ADVANCED STRUCTURES & MATERIALS

Finite Element Analysis Principles – Lecture 4

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From Trusses to Frames

- We now focus on another type of structure with discrete joints: **Frames**.
- Frames, like trusses, are naturally suited to a discretisation process in which the finite element is clearly the **beam**.
- **Beam elements** have the same **1D geometry** as bar elements, but they can **resist shear** and **bending forces** in addition to **axial loads**.
- Like for trusses, the differential equations for the elasticity of beams are known and can be solved in terms of nodal displacements.

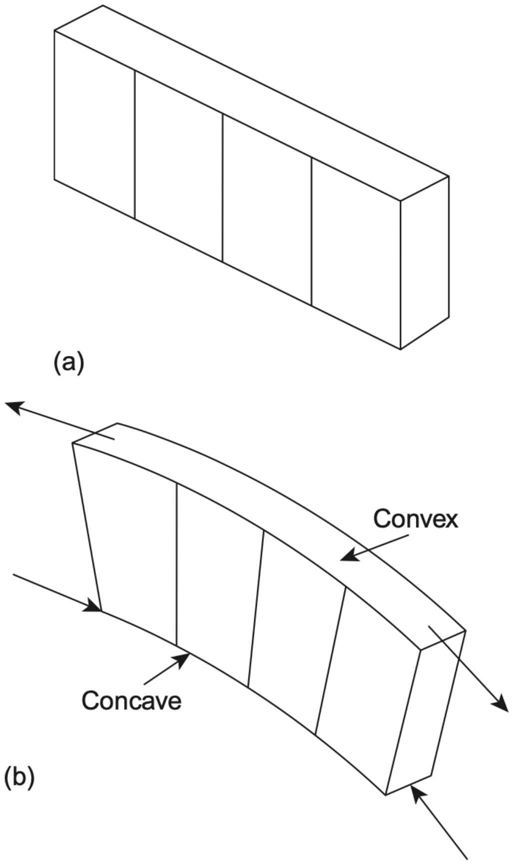
Beam Theory: A Refresher

Assumptions

- 1D geometry
- Small displacements
- Linear elastic material behaviour
- Beam is straight with uniform cross section
- Shear deformations are negligible
- Euler–Bernoulli hypothesis
- Saint-Venant's principle

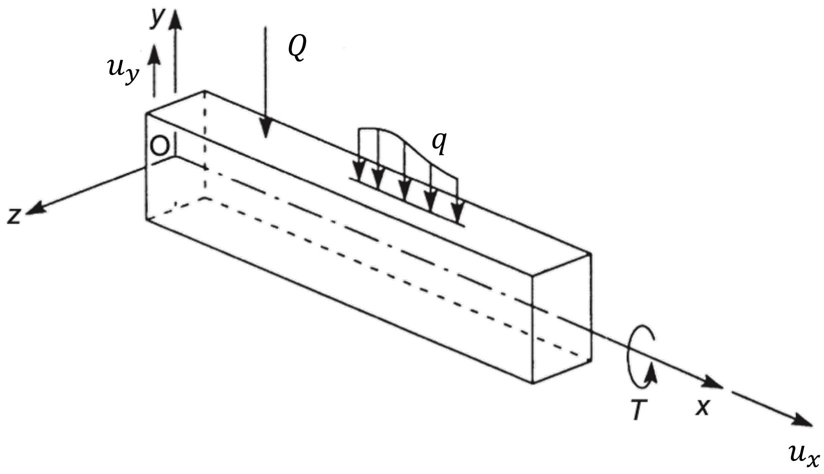
Euler–Bernoulli Hypothesis

Plane cross sections of the beam remain plane and normal to the longitudinal axis of the beam after bending



Local and Global Reference Systems

- We assume that the problem is planar, embedded in the xy-plane.
- The local reference system used for beam elements is identical to that defined for bars.
- It originates at one of the nodes, with the local x-axis directed along the geometric axis of the beam and the local y-axis orthogonal to it, lying in the same plane.
- The displacement field can be defined in terms of the components of deformation along the two local axes.



