CHAPTER

Complex Stress and Strain

14

In Chapters 7, 9, 10 and 11 we determined stress distributions produced separately by axial load, bending moment, shear force and torsion. However, in many practical situations some or all of these force systems act simultaneously so that the various stresses are combined to form complex systems which may include both direct and shear stresses. In such cases it is no longer a simple matter to predict the mode of failure of a structural member, particularly since, as we shall see, the direct and shear stresses at a point due to, say, bending and torsion combined are not necessarily the maximum values of direct and shear stress at that point.

Therefore as a preliminary to the investigation of the theories of elastic failure in Section 14.10 we shall examine states of stress and strain at points in structural members subjected to complex loading systems.

14.1 Representation of stress at a point

We have seen that, generally, stress distributions in structural members vary throughout the member. For example the direct stress in a cantilever beam carrying a point load at its free end varies along the length of the beam and throughout its depth. Suppose that we are interested in the state of stress at a point lying in the vertical plane of symmetry and on the upper surface of the beam mid-way along its span. The direct stress at this point on planes perpendicular to the axis of the beam can be calculated using Eq. (9.9). This stress may be imagined to be acting on two opposite sides of a very small thin element ABCD in the surface of the beam at the point (Fig. 14.1).

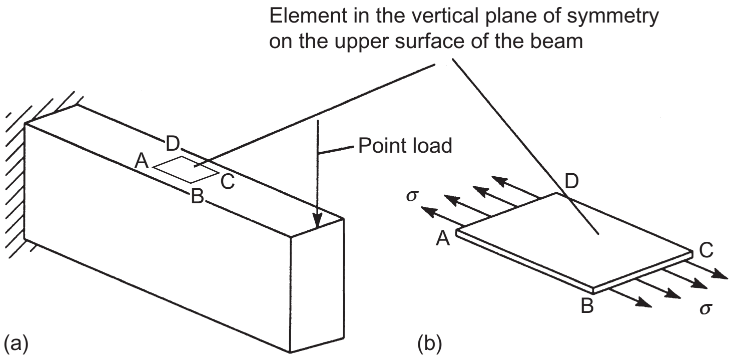

FIGURE 14.1

Representation of stress at a point in a structural member.

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Since the element is thin we can ignore any variation in direct stress across its thickness. Similarly, since the sides of the element are extremely small we can assume that σ has the same value on each opposite side BC and AD of the element and that σ is constant along these sides (in this particular case σ is constant across the width of the beam but the argument would apply if it were not). We are therefore representing the stress at a point in a structural member by a stress system acting on the sides and in the plane of a thin, very small element; such an element is known as a two-dimensional element and the stress system is a plane stress system as we saw in Section 7.11.

14.2 Determination of stresses on inclined planes

Suppose that we wish to determine the direct and shear stresses at the same point in the cantilever beam of Fig. 14.1 but on a plane PQ inclined at an angle to the axis of the beam as shown in Fig. 14.2(a). The direct stress on the sides AD and BC of the element ABCD is σ_x in accordance with the sign convention adopted previously.

Consider the triangular portion PQR of the element ABCD where QR is parallel to the sides AD and BC. On QR there is a direct stress which must also be σ_x since there is no variation of direct stress on planes parallel to QR between the opposite sides of the element. On the side PQ of the triangular element let σ_n be the direct stress and τ the shear stress. Although the stresses are uniformly distributed along the sides of the elements it is convenient to represent them by single arrows as shown in Fig. 14.2(b).

The triangular element PQR is in equilibrium under the action of forces corresponding to the stresses σ_x, σ_n and τ. Thus, resolving forces in a direction perpendicular to PQ and assuming that the element is of unit thickness we have

$$\sigma_n \, PQ = \sigma_x \, QR \cos\theta$$

or

$$\sigma_n = \sigma_x \frac{QR}{PQ} \cos\theta$$

which simplifies to

$$\sigma_n = \sigma_x \cos^2\theta \tag{14.1}$$

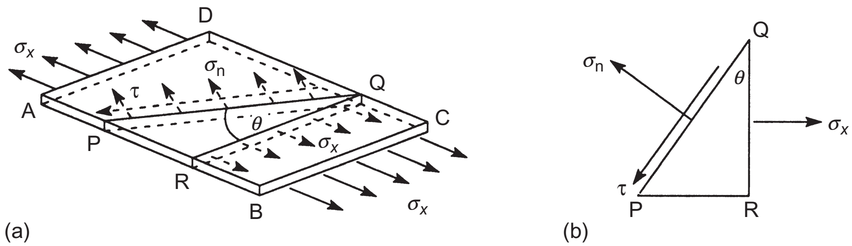

FIGURE 14.2

Determination of stresses on an inclined plane.

Resolving forces parallel to PQ

$$\tau\,\mathrm{PQ} = \sigma_x \mathrm{QR} \sin\theta$$

from which

$$\tau = \sigma_x \cos\theta \sin\theta$$

or

$$\tau = \frac{\sigma_x}{2} \sin 2\theta \tag{14.2}$$

We see from Eqs (14.1) and (14.2) that although the applied load induces direct stresses only on planes perpendicular to the axis of the beam, both direct and shear stresses exist on planes inclined to the axis of the beam. Furthermore it can be seen from Eq. (14.2) that the shear stress τ is a maximum when $\theta = 45°$. This explains the mode of failure of ductile materials subjected to simple tension and other materials such as timber under compression. For example, a flat aluminium alloy test piece fails in simple tension along a line at approximately 45° to the axis of loading as illustrated in Fig. 14.3. This suggests that the crystal structure of the metal is relatively weak in shear and that failure takes the form of sliding of one crystal plane over another as opposed to the tearing apart of two crystal planes. The failure is therefore a shear failure although the test piece is in simple tension.

Biaxial stress system

A more complex stress system may be produced by a loading system such as that shown in Fig. 14.4 where a thin-walled hollow cylinder is subjected to an internal pressure, p. The internal pressure induces circumferential or hoop stresses σ_y, given by Eq. (7.63), on planes parallel to the axis of the cylinder and, in addition, longitudinal stresses, σ_x, on planes perpendicular to the axis of the cylinder (Eq. (7.62)). Thus any two-dimensional element of unit thickness in the wall of the cylinder and having sides perpendicular and parallel to the axis of the cylinder supports a biaxial stress system as shown in Fig. 14.4. In this particular case σ_x and σ_y each have constant values irrespective of the position of the element.

Let us consider the equilibrium of a triangular portion ABC of the element as shown in Fig. 14.5. Resolving forces in a direction perpendicular to AB we have

$$\sigma_\mathrm{n}\mathrm{AB} = \sigma_x \mathrm{BC}\cos\theta + \sigma_y \mathrm{AC}\sin\theta$$

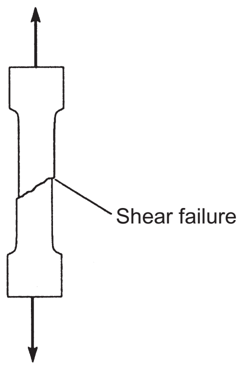

FIGURE 14.3

Mode of failure in an aluminium alloy test piece.

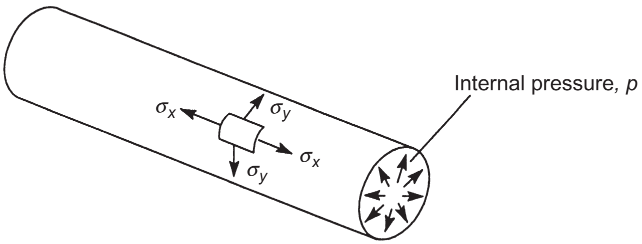

FIGURE 14.4

Generation of a biaxial stress system.

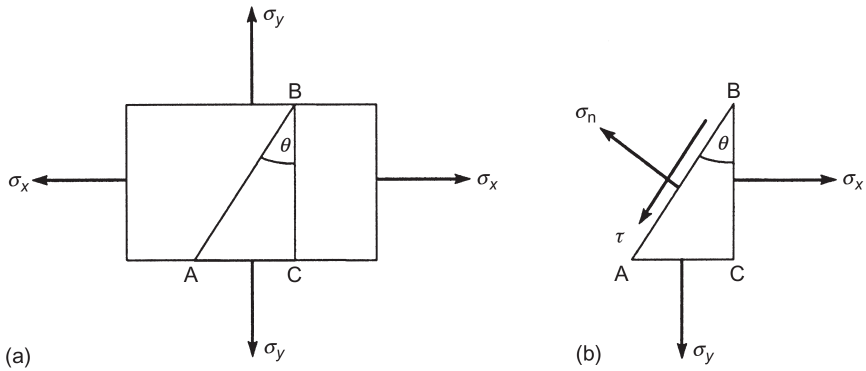

FIGURE 14.5

Determination of stresses on an inclined plane in a biaxial stress system.

or

$$\sigma_n = \sigma_x \frac{BC}{AB} \cos\theta + \sigma_y \frac{AC}{AB} \sin\theta$$

which gives

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta \tag{14.3}$$

Resolving forces parallel to AB

$$\tau\, AB = \sigma_x BC \sin\theta - \sigma_y AC \cos\theta$$

or

$$\tau = \sigma_x \frac{BC}{AB} \sin\theta - \sigma_y \frac{AC}{AB} \cos\theta$$

which gives

$$\tau = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta \tag{14.4}$$

Again we see that although the applied loads produce only direct stresses on planes perpendicular and parallel to the axis of the cylinder, both direct and shear stresses exist on inclined planes. Furthermore, for given values of σ_x and σ_y (i.e. p) the shear stress τ is a maximum on planes inclined at 45° to the axis of the cylinder.

EXAMPLE 14.1

A cylindrical pressure vessel has an internal diameter of 2 m and is fabricated from plates 20 mm thick. If the pressure inside the vessel is 1.5 N/mm^2 and, in addition, the vessel is subjected to an axial tensile load of 2500 kN, calculate the direct and shear stresses on a plane inclined at an angle of 60° to the axis of the vessel. Calculate also the maximum shear stress.

From Eq. (7.63) the circumferential stress is

$$\frac{pd}{2t} = \frac{1.5 \times 2 \times 10^3}{2 \times 20} = 75\,\mathrm{N/mm^2}$$

From Eq. (7.62) the longitudinal stress is

$$\frac{pd}{4t} = 37.5\,\mathrm{N/mm^2}$$

The direct stress due to axial load is, from Eq. (7.1)

$$\frac{2500 \times 10^3}{\pi \times 2000 \times 20} = 19.9\,\mathrm{N/mm^2}$$

Therefore on a rectangular element at any point in the wall of the vessel there is a biaxial stress system as shown in Fig. 14.6. Now considering the equilibrium of the triangular element ABC we have, resolving forces perpendicular to AB

$$\sigma_\mathrm{n}\,\mathrm{AB} \times 20 = 57.4\,\mathrm{BC} \times 20\cos 30° + 75\mathrm{AC} \times 20\cos 60°$$

Since the walls of the vessel are thin the thickness of the two-dimensional element may be taken as 20 mm. However, as can be seen, the thickness cancels out of the above equation so that it is simpler to assume unit thickness for two-dimensional elements in all cases. Then

$$\sigma_\mathrm{n} = 57.4\cos^2 30° + 75\cos^2 60°$$

which gives

$$\sigma_\mathrm{n} = 61.8\,\mathrm{N/mm^2}$$

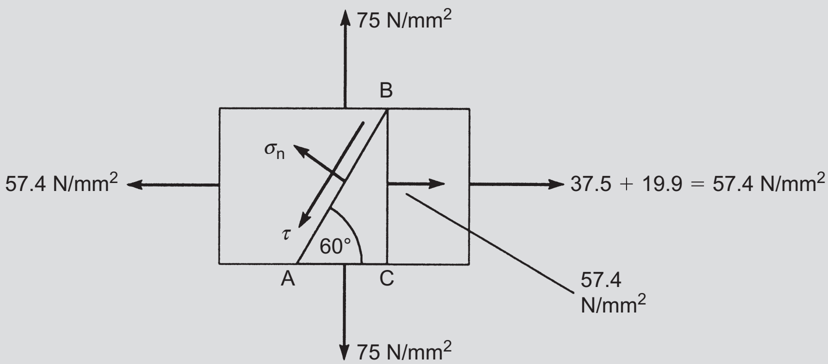

FIGURE 14.6

Biaxial stress system of Ex. 14.1.

Resolving parallel to AB

$$\tau\,\mathrm{AB} = 57.4\,\mathrm{BC}\cos 60^\circ - 75\,\mathrm{AC}\sin 60^\circ$$

or

$$\tau = 57.4\sin 60^\circ\cos 60^\circ - 75\cos 60^\circ\sin 60^\circ$$

from which

$$\tau = -7.6\,\mathrm{N/mm^2}$$

The negative sign of τ indicates that τ acts in the direction AB and not, as was assumed, in the direction BA. From Eq. (14.4) it can be seen that the maximum shear stress occurs on planes inclined at 45° to the axis of the cylinder and is given by

$$\tau_{\max} = \frac{57.4 - 75}{2} = -8.8\,\mathrm{N/mm^2}$$

Again the negative sign of $\tau_{\max}$ indicates that the direction of $\tau_{\max}$ is opposite to that assumed.

General two-dimensional case

If we now apply a torque to the cylinder of Fig. 14.4 in a clockwise sense when viewed from the right-hand end, shear and complementary shear stresses are induced on the sides of the rectangular element in addition to the direct stresses already present. The value of these shear stresses is given by Eq. (11.21) since the cylinder is thin-walled. We now have a general two-dimensional stress system acting on the element as shown in Fig. 14.7(a). The suffixes employed in designating shear stress refer to the plane on which the stress acts and its direction. Thus τ_{xy} is a shear stress acting on an x plane in the y direction. Conversely τ_{yx} acts on a y plane in the x direction. However, since $\tau_{xy} = \tau_{yx}$ we label both shear and complementary shear stresses τ_{xy} as in Fig. 14.7(b).

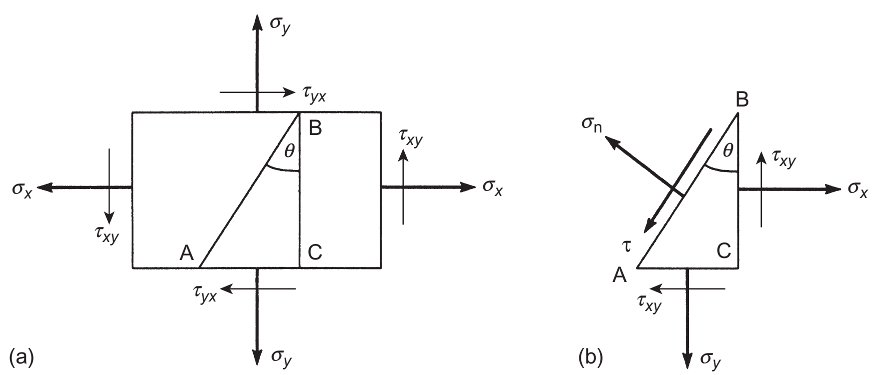

FIGURE 14.7

General two-dimensional stress system.

