Signals, Systems and Control

Contents

1.4 Fourier and Laplace, Correlation

1.4.1 Understanding time frequency transformations

In this section we are revisiting some maths that you should be somewhat familiar with – namely Fourier and Laplace transforms. The goal is to review these, perhaps looking at them from a different angle, and to try and get more of you to really understand what is happening in these integral transforms.

Mathematicians often like to consider these in a theoretical sense, but we will try and give physical significance to the math to try and make it more accessible.

Joseph Fourier is credited with the idea that time domain waveforms can be represented by summations of sinusoids (his basis functions), *although we already know that the maths uses basis functions that are analytic* i.e. rotations - phasors on the complex plane, helixes over time.

Since rotations (and their single-axis sine/cosine projection) have a unique frequency component this concept reveals the mapping between time and frequency domains. i.e. we have a signal that is continuous for all *time* that maps to a single 'point' in the *frequency* domain.

1.4.2 The Fourier Transform

This is the definition of the Fourier transform, hopefully this should be at least slightly familiar. The inverse transform below it.

$$\hat{X}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt$$

The first thing to note is the maths requires the units used in the time and frequency domains to be the reciprocal of each other: i.e. if we have 'seconds' in the time domain, then this will become 'cycles per second' (i.e. frequency in Hertz).

$$x(t) = \int_{-\infty}^{\infty} \hat{X}(f)\, e^{j2\pi f t} df$$

In engineering we often like to use radians per second, rather than cycles per second, as a measure of frequency, hence we need to do a bit of juggling to get the scaling right.

$$\hat{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

If the transform is expressed in radians, the inverse transform has to have an additional 1/2π to make the x-axis units 'square up' – but note there are other conventions such as having a $1/\sqrt{2\pi}$ factor applied to both sides of the transform pairs.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}(\omega)\, e^{j\omega t} d\omega$$

1.4.3 Interpreting the Fourier transform

In this section we will attempt to unpick the transform to understand the what the terms mean and the function of the integral. We will simplify things at first, so it is important to read to the end of the section to get a complete picture. Let's try to work out what each bit of the transform is doing. Looking at the inverse transform (pay close attention to the arguments of functions):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}(\omega)\, e^{j\omega t} d\omega$$

