CHAPTER

Principles of Statics

2

Statics, as the name implies, is concerned with the study of bodies at rest or, in other words, in equilibrium, under the action of a force system. Actually, a moving body is in equilibrium if the forces acting on it are producing neither acceleration nor deceleration. However, in structural engineering, structural members are generally at rest and therefore in a state of *statical equilibrium.*

In this chapter we shall discuss those principles of statics that are essential to structural and stress analysis; an elementary knowledge of vectors is assumed.

2.1 Force

The definition of a force is derived from Newton's First Law of Motion which states that a body will remain in its state of rest or in its state of uniform motion in a straight line unless compelled by an external force to change that state. Force is therefore associated with a *change* in motion, i.e. it causes acceleration or deceleration.

The basic unit of force in structural and stress analysis is the *Newton* (N) which is roughly a tenth of the weight of this book. This is a rather small unit for most of the loads in structural engineering so a more convenient unit, the *kilonewton* (kN) is often used.

$$1\,\text{kN} = 1000\,\text{N}$$

All bodies possess *mass* which is usually measured in *kilograms* (kg). The mass of a body is a measure of the quantity of matter in the body and, for a particular body, is invariable. This means that a steel beam, for example, having a given *weight* (the force due to gravity) on earth would weigh approximately six times less on the moon although its mass would be exactly the same.

We have seen that force is associated with acceleration and Newton's Second Law of Motion tells us that

$$\text{force} = \text{mass} \times \text{acceleration}$$

Gravity, which is the pull of the earth on a body, is measured by the acceleration it imparts when a body falls; this is taken as 9.81 m/s^2 and is given the symbol g. It follows that the force exerted by gravity on a mass of 1 kg is

$$\text{force} = 1 \times 9.81$$

The Newton is defined as the force required to produce an acceleration of 1 m/s^2 in a mass of 1 kg which means that it would require a force of 9.81 N to produce an acceleration of 9.81 m/s^2 in a mass of 1 kg, i.e. the gravitational force exerted on a mass of 1 kg is 9.81 N. Frequently, in everyday usage, mass is taken to mean the weight of a body in kg.

We all have direct experience of force systems. The force of the earth's gravitational pull acts vertically downwards on our bodies giving us weight; wind forces, which can vary in magnitude, tend to push us horizontally. Therefore forces possess magnitude and direction. At the same time the effect of a force depends

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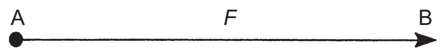

FIGURE 2.1

Representation of a force by a vector.

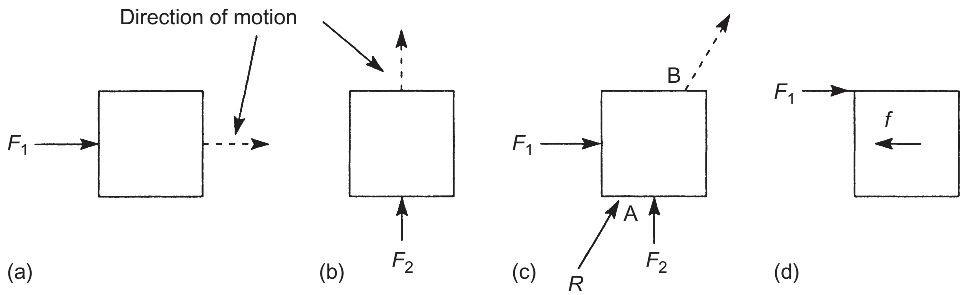

FIGURE 2.2

Action of forces on a cube.

upon its position. For example, a door may be opened or closed by pushing horizontally at its free edge, but if the same force is applied at any point on the vertical line through its hinges the door will neither open nor close. We see then that a force is described by its magnitude, direction and position and is therefore a *vector* quantity. As such it must obey the laws of vector addition, which is a fundamental concept that may be verified experimentally.

Since a force is a vector it may be represented graphically as shown in Fig. 2.1, where the force F is considered to be acting on an infinitesimally small particle at the point A and in a direction from left to right. The magnitude of F is represented, to a suitable scale, by the length of the line AB and its direction by the direction of the arrow. In vector notation the force F is written as $\mathbf{F}$.

Suppose a cube of material, placed on a horizontal surface, is acted upon by a force F_1 as shown in plan in Fig. 2.2(a). If F_1 is greater than the frictional force between the surface and the cube, the cube will move in the direction of F_1. Again if a force F_2 is applied as shown in Fig. 2.2(b) the cube will move in the direction of F_2. It follows that if F_1 and F_2 were applied simultaneously, the cube would move in some inclined direction as though it were acted on by a single inclined force R (Fig. 2.2(c)); R is called the *resultant* of F_1 and F_2.

Note that F_1 and F_2 (and R) are in a horizontal plane and that their lines of action pass through the centre of gravity of the cube, otherwise rotation as well as translation would occur since, if F_1, say, were applied at one corner of the cube as shown in Fig. 2.2(d), the frictional force f, which may be taken as acting at the center of the bottom face of the cube would, with F_1, form a couple (see Section 2.2).

The effect of the force R on the cube would be the same whether it was applied at the point A or at the point B (so long as the cube is rigid). Thus a force may be considered to be applied at any point on its line of action, a principle known as the *transmissibility of a force.*

EXAMPLE 2.1

State the direction of motion of the block of material shown *in plan* in Fig. 2.3 (a)–(f) when it is subjected to the applied force, F, and is supported on a horizontal surface. The frictional force between the surface and the underside of the block is f.

(a) The block moves in the direction DB with no rotation.
(b) The block does not move translationally, possible rotation.

(c) The block moves parallel to AB and rotates in an anticlockwise sense.
(d) The block moves in a direction parallel to DA.
(e) The block moves in a direction parallel to AB with a clockwise rotation.
(f) The block moves in a direction parallel to DB with an anticlockwise rotation.

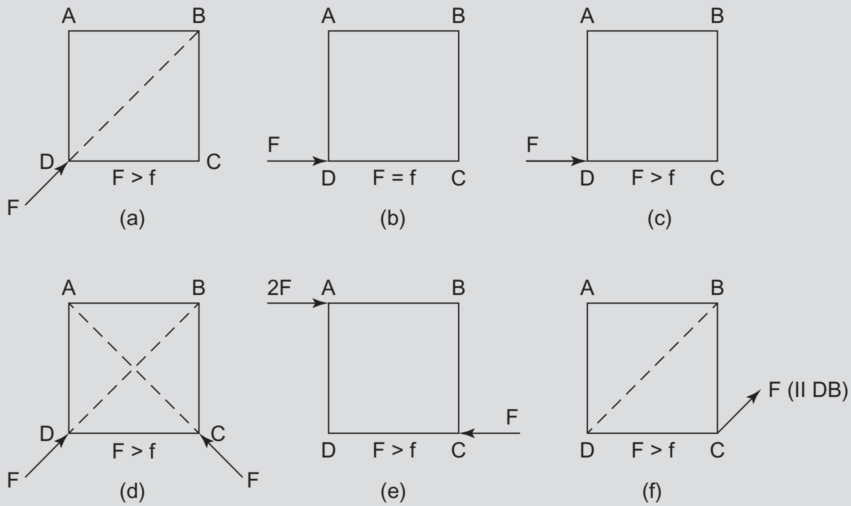

FIGURE 2.3

Force systems on block of Ex. 2.1.

Parallelogram of forces

The resultant of two concurrent and coplanar forces, that is whose lines of action pass through a single point and lie in the same plane (Fig. 2.4(a)), may be found using the theorem of the parallelogram of forces which states that:

> *If two forces acting at a point are represented by two adjacent sides of a parallelogram drawn from that point their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn through the point.*

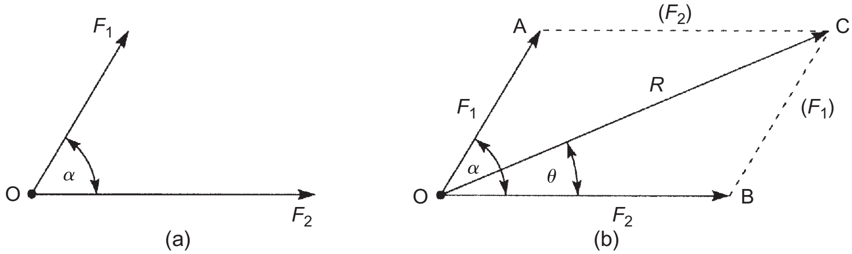

FIGURE 2.4

Resultant of two concurrent forces.

Thus in Fig. 2.4(b) R is the resultant of F_1 and F_2. This result may be verified experimentally or, alternatively, demonstrated to be true using the laws of vector addition. In Fig. 2.4(b) the side BC of the parallelogram is equal in magnitude and direction to the force F_1 represented by the side OA. Therefore, in vector notation

$$\mathbf{R} = \mathbf{F}_2 + \mathbf{F}_1$$

The same result would be obtained by considering the side AC of the parallelogram which is equal in magnitude and direction to the force F_2. Thus

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$$

Note that vectors obey the *commutative law*, i.e.

$$\mathbf{F}_2 + \mathbf{F}_1 = \mathbf{F}_1 + \mathbf{F}_2$$

The actual magnitude and direction of R may be found graphically by drawing the vectors representing F_1 and F_2 to the *same scale* (i.e. OB and BC) and then completing the triangle OBC by drawing in the vector, along OC, representing R. Alternatively, R and θ may be calculated using the trigonometry of triangles, i.e.

$$R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\alpha \tag{2.1}$$

and

$$\tan\theta = \frac{F_1\sin\alpha}{F_2 + F_1\cos\alpha} \tag{2.2}$$

EXAMPLE 2.2

Calculate the magnitude and direction of the resultant of the two forces shown in Fig. 2.5; verify your answer graphically.