Considering the equilibrium of the triangular element ABC in Fig. 14.7(b) and resolving forces in a direction perpendicular to AB

$$\sigma_n \, AB = \sigma_x \, BC \cos\theta + \sigma_y \, AC \sin\theta - \tau_{xy} \, BC \sin\theta - \tau_{xy} \, AC \cos\theta$$

Dividing through by AB and simplifying we obtain

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - \tau_{xy} \sin 2\theta \tag{14.5}$$

Now resolving forces parallel to BA

$$\tau \, AB = \sigma_x \, BC \sin\theta - \sigma_y \, AC \cos\theta + \tau_{xy} \, BC \cos\theta - \tau_{xy} \, AC \sin\theta$$

Again dividing through by AB and simplifying we have

$$\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \tag{14.6}$$

EXAMPLE 14.2

A cantilever of solid, circular cross section supports a compressive load of 50 000 N applied to its free end at a point 1.5 mm below a horizontal diameter in the vertical plane of symmetry together with a torque of 1200 Nm (Fig. 14.8).

Calculate the direct and shear stresses on a plane inclined at 60° to the axis of the cantilever at a point on the lower edge of the vertical plane of symmetry.

The direct loading system is equivalent to an axial load of 50 000 N together with a bending moment of 50 000×1.5 = 75 000 Nmm in a vertical plane. Thus at any point on the lower edge of the vertical plane of symmetry there are direct compressive stresses due to axial load and bending moment which act on planes perpendicular to the axis of the beam and are given, respectively, by Eqs (7.1) and (9.9). Therefore

$$\sigma_x(\text{axial load}) = \frac{50\,000}{\pi \times 60^2/4} = 17.7 \text{ N/mm}^2$$

$$\sigma_x(\text{bending moment}) = \frac{75\,000 \times 30}{\pi \times 60^4/64} = 3.5 \text{ N/mm}^2$$

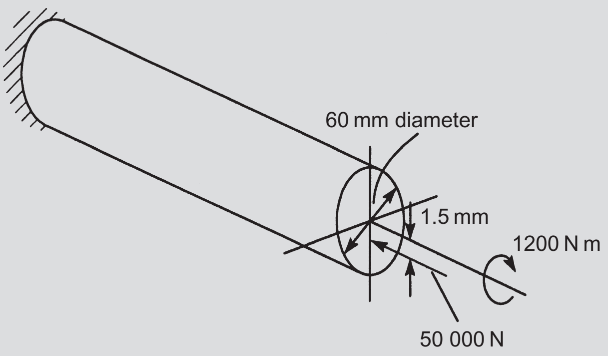

FIGURE 14.8

Cantilever beam of Ex. 14.2.

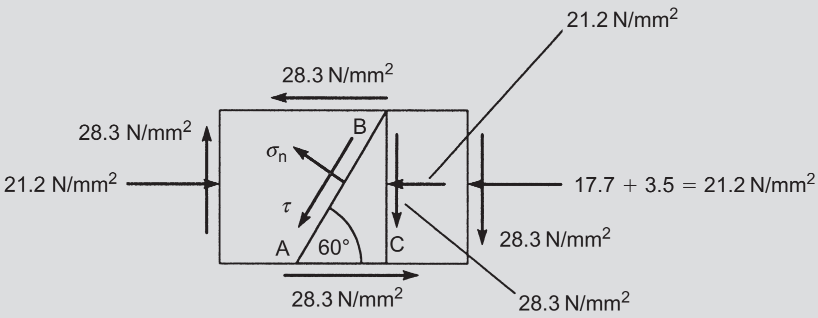

FIGURE 14.9

Two-dimensional stress system in cantilever beam of Ex. 14.2.

The shear stress τ_{xy} at the same point due to the torque is obtained from Eq. (11.4) and is

$$\tau_{xy} = \frac{1200 \times 10^3 \times 30}{\pi \times 60^4/32} = 28.3\,\text{N/mm}^2$$

The stress system acting on a two-dimensional rectangular element at the point is as shown in Fig. 14.9. Note that, in this case, the element is at the bottom of the cylinder so that the shear stress is opposite in direction to that in Fig. 14.7. Considering the equilibrium of the triangular element ABC and resolving forces in a direction perpendicular to AB we have

$$\sigma_n\,\text{AB} = -21.2\,\text{BC}\cos 30° + 28.3\,\text{BC}\sin 30° + 28.3\,\text{AC}\cos 30°$$

Dividing through by AB we obtain

$$\sigma_n = -21.2\cos^2 30° + 28.3\cos 30°\sin 30° + 28.3\sin 30°\cos 30°$$

which gives

$$\sigma_n = 8.6\,\text{N/mm}^2$$

Similarly resolving parallel to AB

$$\tau\,\text{AB} = -21.2\,\text{BC}\cos 60° - 28.3\,\text{BC}\sin 60° + 28.3\,\text{AC}\cos 60°$$

so that

$$\tau = -21.2\sin 60°\cos 60° - 28.3\sin^2 60° + 28.3\cos^2 60°$$

from which

$$\tau = -23.3\,\text{N/mm}^2$$

acting in the direction AB.

14.3 Principal stresses

Equations (14.5) and (14.6) give the direct and shear stresses on an inclined plane at a point in a structural member subjected to a combination of loads which produces a general two-dimensional stress system at that point. Clearly for given values of σ_x, σ_y and τ_{xy}, in other words a given loading system, both σ_{n} and τ vary with the angle θ and will attain maximum or minimum values when $\mathrm{d}\sigma_{\mathrm{n}}/\mathrm{d}\theta = 0$ and $\mathrm{d}\tau/\mathrm{d}\theta = 0$. From Eq. (14.5)

$$\frac{\mathrm{d}\sigma_{\mathrm{n}}}{\mathrm{d}\theta} = -2\sigma_x \cos\theta \sin\theta + 2\sigma_y \sin\theta \cos\theta - 2\tau_{xy} \cos 2\theta = 0$$

then

$$-(\sigma_x - \sigma_y)\sin 2\theta - 2\tau_{xy}\cos 2\theta = 0$$

or

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{14.7}$$

Two solutions, $-\theta$ and $-\theta - \pi/2$, satisfy Eq. (14.7) so that there are two mutually perpendicular planes on which the direct stress is either a maximum or a minimum. Furthermore, by comparison of Eqs (14.7) and (14.6) it can be seen that these planes correspond to those on which $\tau = 0$.

The direct stresses on these planes are called *principal stresses* and the planes are called *principal planes.*

From Eq. (14.7)

$$\sin 2\theta = -\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \qquad \cos 2\theta = \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

and

$$\sin 2\left(\theta + \frac{\pi}{2}\right) = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\left(\theta + \frac{\pi}{2}\right) = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Rewriting Eq. (14.5) as

$$\sigma_{\mathrm{n}} = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) - \tau_{xy}\sin 2\theta$$

and substituting for {sin 2θ, cos 2θ} and {sin $2(\theta + \pi/2)$, cos $2(\theta + \pi/2)$} in turn gives

$$\sigma_{\mathrm{I}} = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \tag{14.8}$$

$$\sigma_{\mathrm{II}} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \tag{14.9}$$

where σ_{I} is the *maximum* or *major principal stress* and σ_{II} is the *minimum* or *minor principal stress*; σ_{I} is algebraically the greatest direct stress at the point while σ_{II} is algebraically the least. Note that when σ_{II} is compressive, i.e. negative, it is possible for σ_{II} to be numerically greater than σ_{I}.

From Eq. (14.6)

$$\frac{d\tau}{d\theta} = (\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

giving

$$\tan 2\theta = \frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} \tag{14.10}$$

It follows that

$$\sin 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sin 2(\theta + \frac{\pi}{2}) = -\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2(\theta + \frac{\pi}{2}) = -\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Substituting these values in Eq. (14.6) gives

$$\tau_{\max,\min} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \tag{14.11}$$

Here, as in the case of the principal stresses, we take the maximum value as being the greater value algebraically.

Comparing Eq. (14.11) with Eqs (14.8) and (14.9) we see that

$$\tau_{\max} = \frac{\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}}}{2} \tag{14.12}$$

Equations (14.11) and (14.12) give alternative expressions for the maximum shear stress acting at the point *in the plane of the given stresses.* This is not necessarily the maximum shear stress in a three-dimensional element subjected to a two-dimensional stress system, as we shall see in Section 14.10.

Since Eq. (14.10) is the negative reciprocal of Eq. (14.7), the angles given by these two equations differ by 90° so that the planes of maximum shear stress are inclined at 45° to the principal planes.

We see now that the direct stresses, σ_x, σ_y, and shear stresses, τ_{xy}, are not, in a general case, the greatest values of direct and shear stress at the point. This fact is clearly important in designing structural members subjected to complex loading systems, as we shall see in Section 14.10. We can illustrate the stresses acting on the various planes at the point by considering a series of elements at the point as shown in Fig. 14.10. Note that generally there will be a direct stress on the planes on which $\tau_{\max}$ acts.

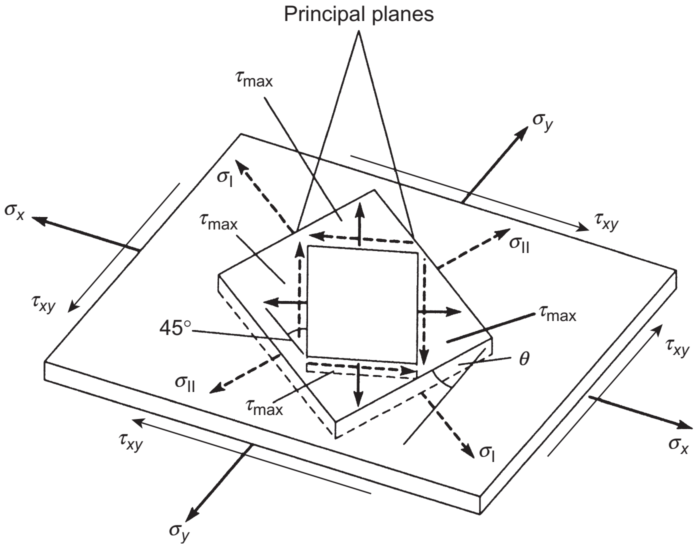

FIGURE 14.10

Stresses acting on different planes at a point in a structural member.

EXAMPLE 14.3

A structural member supports loads which produce, at a particular point, a direct tensile stress of 80 N/mm^2 and a shear stress of 45 N/mm^2 on the same plane. Calculate the values and directions of the principal stresses at the point and also the maximum shear stress, stating on which planes this will act.

Suppose that the tensile stress of 80 N/mm^2 acts in the x direction. Then $\sigma_x = +80$ N/mm^2, $\sigma_y = 0$ and $\tau_{xy} = 45$ N/mm^2. Substituting these values in Eqs (14.8) and (14.9) in turn gives

$$\sigma_{\mathrm{I}} = \frac{80}{2} + \frac{1}{2}\sqrt{80^2 + 4 \times 45^2} = 100.2\,\mathrm{N/mm^2}$$

$$\sigma_{\mathrm{II}} = \frac{80}{2} - \frac{1}{2}\sqrt{80^2 + 4 \times 45^2} = -20.2\,\mathrm{N/mm^2}$$

From Eq. (14.7)

$$\tan 2\theta = -\frac{2 \times 45}{80} = -1.125$$

from which

$$\theta = -24°11' \quad (\text{corresponding to } \sigma_{\mathrm{I}})$$

Also, the plane on which σ_{II} acts corresponds to $\theta = -24°11' - 90° = -114°11'$.

The maximum shear stress is most easily found from Eq. (14.12) and is given by

$$\tau_{max} = \frac{100.2 - (-20.2)}{2} = 60.2\,\mathrm{N/mm^2}$$

The maximum shear stress acts on planes at 45° to the principal planes. Thus $\theta = -69°11'$ and $\theta = -159°11'$ give the planes of maximum shear stress.

14.4 Mohr's circle of stress

The state of stress at a point in a structural member may be conveniently represented graphically by *Mohr's circle of stress.* We have shown that the direct and shear stresses on an inclined plane are given, in terms of known applied stresses, by

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - \tau_{xy} \sin 2\theta \,(\text{Eq. (14.5)})$$

and

$$\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \,(\text{Eq. (14.6)})$$

respectively. The positive directions of these stresses and the angle θ are shown in Fig. 14.7. We now write Eq. (14.5) in the form

$$\sigma_n = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) - \tau_{xy} \sin 2\theta$$

or

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy} \sin 2\theta \tag{14.13}$$

Now squaring and adding Eqs (14.6) and (14.13) we obtain

$$\left[\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y)\right]^2 + \tau^2 = \left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2 \tag{14.14}$$

Equation (14.14) represents the equation of a circle of radius

$$\pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

and having its centre at the point $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$.

The circle may be constructed by locating the points $Q_1(\sigma_x, -\tau_{xy})$ and $Q_2(\sigma_y, +\tau_{xy})$ referred to axes $O\sigma\tau$ as shown in Fig. 14.11. The line Q_1Q_2 is then drawn and intersects the $O\sigma$ axis at C. From Fig. 14.11

$$OC = OP_1 - CP_1 = \sigma_x - \frac{\sigma_x - \sigma_y}{2}$$

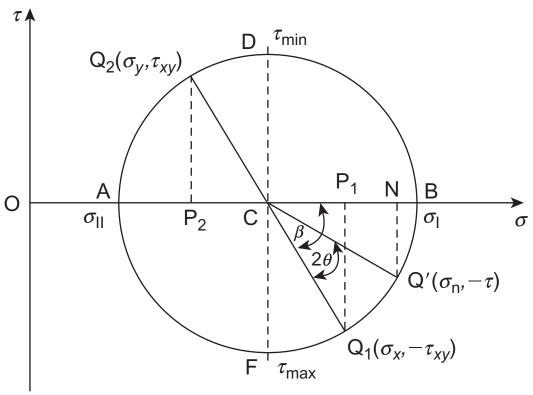

FIGURE 14.11

Mohr's circle of stress.

so that

$$\mathrm{OC} = \frac{\sigma_x + \sigma_y}{2}$$

Thus the point C has coordinates $\left(\frac{\sigma_x+\sigma_y}{2}, 0\right)$ which, as we have seen, is the centre of the circle. Also

$$\mathrm{CQ}_1 = \sqrt{\mathrm{CP}_1^2 + \mathrm{P}_1\mathrm{Q}_1^2} = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

whence

$$\mathrm{CQ}_1 = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

which is the radius of the circle; the circle is then drawn as shown.

Now we set CQ′ at an angle 2θ (positive clockwise) to CQ_1; Q′ is then the point $(\sigma_\mathrm{n}, -\tau)$ as demonstrated below.

