

Virtual Work and Energy Methods

15

The majority of the structural problems we have encountered so far have involved structures in which the support reactions and the internal force systems are statically determinate. These include beams, trusses, cables and three-pinned arches and, in the case of beams, we have calculated displacements. Some statically indeterminate structures have also been investigated. These include the composite structural members in [Section 7.10](#) and the circular section beams subjected to torsion and supported at each end in [Section 11.1](#). These relatively simple problems were solved using a combination of statical equilibrium and compatibility of displacements. Further, in [Section 13.6](#), a statically indeterminate propped cantilever was analysed using the principle of superposition ([Section 3.7](#)) while the support reactions for some cases of fixed beams were determined by combining the conditions of statical equilibrium with the moment-area method ([Section 13.3](#)). These methods are perfectly adequate for the comparatively simple problems to which they have been applied. However, other more powerful methods of analysis are required for more complex structures which may possess a high degree of statical indeterminacy. These methods will, in addition, be capable of providing rapid solutions for some statically indeterminate problems, particularly those involving the calculation of displacements.

The methods fall into two categories and are based on two important concepts; the first, *the principle of virtual work*, is the most fundamental and powerful tool available for the analysis of statically indeterminate structures and has the advantage of being able to deal with conditions other than those in the elastic range, while the second, based on *strain energy*, can provide approximate solutions of complex problems for which exact solutions may not exist. The two methods are, in fact, equivalent in some cases since, although the governing equations differ, the equations themselves are identical.

In modern structural analysis, computer-based techniques are widely used; these include the flexibility and stiffness methods. However, the formulation of, say, stiffness matrices for the elements of a complex structure is based on one of the above approaches, so that a knowledge and understanding of their application is advantageous. We shall examine the flexibility and stiffness methods in [Chapter 16](#) and their role in computer-based analysis.

Other specialist approaches have been developed for particular problems. Examples of these are the slope-deflection method for beams and the moment-distribution method for beams and frames; these will also be described in [Chapter 16](#) where we shall consider statically indeterminate structures. Initially, however, in this chapter, we shall examine the principle of virtual work, the different energy theorems and some of the applications of these two concepts.

15.1 Work

Before we consider the principle of virtual work in detail, it is important to clarify exactly what is meant by *work*. The basic definition of work in elementary mechanics is that ‘work is done when a force moves its point of application’. However, we shall require a more exact definition since we shall be concerned with work

done by both forces and moments and with the work done by a force when the body on which it acts is given a displacement which is not coincident with the line of action of the force.

Consider the force, F , acting on a particle, A , in Fig. 15.1(a). If the particle is given a displacement, Δ , by some external agency so that it moves to A' in a direction at an angle α to the line of action of F , the work, W_F , done by F is given by

$$W_F = F(\Delta \cos \alpha) \quad (15.1)$$

or

$$W_F = (F \cos \alpha) \Delta \quad (15.2)$$

Thus we see that the work done by the force, F , as the particle moves from A to A' may be regarded as either the product of F and the component of Δ in the direction of F (Eq. (15.1)) or as the product of the component of F in the direction of Δ and Δ (Eq. (15.2)).

Now consider the couple (pure moment) in Fig. 15.1(b) and suppose that the couple is given a small rotation of θ radians. The work done by each force F is then $F(a/2)\theta$ so that the total work done, W_C , by the couple is

$$W_C = F \frac{a}{2} \theta + F \frac{a}{2} \theta = Fa\theta$$

It follows that the work done, W_M , by the pure moment, M , acting on the bar AB in Fig. 15.1(c) as it is given a small rotation, θ , is

$$W_M = M\theta \quad (15.3)$$

Note that in the above the force, F , and moment, M , are in position before the displacements take place and are not the cause of them. Also, in Fig. 15.1(a), the component of Δ parallel to the direction of F is in the same direction as F ; if it had been in the opposite direction the work done would have been negative. The same argument applies to the work done by the moment, M , where we see in Fig. 15.1(c) that the rotation, θ , is in the same sense as M . Note also that if the displacement, Δ , had been perpendicular to the force, F , no work would have been done by F .

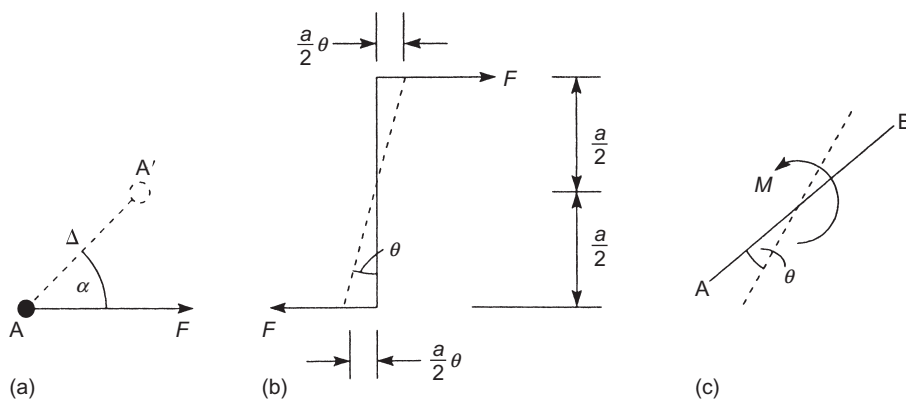


FIGURE 15.1

Work done by a force and a moment.

Finally it should be remembered that work is a scalar quantity since it is not associated with direction (in Fig. 15.1(a) the force F does work if the particle is moved in any direction). Thus the work done by a series of forces is the algebraic sum of the work done by each force.

15.2 Principle of virtual work

The establishment of the principle will be carried out in stages. First we shall consider a particle, then a rigid body and finally a deformable body, which is the practical application we require when analysing structures.

Principle of virtual work for a particle

In Fig. 15.2 a particle, A, is acted upon by a number of concurrent forces, $F_1, F_2, \dots, F_k, \dots, F_r$; the resultant of these forces is R . Suppose that the particle is given a small arbitrary displacement, Δ_v , to A' in some specified direction; Δ_v is an imaginary or *virtual* displacement and is sufficiently small so that the directions of F_1, F_2 , etc., are unchanged. Let θ_R be the angle that the resultant, R , of the forces makes with the direction of Δ_v and $\theta_1, \theta_2, \dots, \theta_k, \dots, \theta_r$ the angles that $F_1, F_2, \dots, F_k, \dots, F_r$ make with the direction of Δ_v , respectively. Then, from either of Eqs (15.1) or (15.2) the total virtual work, W_F , done by the forces F as the particle moves through the virtual displacement, Δ_v , is given by

$$W_F = F_1 \Delta_v \cos \theta_1 + F_2 \Delta_v \cos \theta_2 + \dots + F_k \Delta_v \cos \theta_k + \dots + F_r \Delta_v \cos \theta_r$$

Thus

$$W_F = \sum_{k=1}^r F_k \Delta_v \cos \theta_k$$

or, since Δ_v is a fixed, although imaginary displacement

$$W_F = \Delta_v \sum_{k=1}^r F_k \cos \theta_k \quad (15.4)$$

In Eq. (15.4) $\sum_{k=1}^r F_k \cos \theta_k$ is the sum of all the components of the forces, F , in the direction of Δ_v , and therefore must be equal to the component of the resultant, R , of the forces, F , in the direction of Δ_v , i.e.

$$W_F = \Delta_v \sum_{k=1}^r F_k \cos \theta_k = \Delta_v R \cos \theta_R \quad (15.5)$$

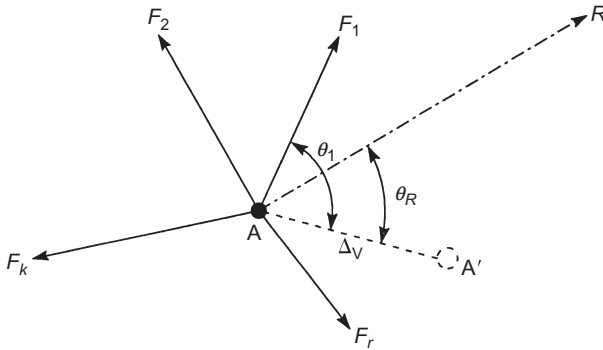


FIGURE 15.2

Virtual work for a system of forces acting on a particle.

If the particle, A , is in equilibrium under the action of the forces, $F_1, F_2, \dots, F_k, \dots, F_r$, the resultant, R , of the forces is zero (Chapter 2). It follows from Eq. (15.5) that the virtual work done by the forces, F , during the virtual displacement, Δ_v , is zero.

We can therefore state the *principle of virtual work* for a particle as follows:

If a particle is in equilibrium under the action of a number of forces the total work done by the forces for a small arbitrary displacement of the particle is zero.

It is possible for the total work done by the forces to be zero even though the particle is not in equilibrium if the virtual displacement is taken to be in a direction perpendicular to their resultant, R . We cannot, therefore, state the converse of the above principle unless we specify that the total work done must be zero for *any* arbitrary displacement. Thus:

A particle is in equilibrium under the action of a system of forces if the total work done by the forces is zero for any virtual displacement of the particle.

Note that in the above, Δ_v is a purely imaginary displacement and is not related in any way to the possible displacement of the particle under the action of the forces, F . Δ_v has been introduced purely as a device for setting up the work–equilibrium relationship of Eq. (15.5). The forces, F , therefore remain unchanged in magnitude and direction during this imaginary displacement; this would not be the case if the displacement were real.

Principle of virtual work for a rigid body

Consider the rigid body shown in Fig. 15.3, which is acted upon by a system of external forces, $F_1, F_2, \dots, F_k, \dots, F_r$. These external forces will induce internal forces in the body, which may be regarded as comprising an infinite number of particles; on adjacent particles, such as A_1 and A_2 , these internal forces will be equal and opposite, in other words self-equilibrating. Suppose now that the rigid body is given a small, imaginary, that is virtual, displacement, Δ_v (or a rotation or a combination of both), in some specified direction. The external and internal forces then do virtual work and the total virtual work done, W_t , is the sum of the virtual work, W_e , done by the external forces and the virtual work, W_i , done by the internal forces. Thus

$$W_t = W_e + W_i \quad (15.6)$$

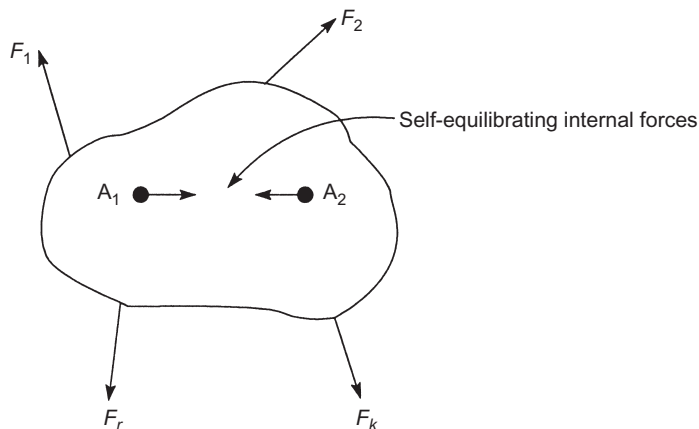


FIGURE 15.3

Virtual work for a rigid body.

Since the body is rigid, all the particles in the body move through the same displacement, Δ_v , so that the virtual work done on all the particles is numerically the same. However, for a pair of adjacent particles, such as A_1 and A_2 in Fig. 15.3, the self-equilibrating forces are in opposite directions, which means that the work done on A_1 is opposite in sign to the work done on A_2 . Thus the sum of the virtual work done on A_1 and A_2 is zero. The argument can be extended to the infinite number of pairs of particles in the body from which we conclude that the internal virtual work produced by a virtual displacement in a rigid body is zero. Equation (15.6) then reduces to

$$W_t = W_c \quad (15.7)$$

Since the body is rigid and the internal virtual work is therefore zero, we may regard the body as a large particle. It follows that if the body is in equilibrium under the action of a set of forces, $F_1, F_2, \dots, F_k, \dots, F_n$, the total virtual work done by the external forces during an arbitrary virtual displacement of the body is zero.

The principle of virtual work is, in fact, an alternative to Eq. (2.10) for specifying the necessary conditions for a system of coplanar forces to be in equilibrium. To illustrate the truth of this we shall consider the calculation of the support reactions in some simple beams.

EXAMPLE 15.1

Calculate the support reactions in the cantilever beam shown in Fig. 15.4(a).

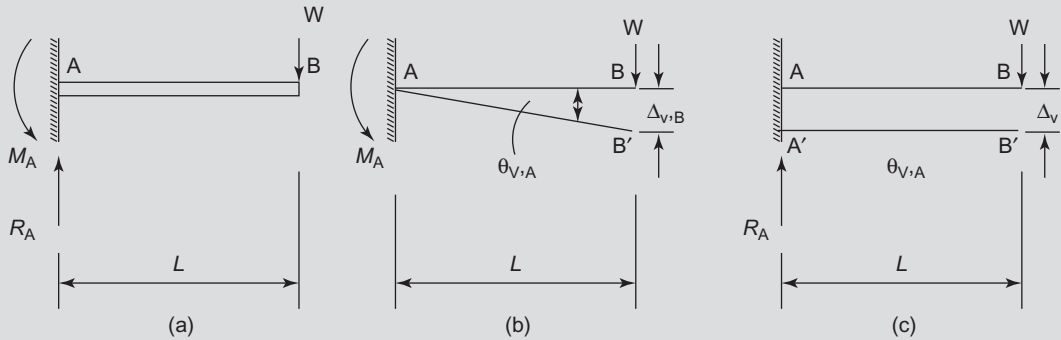


FIGURE 15.4

Beam of Ex. 15.1.

The concentrated load, W , induces a vertical reaction, R_A , and also one of moment, M_A , at A. Suppose that the beam is given a small imaginary, that is virtual, rotation, $\theta_{v,A}$, at A as shown in Fig. 15.4(b). Since we are only concerned here with external forces we may regard the beam as a rigid body so that the beam remains straight and B is displaced to B' . The vertical displacement of B, $\Delta_{v,B}$, is then given by

$$\Delta_{v,B} = \theta_{v,A} L$$

or

$$\theta_{v,A} = \Delta_{v,B} / L \quad (i)$$

The total virtual work, W_t , done by all the forces acting on the beam is given by

$$W_t = W \Delta_{v,B} - M_A \theta_{v,A} \quad (ii)$$

Note that the contribution of M_A to the total virtual work is negative since the assumed direction of M_A is in the opposite sense to the virtual displacement, $\theta_{v,A}$. Note also that there is no linear movement of the beam at A so that R_A does no work. Substituting in Eq. (ii) for $\theta_{v,A}$ from Eq. (i) we have

$$W_t = W\Delta_{v,B} - M_A\Delta_{v,B}/L \quad (\text{iii})$$

Since the beam is in equilibrium, $W_t=0$ from the principle of virtual work. Therefore

$$0 = W\Delta_{v,B} - M_A\Delta_{v,B}/L$$

so that

$$M_A = WL$$

which is the result which would have been obtained from considering the moment equilibrium of the beam about A.

Suppose now that the complete beam is given a virtual displacement, Δ_v , as shown in Fig. 15.4(c). There is no rotation of the beam so that M_A does no work. The total virtual work done is then given by

$$W_t = W\Delta_v - R_A\Delta_v \quad (\text{iv})$$

The contribution of R_A is negative since its direction is opposite to that of Δ_v . The beam is in equilibrium so that $W_t=0$. Therefore, from Eq. (iv)

$$R_A = W$$

which is the result we would have obtained by resolving forces vertically.

EXAMPLE 15.2

Calculate the support reactions in the cantilever beam shown in Fig. 15.5(a).

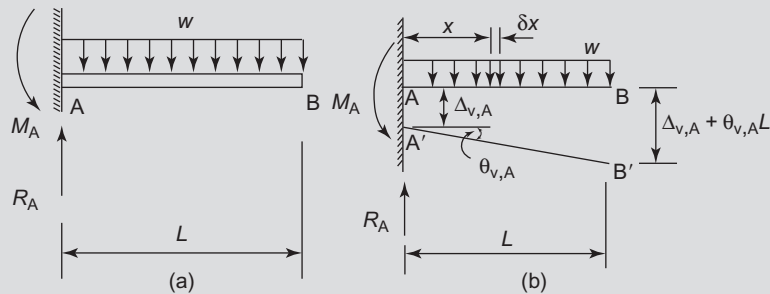


FIGURE 15.5

Beam of Ex. 15.2.

In this case we obtain a solution by simultaneously giving the beam a virtual displacement, $\Delta_{v,A}$, at A and a virtual rotation, $\theta_{v,A}$, at A. The total deflection at B is then $\Delta_{v,A} + \theta_{v,A}L$ and at a distance x from A is $\Delta_{v,A} + \theta_{v,A}x$. Since the beam carries a uniformly distributed load we find the virtual work done by the load by first considering an elemental length, δx , of the load a distance x from A. The load on the element is $w\delta x$ and the virtual work done by this elemental load is given by

$$\delta W = w\delta x(\Delta_{v,A} + \theta_{v,A}x)$$

The total virtual work done on the beam is then

$$W_t = \int_0^L w(\Delta_{v,A} + \theta_{v,A}x)dx - M_A\theta_{v,A} - R_A\Delta_{v,A}$$

which simplifies to

$$W_t = (wL - R_A)\Delta_{v,A} + [(wL^2/2) - M_A]\theta_{v,A} = 0 \quad (i)$$

since the beam is in equilibrium. Equation (i) is valid for all values of $\Delta_{v,A}$ and $\theta_{v,A}$ so that

$$wL - R_A = 0 \text{ and } (wL^2/2) - M_A = 0$$

Therefore

$$R_A = wL \text{ and } M_A = wL^2/2$$

which are the results that would have been obtained by resolving forces vertically and taking moments about A.

EXAMPLE 15.3

Calculate the reactions at the built-in end of the cantilever beam shown in Fig. 15.6.

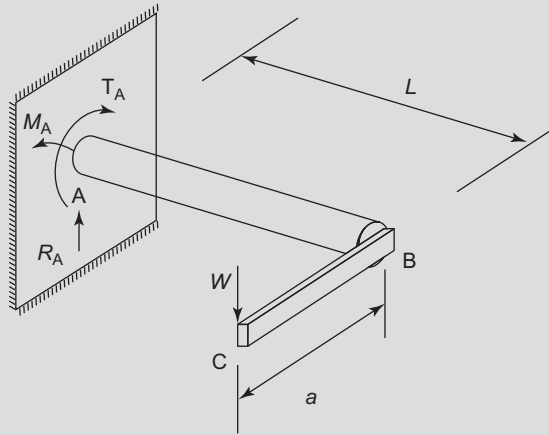


FIGURE 15.6

Beam of Ex. 15.3.

