5. Separation of variables

The separation of variables method

A 'try it and see' technique to solve PDEs

- Separating the variables: PDE $\rightarrow$ ODEs
- We will use Fourier series as part of the method!
- Solution process for wave, heat, and Laplace equations

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Separation of variables: the idea

- To find a solution $u(x,t)$ of a PDE we're going to assume that the solution is *separable*:

$$u(x,t) = X(x)T(t)$$

- Substituting this assumption (guess!) into the PDE will give us two separate ODEs: one for $X(x)$ and one for $T(t)$
- The *homogeneous* boundary conditions will play a crucial role in the solution process

Walk-though example 1: the wave equation

Consider the wave equation on a finite domain

$$\boxed{u_{tt} = c^2 u_{xx}} \qquad \boxed{0 \leqslant x \leqslant L, \quad t \geqslant 0,} \tag{1}$$

subject to homogeneous boundary conditions and initial conditions of known displacement and zero velocity:

$$\boxed{\begin{array}{c} u(0,t) = u(L,t) = 0 \\ \text{for all } t > 0 \end{array}} \qquad \boxed{\begin{array}{c} u(x,0) = f(x), \quad u_t(x,0) = 0 \\ \text{for all } 0 \leqslant x \leqslant L \end{array}} \tag{2}$$

for some given (non-zero) function $f(x)$.

[Note that this means that the solution $u(x,t) \neq 0$, since it's non-zero at $t = 0$]

Separate the variables

First, assume a *separable* solution

$$u(x,t) = X(x)T(t),$$

Substituting this into the PDE we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Big[X(x)T(t)\Big] &= c^2\frac{\partial^2}{\partial x^2}\Big[X(x)T(t)\Big]\\ X(x)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[T(t)\Big] &= c^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big[X(x)\Big]T(t)\\ X(x)T''(t) &= c^2X''(x)T(t)\end{aligned}$$

Then divide both sides by $c^2X(x)T(t)$ to get, for all x and t,

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} \tag{3}$$

The separation constant

The lhs of (3) is a function of time t only, while the rhs is a function of space x only. But x and t are independent variables. This can only be true if both are equal to a constant:

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = \mu \tag{4}$$

Now we can **separate** the equations, into two separate ODEs, one for $X(x)$ and one for $T(t)$

$$\begin{cases} \dfrac{1}{c^2}\dfrac{T''(t)}{T(t)} = \mu \\[2ex] \dfrac{X''(x)}{X(x)} = \mu \end{cases} \quad \Rightarrow \quad \begin{cases} T''(t) = \mu c^2 T(t) \\ X''(x) = \mu X(x) \end{cases}$$

μ is called the **separation constant**.

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Quick Quiz # 1

What are the solutions $Y(y)$ of the following ODEs

1. $$\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} - \alpha^2 Y(y) = 0$$
2. $$\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \alpha^2 Y(y) = 0$$

where $\alpha > 0$ is a constant?

Solving the separated ODEs

We can solve the separated ODEs for $X(x)$ and $T(t)$ if we know the sign of the separation constant μ. There are two choices:

1. If $\mu > 0$, write $\mu = k^2$ for some $k > 0$. Then

$$\begin{cases} T''(t) = (kc)^2 T(t) \\ X''(x) = k^2 X(x) \end{cases} \quad \Rightarrow \quad \begin{cases} T(t) = A\,\mathrm{e}^{kct} + B\,\mathrm{e}^{-kct} \\ X(x) = C\,\mathrm{e}^{kx} + D\,\mathrm{e}^{-kx} \end{cases}$$

2. Alternatively, if $\mu < 0$, write $\mu = -k^2$ for some $k > 0$. Then

$$\begin{cases} T''(t) = -(kc)^2 T(t) \\ X''(x) = -k^2 X(x) \end{cases} \quad \Rightarrow \quad \begin{cases} T(t) = A\cos(kct) + B\sin(kct) \\ X(x) = C\cos(kx) + D\sin(kx) \end{cases}$$

for some constants A, B, C, D

What sign should the separation constant be?

