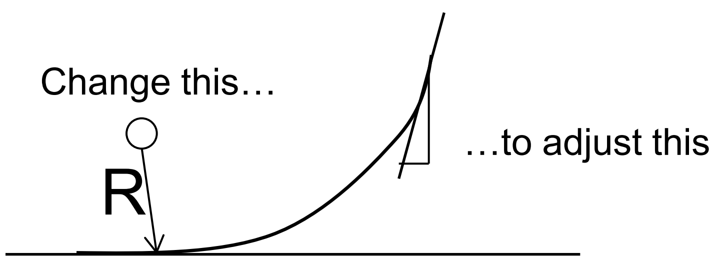
$$\boldsymbol{F'}(\eta) = \boldsymbol{G}(\eta)$$
$$\boldsymbol{G'}(\eta) = F''(\eta) = \boldsymbol{H}(\eta)$$
$$\boldsymbol{H'}(\eta) = F'''(\eta) = -F(\eta)F''(\eta) = -\boldsymbol{F}(\eta)\boldsymbol{H}(\eta)$$

- B.C. gives us $F(0) = F'(0) = G(0) = 0$
- Still stuck because we don't have a value for $H(\eta = 0)$
- Employ an iterative approach known as the shooting method
- ODEs integrated numerically using Euler Predictor-Corrector method, solve by the shooting method

Shooting method

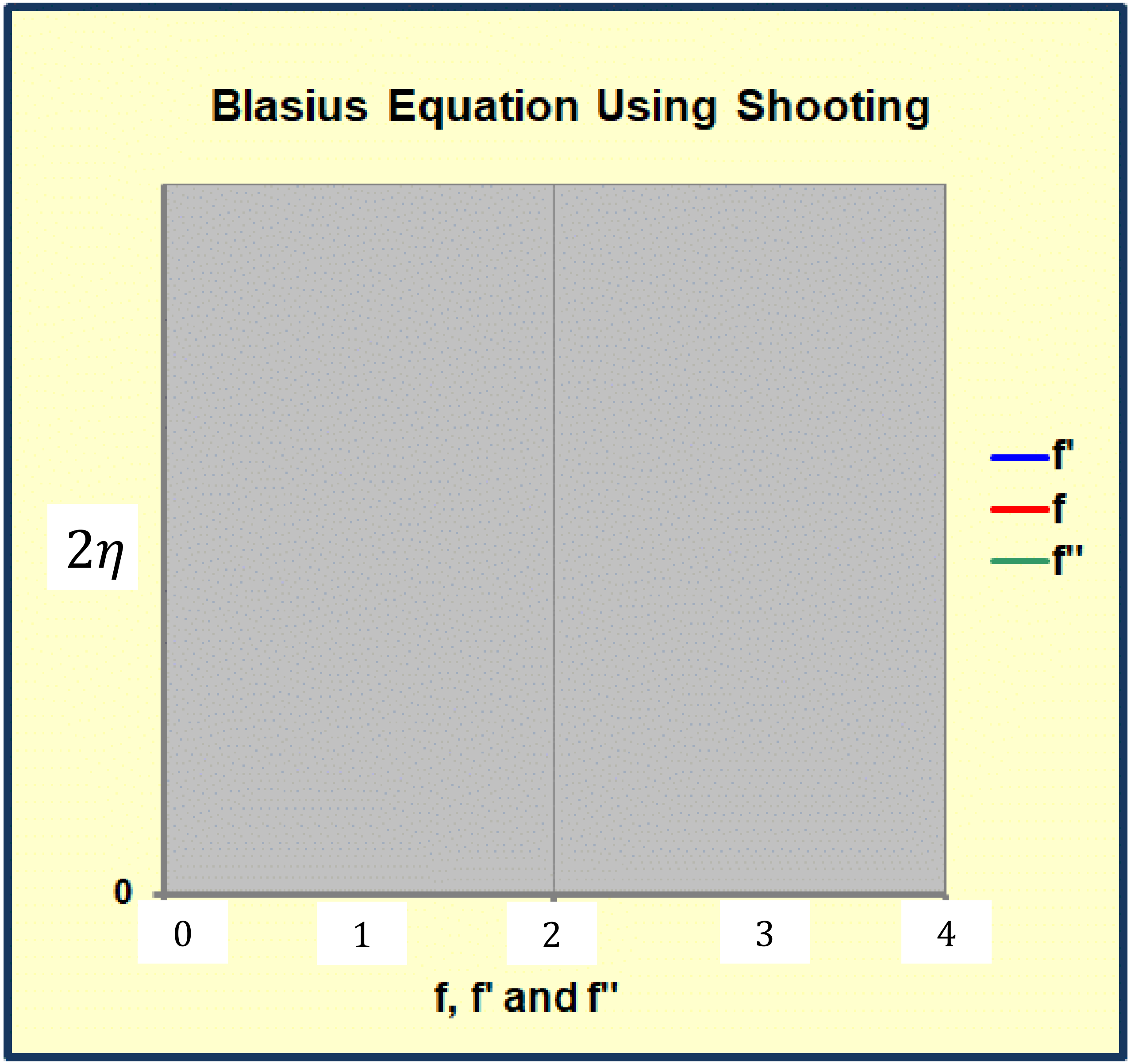
- Find solutions to the initial value problem by changing the initial conditions until we find the solution that also satisfies the boundary conditions of the original boundary value problem

1. Guess $F''(0) = H(0)$ and see if the trajectory allows us to find the correct $F'(\eta \to \infty)$ = 2 at the edge
2. Check the value of F' at the edge and adjust F''
3. Repeat process with new value of F''(0)
4. This is equivalent to iterating on the wall shear stress



$$\frac{\partial u}{\partial y} = u_e \frac{F''}{4}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Kinematic viscosity



$$\frac{u}{u_e} = \frac{F'}{2}$$

$$\frac{\partial u}{\partial y} = u_e \frac{F''}{4} \left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Kinematic viscosity