In this example the load, W , produces reactions of vertical force, moment and torque at the built-in end. The vertical reaction and the moment are the same as in Ex. 15.1. To determine the torque reaction we impose a small virtual displacement, $\Delta_{v,C}$, vertically downwards at C. This causes the beam AB to rotate as a rigid body through an angle, $\theta_{v,AB}$, which is given by

$$\theta_{v,AB} = \Delta_{v,C}/a \quad (i)$$

Alternatively we could have imposed a small virtual rotation, $\theta_{v,AB}$, on the beam which would have resulted in a virtual displacement of C equal to $a\theta_{v,AB}$; clearly the two approaches produce identical results.

The total virtual work done on the beam is then given by

$$W_t = W\Delta_{v,C} - T_A\theta_{v,AB} = 0 \quad (\text{ii})$$

since the beam is in equilibrium. Substituting for $\theta_{v,AB}$ in Eq. (ii) from Eq. (i) we have

$$T_A = Wa$$

which is the result that would have been obtained by considering the statical equilibrium of the beam.

EXAMPLE 15.4

Calculate the support reactions in the simply supported beam shown in Fig. 15.7.

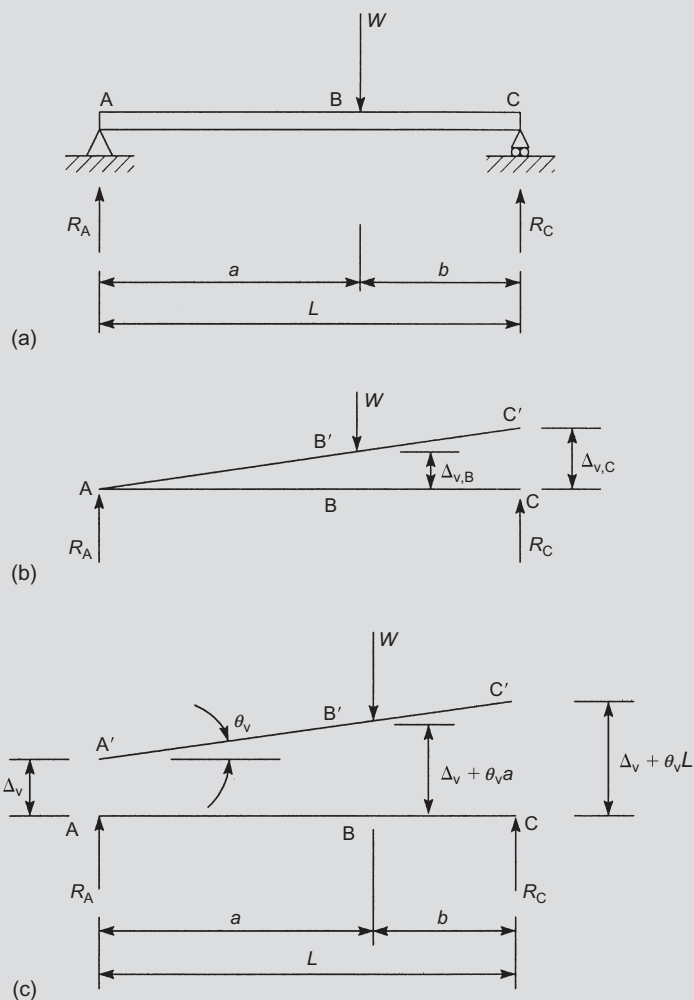


FIGURE 15.7

Use of the principle of virtual work to calculate support reactions.

Only a vertical load is applied to the beam so that only vertical reactions, R_A and R_C , are produced.

Suppose that the beam at C is given a small imaginary, that is a virtual, displacement, $\Delta_{v,C}$, in the direction of R_C as shown in Fig. 15.7(b). Since we are concerned here solely with the *external* forces acting on the beam we may regard the beam as a rigid body. The beam therefore rotates about A so that C moves to C' and B moves to B'. From similar triangles we see that

$$\Delta_{v,B} = \frac{a}{a+b} \Delta_{v,C} = \frac{a}{L} \Delta_{v,C} \quad (\text{i})$$

The total virtual work, W_t , done by all the forces acting on the beam is then given by

$$W_t = R_C \Delta_{v,C} - W \Delta_{v,B} \quad (\text{ii})$$

Note that the work done by the load, W , is negative since $\Delta_{v,B}$ is in the opposite direction to its line of action. Note also that the support reaction, R_A , does no work since the beam only rotates about A. Now substituting for $\Delta_{v,B}$ in Eq. (ii) from Eq. (i) we have

$$W_t = R_C \Delta_{v,C} - W \frac{a}{L} \Delta_{v,C} \quad (\text{iii})$$

Since the beam is in equilibrium, W_t is zero from the principle of virtual work. Hence, from Eq. (iii)

$$R_C \Delta_{v,C} - W \frac{a}{L} \Delta_{v,C} = 0$$

which gives

$$R_C = W \frac{a}{L}$$

which is the result that would have been obtained from a consideration of the moment equilibrium of the beam about A. R_A follows in a similar manner. Suppose now that instead of the single displacement $\Delta_{v,C}$ the complete beam is given a vertical virtual displacement, Δ_v , together with a virtual rotation, θ_v , about A as shown in Fig. 15.7(c). The total virtual work, W_t , done by the forces acting on the beam is now given by

$$W_t = R_A \Delta_v - W(\Delta_v + a\theta_v) + R_C(\Delta_v + L\theta_v) = 0 \quad (\text{iv})$$

since the beam is in equilibrium. Rearranging Eq. (iv)

$$(R_A + R_C - W)\Delta_v + (R_C L - Wa)\theta_v = 0 \quad (\text{v})$$

Equation (v) is valid for all values of Δ_v and θ_v so that

$$R_A + R_C - W = 0 \quad R_C L - Wa = 0$$

which are the equations of equilibrium we would have obtained by resolving forces vertically and taking moments about A.

It is not being suggested here that the application of Eq. (2.10) should be abandoned in favour of the principle of virtual work. The purpose of Exs 15.1–15.4 is to illustrate the application of a virtual displacement and the manner in which the principle can be used.

Virtual work in a deformable body

In structural analysis we are not generally concerned with forces acting on a rigid body. Structures and structural members deform under load, which means that if we assign a virtual displacement to a particular point in a structure, not all points in the structure will suffer the same virtual displacement as would be the case if the structure were rigid. This means that the virtual work produced by the internal forces is not zero as it is in the rigid body case, since the virtual work produced by the self-equilibrating forces on adjacent particles does not cancel out. The total virtual work produced by applying a virtual displacement to a deformable body acted upon by a system of external forces is therefore given by Eq. (15.6).

If the body is in equilibrium under the action of the external force system then every particle in the body is also in equilibrium. Therefore, from the principle of virtual work, the virtual work done by the forces acting on the particle is zero irrespective of whether the forces are external or internal. It follows that, since the virtual work is zero for all particles in the body, it is zero for the complete body and Eq. (15.6) becomes

$$W_e + W_i = 0 \quad (15.8)$$

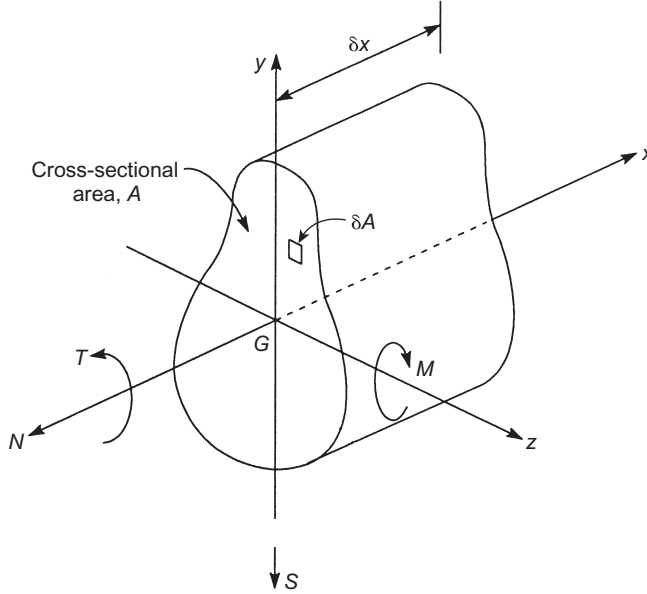
Note that in the above argument only the conditions of equilibrium and the concept of work are employed. Thus Eq. (15.8) does not require the deformable body to be linearly elastic (i.e. it need not obey Hooke's law) so that the principle of virtual work may be applied to any body or structure that is rigid, elastic or plastic. The principle does require that displacements, whether real or imaginary, must be small, so that we may assume that external and internal forces are unchanged in magnitude and direction during the displacements. In addition the virtual displacements must be compatible with the geometry of the structure and the constraints that are applied, such as those at a support. The exception is the situation we have in Exs 15.1–15.4 where we apply a virtual displacement at a support. This approach is valid since we include the work done by the support reactions in the total virtual work equation.

Work done by internal force systems

The calculation of the work done by an external force is straightforward in that it is the product of the force and the displacement of its point of application in its own line of action (Eqs (15.1), (15.2) or (15.3)) whereas the calculation of the work done by an internal force system during a displacement is much more complicated. In Chapter 3 we saw that no matter how complex a loading system is, it may be simplified to a combination of up to four load types: axial load, shear force, bending moment and torsion; these in turn produce corresponding internal force systems. We shall now consider the work done by these internal force systems during arbitrary virtual displacements.

Axial force

Consider the elemental length, δx , of a structural member as shown in Fig. 15.8 and suppose that it is subjected to a positive internal force system comprising a normal force (i.e. axial force), N , a shear force, S , a bending moment, M and a torque, T , produced by some external loading system acting on the structure of which the member is part. Note that the face on which the internal forces act is a negative face, see Fig. 3.7. The stress distributions corresponding to these internal forces have been related in previous chapters to an axis system whose origin coincides with the centroid of area of the cross section. We shall, in fact, be using these stress distributions in the derivation of expressions for internal virtual work in linearly elastic structures so that it is logical to assume the same origin of axes here; we shall also assume that the y axis is an axis of symmetry. Initially we shall consider the normal force, N .

**FIGURE 15.8**

Virtual work due to internal force system.

The direct stress, σ , at any point in the cross section of the member is given by $\sigma = N/A$ (Eq. (7.1)). Therefore the normal force on the element δA at the point (z, y) is

$$\delta N = \sigma \delta A = \frac{N}{A} \delta A$$

Suppose now that the structure is given an arbitrary virtual displacement which produces a virtual axial strain, ϵ_v , in the element. The internal virtual work, $\delta w_{i,N}$, done by the axial force on the elemental length of the member is given by

$$\delta w_{i,N} = \int_A \frac{N}{A} dA \epsilon_v \delta x$$

which, since $\int_A dA = A$, reduces to

$$\delta w_{i,N} = N \epsilon_v \delta x \quad (15.9)$$

In other words, the virtual work done by N is the product of N and the virtual axial displacement of the element of the member. For a member of length L , the virtual work, $w_{i,N}$, done during the arbitrary virtual strain is then

$$w_{i,N} = \int_L N \epsilon_v dx \quad (15.10)$$

For a structure comprising a number of members, the total internal virtual work, $W_{i,N}$, done by axial force is the sum of the virtual work of each of the members. Thus

$$w_{i,N} = \sum \int_L N \epsilon_v dx \quad (15.11)$$

Note that in the derivation of Eq. (15.11) we have made no assumption regarding the material properties of the structure so that the relationship holds for non-elastic as well as elastic materials. However, for a

linearly elastic material, i.e. one that obeys Hooke's law (Section 7.7), we can express the virtual strain in terms of an equivalent virtual normal force. Thus

$$\varepsilon_v = \frac{\sigma_v}{E} = \frac{N_v}{EA}$$

Therefore, if we designate the *actual* normal force in a member by N_A , Eq. (15.11) may be expressed in the form

$$w_{i,N} = \sum \int_L \frac{N_A N_v}{EA} dx \quad (15.12)$$

Shear force

The shear force, S , acting on the member section in Fig. 15.8 produces a distribution of vertical shear stress which, as we saw in Section 10.2, depends upon the geometry of the cross section. However, since the element, δA , is infinitesimally small, we may regard the shear stress, τ , as constant over the element. The shear force, δS , on the element is then

$$\delta S = \tau \delta A \quad (15.13)$$

Suppose that the structure is given an arbitrary virtual displacement which produces a virtual shear strain, γ_v , at the element. This shear strain represents the angular rotation in a vertical plane of the element $\delta A \times \delta x$ relative to the longitudinal centroidal axis of the member. The vertical displacement at the section being considered is therefore $\gamma_v \delta x$. The internal virtual work, $\delta w_{i,S}$, done by the shear force, S , on the elemental length of the member is given by

$$\delta w_{i,S} = \int_A \tau dA \gamma_v \delta x$$

We saw in Section 13.5 that we could assume a uniform shear stress through the cross section of a beam if we allowed for the actual variation by including a form factor, β . Thus the expression for the internal virtual work in the member may be written

$$\delta w_{i,S} = \int_A \beta \left(\frac{S}{A} \right) dA \gamma_v \delta x$$

or

$$\delta w_{i,S} = \beta S \gamma_v \delta x \quad (15.14)$$

Hence the virtual work done by the shear force during the arbitrary virtual strain in a member of length L is

$$w_{i,S} = \beta \int_L S \gamma_v dx \quad (15.15)$$

For a linearly elastic member, as in the case of axial force, we may express the virtual shear strain, γ_v , in terms of an equivalent virtual shear force, S_v . Thus, from Section 7.7

$$\gamma_v = \frac{\tau_v}{G} = \frac{S_v}{GA}$$

so that from Eq. (15.15)

$$w_{i,S} = \beta \int_L \frac{S_A S_v}{GA} dx \quad (15.16)$$

For a structure comprising a number of linearly elastic members the total internal work, $W_{i,S}$, done by the shear forces is

$$W_{i,S} = \sum \beta \int_L \frac{S_A S_v}{GA} dx \quad (15.17)$$

Bending moment

The bending moment, M , acting on the member section in Fig. 15.8 produces a distribution of direct stress, σ , through the depth of the member cross section. The normal force on the element, δA , corresponding to this stress is therefore $\sigma \delta A$. Again we shall suppose that the structure is given a small arbitrary virtual displacement which produces a virtual direct strain, ϵ_v , in the element $\delta A \times \delta x$. Thus the virtual work done by the normal force acting on the element δA is $\sigma \delta A \epsilon_v \delta x$. Hence, integrating over the complete cross section of the member we obtain the internal virtual work, $\delta w_{i,M}$, done by the bending moment, M , on the elemental length of member, i.e.

$$\delta w_{i,M} = \int_A \sigma \delta A \epsilon_v \delta x \quad (15.18)$$

The virtual strain, ϵ_v , in the element $\delta A \times \delta x$ is, from Eq. (9.1), given by

$$\epsilon_v = \frac{y}{R_v}$$

where R_v is the radius of curvature of the member produced by the virtual displacement. Thus, substituting for ϵ_v in Eq. (15.18), we obtain

$$\delta w_{i,M} = \int_A \sigma \frac{y}{R_v} \delta A \delta x$$

or, since $\sigma y \delta A$ is the moment of the normal force on the element, δA , about the z axis,

$$\delta w_{i,M} = \frac{M}{R_v} \delta x$$

Therefore, for a member of length L , the internal virtual work done by an actual bending moment, M_A , is given by

$$w_{i,M} = \int_L \frac{M_A}{R_v} dx \quad (15.19)$$

In the derivation of Eq. (15.19) no specific stress-strain relationship has been assumed, so that it is applicable to a non-linear system. For the particular case of a linearly elastic system, the virtual curvature $1/R_v$ may be expressed in terms of an equivalent virtual bending moment, M_v , using the relationship of Eq. (9.11), i.e.

$$\frac{1}{R_v} = \frac{M_v}{EI}$$

Substituting for $1/R_v$ in Eq. (15.19) we have

$$w_{i,M} = \int_L \frac{M_A M_v}{EI} dx \quad (15.20)$$

so that for a structure comprising a number of members the total internal virtual work, $W_{i,M}$, produced by bending is

$$W_{i,M} = \sum \int_L \frac{M_A M_v}{EI} dx \quad (15.21)$$

In Chapter 9 we used the suffix 'z' to denote a bending moment in a vertical plane about the z axis (M_z) and the second moment of area of the member section about the z axis (I_z). Clearly the bending moments in Eq. (15.21) need not be restricted to those in a vertical plane; the suffixes are therefore omitted.

Torsion

The internal virtual work, $w_{i,T}$, due to torsion in the particular case of a linearly elastic circular section bar may be found in a similar manner and is given by

$$w_{i,T} = \int_L \frac{T_A T_v}{GI_o} dx \quad (15.22)$$

in which I_o is the polar second moment of area of the cross section of the bar (see Section 11.1). For beams of non-circular cross section, I_o is replaced by a torsion constant, J , which, for many practical beam sections is determined empirically (Section 11.5).

Hinges

In some cases it is convenient to impose a virtual rotation, θ_v , at some point in a structural member where, say, the actual bending moment is M_A . The internal virtual work done by M_A is then $M_A \theta_v$, (see Eq. (15.3)); physically this situation is equivalent to inserting a hinge at the point.

Sign of internal virtual work

So far we have derived expressions for internal work without considering whether it is positive or negative in relation to external virtual work.

Suppose that the structural member, AB, in Fig. 15.9(a) is, say, a member of a truss and that it is in equilibrium under the action of two externally applied axial tensile loads, P ; clearly the internal axial, that is normal, force at any section of the member is P . Suppose now that the member is given a virtual extension, δ_v , such that B moves to B'. Then the virtual work done by the applied load, P , is positive since the displacement, δ_v , is in the same direction as its line of action. However, the virtual work done by the internal force, $N(=P)$, is negative since the displacement of B is in the opposite direction to its line of action; in other words work is done *on* the member. Thus, from Eq. (15.8), we see that in this case

$$W_e = W_i \quad (15.23)$$

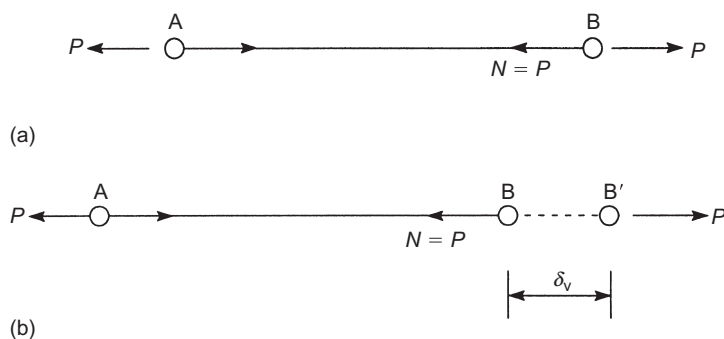


FIGURE 15.9

Sign of the internal virtual work in an axially loaded member.