- We'll need to use the homogeneous boundary conditions to answer this properly
- To cut a long story short:

Sign of the separation constant

For the **wave equation**, and the **heat equation**, the separation constant μ must be **negative**. For **Laplace's equation**, it depends on the boundary conditions (more details later).

Separating the boundary/initial conditions

We can also separate the boundary and initial conditions, but **only if they are homogeneous** (i.e. function value equal to zero).

For example, we have that $u = 0$ at $x = 0$ *for all* $t > 0$

$$0 = u(0, t) = X(0)T(t)$$

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Separate all the homogeneous boundary ($u = 0$ at $x = 0, L$ for all t) & initial conditions ($u_t = 0$ at $t = 0$ for all $0 \leqslant x \leqslant L$) in the same way, to get

$$\boxed{X(0) = X(L) = 0} \quad \boxed{T'(0) = 0} \tag{5}$$

Sign of the separation constant

First guess. If $\mu > 0$, write $\mu = k^2$ for some $k > 0$. Then

$$\begin{cases} T''(t) = (kc)^2 T(t) \\ X''(x) = k^2 X(x) \end{cases} \quad \Rightarrow \quad \begin{cases} T(t) = A\,\mathrm{e}^{kct} + B\,\mathrm{e}^{-kct} \\ X(x) = C\,\mathrm{e}^{kx} + D\,\mathrm{e}^{-kx} \end{cases}$$

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Now apply the separated boundary conditions for X. First, at $x = 0$:

$$0 = X(0) = C + D \quad \Rightarrow \quad D = -C \quad \Rightarrow \quad X(x) = C\,\mathrm{e}^{kx} - C\,\mathrm{e}^{-kx}$$

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and then at $x = L$

$$0 = X(L) = C\,\mathrm{e}^{kL} - C\,\mathrm{e}^{-kL} = C(\mathrm{e}^{kL} - \mathrm{e}^{-kL})$$

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We can't have $\mathrm{e}^{kL} - \mathrm{e}^{-kL} = 0$ (as both $k > 0$ and $L > 0$). So $C = 0$, and hence $D = 0$ too. Thus $X(x) = 0$ for all x, and so $u(x,t) = X(x)T(t) \equiv 0$. Oh dear ☹

So the separation constant must be negative.

Solve separated ODEs, and apply homogeneous boundary conditions

We need $\mu < 0$. Write $\mu = -k^2$ for some $k > 0$. Then

$$\begin{cases} T''(t) = -(kc)^2 T(t) \\ X''(x) = -k^2 X(x) \end{cases} \Rightarrow \begin{cases} T(t) = A\cos(kct) + B\sin(kct) \\ X(x) = C\cos(kx) + D\sin(kx) \end{cases}$$

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$$0 = X(L) = D\sin(kL) \quad \Rightarrow \quad \sin(kL) = 0 \quad \Rightarrow \quad kL = n\pi \quad \text{for some integer } n$$

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So

$$\boxed{k = \frac{n\pi}{L}} \quad \text{and} \quad \boxed{X(x) = D\sin\left(\frac{n\pi x}{L}\right)}$$

Applying the homogeneous initial condition

We also have from (5) that $T'(0) = 0$. Recall that

$$T(t) = A\cos(kct) + B\sin(kct) \quad \Rightarrow \quad T'(t) = -kcA\sin(kct) + kcB\cos(kct)$$

Thus

$$0 = -kcA\sin(0) + kcB\cos(0) = kcB$$

which gives us $B = 0$. Hence $T(t) = A\cos(kct)$ for some arbitrary constant A.

But we know that $k = \dfrac{n\pi}{L}$ for some $n \in \mathbb{Z}$, so

$$\boxed{T(t) = A\cos\left(\frac{n\pi ct}{L}\right)}$$

Putting the pieces together

Hence we have the solution

$$\boxed{u(x,t) = X(x)T(t) = b\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)} \tag{6}$$

where $b = AD$ is any constant, and n is any integer. It satisfies the PDE and the **homogeneous** boundary and initial conditions, i.e.

$$\begin{gathered} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_t(x,0) = 0 \end{gathered} \tag{7}$$

but not the **non-homogeneous** initial condition

$$u(x,0) = f(x)$$

Unless we're very lucky, (6) won't satisfy (14) at $t = 0$. We need one last clever idea...