From Eq. (2.1)

$$R^2 = 10^2 + 15^2 + 2 \times 10 \times 15\cos 45°$$

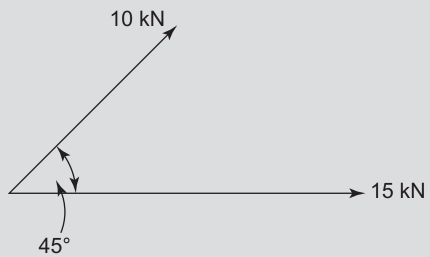

FIGURE 2.5

Force system of Ex. 2.2.

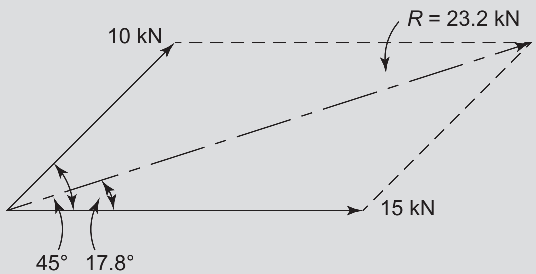

FIGURE 2.6

Graphical solution of Ex. 2.2.

from which

$$R = 23.2\ \text{kN}$$

From Eq. (2.2)

$$\tan\theta = \frac{10\sin 45^\circ}{15 + 10\cos 45^\circ}$$

where θ is the inclination of R to the direction of the 15 kN force. Then

$$\theta = 17.8^\circ$$

The graphical solution is shown in Fig. 2.6.

EXAMPLE 2.3

Calculate the magnitude and direction of the resultant of the two forces shown in Fig. 2.7 and verify your answers graphically.

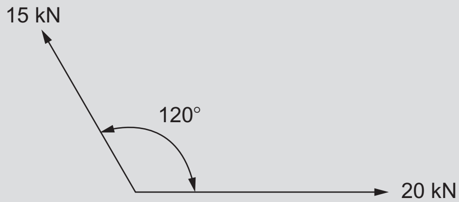

FIGURE 2.7

Force system of Ex. 2.3.

From Eq. (2.1)

$$R^2 = 15^2 + 20^2 + 2 \times 15 \times 20\cos 120^\circ$$

which gives

$$R = 18.03 \text{ kN}$$

From Eq. (2.2)

$$\tan\theta = \frac{15\sin 120^\circ}{20 + 15\cos 120^\circ} = 1.04$$

so that

$$\theta = 46.1^\circ.$$

The graphical solution is shown in Fig. 2.8.

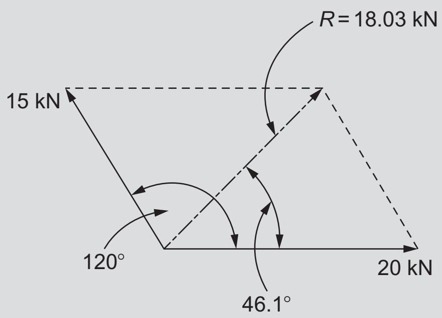

FIGURE 2.8

Graphical solution of Ex. 2.3.

EXAMPLE 2.4

Calculate the magnitude and direction of the resultant of the two forces shown in Fig. 2.9 and verify your answers graphically.

From Eq. (2.1)

$$R^2 = 15^2 + 20^2 + 2 \times 15 \times 20 \cos 240°,$$

which gives

$$R = 18.03 \text{ kN}$$

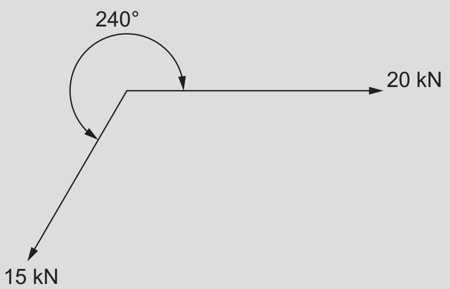

FIGURE 2.9

Force system of Ex. 2.4.

From Eq. (2.2)

$$\tan\theta = \frac{15 \sin 240°}{20 + 15 \cos 240°} = -1.04$$

so that

$$\theta = -46.1°$$

Therefore, we see that R is in a downward direction, to the right and at an angle of 46.1° to the 20-kN force.

The graphical solution is shown in Fig. 2.10.

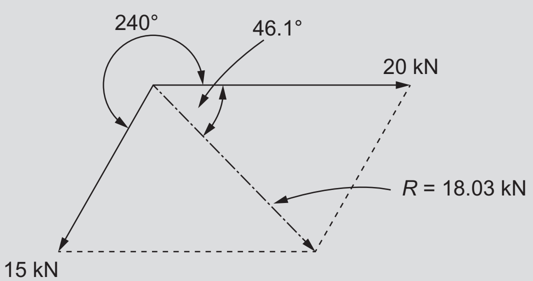

FIGURE 2.10

Graphical solution of Ex. 2.4.

In Exs 2.2–2.4 both F_1 and F_2 are 'pulling away' from the particle at O. In Fig. 2.11(a) F_1 is a 'thrust' whereas F_2 remains a 'pull'. To use the parallelogram of forces the system must be reduced to either two 'pulls' as shown in Fig. 2.11(b) or two 'thrusts' as shown in Fig. 2.11(c). In all three systems we see that the effect on the particle at O is the same.

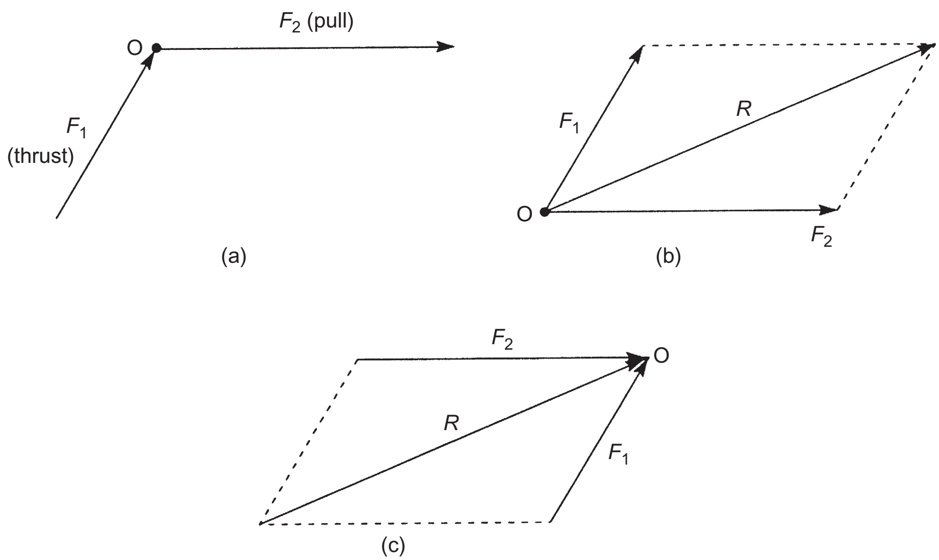

FIGURE 2.11

Reduction of a force system.

As we have seen, the combined effect of the two forces F_1 and F_2 acting simultaneously is the same as if they had been replaced by the single force R. Conversely, if R were to be replaced by F_1 and F_2 the effect would again be the same. F_1 and F_2 may therefore be regarded as the *components* of R in the directions OA and OB; R is then said to have been *resolved* into two components, F_1 and F_2.

Of particular interest in structural analysis is the resolution of a force into two components at right angles to each other. In this case the parallelogram of Fig. 2.4(b) becomes a rectangle in which $\alpha = 90°$ (Fig. 2.12) and, clearly

$$F_2 = R\cos\theta \; F_1 = R\sin\theta \qquad (2.3)$$

It follows from Fig. 2.12, or from Eqs (2.1) and (2.2), that

$$R^2 = F_1^2 + F_2^2 \qquad \tan\theta = \frac{F_1}{F_2} \qquad (2.4)$$

We note, by reference to Fig. 2.2(a) and (b), that a force does not induce motion in a direction perpendicular to its line of action; in other

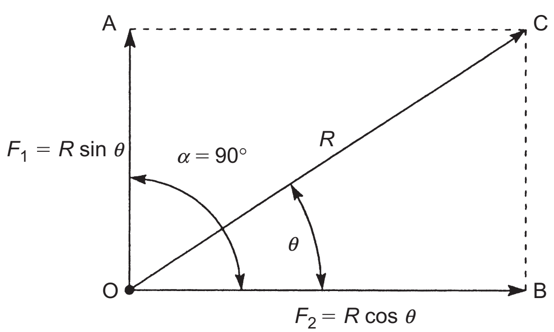

FIGURE 2.12

Resolution of a force into two components at right angles.

words a force has no effect in a direction perpendicular to itself. This may also be seen by setting $\theta = 90°$ in Eq. (2.3), then

$$F_1 = R \quad F_2 = 0$$

and the component of R in a direction perpendicular to its line of action is zero.

The resultant of a system of concurrent forces

So far we have considered the resultant of just two concurrent forces. The method used for that case may be extended to determine the resultant of a system of any number of concurrent coplanar forces such as that shown in Fig. 2.13(a). Thus in the vector diagram of Fig. 2.13(b)

$$\mathbf{R}_{12} = \mathbf{F}_1 + \mathbf{F}_2$$

where $\mathbf{R}_{12}$ is the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$. Further

$$\mathbf{R}_{123} = \mathbf{R}_{12} + \mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

so that $\mathbf{R}_{123}$ is the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$. Finally

$$\mathbf{R} = \mathbf{R}_{123} + \mathbf{F}_4 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$$

where $\mathbf{R}$ is the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$.