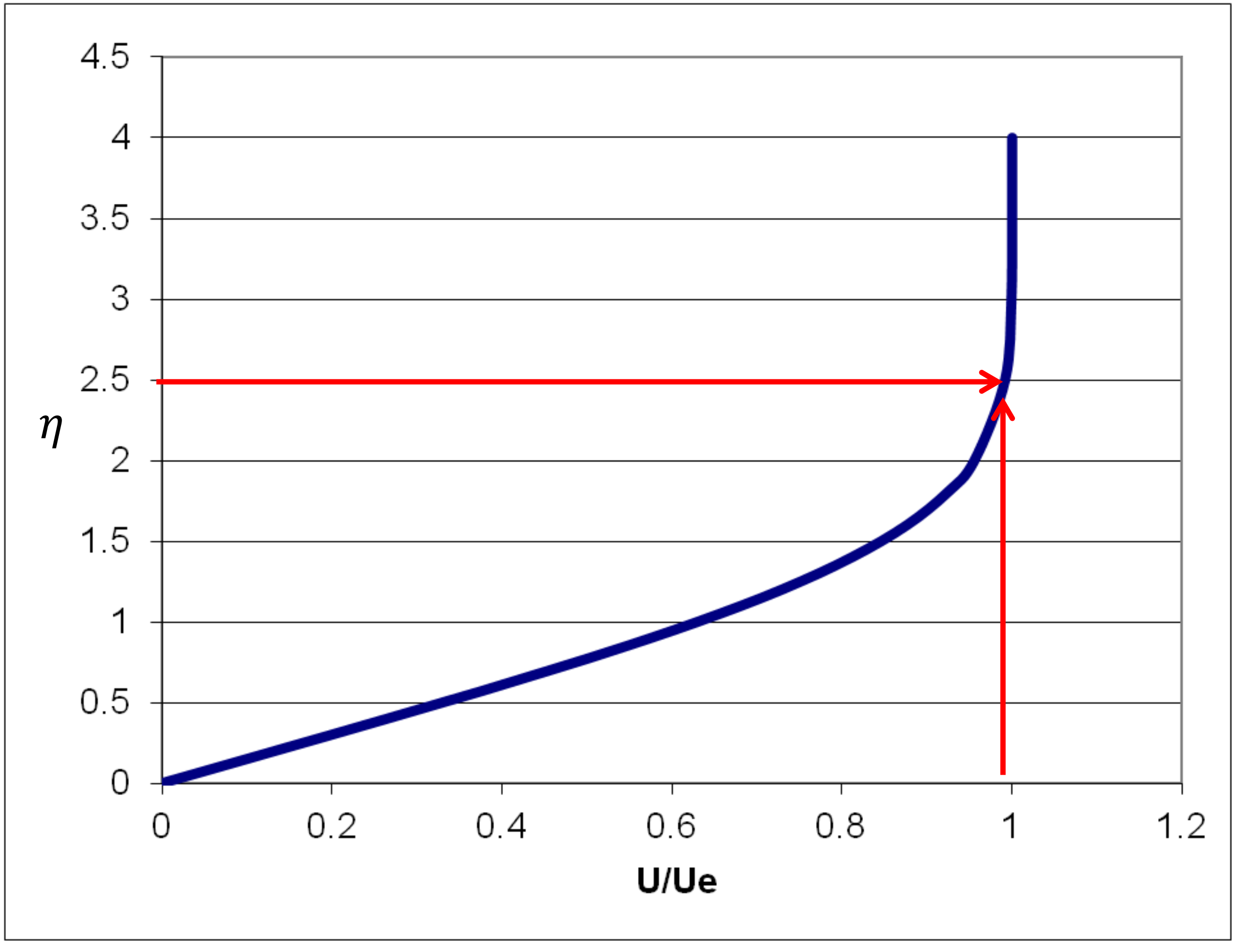
$$\frac{\partial^2 u}{\partial y^2} = \frac{u_e^2 F'''}{8\nu x}$$

$F'(\eta \rightarrow \infty) = 2$

$F'(\eta = 0) = 0$

$F(\eta = 0) = 0$

Adapted from: https://www.robertribando.com/xls/fluid-mechanics/solution-of-blasius-equation/



$$\frac{u}{u_e} = \frac{F'}{2}$$

$$\eta = \frac{1}{2}\frac{y}{\delta} = \frac{1}{2} y \left(\frac{u_e}{x\nu}\right)^{1/2} \Rightarrow 2.5 = \frac{1}{2}\delta\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} = \frac{1}{2}\frac{\delta}{x}\left(\frac{u_e x}{\nu}\right)^{\frac{1}{2}}$$

From last time (slide 16)… $\frac{\delta}{x} = 5\left(\frac{u_e x}{\nu}\right)^{-\frac{1}{2}}$

Kinematic viscosity

This profile can be numerically integrated using definitions of momentum and displacement thickness (slide 21/22) to give:

$$\frac{\delta^*}{x} = \frac{1.721}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \frac{\theta}{x} = \frac{0.664}{\mathrm{Re}_x^{\frac{1}{2}}}$$

Friction coefficient can be computed as:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} = \frac{0.664}{\mathrm{Re}_x^{1/2}} = \frac{\theta}{x} \qquad \tau = \mu \frac{\partial U}{\partial y}$$

Drag coefficient on a (one-sided) plate of length x:

$$c_d = 2c_f = \frac{1.328}{\mathrm{Re}_x^{1/2}}$$