Linearity: general solution

The solution (6) to the PDE + homogeneous boundary and initial conditions is valid for any integer n.

We can **add any linear combination** of them, with any integer values of n, and any multiplicative constant b, and still have a solution*!

So, the **general solution** of the PDE + homogeneous boundary and initial conditions is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right)} \tag{8}$$

The solution has *infinitely many* 'constants of integration' b_n.

*Why? If $u_1(x,t)$ and $u_2(x,t)$ meet the conditions (7), then so does $u_1(x,t) + u_2(x,t)$. Furthermore, because the original PDE (1) is linear, and $u_1(x,t)$ and $u_2(x,t)$ are solutions of the PDE, so is $u_1(x,t) + u_2(x,t)$.

Applying the non-homogeneous initial condition

Finally, we need to apply the initial condition: $u(x,0) = f(x)$ for $0 \leqslant x \leqslant L$, which gives

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{9}$$

We want to find the b_ns, given $f(x)$. **We already know how to do this... it's a half-range Fourier sine series!**

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x} \tag{10}$$

So, the **particular solution** to the PDE (1) + *all* the boundary & initial conditions (2) is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)}$$

where the b_ns are given by (10).

Worked example 5.1

Solve the wave equation on the finite domain

$$\boxed{u_{tt} = c^2 u_{xx}} \qquad \boxed{0 < x < L, \; t > 0}$$

subject to boundary and initial conditions

$$\boxed{\begin{array}{c} u(0,t) = 0, \; u(L,t) = 0 \\ \text{for all } t > 0 \end{array}} \qquad \boxed{\begin{array}{c} u(x,0) = f(x), \; u_t(x,0) = 0 \\ \text{for all } 0 \leqslant x \leqslant L \end{array}}$$

for the specific case $L = 4$ and

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 2 \\ 4 - x, & 2 < x \leqslant 4 \end{cases} \tag{11}$$

Summary: the separation of variables method

Assume a separable solution, e.g. $u(x, t) = X(x)T(t)$, and use it to solve the different parts of the PDE problem step-by-step:

1. Substitute into the PDE, and separate the variables
2. Decide on the sign of the separation constant
3. Solve the separated ODEs
4. Separate, and apply, the *homogeneous* boundary & initial conditions
5. Put the separated pieces back together, and sum to find the general solution (of the PDE + *homogeneous* boundary & initial conditions)
6. Apply the *non-homogeneous* boundary/initial condition, using a half-range Fourier series, to find the particular solution (of the PDE + *all* boundary & initial conditions)

When does separation of variables work?

- The PDE must be **linear**
- It can be used to solve the wave equation, heat equation, and Laplace's equation
- We must have a **finite domain** in space
- There must be a **pair of homogeneous boundary conditions** in space (e.g. $u = 0$ at $x = 0, L$ or $u_x = 0$ at $x = 0, L$)

- We can cope with more general boundary conditions (e.g. $u = 1$ or $u_x = 1$ at one end), but only by modifying the problem
- We can solve other parabolic, hyperbolic, or elliptic PDEs too — as long as it's possible to separate the variables

Quick Quiz #2

Which of the following PDE problems could we use separation of variables to solve?

1. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $0 \leqslant x \leqslant L$, $t > 0$, with $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$, and $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$
2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $-\infty < x < \infty$, $t > 0$, with $u(-\infty,t) = u(\infty,t) = 0$, and $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$
3. $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for $0 \leqslant x \leqslant L$, $t > 0$, with $u(0,t) = u(L,t) = 0$, and $u(x,0) = f(x)$
4. $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial}{\partial x}\left[\mathrm{e}^{-x^2} \frac{\partial u}{\partial x}\right]$ for $0 \leqslant x \leqslant L$, $t > 0$, with $u(0,t) = u(L,t) = 0$, and $u(x,0) = f(x)$