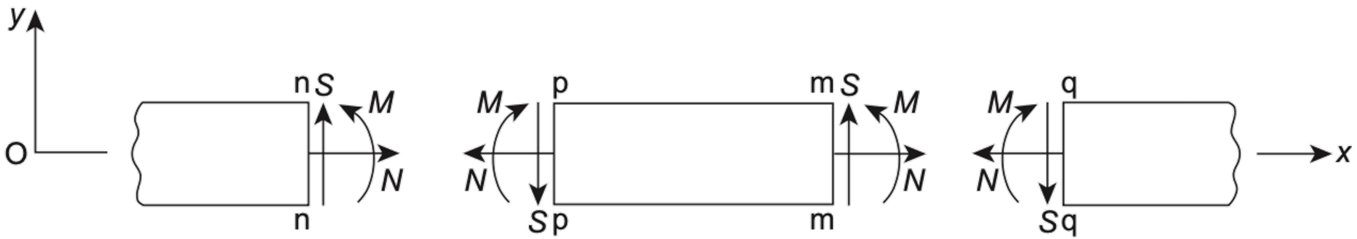
Stress Resultants

Cross-sectional stresses can be integrated to define **stress resultants**, which become the variables of reference for equilibrium.

N = Normal/Axial force

S = Shear force (a.k.a. V)

M = Bending moment

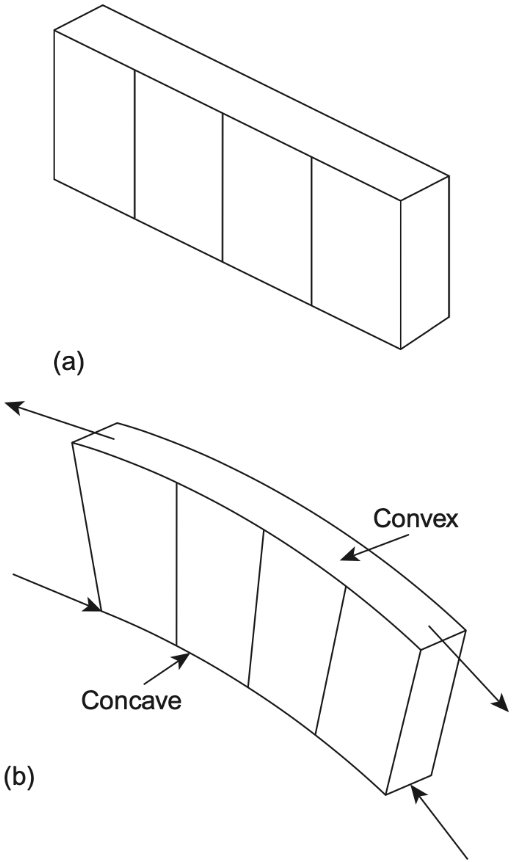


Kinematics of a Beam in Bending

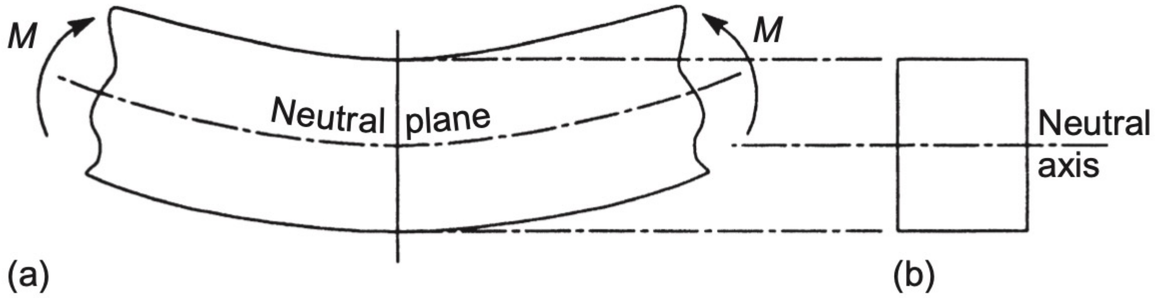
Transverse loads cause transverse deformations.

Longitudinal material lines on the convex side are in extension.

Longitudinal material lines on the concave side are in compression.



