University of BRISTOL

ADVANCED STRUCTURES & MATERIALS

Finite Element Analysis Principles – Lecture 5

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2D Finite Elements

- So far, we considered the finite element analysis of 1D bodies described by one independent variable.

- We'll now consider **2D problems**, in their simplest form: **plane elasticity**.

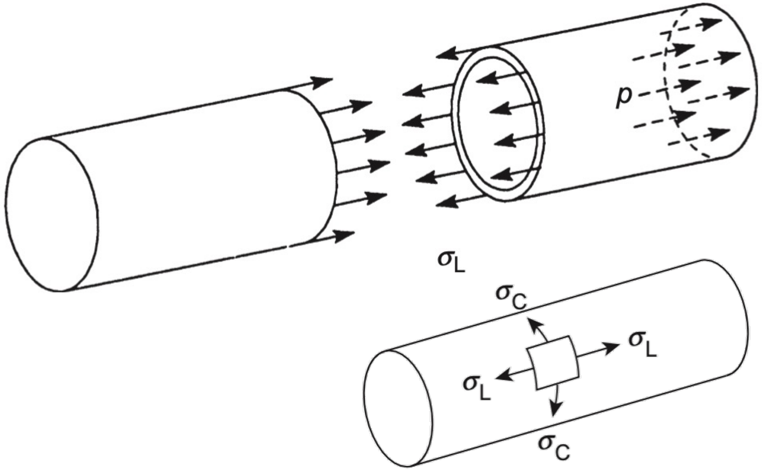
- The elements themselves are not of any particular importance, but there are some **lessons to be learnt** deriving them.

Plane Elasticity

Plane Elasticity

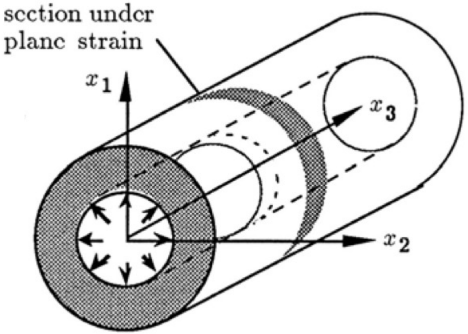
- **Plane Stress**

$$\sigma_{zz} = 0, \qquad \sigma_{xz} = 0, \qquad \sigma_{yz} = 0$$



- **Plane Strain**

$$\varepsilon_{zz} = 0, \qquad \varepsilon_{xz} = 0, \qquad \varepsilon_{yz} = 0$$



https://link.springer.com/chapter/10.1007/978-94-015-8026-7_9

Weak Formulation – Equilibrium

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = - \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = - \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\boldsymbol{D}^{\mathsf{T}} \boldsymbol{\sigma} = -\boldsymbol{b}$$

Weak Formulation – Strain-Displacement

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \Rightarrow \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \Rightarrow \boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{u}$$

Weak Formulation – Constitutive

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}$$

Plane Stress ⇨

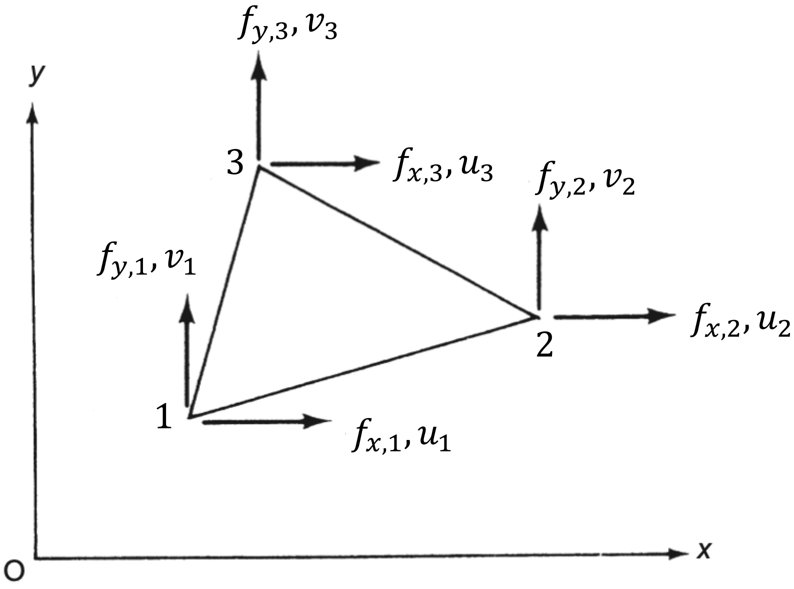
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}$$

Plane Strain ⇨

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}$$

Triangular Elements

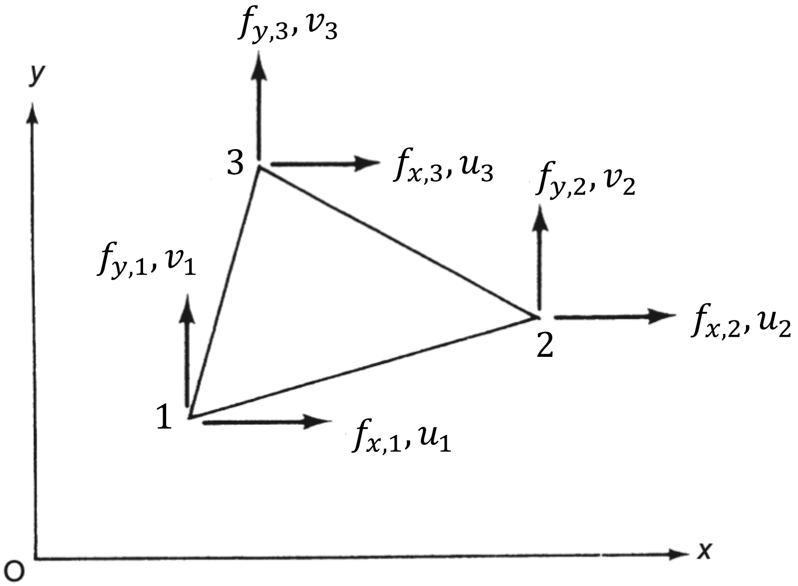
Triangular Finite Element Formulation



6 d.o.f.

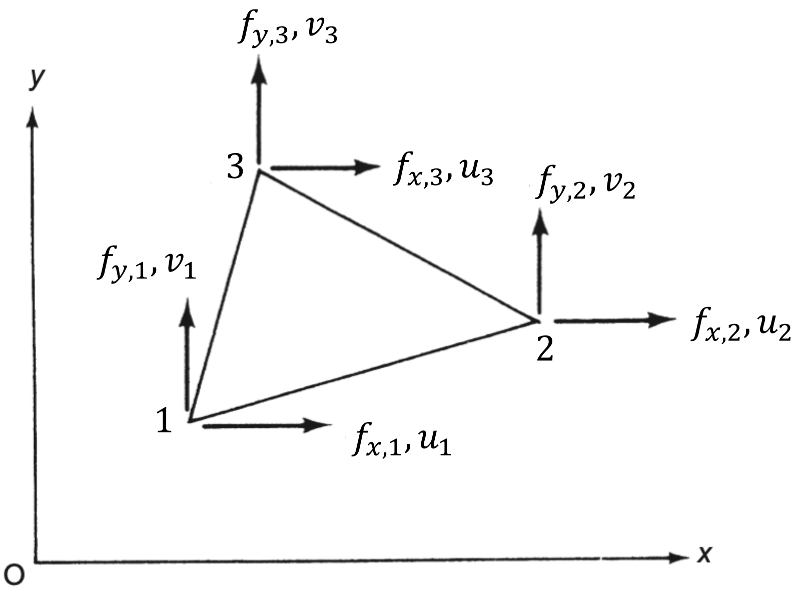
$$\boldsymbol{f} = \begin{bmatrix} f_{x,1} \\ f_{y,1} \\ f_{x,2} \\ f_{y,2} \\ f_{x,3} \\ f_{y,3} \end{bmatrix} \qquad \boldsymbol{\delta} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