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2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $-\infty < x < \infty$, $t > 0$, with $u(-\infty,t) = u(\infty,t) = 0$, and $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$ ✗
3. $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for $0 \leqslant x \leqslant L$, $t > 0$, with $u(0,t) = u(L,t) = 0$, and $u(x,0) = f(x)$ ✓
4. $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial}{\partial x}\left[e^{-x^2} \frac{\partial u}{\partial x}\right]$ for $0 \leqslant x \leqslant L$, $t > 0$, with $u(0,t) = u(L,t) = 0$, and $u(x,0) = f(x)$ ✓

Walk-through example 2: the heat equation

Find the solution $u(x,t)$ of the heat equation on a finite domain

$$\boxed{u_t = \alpha^2 u_{xx}} \qquad \boxed{0 \leqslant x \leqslant L,\ t > 0,} \tag{12}$$

subject to boundary conditions (perfect insulator/no heat flux)

$$\boxed{\begin{array}{c} u_x(0,t) = u_x(L,t) = 0 \\ \text{for all } t > 0 \end{array}} \tag{13}$$

and an initial temperature distribution

$$\boxed{\begin{array}{c} u(x,0) = h(x) \\ \text{for all } 0 \leqslant x \leqslant L \end{array}} \tag{14}$$

Step 1: separate the variables

The basic idea is once again to *try* to find a separable solution. That is, we write

$$u(x,t) = X(x)T(t),$$

Substituting this into the PDE we get

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ \frac{\partial}{\partial t}\Big[X(x)T(t)\Big] &= \alpha^2 \frac{\partial^2}{\partial x^2}\Big[X(x)T(t)\Big] \\ X(x)T'(t) &= \alpha^2 X''(x)T(t), \end{aligned}$$

which, after dividing by $\alpha^2 X(x)T(t)$, simplifies to

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}, \tag{15}$$

Step 1: separate the variables

Again, the left-hand side of (15) is a function of time t only, and the right-hand side is a function of space x only. The only way that this can be true for all x and t is if both functions are equal to a constant. Hence

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const.} = \mu \tag{16}$$

μ is called the **separation constant**. As before, we have to decide what sign μ should be. We proceed by trial and error to see what fits the boundary and initial conditions, and what makes sense physically.

(Sneak preview: μ must be negative)

Steps 2 & 3: sign of separation constant, and solve separated ODEs

Try first a *positive* constant, $\mu = k^2$. Hence we write (16) as:

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = k^2 > 0$$

Then we get two separate linear ODEs, that are easy to solve:

$$\begin{cases} T'(t) = (\alpha k)^2 T(t) \\ X''(x) = k^2 X(x) \end{cases} \quad \Rightarrow \quad \begin{cases} T(t) = A\,\mathrm{e}^{(\alpha k)^2 t} \\ X(x) = B\,\mathrm{e}^{-kx} + C\,\mathrm{e}^{kx} \end{cases}$$

But $T(t) \to +\infty$ as $t \to \infty$. This is not a diffusion-like process (heat decays, not blows up!). Furthermore, applying the homogeneous boundary conditions gives $B = C = 0$. ☹

So μ must be negative.

Steps 2 & 3: sign of separation constant, and solve separated ODEs

We need $\mu < 0$. Setting $\mu = -k^2$ in (16) we get

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -k^2 < 0$$

Thus we get the two separate linear ODEs:

$$\begin{aligned} T'(t) &= -(\alpha k)^2 T(t) \\ X''(x) &= -k^2 X(x) \end{aligned}$$

Both are easy to solve:

$$\begin{aligned} T(t) &= A\, \mathrm{e}^{-(\alpha k)^2 t} \\ X(x) &= B\cos(kx) + C\sin(kx) \end{aligned}$$

Step 4: separate & apply homogeneous boundary conditions

The homogenous boundary conditions (13) become

$$\begin{aligned} 0 &= u_x(0,t) = X'(0)T(t) \quad && \text{for all } t>0 \quad && \Rightarrow \quad && X'(0)=0 \\ 0 &= u_x(L,t) = X'(L)T(t) \quad && \text{for all } t>0 \quad && \Rightarrow \quad && X'(L)=0 \end{aligned}$$