Equation (15.23) would apply if the virtual displacement had been a contraction and not an extension, in which case the signs of the external and internal virtual work in Eq. (15.8) would have been reversed. Clearly the above applies equally if P is a compressive load. The above arguments may be extended to structural members subjected to shear, bending and torsional loads, so that Eq. (15.23) is generally applicable.

Virtual work due to external force systems

So far in our discussion we have only considered the virtual work produced by externally applied concentrated loads. For completeness we must also consider the virtual work produced by moments, torques and distributed loads.

In Fig. 15.10 a structural member carries a distributed load, $w(x)$, and at a particular point a concentrated load, W , a moment, M , and a torque, T . Suppose that at the point a virtual displacement is imposed that has translational components, $\Delta_{v,y}$ and $\Delta_{v,x}$, parallel to the y and x axes, respectively, and rotational components, θ_v and ϕ_v , in the yx and zy planes, respectively.

If we consider a small element, δx , of the member at the point, the distributed load may be regarded as constant over the length δx and acting, in effect, as a concentrated load $w(x) \delta x$. Thus the virtual work, w_e , done by the complete external force system is given by

$$w_e = W\Delta_{v,y} + P\Delta_{v,x} + M\theta_v + T\phi_v + \int_L w(x)\Delta_{v,y} dx$$

For a structure comprising a number of load positions, the total external virtual work done is then

$$W_e = \sum \left[W\Delta_{v,y} + P\Delta_{v,x} + M\theta_v + T\phi_v + \int_L w(x)\Delta_{v,y} dx \right] \quad (15.24)$$

In Eq. (15.24) there need not be a complete set of external loads applied at every loading point so, in fact, the summation is for the appropriate number of loads. Further, the virtual displacements in the above are related to forces and moments applied in a vertical plane. We could, of course, have forces and moments and components of the virtual displacement in a horizontal plane, in which case Eq. (15.24) would be extended to include their contribution.

The internal virtual work equivalent of Eq. (15.24) for a linear system is, from Eqs (15.12), (15.17), (15.21) and (15.22)

$$W_i = \sum \left[\int_L \frac{N_A N_v}{EA} dx + \beta \int_L \frac{S_A S_v}{GA} dx + \int_L \frac{M_A M_v}{EI} dx + \int_L \frac{T_A T_v}{GJ} dx + M_A \theta_v \right] \quad (15.25)$$

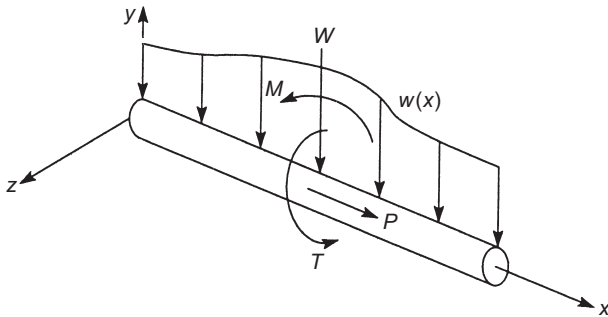


FIGURE 15.10

Virtual work due to externally applied loads.

in which the last term on the right-hand side is the virtual work produced by an actual internal moment at a hinge (see above). Note that the summation in Eq. (15.25) is taken over all the *members* of the structure.

Use of virtual force systems

So far, in all the structural systems we have considered, virtual work has been produced by actual forces moving through imposed virtual displacements. However, the actual forces are not related to the virtual displacements in any way since, as we have seen, the magnitudes and directions of the actual forces are unchanged by the virtual displacements so long as the displacements are small. Thus the principle of virtual work applies for *any* set of forces in equilibrium and *any* set of displacements. Equally, therefore, we could specify that the forces are a set of virtual forces *in equilibrium* and that the displacements are actual displacements. Therefore, instead of relating actual external and internal force systems through virtual displacements, we can relate actual external and internal displacements through virtual forces.

If we apply a virtual force system to a deformable body it will induce an internal virtual force system which will move through the actual displacements; thus, internal virtual work will be produced. In this case, for example, Eq. (15.10) becomes

$$w_{i,N} = \int_L N_v \varepsilon_A dx$$

in which N_v is the internal virtual normal force and ε_A is the actual strain. Then, for a linear system, in which the actual internal normal force is N_A , $\varepsilon_A = N_A/EA$, so that for a structure comprising a number of members the total internal virtual work due to a virtual normal force is

$$W_{i,N} = \sum \int_L \frac{N_v N_A}{EA} dx$$

which is identical to Eq. (15.12). Equations (15.17), (15.21) and (15.22) may be shown to apply to virtual force systems in a similar manner.

Applications of the principle of virtual work

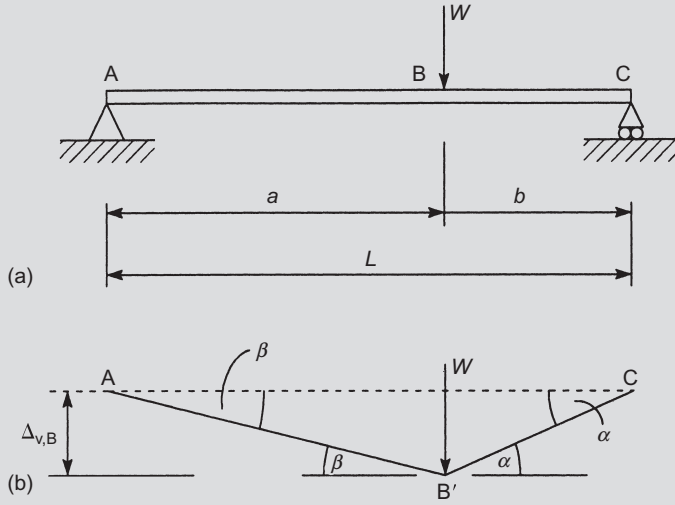
We have now seen that the principle of virtual work may be used either in the form of imposed virtual displacements or in the form of imposed virtual forces. Generally the former approach, as we saw in Exs 15.1–15.4, is used to determine forces, while the latter is used to obtain displacements.

For statically determinate structures the use of virtual displacements to determine force systems is a relatively trivial use of the principle although problems of this type provide a useful illustration of the method. The real power of this approach lies in its application to the solution of statically indeterminate structures, as we shall see in Chapter 16. However, the use of virtual forces is particularly useful in determining actual displacements of structures. We shall illustrate both approaches by examples.

EXAMPLE 15.5

Determine the bending moment at the point B in the simply supported beam ABC shown in Fig. 15.11(a).

We determined the support reactions for this particular beam in Ex. 15.4. In this example, however, we are interested in the actual internal moment, M_B , at the point of application of the load. We must therefore impose a virtual displacement which will relate the internal moment at B to the applied load and which will exclude other unknown external forces such as the support reactions, and unknown internal force systems such as the bending moment distribution along the length of the beam. Therefore, if we

**FIGURE 15.11**

Determination of bending moment at a point in the beam of Ex. 15.5 using virtual work.

imagine that the beam is hinged at B and that the lengths AB and BC are rigid, a virtual displacement, $\Delta_{v,B}$, at B will result in the displaced shape shown in Fig. 15.11(b).

Note that the support reactions at A and C do no work and that the internal moments in AB and BC do no work because AB and BC are rigid links. From Fig. 15.11(b)

$$\Delta_{v,B} = a\beta = b\alpha \quad (i)$$

Hence

$$\alpha = \frac{a}{b}\beta$$

and the angle of rotation of BC relative to AB is then

$$\theta_B = \beta + \alpha = \beta \left(1 + \frac{a}{b} \right) = \frac{L}{b}\beta \quad (ii)$$

Now equating the external virtual work done by W to the internal virtual work done by M_B (see Eq. (15.23)) we have

$$W\Delta_{v,B} = M_B\theta_B \quad (iii)$$

Substituting in Eq. (iii) for $\Delta_{v,B}$ from Eq. (i) and for θ_B from Eq. (ii) we have

$$Wa\beta = M_B \frac{L}{b}\beta$$

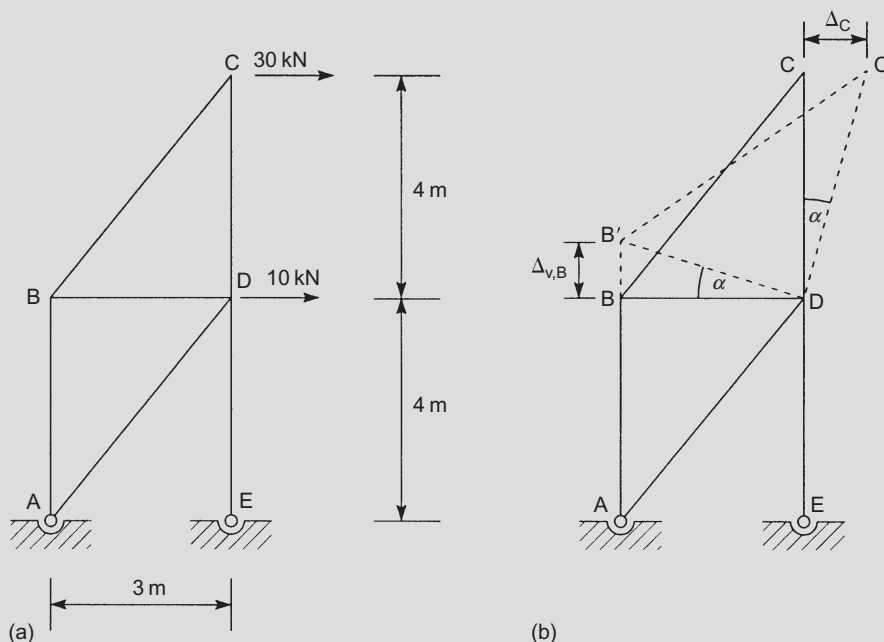
which gives

$$M_B = \frac{Wab}{L}$$

which is the result we would have obtained by calculating the moment of R_C ($= Wa/L$ from Ex. 15.4) about B.

EXAMPLE 15.6

Determine the force in the member AB in the truss shown in Fig. 15.12(a).

**FIGURE 15.12**

Determination of the internal force in a member of a truss using virtual work.

We are required to calculate the force in the member AB, so that again we need to relate this internal force to the externally applied loads without involving the internal forces in the remaining members of the truss. We therefore impose a virtual extension, $\Delta_{v,B}$, at B in the member AB, such that B moves to B'. If we assume that the remaining members are rigid, the forces in them will do no work. Further, the triangle BCD will rotate as a rigid body about D to B'C'D as shown in Fig. 15.12(b). The horizontal displacement of C, Δ_C , is then given by

$$\Delta_C = 4\alpha$$

while

$$\Delta_{v,B} = 3\alpha$$

Hence

$$\Delta_C = \frac{4\Delta_{v,B}}{3} \quad (i)$$

Equating the external virtual work done by the 30 kN load to the internal virtual work done by the force, F_{BA} , in the member, AB, we have (see Eq. (15.23) and Fig. 15.9)

$$30\Delta_C = F_{BA}\Delta_{v,B} \quad (ii)$$

Substituting for Δ_C from Eq. (i) in Eq. (ii),

$$30 \times \frac{4}{3} \Delta_{v,B} = F_{BA} \Delta_{v,B}$$

Whence

$$F_{BA} = +40 \text{ kN} \quad (\text{i.e. } F_{BA} \text{ is tensile})$$

In the above we are, in effect, assigning a positive (i.e. tensile) sign to F_{BA} by imposing a virtual extension on the member AB.

The actual sign of F_{BA} is then governed by the sign of the external virtual work. Thus, if the 30 kN load had been in the opposite direction to Δ_C the external work done would have been negative, so that F_{BA} would be negative and therefore compressive. This situation can be verified by inspection. Alternatively, for the loading as shown in Fig. 15.12(a), a contraction in AB would have implied that F_{BA} was compressive. In this case DC would have rotated in an anticlockwise sense, Δ_C would have been in the opposite direction to the 30 kN load so that the external virtual work done would be negative, resulting in a negative value for the compressive force F_{BA} ; F_{BA} would therefore be tensile as before. Note also that the 10 kN load at D does no work since D remains undisplaced.

We shall now consider problems involving the use of virtual forces. Generally we shall require the displacement of a particular point in a structure, so that if we apply a virtual force to the structure at the point and in the direction of the required displacement the external virtual work done will be the product of the virtual force and the actual displacement, which may then be equated to the internal virtual work produced by the internal virtual force system moving through actual displacements. Since the choice of the virtual force is arbitrary, we may give it any convenient value; the simplest type of virtual force is therefore a unit load and the method then becomes the *unit load method*.

EXAMPLE 15.7

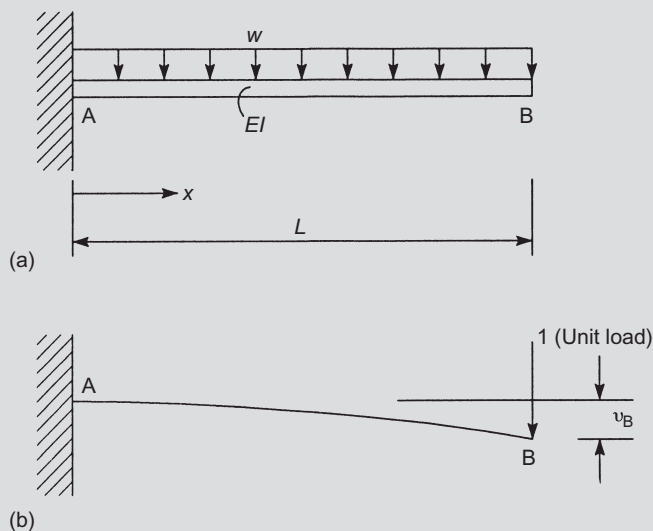
Determine the vertical deflection of the free end of the cantilever beam shown in Fig. 15.13(a).

Let us suppose that the actual deflection of the cantilever at B produced by the uniformly distributed load is v_B and that a vertically downward virtual unit load was applied at B before the actual deflection took place. The external virtual work done by the unit load is, from Fig. 15.13(b), $1v_B$. The deflection, v_B , is assumed to be caused by bending only, i.e. we are ignoring any deflections due to shear. The internal virtual work is given by Eq. (15.21) which, since only one member is involved, becomes

$$W_{i,M} = \int_0^L \frac{M_A M_v}{EI} dx \quad (\text{i})$$

The virtual moments, M_v , are produced by a unit load so that we shall replace M_v by M_1 . Then

$$W_{i,M} = \int_0^L \frac{M_A M_1}{EI} dx \quad (\text{ii})$$

**FIGURE 15.13**

Deflection of the free end of a cantilever beam using the unit load method.

At any section of the beam a distance x from the built-in end

$$M_A = -\frac{w}{2}(L-x)^2 \quad M_1 = -1(L-x)$$

Substituting for M_A and M_1 in Eq. (ii) and equating the external virtual work done by the unit load to the internal virtual work we have

$$1v_B = \int_0^L \frac{w}{2EI}(L-x)^3 dx$$

which gives

$$v_B = -\frac{w}{2EI} \left[\frac{1}{4}(L-x)^4 \right]_0^L$$

so that

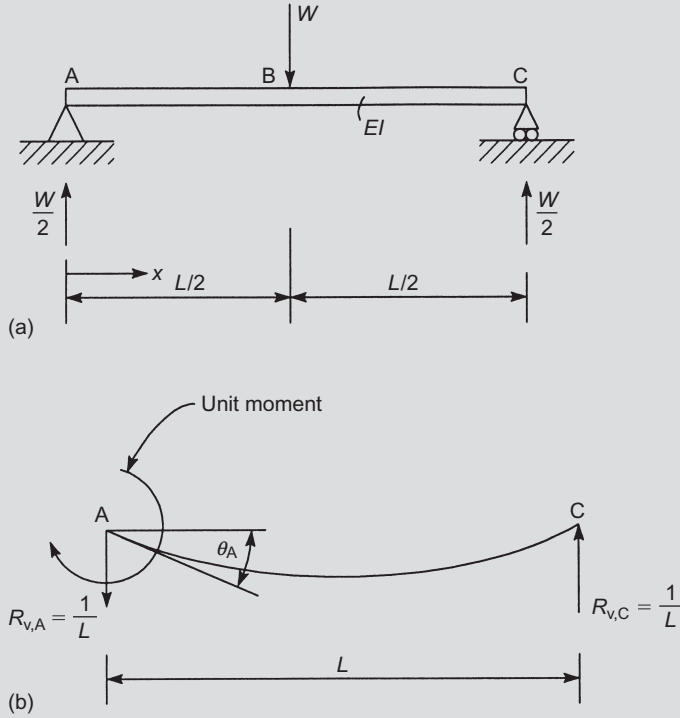
$$v_B = \frac{wL^4}{8EI} \quad (\text{as in Ex. 13.2})$$

Note that v_B is in fact negative but the positive sign here indicates that it is in the same direction as the unit load.

EXAMPLE 15.8

Determine the rotation, i.e. the slope, of the beam ABC shown in Fig. 15.14(a) at A.

The actual rotation of the beam at A produced by the actual concentrated load, W , is θ_A . Let us suppose that a virtual unit moment is applied at A before the actual rotation takes place, as shown in Fig. 15.14(b). The virtual unit moment induces virtual support reactions of $R_{v,A}$ ($= 1/L$) acting downwards and $R_{v,C}$ ($= 1/L$) acting upwards. The actual internal bending moments are

**FIGURE 15.14**

Determination of the rotation of a simply supported beam at a support using the unit load method.

$$M_A = +\frac{W}{2}x \quad 0 \leq x \leq L/2$$

$$M_A = +\frac{W}{2}(L-x) \quad L/2 \leq x \leq L$$

The internal virtual bending moment is

$$M_v = 1 - \frac{1}{L}x \quad 0 \leq x \leq L$$

The external virtual work done is $1\theta_A$ (the virtual support reactions do no work as there is no vertical displacement of the beam at the supports) and the internal virtual work done is given by Eq. (15.21). Hence

$$1\theta_A = \frac{1}{EI} \left[\int_0^{L/2} \frac{W}{2}x \left(1 - \frac{x}{L}\right) dx + \int_{L/2}^L \frac{W}{2}(L-x) \left(1 - \frac{x}{L}\right) dx \right] \quad (i)$$

Simplifying Eq. (i) we have

$$\theta_A = \frac{W}{2EIL} \left[\int_0^{L/2} (Lx - x^2) dx + \int_{L/2}^L (L-x)^2 dx \right] \quad (ii)$$

Hence

$$\theta_A = \frac{W}{2EI} \left\{ \left[L \frac{x^2}{2} - \frac{x^3}{3} \right]_0^{L/2} - \frac{1}{3} \left[(L-x)^3 \right]_{L/2}^L \right\}$$

from which

$$\theta_A = \frac{WL^2}{16EI}$$

which is the result that may be obtained from Eq. (iii) of Ex. 13.5.

