CHAPTER

7

Stress and Strain

We are now in a position to calculate internal force distributions in a variety of structural systems, i.e. normal forces, shear forces and bending moments in beams and arches, axial forces in truss members, the tensions in suspension cables and torque distributions in beams. These internal force systems are distributed throughout the cross section of a structural member in the form of stresses. However, although there are four basic types of internal force, there are only two types of stress: one which acts perpendicularly to the cross section of a member and one which acts tangentially. The former is known as a *direct* or *normal stress*, the latter as a *shear stress*.

The distribution of these stresses over the cross section of a structural member depends upon the internal force system at the section and also upon the geometry of the cross section. In some cases, as we shall see later, these distributions are complex, particularly those produced by the bending and shear of unsymmetrical sections. We can, however, examine the nature of each of these stresses by considering simple loading systems acting on structural members whose cross sections have some degree of symmetry. At the same time we shall define the corresponding strains and investigate the relationships between the two.

7.1 Direct stress in tension and compression

The simplest form of direct stress system is that produced by an axial load. Suppose that a structural member has a uniform 'I' cross section of area A and is subjected to an axial tensile load, P, as shown in Fig. 7.1(a). At any section 'mm' the internal force is a normal force which, from the arguments presented in Chapter 3, is

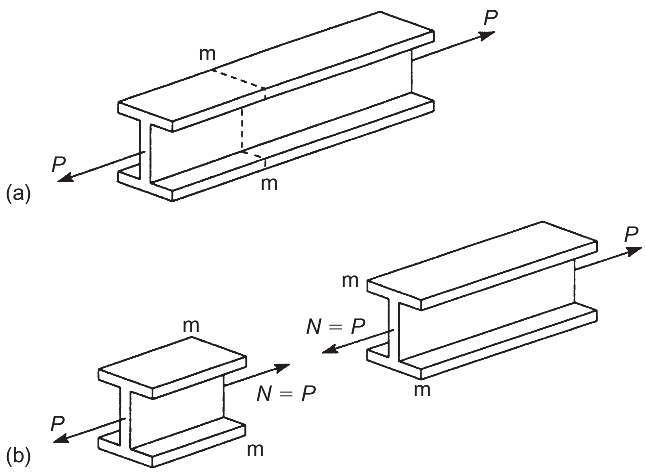

FIGURE 7.1

Structural member with axial load.

Structural and Stress Analysis. https://doi.org/10.1016/B978-0-08-102586-4.00007-X

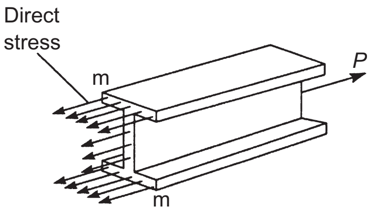

FIGURE 7.2

Internal force distribution in a beam section.

equal to P (Fig. 7.1(b)). It is clear that this normal force is not resisted at just one point on each face of the section as Fig. 7.1(b) indicates but at every point as shown in Fig. 7.2. We assume in fact that P is distributed uniformly over the complete face of the section so that at any point in the cross section there is an intensity of force, i.e. stress, to which we give the symbol σ and which we define as

$$\sigma = \frac{P}{A} \tag{7.1}$$

This direct stress acts in the direction shown in Fig. 7.2 when P is tensile and in the reverse direction when P is compressive. The sign convention for direct stress is identical to that for normal force; a tensile stress is therefore positive while a compressive stress is negative. The SI unit of stress is the pascal (Pa) where 1 Pa is 1 N/m^2. However this is a rather small quantity in many cases so generally we shall use mega-pascals (MPa) where 1 MPa = 1 N/mm^2.

In Fig. 7.1 the section mm is some distance from the point of application of the load. At sections in the proximity of the applied load the distribution of direct stress will depend upon the method of application of the load, and only in the case where the applied load is distributed uniformly over the cross section will the direct stress be uniform over sections in this region. In other cases *stress concentrations* arise which require specialized analysis; this topic is covered in more advanced texts on strength of materials and stress analysis.

We shall see in Chapter 8 that it is the level of stress that governs the behaviour of structural materials. For a given material, failure, or breakdown of the crystalline structure of the material under load, occurs at a constant value of stress. For example, in the case of steel subjected to simple tension failure begins at a stress of about 300 N/mm^2, although variations occur in steels manufactured to different specifications. This stress is independent of size or shape and may therefore be used as the basis for the design of structures fabricated from steel. Failure stress varies considerably from material to material and in some cases depends upon whether the material is subjected to tension or compression.

A knowledge of the failure stress of a material is essential in structural design where, generally, a designer wishes to determine a minimum size for a structural member carrying a given load. For example, for a member fabricated from a given material and subjected to axial load, we would use Eq. (7.1) either to determine a minimum area of cross section for a given load or to check the stress level in a given member carrying a given load.

EXAMPLE 7.1

A short column has a rectangular cross section with sides in the ratio 1:2 (Fig. 7.3). Determine the minimum dimensions of the column section if the column carries an axial load of 800 kN and the failure stress of the material of the column is 400 N/mm^2.

From Eq. (7.1) the minimum area of the cross section is given by

$$A_{\text{min}} = \frac{P}{\sigma_{\text{max}}} = \frac{800 \times 10^3}{400} = 2000\ \text{mm}^2$$

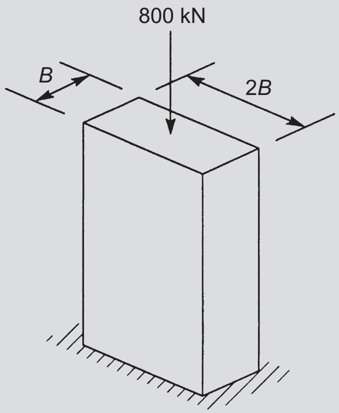

FIGURE 7.3

Column of Ex. 7.1.

But

$$A_{\min} = 2B^2 = 2000\ \text{mm}^2$$

from which

$$B = 31.6\ \text{mm}$$

Therefore the minimum dimensions of the column cross section are 31.6 mm×63.2 mm. In practice these dimensions would be rounded up to 32 mm×64 mm or, if the column were of some standard section, the next section having a cross-sectional area greater than 2000 mm^2 would be chosen. Also the column would not be designed to the limit of its failure stress but to a working or design stress which would incorporate some safety factor (see Section 8.7).

7.2 Shear stress in shear and torsion

An externally applied shear load induces an internal shear force which is tangential to the faces of a beam cross section. Figure 7.4(a) illustrates such a situation for a cantilever beam carrying a shear load W at its free end. We have seen in Chapter 3 that the action of W is to cause sliding of one face of the cross section relative to the other; W also induces internal bending moments which produce internal direct stress systems; these are considered in a later chapter. The internal shear force S ($=W$) required to maintain the vertical equilibrium of the portions of the beam is distributed over each face of the cross section. Thus at any point in the cross section there is a tangential intensity of force which is termed *shear stress*. This shear stress is not distributed uniformly over the faces of the cross section as we shall see in Chapter 10. For the moment, however, we shall define the average shear stress over the faces of the cross section as

$$\tau_{\text{av}} = \frac{W}{A} \tag{7.2}$$

where A is the cross-sectional area of the beam.

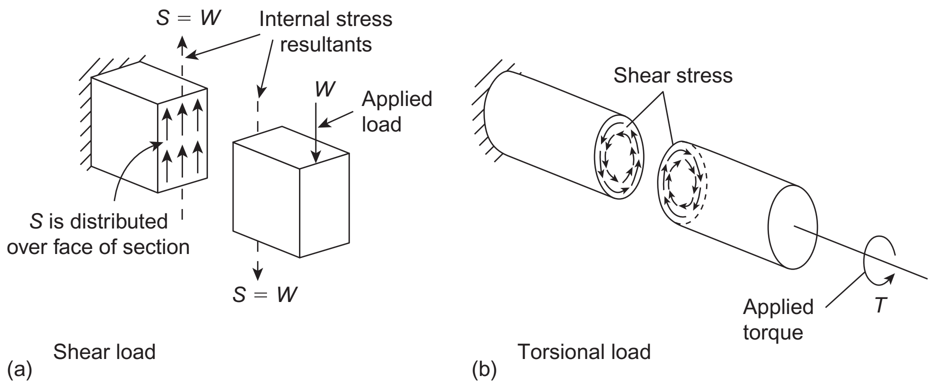

FIGURE 7.4

Generation of shear stresses in beam sections.

Note that the internal shear force S shown in Fig. 7.4(a) is, according to the sign convention adopted in Chapter 3, positive. However, the applied load W would produce an internal shear force in the opposite direction on the positive face of the section so that S would actually be negative.

A system of shear stresses is induced in a different way in the circular-section bar shown in Fig. 7.4(b) where the internal torque (T) tends to produce a relative rotational sliding of the two faces of the cross section. The shear stresses are tangential to concentric circular paths in the faces of the cross section. We shall examine the shear stress due to torsion in various cross sections in Chapter 11.

7.3 Complementary shear stress

Consider the cantilever beam shown in Fig. 7.5(a). Let us suppose that the beam is of rectangular cross section having a depth h and unit thickness; it carries a vertical shear load W at its free end. The internal shear forces on the opposite faces mm and nn of an elemental length δx of the beam are distributed as shear stresses in some manner over each face as shown in Fig. 7.5(b). Suppose now that we isolate a small rectangular element ABCD of depth δh of this elemental length of beam (Fig. 7.5(c)) and consider its equilibrium. Since the element is small, the shear stresses τ on the faces AD and BC may be regarded as constant. The shear force resultants of these shear stresses clearly satisfy vertical equilibrium of the element but rotationally produce an anticlockwise couple. This must be equilibrated by a clockwise couple which can only be produced by shear forces on the horizontal faces AB and CD of the element. Let τ' be the shear stresses induced by these shear forces. The equilibrium of the element is satisfied in both horizontal and vertical directions since the resultant force in either direction is zero. However, the shear forces on the faces BC and AD form a couple which would cause rotation of the element in an anticlockwise sense. We need, therefore, a clockwise balancing couple and this can only be produced by shear forces on the faces AB and CD of the element; the shear stresses corresponding to these shear forces are τ' as shown. Then for rotational equilibrium of the element about the corner D

$$\tau' \times \delta x \times 1 \times \delta h = \tau \times \delta h \times 1 \times \delta x$$

which gives

$$\tau' = \tau \tag{7.3}$$

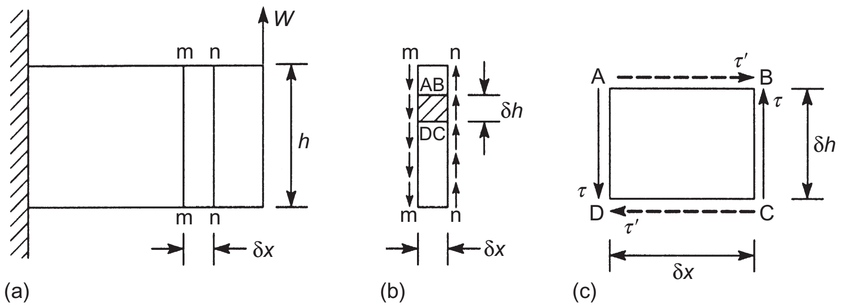

FIGURE 7.5

Complementary shear stress.

We see, therefore, that a shear stress acting on a given plane is always accompanied by an equal *complementary shear stress* acting on planes perpendicular to the given plane and in the opposite sense.

7.4 Direct strain

Since no material is completely rigid, the application of loads produces distortion. An axial tensile load, for example, will cause a structural member to increase in length, whereas a compressive load would cause it to shorten.