Note that Eq.(14.7) shows that the values of θ corresponding to the two principal stresses are both negative so that in Fig. 14.11 when Q′ coincides with B ($\sigma_\mathrm{n} = \sigma_\mathrm{I}$) and A ($\sigma_\mathrm{n} = \sigma_\mathrm{II}$) 2θ then equals β and $\beta + \pi$, respectively.

Therefore, CQ′ has suffered an anticlockwise rotation through the angle 2θ, which, from the above, is negative.

From Fig. 14.11 we see that

$$\mathrm{ON} = \mathrm{OC} + \mathrm{CN}$$

or, since

$$\mathrm{OC} = (\sigma_x + \sigma_y)/2,\ \mathrm{CN} = \mathrm{CQ}' \cos(\beta - 2\theta) \text{ and } \mathrm{CQ}' = \mathrm{CQ}_1,$$

we have

$$\sigma_\mathrm{n} = \frac{\sigma_x + \sigma_y}{2} + \mathrm{CQ}_1(\cos\beta\cos 2\theta + \sin\beta\sin 2\theta)$$

But

$$\mathrm{CQ}_1 = \frac{\mathrm{CP}_1}{\cos\beta} \quad \text{and} \quad \mathrm{CP}_1 = \frac{\sigma_x - \sigma_y}{2}$$

Hence

$$\sigma_\mathrm{n} = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \mathrm{CP}_1 \tan\beta \sin 2\theta$$

which, on rearranging, becomes

$$\sigma_\mathrm{n} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - \tau_{xy}\sin 2\theta$$

as in Eq. (14.5). Similarly it may be shown that

$$\mathrm{Q'N} = -\tau_{xy}\cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta = -\tau$$

as in Eq. (14.6). It must be remembered that the construction of Fig. 14.11 corresponds to the stress system of Fig. 14.7(b); any sign reversal must be allowed for. Also the $\mathrm{O}\sigma$ and $\mathrm{O}\tau$ axes must be constructed to the same scale otherwise the circle would not be that represented by Eq (14.14).

The maximum and minimum values of the direct stress σ_n, that is the major and minor principal stresses σ_I and σ_{II}, occur when N and Q′ coincide with B and A, respectively. Thus

$$\sigma_I = OC + \text{radius of the circle}$$

i.e.

$$\sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \text{(as in Eq. (14.8))}$$

and

$$\sigma_{II} = OC - \text{radius of the circle}$$

so that

$$\sigma_{II} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \text{(as in Eq. (14.9))}$$

The principal planes are then given by $2\theta = \beta(\sigma_I)$ and $2\theta = \beta + \pi\ (\sigma_{II})$.

The maximum and minimum values of the shear stress τ occur when Q’ coincides with F and D at the lower and upper extremities of the circle. At these points $\tau_{max,min}$ are clearly equal to the radius of the circle. Hence

$$\tau_{max,\,min} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \text{(see Eq. (14.11))}$$

The minimum value of shear stress is the algebraic minimum. The planes of maximum and minimum shear stress are given by $2\theta = \beta + \pi/2$ and $2\theta = \beta + 3\pi/2$ and are inclined at 45° to the principal planes.

EXAMPLE 14.4

Direct stresses of 160 N/mm^2, tension, and 120 N/mm^2, compression, are applied at a particular point in an elastic material on two mutually perpendicular planes. The maximum principal stress in the material is limited to 200 N/mm^2, tension. Use a graphical method to find the allowable value of shear stress at the point.

First, axes Oσ τ are set up to a suitable scale. P_1 and P_2 are then located corresponding to $\sigma_x = 160$ N/mm^2 and $\sigma_y = -120$ N/mm^2, respectively; the centre C of the circle is mid-way between P_1 and P_2 (Fig. 14.12). The radius is obtained by locating B($\sigma_1 = 200$ N/mm^2) and the circle then-drawn. The maximum allowable applied shear stress, τ_{xy}, is then obtained by locating Q_1 or Q_2. The maximum shear stress at the point is equal to the radius of the circle and is 180 N/mm^2.

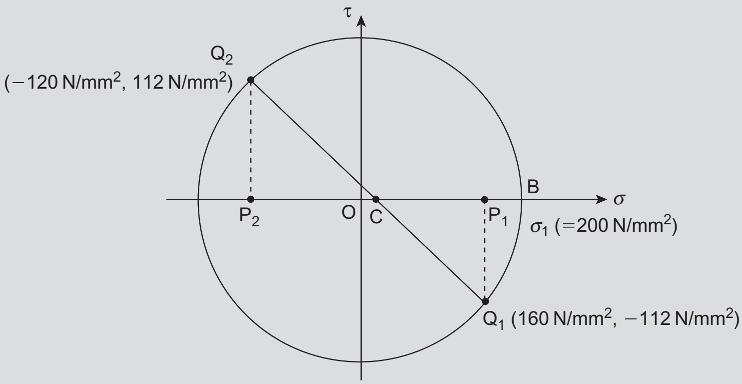

FIGURE 14.12

Mohr’s circle of stress for Ex. 14.4.

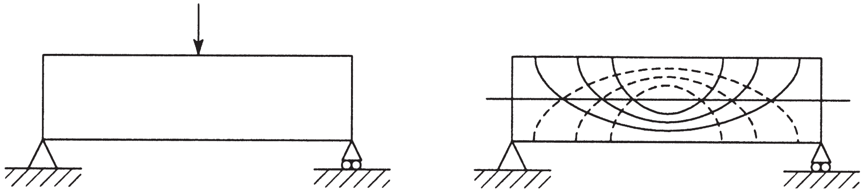

FIGURE 14.13

Stress trajectories in a beam.

14.5 Stress trajectories

We have shown that direct and shear stresses at a point in a beam produced, say, by bending and shear and calculated by the methods discussed in Chapters 9 and 10, respectively, are not necessarily the greatest values of direct and shear stress at the point. In order, therefore, to obtain a more complete picture of the distribution, magnitude and direction of the stresses in a beam we investigate the manner in which the principal stresses vary throughout a beam.

Consider the simply supported beam of rectangular section carrying a central concentrated load as shown in Fig. 14.13(a). Using Eqs (9.9) and (10.4) we can determine the direct and shear stresses at any point in any section of the beam. Subsequently from Eqs (14.8), (14.9) and (14.7) we can find the principal stresses at the point and their directions. If this procedure is followed for very many points throughout the beam, curves, to which the principal stresses are tangential, may be drawn as shown in Fig. 14.13(b). These curves are known as *stress trajectories* and form two orthogonal systems; in Fig. 14.13(b) solid lines represent tensile principal stresses and dotted lines compressive principal stresses. The two sets of curves cross each other at right angles and all curves intersect the neutral axis at 45° where the direct stress (calculated from Eq. (9.9)) is zero. At the top and bottom surfaces of the beam where the shear stress (calculated from Eq. (10.4)) is zero the trajectories have either horizontal or vertical tangents.

Another type of curve that may be drawn from a knowledge of the distribution of principal stress is a *stress contour*. Such a curve connects points of equal principal stress.

14.6 Determination of strains on inclined planes

In Section 14.2 we investigated the two-dimensional state of stress at a point in a structural member and determined direct and shear stresses on inclined planes; we shall now determine the accompanying strains.

Figure 14.14(a) shows a two-dimensional element subjected to a complex direct and shear stress system. The applied stresses will distort the rectangular element of Fig. 14.14(a) into the shape shown in Fig. 14.14(b). In particular, the triangular element ABC will suffer distortion to the shape A′B′C′ with corresponding changes in the length CD and the angle BDC. The strains associated with the stresses σ_x, σ_y and τ_{xy} are ε_x, ε_y and γ_{xy}, respectively. We shall now determine the direct strain ε_n in a direction normal to the plane AB and the shear strain γ produced by the shear stress acting on the plane AB.

To a first order of approximation

$$\left.\begin{aligned} A'C' &= AC(1+\varepsilon_x) \\ C'B' &= CB(1+\varepsilon_y) \\ A'B' &= AB(1+\varepsilon_{n+\pi/2}) \end{aligned}\right\} \qquad (14.15)$$

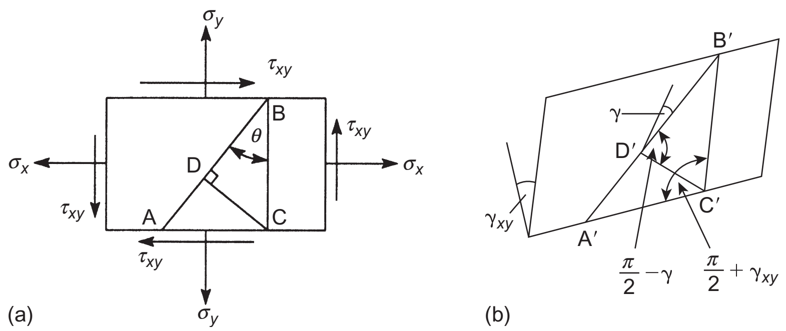

FIGURE 14.14

Determination of strains on an inclined plane.

where $\varepsilon_{n+\pi/2}$ is the direct strain in the direction AB. From the geometry of the triangle A′B′C′ in which angle B′C′A′ $=\pi/2+\gamma_{xy}$

$$(\mathrm{A'B'})^2=(\mathrm{A'C'})^2+(\mathrm{C'B'})^2-2(\mathrm{A'C'})(\mathrm{C'B'})\cos\left(\frac{\pi}{2}+\gamma_{xy}\right)$$

or, substituting from Eq. (14.15)

$$(\mathrm{AB})^2(1+\varepsilon_{n+\pi/2})^2=(\mathrm{AC})^2(1+\varepsilon_x)^2+(\mathrm{CB})^2(1+\varepsilon_y)^2+2(\mathrm{AC})(\mathrm{CB})(1+\varepsilon_x)(1+\varepsilon_y)\sin\gamma_{xy}$$

Noting that $(\mathrm{AB})^2=(\mathrm{AC})^2+(\mathrm{CB})^2$ and neglecting squares and higher powers of small quantities, this equation may be rewritten

$$2(\mathrm{AB})^2\varepsilon_{n+\pi/2}=2(\mathrm{AC})^2\varepsilon_x+2(\mathrm{CB})^2\varepsilon_y+2(\mathrm{AC})(\mathrm{CB})\gamma_{xy}$$

Dividing through by $2(\mathrm{AB})^2$ gives

$$\varepsilon_{n+\pi/2}=\varepsilon_x\sin^2\theta+\varepsilon_y\cos^2\theta+\sin\theta\cos\theta\gamma_{xy} \tag{14.16}$$

The strain ε_n in the direction normal to the plane AB is found by replacing the angle θ in Eq. (14.16) by $\theta-\pi/2$. Hence

$$\varepsilon_n=\varepsilon_x\cos^2\theta+\varepsilon_y\sin^2\theta-\frac{\gamma_{xy}}{2}\sin 2\theta \tag{14.17}$$

Now from triangle C′D′B′ we have

$$(\mathrm{C'B'})^2=(\mathrm{C'D'})^2+(\mathrm{D'B'})^2-2(\mathrm{C'D'})(\mathrm{D'B'})\cos\left(\frac{\pi}{2}-\gamma\right) \tag{14.18}$$

in which

$$\mathrm{C'B'}=\mathrm{CB}(1+\varepsilon_y)$$
$$\mathrm{C'D'}=\mathrm{CD}(1+\varepsilon_n)$$
$$\mathrm{D'B'}=\mathrm{DB}(1+\varepsilon_{n+\pi/2})$$

Substituting in Eq. (14.18) for C′B′,C′D′ and D′B′ and writing cos $(\pi/2-\gamma)=\sin\gamma$ we have

$$
\begin{aligned}
(\mathrm{CB})^2(1+\varepsilon_y)^2 =& (\mathrm{CD})^2(1+\varepsilon_\mathrm{n})^2+(\mathrm{DB})^2(1+\varepsilon_{\mathrm{n}+\pi/2})^2 \\
&-2(\mathrm{CD})(\mathrm{DB})(1+\varepsilon_\mathrm{n})(1+\varepsilon_{\mathrm{n}+\pi/2})\sin\gamma
\end{aligned}
\tag{14.19}
$$

Again ignoring squares and higher powers of strains and writing $\sin\gamma=\gamma$, Eq. (14.19) becomes

$$(\mathrm{CB})^2(1+2\varepsilon_y)=(\mathrm{CD})^2(1+2\varepsilon_\mathrm{n})+(\mathrm{DB})^2(1+2\varepsilon_{\mathrm{n}+\pi/2})-2(\mathrm{CD})(\mathrm{DB})\gamma$$

From Fig. 14.14(a) we see that $(\mathrm{CB})^2=(\mathrm{CD})^2+(\mathrm{DB})^2$ and the above equation simplifies to

$$2(\mathrm{CB})^2\varepsilon_y=2(\mathrm{CD})^2\varepsilon_\mathrm{n}+2(\mathrm{DB})^2\varepsilon_{\mathrm{n}+\pi/2}-2(\mathrm{CD})(\mathrm{DB})\gamma$$

Dividing through by $2(\mathrm{CB})^2$ and rearranging we obtain

$$\gamma=\frac{\varepsilon_\mathrm{n}\sin^2\theta+\varepsilon_{\mathrm{n}+\pi/2}\cos^2\theta-\varepsilon_y}{\sin\theta\cos\theta}$$

Substitution of ε_n and $\varepsilon_{\mathrm{n}+\pi/2}$ from Eqs (14.17) and (14.16) yields

$$\frac{\gamma}{2}=\frac{\varepsilon_x-\varepsilon_y}{2}\sin 2\theta+\frac{\gamma_{xy}}{2}\cos 2\theta \tag{14.20}$$

14.7 Principal strains

From a comparison of Eqs (14.17) and (14.20) with Eqs (14.5) and (14.6) we observe that the former two equations may be obtained from Eqs (14.5) and (14.6) by replacing σ_n by ε_n, σ_x by ε_x, σ_y by ε_y, τ_{xy} by $\gamma_{xy}/2$ and τ by $\gamma/2$. It follows that for each deduction made from Eqs (14.5) and (14.6) concerning σ_n and τ there is a corresponding deduction from Eqs (14.17) and (14.20) regarding ε_n and $\gamma/2$. Thus at a point in a structural member there are two mutually perpendicular planes on which the shear strain γ is zero and normal to which the direct strain is the algebraic maximum or minimum direct strain at the point. These direct strains are the *principal strains* at the point and are given (from a comparison with Eqs (14.8) and (14.9)) by

$$\varepsilon_\mathrm{I}=\frac{\varepsilon_x+\varepsilon_y}{2}+\frac{1}{2}\sqrt{(\varepsilon_x-\varepsilon_y)^2+\gamma_{xy}^2} \tag{14.21}$$

and

$$\varepsilon_\mathrm{II}=\frac{\varepsilon_x+\varepsilon_y}{2}-\frac{1}{2}\sqrt{(\varepsilon_x-\varepsilon_y)^2+\gamma_{xy}^2} \tag{14.22}$$

Since the shear strain γ is zero on these planes it follows that the shear stress must also be zero and we deduce from Section 14.3 that the directions of the principal strains and principal stresses coincide. The related planes are then determined from Eq. (14.7) or from

$$\tan 2\theta=-\frac{\gamma_{xy}}{\varepsilon_x-\varepsilon_y} \tag{14.23}$$

In addition the maximum shear strain at the point is given by

$$\left(\frac{\gamma}{2}\right)_{\max} = \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \tag{14.24}$$

or

$$\left(\frac{\gamma}{2}\right)_{\max} = \frac{\varepsilon_{\mathrm{I}} - \varepsilon_{\mathrm{II}}}{2} \tag{14.25}$$

(cf. Eqs (14.11) and (14.12)).