EXAMPLE 15.9

Calculate the vertical deflection of the joint B and the horizontal movement of the support D in the truss shown in Fig. 15.15(a). The cross-sectional area of each member is 1800 mm^2 and Young's modulus, E , for the material of the members is $200\,000 \text{ N/mm}^2$.

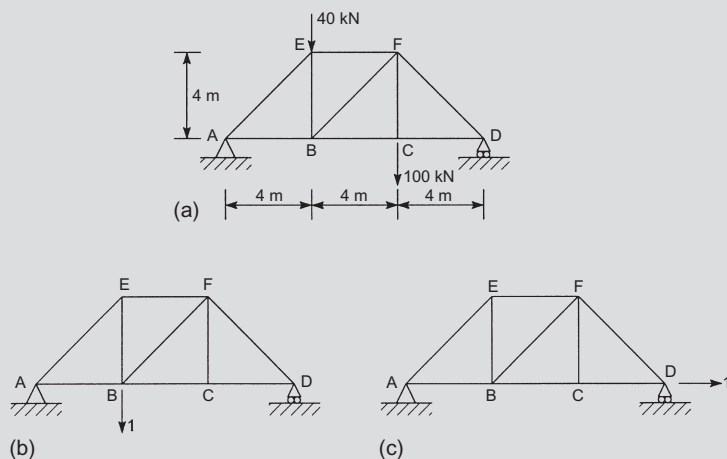


FIGURE 15.15

Deflection of a truss using the unit load method.

The virtual force systems, i.e. unit loads, required to determine the vertical deflection of B and the horizontal deflection of D are shown in Fig. 15.15(b) and (c), respectively. Therefore, if the actual vertical deflection at B is $\delta_{B,v}$ and the horizontal deflection at D is $\delta_{D,h}$ the external virtual work done by the unit loads is $1\delta_{B,v}$ and $1\delta_{D,h}$, respectively. The internal actual and virtual force systems comprise axial forces in all the members. These axial forces are constant along the length of each member so that for a truss comprising n members, Eq. (15.12) reduces to

$$W_{i,N} = \sum_{j=1}^n \frac{F_{A,j} F_{v,j} L_j}{E_j A_j} \quad (i)$$

in which $F_{A,j}$ and $F_{v,j}$ are the actual and virtual forces in the j th member which has a length L_j , an area of cross-section A_j and a Young's modulus E_j .

Since the forces $F_{v,j}$ are due to a unit load, we shall write Eq. (i) in the form

$$W_{i,N} = \sum_{j=1}^n \frac{F_{A,j} F_{1,j} L_j}{E_j A_j} \quad (\text{ii})$$

Also, in this particular example, the area of cross section, A , and Young's modulus, E , are the same for all members so that it is sufficient to calculate $\sum_{j=1}^n F_{A,j} F_{1,j} L_j$ and then divide by EA to obtain $W_{i,N}$.

The forces in the members, whether actual or virtual, may be calculated by the method of joints (Section 4.6). Note that the support reactions corresponding to the three sets of applied loads (one actual and two virtual) must be calculated before the internal force systems can be determined. However, in Fig. 15.15(c), it is clear from inspection that $F_{1,AB} = F_{1,BC} = F_{1,CD} = +1$ while the forces in all other members are zero. The calculations are presented in Table 15.1; note that positive signs indicate tension and negative signs compression.

Table 15.1

Member	L (m)	F_A (kN)	$F_{1,B}$	$F_{1,D}$	$F_A F_{1,B} L$ (kNm)	$F_A F_{1,D} L$ (kNm)
AE	5.7	-84.9	-0.94	0	+451.4	0
AB	4.0	+60.0	+0.67	+1.0	+160.8	+240.0
EF	4.0	-60.0	-0.67	0	+160.8	0
EB	4.0	+20.0	+0.67	0	+53.6	0
BF	5.7	-28.3	+0.47	0	-75.2	0
BC	4.0	+80.0	+0.33	+1.0	+105.6	+320.0
CD	4.0	+80.0	+0.33	+1.0	+105.6	+320.0
CF	4.0	+100.0	0	0	0	0
DF	5.7	-113.1	-0.47	0	+301.0	0
					$\Sigma = +1263.6$	$\Sigma = +880.0$

Thus equating internal and external virtual work done (Eq. (15.23)) we have

$$1\delta_{B,v} = \frac{1263.6 \times 10^6}{200\,000 \times 1800}$$

whence

$$\delta_{B,v} = 3.51 \text{ mm}$$

and

$$1\delta_{D,h} = \frac{880 \times 10^6}{200\,000 \times 1800}$$

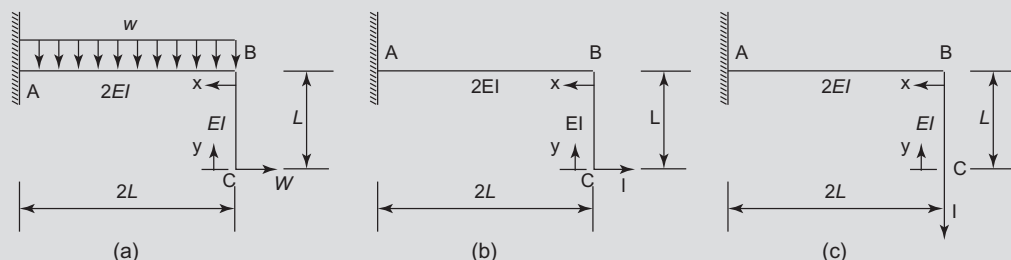
which gives

$$\delta_{D,h} = 2.44 \text{ mm}$$

Both deflections are positive which indicates that the deflections are in the directions of the applied unit loads. Note that in the above it is unnecessary to specify units for the unit load since the unit load appears, in effect, on both sides of the virtual work equation (the internal F_1 forces are directly proportional to the unit load).

EXAMPLE 15.10

Determine the horizontal and vertical components of the deflection of the point C in the frame shown in Fig. 15.16(a); consider the effects of bending only.

**FIGURE 15.16**

Frame of Ex. 15.10.

The components of the deflection of C may be found by applying horizontal and vertical unit loads in turn at C as shown in Figs. 15.16(b) and (c). The internal virtual work done by this virtual force system, that is unit loads acting through real displacements, is given by Eq. (15.20) in which, for AB

$$M_A = WL - (wx^2/2), \quad M_v = 1L \text{ (horiz.)}, \quad M_v = -1x \text{ (vert.)}$$

and for BC

$$M_A = Wy, \quad M_v = 1y \text{ (horiz.)}, \quad M_v = 0 \text{ (vert.)}$$

Considering the horizontal component of deflection first, the total internal work done is

$$W_t = \int_0^L (wy^2/EI) dy + \int_0^{2L} [WL - (wx^2/2)](L/2EI) dx$$

that is

$$W_t = (W/EI) [y^3/3]_0^L + (L/2EI) [WLx - (wx^3/6)]_0^{2L}$$

which gives

$$W_t = 2L^3 (2W - wL)/3EI \quad (i)$$

The virtual external work done by the unit load is $1\delta_{C,h}$ where $\delta_{C,h}$ is the horizontal component of the deflection of C. Equating this to the total internal virtual work given by Eq. (i) gives

$$\delta_{C,h} = 2L^3 (2W - wL)/3EI \quad (ii)$$

Now considering the vertical component of the deflection

$$W_t = (1/2EI) \int_0^{2L} [WL - (wx^2/2)] (-x) dx$$

Note that for BC, $M_v=0$. Integrating this expression and substituting the limits

$$W_t = L^3 (-W + wL)/EI \quad (iii)$$

The external work done is $1\delta_{C,v}$ where $\delta_{C,v}$ is the vertical component of the deflection of C. Equating the internal and external virtual work gives

$$\delta_{C,v} = L^3 (-W + wL) / EI \quad (\text{iv})$$

Note that the components of deflection can be either positive or negative depending on the relative magnitudes of W and w . A positive value indicates a deflection in the direction of the applied unit load while a negative value indicates a deflection in the opposite direction to the applied unit load.

EXAMPLE 15.11

The cantilever beam AB shown in plan in Fig. 15.17 takes the form of a quadrant of a circle and is positioned in a horizontal plane. If the beam supports a vertically downward load, W , at its free end B and its bending and torsional stiffnesses are EI and GJ respectively, calculate the vertical component of the deflection of B.

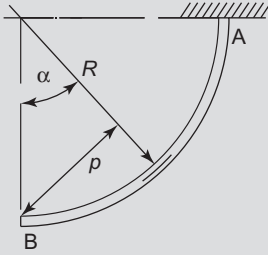


FIGURE 15.17

Cantilever beam of Ex. 15.11.

To determine the vertical displacement we apply a virtual unit load at B vertically downwards (i.e. into the plane of the paper).

At a section of the beam where the radius at the section makes an angle, α , with the radius through B

$$M_A = Wp = WR \sin \alpha, \quad M_v = 1R \sin \alpha$$

$$T_A = W(R - R \cos \alpha), \quad T_v = 1(R - R \cos \alpha)$$

The total internal virtual work done is given by the summation of Eqs. (15.20) and (15.22), that is

$$W_t = (1/EI) \int_0^{\pi/2} WR^2 \sin^2 \alpha R d\alpha + (1/GJ) \int_0^{\pi/2} WR^2 (1 - \cos \alpha)^2 R d\alpha \quad (\text{i})$$

Integrating Eq. (i) and substituting the limits gives

$$W_t = WR^3 \{ (\pi/4EI) + (1/GJ)[(3\pi/4) - 2] \} \quad (\text{ii})$$

The external work done by the unit load is $1\delta_B$ so that equating with Eq. (ii) gives

$$\delta_B = WR^3 \{ (\pi/4EI) + (1/GJ)[(3\pi/4) - 2] \}$$

Examples 15.5–15.11 illustrate the application of the principle of virtual work to the solution of problems involving statically determinate linearly elastic structures. We shall now examine the alternative energy methods but we shall return to the use of virtual work in Chapter 16 when we consider statically indeterminate structures.

15.3 Energy methods

Although it is generally accepted that energy methods are not as powerful as the principle of virtual work in that they are limited to elastic analysis, they possibly find their greatest use in providing rapid approximate solutions of problems for which exact solutions do not exist. Also, many statically indeterminate structures may be conveniently analysed using energy methods while, in addition, they are capable of providing comparatively simple solutions for deflection problems which are not readily solved by more elementary means.

Energy methods involve the use of either the *total complementary energy* or the *total potential energy* (TPE) of a structural system. Either method may be employed to solve a particular problem, although as a general rule displacements are more easily found using complementary energy while forces are more easily found using potential energy.

Strain energy and complementary energy

In Section 7.10 we investigated strain energy in a linearly elastic member subjected to an axial load. Subsequently in Sections 9.4, 10.3 and 11.2 we derived expressions for the strain energy in a linearly elastic member subjected to bending, shear and torsional loads, respectively. We shall now examine the more general case of a member that is not linearly elastic.

Figure 15.18(a) shows the j th member of a structure comprising n members. The member is subjected to a gradually increasing load, P_j , which produces a gradually increasing displacement, Δ_j . If the member possesses non-linear elastic characteristics, the load–deflection curve will take the form shown in Fig. 15.18(b). Let us suppose that the final values of P_j and Δ_j are $P_{j,F}$ and $\Delta_{j,F}$.

As the member extends (or contracts if P_j is a compressive load) P_j does work which, as we saw in Section 7.10, is stored in the member as strain energy. The work done by P_j as the member extends by a small amount $\delta\Delta_j$ is given by

$$\delta W_j = P_j \delta \Delta_j$$

Therefore the total work done by P_j , and therefore the strain energy stored in the member, as P_j increases from zero to $P_{j,F}$ is given by

$$u_j = \int_0^{\Delta_{j,F}} P_j d\Delta_j \quad (15.26)$$

which is clearly the area OBD under the load–deflection curve in Fig. 15.18(b). Similarly the area OAB, which we shall denote by c_j , above the load–deflection curve is given by

$$c_j = \int_0^{P_{j,F}} \Delta_j dP_j \quad (15.27)$$

It may be seen from Fig. 15.18(b) that the area OABD represents the work done by a constant force $P_{j,F}$ moving through the displacement $\Delta_{j,F}$. Thus from Eqs (15.26) and (15.27)

$$u_j + c_j = P_{j,F} \Delta_{j,F} \quad (15.28)$$

It follows that since u_j has the dimensions of work, c_j also has the dimensions of work but otherwise c_j has no physical meaning. It can, however, be regarded as the complement of the work done by P_j in producing the displacement Δ_j and is therefore called the *complementary energy*.

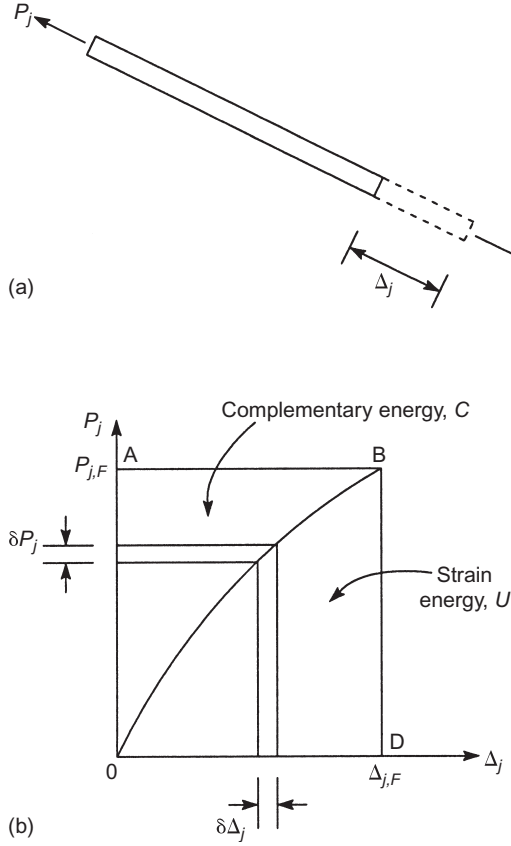


FIGURE 15.18

Load-deflection curve for a non-linearly elastic member.

The total strain energy, U , of the structure is the sum of the individual strain energies of the members. Thus

$$U = \sum_{j=1}^n u_j$$

which becomes, when substituting for u_j from Eq. (15.26)

$$U = \sum_{j=1}^n \int_0^{\Delta_{j,F}} P_j d\Delta_j \quad (15.29)$$

Similarly, the total complementary energy, C , of the structure is given by

$$C = \sum_{j=1}^n c_j$$

whence, from Eq. (15.27)

$$C = \sum_{j=1}^n \int_0^{P_{j,F}} \Delta_j dP_j \quad (15.30)$$

Equation (15.29) may be written in expanded form as

$$U = \int_0^{\Delta_{1,F}} P_1 d\Delta_1 + \int_0^{\Delta_{2,F}} P_2 d\Delta_2 + \dots + \int_0^{\Delta_{j,F}} P_j d\Delta_j + \dots + \int_0^{\Delta_{n,F}} P_n d\Delta_n \quad (15.31)$$

Partially differentiating Eq. (15.31) with respect to a particular displacement, say Δ_j , gives

$$\frac{\partial U}{\partial \Delta_j} = P_j \quad (15.32)$$

Equation (15.32) states that the partial derivative of the strain energy in an elastic structure with respect to a displacement Δ_j is equal to the corresponding force P_j ; clearly U must be expressed as a function of the displacements. This equation is generally known as *Castigliano's first theorem (Part I)* after the Italian engineer who derived and published it in 1879. One of its primary uses is in the analysis of non-linearly elastic structures, which is outside the scope of this book.

Now writing Eq. (15.30) in expanded form we have

$$C = \int_0^{P_{1,F}} \Delta_1 dP_1 + \int_0^{P_{2,F}} \Delta_2 dP_2 + \dots + \int_0^{P_{j,F}} \Delta_j dP_j + \dots + \int_0^{P_{n,F}} \Delta_n dP_n \quad (15.33)$$

The partial derivative of Eq. (15.33) with respect to one of the loads, say P_j , is then

$$\frac{\partial C}{\partial P_j} = \Delta_j \quad (15.34)$$

Equation (15.34) states that the partial derivative of the complementary energy of an elastic structure with respect to an applied load, P_j , gives the displacement of that load in its own line of action; C in this case is expressed as a function of the loads. Equation (15.34) is sometimes called the *Crotti-Engesser theorem* after the two engineers, one Italian, one German, who derived the relationship independently, Crotti in 1879 and Engesser in 1889.

Now consider the situation that arises when the load–deflection curve is linear, as shown in Fig. 15.19. In this case the areas OBD and OAB are equal so that the strain and complementary energies are equal. Thus we may replace the complementary energy, C , in Eq. (15.34) by the strain energy, U . Hence

$$\frac{\partial U}{\partial P_j} = \Delta_j \quad (15.35)$$

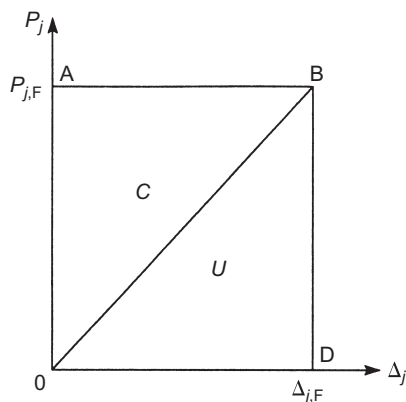


FIGURE 15.19

Load–deflection curve for a linearly elastic member.

Equation (15.35) states that, for a linearly elastic structure, the partial derivative of the strain energy of a structure with respect to a load gives the displacement of the load in its own line of action. This is generally known as *Castigliano's first theorem (Part II)*. Its direct use is limited in that it enables the displacement at a particular point in a structure to be determined *only* if there is a load applied at the point and *only* in the direction of the load. It could not therefore be used to solve for the required displacements at B and D in the truss in Ex. 15.9.