Suppose that δ is the change in length produced by either a tensile or compressive axial load. We now define the *direct strain,* ε, in the member in non-dimensional form as the change in length per unit length of the member. Hence

$$\varepsilon = \frac{\delta}{L_0} \tag{7.4}$$

where L_0 is the length of the member in its unloaded state. Clearly ε may be either a tensile (positive) strain or a compressive (negative) strain. Equation (7.4) is applicable only when distortions are relatively small and can be used for values of strain up to and around 0.001, which is adequate for most structural problems. For larger values, load–displacement relationships become complex and are therefore left for more advanced texts.

We shall see in Section 7.7 that it is convenient to measure distortion in this non-dimensional form since there is a direct relationship between the stress in a member and the accompanying strain. The strain in an axially loaded member therefore depends solely upon the level of stress in the member and is independent of its length or cross-sectional geometry.

7.5 Shear strain

In Section 7.3 we established that shear loads applied to a structural member induce a system of shear and complementary shear stresses on any small rectangular element. The distortion in such an element due to these shear stresses does not involve a change in length but a change in shape as shown in Fig. 7.6. We define the *shear strain*, γ, in the element as the change in angle between two originally mutually perpendicular edges. Thus in Fig. 7.6

$$\gamma = \phi \text{ radians} \tag{7.5}$$

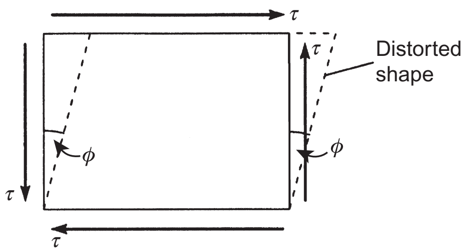

FIGURE 7.6

Shear strain in an element.

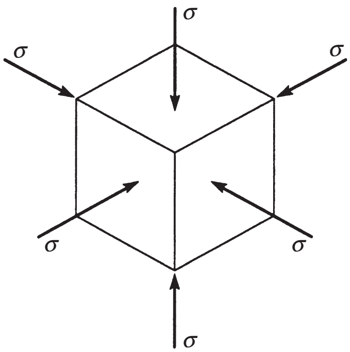

FIGURE 7.7

Cube subjected to hydrostatic pressure.

7.6 Volumetric strain due to hydrostatic pressure

A rather special case of strain which we shall find useful later occurs when a cube of material is subjected to equal compressive stresses, σ, on all six faces as shown in Fig. 7.7. This state of stress is that which would be experienced by the cube if it were immersed at some depth in a fluid, hence the term hydrostatic pressure. The analysis would, in fact, be equally valid if σ were a tensile stress.

Suppose that the original length of each side of the cube is L_0 and that δ is the decrease in length of each side due to the stress. Then, defining the *volumetric strain* as the change in volume per unit volume, we have

$$\text{volumetric strain} = \frac{L_0^3 - (L_0 - \delta)^3}{L_0^3}$$

Expanding the bracketed term and neglecting second- and higher-order powers of δ gives

$$\text{volumetric strain} = \frac{3L_0^2\delta}{L_0^3}$$

from which

$$\text{volumetric strain} = \frac{3\delta}{L_0} \tag{7.6}$$

Thus we see that for this case the volumetric strain is three times the linear strain in any of the three stress directions.

7.7 Stress–strain relationships

Hooke's law and Young's modulus

The relationship between direct stress and strain for a particular material may be determined experimentally by a *tensile test* which is described in detail in Chapter 8. A tensile test consists basically of applying an axial tensile load in known increments to a specimen of material of a given length and cross-sectional area and measuring the corresponding increases in length. The stress produced by each value of load may be calculated from Eq. (7.1) and the corresponding strain from Eq. (7.4). A stress–strain curve is then drawn which, for some materials, would have a shape similar to that shown in Fig. 7.8. Stress–strain curves for other materials differ in detail but, generally, all have a linear portion such as ab in Fig. 7.8. In this region stress is directly proportional to strain, a relationship that was discovered in 1678 by Robert Hooke and which is known as *Hooke's law*. It may be expressed mathematically as

$$\sigma = E\varepsilon \tag{7.7}$$

where E is the constant of proportionality. E is known as *Young's modulus* or the *elastic modulus* of the material and has the same units as stress. For mild steel E is of the order of 200 kN/mm^2. Equation (7.7) may be written in alternative form as

$$\frac{\sigma}{\varepsilon} = E \tag{7.8}$$

For many materials E has the same value in tension and compression.

Shear modulus

By comparison with Eq. (7.8) we can define the *shear modulus* or *modulus of rigidity, G*, of a material as the ratio of shear stress to shear strain; thus

$$G = \frac{\tau}{\gamma} \tag{7.9}$$

Volume or bulk modulus

Again, the *volume modulus* or *bulk modulus, K*, of a material is defined in a similar manner as the ratio of volumetric stress to volumetric strain, i.e.

$$K = \frac{\text{volumetric stress}}{\text{volumetric strain}} \tag{7.10}$$

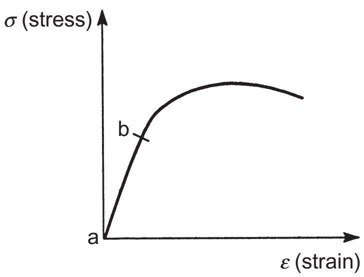

FIGURE 7.8

Typical stress–strain curve.

It is not usual to assign separate symbols to volumetric stress and strain since they may, respectively, be expressed in terms of direct stress and linear strain. Thus in the case of hydrostatic pressure (Section 7.6)

$$K = \frac{\sigma}{3\varepsilon} \tag{7.11}$$

EXAMPLE 7.2

A mild steel column is hollow and circular in cross section with an external diameter of 350 mm and an internal diameter of 300 mm. It carries a compressive axial load of 2000 kN. Determine the direct stress in the column and also the shortening of the column if its initial height is 5 m. Take $E = 200\,000$ N/mm^2.

The cross-sectional area A of the column is given by

$$A = \frac{\pi}{4}(350^2 - 300^2) = 25\,525.4\,\text{mm}^2$$

The direct stress σ in the column is, therefore, from Eq. (7.1)

$$\sigma = -\frac{2000 \times 10^3}{25\,525.4} = -78.4\,\text{N/mm}^2\,(\text{compression})$$

The corresponding strain is obtained from either Eq. (7.7) or Eq. (7.8) and is

$$\varepsilon = \frac{-78.4}{200\,000} = -0.000\,39$$

Finally the shortening, δ, of the column follows from Eq. (7.4), i.e.

$$\delta = 0.000\,39 \times 5 \times 10^3 = 1.95\,\text{mm}$$

EXAMPLE 7.3

A short, deep cantilever beam is 500 mm long by 200 mm deep and is 2 mm thick. It carries a vertically downward load of 10 kN at its free end. Assuming that the shear stress is uniformly distributed over the cross section of the beam, calculate the deflection due to shear at the free end. Take $G = 25\,000$ N/mm^2.

The internal shear force is constant along the length of the beam and equal to 10 kN. Since the shear stress is uniform over the cross section of the beam, we may use Eq. (7.2) to determine its value, i.e.

$$\tau_{\text{av}} = \frac{W}{A} = \frac{10 \times 10^3}{200 \times 2} = 25\,\text{N/mm}^2$$

This shear stress is constant along the length of the beam; it follows from Eq. (7.9) that the shear strain is also constant along the length of the beam and is given by

$$\gamma = \frac{\tau_{\text{av}}}{G} = \frac{25}{25\,000} = 0.001\,\text{rad}$$

This value is in fact the angle that the beam makes with the horizontal. The deflection, Δ_s, due to shear at the free end is therefore

$$\Delta_s = 0.001 \times 500 = 0.5\,\text{mm}$$

In practice, the solution of this particular problem would be a great deal more complex than this since the shear stress distribution is not uniform. Deflections due to shear are investigated in Chapter 13.

7.8 Poisson effect

It is common experience that a material such as rubber suffers a reduction in cross-sectional area when stretched under a tensile load. This effect, known as the *Poisson effect*, also occurs in structural materials subjected to tensile and compressive loads, although in the latter case the cross-sectional area increases. In the region where the stress–strain curve of a material is linear, the ratio of lateral strain to longitudinal strain is a constant which is known as *Poisson's ratio* and is given the symbol v. The effect is illustrated in Fig. 7.9.

Consider now the action of different direct stress systems acting on an elemental cube of material (Fig. 7.10). The stresses are all tensile stresses and are given suffixes which designate their directions in relation to the system of axes specified in Section 3.2. In Fig. 7.10(a) the direct strain, ε_x, in the x direction is obtained directly from either Eq. (7.7) or Eq. (7.8) and is

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Due to the Poisson effect there are accompanying strains in the y and z directions given by

$$\varepsilon_y = -v\varepsilon_x \quad \varepsilon_z = -v\varepsilon_x$$

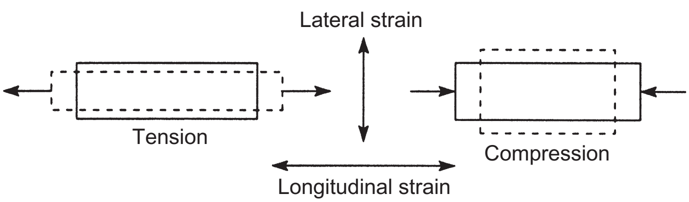

FIGURE 7.9

The Poisson effect.

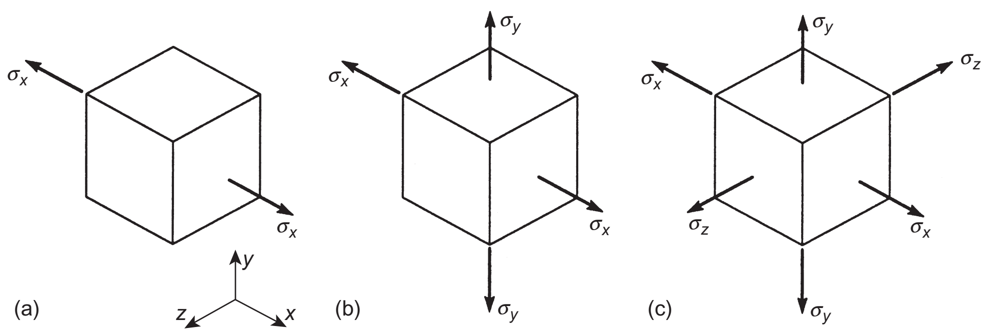

FIGURE 7.10

The Poisson effect in a cube of material.

or, substituting for ε_x in terms of σ_x

$$\varepsilon_y = -v\frac{\sigma_x}{E} \qquad \varepsilon_z = -v\frac{\sigma_x}{E} \tag{7.12}$$

These strains are negative since they are associated with contractions as opposed to positive strains produced by extensions.

In Fig. 7.10(b) the direct stress σ_y has an effect on the direct strain ε_x as does σ_x on ε_y. Thus

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{v\sigma_y}{E} \qquad \varepsilon_y = \frac{\sigma_y}{E} - \frac{v\sigma_x}{E} \qquad \varepsilon_z = -\frac{v\sigma_x}{E} - \frac{v\sigma_y}{E} \tag{7.13}$$

By a similar argument, the strains in the x, y and z directions for the cube of Fig. 7.10(c) are

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{v\sigma_y}{E} - \frac{v\sigma_z}{E} \qquad \varepsilon_y = \frac{\sigma_y}{E} - \frac{v\sigma_x}{E} - \frac{v\sigma_z}{E} \qquad \varepsilon_z = \frac{\sigma_z}{E} - \frac{v\sigma_x}{E} - \frac{v\sigma_y}{E} \tag{7.14}$$

Let us now suppose that the cube of material in Fig. 7.10(c) is subjected to a uniform stress on each face such that $\sigma_x = \sigma_y = \sigma_z = \sigma$. The strain in each of the axial directions is therefore the same and is, from any one of Eq. (7.14)

$$\varepsilon = \frac{\sigma}{E}(1 - 2v)$$

In Section 7.6 we showed that the volumetric strain in a cube of material subjected to equal stresses on all faces is three times the linear strain. Thus in this case

$$\text{volumetric strain} = \frac{3\sigma}{E}(1 - 2v) \tag{7.15}$$

It would be unreasonable to suppose that the volume of a cube of material subjected to tensile stresses on all faces could decrease. It follows that Eq. (7.15) cannot have a negative value. We conclude, therefore, that v must always be less than 0.5. For most metals v has a value in the region of 0.3 while for concrete v can be as low as 0.1.