Kinematics of a Beam in Bending

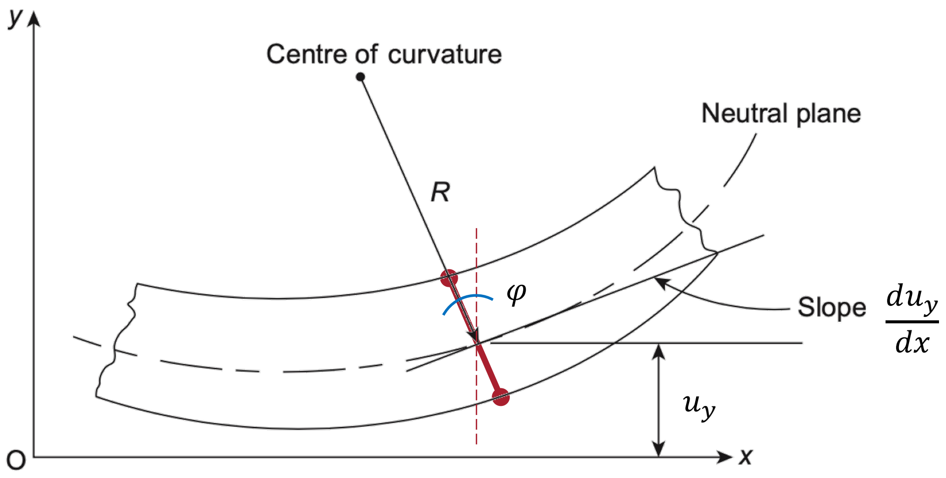


Since strains must be continuous, it follows that a plane will exist that does not change length.

This is called **neutral plane**. Beam equations are referred to this axis, which establishes the **decoupling of flexural and axial behaviour**.

Kinematics of a Beam in Bending

Cross sections deflect by u_y and rotate by φ.

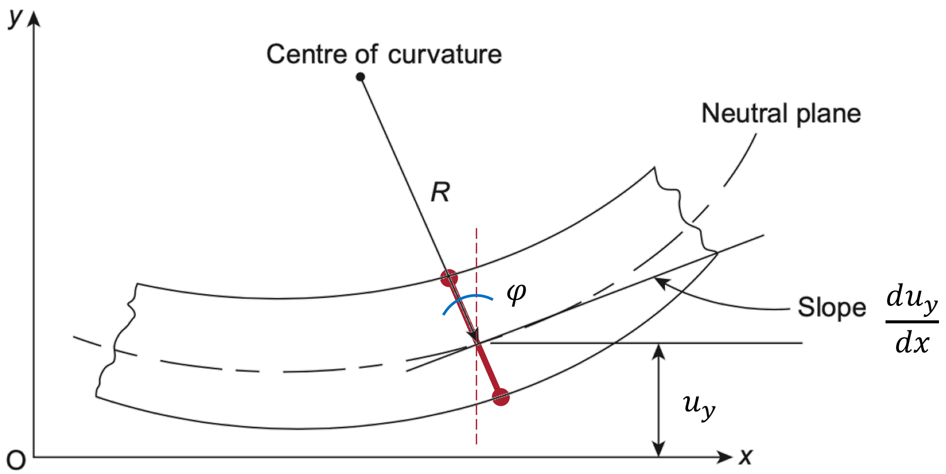


Kinematics of a Beam in Bending

As cross sections remain orthogonal to the neutral axis, φ can be used to represent their angle of rotation.