The principle of the stationary value of the total complementary energy

Suppose that an elastic structure comprising n members is in equilibrium under the action of a number of forces, $P_1, P_2, \dots, P_k, \dots, P_r$, which produce corresponding actual displacements, $\Delta_1, \Delta_2, \dots, \Delta_k, \dots, \Delta_r$, and actual internal forces, $F_1, F_2, \dots, F_j, \dots, F_n$. Now let us suppose that a system of elemental virtual forces, $\delta P_1, \delta P_2, \dots, \delta P_k, \dots, \delta P_r$, are imposed on the structure and act through the actual displacements. The external virtual work, δW_e , done by these elemental virtual forces is, from Section 15.2,

$$\delta W_e = \delta P_1 \Delta_1 + \delta P_2 \Delta_2 + \dots + \delta P_k \Delta_k + \dots + \delta P_r \Delta_r$$

or

$$\delta W_e = \sum_{k=1}^r \Delta_k \delta P_k \quad (15.36)$$

At the same time the elemental external virtual forces are in equilibrium with an elemental internal virtual force system, $\delta F_1, \delta F_2, \dots, \delta F_j, \dots, \delta F_n$, which moves through actual internal deformations, $\delta_1, \delta_2, \dots, \delta_j, \dots, \delta_n$. Hence the internal elemental virtual work done is

$$\delta W_i = \sum_{j=1}^n \delta_j \delta F_j \quad (15.37)$$

From Eq. (15.23)

$$\sum_{k=1}^r \Delta_k \delta P_k = \sum_{j=1}^n \delta_j \delta F_j$$

so that

$$\sum_{j=1}^n \delta_j \delta F_j - \sum_{k=1}^r \Delta_k \delta P_k = 0 \quad (15.38)$$

Equation (15.38) may be written as

$$\delta \left(\sum_{j=1}^n \int_0^{F_j} \delta_j dF_j - \sum_{k=1}^r \Delta_k P_k \right) = 0 \quad (15.39)$$

From Eq. (15.30) we see that the first term in Eq. (15.39) represents the complementary energy, C_i , of the actual internal force system, while the second term represents the complementary energy, C_e , of the external force system. C_i and C_e are opposite in sign since C_e is the complement of the work done *by* the external force system while C_i is the complement of the work done *on* the structure. Rewriting Eq. (15.39), we have

$$\delta(C_i + C_e) = 0 \quad (15.40)$$

In Eq. (15.39) the displacements, Δ_k , and the deformations, δ_j , are the actual displacements and deformations of the elastic structure. They therefore obey the condition of compatibility of displacement so that Eqs (15.39) and (15.40) are equations of geometrical compatibility. Also Eq. (15.40) establishes the *principle of the stationary value of the total complementary energy* which may be stated as:

For an elastic body in equilibrium under the action of applied forces the true internal forces (or stresses) and reactions are those for which the total complementary energy has a stationary value.

In other words the true internal forces (or stresses) and reactions are those that satisfy the condition of compatibility of displacement. This property of the total complementary energy of an elastic structure is particularly useful in the solution of statically indeterminate structures in which an infinite number of stress distributions and reactive forces may be found to satisfy the requirements of equilibrium so that, as we have already seen, equilibrium conditions are insufficient for a solution.

We shall examine the application of the principle in the solution of statically indeterminate structures in Chapter 16. Meanwhile we shall illustrate its application to the calculation of displacements in statically determinate structures.

EXAMPLE 15.12

The calculation of deflections in a truss.

Suppose that we wish to calculate the deflection, Δ_2 , in the direction of the load, P_2 , and at the joint at which P_2 is applied in a truss comprising n members and carrying a system of loads $P_1, P_2, \dots, P_k, \dots, P_r$, as shown in Fig. 15.20. From Eq. (15.39) the total complementary energy, C , of the truss is given by

$$C = \sum_{j=1}^n \int_0^{F_j} \delta_j dF_j - \sum_{k=1}^r \Delta_k P_k \quad (i)$$

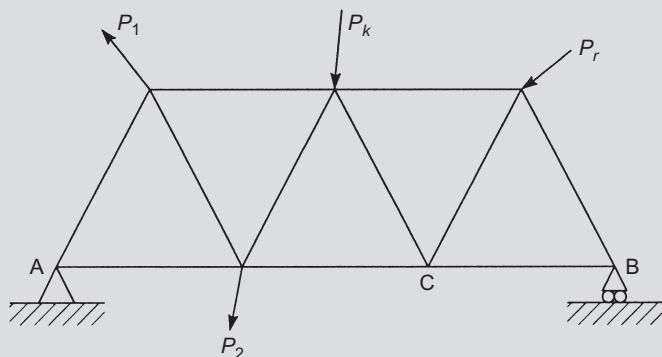


FIGURE 15.20

Deflection of a truss using complementary energy.

From the principle of the stationary value of the total complementary energy with respect to the load P_2 , we have

$$\frac{\partial C}{\partial P_2} = \sum_{j=1}^n \delta_j \frac{\partial F_j}{\partial P_2} - \Delta_2 = 0 \quad (ii)$$

from which

$$\Delta_2 = \sum_{j=1}^n \delta_j \frac{\partial F_j}{\partial P_2} \quad (iii)$$

Note that the partial derivatives with respect to P_2 of the fixed loads, $P_1, P_3, \dots, P_k, \dots, P_r$, vanish. To complete the solution we require the load–displacement characteristics of the structure. For a non-linear system in which, say,

$$F_j = b(\delta_j)^c$$

where b and c are known, Eq. (iii) becomes

$$\Delta_2 = \sum_{j=1}^n \left(\frac{F_j}{b} \right)^{1/c} \frac{\partial F_j}{\partial P_2} \quad (\text{iv})$$

In Eq. (iv) F_j may be obtained from basic equilibrium conditions, e.g. the method of joints, and expressed in terms of P_2 ; hence $\partial F_j / \partial P_2$ is found. The actual value of P_2 is then substituted in the expression for F_j and the product $(F_j/b)^{1/c} \partial F_j / \partial P_2$ calculated for each member. Summation then gives Δ_2 .

In the case of a linearly elastic structure δ_j is, from Sections 7.4 and 7.7, given by

$$\delta_j = \frac{F_j}{E_j A_j} L_j$$

in which E_j , A_j and L_j are Young's modulus, the area of cross section and the length of the j th member. Substituting for δ_j in Eq. (iii) we obtain

$$\Delta_2 = \sum_{j=1}^n \frac{F_j L_j}{E_j A_j} \frac{\partial F_j}{\partial P_2} \quad (\text{v})$$

Equation (v) could have been derived directly from Castigliano's first theorem (Part II) which is expressed in Eq. (15.35) since, for a linearly elastic system, the complementary and strain energies are identical; in this case the strain energy of the j th member is $F_j^2 L_j / 2 A_j E_j$ from Eq. (7.29). Other aspects of the solution merit discussion.

We note that the support reactions at A and B do not appear in Eq. (i). This convenient absence derives from the fact that the displacements, $\Delta_1, \Delta_2, \dots, \Delta_k, \dots, \Delta_r$, are the actual displacements of the truss and fulfil the conditions of geometrical compatibility and boundary restraint. The complementary energy of the reactions at A and B is therefore zero since both of their corresponding displacements are zero.

In Eq. (v) the term $\partial F_j / \partial P_2$ represents the rate of change of the actual forces in the members of the truss with P_2 . This may be found, as described in the non-linear case, by calculating the forces, F_j , in the members in terms of P_2 and then differentiating these expressions with respect to P_2 . Subsequently the actual value of P_2 would be substituted in the expressions for F_j and thus, using Eq. (v), Δ_2 obtained. This approach is rather clumsy. A simpler alternative would be to calculate the forces, F_j , in the members produced by the applied loads including P_2 , then remove all the loads and apply P_2 only as an unknown force and recalculate the forces F_j as functions of P_2 ; $\partial F_j / \partial P_2$ is then obtained by differentiating these functions.

This procedure indicates a method for calculating the displacement of a point in the truss in a direction not coincident with the line of action of a load or, in fact, of a point such as C which carries no load at all. Initially the forces F_j in the members due to $P_1, P_2, \dots, P_k, \dots, P_r$ are calculated. These loads are then removed and a *dummy* or *fictitious* load, P_f , applied at the point and in the direction of the required displacement. A new set of forces, F_j , are calculated in terms of the dummy load, P_f , and thus $\partial F_j / \partial P_f$ is obtained. The required displacement, say Δ_C of C, is then given by

$$\Delta_C = \sum_{j=1}^n \frac{F_j L_j}{E_j A_j} \frac{\partial F_j}{\partial P_f} \quad (\text{vi})$$

The simplification may be taken a stage further. The force F_j in a member due to the dummy load may be expressed, since the system is linearly elastic, in terms of the dummy load as

$$F_j = \frac{\partial F_j}{\partial P_f} P_f \quad (\text{vii})$$

Suppose now that $P_f = 1$, i.e. a *unit load*. Equation (vii) then becomes

$$F_j = \frac{\partial F_j}{\partial P_f} 1$$

so that $\partial F_j / \partial P_f = F_{1,j}$, the load in the j th member due to a unit load applied at the point and in the direction of the required displacement. Thus, Eq. (vi) may be written as

$$\Delta_C = \sum_{j=1}^n \frac{F_j F_{1,j} L_j}{E_j A_j} \quad (\text{viii})$$

in which a unit load has been applied at C in the direction of the required displacement. Note that Eq. (viii) is identical in form to Eq. (ii) of Ex. 15.9.

In the above we have concentrated on members subjected to axial loads. The arguments apply in cases where structural members carry bending moments that produce rotations, shear loads that cause shear deflections and torques that produce angles of twist. We shall now demonstrate the application of the method to structures subjected to other than axial loads.

EXAMPLE 15.13

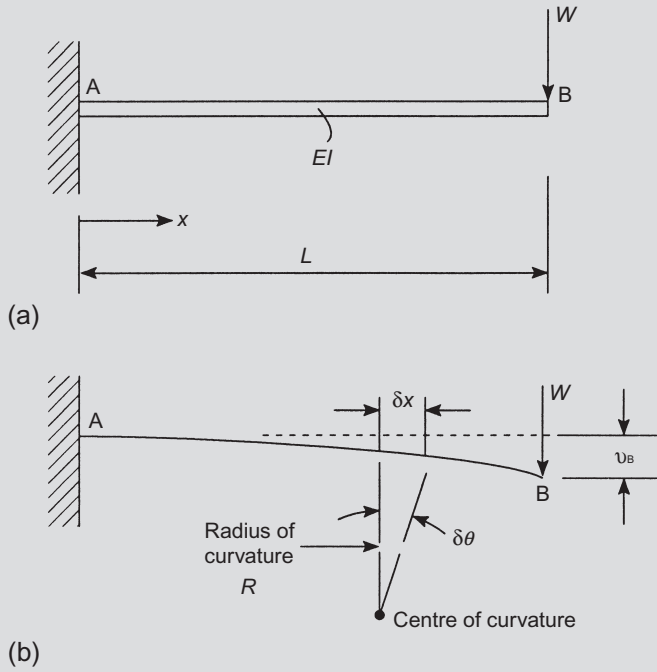
Calculate the deflection, v_B , at the free end of the cantilever beam shown in Fig. 15.21(a).

We shall assume that deflections due to shear are negligible so that v_B is entirely due to bending action in the beam. In this case the total complementary energy of the beam is, from Eq. (15.39)

$$C = \int_0^L \int_0^M d\theta dM - W_{v_B} \quad (\text{i})$$

in which M is the bending moment acting on an element, δx , of the beam; δx subtends a small angle, $\delta\theta$, at the centre of curvature of the beam. The radius of curvature of the beam at the section is R as shown in Fig. 15.21(b) where, for clarity, we represent the beam by its neutral plane. From the principle of the stationary value of the total complementary energy of the beam

$$\frac{\partial C}{\partial W} = \int_0^L \frac{\partial M}{\partial W} d\theta - v_B = 0$$

**FIGURE 15.21**

Deflection of a cantilever beam using complementary energy.

whence

$$v_B = \int_0^L \frac{\partial M}{\partial W} d\theta \quad (\text{ii})$$

In Eq. (ii)

$$\delta\theta = \frac{\delta x}{R}$$

and from Eq. (9.11)

$$-\frac{1}{R} = \frac{M}{EI}$$

(here the curvature is negative since the centre of curvature is below the beam) so that

$$\delta\theta = -\frac{M}{EI} \delta x$$

Substituting in Eq. (ii) for $\delta\theta$ we have

$$v_B = - \int_0^L \frac{M}{EI} \frac{\partial M}{\partial W} dx \quad (\text{iii})$$

From Fig. 15.21(a) we see that

$$M = -W(L-x)$$

Hence

$$\frac{\partial M}{\partial W} = -(L - x)$$

Note: Equation (iii) could have been obtained directly from Eq. (9.21) by using Castigliano's first theorem (Part II).

Equation (iii) then becomes

$$v_B = - \int_0^L \frac{W}{EI} (L - x)^2 dx$$

whence

$$v_B = - \frac{WL^3}{3EI} \quad (\text{as in Ex 13.1})$$

(Note that v_B is downwards and therefore negative according to our sign convention.)

EXAMPLE 15.14

Determine the deflection, v_B , of the free end of a cantilever beam carrying a uniformly distributed load of intensity w . The beam is represented in Fig. 15.22 by its neutral plane; the flexural rigidity of the beam is EI .

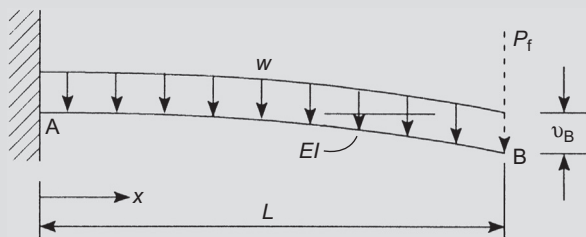


FIGURE 15.22

Deflection of a cantilever beam using the dummy load method.

For this example we use the dummy load method to determine v_B since we require the deflection at a point which does not coincide with the position of a concentrated load; thus we apply a dummy load, P_f , at B as shown. The total complementary energy, C , of the beam includes that produced by the uniformly distributed load; thus

$$C = \int_0^L \int_0^M d\theta dM - P_f v_B - \int_0^L v w dx \quad (i)$$

in which v is the displacement of an elemental length, δx , of the beam at any distance x from the built-in end. Then

$$\frac{\partial C}{\partial P_f} = \int_0^L d\theta \frac{\partial M}{\partial P_f} - v_B = 0$$

so that

$$v_B = \int_0^L d\theta \frac{\partial M}{\partial P_f} \quad (ii)$$

Note that in Eq. (i) v is an actual displacement and w an actual load, so that the last term disappears when C is partially differentiated with respect to P_f . As in Ex. 15.13

$$\delta\theta = -\frac{M}{EI}\delta x$$

Also

$$M = -P_f(L-x) - \frac{w}{2}(L-x)^2$$

in which P_f is imaginary and therefore disappears when we substitute for M in Eq. (ii). Then

$$\frac{\partial M}{\partial P_f} = -(L-x)$$

so that

$$v_B = -\int_0^L \frac{w}{2EI}(L-x)^3 dx$$

whence

$$v_B = -\frac{wL^4}{8EI} \quad (\text{see Ex. 13.2})$$

For a linearly elastic system the bending moment, M_f , produced by a dummy load, P_f , may be written as

$$M_f = \frac{\partial M}{\partial P_f} P_f$$

If $P_f = 1$, i.e. a *unit load*

$$M_f = \frac{\partial M}{\partial P_f} 1$$

so that $\partial M / \partial P_f = M_1$, the bending moment due to a unit load applied at the point and in the direction of the required deflection. Thus we could write an equation for deflection, such as Eq. (ii), in the form

$$v = \int_0^L \frac{M_A M_1}{EI} dx \quad (\text{see Eq. (ii) of Ex. 15.7}) \quad (\text{iii})$$

in which M_A is the actual bending moment at any section of the beam and M_1 is the bending moment at any section of the beam due to a unit load applied at the point and in the direction of the required deflection. Thus, in this example

$$M_A = -\frac{w}{2}(L-x)^2 \quad M_1 = -1(L-x)$$

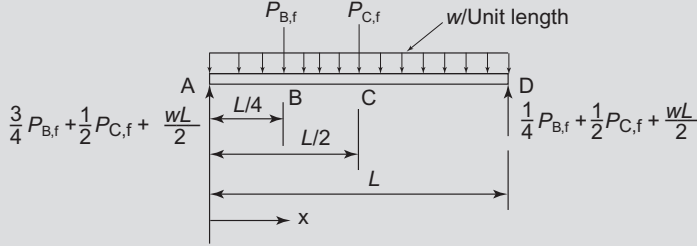
so that

$$v_B = \int_0^L \frac{w}{2EI}(L-x)^3 dx$$

as before. Here v_B is positive which indicates that it is in the same direction as the unit load.

EXAMPLE 15.15

Calculate the vertical displacements at the quarter and mid-span points B and C in the simply supported beam shown in Fig. 15.23. The flexural rigidity of the beam is EI .

**FIGURE 15.23**

Beam deflection of Ex. 15.15.

We apply fictitious loads, $P_{B,f}$ and $P_{C,f}$ at the quarter and mid-span points. The total complementary energy, C , of the system including the fictitious loads is then

$$C = \int_L \int_0^M d\theta dM - P_{B,f} \Delta_B - P_{C,f} \Delta_C - \int_0^L \Delta w dx \quad (i)$$

Hence

$$\frac{\partial C}{\partial P_{B,f}} = \int_L d\theta \frac{\partial M}{\partial P_{B,f}} - \Delta_B = 0 \quad (ii)$$

and

$$\frac{\partial C}{\partial P_{C,f}} = \int_L d\theta \frac{\partial M}{\partial P_{C,f}} - \Delta_C = 0 \quad (iii)$$

Assuming a linearly elastic beam Eqs. (ii) and (iii) become

$$\Delta_B = \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial P_{B,f}} dx \quad (iv)$$

and

$$\Delta_C = \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial P_{C,f}} dx \quad (v)$$

From A to B

$$M = \left(\frac{3}{4} P_{B,f} + \frac{1}{2} P_{C,f} + \frac{wL}{2} \right) x - \frac{wx^2}{2}$$

so that

$$\frac{\partial M}{\partial P_{B,f}} = \frac{3}{4} x, \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2} x$$

From B to C

$$M = \left(\frac{3}{4} P_{B,f} + \frac{1}{2} P_{C,f} + \frac{wL}{2} \right) x - \frac{wx^2}{2} - P_{B,f} \left(x - \frac{L}{4} \right)$$

giving

$$\frac{\partial M}{\partial P_{B,f}} = \frac{1}{4}(L-x), \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2}x$$

From C to D

$$M = \left(\frac{1}{4}P_{B,f} + \frac{1}{2}P_{C,f} + \frac{wL}{2} \right) (L-x) - \frac{w}{2}(L-x)^2$$

so that

$$\frac{\partial M}{\partial P_{B,f}} = \frac{1}{4}(L-x), \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2}(L-x)$$

Substituting these values in Eqs. (iv) and (v) and remembering that $P_{B,f} = P_{C,f} = 0$, we have, from Eq. (iv)

$$\begin{aligned} \Delta_B = \frac{1}{EI} & \left\{ \int_0^{L/4} \left(\frac{wLx}{2} - \frac{wx^2}{2} \right) \frac{3x}{4} dx + \int_{L/4}^{L/2} \left(\frac{wLx}{2} - \frac{wx^2}{2} \right) \frac{1}{4}(L-x) dx \right. \\ & \left. + \int_{L/2}^L \left(\frac{wLx}{2} - \frac{wx^2}{2} \right) \frac{1}{4}(L-x) dx \right\} \end{aligned}$$

from which

$$\Delta_B = \frac{57wL^4}{6144EI}$$

Similarly

$$\Delta_C = \frac{5wL^4}{384EI}$$

which is the result obtained by the double integration method in Ex. 13.4.