Applying them to $X(x) = B\cos(kx) + C\sin(kx) \Rightarrow X'(x) = -Bk\sin(kx) + Ck\cos(kx)$ we get

$$0 = X'(0) = -Bk\sin(0) + Ck\cos(0) = Ck, \tag{17}$$

$$0 = X'(L) = -Bk\sin(kL) + Ck\cos(kL). \tag{18}$$

From (17) we get $C = 0$, hence from (18) we have

$$-Bk\sin(kL) = 0 \quad \Rightarrow \quad kL = n\pi \quad \Rightarrow \quad k = \frac{n\pi}{L}, \quad n \in \mathbb{Z}.$$

Step 5: put the pieces together, and sum

Substitute the value for k into the solutions for $X(t)$ and $T(t)$:

$$\boxed{X(x) = B\cos\left(\frac{n\pi x}{L}\right)} \qquad \boxed{T(t) = A\,\mathrm{e}^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}}$$

Since $u(x,t) = X(x)T(t)$

$$u_n(x,t) = a_n\cos\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

solves the PDE (12) and homogenous boundary conditions (13) for any integer n. Since both are linear, any sum of the u_ns will also be a solution, so the general solution is

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}} \tag{19}$$

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solves the PDE (12) and homogenous boundary conditions (13) for any integer n. Since both are linear, any sum of the u_ns will also be a solution, so the general solution is

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}} \tag{19}$$

Step 6: non-homogeneous initial conditions

The last step is to satisfy the initial condition $u(x, 0) = h(x)$. Setting $t = 0$ in the general solution we get

$$h(x) = u(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Once again, this is just the Fourier half-range cosine series expansion of the function $h(x)$. Hence we know that

$$a_n = \frac{2}{L}\int_0^L h(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \tag{20}$$

Walk-through example 2: the particular solution

So, the particular solution of the heat equation PDE (12) satisfying *all* the boundary and initial conditions, (13) and (14), is:

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\alpha\pi}{L}\right)^2 t}}$$

where the a_ns are the Fourier half-range cosine series coefficients, given by

$$\boxed{a_n = \frac{2}{L}\int_0^L h(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x}$$

Walk-through example 2: finding the a_n

Find the solution $u(x,t)$ of the heat equation

$$\boxed{u_t = \alpha^2 u_{xx}} \qquad \boxed{0 \leqslant x \leqslant L, \ t > 0,}$$

subject to the boundary and initial conditions:

$$\boxed{\begin{array}{c} u_x(0,t) = u_x(L,t) = 0 \\ \text{for all } t > 0 \end{array}} \qquad \boxed{\begin{array}{c} u(x,0) = h(x) \\ \text{for all } 0 \leqslant x \leqslant L \end{array}}$$

for the specific case $L = 4$ and

$$h(x) = \begin{cases} x, & 0 \leqslant x \leqslant 2 \\ 4 - x, & 2 < x \leqslant 4 \end{cases}$$

Walk-through example 2: finding the a_n

The general solution of the PDE + homogeneous boundary conditions is (19):

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

Walk-through example 2: finding the a_n

The general solution of the PDE + homogeneous boundary conditions is (19): with $L = 4$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) e^{-\left(\frac{n\pi\alpha}{4}\right)^2 t}$$

Walk-through example 2: finding the a_n

The general solution of the PDE + homogeneous boundary conditions is (19): with $L = 4$

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Apply the initial (non-homogeneous) condition: set $t = 0$

$$u(x,0) = h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \qquad \text{where } h(x) = \begin{cases} x, & 0 \leqslant x \leqslant 2 \\ 4 - x, & 2 < x \leqslant 4 \end{cases}$$

This is a half-range Fourier cosine series for $h(x)$:

$$a_n = \frac{2}{L} \int_0^L h(x) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x$$

Walk-through example 2: finding the a_n

The general solution of the PDE + homogeneous boundary conditions is (19): with $L = 4$

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This is a half-range Fourier cosine series for $h(x)$: We did this in week 9, Homework #1!

$$a_n = \frac{2}{4}\int_0^4 h(x)\cos\left(\frac{n\pi x}{4}\right)\,\mathrm{d}x$$

Walk-through example 2: finding the a_n

The general solution of the PDE + homogeneous boundary conditions is (19): with $L = 4$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \mathrm{e}^{-\left(\frac{n\pi\alpha}{4}\right)^2 t}$$