The actual value and direction of $\mathbf{R}$ may be found graphically by constructing the vector diagram of Fig. 2.13(b) to scale or by resolving each force into components parallel to two directions at right angles, say the x and y directions shown in Fig. 2.13(a). Then

$$F_x = F_1 + F_2 \cos\alpha - F_3 \cos\beta - F_4 \cos\gamma$$

$$F_y = F_2 \sin\alpha + F_3 \sin\beta - F_4 \sin\gamma$$

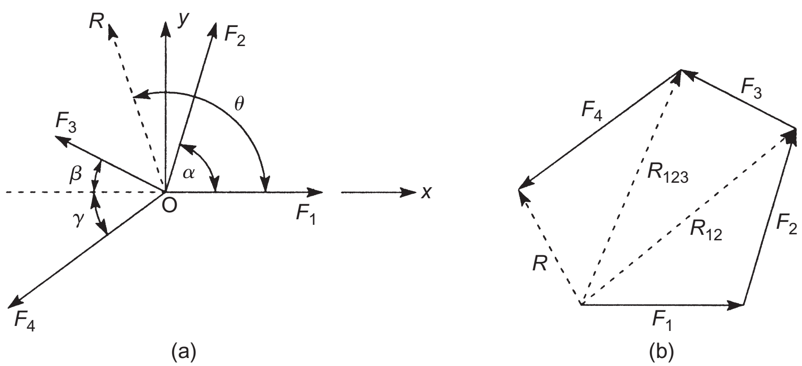

FIGURE 2.13

Resultant of a system of concurrent forces.

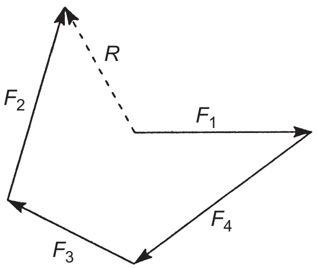

FIGURE 2.14

Alternative construction of force diagram for system of Fig. 2.13(a).

Then

$$R = \sqrt{F_x^2 + F_y^2}$$

and

$$\tan\theta = \frac{F_y}{F_x}$$

The forces F_1, F_2, F_3 and F_4 in Fig. 2.13(a) do not have to be taken in any particular order when constructing the vector diagram of Fig. 2.13(b). Identical results for the magnitude and direction of R are obtained if the forces in the vector diagram are taken in the order F_1, F_4, F_3, F_2 as shown in Fig. 2.14 or, in fact, are taken in any order so long as the directions of the forces are adhered to and one force vector is drawn from the end of the previous force vector.

EXAMPLE 2.5

Calculate the resultant and its direction of the system of forces shown in Fig. 2.15. Verify your answers using a graphical method.

Resolving forces horizontally

$$F_x = 10 + 15\cos 60° - 12\cos 45° - 8\cos 30° = 2.1\text{ kN}$$

Resolving forces vertically

$$F_y = 15\sin 60° + 12\sin 45° - 8\sin 30° = 17.5\text{ kN}$$

The resultant R is then given by

$$R = \sqrt{2.1^2 + 17.5^2} = 17.6\text{kN}$$

and

$$\theta = \tan^{-1}\frac{17.5}{2.1} = 83.2°$$

The graphical solution is shown in Fig. 2.16.

Choosing a different order for the forces as shown in Fig. 2.17 results in an identical solution as shown in Fig. 2.17.

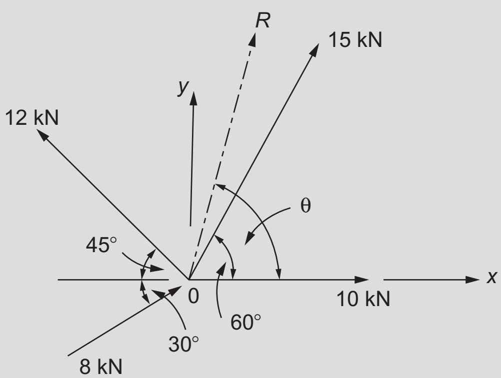

FIGURE 2.15

Force system of Ex. 2.5.

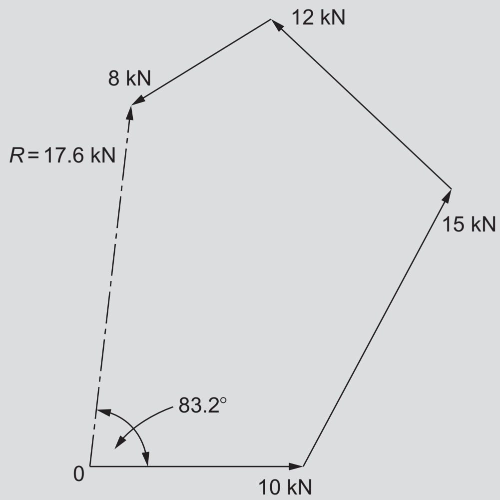

FIGURE 2.16

Graphical solution of Ex. 2.5.

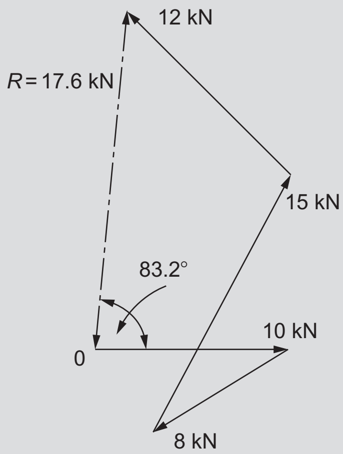

FIGURE 2.17

Alternative graphical solution of Ex. 2.5.

Equilibrant of a system of concurrent forces

In Fig. 2.4(b) the resultant R of the forces F_1 and F_2 represents the combined effect of F_1 and F_2 on the particle at O. It follows that this effect may be eliminated by introducing a force R_E which is equal in magnitude but opposite in direction to R at O, as shown in Fig. 2.18(a). R_E is known at the *equilibrant* of F_1 and F_2 and the particle at O will then be in *equilibrium* and remain stationary. In other words the forces F_1, F_2 and R_E are in equilibrium and, by reference to Fig. 2.18(b), we see that these three forces may be represented by the triangle of vectors OBC as shown in Fig. 2.18(b). This result leads directly to the law of the *triangle of forces* which states that:

> *If three forces acting at a point are in equilibrium they may be represented in magnitude and direction by the sides of a triangle taken in order.*

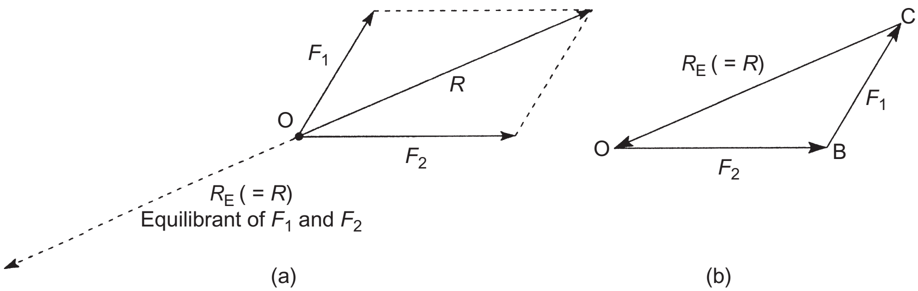

FIGURE 2.18

Equilibrant of two concurrent forces.

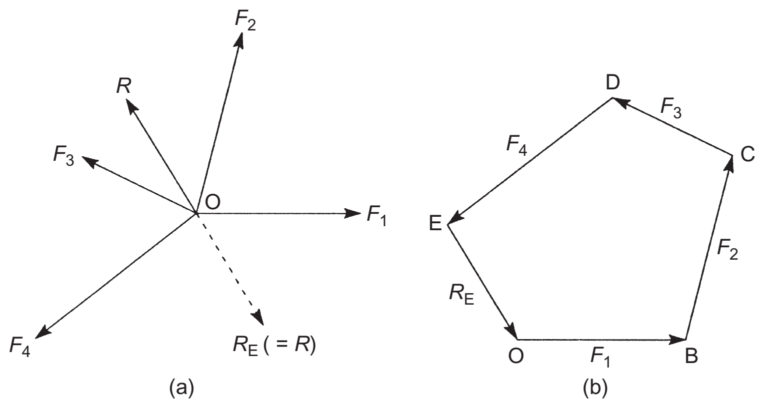

FIGURE 2.19

Equilibrant of a number of concurrent forces.

The law of the triangle of forces may be used in the analysis of a plane, pin-jointed truss in which, say, one of three concurrent forces is known in magnitude and direction but only the lines of action of the other two. The law enables us to find the magnitudes of the other two forces and also the direction of their lines of action.

The above arguments may be extended to a system comprising any number of concurrent forces. In the force system of Fig. 2.13(a), R_E, shown in Fig. 2.19(a), is the equilibrant of the forces F_1, F_2, F_3 and F_4. Then F_1, F_2, F_3, F_4 and R_E may be represented by the force polygon OBCDE as shown in Fig. 2.19(b).

The law of the *polygon of forces* follows:

> *If a number of forces acting at a point are in equilibrium they may be represented in magnitude and direction by the sides of a closed polygon taken in order.*

Again, the law of the polygon of forces may be used in the analysis of plane, pin-jointed trusses where several members meet at a joint but where no more than two forces are unknown in magnitude.

The resultant of a system of non-concurrent forces

In most structural problems the lines of action of the different forces acting on the structure do not meet at a single point; such a force system is non-concurrent.

Consider the system of non-concurrent forces shown in Fig. 2.20(a); their resultant may be found graphically using the parallelogram of forces as demonstrated in Fig. 2.20(b). Produce the lines of action of F_1 and F_2 to their point of intersection, I_1. Measure $I_1A = F_1$ and $I_1B = F_2$ to the same scale, then complete the parallelogram I_1ACB; the diagonal CI_1 represents the resultant, R_{12}, of F_1 and F_2. Now produce the line of action of R_{12} backwards to intersect the line of action of F_3 at I_2. Measure $I_2D = R_{12}$ and $I_2F = F_3$ to the same scale as before, then complete the parallelogram I_2DEF; the diagonal $I_2E = R_{123}$, the resultant of R_{12} and F_3. It follows that $R_{123} = R$, the resultant of F_1, F_2 and F_3. Note that only the line of action and the magnitude of R can be found, not its point of action, since the vectors F_1, F_2 and F_3 in Fig. 2.20(a) define the lines of action of the forces, not their points of action.