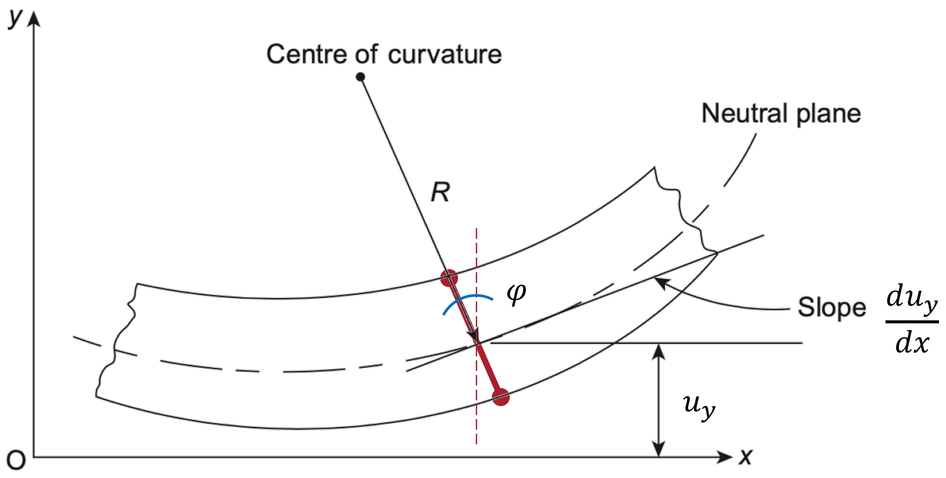
It follows that

$$u_x = -y\varphi$$



Kinematics of a Beam in Bending

How do we quantify φ?

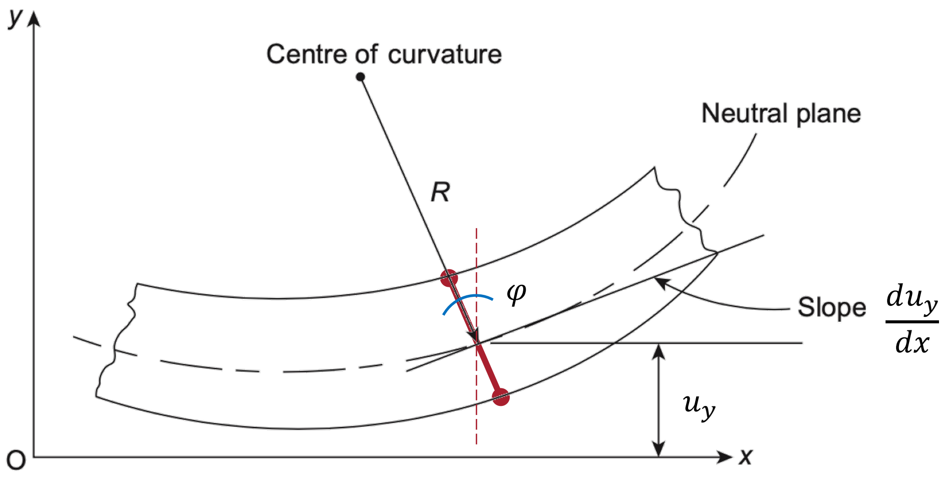


Kinematics of a Beam in Bending

The slope of the deformed neutral axis is $\tan\varphi = du_y/dx$.

Small deflections imply $\tan\varphi \approx \varphi$, so $du_y/dx \approx \varphi$ and

$$u_x = -y\varphi = -y\frac{du_y}{dx}$$



Kinematics of a Beam in Bending

All considered, the **displacement field** for a beam can then be written as

$$
\begin{aligned}
u_x &= u_x(x,y,z) = u_x(x,y) = u_x(x) + u_x(x,y) \\
u_y &= u_y(x,y,z) = u_y(x) \\
u_z &= u_z(x,y,z) = 0
\end{aligned}
$$

Displacements due to axial loads

Displacements due to bending loads

Kinematics of a Beam in Bending

$$
\begin{aligned}
u_x &= u_x(x,y,z) = u_x(x,y) = u_x(x) + u_x(x,y) \\
u_y &= u_y(x,y,z) = u_y(x) \\
u_z &= u_z(x,y,z) = 0
\end{aligned}
$$

Displacements due to axial loads

Displacements due to bending loads

There will be **two components of axial deformation**, one due to the deformation of the beam's axis induced by axial stresses, and the other due to the rotation φ of the cross-sections caused by bending stresses.

Kinematics of a Beam in Bending

$$
\begin{aligned}
u_x &= u_x(x,y,z) = u_x(x,y) = u_x(x) + u_x(x,y) \\
u_y &= u_y(x,y,z) = u_y(x) \\
u_z &= u_z(x,y,z) = 0
\end{aligned}
$$

Displacements due to axial loads

Displacements due to bending loads

This separation is possible because, referring the equations to the neutral axis decouples the bending and axial problems.

