

EQUATIONS OF MOTION 2

Prof. Mark Lowenberg & Prof. Tom Richardson

Email: thomas.richardson@bristol.ac.uk

bristol.ac.uk

<https://thegreatcircle.uk/airducation/spins>



6DOF Equations of Motion

Equations (8), (16), (17) and (18) form the 12 equations of motion for general atmospheric 6 degree-of-freedom flight.

They are all first-order differential equations that can be expressed in nonlinear state-space form as:

$$\dot{x} = f(x, \delta) \quad (19)$$

where $x = [U, V, W, p, q, r, \psi, \theta, \phi, x_E, y_E, z_E]'$ (state vector)

and δ is the vector of input parameters of interest. These are often aerodynamic or propulsion system inputs such as aileron, elevator, rudder deflection or thrust (upon which the formulation of the forces X , Y and Z will depend).

Coordinates and Forces

- The motions that are described are a consequence of the applied forces and moments from outside the body, being:
 - **aerodynamic** (including control forces and gusts)
 - **propulsive** (all additional forces due to propulsion)
 - **gravitational**
- The list above must be all inclusive, e.g. drag due to lowered undercarriage or lift, drag and pitching moment change due to flaps.
- Aerodynamic forces are typically modelled using **aerodynamic data tables** (aerodynamic coefficients and/or stability & control derivatives as function of angle-of-attack, Mach no., etc.)

Coordinates and Forces

- In eqn (8), the three general forces X, Y, Z in the directions of the coordinate axes include those due to aerodynamics, propulsion and gravity.
- When the CG is taken as the axes origin there are no gravity-induced moments so the three moments L, M, N have no gravity terms present.
- Sometimes it is convenient to use only a subset of equations.

For example, for static trim and stability studies, we may assume that the propulsive force is constant and that thrust $F =$ drag; then the aircraft is in equilibrium w.r.t. the x -axis and we may assume that the X -force equation is not perturbed from equilibrium and hence omit this equation.
- Different forms of the equations may be used to address different problems.

Notation

- The steady velocities are U, V, W (upper case).
- For conventional flight conditions, U is relatively large and W and V are small, giving a small angle of attack, α , and small angle of sideslip, β .

Recall that: $\tan \alpha = \frac{W}{U}$ and

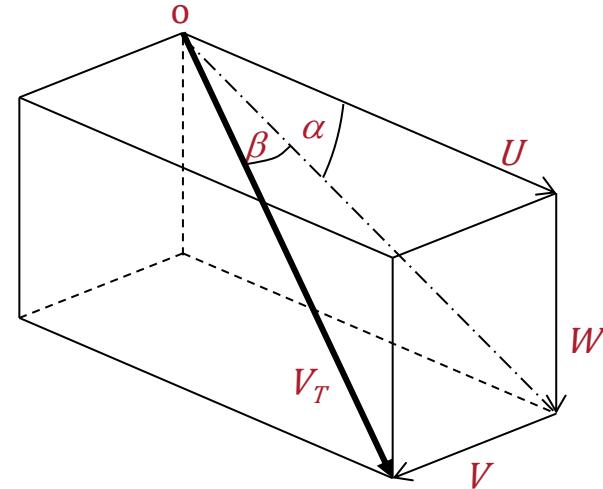
$$\sin \beta = \frac{V}{V_T}$$

where $V_T = \sqrt{U^2 + V^2 + W^2}$

Therefore, for small α and β :

$$\alpha \approx W/U$$

$$\beta \approx V/V_T.$$



- Thus, if preferred, we could write the three translational equations of motion in terms of α, β and V_T instead of U, V and W .

Notation

- Later, when we consider perturbations from the steady state condition, these short-term variations from steady values will be given the symbols u, v, w (lower case) for the U, V and W equations.
- The steady *angular* velocities are written p, q, r . This is the same when considering perturbations from steady state as we usually regard the steady state as having zero angular rate.
- We will often consider flight with body-axis forward speed U and vertical component W but with zero sideslip velocity, $V=0$. And if the perturbation is also longitudinal-only then we will see only U, W, u and w in the perturbation equations, and not V or v .
(If formulated in wind axes, W would also be zero as the x axis would be aligned with the unperturbed total vel.)

Notation

- Do not confuse the notation for dynamic pressure, \bar{q} ($\bar{q} = \frac{1}{2}\rho V_T^2$ or $\frac{1}{2}\rho U^2$) with pitch rate q .
- We shall also run up against the problem of distinguishing between L =lift and L =rolling moment.
- Moments and products of inertia:

$$\begin{array}{ll} I_{xx} = \sum \delta m(y^2 + z^2) & I_{xy} = \sum \delta m.xy \\ I_{yy} = \sum \delta m(x^2 + z^2) & I_{xz} = \sum \delta m.xz \\ I_{zz} = \sum \delta m(x^2 + y^2) & I_{yz} = \sum \delta m.yz \end{array}$$

Learn the notation!