Apply the initial (non-homogeneous) condition: set $t = 0$

$$u(x,0) = h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \qquad \text{where } h(x) = \begin{cases} x, & 0 \leqslant x \leqslant 2 \\ 4 - x, & 2 < x \leqslant 4 \end{cases}$$

This is a half-range Fourier cosine series for $h(x)$: We did this in week 9, Homework #1!

$$a_0 = 2, \qquad a_n = \frac{8}{(n\pi)^2}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right)$$

Walk-through example 2: the particular solution

Finally, put the a_n coefficients into the general solution:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) e^{-\left(\frac{n\alpha\pi}{4}\right)^2 t}$$

Walk-through example 2: the particular solution

Finally, put the a_n coefficients into the general solution:

$$u(x,t) = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right) \cos\left(\frac{n\pi x}{4}\right) e^{-\left(\frac{n\alpha\pi}{4}\right)^2 t}$$

This is the *particular solution* of the entire problem: the PDE + homogeneous boundary conditions + non-homogeneous initial condition.

Walk-through example 2: the particular solution

Finally, put the a_n coefficients into the general solution:

$$u(x,t) = 1 + \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1 \right) \cos\left(\frac{n\pi x}{4}\right) e^{-\left(\frac{n\alpha\pi}{4}\right)^2 t}$$

This is the *particular solution* of the entire problem: the PDE + homogeneous boundary conditions + non-homogeneous initial condition.

Laplace's equation

Since Laplace's equation

$$\boxed{u_{xx} + u_{yy} = 0} \tag{21}$$

involves only spatial co-ordinates, (x, y) (or (x, y, z) in 3D), it is natural to pose Laplace's equation on domains of any shape. E.g. circular (find the shape of a drumskin) or complex curvy shapes (find the incompressible, irrotational flow through a curved river bed).

However, in these lectures we shall concentrate only on the simplest case of a rectangular domain in 2D.

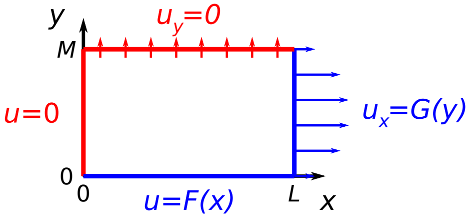
$$\boxed{0 < x < L,\ 0 < y < M}$$

Such a problem could describe, for example, the steady-state temperature $u(x, y)$ of a conductive rectangular sheet. Again, the boundary conditions will play a crucial role.

Boundary conditions

We need one condition on each boundary ($x = 0, L$ and $y = 0, M$). There are two types we can pose:

- **Dirichlet**: u is given on a boundary; e.g.
  - homogeneous: $u(0, y) = 0$ for all $0 \leqslant y \leqslant M$
  - non-homogeneous: $u(x, 0) = F(x)$ for all $0 \leqslant x \leqslant L$
- **Neumann**: the normal derivative of u is given on a boundary; e.g.
  - homogeneous: $u_y(x, M) = 0$ for all $0 \leqslant x \leqslant L$
  - non-homogeneous: $u_x(L, y) = G(y)$ for all $0 \leqslant y \leqslant M$



We'll focus on problems with **three homogeneous** boundary conditions, and **one non-homogeneous** boundary condition.

Sign of separation constant for Laplace's equation

To solve Laplace's equation, we look for a separable solution as usual

$$u(x, y) = X(x)Y(y)$$

Substituting this into the PDE (21) gives us

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

which simplifies to

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const} = \mu$$

What sign should μ take?

Sign of separation constant for Laplace's equation

Choosing the sign of the separation constant isn't quite as easy as for the heat and wave equations. The right choice depends on the homogeneous boundary conditions.