EXAMPLE 14.5

At a point in a structural member the stresses on two mutually perpendicular planes are 60 N/mm^2 tension and 30 N/mm^2 compression together with a shear stress of 15 N/mm^2. Calculate the principal stresses at the point, the maximum shear stress and the angle which the plane of maximum principal stress makes with the plane on which the 60 N/mm^2 stress acts. Verify all your answers using a graphical method. If Young's modulus $E = 200000$ N/mm^2 and Poisson's ratio $\nu = 0.3$ calculate the principal strains and the maximum shear strain.

We shall designate the 60 N/mm^2 as σ_x. Then, from Eq. (14.8)

$$\sigma_{\mathrm{I}} = \frac{60 - 30}{2} + \frac{1}{2}\sqrt{[(60+30)^2 + 4 \times 15^2]}$$

which gives

$$\sigma_{\mathrm{I}} = 62.4\,\mathrm{N/mm^2}$$

Similarly, from Eq. (14.9)

$$\sigma_{\mathrm{II}} = -32.4\,\mathrm{N/mm^2}$$

Then, from Eq. (14.12)

$$\tau_{\max} = \frac{62.4 + 32.4}{2} = 47.4\,\mathrm{N/mm^2}$$

Alternatively, $\tau_{\max}$ could have been obtained from Eq. (14.11) using the given values of stress although this would have involved slightly longer computation.

From Eq. (14.7)

$$\tan 2\theta = -\frac{2 \times 15}{60 + 30} = -0.33$$

so that

$$2\theta = -18.4^\circ \text{ or } -198.4^\circ$$

giving

$$\theta = -9.2^\circ \text{ or } -99.2^\circ$$

From Mohr's circle of stress (see Fig. 14.11) it is clear that the plane on which the maximum principal stress acts is at an angle of 9.2° to the plane on which the 60 N/mm^2 stress acts. For this particular problem the solution using Mohr's circle is shown in Fig. 14.15.

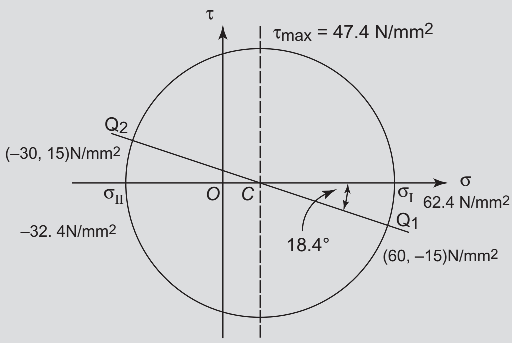

FIGURE 14.15

Mohr's circle for Ex. 14.5.

From Section 7.8

$$\varepsilon_{\mathrm{I}} = \frac{62.4}{200000} - \frac{0.3(-32.4)}{200000} = 360.6 \times 10^{-6}$$

and

$$\varepsilon_{\mathrm{II}} = \frac{-32.4}{200000} - \frac{0.3 \times 62.4}{200000} = -255.6 \times 10^{-6}$$

and from Eq. (14.25)

$$\frac{(\gamma)_{\max}}{2} = \frac{(360.6 + 255.6)}{2} \times 10^{-6}$$

which gives

$$\gamma_{\max} = 616.2 \times 10^{-6}$$

14.8 Mohr's circle of strain

The argument of Section 14.7 may be applied to Mohr's circle of stress described in Section 14.4. A circle of strain, analogous to that shown in Fig. 14.11, may be drawn when σ_x, σ_y, etc., are replaced by ε_x, ε_y, etc., as specified in Section 14.7. The horizontal extremities of the circle represent the principal strains, the radius of the circle half the maximum shear strain, and so on.

EXAMPLE 14.6

A structural member is loaded in such a way that at a particular point in the member a two-dimensional stress system exists consisting of $\sigma_x = +60$ N/mm^2, $\sigma_y = -40$ N/mm^2 and $\tau_{xy} = 50$ N/mm^2.

a. Calculate the direct strain in the x and y directions and the shear strain, γ_{xy}, at the point.

b. Calculate the principal strains at the point and determine the position of the principal planes.

c. Verify your answer using a graphical method. Take $E = 200\,000$ N/mm^2 and Poisson's ratio, $v = 0.3$.

a. From Section 7.8

$$\varepsilon_x = \frac{1}{200\,000}(60 + 0.3 \times 40) = 360 \times 10^{-6}$$

$$\varepsilon_y = \frac{1}{200\,000}(-40 - 0.3 \times 60) = -290 \times 10^{-6}$$

The shear modulus, G, is obtained using Eq. (7.21); thus

$$G = \frac{E}{2(1+v)} = \frac{200\,000}{2(1+0.3)} = 76\,923\ \text{N/mm}^2$$

Hence, from Eq. (7.9)

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{50}{76\,923} = 650 \times 10^{-6}$$

b. Now substituting in Eqs (14.21) and (14.22) for ε_x, ε_y and γ_{xy} we have

$$\varepsilon_{\mathrm{I}} = 10^{-6}\left[\frac{360 - 290}{2} + \frac{1}{2}\sqrt{(360+290)^2 + 650^2}\right]$$

which gives

$$\varepsilon_{\mathrm{I}} = 495 \times 10^{-6}$$

Similarly

$$\varepsilon_{\mathrm{II}} = -425 \times 10^{-6}$$

From Eq. (14.23) we have

$$\tan 2\theta = -\frac{650 \times 10^{-6}}{360 \times 10^{-6} + 290 \times 10^{-6}} = -1$$

Therefore

$$2\theta = -45° \text{ or } -225°$$

so that

$$\theta = -22.5° \text{ or } -112.5°$$

c. Axes Oε and O$_\gamma$ are set up and the points $Q_1(360 \times 10^{-6}, -\frac{1}{2} \times 650 \times 10^{-6})$ and $Q_2(-290 \times 10^{-6}, \frac{1}{2} \times 650 \times 10^{-6})$ located. The centre C of the circle is the intersection of Q_1Q_2 and the Oε axis (Fig. 14.16). The circle is then drawn with radius equal to CQ_1 and the points $B(\varepsilon_{\mathrm{I}})$ and $A(\varepsilon_{\mathrm{II}})$ located. Finally, angle $Q_1CB = -2\theta$ and $Q_1CA = -2\theta - \pi$.

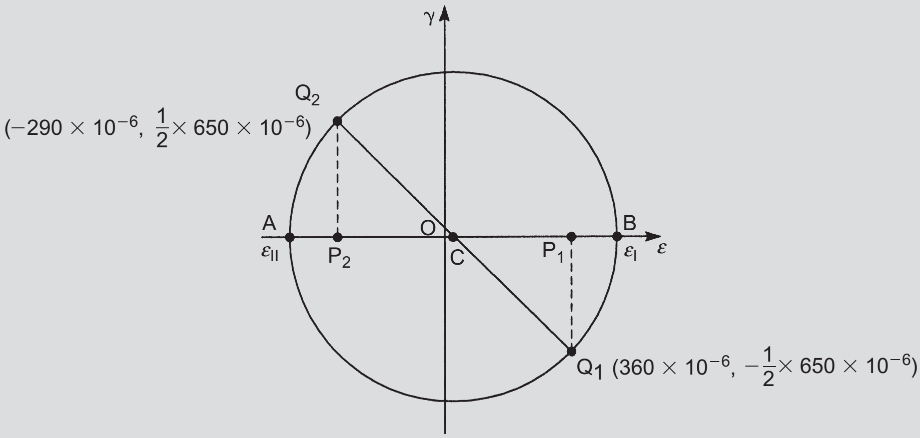

FIGURE 14.16

Mohr's circle of strain for Ex. 14.6.

14.9 Experimental measurement of surface strains and stresses

Stresses at a point on the surface of a structural member may be determined by measuring the strains at the point, usually with electrical resistance strain gauges. These consist of a short length of fine wire sandwiched between two layers of impregnated paper, the whole being glued to the surface of the member. The resistance of the wire changes as the wire stretches or contracts so that as the surface of the member is strained the gauge indicates a change of resistance which is measurable on a Wheatstone bridge.

Strain gauges measure direct strains only, but the state of stress at a point may be investigated in terms of principal stresses by using a strain gauge 'rosette'. This consists of three strain gauges inclined at a given angle to each other. Typical of these is the 45° or 'rectangular' strain gauge rosette illustrated in Fig. 14.17(a). An equiangular rosette has gauges inclined at 60°.

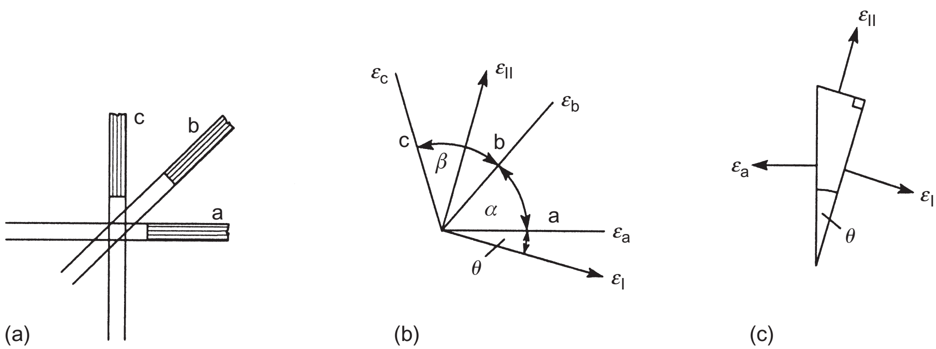

FIGURE 14.17

Electrical resistance strain gauge measurement.

Suppose that a rosette consists of three arms, 'a', 'b' and 'c' inclined at angles α and β as shown in Fig. 14.17(b). Suppose also that ε_{I} and $\varepsilon_{\mathrm{II}}$ are the principal strains at the point and that ε_{I} is inclined at an unknown angle θ to the arm 'a'. Then if ε_{a}, ε_{b} and ε_{c} are the measured strains in the directions θ, $(\theta+\alpha)$ and $(\theta+\alpha+\beta)$ to ε_{I} we have, from Eq. (14.17)

$$\varepsilon_{\mathrm{a}}=\varepsilon_{\mathrm{I}}\cos^2\theta+\varepsilon_{\mathrm{II}}\sin^2\theta \tag{14.26}$$

in which ε_{n} has become ε_{a}, ε_x has become ε_{I}, ε_y has become $\varepsilon_{\mathrm{II}}$ and γ_{xy} is zero since the x and y directions have become principal directions. This situation is equivalent, as far as ε_{a}, ε_{I} and $\varepsilon_{\mathrm{II}}$ are concerned, to the strains acting on a triangular element as shown in Fig. 14.17(c). Rewriting Eq. (14.26) we have

$$\varepsilon_{\mathrm{a}}=\frac{\varepsilon_{\mathrm{I}}}{2}(1+\cos 2\theta)+\frac{\varepsilon_{\mathrm{II}}}{2}(1-\cos 2\theta)$$

or

$$\varepsilon_{\mathrm{a}}=\frac{1}{2}(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{II}})+\frac{1}{2}(\varepsilon_{\mathrm{I}}-\varepsilon_{\mathrm{II}})\cos 2\theta \tag{14.27}$$

Similarly

$$\varepsilon_{\mathrm{b}}=\frac{1}{2}(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{II}})+\frac{1}{2}(\varepsilon_{\mathrm{I}}-\varepsilon_{\mathrm{II}})\cos 2(\theta+\alpha) \tag{14.28}$$

and

$$\varepsilon_{\mathrm{c}}=\frac{1}{2}(\varepsilon_{\mathrm{I}}+\varepsilon_{\mathrm{II}})+\frac{1}{2}(\varepsilon_{\mathrm{I}}-\varepsilon_{\mathrm{II}})\cos 2(\theta+\alpha+\beta) \tag{14.29}$$

Therefore if ε_{a}, ε_{b} and ε_{c} are measured in given directions, i.e. given angles α and β, then ε_{I}, $\varepsilon_{\mathrm{II}}$ and θ are the only unknowns in Eqs (14.27), (14.28) and (14.29).

Having determined the principal strains we obtain the principal stresses using relationships derived in Section 7.8. Thus

$$\varepsilon_{\mathrm{I}}=\frac{1}{E}(\sigma_{\mathrm{I}}-v\sigma_{\mathrm{II}}) \tag{14.30}$$

and

$$\varepsilon_{\mathrm{II}}=\frac{1}{E}(\sigma_{\mathrm{II}}-v\sigma_{\mathrm{I}}) \tag{14.31}$$

Solving Eqs (14.30) and (14.31) for σ_{I} and σ_{II} we have

$$\sigma_{\mathrm{I}}=\frac{E}{1-v^2}(\varepsilon_{\mathrm{I}}+v\varepsilon_{\mathrm{II}}) \tag{14.32}$$

and

$$\sigma_{\mathrm{II}}=\frac{E}{1-v^2}(\varepsilon_{\mathrm{II}}+v\varepsilon_{\mathrm{I}}) \tag{14.33}$$

For a 45° rosette $\alpha=\beta=45°$ and the principal strains may be obtained using the geometry of Mohr's circle of strain. Suppose that the arm 'a' of the rosette is inclined at some unknown angle θ to the maximum principal strain as in Fig. 14.17(b). Then Mohr's circle of strain is as shown in Fig. 14.18; the shear strains γ_{a}, γ_{b} and γ_{c} do not feature in the discussion and are therefore ignored.

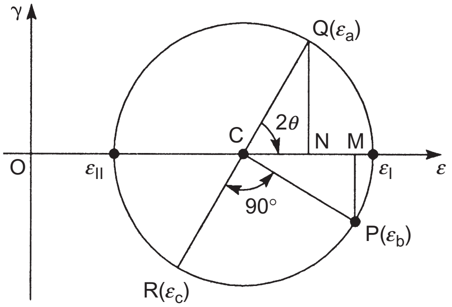

FIGURE 14.18

Mohr's circle of strain for a 45° strain gauge rosette.