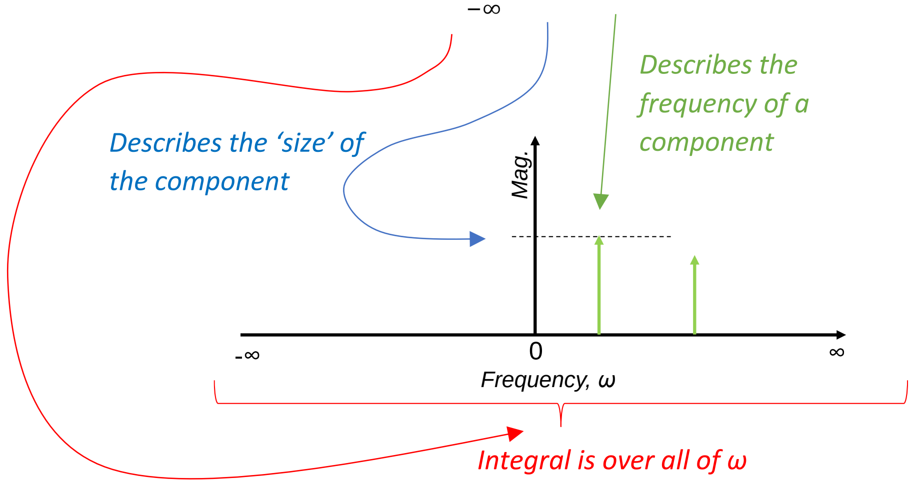


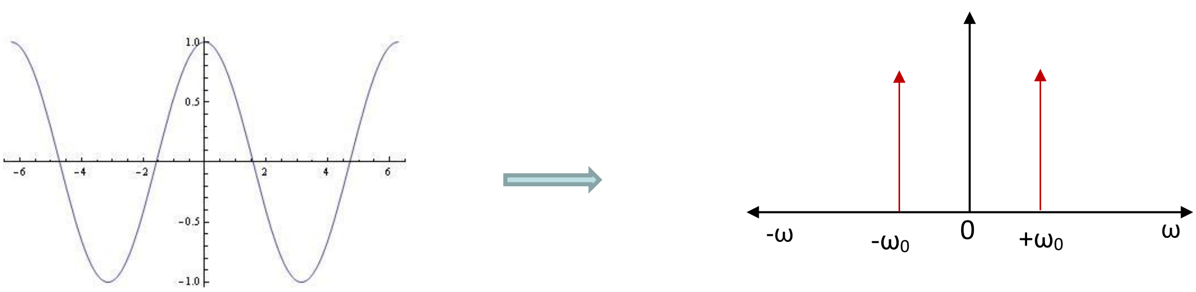
Fig 1. Decomposing the inverse Fourier transform to better understand what the math represents

In words, the inverse transform says the time domain signal, x(t), is made up by summing (the integral) all of the frequency domain components which are themselves made up of all the unity magnitude basis functions, $e^{j\omega t}$, multiplied by the 'size' at that frequency, $\hat{X}(\omega)$.

It is common to use tables to look up frequency domain representations of common functions, here are some transform pairs you may be familiar with:

$\mathrm{x}(t) = \cos(\omega_0 t)$ *in the frequency domain* $x(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

Diagrammatically:



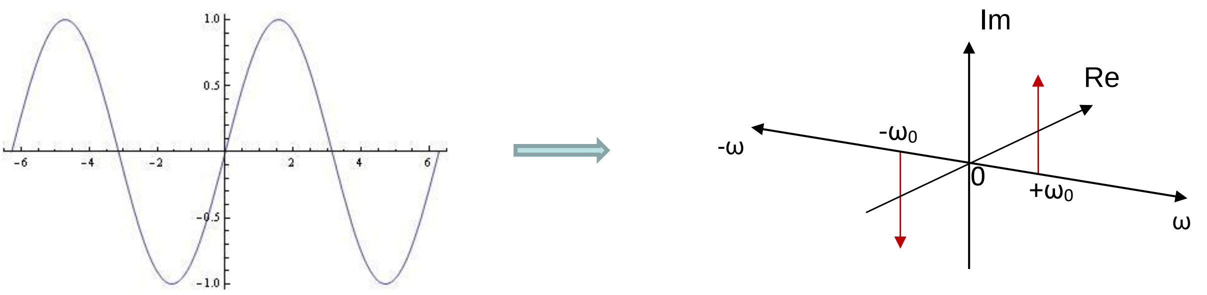
The Fourier transform of Cosine produces two components – one forward and one backward rotating, as we would expect from the discussion so far since our cosine signal is real-valued, not analytical.

Points to note: the delta (impulse) function has infinite magnitude – how can this be? Well, the transform is defined for all time; thus, the integral will be infinite in magnitude if the cosine wave continues for all time – more of this later. Also note the 'π' scaling factor – this is because we are in radians, if we were working in Hertz we would just have a ½ here (remember with real-valued signals the signal is split over forward and backwards rotating components).