The Formal Equations

With the gravitational forces separated from the aerodynamic/propulsive forces, the translational equations can be written as follows (allowing for non-zero values of V , W and pitch and roll angles, θ and ϕ):

Fore/ Aft:

$$m(\dot{U}) - rV + qW = X - mg \sin \theta$$

Lateral

$$m(\dot{V}) - pW + rU = Y + mg \cos \theta \sin \phi$$

Transverse:

$$m(\dot{W}) - qU + pV = Z + mg \cos \theta \cos \phi$$

The Formal Equations

Similarly, the rotational equations are expressed in their complete form as:

Roll:

$$I_{xx}\dot{p} - (I_{yy} - I_{zz})qr - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) = L$$

Pitch:

$$I_{yy}\dot{q} - (I_{zz} - I_{xx})rp - I_{zx}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) = M$$

Yaw:

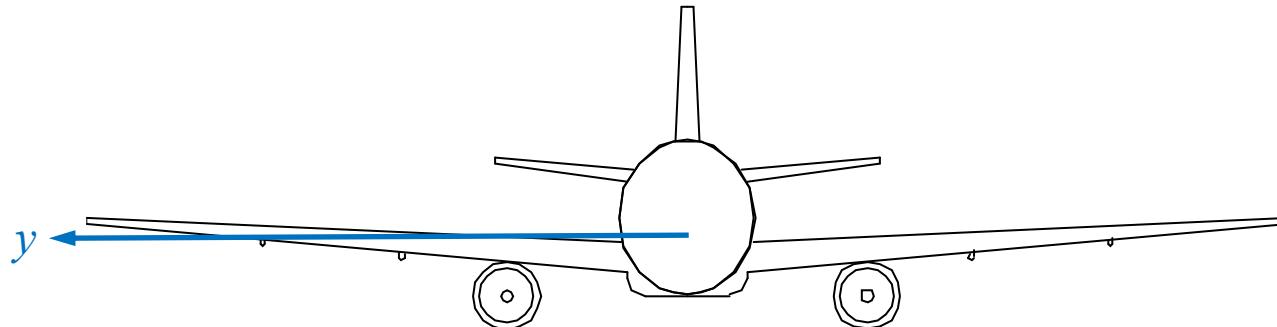
$$I_{zz}\dot{r} - (I_{xx} - I_{yy})pq - I_{xy}(p^2 - q^2) - I_{yz}(\dot{q} + rp) - I_{xz}(\dot{p} - qr) = N$$

(You are not expected to remember these – [see Cook for further reading](#)).

Note: Product terms & cross inertias.

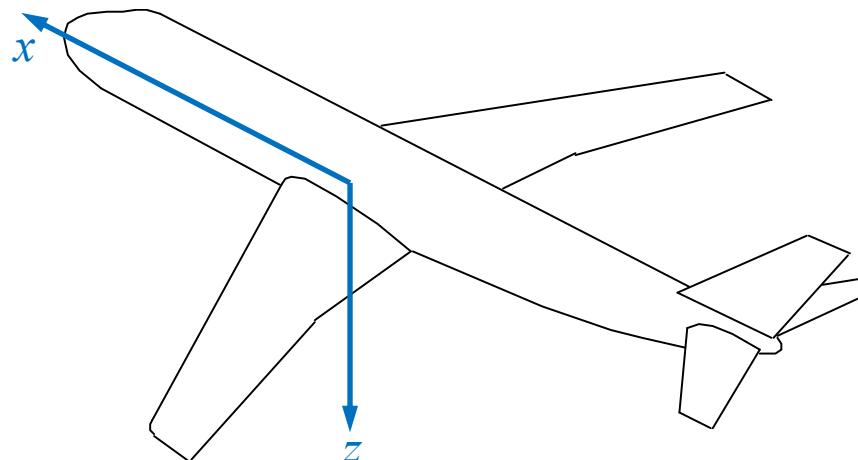
Reminder: the cross-inertias

- As mentioned previously, we usually adopt a simpler form of the equations on the previous slide because we assume the aircraft to have an inertial plane of symmetry. This results in I_{xy} and I_{zy} being zero, yielding eqn (16).
- An interpretation of this is that for any dimension x away from the origin (fore/aft) there are equal masses at $\pm y$ which cancel in the inertia calculations.