From Fig. 14.18

$$\mathrm{OC} = \frac{1}{2}(\varepsilon_a + \varepsilon_c)$$
$$\mathrm{CN} = \varepsilon_a - \mathrm{OC} = \frac{1}{2}(\varepsilon_a - \varepsilon_c)$$
$$\mathrm{QN} = \mathrm{CM} = \varepsilon_b - \mathrm{OC} = \varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)$$

The radius of the circle is CQ and

$$\mathrm{CQ} = \sqrt{\mathrm{CN}^2 + \mathrm{QN}^2}$$

Hence

$$\mathrm{CQ} = \sqrt{\left[\frac{1}{2}(\varepsilon_a - \varepsilon_c)\right]^2 + \left[\varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)\right]^2}$$

which simplifies to

$$\mathrm{CQ} = \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}$$

Therefore ε_{I}, which is given by

$$\varepsilon_{\mathrm{I}} = \mathrm{OC} + \text{radius of the circle}$$

is

$$\varepsilon_{\mathrm{I}} = \frac{1}{2}(\varepsilon_a + \varepsilon_c) + \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2} \tag{14.34}$$

Also

$$\varepsilon_{\mathrm{II}} = \mathrm{OC} - \text{radius of the circle}$$

i.e.

$$\varepsilon_{\mathrm{II}} = \frac{1}{2}(\varepsilon_a + \varepsilon_c) - \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2} \tag{14.35}$$

Finally the angle θ is given by

$$\tan 2\theta = \frac{\mathrm{QN}}{\mathrm{CN}} = \frac{\varepsilon_b - (1/2)(\varepsilon_a + \varepsilon_c)}{(1/2)(\varepsilon_a - \varepsilon_c)}$$

i.e.

$$\tan 2\theta = \frac{2\varepsilon_b - \varepsilon_a - \varepsilon_c}{\varepsilon_a - \varepsilon_c} \tag{14.36}$$

A similar approach can be adopted for a 60° rosette.

EXAMPLE 14.7

A shaft of solid circular cross section has a diameter of 50 mm and is subjected to a torque, T, and axial load, P. A rectangular strain gauge rosette attached to the surface of the shaft recorded the following values of strain: $\varepsilon_a = 1000 \times 10^{-6}$, $\varepsilon_b = -200 \times 10^{-6}$ and $\varepsilon_c = -300 \times 10^{-6}$ where the gauges 'a' and 'c' are in line with and perpendicular to the axis of the shaft, respectively. If the material of the shaft has a Young's modulus of 70 000 N/mm^2 and a Poisson's ratio of 0.3, calculate the values of T and P.

Substituting the values of ε_a, ε_b and ε_c in Eq. (14.34) we have

$$\varepsilon_{\mathrm{I}} = \frac{10^{-6}}{2}(1000 - 300) + \frac{10^{-6}}{\sqrt{2}}\sqrt{(1000 + 200)^2 + (-200 + 300)^2}$$

which gives

$$\varepsilon_{\mathrm{I}} = \frac{10^{-6}}{2}(700 + 1703) = 1202 \times 10^{-6}$$

It follows from Eq. (14.35) that

$$\varepsilon_{\mathrm{II}} = \frac{10^{-6}}{2}(700 - 1703) = -502 \times 10^{-6}$$

Substituting for ε_{I} and $\varepsilon_{\mathrm{II}}$ in Eq. (14.32) we have

$$\sigma_{\mathrm{I}} = \frac{70\,000 \times 10^{-6}}{1 - (0.3)^2}(1202 - 0.3 \times 502) = 80.9\,\mathrm{N/mm^2}$$

Similarly from Eq. (14.33)

$$\sigma_{\mathrm{II}} = \frac{70\,000 \times 10^{-6}}{1 - (0.3)^2}(-502 + 0.3 \times 1202) = -10.9\,\mathrm{N/mm^2}$$

Since $\sigma_y = 0$ (note that the axial load produces σ_x only), Eqs (14.8) and (14.9) reduce to

$$\sigma_{\mathrm{I}} = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2} \tag{i}$$

and

$$\sigma_{\mathrm{II}} = \frac{\sigma_x}{2} - \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2} \tag{ii}$$

respectively. Adding Eqs (i) and (ii) we obtain

$$\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} = \sigma_x$$

Thus

$$\sigma_x = 80.9 - 10.9 = 70\,\text{N/mm}^2$$

Substituting for σ_x in either of Eq. (i) or (ii) gives

$$\tau_{xy} = 29.7\,\text{N/mm}^2$$

For an axial load P

$$\sigma_x = 70\,\text{N/mm}^2 = \frac{P}{A} = \frac{P}{(\pi/4)\times 50^2}\ (\text{Eq. (7.1)})$$

so that

$$P = 137.4\,\text{kN}$$

Also for the torque T and using Eq. (11.4) we have

$$\tau_{xy} = 29.7\,\text{N/mm}^2 = \frac{Tr}{J} = \frac{T\times 25}{(\pi/32)\times 50^4}$$

which gives

$$T = 0.7\,\text{kNm}$$

Note that P could have been found directly in this case from the axial strain ε_a. Thus from Eq. (7.8)

$$\sigma_x = E\varepsilon_a = 70\,000 \times 1000 \times 10^{-6} = 70\,\text{N/mm}^2$$

as before.

EXAMPLE 14.8

The thin-walled cantilever box beam shown in Fig. 14.19 has a bar attached to its free end; the bar carries a vertical load W at a distance r from the vertical plane of symmetry. A rectangular strain gauge rosette is attached to the upper cover of the box beam in the vertical plane of symmetry and at a distance of 1 m from the free end. If the readings from the three arms of the rosette are:

$$\varepsilon_a = 1200 \times 10^{-6},\quad \varepsilon_b = 200 \times 10^{-6},\quad \varepsilon_c = -360 \times 10^{-6}$$

determine the values of W and r. Take $E = 200000\ \text{N/mm}^2$ and $\nu = 0.3$.

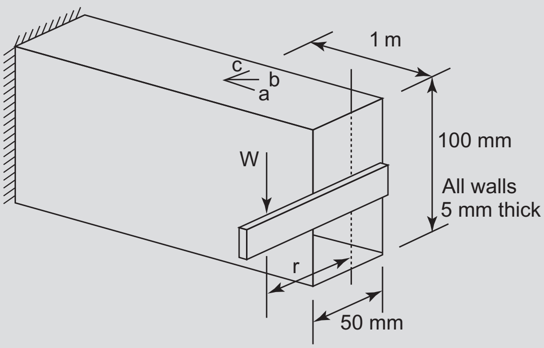

FIGURE 14.19

Cantilever box beam of Ex. 14.8

From Eq. (14.34)

$$\varepsilon_{\mathrm{I}} = \frac{10^{-6}}{2}\left\{(1200-360)+\sqrt{\left[2(1200-200)^2+2(-360-200)^2\right]}\right\}$$

which gives

$$\varepsilon_{\mathrm{I}} = 1230.4\times 10^{-6}$$

Similarly, from Eq. (14.35)

$$\varepsilon_{\mathrm{II}} = -390.4\times 10^{-6}$$

Then, from Eq. (14.32)

$$\sigma_{\mathrm{I}} = \frac{200000}{1-(0.3)^2}(1230.4-0.3\times 390.4)\times 10^{-6}$$

which gives

$$\sigma_{\mathrm{I}} = 244.7\,\mathrm{N/mm^2}$$

Similarly, from Eq. (14.33)

$$\sigma_{\mathrm{II}} = -4.7\,\mathrm{N/mm^2}$$

Since, in this case, $\sigma_y = 0$, Eqs. (14.8) and (14.9) reduce to

$$\sigma_{\mathrm{I}} = \frac{\sigma_x}{2}+\frac{1}{2}\sqrt{({\sigma_x}^2+4{\tau_{xy}}^2)} \tag{i}$$

and

$$\sigma_{\mathrm{II}} = \frac{\sigma_x}{2}-\frac{1}{2}\sqrt{({\sigma_x}^2+4{\tau_{xy}}^2)} \tag{ii}$$

Adding Eqs. (i) and (ii)

$$\sigma_{\mathrm{I}}+\sigma_{\mathrm{II}} = \sigma_x = 244.7-4.7 = 240\,\mathrm{N/mm^2}$$

Substituting for σ_x in Eq. (i) or Eq. (ii) gives

$$\tau_{xy} = 33.9\,\mathrm{N/mm^2}$$

From Eq. (9.9)

$$\sigma_x = \frac{W\times 10^3\times 50}{2(50\times 5\times 50^2+5\times 100^3/12)} = 0.024\,W$$

Therefore

$$W = 240/0.024 = 10000\,\mathrm{N} = 10\,\mathrm{kN}$$

From Eq. (11.21)

$$\tau_{xy} = \frac{10\times 10^3 r}{2\times 50\times 100\times 5} = 33.9$$

which gives

$$r = 169.5\,\text{mm}$$

Note, that as in Ex. 14.7, we could have obtained σ_x directly from the strain gauge reading, that is, from Eq. (7.8)

$$\sigma_x = 200000 \times 1200 \times 10^{-6} = 240\,\text{N/mm}^2$$

However, we would still require a value for either σ_I or σ_II in order to obtain τ_{xy} so that the saving in computation would not have been significant.

14.10 Theories of elastic failure

The direct stress in a structural member subjected to simple tension or compression is directly proportional to strain up to the yield point of the material (Section 7.7). It is therefore a relatively simple matter to design such a member using the direct stress at yield as the design criterion. However, as we saw in Section 14.3, the direct and shear stresses at a point in a structural member subjected to a complex loading system are not necessarily the maximum values at the point. In such cases it is not clear how failure occurs, so that it is difficult to determine limiting values of load or alternatively to design a structural member for given loads. An obvious method, perhaps, would be to use direct experiment in which the structural member is loaded until deformations are no longer proportional to the applied load; clearly such an approach would be both time-wasting and uneconomical. Ideally a method is required that relates some parameter representing the applied stresses to, say, the yield stress in simple tension which is a constant for a given material.

In Section 14.3 we saw that a complex two-dimensional stress system comprising direct and shear stresses could be represented by a simpler system of direct stresses only, in other words, the principal stresses. The problem is therefore simplified to some extent since the applied loads are now being represented by a system of direct stresses only. Clearly this procedure could be extended to the three-dimensional case so that no matter how complex the loading and the resulting stress system, there would remain at the most just three principal stresses, σ_I, σ_II and σ_III, as shown, for a three-dimensional element, in Fig. 14.20.

It now remains to relate, in some manner, these principal stresses to the yield stress in simple tension, σ_Y, of the material.

Ductile materials

A number of theories of elastic failure have been proposed in the past for ductile materials but experience and experimental evidence have led to all but two being discarded.

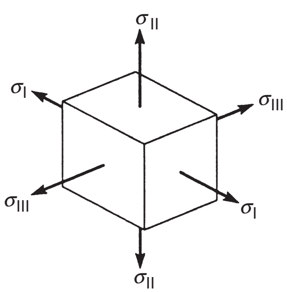

FIGURE 14.20

Reduction of a complex three-dimensional stress system.

Maximum shear stress theory

This theory is usually linked with the names of Tresca and Guest, although it is more widely associated with the former. The theory proposes that:

Failure (i.e. yielding) will occur when the maximum shear stress in the material is equal to the maximum shear stress at failure in simple tension.

For a two-dimensional stress system the maximum shear stress is given in terms of the principal stresses by Eq. (14.12). For a three-dimensional case the maximum shear stress is given by

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \tag{14.37}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the algebraic maximum and minimum principal stresses. At failure in simple tension the yield stress σ_Y is in fact a principal stress and since there can be no direct stress perpendicular to the axis of loading, the maximum shear stress is, therefore, from either of Eqs. (14.12) or (14.37)

$$\tau_{\max} = \frac{\sigma_Y}{2} \tag{14.38}$$

Thus the theory proposes that failure in a complex system will occur when

$$\frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{\sigma_Y}{2}$$

or

$$\sigma_{\max} - \sigma_{\min} = \sigma_Y \tag{14.39}$$

Let us now examine stress systems having different relative values of σ_I, σ_{II} and σ_{III}. First suppose that $\sigma_I > \sigma_{II} > \sigma_{III} > 0$. From Eq. (14.39) failure occurs when

$$\sigma_I - \sigma_{III} = \sigma_Y \tag{14.40}$$

Second, suppose that $\sigma_I > \sigma_{II} > 0$ but $\sigma_{III} = 0$. In this case the three-dimensional stress system of Fig. 14.20 reduces to a two-dimensional stress system but *is still acting on a three-dimensional element.* Thus Eq. (14.39) becomes

$$\sigma_I - 0 = \sigma_Y$$

or

$$\sigma_I = \sigma_Y \tag{14.41}$$

Here we see an apparent contradiction of Eq. (14.12) where the maximum shear stress in a two-dimensional stress system is equal to half the difference of σ_I and σ_{II}. However, the maximum shear stress in that case occurs in the plane of the two-dimensional element, i.e. in the plane of σ_I and σ_{II}. In this case we have a three-dimensional element so that the maximum shear stress will lie in the plane of σ_I and σ_{III}.

Finally, let us suppose that $\sigma_I > 0$, $\sigma_{II} < 0$ and $\sigma_{III} = 0$. Again we have a two-dimensional stress system acting on a three-dimensional element but now σ_{II} is a compressive stress and algebraically less than σ_{III}. Thus Eq. (14.39) becomes

$$\sigma_I - \sigma_{II} = \sigma_Y \tag{14.42}$$

Shear strain energy theory

This particular theory of elastic failure was established independently by von Mises, Maxwell and Hencky but is now generally referred to as the von Mises criterion. The theory proposes that:

> *Failure will occur when the shear or distortion strain energy in the material reaches the equivalent value at yielding in simple tension.*

In 1904 Huber proposed that the total strain energy, U_t, of an element of material could be regarded as comprising two separate parts: that due to change in volume and that due to change in shape. The former is termed the volumetric strain energy, U_v, the latter the distortion or shear strain energy, U_s. Thus

$$U_t = U_v + U_s \tag{14.43}$$

Since it is relatively simple to determine U_t and U_v, we obtain U_s by transposing Eq. (14.43). Hence

$$U_s = U_t - U_v \tag{14.44}$$

Initially, however, we shall demonstrate that the deformation of an element of material may be separated into change of volume and change in shape.