Now look at the familiar transform pair for sine:

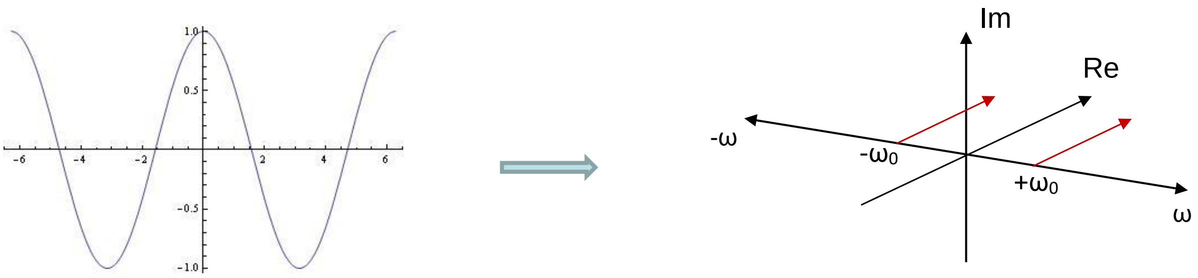
$\mathrm{x(t)} = \sin\,(\omega_0 t)$ *in the frequency domain* $x(\omega) = \frac{\pi}{j}\,[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

The first thing you should notice is the 'j' in the frequency domain expression – this is a complex number! We need to rethink our frequency domain representation to cope with complex numbers, we can represent it like this (recall that 1/j = -j):



Do you start to see some similarity between the frequency domain space above and the (3D) space in which we drew our complex exponentials in the time domain?

Now we have seen that the components in the frequency domain are actually complex numbers, we can revisit cosine and draw it in 3D space:



But what information are we capturing by having a complex plane in the frequency domain? It turns out that $\hat{X}(\omega)$ is representing more than just the magnitude of the component as a function of frequency (hence why 'size' was used earlier): it also tells us about the *phase* of the frequency components.

1.4.4 Visualising the Fourier transform

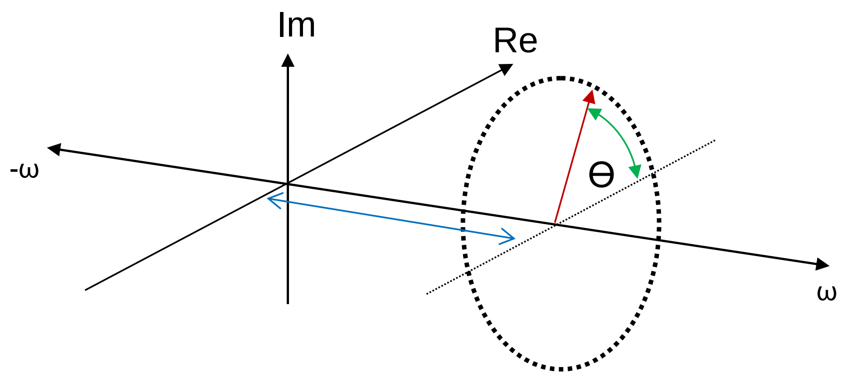


Fig 2. Representing the Fourier domain in three dimensions we capture the frequency, the magnitude and the phase (relative to the positive real axis) of each component.

From our time-centric way of looking at the world we can think of our time-domain rotating vector as the definition of an individual frequency component (point) in the frequency domain. The associated frequency domain component is fixed for all time, i.e. it doesn't rotate, it's just a vector.

Thus, a phasor on the complex plane in the time domain maps to a vector on the complex plane in the frequency domain.

So far we have just looked at time domain signals that exist for all time, and mapped them to a single point along the frequency axis, but take a moment to think – what would the time domain signal look like if the signal was a helix in the frequency domain?

The clue here is that the Fourier transform and the inverse transform are pretty much identical from a mathematics point of view. You will also likely have already seen a function (time domain) whose transform pair is a complex exponential…… If you are struggling here, don't worry, we will return to this later.

1.4.5 The Laplace Transform

Laplace went one stage further than Fourier and represented his time domain functions with sinusoidal basis functions which could grow or shrink. Laplace transforms map the time domain to the complex frequency domain, denoted ‘S’. The physical significance of this is that the S domain can model systems that gain or lose energy over time – e.g. the free response of a mass/spring/damper oscillator as it ‘rings down’.

This idea is central to the control theory you will study next term – stable systems – because any system that gains energy over time will eventually become unstable.