Triangular Finite Element Formulation



$$\boldsymbol{u} = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \boldsymbol{\Phi a}$$

Triangular Finite Element Formulation



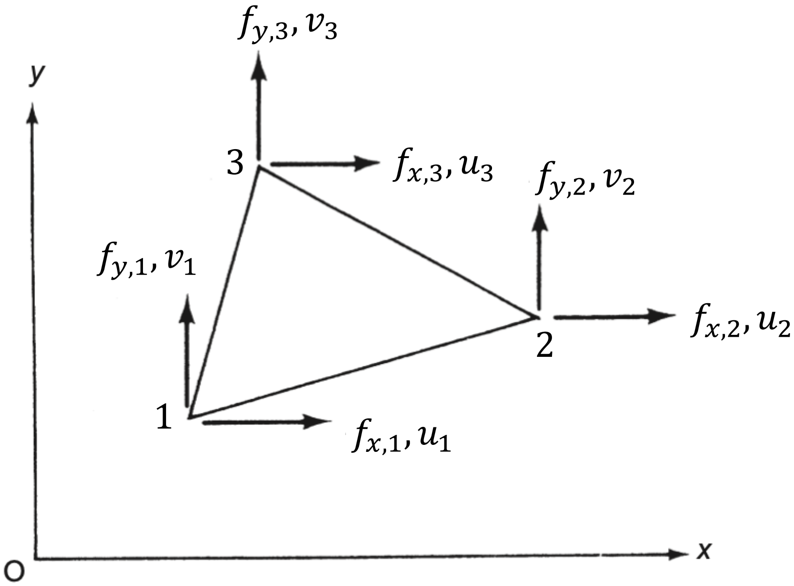
- The constant terms, a_1 and a_4, capture in-plane rigid body motion
- The linear terms enable states of constant strain to be specified

Recall

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

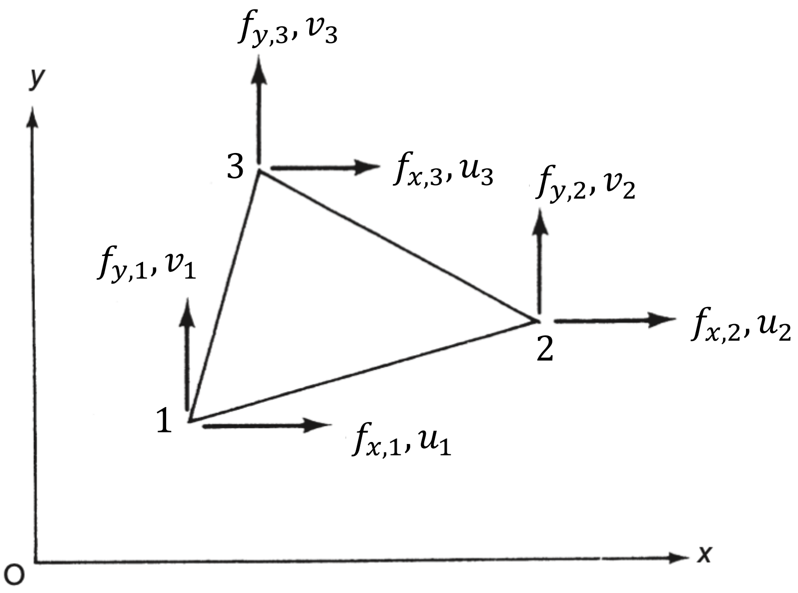
Constant strain element

Triangular Finite Element Formulation



$$\boldsymbol{\delta} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \bar{\boldsymbol{\Phi}} \boldsymbol{a}$$

Triangular Finite Element Formulation



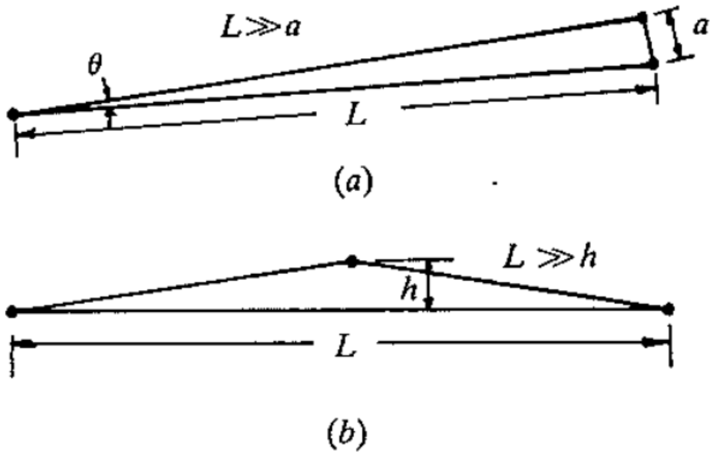
$$\boldsymbol{u} = \boldsymbol{\Phi}\bar{\boldsymbol{\Phi}}^{-1}\boldsymbol{\delta} = \boldsymbol{N}\boldsymbol{\delta}$$

Note $\bar{\boldsymbol{\Phi}}$ may be ill-conditioned.

Element geometry matters.

$$\bar{\boldsymbol{\Phi}} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}$$

Triangular Finite Element Formulation

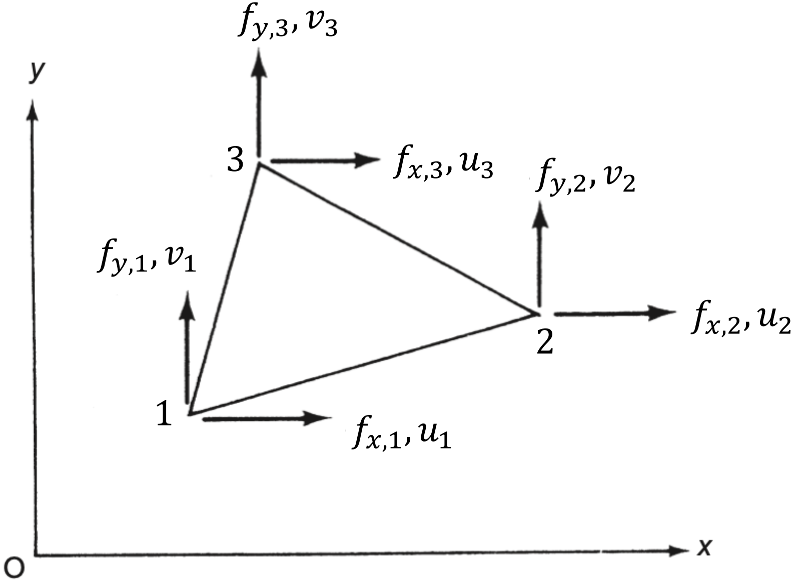


The inverse of $\bar{\mathbf{\Phi}}$ does not exist when the matrix is **singular**.