The principal stresses σ_{I}, σ_{II} and σ_{III} acting on the element of Fig. 14.20 may be written as

$$\sigma_{\text{I}} = \frac{1}{3}(\sigma_{\text{I}} + \sigma_{\text{II}} + \sigma_{\text{III}}) + \frac{1}{3}(2\sigma_{\text{I}} - \sigma_{\text{II}} - \sigma_{\text{III}})$$

$$\sigma_{\text{II}} = \frac{1}{3}(\sigma_{\text{I}} + \sigma_{\text{II}} + \sigma_{\text{III}}) + \frac{1}{3}(2\sigma_{\text{II}} - \sigma_{\text{I}} - \sigma_{\text{III}})$$

$$\sigma_{\text{III}} = \frac{1}{3}(\sigma_{\text{I}} + \sigma_{\text{II}} + \sigma_{\text{III}}) + \frac{1}{3}(2\sigma_{\text{III}} - \sigma_{\text{II}} - \sigma_{\text{I}})$$

or

$$\left.\begin{aligned} \sigma_{\text{I}} &= \bar{\sigma} + \sigma_{\text{I}}^1 \\ \sigma_{\text{II}} &= \bar{\sigma} + \sigma_{\text{II}}^1 \\ \sigma_{\text{III}} &= \bar{\sigma} + \sigma_{\text{III}}^1 \end{aligned}\right\} \tag{14.45}$$

Thus the stress system of Fig. 14.20 may be represented as the sum of two separate stress systems as shown in Fig. 14.21. The $\bar{\sigma}$ stress system is clearly equivalent to a hydrostatic or volumetric stress which will produce a

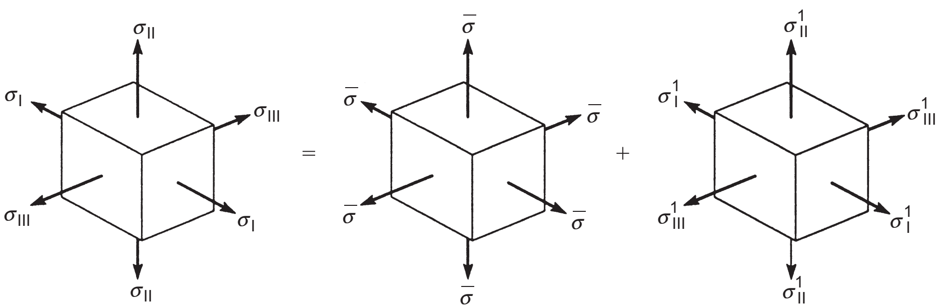

FIGURE 14.21

Representation of principal stresses as volumetric and distortional stresses.

change in volume but not a change in shape. The effect of the σ^1 stress system may be determined as follows. Adding together Eqs (14.45) we obtain

$$\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}} = 3\overline{\sigma} + \sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1$$

but

$$\overline{\sigma} = \frac{1}{3}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})$$

so that

$$\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1 = 0 \tag{14.46}$$

From the stress–strain relationships of Section 7.8 we have

$$\left.\begin{aligned}
\varepsilon_{\mathrm{I}}^1 &= \frac{\sigma_{\mathrm{I}}^1}{E} - \frac{v}{E}(\sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1)\\
\varepsilon_{\mathrm{II}}^1 &= \frac{\sigma_{\mathrm{II}}^1}{E} - \frac{v}{E}(\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{III}}^1)\\
\varepsilon_{\mathrm{III}}^1 &= \frac{\sigma_{\mathrm{III}}^1}{E} - \frac{v}{E}(\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1)
\end{aligned}\right\} \tag{14.47}$$

The volumetric strain ε_{v} corresponding to σ_{I}^1, σ_{II}^1, and σ_{III}^1 is equal to the sum of the linear strains. Thus from Eqs (14.47)

$$\varepsilon_{\mathrm{v}} = \varepsilon_{\mathrm{I}}^1 + \varepsilon_{\mathrm{II}}^1 + \varepsilon_{\mathrm{III}}^1 = \frac{(1-2v)}{E}(\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1)$$

which, from Eq. (14.46), gives

$$\varepsilon_{\mathrm{v}} = 0$$

It follows that σ_{I}^1, σ_{II}^1 and σ_{III}^1 produce no change in volume but only change in shape. We have therefore successfully divided the σ_{I}, σ_{II}, σ_{III} stress system into stresses ($\overline{\sigma}$) producing changes in volume and stresses (σ^1) producing changes in shape.

In Section 7.10 we derived an expression for the strain energy, U, of a member subjected to a direct stress, σ (Eq. (7.30)), i.e.

$$U = \frac{1}{2} \times \frac{\sigma^2}{E} \times \text{volume}$$

This equation may be rewritten

$$U = \frac{1}{2} \times \sigma \times \varepsilon \times \text{volume}$$

since $E = \sigma/\varepsilon$. The strain energy per unit volume is then $\sigma\varepsilon/2$. Thus for a three-dimensional element subjected to a stress $\overline{\sigma}$ on each of its six faces the strain energy in one direction is

$$\frac{1}{2}\overline{\sigma}\overline{\varepsilon}$$

where $\bar{\varepsilon}$ is the strain due to $\bar{\sigma}$ in each of the three directions. The total or volumetric strain energy per unit volume, U_{v}, of the element is then given by

$$U_{\mathrm{v}} = 3\left(\frac{1}{2}\bar{\sigma}\bar{\varepsilon}\right)$$

or, since

$$\bar{\varepsilon} = \frac{\bar{\sigma}}{E} - 2v\frac{\bar{\sigma}}{E} = \frac{\bar{\sigma}}{E}(1-2v)$$

$$U_{\mathrm{v}} = \frac{1}{2}\bar{\sigma}\frac{3\bar{\sigma}}{E}(1-2v) \tag{14.48}$$

But

$$\bar{\sigma} = \frac{1}{3}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})$$

so that Eq. (14.48) becomes

$$U_{\mathrm{v}} = \frac{(1-2v)}{6E}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})^2 \tag{14.49}$$

By a similar argument the total strain energy per unit volume, U_{t}, of an element subjected to stresses σ_{I}, σ_{II} and σ_{III} is

$$U_{\mathrm{t}} = \frac{1}{2}\sigma_{\mathrm{I}}\varepsilon_{\mathrm{I}} + \frac{1}{2}\sigma_{\mathrm{II}}\varepsilon_{\mathrm{II}} + \frac{1}{2}\sigma_{\mathrm{III}}\varepsilon_{\mathrm{III}} \tag{14.50}$$

where

$$\left.\begin{aligned} \varepsilon_{\mathrm{I}} &= \frac{\sigma_{\mathrm{I}}}{E} - \frac{v}{E}(\sigma_{\mathrm{II}} + \sigma_{\mathrm{III}}) \\ \varepsilon_{\mathrm{II}} &= \frac{\sigma_{\mathrm{II}}}{E} - \frac{v}{E}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{III}}) \\ \text{and}\qquad \varepsilon_{\mathrm{III}} &= \frac{\sigma_{\mathrm{III}}}{E} - \frac{v}{E}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) \end{aligned}\right\} \text{(see Eq. (14.47))} \tag{14.51}$$

Substituting for ε_{I}, etc. in Eq. (14.50) and then for U_{v} from Eq. (14.49) and U_{t} in Eq. (14.44) we have

$$U_{\mathrm{s}} = \frac{1}{2E}\left[\sigma_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2 + \sigma_{\mathrm{III}}^2 - 2v(\sigma_{\mathrm{I}}\sigma_{\mathrm{II}} + \sigma_{\mathrm{II}}\sigma_{\mathrm{III}} + \sigma_{\mathrm{III}}\sigma_{\mathrm{I}}) - \frac{(1-2v)}{6E}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})^2\right]$$

which simplifies to

$$U_{\mathrm{s}} = \frac{(1+v)}{6E}\left[(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})^2 + (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})^2 + (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})^2\right]$$

per unit volume.

From Eq. (7.21)

$$E = 2G(1+v)$$

Thus

$$U_s = \frac{1}{12G}\left[(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})^2 + (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})^2 + (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})^2\right] \tag{14.52}$$

The shear or distortion strain energy per unit volume at failure in simple tension corresponds to $\sigma_{\mathrm{I}} = \sigma_{\mathrm{Y}}$, $\sigma_{\mathrm{II}} = \sigma_{\mathrm{III}} = 0$. Hence from Eq. (14.52)

$$U_s\,(\text{at failure in simple tension}) = \frac{\sigma_{\mathrm{Y}}^2}{6G} \tag{14.53}$$

According to the von Mises criterion, failure occurs when U_s, given by Eq. (14.52), reaches the value of U_s, given by Eq. (14.53), i.e. when

$$(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})^2 + (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})^2 + (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})^2 = 2\sigma_{\mathrm{Y}}^2 \tag{14.54}$$

For a two-dimensional stress system in which $\sigma_{\mathrm{III}} = 0$, Eq. (14.54) becomes

$$\sigma_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2 - \sigma_{\mathrm{I}}\sigma_{\mathrm{II}} = \sigma_{\mathrm{Y}}^2 \tag{14.55}$$

Design application

Codes of Practice for the use of structural steel in building use the von Mises criterion for a two-dimensional stress system (Eq. (14.55)) in determining an equivalent allowable stress for members subjected to bending and shear. Thus if σ_x and τ_{xy} are the direct and shear stresses, respectively, at a point in a member subjected to bending and shear, then the principal stresses at the point are, from Eqs (14.8) and (14.9)

$$\sigma_{\mathrm{I}} = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

$$\sigma_{\mathrm{II}} = \frac{\sigma_x}{2} - \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

Substituting these expressions in Eq. (14.55) and simplifying we obtain

$$\sigma_{\mathrm{Y}} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} \tag{14.56}$$

In Codes of Practice σ_{Y} is termed an equivalent stress and allowable values are given for a series of different structural members.

Yield loci

Equations (14.39) and (14.54) may be plotted graphically for a two-dimensional stress system in which $\sigma_{\mathrm{III}} = 0$ and in which it is assumed that the yield stress, σ_{Y}, is the same in tension and compression.

Figure 14.22 shows the yield locus for the maximum shear stress or Tresca theory of elastic failure. In the first and third quadrants, when σ_{I} and σ_{II} have the same sign, failure occurs when either $\sigma_{\mathrm{I}} = \sigma_{\mathrm{Y}}$ or $\sigma_{\mathrm{II}} = \sigma_{\mathrm{Y}}$ (see Eq. (14.41)) depending on which principal stress attains the value σ_{Y} first. For example, a structural member may be subjected to loads that produce a given value of σ_{II} ($< \sigma_{\mathrm{Y}}$) and varying values of σ_{I}. If the loads were increased, failure would occur when σ_{I} reached the value σ_{Y}. Similarly for a fixed value of σ_{I} and varying σ_{II}. In the second and third quadrants where σ_{I} and σ_{II} have opposite signs, failure occurs when $\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}} = \sigma_{\mathrm{Y}}$ or $\sigma_{\mathrm{II}} - \sigma_{\mathrm{I}} = \sigma_{\mathrm{Y}}$ (see Eq. (14.42)). Both these equations represent straight lines, each having a gradient of 45° and an intercept on the σ_{II} axis of σ_{Y}. Clearly all combinations of σ_{I} and σ_{II} that lie inside the locus will not cause failure, while all combinations of σ_{I} and σ_{II} on or outside the locus will.

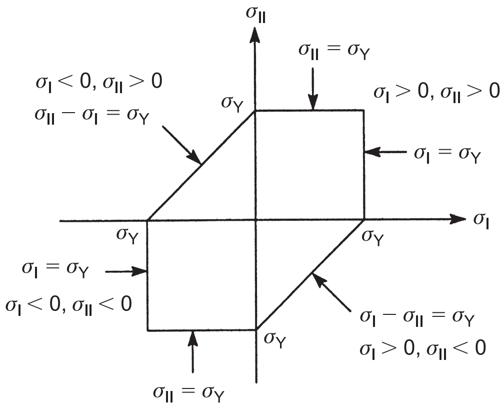

FIGURE 14.22

Yield locus for the Tresca theory of elastic failure.

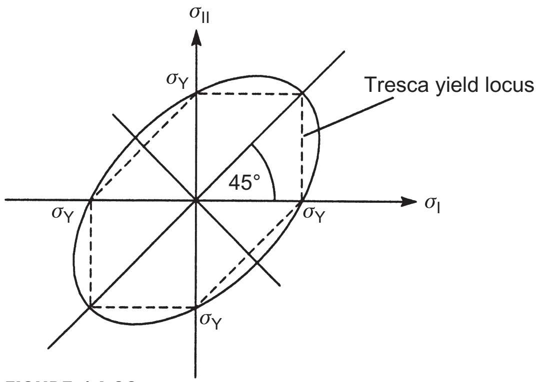

FIGURE 14.23

Yield locus for the von Mises theory.

Thus the inside of the locus represents elastic conditions while the outside represents plastic conditions. Note that for the purposes of a yield locus, σ_I and σ_{II} are interchangeable.

The shear strain energy (von Mises) theory for a two-dimensional stress system is represented by Eq. (14.55). This equation may be shown to be that of an ellipse whose major and minor axes are inclined at 45° to the axes of σ_I and σ_{II} as shown in Fig. 14.23. It may also be shown that the ellipse passes through the six corners of the Tresca yield locus so that at these points the two theories give identical results. However, for other combinations of σ_I and σ_{II} the Tresca theory predicts failure where the von Mises theory does not so that the Tresca theory is the more conservative of the two.

The value of the yield loci lies in their use in experimental work on the validation of the different theories. Structural members fabricated from different materials may be subjected to a complete range of combinations of σ_I and σ_{II} each producing failure. The results are then plotted on the yield loci and the accuracy of each theory is determined for different materials.

EXAMPLE 14.9

The state of stress at a point in a structural member is defined by a two-dimensional stress system as follows: $\sigma_x = +140$ N/mm^2, $\sigma_y = -70$ N/mm^2 and $\tau_{xy} = +60$ N/mm^2. If the material of the member has a yield stress in simple tension of 225 N/mm^2, determine whether or not yielding has occurred according to the Tresca and von Mises theories of elastic failure.

The first step is to determine the principal stresses σ_I and σ_{II}. From Eqs (14.8) and (14.9)

$$\sigma_I = \frac{1}{2}(140 - 70) + \frac{1}{2}\sqrt{(140+70)^2 + 4 \times 60^2}$$

i.e.

$$\sigma_I = 155.9\,\text{N/mm}^2$$

and

$$\sigma_{II} = \frac{1}{2}(140 - 70) - \frac{1}{2}\sqrt{(140+70)^2 + 4 \times 60^2}$$

i.e.

$$\sigma_{II} = -85.9\,\text{N/mm}^2$$

Since σ_{II} is algebraically less than σ_{III} $(=0)$, Eq. (14.42) applies.
Thus

$$\sigma_I - \sigma_{II} = 241.8\,\text{N/mm}^2$$

This value is greater than σ_Y $(=225\ \text{N/mm}^2)$ so that according to the Tresca theory failure has, in fact, occurred.

Substituting the above values of σ_I and σ_{II} in Eq. (14.55) we have

$$(155.9)^2 + (-85.9)^2 - (155.9)(-85.9) = 45\,075.4$$

The square root of this expression is 212.3 N/mm^2 so that according to the von Mises theory the material has not failed.

EXAMPLE 14.10

The rectangular cross section of a thin-walled box girder (Fig. 14.24) is subjected to a bending moment of 250 kNm and a torque of 200 kNm. If the allowable equivalent stress for the material of the box girder is 180 N/mm^2, determine whether or not the design is satisfactory using the requirement of Eq. (14.56).