EXAMPLE 15.16

Use the principle of the stationary value of the total complementary energy of a system to calculate the horizontal displacement of the point C in the frame of Ex. 15.10.

Referring to Fig. 15.16 we apply a horizontal fictitious load, $P_{C,f}$ at C. The total complementary energy of the system, including the fictitious load, is given by

$$C = \int_L \int_0^M d\theta dM - P_{C,f} \Delta_{C,h} - \int_0^{2L} \Delta w dx - W \Delta_{C,h} \quad (i)$$

where $\Delta_{C,h}$ is the actual horizontal displacement of C. Then

$$\frac{\partial C}{\partial P_{C,f}} = \int_L d\theta \frac{\partial M}{\partial P_{C,f}} - \Delta_{C,h} = 0 \quad (ii)$$

Then

$$\Delta_{C,h} = \frac{1}{EI} \int_0^{2L} M \frac{\partial M}{\partial P_{C,f}} dx + \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial P_{C,f}} dy \quad (\text{iii})$$

In AB

$$M = (W + P_{C,f})L - (wx^2/2)$$

so that

$$\frac{\partial M}{\partial P_{C,f}} = L$$

In BC

$$M = (W + P_{C,f})y$$

so that

$$\frac{\partial M}{\partial P_{C,f}} = y$$

Substituting these expressions in Eq. (iii) and remembering that $P_{C,f} = 0$ we have

$$\Delta_{C,h} = \frac{1}{2EI} \int_0^{2L} [WL - (wx^2/2)]L dx + \frac{1}{EI} \int_0^L Wy^2 dy \quad (\text{iv})$$

Integrating Eq. (iv) and substituting the limits gives

$$\Delta_{C,h} = (2L^3/3EI)(2W - wL)$$

which is the solution produced in Ex. 15.10.

Temperature effects

The principle of the stationary value of the total complementary energy in conjunction with the unit load method may be used to determine the effect of a temperature gradient through the depth of a beam.

Normally, if a structural member is subjected to a uniform temperature rise, t , it will expand as shown in Fig. 15.24. However, a variation in temperature through the depth of the member such as the linear variation shown in Fig. 15.25(b) causes the upper fibres to expand more than the lower ones so that bending strains, without bending stresses, are induced as shown in Fig. 15.25(a). Note that the undersurface of the member is unstrained since the change in temperature in this region is zero.

Consider an element, δx , of the member. The upper surface will increase in length to $\delta x(1 + \alpha t)$, while the length of the lower surface remains equal to δx as shown in Fig. 15.25(c); α is the coefficient of linear expansion of the material of the member. Thus, from Fig. 15.25(c)

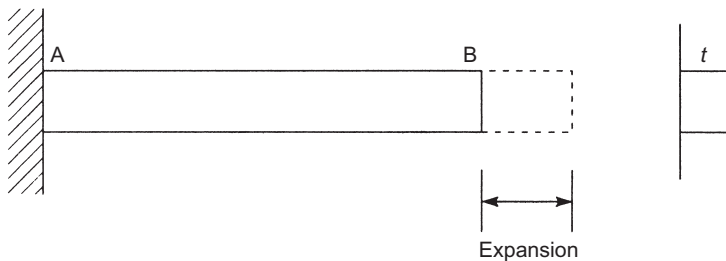


FIGURE 15.24

Expansion of a member due to a uniform temperature rise.

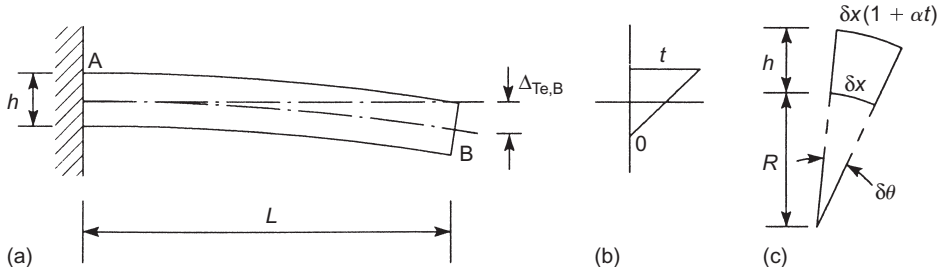


FIGURE 15.25

Bending of a beam due to a linear temperature gradient.

$$\frac{R}{\delta x} = \frac{R + h}{\delta x(1 + \alpha t)}$$

so that

$$R = \frac{h}{\alpha t}$$

Also

$$\delta \theta = \frac{\delta x}{R}$$

whence

$$\delta \theta = \frac{\alpha t \delta x}{h} \quad (15.41)$$

If we require the deflection, $\Delta_{T_e, B}$, of the free end of the member due to the temperature rise, we can employ the unit load method as in Ex. 15.14. Thus, by comparison with Eq. (ii) in Ex. 15.14

$$\Delta_{T_e, B} = \int_0^L d\theta \frac{\partial M}{\partial P_f} \quad (15.42)$$

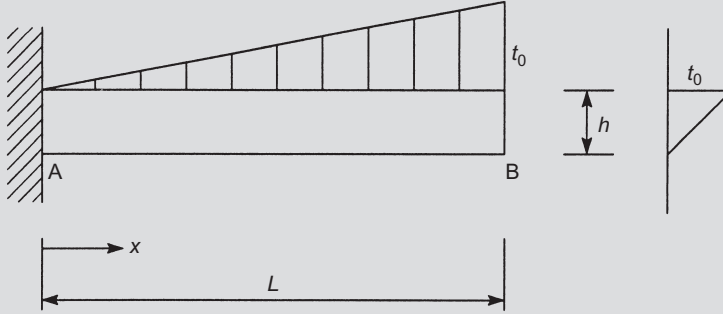
in which, as we have seen, $\partial M / \partial P_f = M_1$, the bending moment at any section of the member produced by a unit load acting vertically downwards at B. Now substituting for $\delta \theta$ in Eq. (15.42) from Eq. (15.41)

$$\Delta_{T_e, B} = - \int_0^L M_1 \frac{\alpha t}{h} dx \quad (15.43)$$

In the case of a beam carrying actual external loads the total deflection is, from the principle of superposition (Section 3.7), the sum of the bending, shear (unless neglected) and temperature deflections. Note that in Eq. (15.43) t can vary arbitrarily along the length of the beam but only linearly with depth. Note also that the temperature gradient shown in Fig. 15.25(b) produces a hogging deflected shape for the member. Thus, strictly speaking, the radius of curvature, R , in the derivation of Eq. (15.41) is negative (compare with Fig. 9.4) so that we must insert a minus sign in Eq. (15.43) as shown.

EXAMPLE 15.17

Determine the deflection of the free end of the cantilever beam in Fig. 15.26 when subjected to the temperature gradients shown.

**FIGURE 15.26**

Deflection of a cantilever beam having linear lengthwise and depthwise temperature gradients.

The temperature, t , at any section x of the beam is given by

$$t = \frac{x}{L} t_0$$

Thus, substituting for t in Eq. (15.43), which applies since the variation of temperature through the depth of the beam is identical to that in Fig. 15.25(b), and noting that $M_1 = -1(L-x)$ we have

$$\Delta_{T_e, B} = - \int_0^L [-1(L-x)] \frac{\alpha x}{hL} t_0 dx$$

which simplifies to

$$\Delta_{T_e, B} = \frac{\alpha t_0}{hL} \int_0^L (Lx - x^2) dx$$

whence

$$\Delta_{T_e, B} = \frac{\alpha t_0 L^2}{6h}$$

Potential energy

In the spring-mass system shown in its unstrained position in Fig. 15.27(a) the *potential energy* of the mass, m , is defined as the product of its weight and its height, h , above some arbitrary fixed datum. In other words, it possesses energy by virtue of its position. If the mass is allowed to move to the equilibrium position shown in Fig. 15.27(b) it has lost an amount of potential energy $mg \Delta_F$. Thus, deflection is associated with a loss of potential energy or, alternatively, we could say that the loss of potential energy of the mass represents a *negative gain* in potential energy. Thus, if we define the potential energy of the mass as zero in its undeflected position in Fig. 15.27(a), which is the same as taking the position of the datum such that $h=0$, its actual

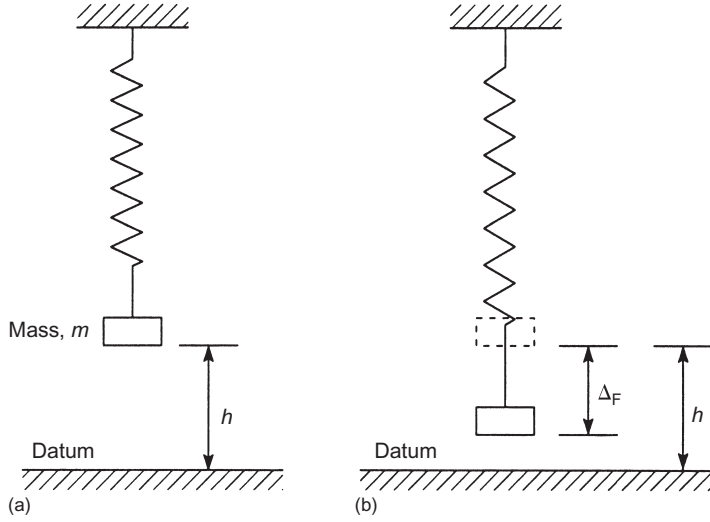


FIGURE 15.27

Potential energy of a spring-mass system.

potential energy in its deflected state in Fig. 15.27(b) is $-mg\Delta_F$. In the deflected state, the total energy of the spring-mass system is the sum of the potential energy of the mass ($-mg\Delta_F$) and the strain energy of the spring.

Applying the above argument to the elastic member in Fig. 15.18(a) and defining the ‘total potential energy’ (TPE) of the member as the sum of the strain energy, U , of the member and the potential energy, V , of the load, we have

$$\text{TPE} = U + V = \int_0^{\Delta_{j,F}} P_j d\Delta_j - P_{j,F} \Delta_{j,F} \quad (\text{see Eq. (15.24)}) \quad (15.44)$$

Thus, for a structure comprising n members and subjected to a system of loads, $P_1, P_2, \dots, P_k, \dots, P_r$, the TPE is given by

$$\text{TPE} = U + V = \sum_{j=1}^n \int_0^{\Delta_{j,F}} P_j d\Delta_j - \sum_{k=1}^r P_k \Delta_k \quad (15.45)$$

in which P_j is the internal force in the j th member, $\Delta_{j,F}$ is its extension or contraction and Δ_k is the displacement of the load, P_k , in its line of action.

The principle of the stationary value of the total potential energy

Let us now consider an elastic body in equilibrium under a series of loads, $P_1, P_2, \dots, P_k, \dots, P_r$, and let us suppose that we impose infinitesimally small virtual displacements, $\delta\Delta_1, \delta\Delta_2, \dots, \delta\Delta_k, \dots, \delta\Delta_r$, at the points of application and in the directions of the loads. The virtual work done by the loads is then

$$\delta W_e = \sum_{k=1}^r P_k \delta\Delta_k \quad (15.46)$$

This virtual work will be accompanied by an increment of virtual strain energy, δU , or internal virtual work since, by imposing virtual displacements at the points of application of the loads we induce

accompanying virtual strains in the body itself. Therefore, from the principle of virtual work (Eq. (15.23)) we have

$$\delta W_e = \delta U$$

or

$$\delta U - \delta W_e = 0$$

Substituting for δW_e from Eq. (15.46) we obtain

$$\delta U - \sum_{k=1}^r P_k \delta \Delta_k = 0 \quad (15.47)$$

which may be written in the form

$$\delta \left(U - \sum_{k=1}^r P_k \Delta_k \right) = 0$$

in which we see that the second term is the potential energy, V , of the applied loads. Hence the equation becomes

$$\delta(U + V) = 0 \quad (15.48)$$

and we see that the TPE of an elastic system has a stationary value for all small displacements if the system is in equilibrium.

It may also be shown that if the stationary value is a minimum, the equilibrium is stable. This may be demonstrated by examining the states of equilibrium of the particle at the positions A, B and C in Fig. 15.28. The TPE of the particle is proportional to its height, h , above some arbitrary datum, u ; note that a single particle does not possess strain energy, so that in this case $\text{TPE} = V$. Clearly, at each position of the particle, the first-order variation, $\partial(U + V)/\partial u$, is zero (indicating equilibrium) but only at B, where the TPE is a minimum, is the equilibrium stable; at A the equilibrium is unstable while at C the equilibrium is neutral.

The *principle of the stationary value of the TPE* may therefore be stated as:

The TPE of an elastic system has a stationary value for all small displacements when the system is in equilibrium; further, the equilibrium is stable if the stationary value is a minimum.

Potential energy can often be used in the approximate analysis of structures in cases where an exact analysis does not exist. We shall illustrate such an application for a simple beam in Ex. 15.18 below and in

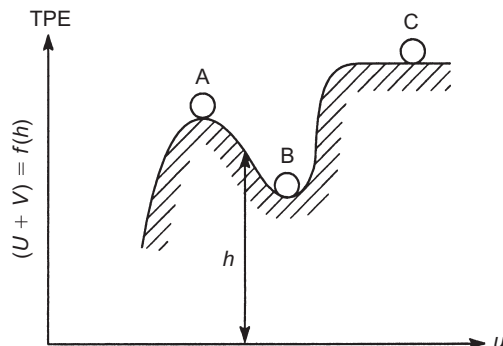


FIGURE 15.28

States of equilibrium of a particle.

Chapter 21 in the case of a buckled column; in both cases we shall suppose that the deflected form is unknown and has to be initially assumed (this approach is called the *Rayleigh–Ritz method*). For a linearly elastic system, of course, the methods of complementary energy and potential energy are identical.

EXAMPLE 15.18

Determine the deflection of the mid-span point of the linearly elastic, simply supported beam ABC shown in Fig. 15.29(a).

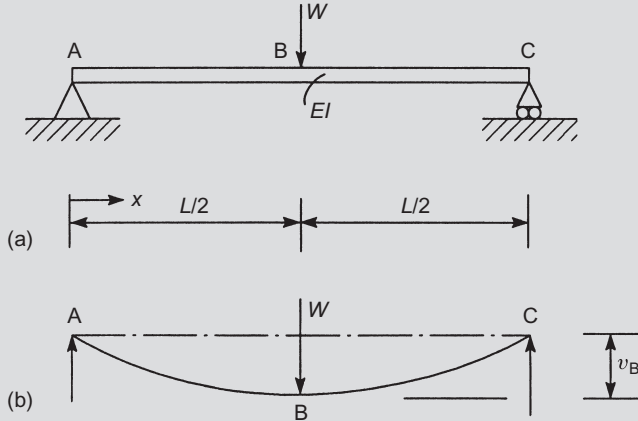


FIGURE 15.29

Approximate value for beam deflection using TPE.

We shall suppose that the deflected shape of the beam is unknown. Initially, therefore, we shall assume a deflected shape that satisfies the boundary conditions for the beam. Generally, trigonometric or polynomial functions have been found to be the most convenient where the simpler the function the less accurate the solution. Let us suppose that the displaced shape of the beam is given by

$$v = v_B \sin \frac{\pi x}{L} \quad (\text{i})$$

in which v_B is the deflection at the mid-span point. From Eq. (i) we see that when $x=0$ and $x=L$, $v=0$ and that when $x=L/2$, $v=v_B$. Furthermore, $dv/dx = (\pi/L)v_B \cos(\pi x/L)$ which is zero when $x=L/2$. Thus the displacement function satisfies the boundary conditions of the beam.

The strain energy due to bending of the beam is given by Eq. (9.21), i.e.

$$U = \int_0^L \frac{M^2}{2EI} dx \quad (\text{ii})$$

Also, from Eq. (13.3)

$$M = EI \frac{d^2 v}{dx^2} \quad (\text{iii})$$

Substituting in Eq. (iii) for v from Eq. (i), and for M in Eq. (ii) from Eq. (iii), we have

$$U = \frac{EI}{2} \int_0^L \frac{v_B^2 \pi^4}{L^4} \sin^2 \frac{\pi x}{L} dx$$

which gives

$$U = \frac{\pi^4 EI v_B^2}{4L^3}$$

The TPE of the beam is then given by

$$\text{TPE} = U + V = \frac{\pi^4 EI v_B^2}{4L^3} - W v_B$$

Hence, from the principle of the stationary value of the TPE

$$\frac{\partial(U + V)}{\partial v_B} = \frac{\pi^4 EI v_B}{2L^3} - W = 0$$

whence

$$v_B = \frac{2WL^3}{\pi^4 EI} = 0.02053 \frac{WL^3}{EI} \quad (\text{iv})$$

The exact expression for the deflection at the mid-span point was found in Ex. 13.5 and is

$$v_B = \frac{WL^3}{48EI} = 0.02083 \frac{WL^3}{EI} \quad (\text{v})$$

Comparing the exact and approximate results we see that the difference is less than 2%. Furthermore, the approximate deflection is less than the exact deflection because, by assuming a deflected shape, we have, in effect, forced the beam into that shape by imposing restraints; the beam is therefore stiffer.

15.4 Reciprocal theorems

There are two reciprocal theorems: one, attributed to Maxwell, is the *theorem of reciprocal displacements* (often referred to as Maxwell's reciprocal theorem) and the other, derived by Betti and Rayleigh, is the *theorem of reciprocal work*. We shall see, in fact, that the former is a special case of the latter. We shall also see that their proofs rely on the principle of superposition (Section 3.7) so that their application is limited to linearly elastic structures.