We are not going to spend as long with the Laplace transform pair, but it is worth looking at them as a reminder. The Laplace transform itself is deceptively similar to the Fourier transform; it is the inverse transform that reveals the greater complexity of the approach. The limits of integration are such that growing (or decaying) exponentials are captured and we are now transforming between ‘t’ and ‘s’, hence the 1/2πj unit correction.

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st} dt$$

Where: s = r +jω

$$f(t) = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{\gamma-jT}^{\gamma+jT} F(s)\, e^{-st} ds$$

Visualising the Laplace transform

This is a little tricky. In the frequency domain our Fourier transform resulted in 3 variables to represent – frequency, phase and magnitude. In the complex frequency domain we now have an additional factor associated with the growth/decay of our basis functions; we would need 4 dimensions!

However, some people have taken on this challenge – for example in this youtube video they use colour to represent phase. Be warned, you still need to use a bit of mental gymnastics to see what is going on!

https://www.youtube.com/watch?v=6MXMDrs6ZmA

1.4.6 Beyond Fourier and Laplace

We have refreshed your knowledge of Laplace and Fourier as a lead-in to frequency domain analysis, but there is much, much more to this field, and it is well supported by easily accessible theory.

Fourier analysis is in some ways a 'pure' transform because the time and frequency domain are entirely orthogonal. In the Laplace transform you start to see there is some coupling – because it matters *when* our basis functions start to grow or shrink. If we flex our minds a little more, we can admit the possibility that there is a continuum of domains between the pure time and pure frequency and these are really useful – in fact they are essential for dealing with real signals.

At a simple level you could take a few seconds of a time domain signal, perform a Fourier transform on this time limited sequence, then a little bit later capture a few more seconds of the time signal, perform a Fourier transform etc. You would end up with a 3D signal that had both frequency and time dimensions – this is called a spectrogram

> *You can get spectrogram apps for Andriod and iOS that will produce a spectrogram from the microphone input on your device – give it a try.*

This technique is often called the Short-Time Fourier Transform (STFT). One way of thinking of the STFT is that the basis functions have been 'windowed' i.e. we have captured a short time period of the basis function. The shape of these windows is defined and they have names such as 'boxcar' and 'Hanning'. This is a very big field and outside the scope of this course, but online resources are very good – if you are interested try looking at:
https://en.wikipedia.org/wiki/Window_function#Rectangular_window

Stretching your powers of imagining a little further, we can consider another time/frequency transformation you may come across - the Wavelet Transform. Wavelets use basis functions similar to the function below – however they do not exist for all-time (as a sine wave does), and although they have a constant zero crossing, they do not correspond to a single frequency in the frequency domain defined by Fourier. They are example of a transform somewhere between the two orthogonal domains of time and frequency.

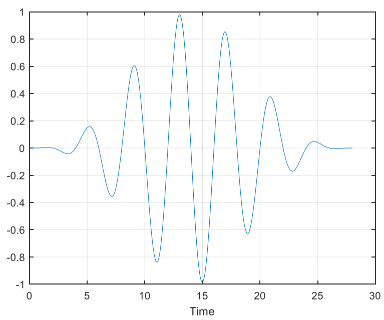


Fig. 3. A Wavelet in the time domain. A wavelet is similar to a note played on a musical instrument.

1.4.7 Units of the Fourier transform

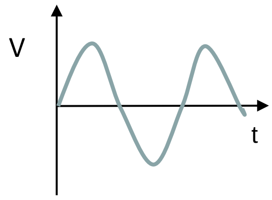
$$\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt = F(f)$$

*Units of time*signal units*

Units of the signal

Units of time

Dimensionless



If the time domain signal is in volts…

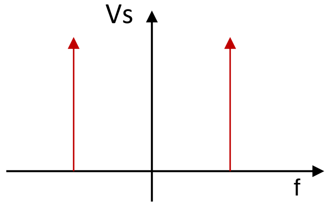
$$\int_{-\infty}^{\infty} \hat{F}(f)\, e^{j2\pi f t} df = x(t)$$

Units of the signal

*Units of time*signal units*

Units of frequency, i.e. 1/time

Dimensionless



…the frequency domain 'signal' is in volts*seconds

By considering the units of the Fourier transform we can start to see why we end up with an impulse functions (with infinite amplitude) – the transform has units of the signal * time, hence if the integral is evaluated for all time, the amplitude will be infinite.