Collectively E, G, K and v are known as the *elastic constants* of a material.

EXAMPLE 7.4

A cube of material is subjected to the following direct stress system: $\sigma_x = +120\ \text{N/mm}^2$, $\sigma_y = +80\ \text{N/mm}^2$ and $\sigma_z = -100\ \text{N/mm}^2$. If Young's modulus, E, is 200 000 N/mm^2 and Poisson's ratio, ν, is 0.3 calculate the direct strain in the x, y and z directions and hence the volumetric strain in the cube.

The strain in the x direction is given by the first of Eqs (7.14) and is

$$\varepsilon_x = (120 - 0.3 \times 80 + 0.3 \times 100)/200\,000 = 6.3 \times 10^{-4}$$

Similarly, from the second of Eqs (7.14)

$$\varepsilon_y = (80 - 0.3 \times 120 + 0.3 \times 100)/200\,000 = 3.7 \times 10^{-4}$$

and from the third of Eqs (7.14)

$$\varepsilon_z = (-100 - 0.3 \times 120 - 0.3 \times 80)/200\,000 = -8.0 \times 10^{-4}$$

If L_0 is the initial length of each side of the cube then the final lengths are

$$L_x = L_0 + 6.3 \times 10^{-4} L_0 = 1.00063 L_0$$
$$L_y = L_0 + 3.7 \times 10^{-4} L_0 = 1.00037 L_0$$
$$L_z = L_0 - 8.0 \times 10^{-4} L_0 = 0.9992 L_0$$

The volumetric strain in the cube is then

$$\text{Vol. Strain} = [L_0{}^3 - (1.00063)(1.00037)(0.9992) L_0{}^3]/L_0{}^3$$

i.e

$$\text{Vol. Strain} = -1.99 \times 10^{-4}$$

7.9 Relationships between the elastic constants

There are different methods for determining the relationships between the elastic constants. The one presented here is relatively simple in approach and does not require a knowledge of topics other than those already covered.

In Fig. 7.11(a), ABCD is a square element of material of unit thickness and is in equilibrium under a shear and complementary shear stress system τ. Imagine now that the element is 'cut' along the diagonal AC as shown in Fig. 7.11(b). In order to maintain the equilibrium of the triangular portion ABC it is possible that a direct force and a shear force are required on the face AC. These forces, if they exist, will be distributed over the face of the element in the form of direct and shear stress systems, respectively. Since the element is small, these stresses may be assumed to be constant along the face AC. Let the direct stress on AC in the direction BD be σ_{BD} and the shear stress on AC be τ_{AC}. Then resolving forces on the element in the direction BD we have

$$\sigma_{\text{BD}}\text{AC} \times 1 - \tau\text{AB} \times 1 \times \cos 45° - \tau\text{BC} \times 1 \times \cos 45° = 0$$

Dividing through by AC

$$\sigma_{\text{BD}} = \tau \frac{\text{AB}}{\text{AC}} \cos 45° + \tau \frac{\text{BC}}{\text{AC}} \cos 45°$$

or

$$\sigma_{\text{BD}} = \tau \cos^2 45° + \tau \cos^2 45°$$

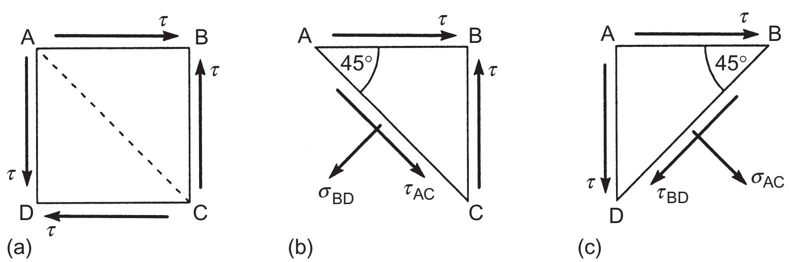

FIGURE 7.11

Determination of the relationships between the elastic constants.

from which

$$\sigma_{BD} = \tau \tag{7.16}$$

The positive sign indicates that σ_{BD} is a tensile stress. Similarly, resolving forces in the direction AC

$$\tau_{AC}AC \times 1 + \tau AB \times 1 \times \cos 45^\circ - \tau BC \times 1 \times \cos 45^\circ = 0$$

Again dividing through by AC we obtain

$$\tau_{AC} = -\tau \cos^2 45^\circ + \tau \cos^2 45^\circ = 0$$

A similar analysis of the triangular element ABD in Fig. 7.11(c) shows that

$$\sigma_{AC} = -\tau \tag{7.17}$$

and

$$\tau_{BD} = 0$$

Hence we see that on planes parallel to the diagonals of the element there are direct stresses σ_{BD} (tensile) and σ_{AC} (compressive) both numerically equal to τ as shown in Fig. 7.12. It follows from Section 7.8 that the direct strain in the direction BD is given by

$$\varepsilon_{BD} = \frac{\sigma_{BD}}{E} + \frac{\nu\sigma_{AC}}{E} = \frac{\tau}{E}(1+\nu) \tag{7.18}$$

Note that the compressive stress σ_{AC} makes a positive contribution to the strain ε_{BD}.

In Section 7.5 we defined shear strain and saw that under pure shear, only a change of shape is involved. Thus the element ABCD of Fig. 7.11(a) distorts into the shape A′B′CD shown in Fig. 7.13. The shear strain γ produced by the shear stress τ is then given by

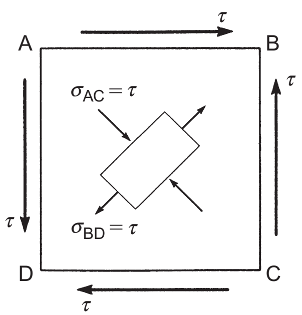

FIGURE 7.12

Stresses on diagonal planes in element.

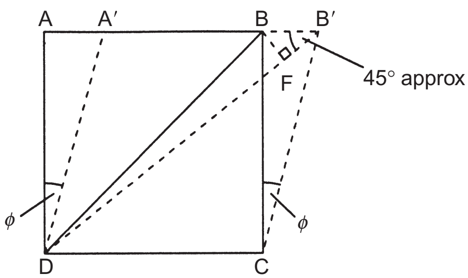

FIGURE 7.13

Distortion due to shear in element.

$$\gamma = \phi \text{ radians} = \frac{\mathrm{B}'B}{\mathrm{BC}} \tag{7.19}$$

since ϕ is a small angle. The increase in length of the diagonal DB to DB′ is approximately equal to FB′ where BF is perpendicular to DB′. Thus

$$\varepsilon_{\mathrm{DB}} = \frac{\mathrm{DB}' - \mathrm{DB}}{\mathrm{DB}} = \frac{\mathrm{FB}'}{\mathrm{DB}}$$

Again, since ϕ is a small angle, $\mathrm{B}\hat{\mathrm{B}}'\mathrm{F} \simeq 45°$ so that

$$\mathrm{FB}' = \mathrm{BB}' \cos 45°$$

Also

$$\mathrm{DB} = \frac{\mathrm{BC}}{\cos 45°}$$

Hence

$$\varepsilon_{\mathrm{DB}} = \frac{\mathrm{B}'\mathrm{B} \cos^2 45°}{\mathrm{BC}} = \frac{1}{2}\frac{\mathrm{B}'\mathrm{B}}{\mathrm{BC}}$$

Therefore, from Eq. (7.19)

$$\varepsilon_{\mathrm{DB}} = \frac{1}{2}\gamma \tag{7.20}$$

Substituting for $\varepsilon_{\mathrm{DB}}$ in Eq. (7.18) we obtain

$$\frac{1}{2}\gamma = \frac{\tau}{E}(1 + v)$$

or, since $\tau/\gamma = G$ from Eq. (7.9)

$$G = \frac{E}{2(1+v)} \quad \text{or} \quad E = 2G(1+v) \tag{7.21}$$

The relationship between Young's modulus E and bulk modulus K is obtained directly from Eqs (7.10) and (7.15). Thus, from Eq. (7.10)

$$\text{volumetric strain} = \frac{\sigma}{K}$$

where σ is the volumetric stress. Substituting in Eq. (7.15)

$$\frac{\sigma}{K} = \frac{3\sigma}{E}(1 - 2v)$$

from which

$$K = \frac{E}{3(1-2v)} \tag{7.22}$$

Eliminating E from Eqs (7.21) and (7.22) gives

$$K = \frac{2G(1+v)}{3(1-2v)} \tag{7.23}$$

EXAMPLE 7.5

If the cube in Ex. 7.4 carries shear and complementary shear stresses of 60 N/mm^2 in the xy plane calculate the corresponding shear strains.

From Eqs (7.21)

$$G = 200\,000/2(1+0.3) = 76923\,\text{N/mm}^2$$

Then, from Eq. (7.9)

$$\gamma_{xy} = 60/76923 = 7.8 \times 10^{-4}$$

EXAMPLE 7.6

A cube of material is subjected to a compressive stress σ on each of its faces. If $v = 0.3$ and $E = 200000$ N/mm^2, calculate the value of this stress if the volume of the cube is reduced by 0.1%. Calculate also the percentage reduction in length of one of the sides.

From Eq. (7.22)

$$K = \frac{200\,000}{3(1 - 2 \times 0.3)} = 167\,000\,\text{N/mm}^2$$

The volumetric strain is 0.001 since the volume of the block is reduced by 0.1%.

Therefore, from Eq. (7.10)

$$0.001 = \frac{\sigma}{K}$$

or

$$\sigma = 0.001 \times 167\,000 = 167\,\text{N/mm}^2$$

In Section 7.6 we established that the volumetric strain in a cube subjected to a uniform stress on all six faces is three times the linear strain. Thus in this case

$$\text{linear strain} = \frac{1}{3} \times 0.001 = 0.00033$$

The length of one side of the cube is therefore reduced by 0.033%.

EXAMPLE 7.7

A mild steel column of height 5 m is hollow and circular in cross section with an external diameter of 400 mm and a wall thickness of 20 mm. If the column carries a compressive load of 2500 kN calculate the direct stress in the column and also the shortening of the column. Take $E = 200\,000$ N/mm^2. If the ends of the column are then fixed so that no further axial movement is possible calculate the total direct stress in the column when it is subjected to a temperature rise of 20 K. The coefficient of linear expansion of the material of the column is $0.00005/K$ and with the usual notation $L = L_0(1 + \alpha T)$.