As usual there are two possibilities: either $\mu = -k^2 < 0$, giving

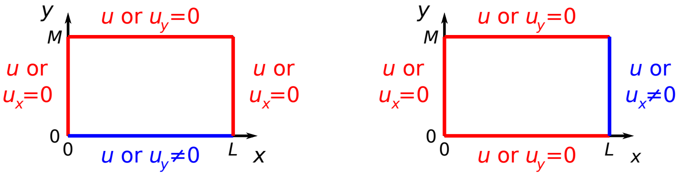
$$\begin{cases} X''(x) = -k^2 X(x) \\ Y''(y) = k^2 Y(y) \end{cases} \quad \Rightarrow \quad \begin{cases} X(x) = A\cos(kx) + B\sin(kx) \\ Y(y) = C\,\mathrm{e}^{ky} + D\,\mathrm{e}^{-ky} \end{cases} \tag{22}$$

or $\mu = +k^2 > 0$, so that

$$\begin{cases} X''(x) = k^2 X(x) \\ Y''(y) = -k^2 Y(y) \end{cases} \quad \Rightarrow \quad \begin{cases} X(x) = A\,\mathrm{e}^{kx} + B\,\mathrm{e}^{-kx} \\ Y(y) = C\cos(ky) + D\sin(ky) \end{cases} \tag{23}$$

Sign of separation constant for Laplace's equation (2)

Since there is exactly one non-homogeneous boundary condition, that means that two opposite sides of the domain have homogeneous boundary conditions: u, or its normal derivative, equal to zero:



In case **(i)**, separating the boundary conditions gives $X = 0$ or $X' = 0$ at *both* $x = 0$ and $x = L$. So $X(x)$ must be sin & cos (as in (22)).

In case **(ii)**, separating the boundary conditions gives $Y = 0$ or $Y' = 0$ at *both* $y = 0$ and $y = M$. So $Y(y)$ must be sin & cos (as in (23)).

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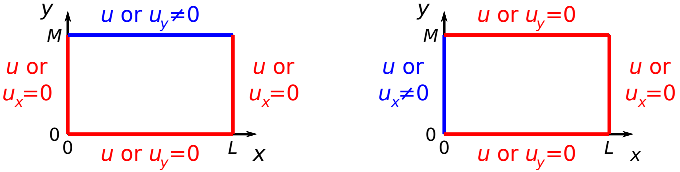

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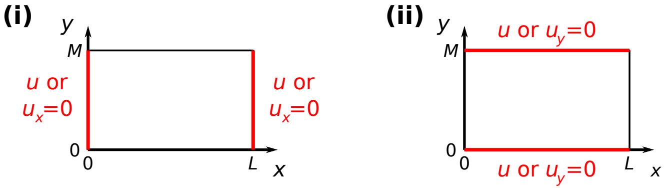

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Worked example 5.2

Solve Laplace's equation on a rectangular domain

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < L,\ 0 < y < M$$

subject to the Dirichlet boundary conditions

$$u(0,y) = u(L,y) = 0 \quad \text{for all } 0 \leqslant y \leqslant M \qquad u(x,0) = 0,\ u(x,M) = f(x) \quad \text{for all } 0 \leqslant x \leqslant L$$

for the particular case $L = 4$ and $M = 2$ and

$$f(x) = \sin\left(\frac{\pi x}{4}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{4}\right)$$

Homework #5

Use separation of variables to solve the wave equation on a finite domain

$$u_{tt} = c^2 u_{xx}, \qquad 0 \leqslant x \leqslant L,\ t > 0,$$

with

$$u_x(0,t) = u_x(L,t) = 0, \qquad u(x,0) = 0,\ u_t(x,0) = x(L-x) - \frac{L^2}{6}.$$

You may assume, without proof, that the Fourier half-range cosine and sine series of $f(x)$ are:

$$f(x) \sim -\sum_{n=1}^{\infty} \frac{2L^2(1+(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right),$$

$$f(x) \sim \sum_{n=1}^{\infty} \frac{L^2(12-n^2\pi^2)(1-(-1)^n)}{3n^3\pi^3} \sin\left(\frac{n\pi x}{L}\right).$$

Summary

- The separation of variables is a trial and error method. We **try** $u(x,t) = X(x)T(t)$, or $u(x,y) = X(x)Y(y)$ as appropriate.
- The choice of the *sign* of the separation constant is crucial.
  - negative for heat and wave equations
  - depends on boundary conditions for Laplace's equation
- The solution of the PDE is an infinite sum, with infinitely many 'constants of integration'
- We use half-range Fourier series to compute them