This is sometimes not very useful in practice, hence the transform is often evaluated over a fixed time and then the result *normalised* to that length of time – we shall see this later in the lecture.

1.4.8 Power and energy signals

The concept we discussed in section 1.5.7 leads nicely on to another classification of signals – they can be classed 'power signals' and 'energy signals' depending on which property is bounded.

Signals that are finite, such as exponential decay are described as 'energy signals' because the total energy in the signal is finite, defined as:

$$E(t) = \int_{t_s}^{t_s+T} |f(t)|^2 dt \qquad\qquad 0 < \lim_{T\to\infty} E(t) < \infty$$

Alternatively, signals that go on indefinitely, sine, unit step, etc. are sometimes described as 'power signals' because we need to work with the energy of the signal per unit time. The average power and definition of a power signal is:

$$P(t) = \frac{1}{T}\int_0^T |f(t)|^2 dt \qquad\qquad 0 < \lim_{T\to\infty} P(t) < \infty$$

These definitions are useful as they guide the approach required to analyse a particular signal.

1.4.9 Practical usage

We have outlined the theory of Laplace and Fourier transforms, and hinted at some of the steps we may have to take to make them practical, but here are the 'take-home' messages:

1) *The physical signals we measure will almost always be single (real) valued signals and they are not going to be continuous for all time.*

2) *We normally use some form of Fourier type transform on signals and we need to manage the negative frequency components we get form the math, and consider how the magnitude of the signal in the frequency domain can be usefully captured.*

3) *When we come to model the response of our linear systems, we will use the Laplace transform. We may set s = jω to extract the 'sinusoidal steady state' behaviour i.e. the behaviour of the system 'forced' with a sinusoid.*

1.4.10 Correlation

The Fourier and Laplace Transforms are examples of a process you should have seen before - *correlation*.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

In words, the Fourier Transform is saying: 'The value in frequency domain at 'ω' = integral of: (the time domain waveform * the rotating vector at 'ω')'

This is a correlation of $f(t)$ with $e^{-j\omega t}$, and correlation is a measure of similarity. Here it is telling us how much of the basis function $e^{-j\omega t}$ is in the time domain signal $f(t)$. Of course, that makes sense because that is what we want the transform to do!

In general, a correlation of *f(t)* and *g(t)* is given by:

$$corr(f,g) = \int_{-\infty}^{\infty} f(t)g(t)dt$$

Simple examples

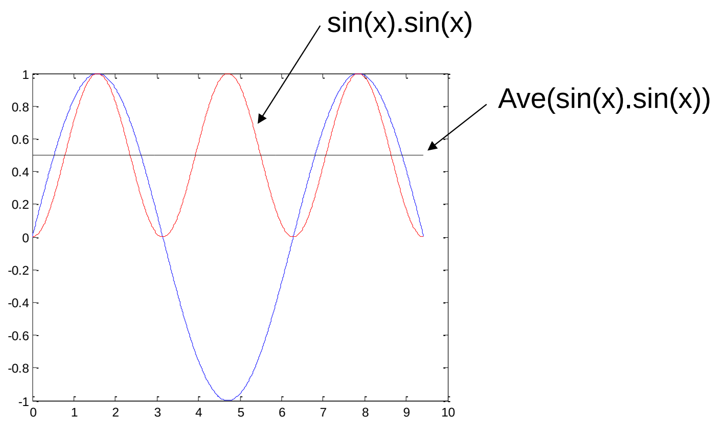


Fig. 4. Correlating sin(x) with sin(x)

If we correlate sin(x) with sin(x) we get a value of 0.5. Let's consider this result in the context of the Fourier Transforms we have already considered.