The cross sectional area, A, of the column is given by

$$A = \pi(400^2 - 360^2)/4 = 23876.1\ \text{mm}^2$$

Then, from Eq. (7.1)

$$\sigma = 2500 \times 10^3/23876.1 = 104.7\ \text{N/mm}^2 (\text{compression})$$

The direct strain in the column is obtained from either of Eqs (7.7) or (7.8) and is

$$\varepsilon = \sigma/E = 104.7/200\,000 = 0.00052$$

so that the shortening, δ, of the column is, from Eq. (7.4), given by

$$\delta = 0.00052 \times 5 \times 10^3 = 2.62\ \text{mm}$$

Due to the temperature rise the column would, if not prevented, increase in height with no corresponding change in stress. However, this change in height is prevented thereby causing, in effect, a strain and an accompanying stress. Now $L = L_0(1 + \alpha T)$ so that the change in height due to a temperature increase would be $L - L_0$ which is equal to $L_0 \alpha T$. The strain corresponding to the suppression of this change in height is then $(L - L_0)/L_0 = \alpha T$. The accompanying direct stress is then, from Eq. (7.7), given by

$$\sigma = 0.00005 \times 20 \times 200\,000 = 200\ \text{N/mm}^2$$

which is compressive since the increase in height is prevented. The total stress in the column produced by the load and the temperature rise is then

$$\sigma(\text{total}) = 200 + 104.7 = 304.7\ \text{N/mm}^2$$

7.10 Strain energy in simple tension or compression

An important concept in the analysis of structures is that of *strain energy*. The total strain energy of a structural member may comprise the separate strain energies due to axial load, bending moment, shear and torsion. In this section we shall concentrate on the strain energy due to tensile or compressive loads; the strain energy produced by each of the other loading systems is considered in the relevant, later chapters.

A structural member subjected to a gradually increasing tensile load P gradually increases in length (Fig. 7.14(a)). The load–extension curve for the member is linear until the limit of proportionality is exceeded, as shown in Fig. 7.14(b). The geometry of the non-linear portion of the curve depends upon the properties of the material of the member (see Chapter 8). Clearly the load P moves through small displacements Δ and therefore does work on the member. This work, which causes the member to extend, is stored in the member as strain energy. If the value of P is restricted so that the limit of proportionality is not exceeded, the gradual removal of P results in the member returning to its original length and the strain energy stored in the member may be recovered in the form of work. When the limit of proportionality is exceeded, not all of the work done by P is recoverable; some is used in producing a permanent distortion of the member (see Chapter 8), the related energy appearing largely as heat.

Suppose the structural member of Fig. 7.14(a) is gradually loaded to some value of P within the limit of proportionality of the material of the member, the corresponding elongation being Δ. Let the elongation

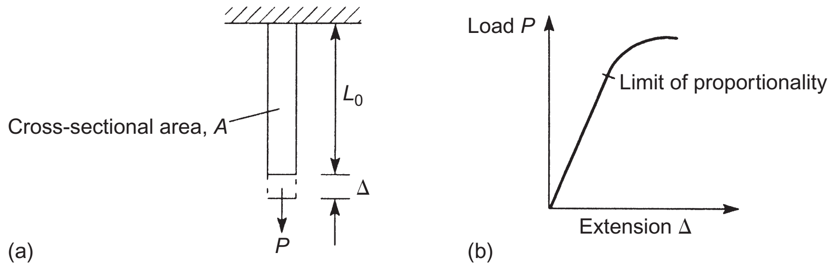

FIGURE 7.14

Load–extension curve for an axially loaded member.

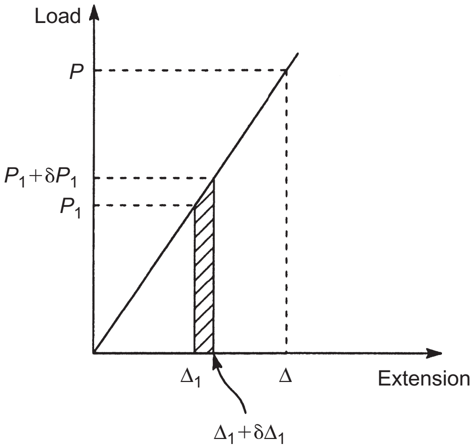

FIGURE 7.15

Work done by a gradually applied load.

corresponding to some intermediate value of load, say P_1, be Δ_1 (Fig. 7.15). Then a small increase in load of δP_1 will produce a small increase, $\delta\Delta_1$, in elongation. The incremental work done in producing this increment in elongation may be taken as equal to the average load between P_1 and $P_1+\delta P_1$ multiplied by $\delta\Delta_1$. Thus

$$\text{incremental work done} = \left[\frac{P_1 + (P_1 + \delta P_1)}{2}\right]\delta\Delta_1$$

which, neglecting second-order terms, becomes

$$\text{incremental work done} = P_1\delta\Delta_1$$

The total work done on the member by the load P in producing the elongation Δ is therefore given by

$$\text{total work done} = \int_0^{\Delta} P_1\,\mathrm{d}\Delta_1 \tag{7.24}$$

Since the load–extension relationship is linear, then

$$P_1 = K\Delta_1 \tag{7.25}$$

where K is some constant whose value depends upon the material properties of the member. Substituting the particular values of P and Δ in Eq. (7.25), we obtain

$$K = \frac{P}{\Delta}$$

so that Eq. (7.25) becomes

$$P_1 = \frac{P}{\Delta}\Delta_1$$

Now substituting for P_1 in Eq. (7.24) we have

$$\text{total work done} = \int_0^{\Delta} \frac{P}{\Delta}\Delta_1 \, \mathrm{d}\Delta_1$$

Integration of this equation yields

$$\text{total work done} = \frac{1}{2}P\Delta \tag{7.26}$$

Alternatively, we see that the right-hand side of Eq. (7.24) represents the area under the load–extension curve, so that again we obtain

$$\text{total work done} = \frac{1}{2}P\Delta$$

By the law of conservation of energy, the total work done is equal to the strain energy, U, stored in the member. Thus

$$U = \frac{1}{2}P\Delta \tag{7.27}$$

The direct stress, σ, in the member of Fig. 7.14(a) corresponding to the load P is given by Eq. (7.1), i.e.

$$\sigma = \frac{P}{A}$$

Also the direct strain, ε, corresponding to the elongation Δ is, from Eq. (7.4)

$$\varepsilon = \frac{\Delta}{L_0}$$

Furthermore, since the load–extension curve is linear, the direct stress and strain are related by Eq. (7.7), so that

$$\frac{P}{A} = E\frac{\Delta}{L_0}$$

from which

$$\Delta = \frac{PL_0}{AE} \tag{7.28}$$

In Eq. (7.28) the quantity L_0/AE determines the magnitude of the displacement produced by a given load; it is therefore known as the *flexibility* of the member. Conversely, by transposing Eq. (7.28) we see that

$$P = \frac{AE}{L_0}\Delta$$

in which the quantity AE/L_0 determines the magnitude of the load required to produce a given displacement. The term AE/L_0 is then the *stiffness* of the member.

Substituting for Δ in Eq. (7.27) gives

$$U = \frac{P^2 L_0}{2AE} \tag{7.29}$$

It is often convenient to express strain energy in terms of the direct stress σ. Rewriting Eq. (7.29) in the form

$$U = \frac{1}{2}\frac{P^2}{A^2}\frac{AL_0}{E}$$

we obtain

$$U = \frac{\sigma^2}{2E} \times AL_0 \tag{7.30}$$

in which we see that AL_0 is the volume of the member. The strain energy per unit volume of the member is then

$$\frac{\sigma^2}{2E}$$

The greatest amount of strain energy per unit volume that can be stored in a member without exceeding the limit of proportionality is known as the *modulus of resilience* and is reached when the direct stress in the member is equal to the direct stress corresponding to the elastic limit of the material of the member.

The strain energy, U, may also be expressed in terms of the elongation, Δ, or the direct strain, ε. Thus, substituting for P in Eq. (7.29)

$$U = \frac{EA\Delta^2}{2L_0} \tag{7.31}$$

or, substituting for σ in Eq. (7.30)

$$U = \frac{1}{2}E\varepsilon^2 \times AL_0 \tag{7.32}$$

The above expressions for strain energy also apply to structural members subjected to compressive loads since the work done by P in Fig. 7.14(a) is independent of the direction of movement of P. It follows that strain energy is always a positive quantity.

The concept of strain energy has numerous and wide ranging applications in structural analysis particularly in the solution of statically indeterminate structures. We shall examine in detail some of the uses of strain energy later but here we shall illustrate its use by applying it to some relatively simple structural problems.

Deflection of a simple truss

The truss shown in Fig. 7.16 carries a gradually applied load W at the joint A. Considering the vertical equilibrium of joint A

$$P_{AB} \cos 45° - W = 0$$

so that

$$P_{AB} = 1.41\, W \text{ (tension)}$$

Now resolving forces horizontally at A

$$P_{AC} + P_{AB} \cos 45° = 0$$

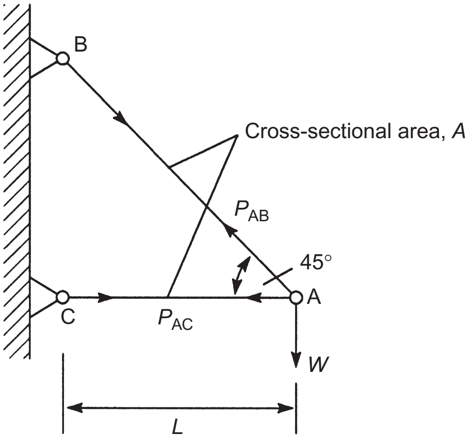

FIGURE 7.16

Deflection of a simple truss.

which gives

$$P_{AC} = -W \text{ (compression)}$$

It is obvious from inspection that P_{AC} is a compressive force but, for consistency, we continue with the convention adopted in Chapter 4 for solving trusses where all members are assumed, initially, to be in tension.

The strain energy of each member is then, from Eq. (7.29)

$$U_{AB} = \frac{(1.41W)^2 \times 1.41L}{2AE} = \frac{1.41W^2L}{AE}$$
$$U_{AC} = \frac{W^2L}{2AE}$$

If the *vertical* deflection of A is Δ_v, the work done by the gradually applied load, W, is

$$\frac{1}{2}W\Delta_v$$

Then equating the work done to the total strain energy of the truss we have

$$\frac{1}{2}W\Delta_v = \frac{1.41W^2L}{AE} + \frac{W^2L}{2AE}$$

so that

$$\Delta_v = \frac{3.82WL}{AE}$$

Using strain energy to calculate deflections in this way has limitations. In the above example Δ_V, is, in fact, only the vertical component of the actual deflection of the joint A since A moves horizontally as well as vertically. Therefore we can only find the deflection of a load *in its own line of action* by this method. Furthermore, the method cannot be applied to structures subjected to more than one applied load as each load would contribute to the total work done by moving through an unknown displacement in its own line of action. There would, therefore, be as many unknown displacements as loads in the work–energy equation. We shall return to examine energy methods in much greater detail in Chapter 15.

EXAMPLE 7.8

Calculate the vertical displacement of the joint C in the truss shown in Fig. 7.17. All members have a cross sectional area of 500 mm^2 and a Young's modulus of 200 000 N/mm^2.

The forces in the members of the truss may be found using the method of joints and are as follows:

$$F_{BC} = +282.8\,kN,\ F_{CD} = -200.0\,kN,\ F_{BD} = 0,\ F_{DE} = -200.0\,kN,$$
$$F_{BE} = -282.8\,kN,\ F_{AB} = +400\,kN.$$

The lengths of BE and BC are each $= \sqrt{(1.0^2 + 1.0^2)} = 1.41$ m. Then, from Eq. (7.29) the total strain energy of the truss is given by

$$U = (282.8^2 \times 1.41 + 200.0^2 \times 1.0 + 200.0^2 \times 1.0 + 282.8^2 \times 1.41$$
$$+400.0^2 \times 1.0) \times 10^9/2 \times 500 \times 200\,000$$

which gives

$$U = 2.33 \times 10^6\ \text{Nmm}.$$

This is equal to the work done by the load as it moves through the vertical displacement, Δ_V. Then

$$200 \times 10^3 \Delta_V/2 = 2.33 \times 10^6$$

so that

$$\Delta_V = 23.3\ \text{mm}.$$

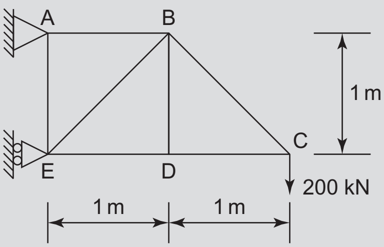

FIGURE 7.17

Truss of Ex. 7.8.

