Appendix 10: Transformation of the Moments and Products of Inertia from a Body Axes Reference to a Wind Axes Reference

A10.1 Introduction

In the same way that it is sometimes necessary to transform the aerodynamic stability and control derivatives from a body axes reference to a wind axes reference and vice versa, it is necessary to transform the corresponding moments and products of inertia. Again, the procedure is very straightforward and makes use of the transformation relationships discussed in Chapter 2. It is assumed that the body axes and wind axes in question have a common origin at the *cg* of the aeroplane, and that the aeroplane is in steady level symmetric flight. Thus the axes differ only by the steady body incidence α_e as shown in Fig. 2.2.

A10.2 Coordinate Transformation

A10.2.1 Body Axes to Wind Axes

A set of coordinates in a body axes system x_b, y_b, z_b may be transformed into the equivalent set in a wind axes system x_w, y_w, z_w by application of the inverse direction cosine matrix given by equation (2.13). Writing $\theta = \alpha_e$ and $\phi = \psi = 0$, since level symmetric flight is assumed,

$$\begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & \sin\alpha_e \\ 0 & 1 & 0 \\ -\sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \tag{A10.1}$$

or

$$\begin{aligned} x_w &= x_b\cos\alpha_e + z_b\sin\alpha_e \\ y_w &= y_b \\ z_w &= z_b\cos\alpha_e - x_b\sin\alpha_e \end{aligned} \tag{A10.2}$$

A10.2.2 Wind Axes to Body Axes

A set of coordinates in a wind axes system x_w, y_w, z_w may be transformed into the equivalent set in a body axes system x_b, y_b, z_b by application of the direction cosine matrix given by equation (2.12). Again, writing $\theta = \alpha_e$ and $\phi = \psi = 0$, since level symmetric flight is assumed,

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & -\sin\alpha_e \\ 0 & 1 & 0 \\ \sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} \tag{A10.3}$$

which is simply the inverse of equation (A10.1). Alternatively,

$$\begin{aligned} x_b &= x_w\cos\alpha_e - z_w\sin\alpha_e \\ y_b &= y_w \\ z_b &= z_w\cos\alpha_e + x_w\sin\alpha_e \end{aligned} \tag{A10.4}$$

A10.3 Transformation of the Moment of Inertia in Roll from a Body Axes Reference to a Wind Axes Reference

The moment of inertia in roll is defined in Chapter 4, Table 4.1, and may be written when referenced to wind axes as

$$I_{x_w} = \sum \delta m(y_w^2 + z_w^2) \tag{A10.5}$$

Substitute for y_w and z_w from equations (A10.2) to obtain

$$I_{x_w} = \sum \delta m(y_b^2 + z_b^2) + \sum \delta m(x_b^2 - z_b^2)\sin^2\alpha_e - 2\sum \delta m x_b z_b \sin\alpha_e \cos\alpha_e \tag{A10.6}$$

Add the following null expression to the right-hand side of equation (A10.6):

$$\sum \delta m(y_b^2 + z_b^2)\sin^2\alpha_e - \sum \delta m(y_b^2 + z_b^2)\sin^2\alpha_e$$

Rearrange to obtain

$$I_{x_w} = \sum \delta m(y_b^2 + z_b^2)\cos^2\alpha_e + \sum \delta m(x_b^2 + y_b^2)\sin^2\alpha_e - 2\sum \delta m x_b z_b \sin\alpha_e \cos\alpha_e \tag{A10.7}$$

Referring to the definitions of moments and products of inertia in Chapter 4, Table 4.1, equation (A10.7) may be rewritten as

$$I_{x_w} = I_{x_b}\cos^2\alpha_e + I_{z_b}\sin^2\alpha_e - 2I_{xz_b}\sin\alpha_e \cos\alpha_e \tag{A10.8}$$

Equation (A10.8) therefore describes the inertia transformation from a body axes reference to a wind axes reference.

This simple procedure may be repeated to obtain all transformations of moment and product of inertia from a body axes reference to a wind axes reference. The inverse procedure, using the coordinate transformations given by equations (A10.4), is equally straightforward for obtaining the corresponding transformations from a wind axes reference to a body axes reference.

A10.4 Summary

The *body axes to wind axes* moments and products of inertia transformations are summarised in Table A10.1. The corresponding transformations from *wind axes to body axes* obtained by the inverse procedure are summarised in Table A10.2.

Table A10.1 Moment and Product of Inertia Transformations from a Body Axes Reference to a Wind Axes Reference

Wind Axes	Body Axes
I_{x_w}	$I_{x_b}\cos^2\alpha_e + I_{z_b}\sin^2\alpha_e - 2I_{xz_b}\sin\alpha_e\cos\alpha_e$
I_{y_w}	I_{y_b}
I_{z_w}	$I_{z_b}\cos^2\alpha_e + I_{x_b}\sin^2\alpha_e + 2I_{xz_b}\sin\alpha_e\cos\alpha_e$
I_{xy_w}	$I_{xy_b}\cos\alpha_e + I_{yz_b}\sin\alpha_e$
I_{xz_w}	$I_{xz_b}(\cos^2\alpha_e - \sin^2\alpha_e) + (I_{x_b} - I_{z_b})\sin\alpha_e\cos\alpha_e$
I_{yz_w}	$I_{yz_b}\cos\alpha_e - I_{xy_b}\sin\alpha_e$

Table A10.2 Moment and Product of Inertia Transformations from a Wind Axes Reference to a Body Axes Reference

Body Axes	Wind Axes
I_{x_b}	$I_{x_w}\cos^2\alpha_e + I_{z_w}\sin^2\alpha_e + 2I_{xz_w}\sin\alpha_e\cos\alpha_e$
I_{y_b}	I_{y_w}
I_{z_b}	$I_{z_w}\cos^2\alpha_e + I_{x_w}\sin^2\alpha_e - 2I_{xz_w}\sin\alpha_e\cos\alpha_e$
I_{xy_b}	$I_{xy_w}\cos\alpha_e - I_{yz_w}\sin\alpha_e$
I_{xz_b}	$I_{xz_w}(\cos^2\alpha_e - \sin^2\alpha_e) + (I_{z_w} - I_{x_w})\sin\alpha_e\cos\alpha_e$
I_{yz_b}	$I_{yz_w}\cos\alpha_e + I_{xy_w}\sin\alpha_e$