Reminder: the cross-inertias

- Similarly, for a choice of some fixed position z away from the origin (vertically) there will be equal masses at $\pm y$.
- However, I_{zx} is not zero for a typical aircraft geometry. The vertical tail (fin) configuration, for example, means that at aft positions x we do not have equal masses at $\pm z$ as there is no fin below the fuselage.



Alternative Form of the Equations

TABLE 2.4-1. The Flat-Earth, Body Axes 6-DOF Equations

Force Equations

$$\begin{aligned}\dot{U} &= RV - QW - g'_0 \sin \theta + \frac{F_x}{m} \\ \dot{V} &= -RU + PW + g'_0 \sin \phi \cos \theta + \frac{F_y}{m} \\ \dot{W} &= QU - PV + g'_0 \cos \phi \cos \theta + \frac{F_z}{m}\end{aligned}\tag{2.4-2}$$

Kinematic Equations

$$\begin{aligned}\dot{\phi} &= P + \tan \theta(Q \sin \phi + R \cos \phi) \\ \dot{\theta} &= Q \cos \phi - R \sin \phi \\ \dot{\psi} &= \frac{Q \sin \phi + R \cos \phi}{\cos \theta}\end{aligned}\tag{2.4-3}$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)

Alternative Form of the Equations

Moment Equations

$$\begin{aligned}\dot{P} &= (c_1 R + c_2 P)Q + c_3 \bar{L} + c_4 N \\ \dot{Q} &= c_5 PR - c_6(P^2 - R^2) + c_7 M \\ \dot{R} &= (c_8 P - c_2 R)Q + c_4 \bar{L} + c_9 N\end{aligned}\tag{2.4-4}$$

Navigation Equations

$$\begin{aligned}\dot{p}_N &= U \cos \theta \cos \psi + V(-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi) \\ &\quad + W(\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi) \\ \dot{p}_E &= U \cos \theta \sin \psi + V(\cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi) \\ &\quad + W(-\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi) \\ \dot{h} &= U \sin \theta - V \sin \phi \cos \theta - W \cos \phi \cos \theta\end{aligned}\tag{2.4-5}$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)

Alternative Form of the Equations – mass properties

$$\begin{aligned}\Gamma c_1 &= (J_y - J_z)J_z - J_{xz}^2, & \Gamma c_2 &= (J_x - J_y + J_z)J_{xz} \\ \Gamma c_3 &= J_z, & \Gamma c_4 &= J_{xz} \\ c_5 &= \frac{J_z - J_x}{J_y}, & c_6 &= \frac{J_{xz}}{J_y} \end{aligned} \tag{2.4-6}$$

$$c_7 = \frac{1}{J_y}, \quad \Gamma c_8 = J_x(J_x - J_y) + J_{xz}^2,$$

$$\Gamma c_9 = J_x,$$

where

$$\Gamma = J_x J_z - J_{xz}^2 \quad [\text{as in (1.3-19b)}].$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)