The maximum shear stress in the cross section occurs in the vertical walls of the section and is given by Eq. (11.22), i.e.

$$\tau_{max} = \frac{T_{\text{max}}}{2At_{\text{min}}} = \frac{200 \times 10^6}{2 \times 500 \times 250 \times 10} = 80\,\text{N/mm}^2$$

The maximum stress due to bending occurs at the top and bottom of each vertical wall and is given by Eq. (9.9), i.e.

$$\sigma = \frac{My}{I}$$

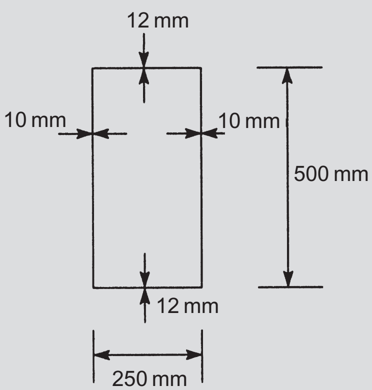

FIGURE 14.24

Box girder beam section of Ex. 14.10.

where

$$I = 2 \times 12 \times 250 \times 250^2 + \frac{2 \times 10 \times 500^3}{12} \quad \text{(see Section 9.6)}$$

i.e.

$$I = 583.3 \times 10^6 \text{ mm}^4$$

Thus

$$\sigma = \frac{250 \times 10^6 \times 250}{583.3 \times 10^6} = 107.1 \text{ N/mm}^2$$

Substituting these values in Eq. (14.56) we have

$$\sqrt{\sigma_x^2 + 3\tau_{xy}^2} = \sqrt{107.1^2 + 3 \times 80^2} = 175.1 \text{ N/mm}^2$$

This equivalent stress is less than the allowable value of 180 N/mm^2 so that the box girder section is satisfactory.

EXAMPLE 14.11

A beam of rectangular cross section 60 mm×100 mm is subjected to an axial tensile load of 60 000 N. If the material of the beam fails in simple tension at a stress of 150 N/mm^2 determine the maximum shear force that can be applied to the beam section in a direction parallel to its longest side using the Tresca and von Mises theories of elastic failure.

The direct stress σ_x due to the axial load is uniform over the cross section of the beam and is given by

$$\sigma_x = \frac{60\,000}{60 \times 100} = 10 \text{ N/mm}^2$$

The maximum shear stress $\tau_{\max}$ occurs at the horizontal axis of symmetry of the beam section and is, from Eq. (10.7)

$$\tau_{\max} = \frac{3}{2} \times \frac{S_y}{60 \times 100} \tag{i}$$

Thus from Eqs (14.8) and (14.9)

$$\sigma_{\mathrm{I}} = \frac{10}{2} + \frac{1}{2}\sqrt{10^2 + 4\tau_{\max}^2} \quad \sigma_{\mathrm{II}} = \frac{10}{2} - \frac{1}{2}\sqrt{10^2 + 4\tau_{\max}^2}$$

or

$$\sigma_{\mathrm{I}} = 5 + \sqrt{25 + \tau_{\max}^2} \quad \sigma_{\mathrm{II}} = 5 - \sqrt{25 + \tau_{\max}^2} \tag{ii}$$

It is clear from the second of Eq. (ii) that σ_{II} is negative since $|\sqrt{25+\tau_{max}^2}| > 5$. Thus in the Tresca theory Eq. (14.42) applies and

$$\sigma_I - \sigma_{II} = 2\sqrt{25+\tau_{max}^2} = 150\,\text{N/mm}^2$$

from which

$$\tau_{max} = 74.8\,\text{N/mm}^2$$

Thus from Eq. (i)

$$S_y = 299.3\,\text{kN}$$

Now substituting for σ_I and σ_{II} in Eq. (14.55) we have

$$\left(5+\sqrt{25+\tau_{max}^2}\right)^2 + \left(5-\sqrt{25+\tau_{max}^2}\right)^2 - \left(5+\sqrt{25+\tau_{max}^2}\right)\left(5-\sqrt{25+\tau_{max}^2}\right) = 150^2$$

which gives

$$\tau_{max} = 86.4\,\text{N/mm}^2$$

Again from Eq. (i)

$$S_y = 345.6\,\text{kN}$$

Brittle materials

When subjected to tensile stresses brittle materials such as cast iron, concrete and ceramics fracture at a value of stress very close to the elastic limit with little or no permanent yielding on the planes of maximum shear stress. In fact the failure plane is generally flat and perpendicular to the axis of loading, unlike ductile materials which have failure planes inclined at approximately 45° to the axis of loading; in the latter case failure occurs on planes of maximum shear stress (see Sections 8.3 and 14.2). This would suggest, therefore, that shear stresses have no effect on the failure of brittle materials and that a direct relationship exists between the principal stresses at a point in a brittle material subjected to a complex loading system and the failure stress in simple tension or compression. This forms the basis for the most widely accepted theory of failure for brittle materials.

Maximum normal stress theory

This theory, frequently attributed to Rankine, states that:

> *Failure occurs when one of the principal stresses reaches the value of the yield stress in simple tension or compression.*

For most brittle materials the yield stress in tension is very much less than the yield stress in compression, e.g. for concrete σ_Y (compression) is approximately 20 σ_Y (tension). Thus it is essential in any particular problem to know which of the yield stresses is achieved first.

Suppose that a brittle material is subjected to a complex loading system which produces principal stresses σ_I, σ_{II} and σ_{III} as in Fig. 14.20. Thus for $\sigma_I > \sigma_{II} > \sigma_{III} > 0$ failure occurs when

$$\sigma_I = \sigma_Y \quad \text{(tension)} \tag{14.57}$$

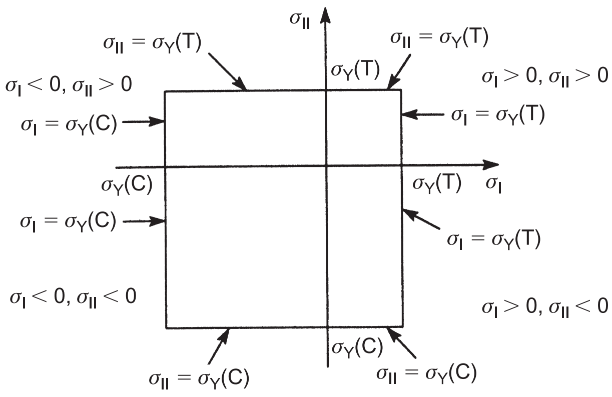

FIGURE 14.25

Yield locus for a brittle material.

Alternatively, for $\sigma_{\rm I} > \sigma_{\rm II} > 0$, $\sigma_{\rm III} < 0$ and $\sigma_{\rm I} < \sigma_{\rm Y}$ (tension) failure occurs when

$$\sigma_{\rm III} = \sigma_{\rm Y} \quad \text{(compression)} \tag{14.58}$$

and so on.

A yield locus may be drawn for the two-dimensional case, as for the Tresca and von Mises theories of failure for ductile materials, and is shown in Fig. 14.25. Note that since the failure stress in tension, $\sigma_{\rm Y}$(T), is generally less than the failure stress in compression, $\sigma_{\rm Y}$(C), the yield locus is not symmetrically arranged about the $\sigma_{\rm I}$ and $\sigma_{\rm II}$ axes. Again combinations of stress corresponding to points inside the locus will not cause failure, whereas combinations of $\sigma_{\rm I}$ and $\sigma_{\rm II}$ on or outside the locus will.

EXAMPLE 14.12

A concrete beam has a rectangular cross section 250 mm×500 mm and is simply supported over a span of 4 m. Determine the maximum mid-span concentrated load the beam can carry if the failure stress in simple tension of concrete is 1.5 N/mm^2. Neglect the self-weight of the beam.

If the central concentrated load is W N the maximum bending moment occurs at mid-span and is

$$\frac{4W}{4} = W\,\text{Nm} \quad \text{(see Ex. 3.7)}$$

The maximum direct tensile stress due to bending occurs at the soffit of the beam and is

$$\sigma = \frac{W \times 10^3 \times 250 \times 12}{250 \times 500^3} = W \times 9.6 \times 10^{-5}\,\text{N/mm}^2 \text{ (Eq.9.9)}$$

At this point the maximum principal stress is, from Eq. (14.8)

$$\sigma_{\rm I} = W \times 9.6 \times 10^{-5}\,\text{N/mm}^2$$

Thus from Eq. (14.57) the maximum value of W is given by

$$\sigma_{\rm I} = W \times 9.6 \times 10^{-5} = \sigma_Y\,\text{(tension)} = 1.5\,\text{N/mm}^2$$

from which $W = 15.6$ kN.

The maximum shear stress occurs at the horizontal axis of symmetry of the beam section over each support and is, from Eq. (10.7)

$$\tau_{\max} = \frac{3}{2} \times \frac{W/2}{250 \times 500}$$

i.e.

$$\tau_{\max} = W \times 0.6 \times 10^{-5}\ \mathrm{N/mm^2}$$

Again, from Eq. (14.8), the maximum principal stress is

$$\sigma_1 = W \times 9.6 \times 10^{-5}\ \mathrm{N/mm^2} = \sigma_{\mathrm{Y}}(\text{tension}) = 1.5\ \mathrm{N/mm^2}$$

from which

$$W = 250\ \mathrm{kN}$$

Thus the maximum allowable value of W is 15.6 kN.

PROBLEMS

P.14.1 At a point in an elastic material there are two mutually perpendicular planes, one of which carries a direct tensile stress of 50 N/mm^2 and a shear stress of 40 N/mm^2 while the other plane is subjected to a direct compressive stress of 35 N/mm^2 and a complementary shear stress of 40 N/mm^2. Determine the principal stresses at the point, the position of the planes on which they act and the position of the planes on which there is no direct stress.

Ans. $\sigma_{\mathrm{I}} = 65.9$ N/mm^2, $\theta = -21.6°$, $\sigma_{\mathrm{II}} = -50.9$ N/mm^2, $\theta = -111.6°$.
No direct stress on planes at 27.1° and 117.1° to the plane on which the 50 N/mm^2 stress acts.

P.14.2 One of the principal stresses in a two-dimensional stress system is 139 N/mm^2 acting on a plane A. On another plane B normal and shear stresses of 108 and 62 N/mm^2, respectively, act. Determine

a. the angle between the planes A and B,
b. the other principal stress,
c. the direct stress on the plane perpendicular to plane B.

Ans. (a) 26°34′, (b) −16 N/mm^2, (c) 15 N/mm^2.

P.14.3 The state of stress at a point in a structural member may be represented by a two-dimensional stress system in which $\sigma_x = 100$ N/mm^2, $\sigma_y = -80$ N/mm^2 and $\tau_{xy} = 45$ N/mm^2. Determine the direct stress on a plane inclined at 60° to the positive direction of σ_x and also the principal stresses. Calculate also the inclination of the principal planes to the plane on which σ_x acts. Verify your answers by a graphical method.

Ans. $\sigma_{\mathrm{n}} = 16$ N/mm^2, $\sigma_{\mathrm{I}} = 110.6$ N/mm^2, $\sigma_{\mathrm{II}} = -90.6$ N/mm^2, $\theta = -13.3°$ and $-103.3°$.

P.14.4 Determine the normal and shear stress on the plane AB shown in Fig. P.14.4 when

i. $\alpha = 60°$, $\sigma_x = 54$ N/mm^2, $\sigma_y = 30$ N/mm^2, $\tau_{xy} = 5$ N/mm^2;
ii. $\alpha = 120°$, $\sigma_x = -60$ N/mm^2, $\sigma_y = -36$ N/mm^2, $\tau_{xy} = 5$ N/mm^2.

Ans. (i) $\sigma_{\mathrm{n}} = 52.3$ N/mm^2, $\tau = 7.9$ N/mm^2;
(ii) $\sigma_{\mathrm{n}} = -58.3$ N/mm^2, $\tau = 7.9$ N/mm^2.

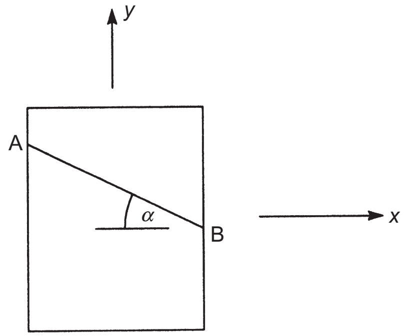

FIGURE P.14.4

P.14.5 A shear stress τ_{xy} acts in a two-dimensional field in which the maximum allowable shear stress is denoted by $\tau_{\max}$ and the major principal stress by σ_{I}. Derive, using the geometry of Mohr's circle of stress, expressions for the maximum values of direct stress which may be applied to the x and y planes in terms of the parameters given.

Ans.

$$\sigma_x = \sigma_{\mathrm{I}} - \tau_{\max} + \sqrt{\tau_{\max}^2 - \tau_{xy}^2}, \quad \sigma_y = \sigma_{\mathrm{I}} - \tau_{max} - \sqrt{\tau_{max}^2 - \tau_{xy}^2}.$$

P.14.6 In an experimental determination of principal stresses a cantilever of hollow circular cross section is subjected to a varying bending moment and torque; the internal and external diameters of the cantilever are 40 and 50 mm, respectively. For a given loading condition the bending moment and torque at a particular section of the cantilever are 100 and 50 Nm, respectively. Calculate the maximum and minimum principal stresses at a point on the outer surface of the cantilever at this section where the direct stress produced by the bending moment is tensile. Determine also the maximum shear stress at the point and the inclination of the principal stresses to the axis of the cantilever.

The experimental values of principal stress are estimated from readings obtained from a 45° strain gauge rosette aligned so that one of its three arms is parallel to and another perpendicular to the axis of the cantilever. For the loading condition of zero torque and varying bending moment, comment on the ratio of these strain gauge readings.

Ans. $\sigma_{\mathrm{I}} = 14.6\ \mathrm{N/mm^2}$, $\sigma_{\mathrm{II}} = -0.8\ \mathrm{N/mm^2}$
$\tau_{\max} = 7.7\ \mathrm{N/mm^2}$, $\theta = -13.3°$ and $-103.3°$.

P.14.7 A thin-walled cylinder has an internal diameter of 1200 mm and has walls 1.2 mm thick. It is subjected to an internal pressure of $0.7\ \mathrm{N/mm^2}$ and a torque, about its longitudinal axis, of 500 kNm. Determine the principal stresses at a point in the wall of the cylinder and also the maximum shear stress.

Ans. $466.4\ \mathrm{N/mm^2}$, $58.6\ \mathrm{N/mm^2}$, $203.9\ \mathrm{N/mm^2}$.

P.14.8 A rectangular piece of material is subjected to tensile stresses of 83 and 65 $\mathrm{N/mm^2}$ on mutually perpendicular faces. Find the strain in the direction of each stress and in the direction perpendicular to both stresses. Determine also the maximum shear strain in the plane of the stresses, the maximum shear stress and their directions. Take $E = 200\,000\ \mathrm{N/mm^2}$ and $v = 0.3$.