Theorem of reciprocal displacements

In a linearly elastic body a load, P_1 , applied at a point 1 will produce a displacement, Δ_1 , at the point and in its own line of action given by

$$\Delta_1 = a_{11}P_1$$

in which a_{11} is a *flexibility coefficient* which is defined as the displacement at the point 1 in the direction of P_1 produced by a unit load at the point 1 in the direction of P_1 . It follows that if the elastic body is subjected to a series of loads, $P_1, P_2, \dots, P_k, \dots, P_r$, each of the loads will contribute to the displacement of point 1. Thus the corresponding displacement, Δ_1 , at the point 1 (i.e. the total displacement in the direction of P_1 produced by all the loads) is then

$$\Delta_1 = a_{11}P_1 + a_{12}P_2 + \dots + a_{1k}P_k + \dots + a_{1r}P_r$$

in which a_{12} is the displacement at the point 1 in the direction of P_1 produced by a unit load at 2 in the direction of P_2 , and so on. The corresponding displacements at the points of application of the loads are then

$$\left. \begin{aligned} \Delta_1 &= a_{11}P_1 + a_{12}P_2 + \dots + a_{1k}P_k + \dots + a_{1r}P_r \\ \Delta_2 &= a_{21}P_1 + a_{22}P_2 + \dots + a_{2k}P_k + \dots + a_{2r}P_r \\ &\vdots \\ \Delta_k &= a_{k1}P_1 + a_{k2}P_2 + \dots + a_{kk}P_k + \dots + a_{kr}P_r \\ &\vdots \\ \Delta_r &= a_{r1}P_1 + a_{r2}P_2 + \dots + a_{rk}P_k + \dots + a_{rr}P_r \end{aligned} \right\} \quad (15.49)$$

or, in matrix form

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_k \\ \vdots \\ \Delta_r \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2k} & \dots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} & \dots & a_{kr} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rk} & \dots & a_{rr} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \\ \vdots \\ P_r \end{Bmatrix} \quad (15.50)$$

which may be written in matrix shorthand notation as

$$\{\Delta\} = [A]\{P\}$$

Suppose now that a linearly elastic body is subjected to a gradually applied load, P_1 , at a point 1 and then, while P_1 remains in position, a load P_2 is gradually applied at another point 2. The total strain energy, U_1 , of the body is equal to the external work done by the loads; thus

$$U_1 = \frac{P_1}{2}(a_{11}P_1) + \frac{P_2}{2}(a_{22}P_2) + P_1(a_{12}P_2) \quad (15.51)$$

The third term on the right-hand side of Eq. (15.51) results from the additional work done by P_1 as it is displaced through a further distance $a_{12}P_2$ by the action of P_2 . If we now remove the loads and then apply P_2 followed by P_1 , the strain energy, U_2 , is given by

$$U_2 = \frac{P_2}{2}(a_{22}P_2) + \frac{P_1}{2}(a_{11}P_1) + P_2(a_{21}P_1) \quad (15.52)$$

By the principle of superposition the strain energy of the body is independent of the order in which the loads are applied. Hence

$$U_1 = U_2$$

so that

$$a_{12} = a_{21} \quad (15.53)$$

Thus, in its simplest form, the theorem of reciprocal displacements states that:

The displacement at a point 1 in a given direction due to a unit load at a point 2 in a second direction is equal to the displacement at the point 2 in the second direction due to a unit load at the point 1 in the given direction.

The theorem of reciprocal displacements may also be expressed in terms of moments and rotations. Thus:

The rotation at a point 1 due to a unit moment at a point 2 is equal to the rotation at the point 2 produced by a unit moment at the point 1.

Finally we have:

The rotation in radians at a point 1 due to a unit load at a point 2 is numerically equal to the displacement at the point 2 in the direction of the unit load due to a unit moment at the point 1.

EXAMPLE 15.19

A cantilever 800 mm long with a prop 500 mm from its built-in end deflects in accordance with the following observations when a concentrated load of 40 kN is applied at its free end:

Distance from fixed end (mm)	0	100	200	300	400	500	600	700	800
Deflection (mm)	0	0.3	1.4	2.5	1.9	0	-2.3	-4.8	-10.6

What will be the angular rotation of the beam at the prop due to a 30 kN load applied 200 mm from the built-in end together with a 10 kN load applied 350 mm from the built-in end?

The initial deflected shape of the cantilever is plotted to a suitable scale from the above observations and is shown in Fig. 15.30(a). Thus, from Fig. 15.30(a) we see that the deflection at D due to a 40 kN load at C is 1.4 mm. Hence the deflection at C due to a 40 kN load at D is, from the reciprocal theorem, 1.4 mm. It follows that the deflection at C due to a 30 kN load at D is equal to $(3/4) \times (1.4) = 1.05$ mm. Again, from Fig. 15.30(a), the deflection at E due to a 40 kN load at C is 2.4 mm. Thus the deflection at C due to a 10 kN load at E is equal to $(1/4) \times (2.4) = 0.6$ mm. Therefore the total deflection at C due to a 30 kN load at D and a 10 kN load at E is $1.05 + 0.6 = 1.65$ mm. From Fig. 15.30(b) we see that the rotation of the beam at B is given by

$$\theta_B = \tan^{-1} \left(\frac{1.65}{300} \right) = \tan^{-1} (0.0055)$$

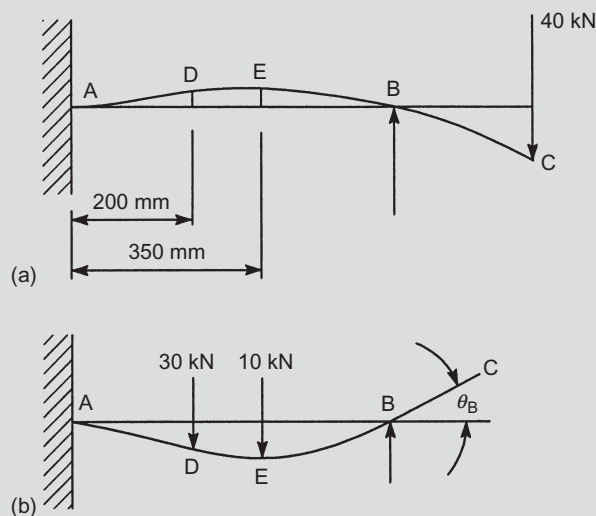


FIGURE 15.30

Deflection of a propped cantilever using the reciprocal theorem.

or

$$\theta_B = 0^\circ 19'$$

EXAMPLE 15.20

An elastic member is pinned to a drawing board at its ends A and B. When a moment, M , is applied at A, A rotates by θ_A , B rotates by θ_B and the centre deflects by δ_1 . The same moment, M , applied at B rotates B by θ_C and deflects the centre through δ_2 . Find the moment induced at A when a load, W , is applied to the centre in the direction of the measured deflections, and A and B are restrained against rotation.

The three load conditions and the relevant displacements are shown in Fig. 15.31. Thus, from Fig. 15.31(a) and (b) the rotation at A due to M at B is, from the reciprocal theorem, equal to the rotation at B due to M at A.

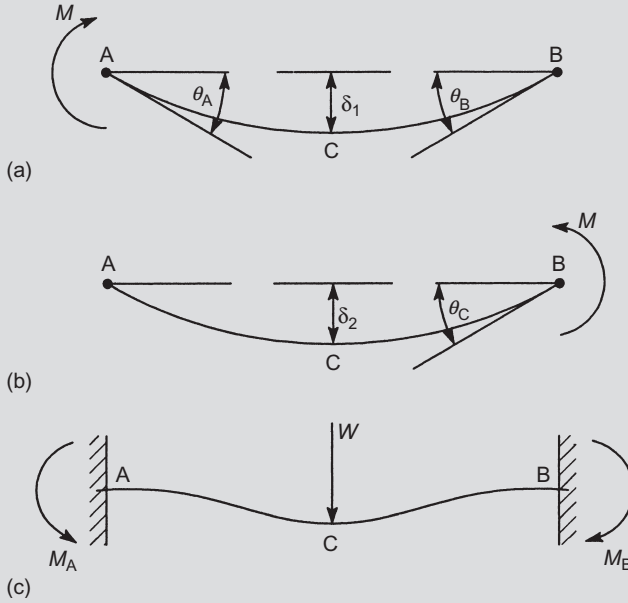


FIGURE 15.31

Model analysis of a fixed beam.

Thus

$$\theta_{A(b)} = \theta_B$$

It follows that the rotation at A due to M_B at B is

$$\theta_{A(c),1} = \frac{M_B}{M} \theta_B \quad (i)$$

where (b) and (c) refer to (b) and (c) in Fig. 15.31.

Also, the rotation at A due to a unit load at C is equal to the deflection at C due to a unit moment at A. Therefore

$$\frac{\theta_{A(c),2}}{W} = \frac{\delta_1}{M}$$

or

$$\theta_{A(c),2} = \frac{W}{M} \delta_1 \quad (ii)$$

in which $\theta_{A(c),2}$ is the rotation at A due to W at C. Finally the rotation at A due to M_A at A is, from Fig. 15.31(a) and (c)

$$\theta_{A(c),3} = \frac{M_A}{M} \theta_A \quad (\text{iii})$$

The total rotation at A produced by M_A at A, W at C and M_B at B is, from Eqs (i), (ii) and (iii)

$$\theta_{A(c),1} + \theta_{A(c),2} + \theta_{A(c),3} = \frac{M_B}{M} \theta_B + \frac{W}{M} \delta_1 + \frac{M_A}{M} \theta_A = 0 \quad (\text{iv})$$

since the end A is restrained against rotation. In a similar manner the rotation at B is given by

$$\frac{M_B}{M} \theta_C + \frac{W}{M} \delta_2 + \frac{M_A}{M} \theta_B = 0 \quad (\text{v})$$

Solving Eqs (iv) and (v) for M_A gives

$$M_A = W \left(\frac{\delta_2 \theta_B - \delta_1 \theta_C}{\theta_A \theta_C - \theta_B^2} \right)$$

The fact that the arbitrary moment, M , does not appear in the expression for the restraining moment at A (similarly it does not appear in M_B) produced by the load W indicates an extremely useful application of the reciprocal theorem, namely the model analysis of statically indeterminate structures. For example, the fixed beam of Fig. 15.31(c) could possibly be a full-scale bridge girder. It is then only necessary to construct a model, say, of perspex, having the same flexural rigidity, EI , as the full-scale beam and measure rotations and displacements produced by an arbitrary moment, M , to obtain the fixed-end moments in the full-scale beam supporting a full-scale load.

Theorem of reciprocal work

Let us suppose that a linearly elastic body is to be subjected to two systems of loads, $P_1, P_2, \dots, P_k, \dots, P_r$, and, $Q_1, Q_2, \dots, Q_i, \dots, Q_m$, which may be applied simultaneously or separately. Let us also suppose that corresponding displacements are $\Delta_{P,1}, \Delta_{P,2}, \dots, \Delta_{P,k}, \dots, \Delta_{P,r}$ due to the loading system, P , and $\Delta_{Q,1}, \Delta_{Q,2}, \dots, \Delta_{Q,i}, \dots, \Delta_{Q,m}$ due to the loading system, Q . Finally, let us suppose that the loads, P , produce displacements $\Delta'_{Q,1}, \Delta'_{Q,2}, \dots, \Delta'_{Q,i}, \dots, \Delta'_{Q,m}$ at the points of application and in the direction of the loads, Q , while the loads, Q , produce displacements $\Delta'_{P,1}, \Delta'_{P,2}, \dots, \Delta'_{P,k}, \dots, \Delta'_{P,r}$ at the points of application and in the directions of the loads, P .

Now suppose that the loads P and Q are applied to the elastic body gradually and simultaneously. The total work done, and hence the strain energy stored, is then given by

$$\begin{aligned} U_1 = & \frac{1}{2} P_1 (\Delta_{P,1} + \Delta'_{P,1}) + \frac{1}{2} P_2 (\Delta_{P,2} + \Delta'_{P,2}) + \dots + \frac{1}{2} P_K (\Delta_{P,k} + \Delta'_{P,k}) \\ & + \dots + \frac{1}{2} P_r (\Delta_{P,r} + \Delta'_{P,r}) + \frac{1}{2} Q_1 (\Delta_{Q,1} + \Delta'_{Q,1}) + \frac{1}{2} Q_2 (\Delta_{Q,2} + \Delta'_{Q,2}) \\ & + \dots + \frac{1}{2} Q_i (\Delta_{Q,i} + \Delta'_{Q,i}) + \dots + \frac{1}{2} Q_m (\Delta_{Q,m} + \Delta'_{Q,m}) \end{aligned} \quad (15.54)$$

If now we apply the P -loading system followed by the Q -loading system, the total strain energy stored is

$$U_2 = \frac{1}{2}P_1\Delta_{P,1} + \frac{1}{2}P_2\Delta_{P,2} + \dots + \frac{1}{2}P_k\Delta_{P,k} + \dots + \frac{1}{2}P_r\Delta_{P,r} + \frac{1}{2}Q_1\Delta_{Q,1} + \frac{1}{2}Q_2\Delta_{Q,2} \\ + \dots + \frac{1}{2}Q_i\Delta_{Q,i} + \dots + \frac{1}{2}Q_m\Delta_{Q,m} + P_1\Delta'_{P,1} + P_2\Delta'_{P,2} + P_k\Delta'_{P,k} + \dots + P_r\Delta'_{P,r} \quad (15.55)$$

Since, by the principle of superposition, the total strain energies, U_1 and U_2 , must be the same, we have from Eqs (15.54) and (15.55)

$$-\frac{1}{2}P_1\Delta'_{P,1} - \frac{1}{2}P_2\Delta'_{P,2} - \dots - \frac{1}{2}P_k\Delta'_{P,k} - \dots - \frac{1}{2}P_r\Delta'_{P,r} \\ = -\frac{1}{2}Q_1\Delta'_{Q,1} - \frac{1}{2}Q_2\Delta'_{Q,2} - \dots - \frac{1}{2}Q_i\Delta'_{Q,i} - \dots - \frac{1}{2}Q_m\Delta'_{Q,m}$$

In other words

$$\sum_{k=1}^r P_k\Delta'_{P,k} = \sum_{i=1}^m Q_i\Delta'_{Q,i} \quad (15.56)$$

The expression on the left-hand side of Eq. (15.56) is the sum of the products of the P loads and their corresponding displacements produced by the Q loads. The right-hand side of Eq. (15.56) is the sum of the products of the Q loads and their corresponding displacements produced by the P loads. Thus the *theorem of reciprocal work* may be stated as:

The work done by a first loading system when moving through the corresponding displacements produced by a second loading system is equal to the work done by the second loading system when moving through the corresponding displacements produced by the first loading system.

Again, as in the theorem of reciprocal displacements, the loading systems may be either forces or moments and the displacements may be deflections or rotations.

If, in the above, the P - and Q -loading systems comprise just two loads, say P_1 and Q_2 , then, from Eq. (15.56), we see that

$$P_1(a_{12}Q_2) = Q_2(a_{21}P_1)$$

so that

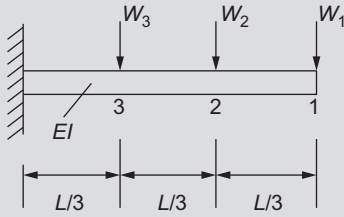
$$a_{12} = a_{21}$$

as in the theorem of reciprocal displacements. Therefore, as stated initially, we see that the theorem of reciprocal displacements is a special case of the theorem of reciprocal work.

In addition to the use of the reciprocal theorems in the model analysis of structures as described in Ex. 15.20, they are used to establish the symmetry of, say, the stiffness matrix in the matrix analysis of some structural systems. We shall examine this procedure in Chapter 16.

EXAMPLE 15.21

The cantilever beam shown in Fig. 15.32 carries load W_2 and W_3 at the two-thirds and one-third span points, respectively. Use the reciprocal theorem and the principle of superposition to determine the value of a load W_1 positioned at the end of the cantilever, which would produce the same vertical displacement at the end.

**FIGURE 15.32**

Cantilever beam of Ex 15.21.

From Eq. (iv) of Ex. 13.1, the equation of the deflection curve of the beam due to a concentrated load W at the free end of the beam is

$$v = -\frac{W}{6EI} (3Lx^2 - x^3) \quad (\text{i})$$

Therefore, the deflection at the free end of the beam due to W_1 is given by

$$\Delta_1 = -\frac{W_1 L^3}{3EI}$$

For convenience (as will be seen), we express this in terms of a denominator 81. Then

$$\Delta_1 = -\frac{27 W_1 L^3}{81EI} = a_{11} W_1 \quad (\text{ii})$$

The deflection at the point 2 due to W_1 is, from Eq. (i)

$$\Delta_2 = -\frac{W_1}{6EI} \left(3L \frac{4L^2}{9} - 8L^3 \right) = -\frac{14 W_1 L^3}{81EI} = a_{21} W_1 \quad (\text{iii})$$

The deflection at the point 3 due to W_1 is, from Eq. (i)

$$\Delta_3 = -\frac{W_1}{6EI} \left(3L \frac{L^2}{9} - \frac{L^3}{27} \right) = \frac{-4 W_1 L^3}{81EI} = a_{31} W_1 \quad (\text{iv})$$

From Section 15.4, the deflection at the point 1 due to W_2 and W_3 is

$$\Delta_1 = a_{12} W_2 + a_{13} W_3 \quad (\text{v})$$

From the reciprocal theorem $a_{12} = a_{21}$ and $a_{13} = a_{31}$. Therefore, from Eqs (ii), (iii), (iv), and (v)

$$-\frac{27 L^3}{81EI} W_1 = -\frac{14 L^3}{81EI} W_2 - \frac{4 L^3}{81EI} W_3,$$

which gives $W_1 = \frac{2}{27} (7 W_2 + 2 W_3)$

PROBLEMS

P.15.1 Use the principle of virtual work to determine the support reactions in the beam ABCD shown in Fig. P.15.1.

Ans. $R_A = 1.25W$, $R_D = 1.75W$.

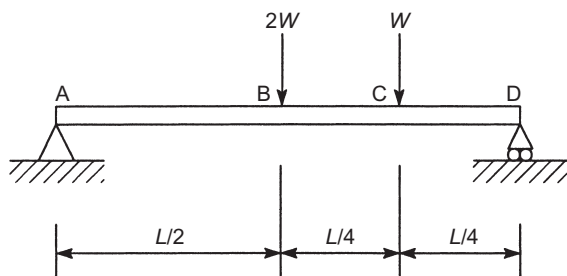


FIGURE P.15.1

P.15.2 Find the support reactions in the beam ABC shown in Fig. P.15.2 using the principle of virtual work.