This happens only when the three nodes lie on the **same line**.

However, if two nodes are close relative to the third or if the nodes are nearly colinear, $\bar{\mathbf{\Phi}}$ will be **nearly singular** and **numerically noninvertible**.

Triangular Finite Element Formulation



$$\boldsymbol{u} = \boldsymbol{\Phi}\bar{\boldsymbol{\Phi}}^{-1}\boldsymbol{\delta} = \boldsymbol{N}\boldsymbol{\delta}$$

$$N_1 = \frac{1}{2A_{123}}\left[y_{32}\left(x - x_2\right) - x_{32}\left(y - y_2\right)\right]$$

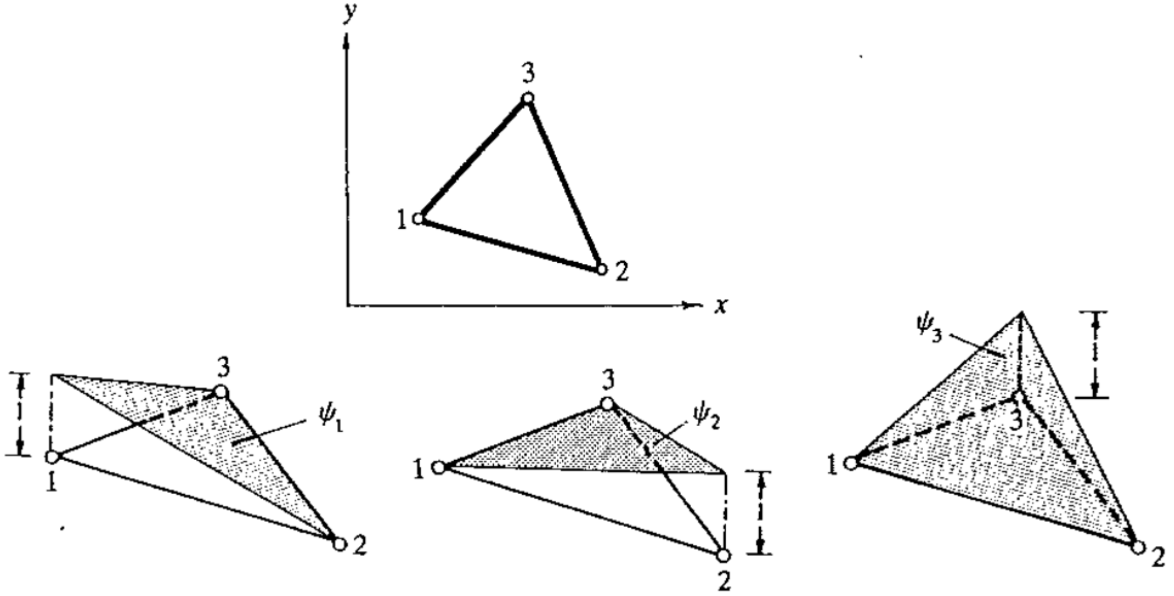
$$N_2 = \frac{1}{2A_{123}}\left[x_{31}\left(y - y_3\right) - y_{31}\left(x - x_3\right)\right]$$

$$N_3 = \frac{1}{2A_{123}}\left[y_{21}\left(x - x_1\right) - x_{21}\left(y - y_1\right)\right]$$

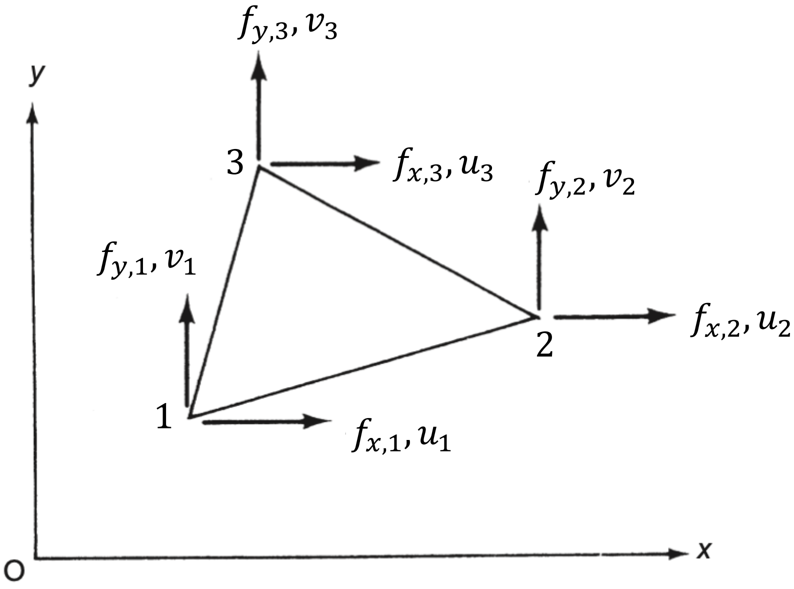
$$2A_{123} = x_{32}y_{21} - x_{21}y_{32}$$

$$x_{ij} = x_i - x_j \qquad y_{ij} = y_i - y_j$$

Triangular Finite Element Formulation



Triangular Finite Element Formulation



The rest follows as

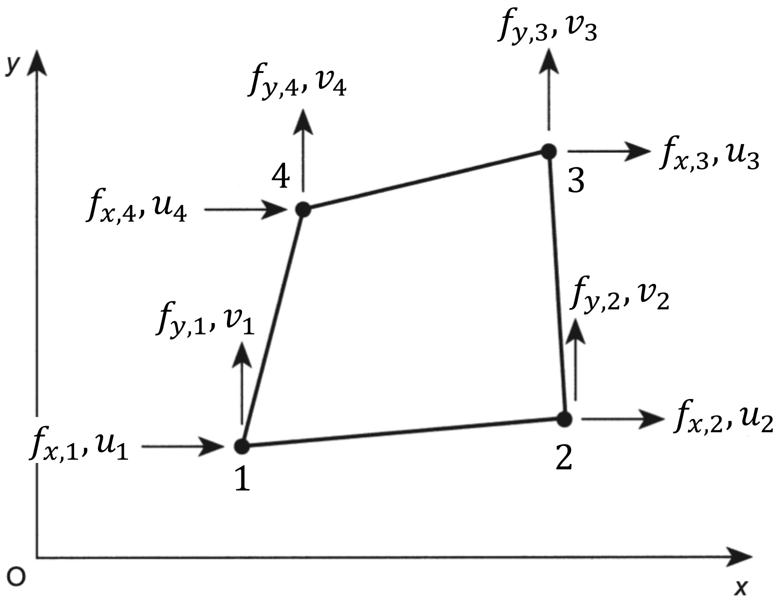
$$\boldsymbol{K} = \int_V \boldsymbol{B}^\mathsf{T} \boldsymbol{E} \boldsymbol{B} \, dV$$

$$\boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T} \boldsymbol{p} \, dS + \int_V \boldsymbol{N}^\mathsf{T} \boldsymbol{b} \, dV$$

$$\boldsymbol{B} = \boldsymbol{D}\boldsymbol{N} \qquad \boldsymbol{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