Composite structural members

Axially loaded composite members are of direct interest in civil engineering where concrete columns are reinforced by steel bars and steel columns are frequently embedded in concrete as a fire precaution.

In Fig. 7.18 a concrete column of cross-sectional area A_C is reinforced by two steel bars having a combined cross-sectional area A_S. The modulus of elasticity of the concrete is E_C and that of the steel E_S. A load P is transmitted to the column through a plate which we shall assume is rigid so that the deflection of the concrete is equal to that of the steel. It follows that their respective strains are equal since both have the same original length. Since E_C is not equal to E_S we see from Eq. (7.7) that the compressive stresses, σ_C and σ_S, in the concrete and steel, respectively, must have different values. This also means that unless A_C and A_S have particular values, the compressive loads, P_C and P_S, in the concrete and steel are also different. The problem is therefore statically indeterminate since we can write down only one equilibrium equation, i.e.

$$P_C + P_S = P \tag{7.33}$$

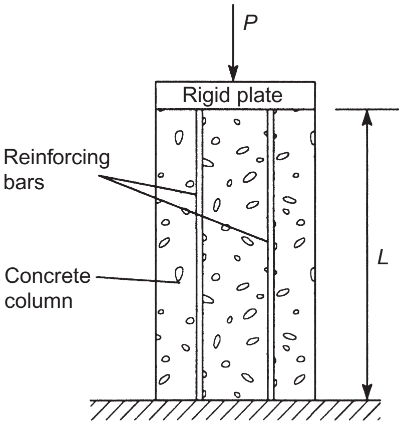

FIGURE 7.18

Composite concrete column.

The second required equation derives from the fact that the displacements of the steel and concrete are identical since, as noted above, they are connected by the rigid plate; this is a *compatibility of displacement* condition. Then, from Eq. (7.28)

$$\frac{P_C L}{A_C E_C} = \frac{P_S L}{A_S E_S} \tag{7.34}$$

Substituting for P_C from Eq. (7.34) in Eq. (7.33) gives

$$P_S\left(\frac{A_C E_C}{A_S E_S} + 1\right) = P$$

from which

$$P_S = \frac{A_S E_S}{A_C E_C + A_S E_S} P \tag{7.35}$$

P_C follows directly from Eqs (7.34) and (7.35), i.e.

$$P_C = \frac{A_C E_C}{A_C E_C + A_S E_S} P \tag{7.36}$$

The vertical displacement, δ, of the column is obtained using either side of Eq. (7.34) and the appropriate compressive load, P_C or P_S. Thus

$$\delta = \frac{PL}{A_C E_C + A_S E_S} \tag{7.37}$$

The direct stresses in the steel and concrete are obtained from Eqs (7.35) and (7.36), thus

$$\sigma_S = \frac{E_S}{A_C E_C + A_S E_S} P \quad \sigma_C = \frac{E_C}{A_C E_C + A_S E_S} P \tag{7.38}$$

We could, in fact, have solved directly for the stresses by writing Eqs (7.33) and (7.34) as

$$\sigma_C A_C + \sigma_S A_S = P \tag{7.39}$$

and

$$\frac{\sigma_C L}{E_C} = \frac{\sigma_S L}{E_S} \tag{7.40}$$

respectively.

EXAMPLE 7.9

A reinforced concrete column, 5 m high, has the cross section shown in Fig. 7.19. It is reinforced by four steel bars each 20 mm in diameter and carries a load of 1000 kN. If Young's modulus for steel is 200000 $\mathrm{N/mm^2}$ and that for concrete is 15 000 $\mathrm{N/mm^2}$, calculate the stress in the steel and in the concrete and also the shortening of the column.

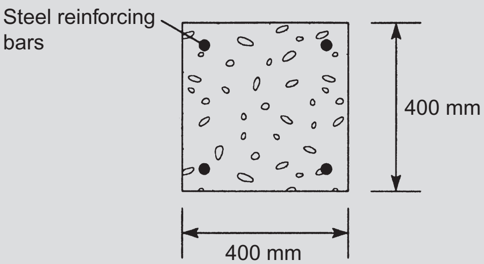

FIGURE 7.19

Reinforced concrete column of Ex. 7.9.

The total cross-sectional area, A_S, of the steel reinforcement is

$$A_S = 4 \times \frac{\pi}{4} \times 20^2 = 1257\ \mathrm{mm}^2$$

The cross-sectional area, A_C, of the concrete is reduced due to the presence of the steel and is given by

$$A_C = 400^2 - 1257 = 158\,743\ \mathrm{mm}^2$$

Equations (7.38) then give

$$\sigma_S = \frac{200\,000 \times 1000 \times 10^3}{158\,743 \times 15\,000 + 1257 \times 200\,000} = 76.0\ \mathrm{N/mm}^2$$

$$\sigma_C = \frac{15\,000 \times 1000 \times 10^3}{158\,743 \times 15\,000 + 1257 \times 200\,000} = 5.7\ \mathrm{N/mm}^2$$

The deflection, δ, of the column is obtained using either side of Eq. (7.40). Thus

$$\delta = \frac{\sigma_C L}{E_C} = \frac{5.7 \times 5 \times 10^3}{15\,000} = 1.9\ \mathrm{mm}$$

Thermal effects

It is possible for stresses to be induced by temperature changes in composite members which are additional to those produced by applied loads. These stresses arise when the components of a composite member have different rates of thermal expansion and contraction.

First, let us consider a member subjected to a uniform temperature rise, ΔT, along its length. The member expands from its original length, L_0, to a length, L_T, given by

$$L_T = L_0(1 + \alpha \Delta T)$$

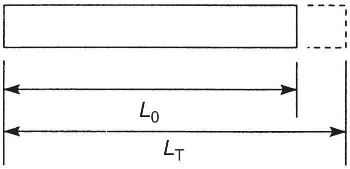

FIGURE 7.20

Expansion due to temperature rise.

where α is the coefficient of linear expansion of the material of the member. In the condition shown in Fig. 7.20 the member has been allowed to expand freely so that no stresses are induced. The increase in the length of the member is then

$$L_T - L_0 = L_0 \alpha \Delta T$$

Suppose now that expansion is completely prevented so that the final length of the member after the temperature rise is still L_0. The member has, in effect, been compressed by an amount $L_0\alpha\Delta T$, thereby producing a compressive strain, ε, which is given by (see Eq. (7.4))

$$\varepsilon = \frac{L_0 \alpha \Delta T}{L_0} = \alpha \Delta T \tag{7.41}$$

The corresponding compressive stress, σ, is from Eq. (7.7)

$$\sigma = E \alpha \Delta T \tag{7.42}$$

In composite members the restriction on expansion or contraction is usually imposed by the attachment of one component to another. For example, in a reinforced concrete column, the bond between the reinforcing steel and the concrete prevents the free expansion or contraction of either.

Consider the reinforced concrete column shown in Fig. 7.21(a) which is subjected to a temperature rise, ΔT. For simplicity we shall suppose that the reinforcement consists of a single steel bar of cross-sectional area, A_S, located along the axis of the column; the actual cross-sectional area of concrete is A_C. Young's modulus and the coefficient of linear expansion of the concrete are E_C and α_C, respectively, while the corresponding values for the steel are E_S and α_S. We shall assume that $\alpha_S > \alpha_C$.

Figure 7.21(b) shows the positions the concrete and steel would attain if they were allowed to expand freely; in this situation neither material is stressed. The displacements $L_0\alpha_C\Delta T$ and $L_0\alpha_S\Delta T$ are obtained directly from Eq. (7.41). However, since they are attached to each other, the concrete prevents the steel from

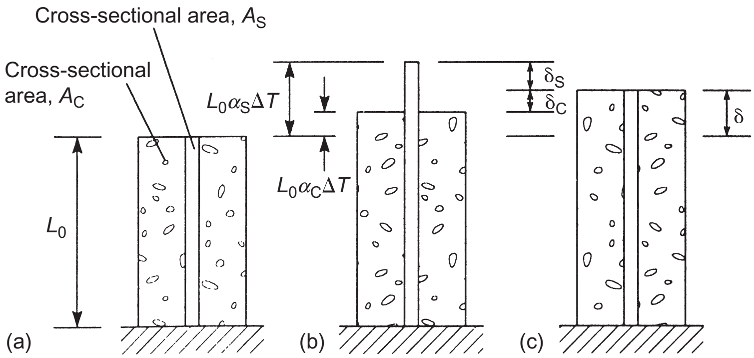

FIGURE 7.21

Reinforced concrete column subjected to a temperature rise.

expanding this full amount while the steel forces the concrete to expand further than it otherwise would; their final positions are shown in Fig. 7.21(c). It can be seen that δ_C is the effective elongation of the concrete which induces a direct tensile load, P_C. Similarly δ_S is the effective contraction of the steel which induces a compressive load, P_S. There is no externally applied load so that the resultant axial load at any section of the column is zero so that

$$P_C\,(\text{tension}) = P_S\,(\text{compression}) \tag{7.43}$$

Also, from Fig. 7.21(b) and (c) we see that

$$\delta_C + \delta_S = L_0\alpha_S\Delta T - L_0\alpha_C\Delta T$$

or

$$\delta_C + \delta_S = L_0\Delta T(\alpha_S - \alpha_C) \tag{7.44}$$

From Eq. (7.28)

$$\delta_C = \frac{P_C L_0}{A_C E_C} \quad \delta_S = \frac{P_S L_0}{A_S E_S} \tag{7.45}$$

Substituting for δ_C and δ_S in Eq. (7.44) we obtain

$$\frac{P_C}{A_C E_C} + \frac{P_S}{A_S E_S} = \Delta T(\alpha_S - \alpha_C) \tag{7.46}$$

Simultaneous solution of Eqs (7.43) and (7.46) gives

$$P_C\,(\text{tension}) = P_S\,(\text{compression}) = \frac{\Delta T(\alpha_S - \alpha_C)}{\left(\dfrac{1}{A_C E_C} + \dfrac{1}{A_S E_S}\right)} \tag{7.47}$$

or

$$P_C\,(\text{tension}) = P_S\,(\text{compression}) = \frac{\Delta T(\alpha_S - \alpha_C)A_C E_C A_S E_S}{A_C E_C + A_S E_S} \tag{7.48}$$

The tensile stress, σ_C, in the concrete and the compressive stress, σ_S, in the steel follow directly from Eq. (7.48).

$$\begin{aligned} \sigma_C &= \frac{P_C}{A_C} = \frac{\Delta T(\alpha_S - \alpha_C)E_C A_S E_S}{A_C E_C + A_S E_S} \\ \sigma_S &= \frac{P_S}{A_S} = \frac{\Delta T(\alpha_S - \alpha_C)A_C E_C E_S}{A_C E_C + A_S E_S} \end{aligned} \tag{7.49}$$

From Fig. 7.21(b) and (c) it can be seen that the actual elongation, δ, of the column is given by either

$$\delta = L_0\alpha_C\Delta T + \delta_C \quad \text{or} \quad \delta = L_0\alpha_S\Delta T - \delta_S \tag{7.50}$$

Using the first of Eq. (7.50) and substituting for δ_C from Eq. (7.45) then P_C from Eq. (7.48) we have

$$\delta = L_0\alpha_C\Delta T + \frac{\Delta T(\alpha_S - \alpha_C)A_C E_C A_S E_S L_0}{A_C E_C(A_C E_C + A_S E_S)}$$

which simplifies to

$$\delta = L_0\Delta T\left(\frac{\alpha_C A_C E_C + \alpha_S A_S E_S}{A_C E_C + A_S E_S}\right) \tag{7.51}$$

Clearly when $\alpha_C = \alpha_S = \alpha$, say, $P_C = P_S = 0$, $\sigma_C = \sigma_S = 0$ and $\delta = L_0\alpha\Delta T$ as for unrestrained expansion.