If the points of action of the forces are known, defined, say, by coordinates referred to a convenient xy axis system, the magnitude, direction and point of action of their resultant may be found by resolving each force into components parallel to the x and y axes and then finding the magnitude and position of the resultants R_x and R_y of each set of components using the method described in Section 2.3 for a system of parallel forces. The resultant R of the force system is then given by

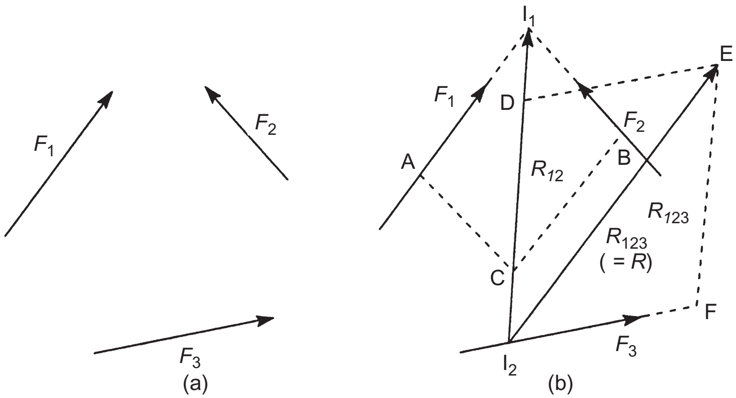

FIGURE 2.20

Resultant of a system of non-concurrent forces.

$$R = \sqrt{R_x^2 + R_y^2}$$

and its point of action is the point of intersection of R_x and R_y; finally, its inclination θ to the x axis, say, is

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

EXAMPLE 2.6

Calculate the resultant of the force system shown in Fig. 2.21 and determine its inclination to the horizontal. Verify your solution using a graphical method and, finally, calculate the coordinates of the point at which it acts.

Resolving forces horizontally

$$F_x = 20\cos 45° - 10\cos 60° + 18\cos 15° = 26.5\text{ kN}$$

Resolving forces vertically

$$F_y = 20\sin 45° + 10\sin 60° + 18\sin 15° = 27.5\text{ kN}$$

The resultant R is then given by

$$R = \sqrt{26.5^2 + 27.5^2} = 38.2\,\text{kN}$$

The inclination of R to the horizontal is then given by

$$\theta = \tan^{-1}\frac{27.5}{26.5} = 46.1^{\circ}$$

The graphical solution proceeds as follows and is shown in Fig. 2.22.

Extend the line of action of the 10-kN force to intersect the line of action of the 20-kN force at I_1. From I_1 measure $I_1A = 20$ kN and $I_1B = 10$ kN to the same scale. Construct the parallelogram $ACBI_1$. Then the resultant of the 20-kN force and the 10-kN force is $R_{20,10}$ = the diagonal CI_1. Now extend the lines of action of $R_{20,10}$ and the 18-kN force backwards to intersect at I_2. Measure $I_2D = R_{20,10}$ and

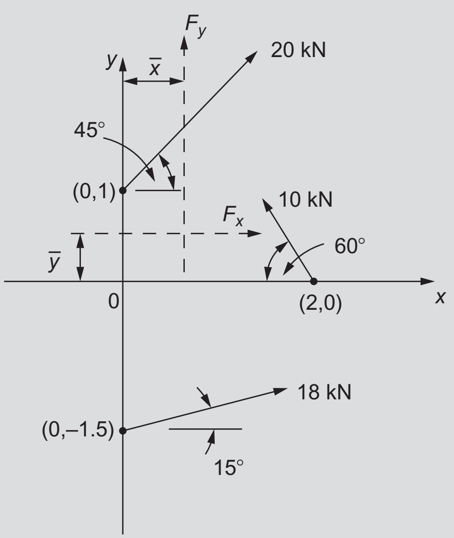

FIGURE 2.21

Force system of Ex. 2.6.

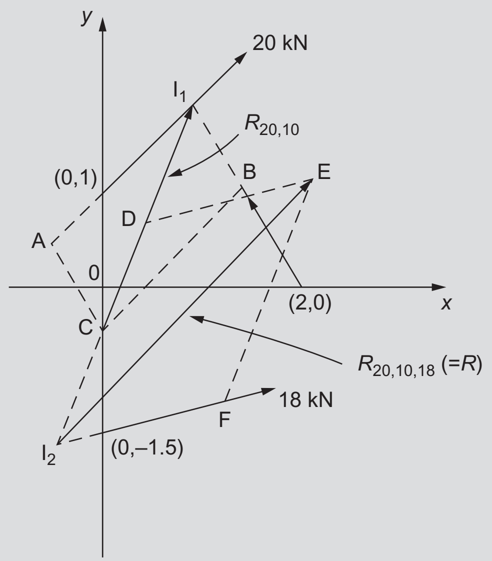

FIGURE 2.22

Graphical solution of Ex. 2.6.

$I_2F = 18$ kN to the same scale as before. Construct the parallelogram I_2DEF. The resultant of $R_{20,10}$ and the 18-kN force is then the diagonal I_2E and is the resultant of the complete force system.

The point of application of the resultant R may be taken to be the point of intersection of F_x and F_y. The line of action of F_x is found as follows. Taking moments of forces about Ox (see Section 2.2 and Ex. 2.8)

$$F_x\bar{y} = 20 \cos 45^\circ \times (1) - 18 \cos 15^\circ \times (1.5)$$

that is

$$26.5\bar{y} = -11.9,$$

which gives

$$\bar{y} = -0.5.$$

Now taking moments about Oy

$$F_y\bar{x} = 10 \sin 60^\circ \times (2)$$

that is

$$27.5\bar{x} = 17.3$$

so that

$$\bar{x} = 0.6$$

Therefore, the point of application of R is (0.6,−0.5).

Note that the point of application of the resultant of the force system cannot be obtained graphically. However, it may be verified from Fig. 2.22 that the line of action of the resultant passes through the point (0.6,−0.5). The actual point of application of the resultant of a system of forces acting on a body is not of particular significance since, from the principle of the transmissibility of forces, a force may be regarded as acting at any point on its line of action.

EXAMPLE 2.7

Use a graphical method to determine the resultant and its line of action of the system of coplanar forces acting on the block of material shown *in plan* in Fig. 2.23.

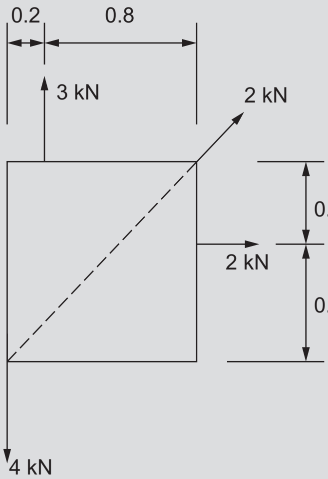

FIGURE 2.23

Force system of Ex. 2.7.

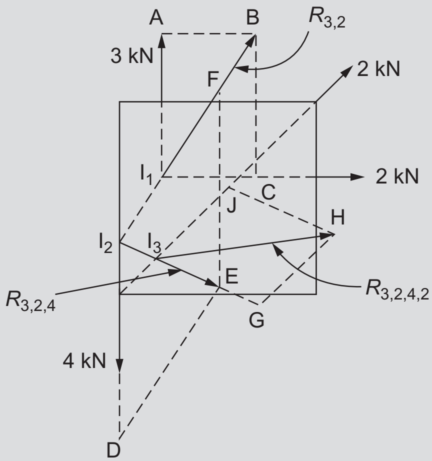

FIGURE 2.24

Graphical solution of Ex. 2.7.

The solution proceeds as follows and is shown in Fig. 2.24. Extend the lines of action of the horizontal 2-kN force and the vertical 3-kN force backwards to intersect at I_1. (Note: All the forces acting on the block are in a horizontal plane, but for descriptive purposes, it is convenient to refer to horizontal and vertical as they appear in Fig. 2.24). Measure $I_1A = 3$ kN and $I_1C = 2$ kN to the same scale. Complete the parallelogram I_1ABC. The resultant, $R_{3,2}$, of the vertical 3-kN force and the horizontal 2-kN force is then the diagonal I_1B. Now extend the line of action of $R_{3,2}$ backwards to intersect the line of action of the 4-kN force at I_2. Measure $I_2F = R_{3,2}$ and $I_2D = 4$ kN to the same scale as before. (Note that both forces are pulling away from I_2). Complete the parallelogram I_2DEF. The diagonal I_2E ($= R_{3,2,4}$) is then the resultant of $R_{3,2}$ and the 4-kN force. Now extend the line of action of the inclined 2-kN force backwards to intersect the line of action of $R_{3,2,4}$ at I_3. Measure $I_3G = R_{3,2,4}$ and $I_3J = 2$ kN. Complete the parallelogram I_3GHJ. The diagonal I_3H is then the resultant of the complete force system acting on the block.

Note again that the point of application of the resultant force is not determined in this graphical solution. However, from the principle of the transmissibility of a force, the resultant may be considered to act at any point in its line of action so that its actual point of application does not affect its action on the block.

It is clear from Ex. 2.7 that if a large number of forces are involved, this graphical method of solution will become rather cumbersome. An alternative is the use of Bow's Notation. The block and force system are drawn accurately as shown in Fig. 2.25(a), although the magnitude of each force need not be drawn to scale.

The spaces between the forces are labelled A, B, C, and D and a force polygon is drawn as shown in Fig. 2.25(b) as though all the forces act at the same point. Suppose the space to the left of the 4-kN force is labelled F; the 4-kN force is then represented in the force polygon by the vector df. The closing vector fa would then be the equilibrant of the applied force system so that the vector af is the resultant in magnitude and direction of the force system. This, however, does not give its position.

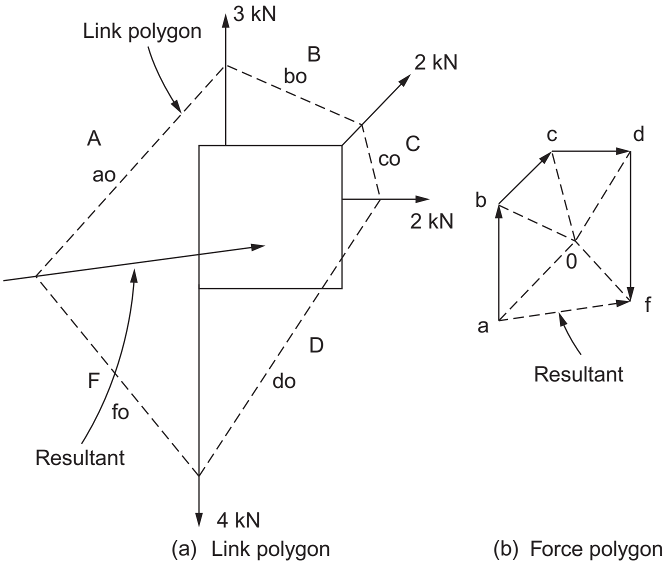

FIGURE 2.25

Solution of Ex. 2.7 using Bow's Notation.