Quadrilateral Elements

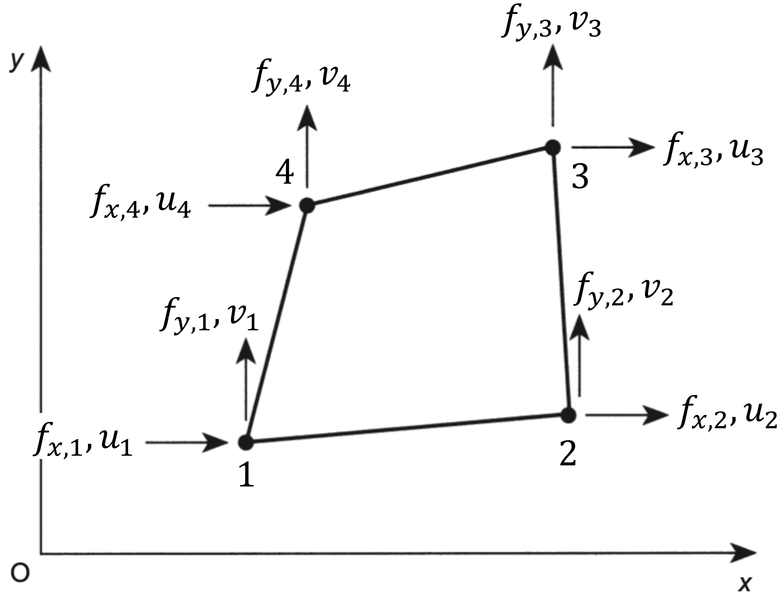
Quadrilateral Finite Element Formulation



8 d.o.f.

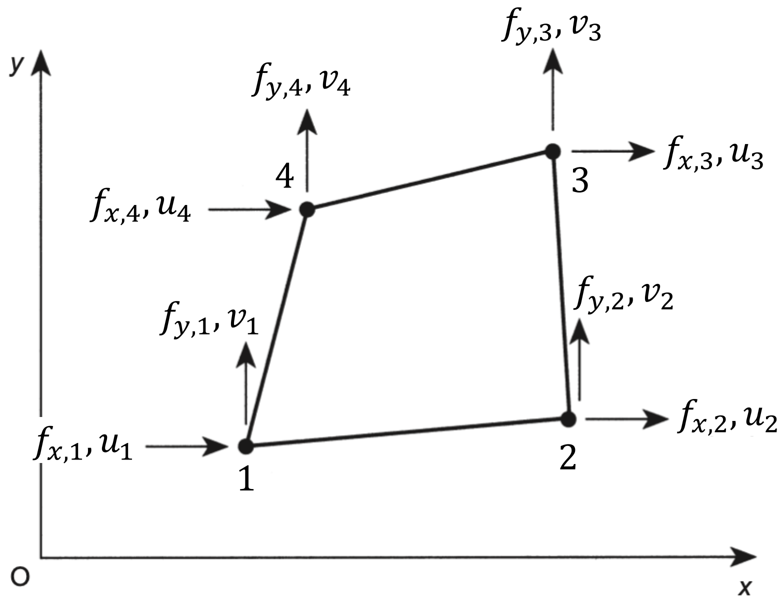
$$\boldsymbol{f} = \begin{bmatrix} f_{x,1} \\ f_{y,1} \\ f_{x,2} \\ f_{y,2} \\ f_{x,3} \\ f_{y,3} \\ f_{x,4} \\ f_{y,4} \end{bmatrix} \qquad \boldsymbol{\delta} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

Quadrilateral Finite Element Formulation



$$\boldsymbol{u} = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \boldsymbol{\Phi a}$$

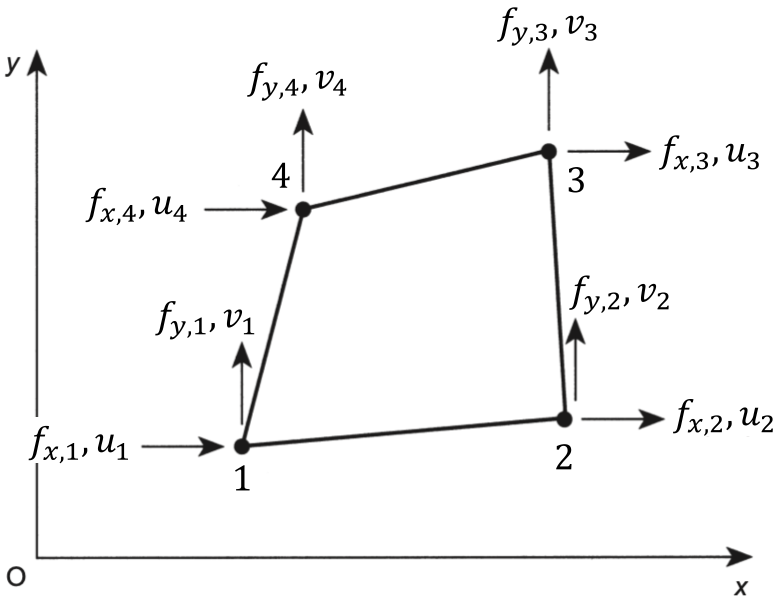
Quadrilateral Finite Element Formulation



- The constant terms, a_1 and a_5, capture in-plane rigid body motion
- The linear terms enable states of constant strain to be specified
- The mixed terms (xy) enable linear variations of strain through the element

Linear strain element

Quadrilateral Finite Element Formulation

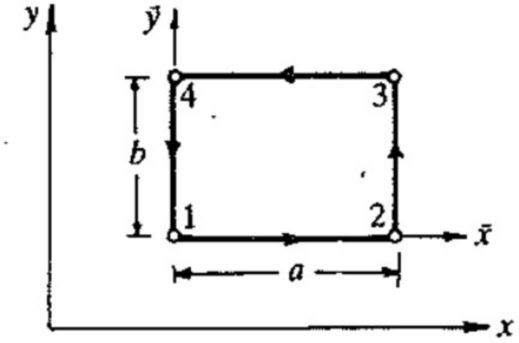


$$\bar{\mathbf{\Phi}} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & x_4y_4 \end{bmatrix}$$

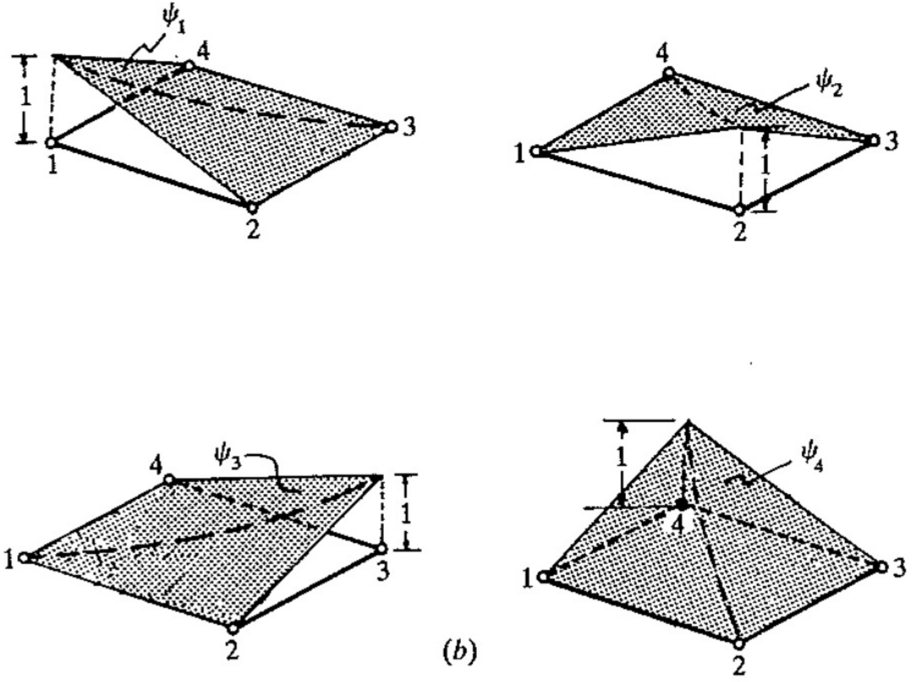
Element geometry matters again!