The above analysis also applies to the case, $\alpha_C > \alpha_S$, when, as can be seen from Eqs (7.48) and (7.49) the signs of P_C, P_S, σ_C and σ_S are reversed. Thus the load and stress in the concrete become compressive, while those in the steel become tensile. A similar argument applies when ΔT specifies a temperature reduction.

Equation (7.44) is an expression of the compatibility of displacement of the concrete and steel. Also note that the stresses could have been obtained directly by writing Eqs (7.43) and (7.44) as

$$\sigma_C A_C = \sigma_S A_S$$

and

$$\frac{\sigma_C L_0}{E_C} + \frac{\sigma_S L_0}{E_S} = L_0 \Delta T(\alpha_S - \alpha_C)$$

respectively.

EXAMPLE 7.10

A rigid slab of weight 100 kN is supported on three columns each of height 4 m and cross-sectional area 300 mm^2 arranged in line. The two outer columns are fabricated from material having a Young's modulus of 80 000 N/mm^2 and a coefficient of linear expansion of 1.85×10^{-5}/°C; the corresponding values for the inner column are 200 000 N/mm^2 and 1.2×10^{-5}/°C. If the slab remains firmly attached to each column, determine the stress in each column and the displacement of the slab if the temperature is increased by 100°C.

The problem may be solved by determining separately the stresses and displacements produced by the applied load and the temperature rise; the two sets of results are then superimposed. Let subscripts o and i refer to the outer and inner columns, respectively. Using Eq. (7.38) we have

$$\sigma_i\,(\text{load}) = \frac{E_i}{A_o E_o + A_i E_i} P \quad \sigma_o\,(\text{load}) = \frac{E_o}{A_o E_o + A_i E_i} P \tag{i}$$

In Eq. (i)

$$A_o E_o + A_i E_i = 2 \times 300 \times 80\,000 + 300 \times 200\,000 = 108.0 \times 10^6$$

Then

$$\sigma_i\,(\text{load}) = \frac{200\,000 \times 100 \times 10^3}{108.0 \times 10^6} = 185.2\,\text{N/mm}^2\ (\text{compression})$$

$$\sigma_o\,(\text{load}) = \frac{80\,000 \times 100 \times 10^3}{108.0 \times 10^6} = 74.1\,\text{N/mm}^2\ (\text{compression})$$

Equation (7.49) give the values of σ_i (temp.) and σ_o (temp.) produced by the temperature rise, i.e.

$$\begin{aligned}\sigma_o\,(\text{temp.}) &= \frac{\Delta T(\alpha_i - \alpha_o) E_o A_i E_i}{A_o E_o + A_i E_i} \\ \sigma_i\,(\text{temp.}) &= \frac{\Delta T(\alpha_i - \alpha_o) A_o E_o E_i}{A_o E_o + A_i E_i}\end{aligned} \tag{ii}$$

In Eq. (ii) $\alpha_o > \alpha_i$ so that σ_o (temp.) is a compressive stress while σ_i (temp.) is a tensile stress. Hence

Continued

EXAMPLE 7.10 CONT'D

$$\sigma_o\,(\text{temp.}) = \frac{100(1.2-1.85)\times 10^{-5}\times 80\,000\times 300\times 200\,000}{105.0\times 10^6}$$

$$= -28.9\,\text{N/mm}^2\ (\text{i.e.compression})$$

$$\sigma_i\,(\text{temp.}) = \frac{100(1.2-1.85)\times 10^{-5}\times 2\times 300\times 80\,000\times 200\,000}{108.0\times 10^6}$$

$$= -57.8\,\text{N/mm}^2\ (\text{i.e.tension})$$

Superimposing the sets of stresses, we obtain the final values of stress, σ_i and σ_o, due to load and temperature change combined. Hence

$$\sigma_i = 185.2 - 57.8 = 127.4\,\text{N/mm}^2\ (\text{compression})$$

$$\sigma_o = 74.1 + 28.9 = 103.0\,\text{N/mm}^2\ (\text{compression})$$

The displacements due to the load and temperature change are found using Eqs (7.37) and (7.51), respectively. Hence

$$\delta\,(\text{load}) = \frac{100\times 10^3\times 4\times 10^3}{108.0\times 10^6} = 3.7\,\text{mm}\ (\text{contraction})$$

$$\delta\,(\text{temp.}) = 4\times 10^3\times 100 \times\left(\frac{1.85\times 10^{-5}\times 2\times 300\times 80\,000 + 1.2\times 10^{-5}\times 300\times 200\,000}{108.0\times 10^6}\right)$$

$$= 6.0\,\text{mm}\ (\text{elongation})$$

The final displacement of the slab involves an overall elongation of the columns of $6.0-3.7 = 2.3$ mm.

Initial stresses and prestressing

The terms initial stress and prestressing refer to structural situations in which some or all of the components of a structure are in a state of stress *before* external loads are applied. In some cases, for example welded connections, this is an unavoidable by-product of fabrication and unless the whole connection is stress-relieved by suitable heat treatment the initial stresses are not known with any real accuracy. On the other hand, the initial stress in a component may be controlled as in a bolted connection; the subsequent applied load may or may not affect the initial stress in the bolt.

Initial stresses may be deliberately induced in a structural member so that the adverse effects of an applied load are minimized. In this category is the prestressing of beams fabricated from concrete which is particularly weak in tension. An overall state of compression is induced in the concrete so that tensile stresses due to applied loads merely reduce the level of compressive stress in the concrete rather than cause tension. Two methods of prestressing are employed, pre- and post-tensioning. In the former the prestressing tendons are positioned in the mould before the concrete is poured and loaded to the required level of tensile stress. After the concrete has set, the tendons are released and the tensile load in the tendons is transmitted, as a compressive load, to the concrete. In a post-tensioned beam, metal tubes or conduits are located in the mould at points where reinforcement is required, the concrete is poured and allowed to set. The reinforcing tendons are then passed through the conduits, tensioned and finally attached to end plates which transmit the tendon tensile load, as a compressive load, to the concrete.

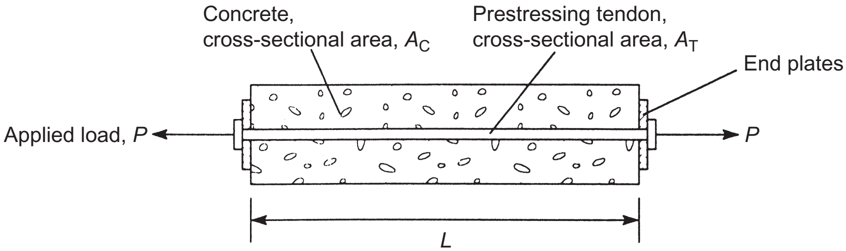

FIGURE 7.22

Prestressed concrete beam.

Usually the reinforcement in a concrete beam supporting vertical shear loads is placed closer to either the upper or the lower surface since such a loading system induces tension in one part of the beam and compression in the other; clearly the reinforcement is placed in the tension zone. To demonstrate the basic principle, however, we shall investigate the case of a post-tensioned beam containing one axially loaded prestressing tendon.

Suppose that the initial load in the prestressing tendon in the concrete beam shown in Fig. 7.22 is F. In the absence of an applied load the resultant load at any section of the beam is zero so that the load in the concrete is also F but compressive. If now a tensile load, P, is applied to the beam, the tensile load in the prestressing tendon will increase by an amount ΔP_T while the compressive load in the concrete will decrease by an amount ΔP_C. From a consideration of equilibrium

$$\Delta P_T + \Delta P_C = P \tag{7.52}$$

Furthermore, the total tensile load in the tendon is $F + \Delta P_T$ while the total compressive load in the concrete is $F - \Delta P_C$.

The tendon and concrete beam are interconnected through the end plates so that they both suffer the same elongation, δ, due to P. Then, from Eq. (7.28)

$$\delta = \frac{\Delta P_T L}{A_T E_T} = \frac{\Delta P_C L}{A_C E_C} \tag{7.53}$$

where E_T and E_C are Young's modulus for the tendon and the concrete, respectively. From Eq. (7.53)

$$\Delta P_T = \frac{A_T E_T}{A_C E_C} \Delta P_C \tag{7.54}$$

Substituting in Eq. (7.52) for ΔP_T we obtain

$$\Delta P_C \left(\frac{A_T E_T}{A_C E_C} + 1 \right) = P$$

which gives

$$\Delta P_C = \frac{A_C E_C}{A_C E_C + A_T E_T} P \tag{7.55}$$

Substituting now for ΔP_C in Eq. (7.54) from Eq. (7.55) gives

$$\Delta P_T = \frac{A_T E_T}{A_C E_C + A_T E_T} P \tag{7.56}$$

The final loads, P_C and P_T, in the concrete and tendon, respectively, are then

$$P_C = F - \frac{A_C E_C}{A_C E_C + A_T E_T} P \quad \text{(compression)} \tag{7.57}$$

and

$$P_T = F + \frac{A_T E_T}{A_C E_C + A_T E_T} P \quad \text{(tension)} \tag{7.58}$$

The corresponding final stresses, σ_C and σ_T, follow directly and are given by

$$\sigma_C = \frac{P_C}{A_C} = \frac{1}{A_C}\left(F - \frac{A_C E_C}{A_C E_C + A_T E_T} P\right) \quad \text{(compression)} \tag{7.59}$$

and

$$\sigma_T = \frac{P_T}{A_T} = \frac{1}{A_T}\left(F + \frac{A_T E_T}{A_C E_C + A_T E_T} P\right) \quad \text{(tension)} \tag{7.60}$$

Obviously if the bracketed term in Eq. (7.59) is negative then σ_C will be a tensile stress.
Finally the elongation, δ, of the beam due to P is obtained from either of Eq. (7.53) and is

$$\delta = \frac{L}{A_C E_C + A_T E_T} P \tag{7.61}$$

EXAMPLE 7.11

A concrete beam of rectangular cross section, 120 mm × 300 mm, is to be reinforced by six high-tensile steel prestressing tendons each having a cross-sectional area of 300 mm^2. If the level of prestress in the tendons is 150 N/mm^2, determine the corresponding compressive stress in the concrete. If the reinforced beam is subjected to an axial tensile load of 150 kN, determine the final stress in the steel and in the concrete assuming that the ratio of the elastic modulus of steel to that of concrete is 15.