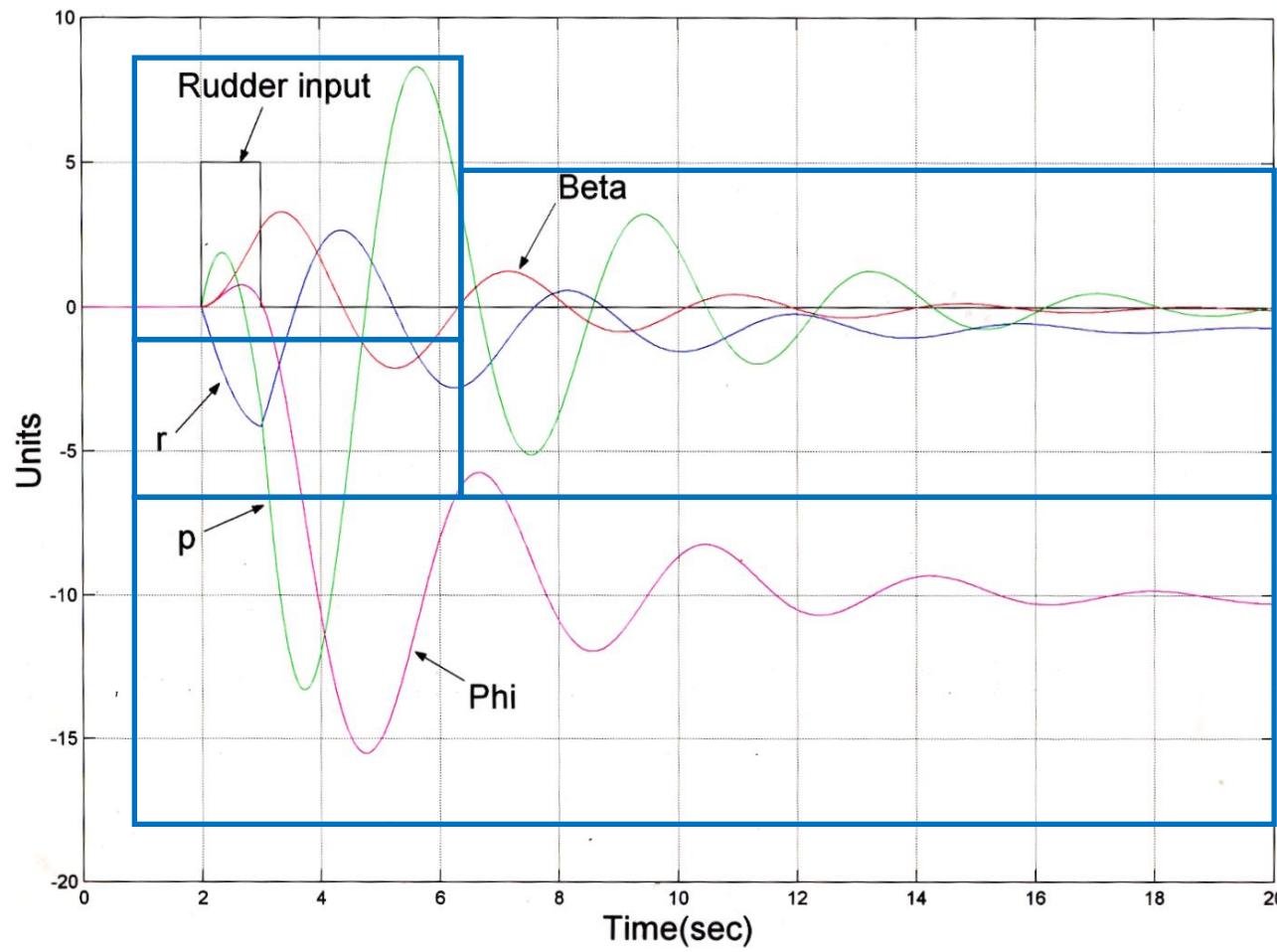
Alternative Form of the Equations

$$\begin{aligned}\dot{V}_T &= \frac{U\dot{U} + V\dot{V} + W\dot{W}}{V_T} \\ \dot{\beta} &= \frac{\dot{V}V_T - V\dot{V}_T}{V_T^2 \cos \beta} \\ \dot{\alpha} &= \frac{U\dot{W} - W\dot{U}}{U^2 + W^2}.\end{aligned}\tag{2.4-8}$$