Now choose any point O inside (or outside) the force polygon and join a, b, c, d, and f to O. In the space A in Fig. 2.25(a), draw a line ao parallel to aO in Fig. 2.25(b) to cut the line of action of the 3-kN force. (The line ao may be drawn in any convenient position). From the point of intersection of ao and the line of action of the 3-kN force, draw a line bo parallel to bO in Fig. 2.25(b) to intersect the line of action of the inclined 2-kN force. Continue in this manner across the spaces C and D. The final line, fo, intersects ao and gives the position of the resultant of the force system and is represented in the force polygon by the vector af. The polygon in Fig. 2.25(a) is known as the *link polygon.*

The validity of this method, i.e. the use of Bow's Notation, is based on the triangle of forces and may be found in standard applied mathematical texts.

2.2 Moment of a force

So far we have been concerned with the translational effect of a force, i.e. the tendency of a force to move a body in a straight line from one position to another. A force may, however, exert a rotational effect on a body so that the body tends to turn about some given point or axis.

Figure 2.26(a) shows the cross section of, say, a door that is attached to a wall by a pivot and bracket arrangement which allows it to rotate in a horizontal plane. A horizontal force, F, whose line of action passes through the pivot, will have no rotational effect on the door but when applied at some distance along the door (Fig. 2.26(b)) will cause it to rotate about the pivot. It is common experience that the nearer the pivot the force F is applied the greater must be its magnitude to cause rotation. At the same time its effect will be greatest when it is applied at right angles to the door.

In Fig. 2.26(b) F is said to exert a *moment* on the door about the pivot. Clearly the rotational effect of F depends upon its magnitude and also on its distance from the pivot. We therefore define the moment of a force, F, about a given point O (Fig. 2.27) as the product of the force and the perpendicular distance of its line of action from the point. Thus, in Fig. 2.27, the moment, M, of F about O is given by

$$M = Fa \tag{2.5}$$

where 'a' is known as the *lever arm* or *moment arm* of F about O; note that the units of a moment are the units of force $\times$ distance.

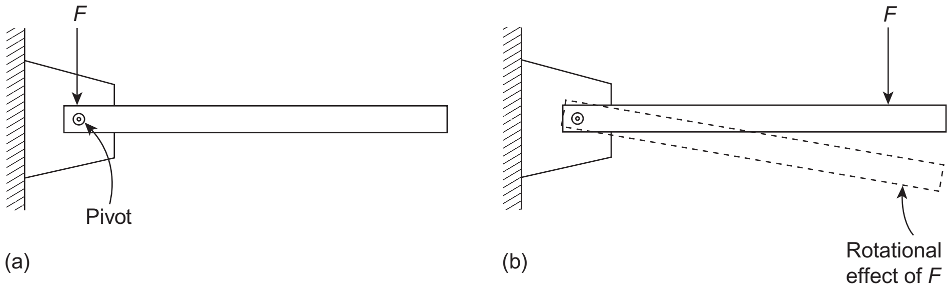

FIGURE 2.26

Rotational effect of a force.

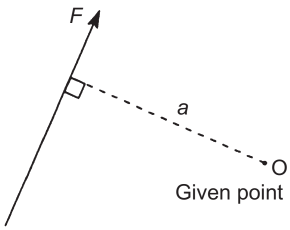

FIGURE 2.27

Moment of a force about a given point.

It can be seen from the above that a moment possesses both magnitude and a rotational sense. For example, in Fig. 2.27, F exerts a clockwise moment about O. A moment is therefore a vector (an alternative argument is that the product of a vector, F, and a scalar, a, is a vector). It is conventional to represent a moment vector graphically by a double-headed arrow, where the direction of the arrow designates a clockwise moment when looking in the direction of the arrow. Therefore, in Fig. 2.27, the moment $M(= Fa)$ would be represented by a double-headed arrow through O with its direction into the plane of the paper.

Moments, being vectors, may be resolved into components in the same way as forces. Consider the moment, M (Fig. 2.28(a)), in a plane inclined at an angle θ to the xz plane. The component of M in the xz plane, M_{xz}, may be imagined to be produced by rotating the plane containing M through the angle θ into the xz plane. Similarly, the component of M in the yz plane, M_{yz}, is obtained by rotating the plane containing M through the angle $90 - \theta$. Vectorially, the situation is that

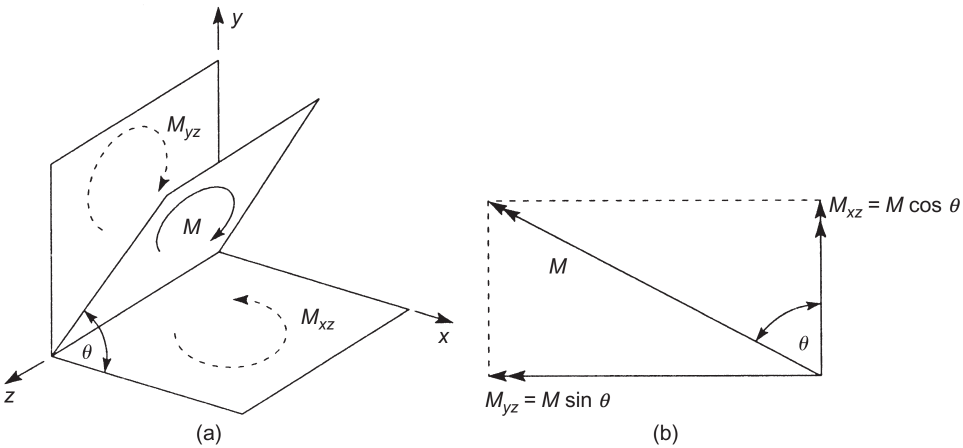

FIGURE 2.28

Resolution of a moment.

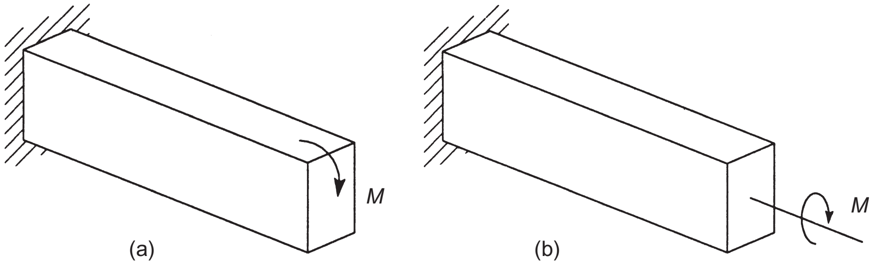

FIGURE 2.29

Action of a moment in different planes.

shown in Fig. 2.28(b), where the directions of the arrows represent clockwise moments when viewed in the directions of the arrows. Then

$$M_{xz} = M\cos\theta \quad M_{yz} = M\sin\theta$$

The action of a moment on a structural member depends upon the plane in which it acts. For example, in Fig. 2.29(a), the moment, M, which is applied in the longitudinal vertical plane of symmetry, will cause the beam to bend in a vertical plane. In Fig. 2.29(b) the moment, M, is applied in the plane of the cross section of the beam and will therefore produce twisting; in this case M is called a *torque*.

Couples

Consider the two coplanar, equal and parallel forces F which act in opposite directions as shown in Fig. 2.30. The sum of their moments, M_O, about *any* point O in their plane is

$$M_O = F \times BO - F \times AO$$

where OAB is perpendicular to both forces. Then

$$M_O = F(BO - AO) = F \times AB$$

and we see that the sum of the moments of the two forces F about any point in their plane is equal to the product of one of the forces and the perpendicular distance between their lines of action; this system is termed a *couple* and the distance AB is the *arm* or *lever arm* of the couple.

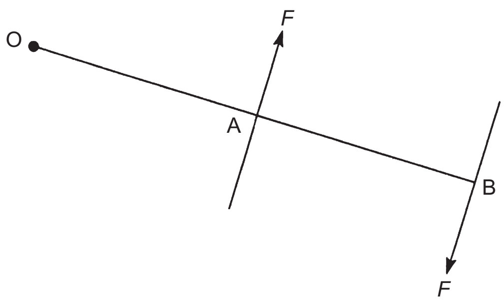

FIGURE 2.30

Moment of a couple.

Since a couple is, in effect, a pure moment (not to be confused with the moment of a force about a specific point which varies with the position of the point) it may be resolved into components in the same way as the moment M in Fig. 2.28.

Equivalent force systems

In structural analysis it is often convenient to replace a force system acting at one point by an equivalent force system acting at another. For example, in Fig. 2.31(a), the effect on the cylinder of the force F acting at A on the arm AB may be determined as follows.

If we apply equal and opposite forces F at B as shown in Fig. 2.31(b), the overall effect on the cylinder is unchanged. However, the force F at A and the equal and opposite force F at B form a couple which, as we have seen, has the same moment (Fa) about any point in its plane. Thus the single force F at A may be replaced by a single force F at B together with a moment equal to Fa as shown in Fig. 2.31(c). The effects of the force F at B and the moment (actually a torque) Fa may be calculated separately and then combined using the principle of superposition (see Section 3.7).

2.3 The resultant of a system of parallel forces

Since, as we have seen, a system of forces may be replaced by their resultant, it follows that a particular action of a force system, say the combined moments of the forces about a point, must be identical to the same action of their resultant. This principle may be used to determine the magnitude and line of action of a system of parallel forces such as that shown in Fig. 2.32(a).