You will noticed I have cheated a little here to illustrate some of the approaches need to take with our practical Fourier type transforms. Firstly, I have used an average (the mean) rather than the integral – I don't want the correlation to get bigger if I use a longer time sequence (this is a power signal). Averaging, which divides the sum by the time, prevents this (It is also worth noting that I need to choose a window equal to a multiple of the period). Secondly, the value of the correlation is '0.5' but surely if the question I'm asking is "how much of the function sin(x) is in the function sin(x)", then the answer should be '1'? - We will need to introduce some scaling.

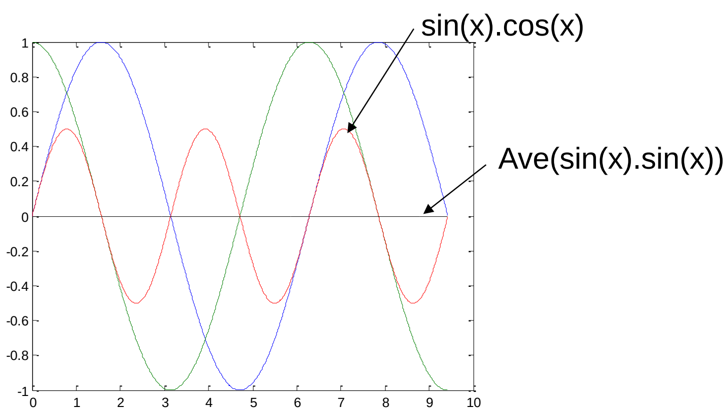


Fig. 5. Correlating sin(x) with sin(x)

If we correlate sin(x) with cos(x) we get '0'. This is expected since the waveforms are *orthogonal*, but again note we only get '0' if we average over a multiple of the period.

1.4.11 Cross Correlation

In the previous section the waveforms we correlated were 'static'. The operation of cross-correlation 'slides' one waveform over another an comes up with a range of correlation values as a function of the sliding value. Cross correlation is given by:

$$(f * g)(\tau) = \int_{-\infty}^{\infty} f^*(t)g(t+\tau)dt$$

Where τ is the sliding variable which acts long the same axis as t (time).

If g(t) and f(t) are two rectangular functions then the cross-correlation is a triangular shape, as shown below.

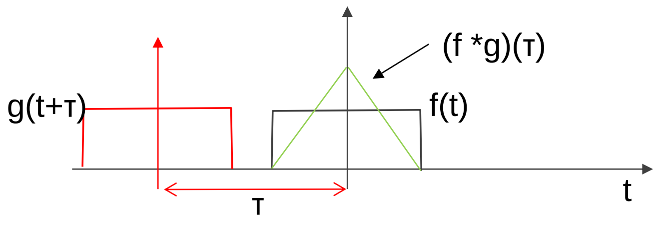


Fig. 6. Cross correlation involves expressing the correlation of two signals as a function of their offset.

1.4.12 Auto-correlation

In the cross-correlation example we used two waveforms with the same shape (they didn't need to be), and in that special case where the two signals are the same the process is known as auto-correlation – i.e. correlating a signal with itself.

Mathematically auto correlation is described by:

$$(f * f)(\tau) = \int_{-\infty}^{\infty} f^*(t)f(t+\tau)dt$$

1.4.13 Uses of Correlation

- Can find the time differences between similar signals

Performing a cross-correlation and looking for the value of offset that produces the peak value, indicates the time difference.

- Can find particular components in a signal

As used in the Fourier transform correlation can find components within a signal. There is no restrictions on the components we search for – e.g. we can look for complex exponential, as in the fully Fourier transform, or we can look for sine or cosine components (which we will see next in the Fourier series), or we could look for an arbitrary waveform.