The new state vector is

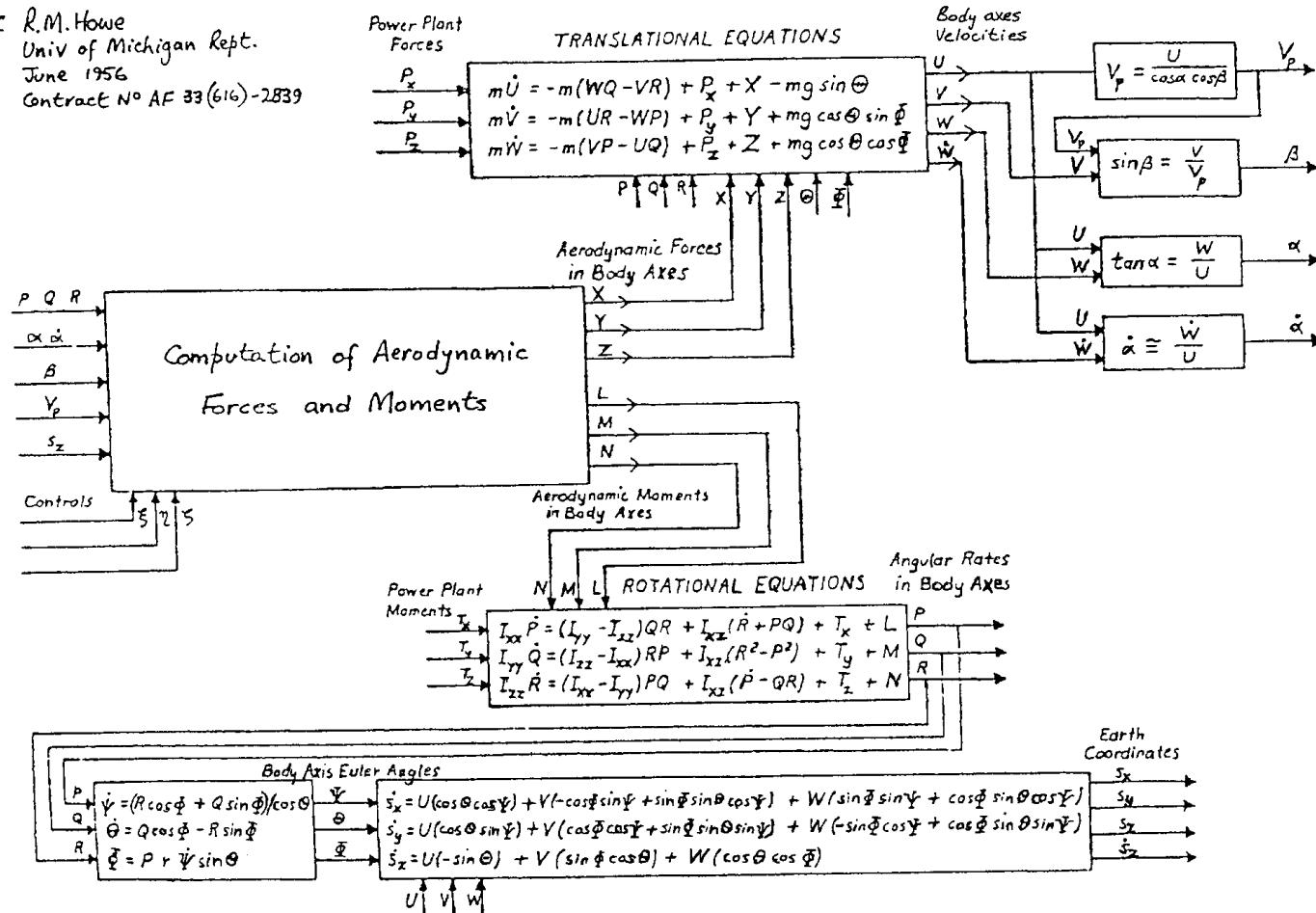
$$X^T = [V_T \quad \beta \quad \alpha \quad \phi \quad \theta \quad \psi \quad P \quad Q \quad R \quad p_N \quad p_E \quad h]. \tag{2.4-9}$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)



Equations of motion in simulation model

Ref R.M. Howe
 Univ of Michigan Rept.
 June 1956
 Contract No AF 33(616)-2839



Coupled Motion

- In general, the **motions are coupled**, and, for example, an impulsive force (moment) applied in **roll** will eventually disturb the motions in all **6 degrees of freedom** and there will be resultant non-zero values for all 6 motion variables.
- However, for many applications it is standard practice, and sufficient, to separate the equations into **two de-coupled sets** of three freedoms each to provide what is called:
 - the longitudinal equations in U, W, q
 - the lateral-directional equations in V, p, r .

Coupled Motion

In the form quoted above, the equations are not yet linearised and they show for example, products of the variables. However, after linearisation we shall find that the form of the full set of six can be displayed as follows:

$$\begin{array}{l|l} \text{Translational} & \text{equations} \\ \text{in upper} & \text{portion} \\ \hline \text{Rotational} & \text{equations} \\ \text{in lower} & \text{portion} \end{array} \quad \boxed{\begin{matrix} u \\ v \\ w \\ p \\ q \\ r \end{matrix}} \quad \text{or} \quad \begin{matrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{matrix} = \begin{matrix} \text{Fwd. force} \\ \text{Lateral force} \\ \text{Transv. force} \\ \text{Roll.mom.} \\ \text{Pitch.mom.} \\ \text{Yaw.mom.} \end{matrix}$$

Coupled Motion

In this form, all four partitions of the LHS matrix will have non-zero terms. If we change the order of the forces and the order of the variables to allow for the lateral/longitudinal split, namely to display

- **longitudinal actions**: forces and motions which are within the plane of symmetry, and
- **lateral-directional actions**: forces which act out of the plane of symmetry and consequent motions of that plane away from its normal vertical position,

we can re-form the equations to display them as:

Coupled Motion

$$\begin{bmatrix} \text{Longitudinal} \\ \text{equations} \\ \hline \text{Lateral} \\ \text{equations} \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ v \\ p \\ r \end{bmatrix} = \begin{bmatrix} \text{Fwd. force} \\ \text{Transv. force} \\ \text{Pitch. mom.} \\ \hline \text{Lat. force} \\ \text{Roll. mom.} \\ \text{Yaw. mom.} \end{bmatrix}$$

both of these partitions are nominally null

Next Session

Equations of Motion 3

bristol.ac.uk

<https://www.redbull.com/>



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Email: thomas.richardson@bristol.ac.uk