The point of intersection of the lines of action of F_1 and F_2 is at infinity so that the parallelogram of forces (Fig. 2.4(b)) degenerates into a straight line as shown in Fig. 2.32(b) where, clearly

$$R = F_1 + F_2 \tag{2.6}$$

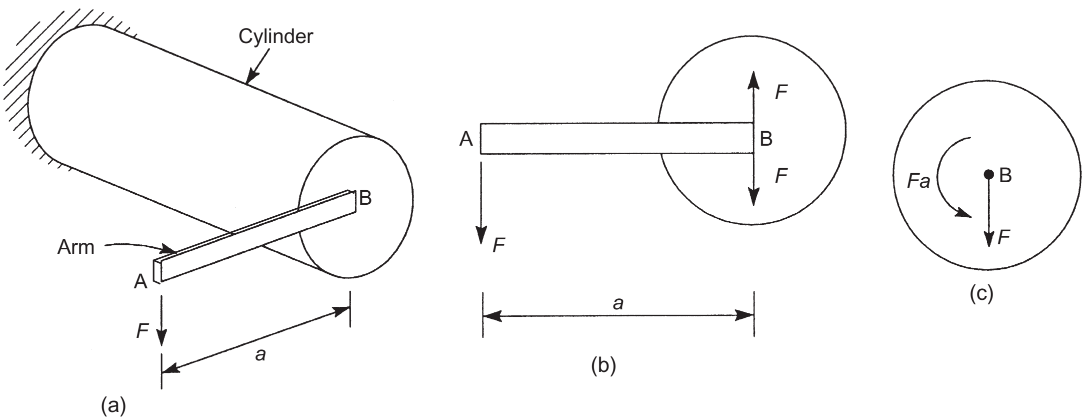

FIGURE 2.31

Equivalent force system.

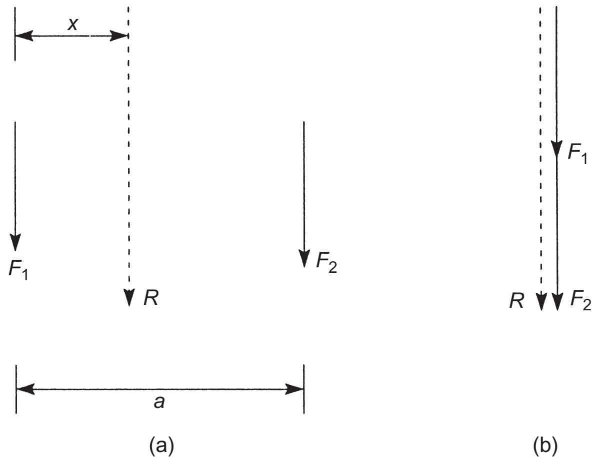

FIGURE 2.32

Resultant of a system of parallel forces.

The position of the line of action of R may be found using the principle stated above, i.e. the sum of the moments of F_1 and F_2 about any point must be equivalent to the moment of R about the same point. Thus from Fig. 2.32(a) and taking moments about, say, the line of action of F_1 we have

$$F_2 a = Rx = (F_1 + F_2)x$$

Hence

$$x = \frac{F_2}{F_1 + F_2} a \tag{2.7}$$

Note that the action of R is *equivalent* to that of F_1 and F_2, so that, in this case, we equate clockwise to clockwise moments.

The principle of equivalence may be extended to any number of parallel forces irrespective of their directions and is of particular use in the calculation of the position of centroids of area, as we shall see in Section 9.6.

EXAMPLE 2.8

Find the magnitude and position of the line of action of the resultant of the force system shown in Fig. 2.33.

In this case the polygon of forces (Fig. 2.13(b)) degenerates into a straight line and

$$R = 2 - 3 + 6 + 1 = 6\,\text{kN} \tag{i}$$

Suppose that the line of action of R is at a distance x from the 2 kN force, then, taking moments about the 2 kN force

$$Rx = -3 \times 0.6 + 6 \times 0.9 + 1 \times 1.2$$

Substituting for R from Eq. (i) we have

$$6x = -1.8 + 5.4 + 1.2$$

which gives

$$x = 0.8\,\text{m}$$

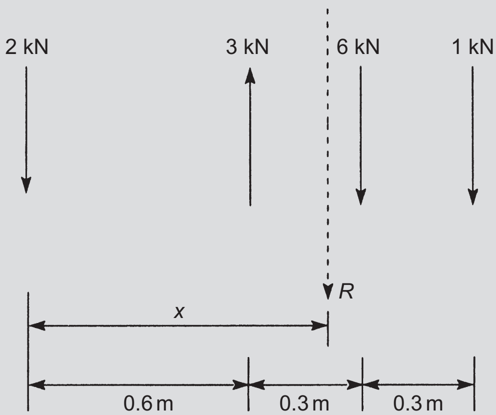

FIGURE 2.33

Force system of Ex. 2.8.

We could, in fact, take moments about any point, say now the 6 kN force. Then

$$R(0.9 - x) = 2 \times 0.9 - 3 \times 0.3 - 1 \times 0.3$$

so that

$$x = 0.8\,\text{m as before}$$

Note that in the second solution, anticlockwise moments have been selected as positive.

2.4 Equilibrium of force systems

We have seen in Section 2.1 that, for a particle or a body to remain stationary, i.e. in statical equilibrium, the resultant force on the particle or body must be zero. It follows that if a body (generally in structural analysis we are concerned with bodies, i.e. structural members, not particles) is not to move in a particular direction, the resultant force in that direction must be zero. Furthermore, the prevention of the movement of a body in two directions at right angles ensures that the body will not move in any direction at all. Then, for such a body to be in equilibrium, the sum of the components of all the forces acting on the body in any two mutually perpendicular directions must be zero. In mathematical terms and choosing, say, the x and y directions as the mutually perpendicular directions, the condition may be written

$$\sum F_x = 0 \quad \sum F_y = 0 \tag{2.8}$$

However, the condition specified by Eq. (2.8) is not sufficient to guarantee the equilibrium of a body acted on by a system of coplanar forces. For example, in Fig. 2.34 the forces F acting on a plate resting on a horizontal surface satisfy the condition $\Sigma F_x = 0$ (there are no forces in the y direction so that $\Sigma F_y = 0$ is automatically satisfied), but form a couple Fa which will cause the plate to rotate in an anticlockwise sense so long as its magnitude is sufficient to overcome the frictional resistance between the plate and the surface. We have also seen that a couple exerts the same moment about any point in its plane so that we may deduce a further condition for the statical equilibrium of a body acted upon by a system of coplanar forces, namely, that the sum of the moments of all the forces acting on the body about *any* point in their plane must be zero. Therefore, designating a moment in the xy plane about the z axis as M_z, we have

$$\sum M_z = 0 \tag{2.9}$$

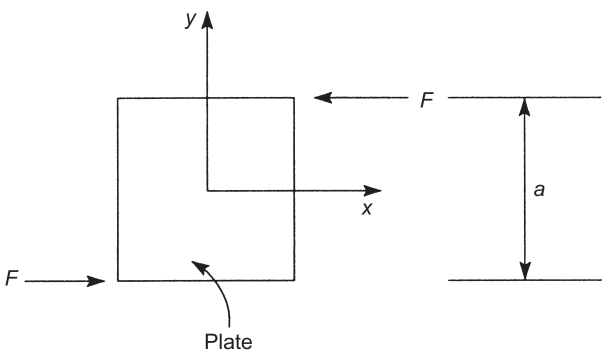

FIGURE 2.34

Couple produced by out-of-line forces.

Combining Eqs (2.8) and (2.9) we obtain the necessary conditions for a system of coplanar forces to be in equilibrium, i.e.

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_z = 0 \tag{2.10}$$

The above arguments may be extended to a three-dimensional force system which is, again, referred to an xyz axis system. Thus for equilibrium

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0 \tag{2.11}$$

and

$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0 \tag{2.12}$$

2.5 Calculation of support reactions

The conditions of statical equilibrium, Eq. (2.10), are used to calculate reactions at supports in structures so long as the support system is statically determinate (see Section 1.5). Generally the calculation of support reactions is a necessary preliminary to the determination of internal force and stress distributions and displacements.

EXAMPLE 2.9

Calculate the support reactions in the simply supported beam ABCD shown in Fig. 2.35.

The different types of support have been discussed in Section 1.4. In Fig. 2.35 the support at A is a pinned support which allows rotation but no translation in any direction, while the support at D allows rotation and translation in a horizontal direction but not in a vertical direction. Therefore there will be no moment reactions at A or D and only a vertical reaction at D, R_D. It follows that the horizontal component of the 5 kN load can only be resisted by the support at A, $R_{A,H}$, which, in addition, will provide a vertical reaction, $R_{A,V}$.

Since the forces acting on the beam are coplanar, Eqs. (2.10) are used. From the first of these, i.e. $\sum F_x = 0$, we have

$$R_{A,H} - 5\cos 60° = 0$$

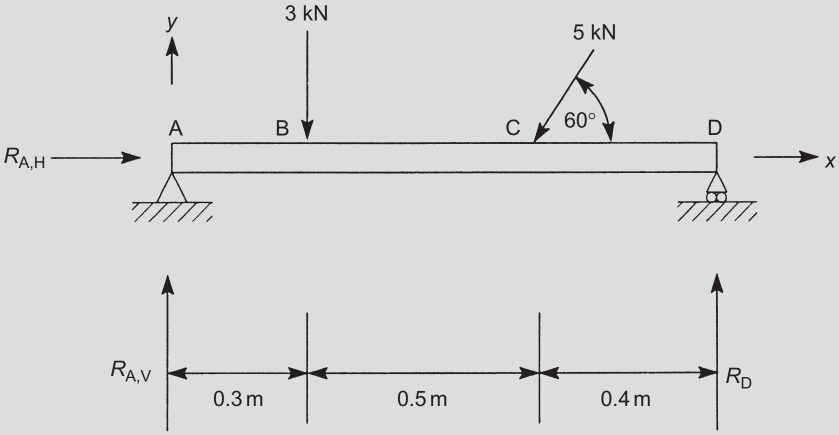

FIGURE 2.35

Beam of Ex. 2.9.

which gives

$$R_{A,H} = 2.5\,\text{kN}$$

The use of the second equation, $\Sigma F_y = 0$, at this stage would not lead directly to either $R_{A,V}$ or R_D since both would be included in the single equation. A better approach is to use the moment equation, $\Sigma M_z = 0$, and take moments about either A or D (it is immaterial which), thereby eliminating one of the vertical reactions. Taking moments, say, about D, we have

$$R_{A,V} \times 1.2 - 3 \times 0.9 - (5 \sin 60°) \times 0.4 = 0 \tag{i}$$

Note that in Eq. (i) the moment of the 5 kN force about D may be obtained either by calculating the perpendicular distance of its line of action from D (0.4 sin 60°) or by resolving it into vertical and horizontal components (5 sin 60° and 5 cos 60°, respectively) where only the vertical component exerts a moment about D. From Eq. (i)

$$R_{A,V} = 3.7\,\text{kN}$$

The vertical reaction at D may now be found using $\sum F_y = 0$ or by taking moments about A, which would be slightly lengthier. Thus

$$R_D + R_{A,V} - 3 - 5 \sin 60° = 0$$

so that

$$R_D = 3.6\,\text{kN}$$

EXAMPLE 2.10

Calculate the reactions at the support in the cantilever beam shown in Fig. 2.36.