Internal angles never close to 0° or 180°.

Quadrilateral Finite Element Formulation



(a)



(b)

3D Elements

Application of FEM to 3D Solid Bodies

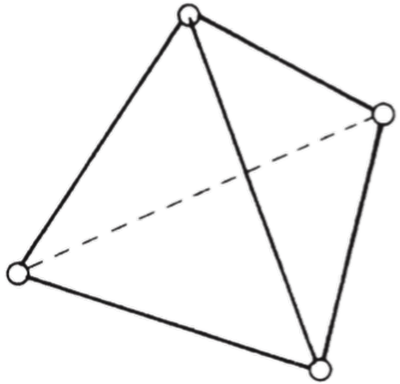
- Straightforward extension of the analysis of 2D structures.

- Basic 3D elements are the **tetrahedron** and the **hexahedron**.

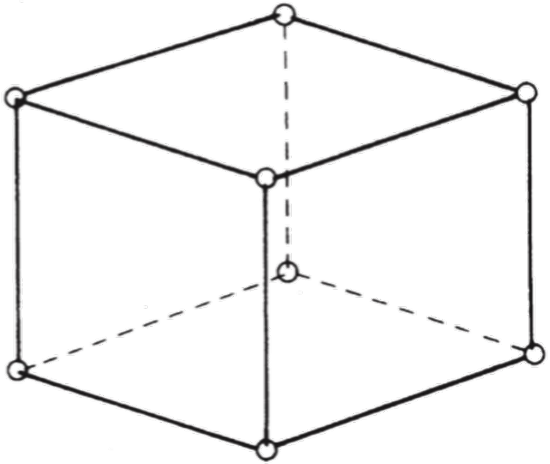
- Displacement functions require polynomials in x, y, z.

Application of FEM to 3D Solid Bodies

- The **tetrahedron** has **4 nodes** each possessing 3 d.o.f., a total of **12 for the element**.



- The **hexahedron** has **8 nodes** and a total of **24 d.o.f.**



General Element Formulation Procedure

General Element Formulation Procedure

The procedure for element formulation described so far becomes increasingly complicated for increasingly complex elements.

Elements can be higher order and have more nodes.

They can have irregular shape and curved boundaries.

$\bar{\mathbf{\Phi}}$ becomes difficult to invert symbolically.

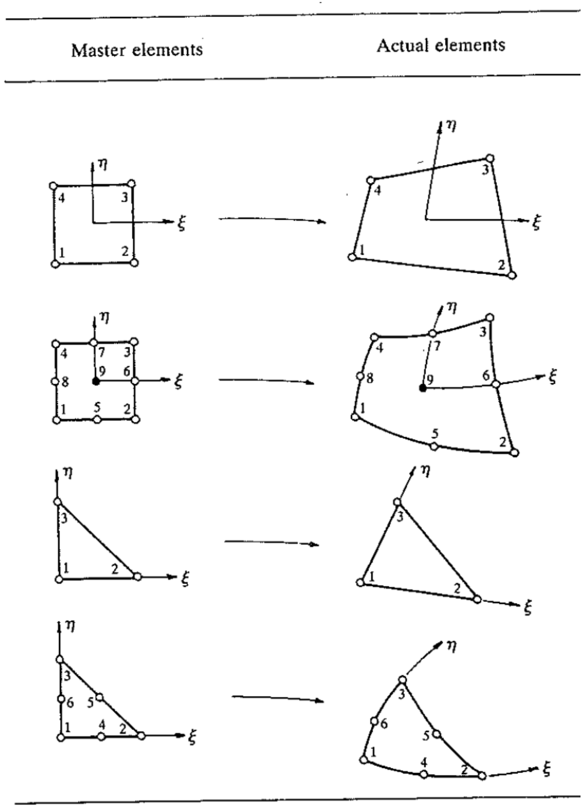
Volume and boundary integrals complex to compute analytically.

General Element Formulation Procedure

- Generally applicable **interpolation functions** developed for regularly shaped elements, called **master elements**.

- **Mapping** from **actual to master** element and from **master to actual** defined.

- Numerical integration required.



Lagrange Shape Functions

Lagrange Shape Functions

The most common interpolation functions for finite element formulation are the **Lagrange shape functions**. → LAGRANGE FINITE ELEMENTS

For a 1D element with n nodes, n Lagrange shape functions, $N_i(\xi)$, can be defined:

$$N_i(\xi) = \frac{\xi - \xi_1}{\xi_i - \xi_1} \cdot \frac{\xi - \xi_2}{\xi_i - \xi_2} \cdots \frac{\xi - \xi_{i-1}}{\xi_i - \xi_{i-1}} \cdot \frac{\xi - \xi_{i+1}}{\xi_i - \xi_{i+1}} \cdots \frac{\xi - \xi_n}{\xi_i - \xi_n} = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{\xi - \xi_j}{\xi_i - \xi_j}$$

where ξ_j is the coordinate of the j^{th} node.

Note $N_i(\xi)$ are functions of degree $n - 1$ and zero at all nodes except the i^{th}.

Numerical Integration

Numerical Integration

Exact evaluation of the integrals

$$\boldsymbol{K} = \int_V \boldsymbol{B}^\mathsf{T} \boldsymbol{E} \boldsymbol{B} \, dV \qquad \boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T} \boldsymbol{p} \, dS + \int_V \boldsymbol{N}^\mathsf{T} \boldsymbol{b} \, dV$$

is not always possible.

Numerical Integration

Most commonly the integrals are evaluated numerically, with procedures known as numerical integration, numerical quadrature or Gaussian quadrature.

These involve approximating the integrand by a polynomial of sufficient degree. In 1D,

$$\mathcal{I} = \int_{x_a}^{x_b} F(x)\, dx \qquad\qquad F(x) = \sum_{i}^{N} F(x_i)\psi_i(x)$$

Numerical Integration

$$\mathcal{I} = \int_{x_a}^{x_b} F(x)\, dx \qquad\qquad F(x) = \sum_{i}^{N} F(x_i)\psi_i(x)$$

This should look familiar. It is a finite element interpolation of $F(x)$, where:

- $F(x_i)$ denotes the values of $F(x)$ at the i^{th} evaluation point x_i.
- $\psi_i(x)$ are polynomials of degree $N-1$.
  - Linear interpolation: polynomial order 1 – Evaluation points required: $N = 2$
  - Quadratic interpolation: polynomial order 2 – Evaluation points required: $N = 3$

Numerical Integration

In general, quadrature formulae have the form

$$\mathcal{I} = \int_{x_a}^{x_b} F(x)\, dx \approx \sum_i^N F(x_i) W_i$$

Evaluation points, x_i, are called **quadrature points** or integration points.