- Can extract signals from noise or other unwanted signals

A signal that is hidden by unwanted noise can be extracted by correlation – e,g, radio signals from space.

Also GPS satellites – they all transmit of the same frequency. The messages are 'overlaid' when received but are separated by correlation of the received signal with 'gold' codes – orthogonal binary sequences – that allows the information from a particular satellite to be recovered. This is termed Code Division Multiple Access (CDMA) and allows lots of signals to be broadcast on top of each other and then extracted using correlation.

Run the MATLAB M-files attached to this lecture – they will help illustrate cross and auto correlation.

1.4.14 Fourier Series

A signal made up of summed sinusoids is called a Fourier series. You will have come across Fourier series before, but we are going to revisit it again to try and solidify some of the ideas we have talked about in this lecture.

The Fourier series is a particular implementation of the ideas of Fourier – you might like to think of it a simplified version as it avoids dealing with analytical signals - and is especially useful for practical, engineering applications.

In Fourier series the individual frequency components are harmonically related i.e. integer multiples of the lowest frequency (sometimes called the base or fundamental frequency), and these are very useful for describing periodic signals; you may well know the series of harmonics that makes up square, triangle or sawtooth waveforms, e.g. a square wave contains: $\omega_b + 3\omega_b/3 + 5\omega_b/5 + 7\omega_b/7$ + ……etc.

The Fourier series can be described as the sum of weighted sine and cosine harmonic sets and a DC (zero frequency) constant term:

$$f(t) = A_0 + \sum_{n=1}^{\infty} a_n \sin n\omega t + \sum_{n=1}^{\infty} b_n \cos n\omega t$$

The coefficients A_0, a_n, b_n are found by evaluating the mean on the time domain function (for A_n) and two correlations (for a_n & b_n).

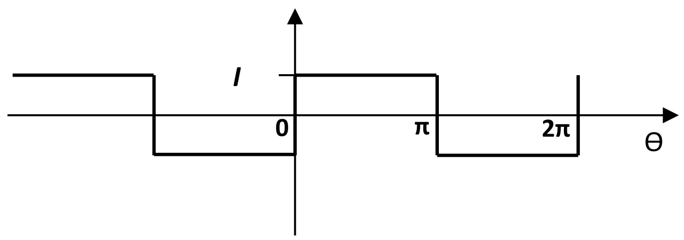
$$A_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(t)\delta t \qquad a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\, sin\, n\, \omega t\delta t \qquad b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\, cos\, n\, \omega t\delta t$$

So, what are we looking for here?

1. We only have a finite number of frequencies to look for (if we limit n)
2. We have dropped analytical signals and in place use orthogonal sine/cosine components – we don't have to worry about forward/backward components. In fact, we can use the symmetry of many periodic waveforms to ignore whole groups of terms and simplify the number of integrals we need to evaluate.
3. The correlations are performed over the period of the fundamental frequency; we can evaluate the integral of sines and cosines between defined limits relatively easily.

Example

Consider a square wave of peak magnitude 'I', to simplify things we have normalised the frequency:



By inspection of the waveform we see that

- The average value is zero, hence $A_0=0$.
- It is an odd function hence will only contain sine terms. *($b_n=0$)*
- Because the waveform has quarter-wave symmetry there are no even harmonics. *($a_n=0$, for n=2,4,6 etc)*
- *Because of symmetry we only need to solve over a 0 – π/2 range*

We therefore only need to solve the integral:

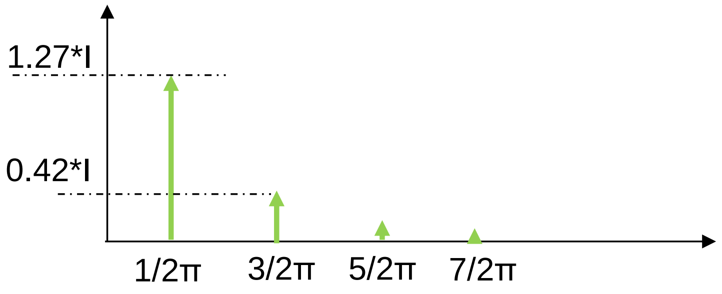
$$a_n = \frac{4}{\pi}\int_0^{\pi/2} i(\theta)\sin n\,\theta\delta\theta = \frac{4}{\pi}\int_0^{\pi/2} I\sin n\,\theta\delta\theta$$

for n=1,3,5 etc

Which gives the result:

$$a_n = \frac{4I}{\pi n}$$

Harmonic	Magnitude
1st	1.27*I
3rd	0.42*I
5th	0.25*I
7th	0.18*I



Fun fact: you will notice that the magnitude of the fundamental (1st harmonic) is bigger than the magnitude of the square wave. This ability of higher harmonics to suppress the peak value of a waveform was exploited by a now-retired Bristol Academic (Duncan Grant) when he and a collaborator invented 3rd harmonic injection – a technique to get more power from a 3 phase motor with limited voltage supply and became an industry standard.
https://ieeexplore.ieee.org/abstract/document/4504587

1.4.15 Polar form of the Fourier Series

$$f(t) = A_0 + \sum_{n=1}^{\infty} a_n \, sin \, n \, \omega t + \sum_{n=1}^{\infty} b_n \, cos \, n \, \omega t$$

The Fourier series we have seen so far is in rectangular form - this resolves any frequency component with a phase offset (i.e. not a sine or cosine) into sine and cosine components, however it may be more intuitive to describe the Fourier series in a polar form;

$$f(t) = A_0 + \sum_{n=1}^{\infty} C_n \, sin(n\omega t + \theta_n)$$

$$\textit{Where} \qquad C_n = \sqrt{{a_n}^2 + {b_n}^2}$$

$$\textit{and} \qquad \theta_n = tan^{-1}\left(\frac{a_n}{b_n}\right)$$

1.4.16 Coefficients of the Fourier series

You will notice some formulations of the inverse Fourier transform/series multiply the correlation integral by some factor divided by the period over which the integral is taken. We touched on this earlier when we looked at expressing Fourier Transforms in radians per second, rather than frequency in Hertz. They are connected with the same issue – making sure the correlation integral gives us the value we expect. For example:

$$A_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f(t) dt \qquad\qquad a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \, sin(\, n\omega t) dt$$

A_n it follows the definition of mean, i.e. *1/(period of integration)* but for a_n and b_n the factor is different. Why is this? And how do you work it out?

First remember the correlation is telling us "how much of the component *sin(nωt)* is in *f(t)* ", and if the signal continues for all time, the integral becomes larger as the period over which it is taken increases. If a signal is infinite in time, then all components will tend to infinity – and this is not really what we want to know. Also, as we discussed when correlating sin(x) with sin(x), the result isn't unity, as we might want or expect.

It is easy (but somewhat longwinded) to work out what we must multiply the integral by, from the simple consideration that if f(t) = sin(ωt) and n =1, we require $a_n = 1$.

1.4.17 Test Yourself

1) Why does the does the FT of a sine wave result in an impulse function – a function with infinite magnitude?
2) We need 3 dimensions to represent signals in the Fourier (frequency) domain – what characteristic of the signal does each dimension represent?
3) Why can we not represent signals the Laplace (complex frequency) domain in 3 dimensions?
4) What is a wavelet?
5) If a signal has a magnitude measured in units 'v' in the time domain, what are the units of the Fourier transform of the signal?
6) Name an example of a power signal.
7) Name an example of an energy signal.
8) How is correlation used in the Fourier and Laplace transforms?
9) Ensure you know the operations: a) Correlation, b) Cross-correlation; c) Auto-correlation
10) What is a 'Fourier series'?