The cross-sectional area, A_C, of the concrete in the beam is given by

$$A_C = 120 \times 300 - 6 \times 300 = 34200 \text{ mm}^2$$

The initial compressive load in the concrete is equal to the initial tensile load in the steel; thus

$$\sigma_{Ci} \times 34\,200 = 150 \times 6 \times 300 \tag{i}$$

where σ_{Ci} is the initial compressive stress in the concrete. Hence

$$\sigma_{Ci} = 7.9 \text{ N/mm}^2$$

The final stress in the concrete and in the steel are given by Eqs (7.59) and (7.60), respectively. From Eq. (7.59)

$$\sigma_C = \frac{F}{A_C} - \frac{E_C}{A_C E_C + A_T E_T} P \tag{ii}$$

in which $F/A_C = \sigma_{Ci} = 7.9$ N/mm^2. Rearranging Eq. (ii) we have

$$\sigma_C = 7.9 - \frac{1}{A_C + \left(\frac{E_T}{E_C}\right) A_T} P$$

or

$$\sigma_C = 7.9 - \frac{150 \times 10^3}{34\,200 + 15 \times 6 \times 300} = 5.4 \text{ N/mm}^2 \quad \text{(compression)}$$

Similarly, from Eq. (7.60)

$$\sigma_T = 150 + \frac{1}{\left(\frac{E_C}{E_T}\right)A_C + A_T} P$$

from which

$$\sigma_T = 150 + \frac{150 \times 10^3}{\frac{1}{15} \times 34\,200 + 6 \times 300} = 186.8 \text{ N/mm}^2 \quad \text{(tension)}$$

7.11 Plane stress

In some situations the behaviour of a structure, or part of it, can be regarded as two-dimensional. For example, the stresses produced in a flat plate which is subjected to loads solely in its own plane would form a two-dimensional stress system; in other words, a *plane stress* system. These stresses would, however, produce strains perpendicular to the surfaces of the plate due to the Poisson effect (Section 7.8).

An example of a plane stress system is that produced in the walls of a thin cylindrical shell by internal pressure. Figure 7.23 shows a long, thin-walled cylindrical shell subjected to an internal pressure p. This internal pressure has a dual effect; it acts on the sealed ends of the shell thereby producing a *longitudinal* direct stress in cross sections of the shell and it also tends to separate one-half of the shell from the other along a diametral plane causing *circumferential* or *hoop* stresses. These two situations are shown in Figs. 7.24 and 7.25, respectively.

Suppose that d is the internal diameter of the shell and t the thickness of its walls. In Fig. 7.24 the axial load on each end of the shell due to the pressure p is

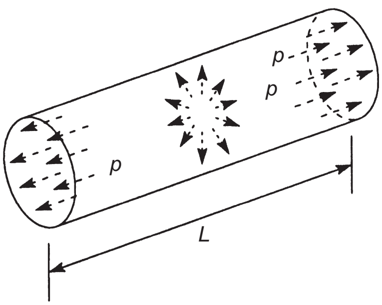

FIGURE 7.23

Thin cylindrical shell under internal pressure.

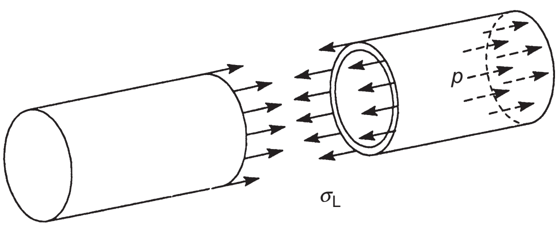

FIGURE 7.24

Longitudinal stresses due to internal pressure.

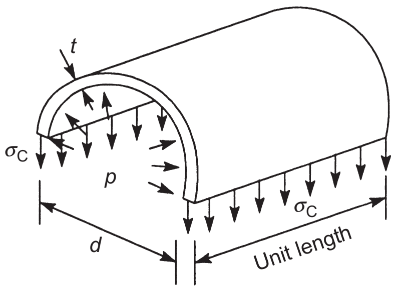

FIGURE 7.25

Circumferential stress due to internal pressure.

$$p \times \frac{\pi d^2}{4}$$

This load is equilibrated by an internal force corresponding to the longitudinal direct stress, σ_L, so that

$$\sigma_L \pi\, dt = p\frac{\pi\, d^2}{4}$$

which gives

$$\sigma_L = \frac{pd}{4t} \tag{7.62}$$

Now consider a unit length of the half shell formed by a diametral plane (Fig. 7.25). The force on the shell, produced by p, in the opposite direction to the circumferential stress, σ_C, is given by

$$p \times \text{projected area of the shell in the direction of } \sigma_C$$

Therefore for equilibrium of the unit length of shell

$$2\sigma_C \times (1 \times t) = p \times (1 \times d)$$

which gives

$$\sigma_C = \frac{pd}{2t} \tag{7.63}$$

We can now represent the state of stress at any point in the wall of the shell by considering the stress acting on the edges of a very small element of the shell wall as shown in Fig. 7.26(a). The stresses comprise the longitudinal stress, σ_L, (Eq. (7.62)) and the circumferential stress, σ_C, (Eq. (7.63)). Since the element is very small, the effect of the curvature of the shell wall can be neglected so that the state of stress may be represented as a *two-dimensional* or *plane* stress system acting on a plane element of thickness, t (Fig. 7.26(b)).

In addition to stresses, the internal pressure produces corresponding strains in the walls of the shell which lead to a change in volume. Consider the element of Fig. 7.26(b). The longitudinal strain, ε_L, is, from Eq. (7.13)

$$\varepsilon_L = \frac{\sigma_L}{E} - v\frac{\sigma_C}{E}$$

or, substituting for σ_L and σ_C from Eqs (7.62) and (7.63), respectively

$$\varepsilon_L = \frac{pd}{2tE}\left(\frac{1}{2} - v\right) \tag{7.64}$$

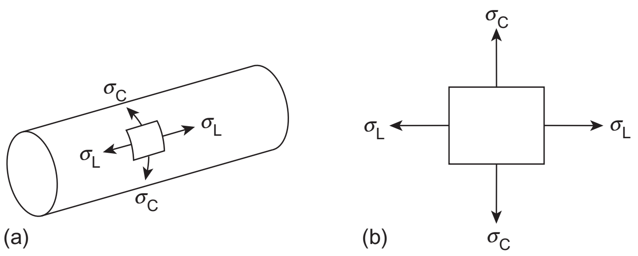

FIGURE 7.26

Two-dimensional stress system.

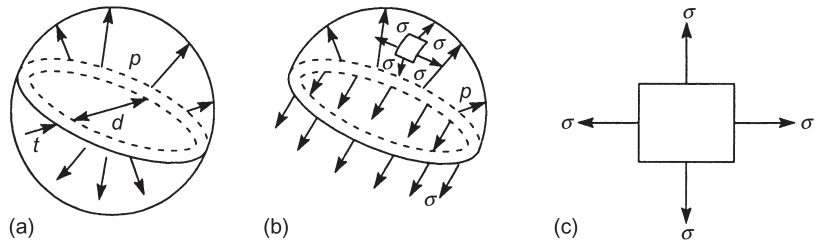

FIGURE 7.27

Stress in a spherical shell.

Similarly, the circumferential strain, ε_C, is given by

$$\varepsilon_C = \frac{pd}{2tE}\left(1 - \frac{1}{2}v\right) \tag{7.65}$$

The increase in length of the shell is $\varepsilon_L L$ while the increase in circumference is $\varepsilon_C \pi d$. We see from the latter expression that the increase in circumference of the shell corresponds to an increase in diameter, $\varepsilon_C d$, so that the circumferential strain is equal to diametral strain (and also radial strain). The increase in volume, ΔV, of the shell is then given by

$$\Delta V = \frac{\pi}{4}(d + \varepsilon_C d)^2 (L + \varepsilon_L L) - \frac{\pi}{4} d^2 L$$

which, when second-order terms are neglected, simplifies to

$$\Delta V = \frac{\pi d^2 L}{4}(2\varepsilon_C + \varepsilon_L) \tag{7.66}$$

Substituting for ε_L and ε_C in Eq. (7.66) from Eqs (7.64) and (7.65) we obtain

$$\Delta V = \frac{\pi d^2 L}{4}\frac{pd}{tE}\left(\frac{5}{4} - v\right)$$

so that the volumetric strain is

$$\frac{\Delta V}{(\pi d^2 L/4)} = \frac{pd}{tE}\left(\frac{5}{4} - v\right) \tag{7.67}$$

The analysis of a spherical shell is somewhat simpler since only one direct stress is involved. It can be seen from Fig. 7.27(a) and (b) that no matter which diametral plane is chosen, the tensile stress, σ, in the walls of the shell is constant. Thus for the equilibrium of the hemispherical portion shown in Fig. 7.27(b)

$$\sigma \times \pi\, \mathrm{d}t = p \times \frac{\pi\, d^2}{4}$$

from which

$$\sigma = \frac{pd}{4t} \tag{7.68}$$

Again we have a two-dimensional state of stress acting on a small element of the shell wall (Fig. 7.27(c)) but in this case the direct stresses in the two directions are equal. Also the volumetric strain is determined in an identical manner to that for the cylindrical shell and is

$$\frac{3pd}{4tE}(1 - v) \tag{7.69}$$

EXAMPLE 7.12

A thin-walled, cylindrical shell has an internal diameter of 2 m and is fabricated from plates 20 mm thick. Calculate the safe pressure in the shell if the tensile strength of the plates is 400 N/mm^2 and the factor of safety is 6. Determine also the percentage increase in the volume of the shell when it is subjected to this pressure. Take Young's modulus $E = 200\ 000$ N/mm^2 and Poisson's ratio $v = 0.3$.

The maximum tensile stress in the walls of the shell is the circumferential stress, σ_C, given by Eq. (7.63). Then

$$\frac{400}{6} = \frac{p \times 2 \times 10^3}{2 \times 20}$$

from which

$$p = 1.33\,\text{N/mm}^2$$

The volumetric strain is obtained from Eq. (7.67) and is

$$\frac{1.33 \times 2 \times 10^3}{20 \times 200\,000}\left(\frac{5}{4} - 0.3\right) = 0.00063$$

Hence the percentage increase in volume is 0.063%.

7.12 Plane strain

The condition of *plane strain* occurs when all the strains in a structure, or part of a structure, are confined to a single plane. This does not necessarily coincide with a plane stress system as we noted in Section 7.11. Conversely, it generally requires a three-dimensional stress system to produce a condition of plane strain.

Practical examples of plane strain situations are retaining walls or dams where the ends of the wall or dam are constrained against movement and the loading is constant along its length. All cross sections are then in the same condition so that any thin slice of the wall or dam taken perpendicularly to its length would only be subjected to strains in its own plane.

We shall examine more complex cases of plane stress and plane strain in Chapter 14.

PROBLEMS

P.7.1 A column 3 m high has a hollow circular cross section of external diameter 300 mm and carries an axial load of 5000 kN. If the stress in the column is limited to 150 N/mm^2 and the shortening of the column under load must not exceed 2 mm calculate the maximum allowable internal diameter. Take $E = 200\ 000$ N/mm^2.

Ans. 205.6 mm.

P.7.2 A steel girder is firmly attached to a wall at each end so that changes in its length are prevented. If the girder is initially unstressed, calculate the stress induced in the girder when it is subjected to a uniform temperature rise of 30 K. The coefficient of linear expansion of the steel is 0.000 05/K and Young's modulus $E = 180\ 000$ N/mm^2. (Note $L = L_0(1 + \alpha T)$.)

Ans. 270 N/mm^2 (compression).

P.7.3 A column 3 m high has a solid circular cross section and carries an axial load of 10 000 kN. If the direct stress in the column is limited to 150 N/mm^2 determine the minimum allowable diameter. Calculate also the shortening of the column due to this load and the increase in its diameter. Take $E = 200\ 000$ N/mm^2 and $\nu = 0.3$.

Ans. 291.3 mm, 2.25 mm, 0.066 mm.

P.7.4 A structural member has a rectangular cross section of side 100×300 mm and carries an axial tensile load of 5000 kN. Calculate the direct stress in the member, its increase in length over a span of 5 m and the percentage change in its cross sectional area under the load, Take $E = 200\ 000$ N/mm^2 and $\nu = 0.3$.