Ans. $R_A = (W + 2wL)/4$, $R_C = (3W + 2wL)/4$.

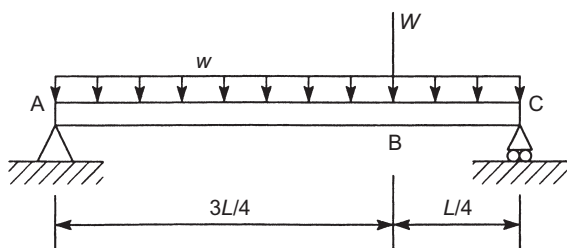


FIGURE P.15.2

P.15.3 Determine the reactions at the built-in end of the cantilever beam ABC shown in Fig. P.15.3 using the principle of virtual work.

Ans. $R_A = 3W$, $M_A = 2.5WL$.

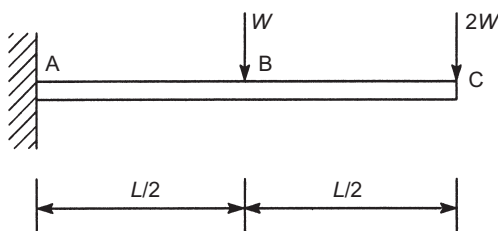


FIGURE P.15.3

P.15.4 Find the bending moment at the three-quarter-span point in the beam shown in Fig. P.15.4. Use the principle of virtual work.

Ans. $3wL^2/32$.

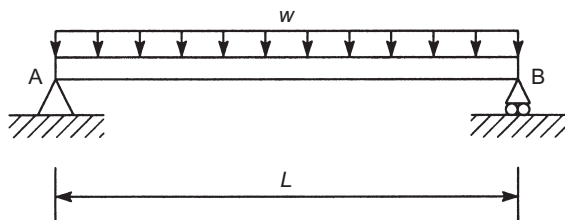


FIGURE P.15.4

P.15.5 Calculate the forces in the members FG, GD and CD of the truss shown in Fig. P.15.5 using the principle of virtual work. All horizontal and vertical members are 1 m long.

Ans. $FG = +20$ kN, $GD = +28.3$ kN, $CD = -20$ kN.

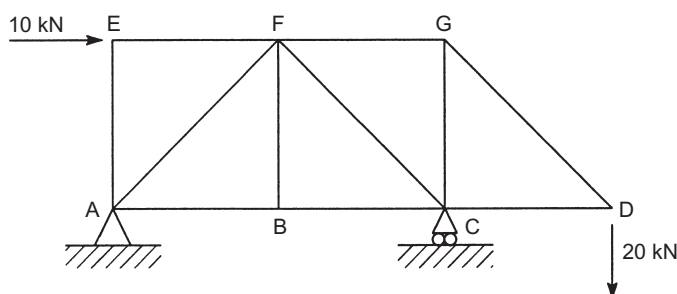


FIGURE P.15.5

P.15.6 The frame shown in Fig. P.15.6 consists of a cranked beam simply supported at A and F and reinforced by a tie bar pinned to the beam at B and E. The beam carries a uniformly distributed load of intensity, w , over the outer parts AB and EF. Considering the effects of bending and axial load determine the axial force in the tie bar and the bending moments at B and C. The bending stiffness of the beam is EI and its cross sectional area is $3A$ while the corresponding values for the tie bar are EI and A .

Ans. Force in tie bar $T = 9wL^3/8[L^2 + (4I/A)]$. M (at B) $= wL^2$ (anticlockwise),
 M (at C) $= (TL - wL^2)/2$ (clockwise).

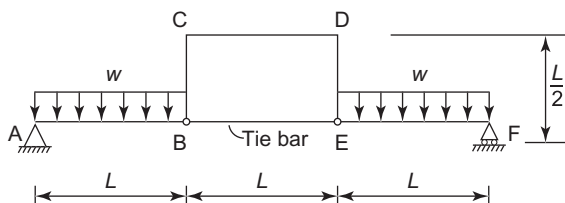


FIGURE P.15.6

P.15.7 The flat tension spring shown in Fig. P.15.7 consists of a length of wire of circular cross section, having a diameter, d , and Young's modulus, E . The spring consists of n open loops each of which subtends an angle of $3\pi/2$ radians at its centre; the length between the ends of the spring is L . Considering bending and axial strains only calculate the stiffness of the spring.

Ans. $(\sqrt{2}) \pi E d^2 / L \{ [48L^2(\pi+1)/n^2 d^2] + (3\pi-2) \}$.

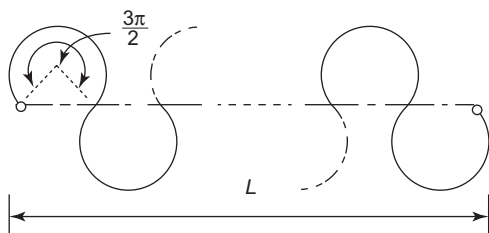


FIGURE P.15.7

P.15.8 Use the unit load method to calculate the vertical displacements at the quarter and mid-span points in the beam shown in Fig. P.15.8.

Ans. $57wL^4/6144EI$, $5wL^4/384EI$. (both downwards)

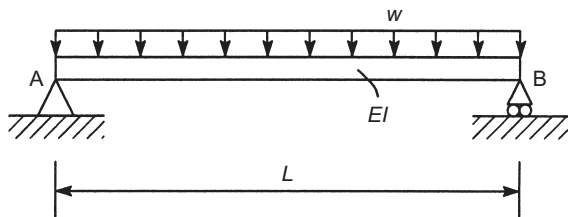


FIGURE P.15.8

P.15.9 Calculate the deflection of the free end C of the cantilever beam ABC shown in Fig. P.15.9 using the unit load method.

Ans. $wa^3(4L-a)/24EI$. (downwards)

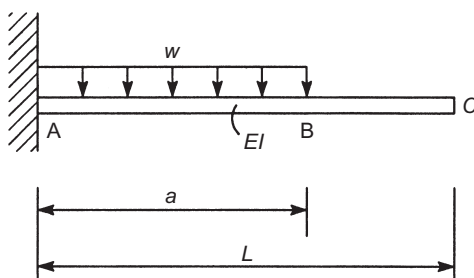


FIGURE P.15.9

P.15.10 Use the unit load method to calculate the deflection at the free end of the cantilever beam ABC shown in Fig. P.15.10.

Ans. $3WL^3/8EI$. (downwards)

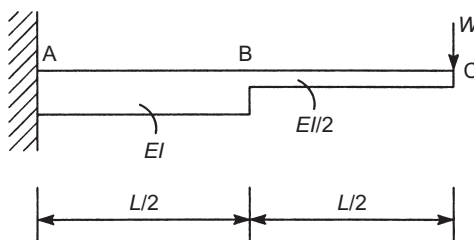


FIGURE P.15.10

- P.15.11** Use the unit load method to find the magnitude and direction of the deflection of the joint C in the truss shown in Fig. P.15.11. All members have a cross-sectional area of 500 mm^2 and a Young's modulus of $200\,000 \text{ N/mm}^2$.

Ans. 23.4 mm, 9.8° to left of vertical.

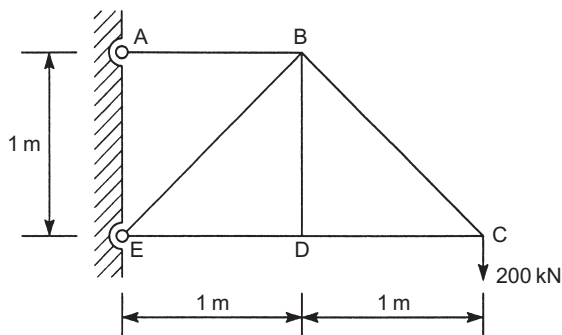


FIGURE P.15.11

- P.15.12** Calculate the magnitude and direction of the deflection of the joint A in the truss shown in Fig. P.15.12. The cross-sectional area of the compression members is 1000 mm^2 while that of the tension members is 750 mm^2 . Young's modulus is $200\,000 \text{ N/mm}^2$.

Ans. 30.3 mm, 10.5° to right of vertical.

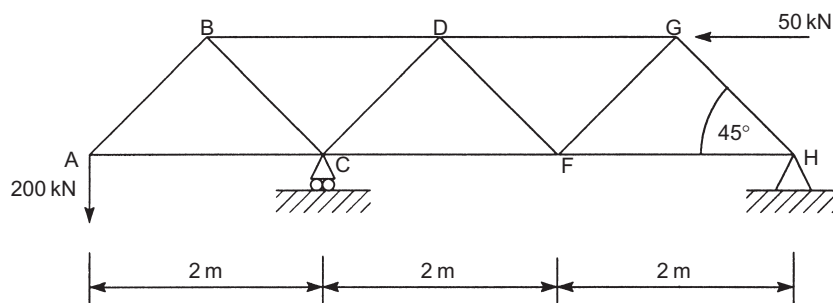


FIGURE P.15.12

- P.15.13** A rigid triangular plate is suspended from a horizontal plane by three vertical wires attached to its corners. The wires are each 1 mm diameter, 1440 mm long with a modulus of elasticity of $196\,000 \text{ N/mm}^2$. The ratio of the lengths of the sides of the plate is 3:4:5. Calculate the deflection at the point of application of a load of 100 N placed at a point equidistant from the three sides of the plate.

Ans. 0.33 mm.

- P.15.14** The pin-jointed space truss shown in Fig. P.15.14 is pinned to supports 0, 4, 5 and 9 and is loaded by a force P in the x direction and a force $3P$ in the negative y direction at the point 7. Find the rotation of the member 27 about the z axis due to this loading. All members have the same cross-sectional area, A , and Young's modulus, E . (Hint. Calculate the deflections in the x direction of joints 2 and 7.)

Ans. $382P/9AE$.

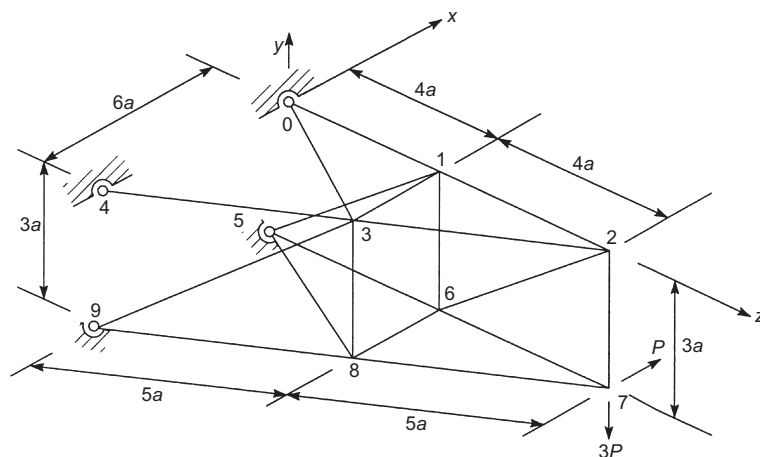


FIGURE P.15.14

- P.15.15** The tubular steel post shown in Fig. P.15.15 carries a load of 250 N at the free end C. The outside diameter of the tube is 100 mm and its wall thickness is 3 mm. If the modulus of elasticity of the steel is $206\,000\text{ N/mm}^2$, calculate the horizontal movement of C.

Ans. 53.3 mm.

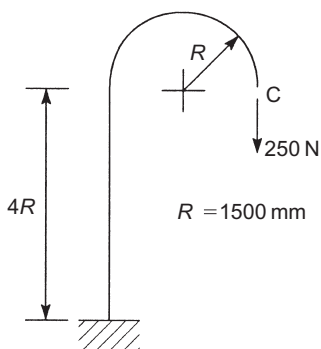


FIGURE P.15.15

- P.15.16** Figure P.15.16 shows a cantilever beam subjected to linearly varying temperature gradients along its length and through its depth. Calculate the deflection at the free end of the beam.

Ans. $2\alpha_0 L^2/3h$.

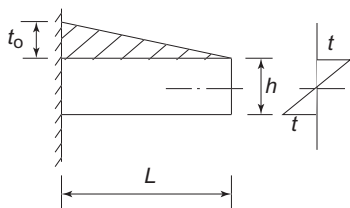


FIGURE P.15.16

- P.15.17** A cantilever beam of length L and depth h is subjected to a uniform temperature rise along its length. At any section, however, the temperature increases linearly from t_1 on the undersurface of the beam to t_2 on its upper surface. If the coefficient of linear expansion of the material of the beam is α , calculate the deflection at its free end.

Ans. $\alpha(t_2 - t_1)L^2/2h$.

- P.15.18** A simply supported beam of span L is subjected to a temperature gradient which increases linearly from zero at the left-hand support to t_0 at the right-hand support. If the temperature gradient also varies linearly through the depth, h , of the beam and is zero on its undersurface, calculate the deflection of the beam at its mid-span point. The coefficient of linear expansion of the material of the beam is α .

Ans. $-\alpha t_0 L^2/48h$.

- P.15.19** Figure P.15.19 shows a frame pinned to supports at A and B. The frame centreline is a circular arc and its section is uniform, of bending stiffness EI and depth d . Find the maximum stress in the frame produced by a uniform temperature gradient through the depth, the temperature on the outer and inner surfaces being raised and lowered by an amount T . The coefficient of linear expansion of the material of the frame is α . (Hint. Treat half the frame as a curved cantilever built-in on its axis of symmetry and determine the horizontal reaction at a support by equating the horizontal deflection produced by the temperature gradient to the horizontal deflection produced by the reaction).

Ans. $1.29ET\alpha$.

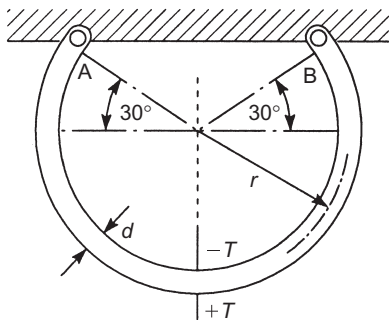


FIGURE P.15.19

- P.15.20** A simply supported beam AB of span L and uniform section carries a distributed load of intensity varying from zero at A to w_0 /unit length at B according to the law

$$w = \frac{2w_0x}{L} \left(1 - \frac{x}{2L} \right)$$

per unit length. If the deflected shape of the beam is given approximately by the expression

$$v = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L}$$

evaluate the coefficients a_1 and a_2 and find the deflection of the beam at mid-span.

Ans. $a_1 = 2w_0L^4(\pi^2 + 4)/EI\pi^7$, $a_2 = -w_0L^4/16EI\pi^5$, $0.00918w_0L^4/EI$.

- P.15.21** A uniform simply supported beam, span L , carries a distributed loading which varies according to a parabolic law across the span. The load intensity is zero at both ends of the beam and w_0 at its mid-point. The loading is normal to a principal axis of the beam cross section and the relevant flexural rigidity is EI . Assuming that the deflected shape of the beam can be represented by the series

$$v = \sum_{i=1}^{\infty} a_i \sin \frac{i\pi x}{L}$$

find the coefficients a_i and the deflection at the mid-span of the beam using only the first term in this series.

Ans. $a_1 = 32w_0L^4/EI\pi^7$ (i odd), $w_0L^4/94.4EI$.

- P.15.22** Calculate the deflection at the mid-span point of the beam of Ex. 15.18 by assuming a deflected shape function of the form

$$v = v_1 \sin \frac{\pi x}{L} + v_3 \sin \frac{3\pi x}{L}$$

in which v_1 and v_3 are unknown displacement parameters. Note:

$$\int_0^L \sin^2 \left(\frac{n\pi x}{L} \right) dx = \frac{L}{2} \quad \int_0^L \sin \left(\frac{m\pi x}{L} \right) \sin \left(\frac{n\pi x}{L} \right) dx = 0$$

Ans. $0.02078 WL^3/EI$.

- P.15.23** A beam is supported at both ends and has the central half of its span reinforced such that its flexural rigidity is $2EI$; the flexural rigidity of the remaining parts of the beam is EI . The beam has a span L and carries a vertically downward concentrated load, W , at its mid-span point. Assuming a deflected shape function of the form

$$v = \frac{4v_m x^2}{L^3} (3L - 4x) \quad (0 \leq x \leq L/2)$$

in which v_m is the deflection at the mid-span point, determine the value of v_m .

Ans. $0.00347 WL^3/EI$.

- P.15.24** Figure P.15.24 shows two cantilevers, the end of one being vertically above the end of the other and connected to it by a spring AB. Initially the system is unstrained. A weight, W , placed at A causes a vertical deflection at A of δ_1 and a vertical deflection at B of δ_2 . When the spring is removed the weight W at A causes a deflection at A of δ_3 . Find the extension of the spring when it is replaced and the weight, W , is transferred to B.

Ans. $\delta_2(\delta_1 - \delta_2)/(\delta_3 - \delta_1)$

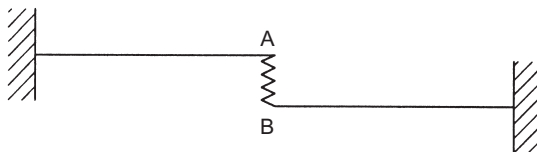


FIGURE P.15.24

- P.15.25** A beam 2.4 m long is simply supported at two points A and B which are 1.44 m apart; point A is 0.36 m from the left-hand end of the beam and point B is 0.6 m from the right-hand end; the value of EI for the beam is $240 \times 10^8 \text{ Nmm}^2$. Find the slope at the supports due to a load of 2 kN applied at the mid-point of AB.

Use the reciprocal theorem in conjunction with the above result to find the deflection at the mid-point of AB due to loads of 3 kN applied at each end of the beam.

Ans. 0.011, 15.8 mm.

- P.15.26** The displacements at three points, 1, 2, and 3 on a beam were measured when loads were applied separately at each point; the flexural rigidity of the beam is EI . Initially, when a load $W_1 = 400 \text{ N}$ was applied at the point 1 the displacements were $\Delta_1 = 0.75 \text{ mm}$, $\Delta_2 = -0.25 \text{ mm}$, and $\Delta_3 = -1.25 \text{ mm}$. Find

- (i) Δ_1 given that $\Delta_2 = 1 \text{ mm}$, $\Delta_3 = 0.875 \text{ mm}$ for $W_2 = 300 \text{ N}$.
- (ii) Δ_1 and Δ_2 given that $\Delta_3 = 2.5 \text{ mm}$ for $W_3 = 500 \text{ N}$.
- (iii) Δ_1 , Δ_2 , and Δ_3 when $W_1 = W_2 = W_3 = 600 \text{ N}$ are applied simultaneously.

Ans. (i) -0.1875 mm , (ii) -1.5625 mm , 1.4583 mm , (iii) -1.125 mm , 3.375 mm , 2.875 mm .