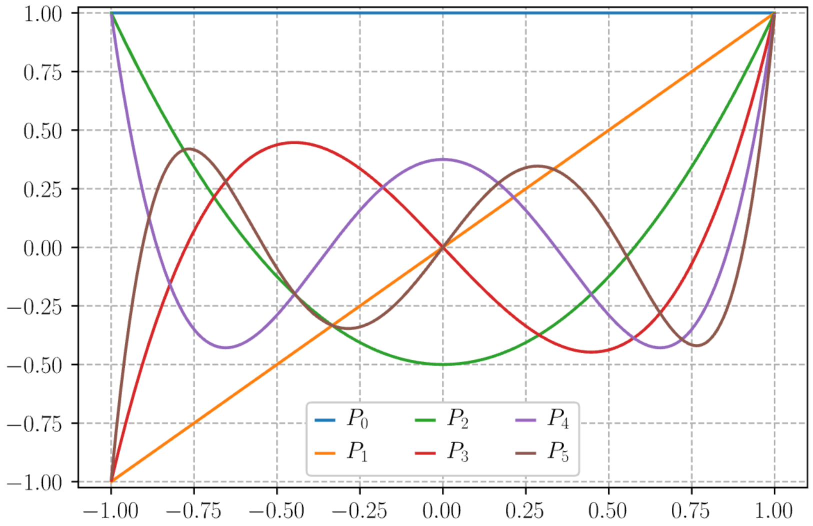
The coefficients W_i are called **quadrature weights**.

They are usually **tabulated** as they depend on the functions ψ_i and on the domain of integration, not on $F(x)$.

Numerical Integration

Of all options, the **Gauss-Legendre quadrature** is the most commonly used.

Gauss-Legendre quadrature is based on **Legendre polynomials**.

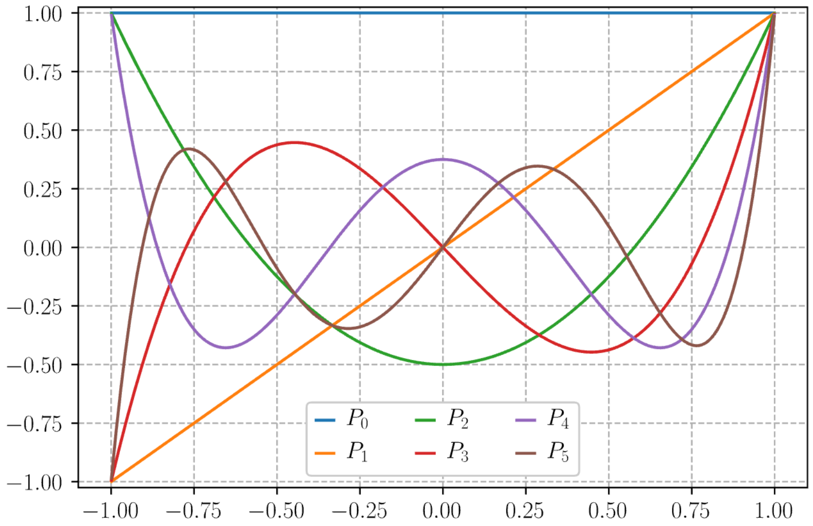


Numerical Integration

Legendre polynomials are complete and orthogonal, but defined in $[-1, 1]$.

This requires **mapping** of the domain of integration from the **global** to a '**normal**', a.k.a. '**natural**' or **master**, coordinate system. In 1D,

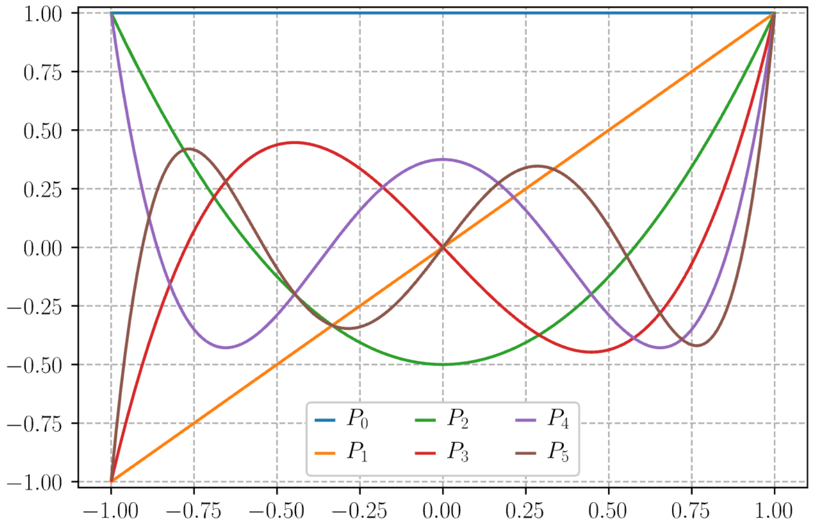
$$x \in [x_a, x_b] \quad \mapsto \quad \xi \in [-1, 1]$$



Numerical Integration

Essentially, we need to transform the integral expression as follows.

$$\mathcal{I} = \int_{x_a}^{x_b} F(x)\, dx = \int_{-1}^{1} \hat{F}(\xi)\, d\xi$$



In conclusion, numerical integration needs dictate master element and mapping requirements.

Master Elements

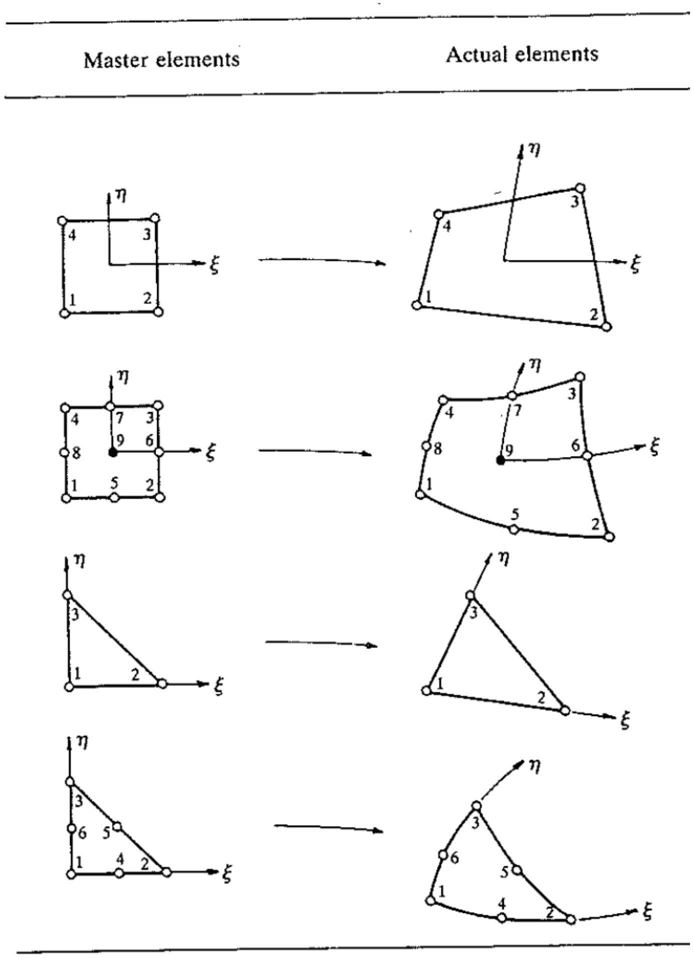
Master Elements

Master elements are defined in natural coordinates.