Ans. 166.7 N/mm^2, 4.17 mm, 0.043%."

P.7.5 The block of material shown in Fig. P.7.5 is subjected to the stress system shown. If Young's modulus, E, is 200 000 N/mm^2 and Poisson's ratio, ν, is 0.3 calculate the percentage change in volume in the block.

Ans. 0.049%.

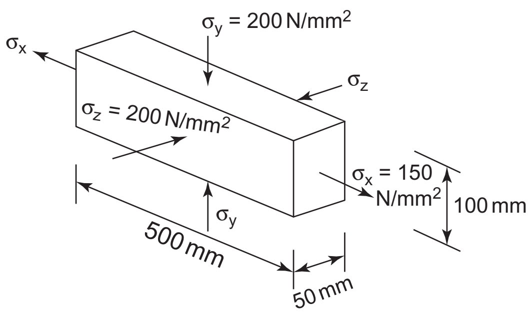

FIGURE P.7.5

P.7.6 A structural member, 2 m long, is found to be 1.5 mm short when positioned in a framework. To enable the member to be fitted it is heated uniformly along its length. Determine the necessary temperature rise. Calculate also the residual stress in the member when it cools to its original temperature if movement of the ends of the member is prevented.

If the member has a rectangular cross section, determine the percentage change in cross-sectional area when the member is fixed in position and at its original temperature.

Young's modulus $E = 200\ 000$ N/mm^2, Poisson's ratio $v = 0.3$ and the coefficient of linear expansion of the material of the member is 0.000 012/K.

Ans. 62.5 K, 150 N/mm^2 (tension), 0.045% (reduction).

P.7.7 A member of a framework is required to carry an axial tensile load of 100 kN. It is proposed that the member be comprised of two angle sections back to back in which one 18 mm diameter hole is allowed per angle for connections. If the allowable stress is 155 N/mm^2, suggest suitable angles.

Ans. Required minimum area of cross section = 645.2 mm^2. From steel tables, two equal angles $50 \times 50 \times 5$ mm are satisfactory.

P.7.8 A vertical hanger supporting the deck of a suspension bridge is formed from a steel cable 25 m long and having a diameter of 7.5 mm. If the density of the steel is 7850 kg/m^3 and the load at the lower end of the hanger is 5 kN, determine the maximum stress in the cable and its elongation. Young's modulus $E = 200\ 000$ N/mm^2.

Ans. 115.1 N/mm^2, 14.3 mm.

P.7.9 A concrete chimney 40 m high has a cross-sectional area (of concrete) of 0.15 m^2 and is stayed by three groups of four cables attached to the chimney at heights of 15, 25 and 35 m respectively. If each cable is anchored to the ground at a distance of 20 m from the base of the chimney and tensioned to a force of 15 kN, calculate the maximum stress in the chimney and the shortening of the chimney including the effect of its own weight. The density of concrete is 2500 kg/m^3 and Young's modulus $E = 20\ 000$ N/mm^2.

Ans. 1.9 N/mm^2, 2.2 mm.

P.7.10 A column of height h has a rectangular cross section which tapers linearly in width from b_1 at the base of the column to b_2 at the top. The breadth of the cross section is constant and equal to a. If Young's modulus for the material of the column is E determine the shortening of the column due to an axial load P.

Ans. $(Ph/[aE(b_1 - b_2)])\ \log_e(b_1/b_2)$.

P.7.11 Determine the vertical deflection of the 20 kN load in the truss shown in Fig. P.7.11. The cross-sectional area of the tension members is 100 mm^2 while that of the compression members is 200 mm^2. Young's modulus $E = 205\ 000$ N/mm^2.

Ans. 4.5 mm.

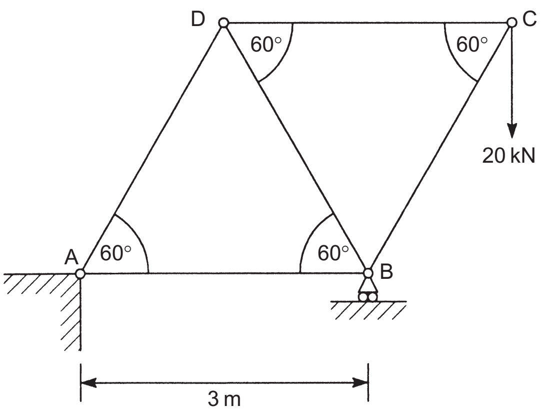

FIGURE P.7.11

P.7.12 The truss shown in Fig. P.7.12 has members of cross-sectional area 1200 mm^2 and Young's modulus 205 000 N/mm^2. Determine the vertical deflection of the load.

Ans. 10.3 mm.

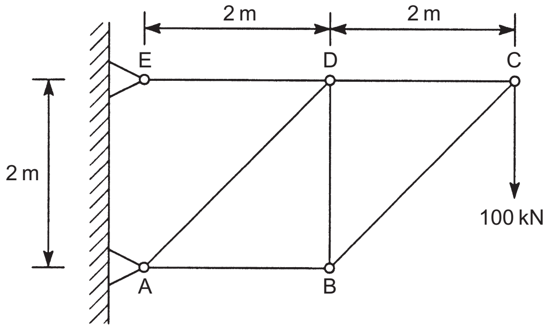

FIGURE P.7.12

P.7.13 Three identical bars of length L are hung in a vertical position as shown in Fig. P.7.13. A rigid, weightless beam is attached to their lower ends and this in turn carries a load P. Calculate the load in each bar.

Ans. $P_1=P/12$, $P_2=P/3$, $P_3=7P/12$.

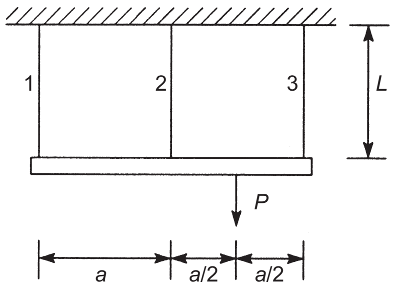

FIGURE P.7.13

P.7.14 A composite column is formed by placing a steel bar, 20 mm in diameter and 200 mm long, inside an alloy cylinder of the same length whose internal and external diameters are 20 and 25 mm, respectively. The column is then subjected to an axial load of 50 kN. If E for steel is 200 000 N/mm^2 and E for the alloy is 70 000 N/mm^2, calculate the stress in the cylinder and in the bar, the shortening of the column and the strain energy stored in the column.

Ans. 46.5 N/mm^2 (cylinder), 132.9 N/mm^2 (bar), 0.13 mm, 3.3 Nm.

P.7.15 A timber column, 3 m high, has a rectangular cross section, 100 mm$\times$200 mm, and is reinforced over its complete length by two steel plates each 200 mm wide and 10 mm thick attached to its 200 mm wide faces. The column is designed to carry a load of 100 kN. If the failure stress of the timber is 55 N/mm^2 and that of the steel is 380 N/mm^2, check the design using a factor of safety of 3 for the timber and 2 for the steel. E (timber) = 15 000 N/mm^2, E (steel) = 200 000 N/mm^2.

Ans. σ (timber) = 13.6 N/mm^2 (allowable stress = 18.3 N/mm^2),
σ (steel) = 181.8 N/mm^2 (allowable stress = 190 N/mm^2).

P.7.16 The composite bar shown in Fig. P.7.16 is initially unstressed. If the temperature of the bar is reduced by an amount T uniformly along its length, find an expression for the tensile stress induced. The coefficients of linear expansion of steel and aluminium are α_S and α_A per unit temperature change, respectively, while the corresponding values of Young's modulus are E_S and E_A.

Ans. $T(\alpha_S L_1 + \alpha_A L_2)/(L_1/E_S + L_2/E_A)$.

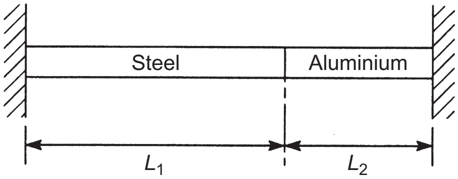

FIGURE P.7.16

P.7.17 A short bar of copper, 25 mm in diameter, is enclosed centrally within a steel tube of external diameter 36 mm and thickness 3 mm. At 0°C the ends of the bar and tube are rigidly fastened together and the complete assembly heated to 80°C. Calculate the stress in the bar and in the tube if E for copper is 100 000 N/mm^2, E for steel is 200 000 N/mm^2 and the coefficients of linear expansion of copper and steel are 0.000 01/°C and 0.000 006/°C, respectively.

Ans. σ (steel) = 28.3 N/mm^2 (tension),
σ (copper) = 17.9 N/mm^2 (compression).

P.7.18 A bar of mild steel of diameter 75 mm is placed inside a hollow aluminium cylinder of internal diameter 75 mm and external diameter 100 mm; both bar and cylinder are the same length. The resulting composite bar is subjected to an axial compressive load of 10^6 N. If the bar and cylinder contract by the same amount, calculate the stress in each.

The temperature of the compressed composite bar is then reduced by 150°C but no change in length is permitted. Calculate the final stress in the bar and in the cylinder. Take E (steel) = 200 000 N/mm^2, E (aluminium) = 80 000 N/mm^2, α (steel) = 0.000 012/°C, α (aluminium) = 0.000 005/°C.

Ans. Due to load: σ (steel) = 172.6 N/mm^2 (compression),
σ (aluminium) = 69.1 N/mm^2 (compression).
Final stress: σ (steel) = 187.4 N/mm^2 (tension),
σ (aluminium) = −9.1 N/mm^2 (compression).

P.7.19 Two structural members are connected together by a hinge which is formed as shown in Fig. P.7.19. The bolt is tightened up onto the sleeve through rigid end plates until the tensile force in the bolt is 10 kN. The distance between the head of the bolt and the nut is then 100 mm and the sleeve is 80 mm in length. If the diameter of the bolt is 15 mm and the internal and outside diameters of the sleeve are 20 and 30 mm, respectively, calculate the final stresses in the bolt and sleeve when an external tensile load of 5 kN is applied to the bolt.

Ans. σ (bolt) = 65.4 N/mm^2 (tension),
σ (sleeve) = 16.7 N/mm^2 (compression).

P.7.20 Calculate the minimum wall thickness of a cast iron water pipe having an internal diameter of 1 m under a head of 120 m. The limiting tensile strength of cast iron is 20 N/mm^2 and the density of water is 1000 kg/m^3.

Ans. 29.4 mm.

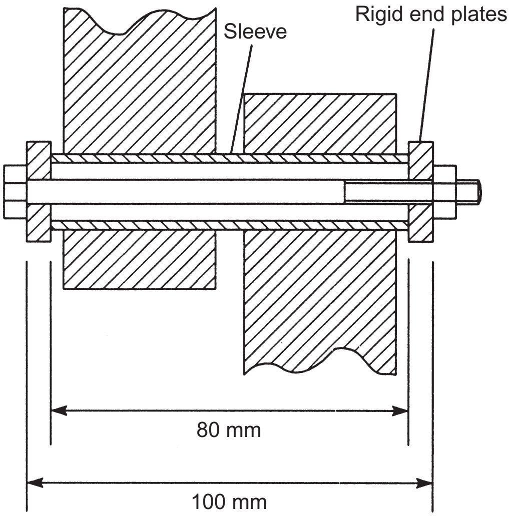

FIGURE P.7.19

P.7.21 A thin-walled spherical shell is fabricated from steel plates and has to withstand an internal pressure of 0.75 N/mm^2. The internal diameter is 3 m and the joint efficiency 80%. Calculate the thickness of plates required using a working stress of 80 N/mm^2. (Note, effective thickness of plates $= 0.8 \times$ actual thickness).

Ans. 8.8 mm.