Ans. 3.18×10^{-4}, 2.01×10^{-4}, -2.22×10^{-4}, $\gamma_{\max} = 1.17 \times 10^{-4}$, $\tau_{\max} = 9.0\ \mathrm{N/mm^2}$ at 45° to the direction of the given stresses.

P.14.9 At a particular point in a structural member a direct tensile stress of 60 N/mm^2 exists on a plane perpendicular to the longitudinal axis of the member while a direct compressive stress of 40 N/mm^2 occurs on a plane parallel to this axis; in addition shear and complementary shear stresses of 50 N/mm^2 act on these planes. If Young's modulus E is 200000 N/mm^2 and Poisson's ratio ν is 0.3 calculate the direct and shear strains on these planes and hence, using a graphical method, determine the principal strains at the point, the maximum shear strain and the angle the plane of maximum principal strain makes with the longitudinal axis of the member. Finally, calculate the principal stresses at the point.

Ans. $\varepsilon_x = 3.6\times10^{-4}$, $\varepsilon_y = -2.9\times10^{-4}$, $\gamma_{xy} = 6.5\times10^{-4}$
$\varepsilon_{\mathrm{I}} = 4.95\times10^{-4}$, $\varepsilon_{\mathrm{II}} = -4.25\times10^{-4}$, $\gamma_{max} = 9.2\times10^{-4}$, 67.5°
$\sigma_{\mathrm{I}} = 80.8$ N/mm^2, $\sigma_{\mathrm{II}} = -60.8$ N/mm^2.

P.14.10 The thin-walled cantilever box beam shown in Fig. P.14.10 carries a vertically downward load of 100 kN at its free end. Calculate the principal strains at the point A which lies at the edge of the top cover of the beam at the built-in end. Take $E = 200000$ N/mm^2 and $\nu = 0.3$.

Ans. $\varepsilon_{\mathrm{I}} = 9.33\times10^{-4}$, $\varepsilon_{\mathrm{II}} = -2.87\times10^{-4}$.

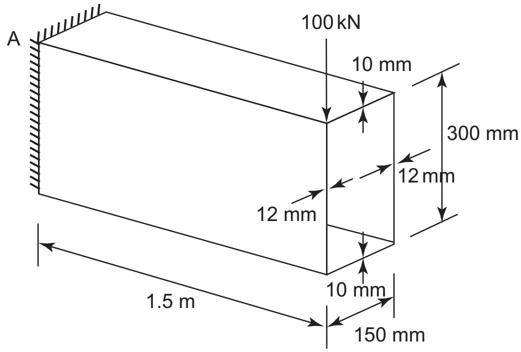

FIGURE P.14.10

P.14.11 A cantilever beam of length 2 m has a rectangular cross section 100 mm wide and 200 mm deep. The beam is subjected to an axial tensile load, P, and a vertically downward uniformly distributed load of intensity w. A rectangular strain gauge rosette attached to a vertical side of the beam at the built-in end and in the neutral plane of the beam recorded the following values of strain: $\varepsilon_a = 1000\times10^{-6}$, $\varepsilon_b, = 100\times10^{-6}$, $\varepsilon_c = -300\times10^{-6}$. The arm 'a' of the rosette is aligned with the longitudinal axis of the beam while the arm 'c' is perpendicular to the longitudinal axis.

Calculate the value of Poisson's ratio, the principal strains at the point and hence the values of P and w. Young's modulus, $E = 200\ 000$ N/mm^2.

Ans. $\nu = 0.3$, $\varepsilon_{\mathrm{I}} = 1046.4 \times 10^{-6}$, $\varepsilon_{\mathrm{II}} = -346.4 \times 10^{-6}$, $w = 255.3$ kN/m.

P.14.12 A beam has a rectangular thin-walled box section 50 mm wide by 100 mm deep and has walls 2 mm thick. At a particular section the beam carries a bending moment M and a torque T. A rectangular strain gauge rosette positioned on the top horizontal wall of the beam at this section recorded the following values of strain: $\varepsilon_a = 1000\times10^{-6}$, $\varepsilon_b = -200\times10^{-6}$, $\varepsilon_c = -300\times10^{-6}$. If the strain gauge 'a' is aligned with the longitudinal axis of the beam and the strain gauge 'c' is perpendicular to the longitudinal axis, calculate the values of M and T. Take $E = 200\ 000$ N/mm^2 and $\nu = 0.3$.

Ans. $M = 3333$ Nm, $T = 1692$ Nm.

P.14.13 The simply supported beam shown in Fig. P.14.13 carries two symmetrically placed transverse loads, W. A rectangular strain gauge rosette positioned at the point P gave strain readings as follows: $\varepsilon_a = -222\times10^{-6}$, $\varepsilon_b = -213\times10^{-6}$, $\varepsilon_c = 45\times10^{-6}$. Also the direct stress at P due to an external axial compressive load is 7 N/mm^2. Calculate the magnitude of the transverse load. Take $E = 31000$ N/mm^2, $v = 0.2$.

Ans. W = 98.1 kN

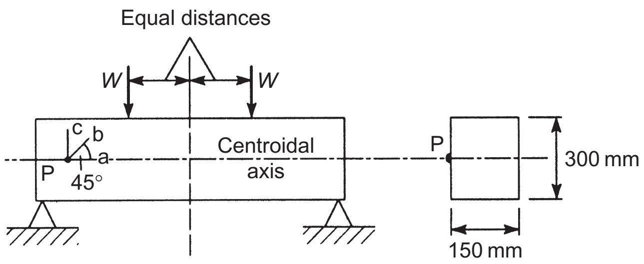

FIGURE P.14.13

P.14.14 A cantilever beam of solid circular cross section is 1 m long and has a diameter of 100 mm. Attached to its free end is a horizontal arm which carries a vertically downward load W at a distance r from the beam's vertical plane of symmetry. On the beam's upper surface, halfway along its length and positioned in its vertical plane of symmetry, is a rectangular strain gauge rosette which gave the following readings for particular values of W and r.

$$\varepsilon_a = 1500 \times 10^{-6},\ \varepsilon_b = -300 \times 10^{-6},\ \varepsilon_c = -450 \times 10^{-6}$$

where the gauges "a" and "c" are aligned with and perpendicular to the axis of the beam respectively. Calculate the values of W and r. Take $E = 200000$ N/mm^2 and $\nu = 0.3$.

Ans. $W = 58.9$ kN, $r = 423$ mm.

P.14.15 A simply supported beam has a span of 4 m, a rectangular cross section 100 mm wide by 200 mm deep and carries a uniformly distributed load of intensity w N/mm over its complete span. A rectangular strain gauge rosette positioned at mid-span on the upper surface of the beam and in the vertical plane of symmetry recorded the following values of strain:

$$\varepsilon_a = -900 \times 10^{-6},\ \varepsilon_b = -200 \times 10^{-6},\ \varepsilon_c = +300 \times 10^{-6}$$

If Young's modulus E is 200000 N/mm^2 calculate the value of Poisson's ratio, the principal stresses at the point and hence the load intensity.

Ans. $\nu = 0.33$, $\sigma_I = 0$, $\sigma_{II} = -180$ N/mm^2, $w = 60$ N/mm.

P.14.16 In a tensile test on a metal specimen having a cross section 20 mm by 10 mm elastic breakdown occurred at a load of 70 000 N.

A thin plate made from the same material is to be subjected to loading such that at a certain point in the plate the stresses are $\sigma_y = -70$ N/mm^2, $\tau_{xy} = 60$ N/mm^2 and σ_x. Determine the maximum allowable values of σ_x using the Tresca and von Mises theories of elastic breakdown.

Ans. 259 N/mm^2 (Tresca), 294 N/mm^2 (von Mises).

P.14.17 A beam of circular cross section is 3000 mm long and is attached at each end to supports which allow rotation of the ends of the beam in the longitudinal vertical plane of symmetry but prevent rotation of the ends in vertical planes perpendicular to the axis of the beam (Fig. P.14.17). The beam supports an offset load of 40 000 N at mid-span.

If the material of the beam suffers elastic breakdown in simple tension at a stress of 145 N/mm^2, calculate the minimum diameter of the beam on the basis of the Tresca and von Mises theories of elastic failure.

Ans. 136 mm (Tresca), 135 mm (von Mises).

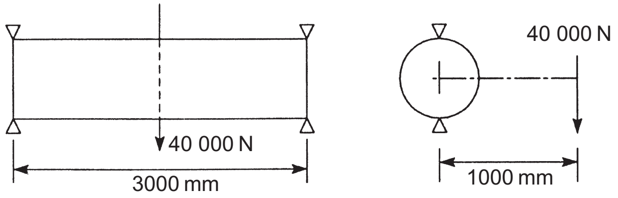

FIGURE P.14.17

P.14.18 A cantilever of circular cross section has a diameter of 150 mm and is made from steel, which, when subjected to simple tension suffers elastic breakdown at a stress of 150 N/mm^2.

The cantilever supports a bending moment and a torque, the latter having a value numerically equal to twice that of the former. Calculate the maximum allowable values of the bending moment and torque on the basis of the Tresca and von Mises theories of elastic failure.

Ans. $M = 22.2$ kNm, $T = 44.4$ kNm (Tresca).
$M = 24.9$ kNm, $T = 49.8$ kNm (von Mises).

P.14.19 A certain material has a yield stress limit in simple tension of 387 N/mm^2. The yield limit in compression can be taken to be equal to that in tension. The material is subjected to three stresses in mutually perpendicular directions, the stresses being in the ratio 3 : 2 : −1.8. Determine the stresses that will cause failure according to the von Mises and Tresca theories of elastic failure.

Ans. Tresca: $\sigma_{\text{I}} = 241.8$ N/mm^2, $\sigma_{\text{II}} = 161.2$ N/mm^2, $\sigma_{\text{III}} = -145.1$ N/mm^2.
von Mises: $\sigma_{\text{I}} = 264.0$ N/mm^2, $\sigma_{\text{II}} = 176.0$ N/mm^2, $\sigma_{\text{III}} = -158.4$ N/mm^2.

P.14.20 A thin-walled column has a circular cross section of diameter 200 mm and thickness 5 mm. It carries an axial load P and a torque of 20 kNm. If the material of the column fails in simple tension at a stress of 240 N/mm^2 find the maximum allowable value of P using the Tresca and von Mises theories of elastic failure.

Ans. 640 kN (Tresca), 670.6 kN (von Mises).

P.14.21 A thin-walled box beam has the cross section shown in Fig. P.14.21 and is simply supported over a span of 3 m. The beam carries a concentrated torque of 10 kNm at mid-span together with a vertical uniformly distributed load of intensity w N/mm over its complete span. If the yield stress in simple tension of the material of the beam is 160 N/mm^2 find the maximum allowable value of w using the Tresca and von Mises theories of elastic failure.

Ans. 9.7 N/mm (Tresca), 10.6 N/mm (von Mises).

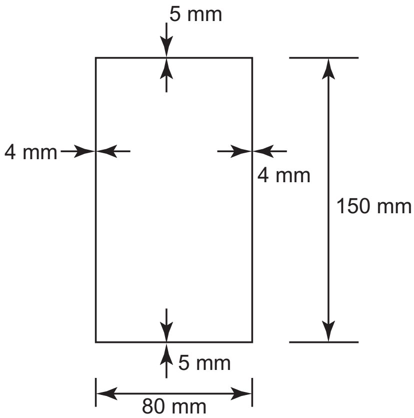

FIGURE P.14.21

P.14.22 The hollow cylinder shown in Fig. P.14.22 is built-in at one end and carries a vertical load of 2 kN offset a distance *e* from its vertical plane of symmetry. If the material of the cylinder fails in simple tension at a stress of 150 N/mm^2 calculate the maximum allowable value of *e* using the Tresca and von Mises theories of elastic failure.

Ans. 0.8 m (Tresca), 0.9 m (von Mises).

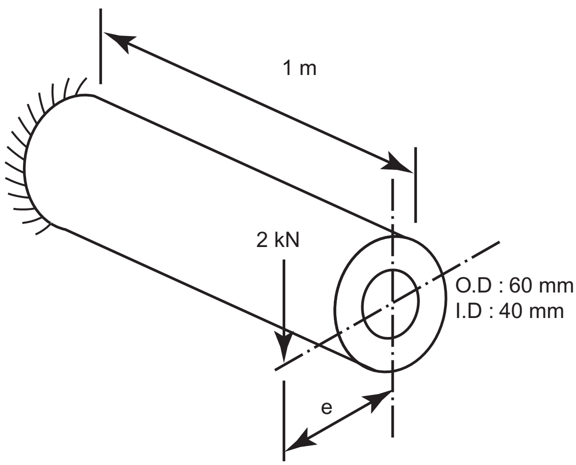

FIGURE P.14.22

P.14.23 A thin-walled cylinder of diameter 40 mm is subjected to an internal pressure of 5 N/mm^2 and a torque of 100 Nm. If the material of the cylinder has a yield stress in simple tension of 150 N/mm^2 determine the required wall thickness using the Tresca and von Mises theories of elastic failure.

Ans. 0.81 mm (Tresca), 0.74 mm (von Mises).

P.14.24 A hollow cylindrical shaft has an internal diameter of 100 mm, an external diameter of 130 mm and is required to support a bending moment and a torque each having a value of 20 kNm. If the equivalent stress in the shaft must not exceed 230 N/mm^2 verify that the dimensions of the shaft are satisfactory and also determine the minimum thickness the shaft must have for the same external diameter and loading.

Ans. $\sigma_e = 188.8$ N/mm^2 therefore dimensions are satisfactory.
Minimum thickness = 11.3 mm.

P.14.25 A column has the cross section shown in Fig. P.14.25 and carries a compressive load P parallel to its longitudinal axis. If the failure stresses of the material of the column are 4 and 22 N/mm^2 in simple tension and compression, respectively, determine the maximum allowable value of P using the maximum normal stress theory.

Ans. 634.9 kN.

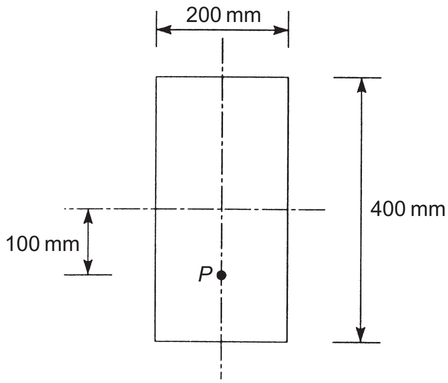

FIGURE P.14.25

P.14.26 A cast iron pipe of external diameter 300 mm has walls 10 mm thick and is required to carry water to a maximum of half its internal depth. If the failure stress in tension of cast iron is 138 N/mm^2 calculate the maximum allowable simply supported span the pipe can have. Take the density of cast iron as 72.3 kN/m^3 and ignore the negligibly small shear stresses.

Ans. 27.0 m.