These coordinates span unit domains, e.g.

$$-1 \leq \xi \leq 1$$

$$-1 \leq (\xi, \eta) \leq 1$$



Mapping

Mapping

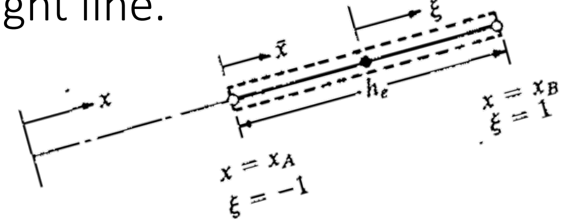
To map the element's domain from natural coordinates onto the global reference system and vice versa, we require an invertible transformation of the form

$$\boldsymbol{x} = f(\boldsymbol{\xi})$$

An example for a straight 1D element of length L, with two nodes, is

$$f(\xi) = x_a + \tfrac{1}{2}L(1+\xi)$$

$f(\xi)$ is linear. Hence, a straight line is transformed into a straight line.



Mapping

In general, the the domain of integration, *i.e.* the element geometry, can have more complex shapes than just a line.

If the element's shape is curved, a nonlinear $f(\xi)$ can map the $\xi \in [-1, 1]$ segment onto the more complex geometry.

Mapping

A systematic way of creating the map involves approximating the geometry of the element as we approximated the dependent variables. Recall

$$u = \sum_{j=1}^{n} N_j^e(x) \cdot u_j^e$$

So,

$$\Downarrow$$

$$x = \sum_{i=1}^{m} \hat{N}_i^e(\xi) \cdot x_i^e$$

Mapping

$$x = \sum_{i=1}^{m} \hat{N}_i^e(\xi) \cdot x_i^e$$ maps ξ space onto $\boldsymbol{x}$ space.

- $\boldsymbol{x_i^e}$ is the global coordinate of the i^{th} (of m) node of an element.
- $\widehat{N}_i^e(\xi)$ are Lagrange interpolation functions of degree $m-1$.

Isoparametric Formulations

Let's consider

$$u = \sum_{j=1}^{n} N_j^e(x) \cdot u_j^e \qquad\qquad x = \sum_{i=1}^{m} \hat{N}_i^e(\xi) \cdot x_i^e$$

In general, $\widehat{N}_i^e(\xi)$ need not be equal to $N_i^e(\xi)$, and m need not be equal to n.

When the shape function are the same:

1. Subparametric formulation: $m < n$
2. Isoparametric formulation: $m = n$
3. Superparametric formulation: $m > n$

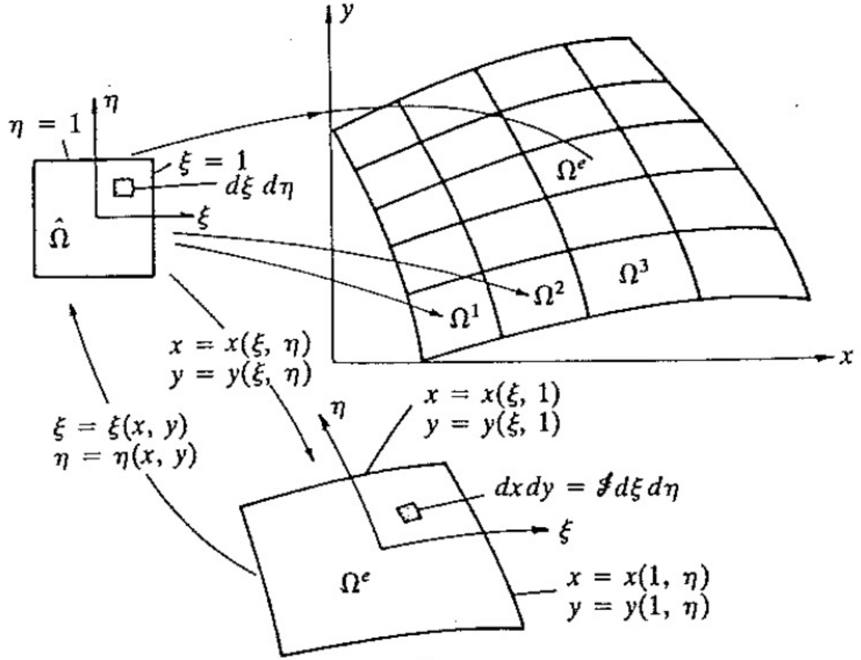
Mapping

In 2D, equations

$$x = \sum_{i=1}^{m} \hat{N}_i^e(\xi, \eta) \cdot x_i^e$$

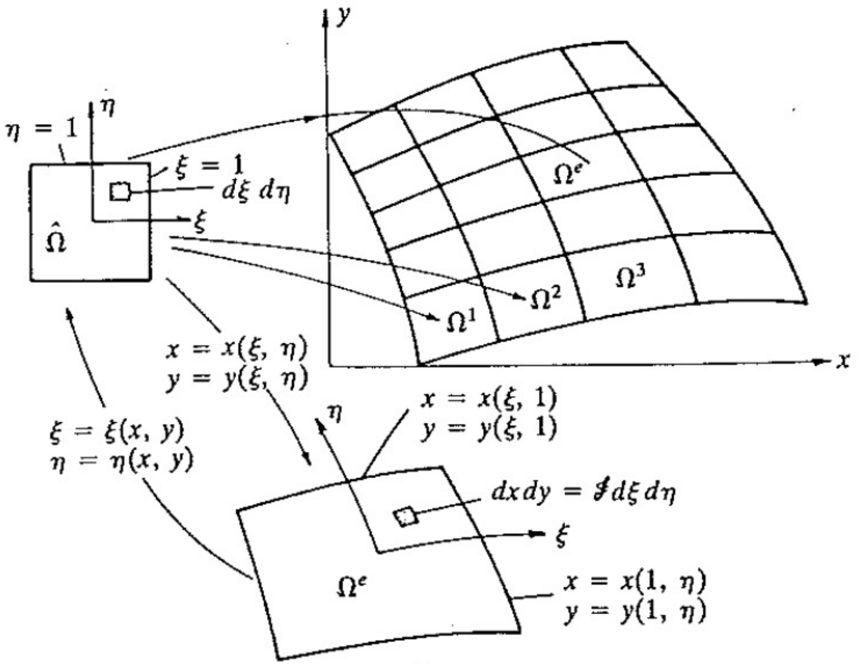
$$y = \sum_{i=1}^{m} \hat{N}_i^e(\xi, \eta) \cdot y_i^e$$

map (ξ, η) space onto (x, y) space.