The beam has a fixed support at A which prevents translation in any direction and also rotation. The loads applied to the beam will therefore induce a horizontal reaction, $R_{A,H}$, at A and a vertical reaction, $R_{A,V}$, together with a moment reaction M_A. Using the first of Eqs. (2.10), $\Sigma F_x = 0$, we obtain

$$R_{A,H} - 2 \cos 45° = 0$$

whence

$$R_{A,H} = 1.4\,\text{kN}$$

From the second of Eqs. (2.10), $\sum F_y = 0$

$$R_{A,V} - 5 - 2 \sin 45° = 0$$

which gives

$$R_{A,V} = 6.4\,\text{kN}$$

Finally from the third of Eqs. (2.10), $\Sigma M_z = 0$, and taking moments about A, thereby eliminating $R_{\mathrm{A,H}}$ and $R_{\mathrm{A,V}}$

$$M_{\mathrm{A}} - 5 \times 0.4 - (2 \sin 45°) \times 1.0 = 0$$

from which

$$M_{\mathrm{A}} = 3.4\,\mathrm{kN\,m}$$

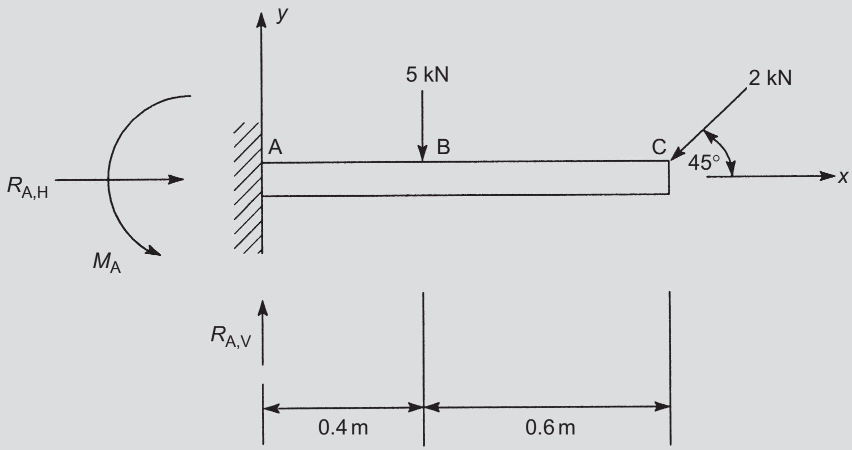

FIGURE 2.36

Beam of Ex. 2.10.

In Exs 2.9 and 2.10, the directions or sense of the support reactions is reasonably obvious. However, where this is not the case, a direction or sense is assumed which, if incorrect, will result in a negative value.

Occasionally the resultant reaction at a support is of interest. In Ex. 2.9 the resultant reaction at A is found using the first of Eqs. (2.4), i.e.

$$R_{\mathrm{A}}^2 = R_{\mathrm{A,H}}^2 + R_{\mathrm{A,V}}^2$$

which gives

$$R_{\mathrm{A}}^2 = 2.5^2 + 3.7^2$$

so that

$$R_{\mathrm{A}} = 4.5\,\mathrm{kN}$$

The inclination of R_{A} to, say, the vertical is found from the second of Eqs. (2.4). Thus

$$\tan\theta = \frac{R_{\mathrm{A,H}}}{R_{\mathrm{A,V}}} = \frac{2.5}{3.7} = 0.676$$

from which

$$\theta = 34.0°$$

EXAMPLE 2.11

Calculate the reactions at the supports in the plane truss shown in Fig. 2.37.

The truss is supported in the same manner as the beam in Ex. 2.9 so that there will be horizontal and vertical reactions at A and only a vertical reaction at B.

The angle of the truss, α, is given by

$$\alpha = \tan^{-1}\left(\frac{2.4}{3}\right) = 38.7°$$

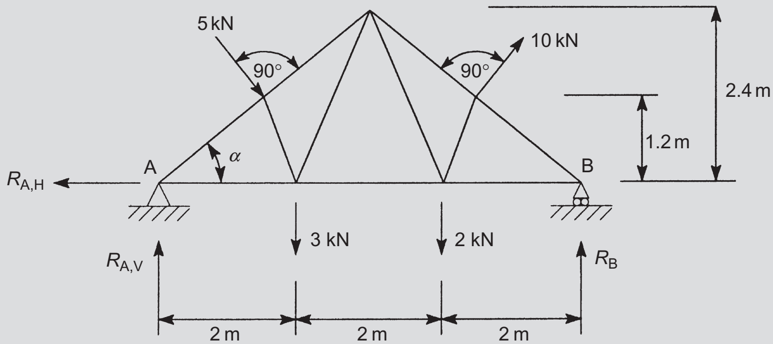

FIGURE 2.37

Truss of Ex. 2.11.

From the first of Eqs. (2.10) we have

$$R_{A,H} - 5\sin 38.7° - 10\sin 38.7° = 0$$

from which

$$R_{A,H} = 9.4\,\text{kN}$$

Now taking moments about B, say, $(\sum M_B = 0)$

$$R_{A,V} \times 6 - (5\cos 38.7°) \times 4.5 + (5\sin 38.7°) \times 1.2 + (10\cos 38.7°)$$
$$\times 1.5 + (10\sin 38.7) \times 1.2 - 3 \times 4 - 2 \times 2 = 0$$

which gives

$$R_{A,V} = 1.8\,\text{kN}$$

The resultant reaction at A is then

$$R_A = \sqrt{R_{A,V}{}^2 + R_{A,H}{}^2} = \sqrt{1.8^2 + 9.4^2} = 9.6\,\text{kN}$$

and its angle to the horizontal is given by

$$\tan^{-1} = \frac{1.8}{9.4} = 10.8°$$

Note that in the moment equation it is simpler to resolve the 5 kN and 10 kN loads into horizontal and vertical components at their points of application and then take moments rather than calculate the perpendicular distance of each of their lines of action from B.

The reaction at B, R_B, is now most easily found by resolving vertically ($\sum F_y = 0$), i.e.

$$R_B + R_{A,V} - 5\cos 38.7° + 10\cos 38.7° - 3 - 2 = 0$$

which gives

$$R_B = -0.7\ \text{kN}$$

In this case the negative sign of R_B indicates that the reaction is downward, not upward, as initially assumed.

PROBLEMS

P.2.1. State the direction of motion of the cube of material shown *in plan* in Fig. P.2.1(a)–(d) which is subjected to an applied force, F, and which is supported on a horizontal surface where the frictional force between the surface and the underside of the cube is f.

Ans.
(a) Translation parallel to BA.
(b) Translation parallel to BD, clockwise rotation.
(c) No translation, possible clockwise rotation.
(d) Translation at an angle of 28.7° to AD, anticlockwise rotation.

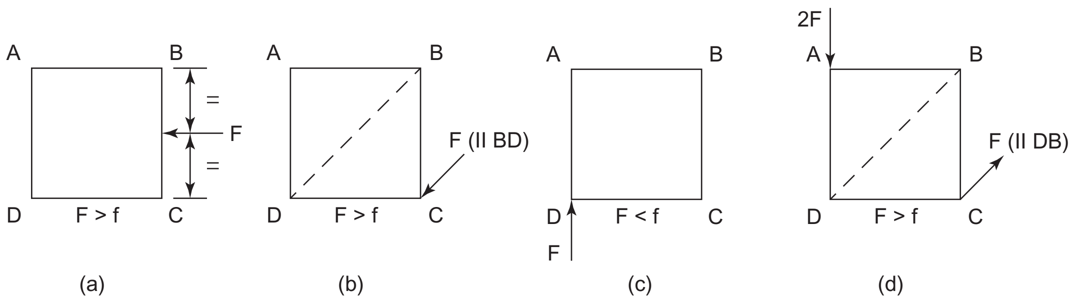

FIGURE P.2.1

P.2.2. Determine the magnitude and inclination of the resultant of the two forces acting at the point O in Fig. P.2.2 (a) by a graphical method and (b) by calculation.

Ans. 21.8 kN, 23.4° to the direction of the 15 kN load.

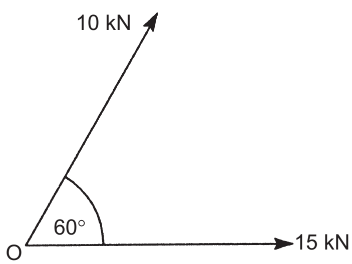

FIGURE P.2.2

P.2.3. Determine the magnitude and inclination of the resultant of the system of concurrent forces shown in Fig. P.2.3, (a) by a graphical method and (b) by calculation.

Ans. 8.6 kN, 23.9° down and to the left.