Mapping

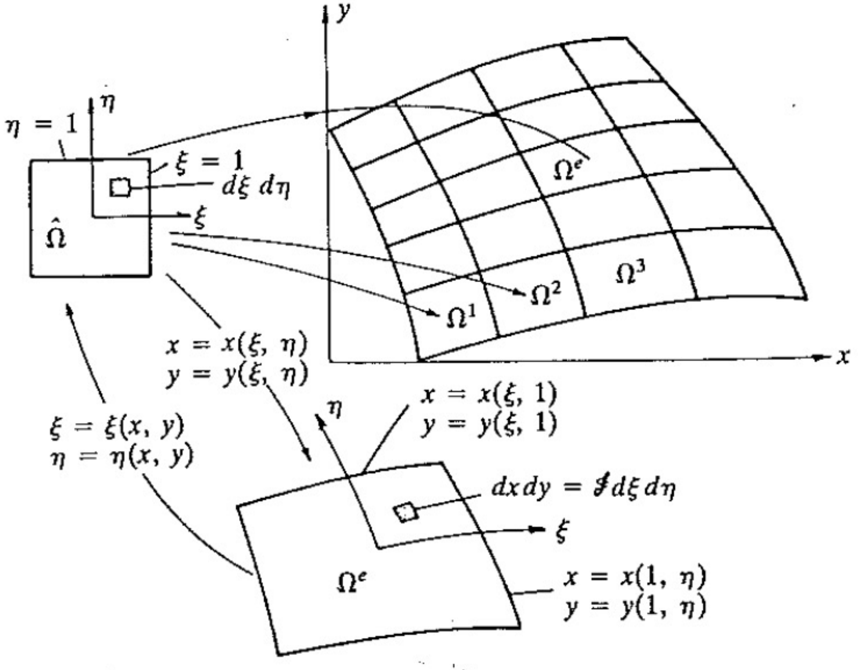
The $(\xi, \eta) \longmapsto (x, y)$ map allow **master elements** with regular geometries (triangular, square, cubic, etc.) to be mapped **onto elements in the global mesh**, which may have more complex shapes.



Mapping

For numerical integration, we need the inverse map: $(x, y) \longmapsto (\xi, \eta)$.

That's to cast the weak form integrals onto the unit domains upon which quadrature rules apply.



Mapping

For the sake of argument, let's consider a generic K_{ij}^{e} term

$$K_{ij}^{e} = \int_{\Omega^e} \left[a(x,y) \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + b(x,y) \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} + c(x,y) N_i^e N_j^e \right] dxdy$$

where, remember, $N_i^e = N_i^e(x,y)$.

Coefficients and N_i^e terms transform using the geometric map.

What about the derivatives?

Mapping

Using

$$x = \sum_{i=1}^{m} \hat{N}_i^e(\xi, \eta) \cdot x_i^e \qquad\qquad y = \sum_{i=1}^{m} \hat{N}_i^e(\xi, \eta) \cdot y_i^e$$

and the chain rule of partial differentiation. We can write

$$\frac{\partial N_i^e}{\partial \xi} = \frac{\partial N_i^e}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i^e}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial N_i^e}{\partial \eta} = \frac{\partial N_i^e}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i^e}{\partial y}\frac{\partial y}{\partial \eta}$$

Mapping

In matrix form

$$\begin{bmatrix} \dfrac{\partial N_i^e}{\partial \xi} \\ \dfrac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^e \begin{bmatrix} \dfrac{\partial N_i^e}{\partial x} \\ \dfrac{\partial N_i^e}{\partial y} \end{bmatrix} = \mathcal{J} \begin{bmatrix} \dfrac{\partial N_i^e}{\partial x} \\ \dfrac{\partial N_i^e}{\partial y} \end{bmatrix}$$

Jacobian Matrix of
the transformation

Mapping

$$\begin{bmatrix} \dfrac{\partial N_i^e}{\partial \xi} \\ \dfrac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \boldsymbol{\mathcal{J}} \begin{bmatrix} \dfrac{\partial N_i^e}{\partial x} \\ \dfrac{\partial N_i^e}{\partial y} \end{bmatrix}$$

This transformation is the opposite of what we need.

Assuming $\boldsymbol{\mathcal{J}}$ is nonsingular, we can invert the matrix.

$$\begin{bmatrix} \dfrac{\partial N_i^e}{\partial x} \\ \dfrac{\partial N_i^e}{\partial y} \end{bmatrix} = \boldsymbol{\mathcal{J}}^{-1} \begin{bmatrix} \dfrac{\partial N_i^e}{\partial \xi} \\ \dfrac{\partial N_i^e}{\partial \eta} \end{bmatrix}$$

Mapping

For $\boldsymbol{\mathcal{J}}^{-1}$ to be usable, the determinant of $\boldsymbol{\mathcal{J}}$ must be positive.

$$\boldsymbol{\mathcal{J}} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \qquad \mathcal{J} = \det \boldsymbol{\mathcal{J}} = J_{11}J_{22} - J_{12}J_{21} > 0$$

$$\boldsymbol{\mathcal{J}}^{-1} = \boldsymbol{\mathcal{J}}^{*} = \begin{bmatrix} J_{11}^{*} & J_{12}^{*} \\ J_{21}^{*} & J_{22}^{*} \end{bmatrix} = \frac{1}{\mathcal{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

Mapping

$$\mathcal{J} = \det \boldsymbol{\mathcal{J}} = J_{11}J_{22} - J_{12}J_{21} > 0$$

The determinant, known as the **Jacobian** of the transformation, represents the scaling factor between areas in the two coordinate systems.

$$dA = dxdy = \mathcal{J}\, d\xi d\eta$$

$\mathcal{J} \approx \mathbf{0}$ when the area of the element is squashed flat, when nodes are too close. $\mathcal{J} < \mathbf{0}$ locally, when the element is non-convex.

Element geometry matters again. This is what makes FEA software complain about distorted meshes.

Mapping

How is the Jacobian computed?

Using $$x = \sum_{i=1}^{m} \hat{N}_i^e(\xi, \eta) \cdot x_i^e \qquad\qquad y = \sum_{i=1}^{m} \hat{N}_i^e(\xi, \eta) \cdot y_i^e$$

$$\boldsymbol{\mathcal{J}} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} x_i^e \dfrac{\partial \hat{N}_i^e}{\partial \xi} & \sum_{i=1}^{m} y_i^e \dfrac{\partial \hat{N}_i^e}{\partial \xi} \\ \sum_{i=1}^{m} x_i^e \dfrac{\partial \hat{N}_i^e}{\partial \eta} & \sum_{i=1}^{m} y_i^e \dfrac{\partial \hat{N}_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \hat{N}_1^e}{\partial \xi} & \dfrac{\partial \hat{N}_2^e}{\partial \xi} & \cdots & \dfrac{\partial \hat{N}_m^e}{\partial \xi} \\ \dfrac{\partial \hat{N}_1^e}{\partial \eta} & \dfrac{\partial \hat{N}_2^e}{\partial \eta} & \cdots & \dfrac{\partial \hat{N}_m^e}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$$

Numerical Integration

Finally,

$$
\begin{aligned}
K_{ij}^e &= \int_{\Omega^e} \left[a(x,y) \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + b(x,y) \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} + c(x,y) N_i^e N_j^e \right] dx dy = \\
&= \int_{\hat{\Omega}} \left[\hat{a}(\xi,\eta) \left(J_{11}^* \frac{\partial N_i^e}{\partial \xi} + J_{12}^* \frac{\partial N_i^e}{\partial \eta} \right) \left(J_{11}^* \frac{\partial N_j^e}{\partial \xi} + J_{12}^* \frac{\partial N_j^e}{\partial \eta} \right) + \right. \\
&\qquad \left. \hat{b}(\xi,\eta) \left(J_{21}^* \frac{\partial N_i^e}{\partial \xi} + J_{22}^* \frac{\partial N_i^e}{\partial \eta} \right) \left(J_{21}^* \frac{\partial N_j^e}{\partial \xi} + J_{22}^* \frac{\partial N_j^e}{\partial \eta} \right) + \hat{c}(\xi,\eta) N_i^e N_j^e \right] \mathcal{J} \, d\xi d\eta
\end{aligned}
$$