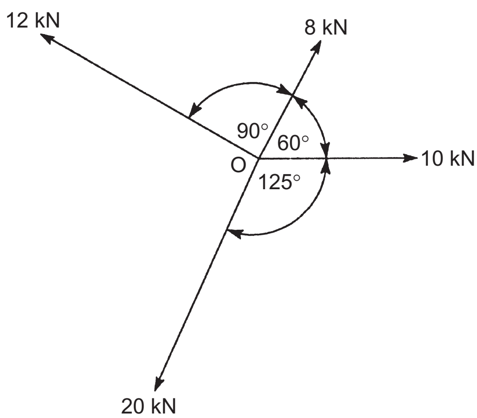

FIGURE P.2.3

P.2.4. A loading crane is attached to the outside wall of a warehouse and comprises two steel beams AB and BC as shown in Fig. P.2.4. The beams are pinned at B and attached to the warehouse wall via hinges in brackets at A and C. If the crane carries a load of 10 kN, calculate the forces in the beams AB and BC. Verify your answer using a graphical method.

Ans. AB = 5.0 kN (tension), BC = 11.2 kN (compression).

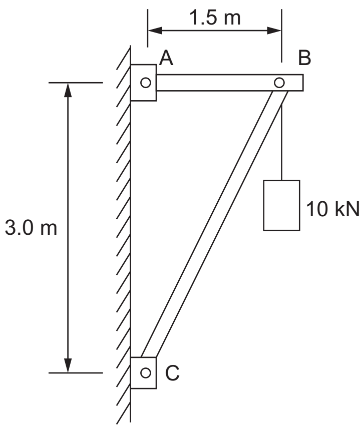

FIGURE P.2.4

P.2.5. Use a graphical method to verify that the result reaction at A in the truss of Ex. 2.11 is 9.6 kN at an angle of 10.8° to the horizontal.

P.2.6. Determine the resultant of the system of parallel forces shown in Fig. P.2.6 and hence find its components parallel to the x and y axes. At what point does the resultant cut the x axis?

Ans. $R = 11.0$ kN, $R_x = 5.5$ kN, $R_y = 9.5$ kN, (0.94, 0).

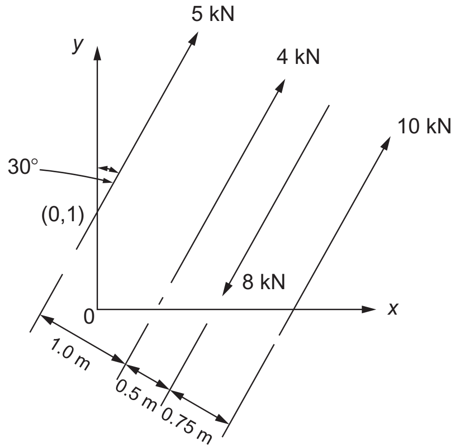

FIGURE P.2.6

P.2.7. The circular section cylinder shown in Fig. P.2.7 is built-in at one end and carries a series of loads applied via a horizontal bar at its free end. Calculate the resultant downward force on the cylinder, the applied torque and the bending moment at its built-in end.

Ans. 21 kN, 15.5 kNm anticlockwise, 52.5 kNm.

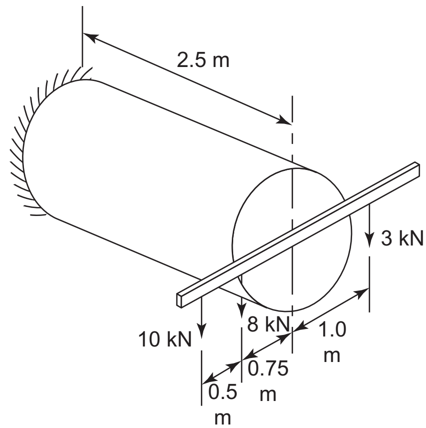

FIGURE P.2.7

P.2.8. Determine the line of action of the resultant of the loads on the free end of the cylinder of P.2.7 and then show that this may be replaced by a vertically downward load acting through the centre of the cylinder together with an anticlockwise torque of 15.5 kNm.

P.2.9. Calculate the magnitude, inclination and point of action of the resultant of the system of non-concurrent forces shown in Fig. P.2.9. The coordinates of the points of action are given in metres.

Ans. 130.4 kN, 49.5° to the x direction at the point (0.81, 1.22).

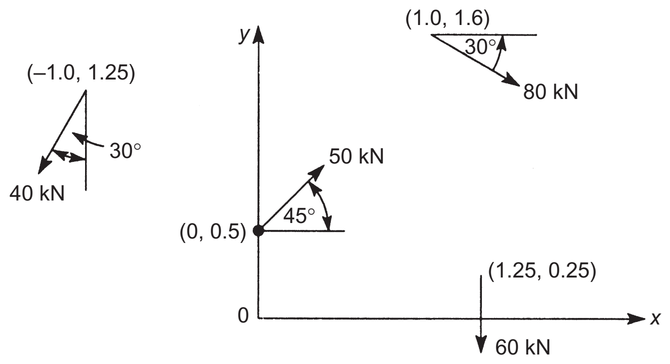

FIGURE P.2.9

P.2.10. Calculate the support reactions in the beams shown in Fig. P.2.10(a)–(d).

Ans.

(a) $R_{A,H}=9.2$ kN to left, $R_{A,V}=6.9$ kN upwards, $R_B=7.9$ kN upwards.
(b) $R_A=65$ kN, $M_A=400$ kNm anticlockwise.
(c) $R_{A,H}=20$ kN to right, $R_{A,V}=22.5$ kN upwards, $R_B=12.5$ kN upwards.
(d) $R_A=41.8$ kN upwards, $R_B=54.2$ kN upwards.

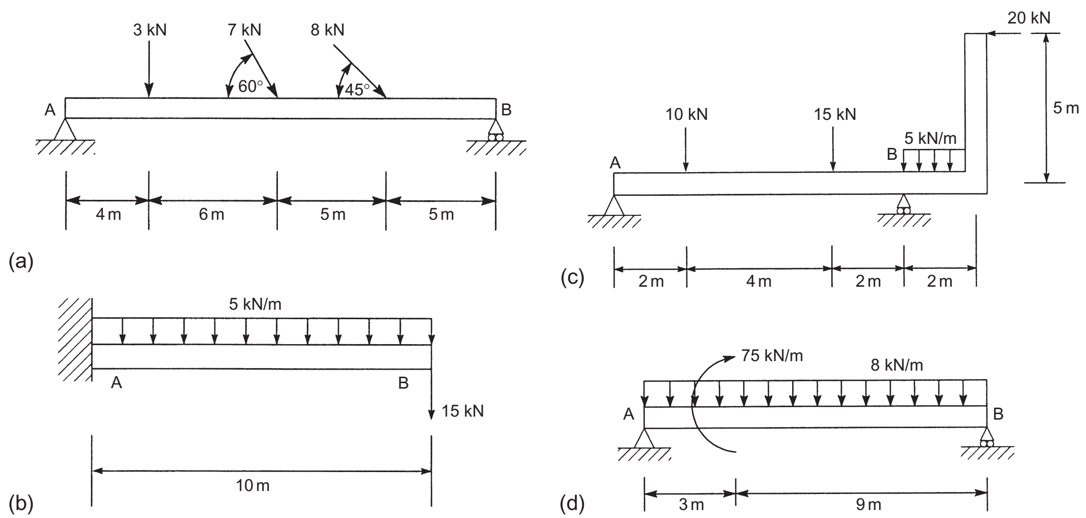

FIGURE P.2.10

P.2.11. Calculate the support reactions in the plane trusses shown in Fig. P.2.11(a) and (b).

Ans.

(a) $R_A=57$ kN upwards, $R_B=2$ kN downwards.
(b) $R_{A,H}=3713.6$ N to left, $R_{A,V}=835.6$ N downwards, $R_B=4735.3$ N downwards.

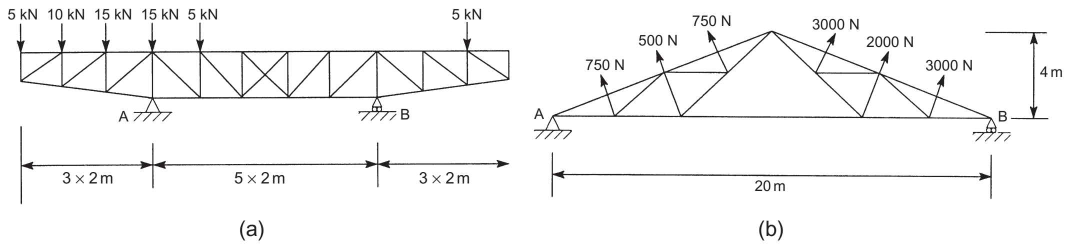

FIGURE P.2.11

P.2.12. The position of the lamp post shown in Fig. P.2.12 is referred to an *xyz* axis system. It lies in the *yz* plane and is fabricated from material weighing 500 N/m. If the worst case wind loads on the lamp post are represented by 600 N on the vertical portion and 250 N on the semi-circular portion, both acting horizontally in the *x* direction, determine the reactions at the built-in base of the lamp post.

Ans. Vertical reaction = 3442.5 N. Bending moment = 2947.6 Nm acting in a vertical plane at an angle of 79.7° to the vertical *yz* plane.
Torque = 150 Nm.

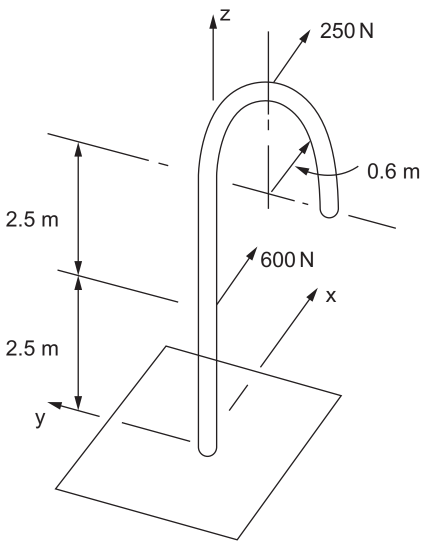

FIGURE P.2.12