Kinematics of a Beam in Bending

In matrix form

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x(x,y) \\ u_y(x) \\ 0 \end{bmatrix} = \begin{bmatrix} u_x(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_x(x,y) \\ u_y(x) \\ 0 \end{bmatrix} = \begin{bmatrix} u_x(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -y\frac{du_y(x)}{dx} \\ u_y(x) \\ 0 \end{bmatrix}$$

Displacements due to axial loads

Displacements due to bending loads

Kinematics of a Beam in Bending

In matrix form

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x(x,y) \\ u_y(x) \\ 0 \end{bmatrix} = \begin{bmatrix} u_x(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_x(x,y) \\ u_y(x) \\ 0 \end{bmatrix} = \begin{bmatrix} u_x(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -y\frac{du_y(x)}{dx} \\ u_y(x) \\ 0 \end{bmatrix}$$

Exactly the displacement vector obtained for bar elements.
We can focus on the second term and then combine the results.

Displacements due to axial loads

Displacements due to bending loads

Beam Finite Element Formulation

Beam Finite Element Formulation

It is customary to proceed with the displacement vector not in the form indicated in the previous slide, but as

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x(x) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -y\frac{du_y(x)}{dx} \\ u_y(x) \\ 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{u} = \begin{bmatrix} u_x(x) \\ u_y(x) \\ \varphi(x) \end{bmatrix}$$

because these three quantities refer to the reference axis of the beam, are fully sufficient to determine the deformed configuration, and enable us to find suitable relationships of the form $\boldsymbol{f} = \boldsymbol{K\delta}$ between generalised nodal forces, $\boldsymbol{f}$, and generalised nodal displacements, $\boldsymbol{\delta}$.

Beam Finite Element Formulation

Note, indeed, that

$$\boldsymbol{u} = \begin{bmatrix} u_x(x) \\ u_y(x) \\ \varphi(x) \end{bmatrix} = \begin{bmatrix} u_x(\delta_{1x}, \delta_{2x}) \\ u_y(\delta_{1y}, \varphi_1, \delta_{2y}, \varphi_2) \\ \varphi(\delta_{1y}, \varphi_1, \delta_{2y}, \varphi_2) \end{bmatrix}$$

where the δ_{ix}, δ_{iy}, are nodal axial and transverse displacements, and φ_i represent nodal rotations.

Beam Finite Element Formulation

Grouping the generalised nodal displacements as

$$\boldsymbol{\delta}^{\mathsf{T}} = \left[\delta_{1x}, \delta_{1y}, \varphi_1, \delta_{2x}, \delta_{2y}, \varphi_2\right]$$

We now need to find suitable shape functions, $\boldsymbol{N}$, relating $\boldsymbol{u}$ and $\boldsymbol{\delta}$.
Actually, of a reduced version of $\boldsymbol{\delta}$ without δ_{1x} and δ_{2x}.

Beam Finite Element Formulation

How do we choose suitable shape functions?

We’ll follow a different and more general procedure than the one used for bar elements.

Beam Finite Element Formulation

Recall

$$\frac{\partial^4 u_y}{\partial x^4} = -\frac{q}{EI}$$

This means u_y will need to be **4 times differentiable**. Additionally, we need a function with **four coefficients to fit the four nodal parameters***.

So, we can use the expression

$$u_y(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

*Look up Polynomial Interpolation

Beam Finite Element Formulation

Since $\varphi = du_y/dx$

$$u_y(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

$$\Downarrow$$

$$\varphi(x) = 3a_1x^2 + 2a_2x + a_3$$

Beam Finite Element Formulation

Rearranging the relationships found in matrix form, the vector $\boldsymbol{u}$ (excluding the first row, as well as the nodal parameters $\boldsymbol{\delta}_{x1}$ and $\boldsymbol{\delta}_{x2}$ that determine it) can be written as:

$$\boldsymbol{u} = \begin{bmatrix} u_y \\ \varphi \end{bmatrix} = \begin{bmatrix} x^3 & x^2 & x & 1 \\ 3x^2 & 2x & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \boldsymbol{\Phi a}$$

Beam Finite Element Formulation

To derive the functional relationship between $\boldsymbol{u}$ and the nodal displacement vector $\boldsymbol{\delta}$, we evaluate equation $\boldsymbol{u} = \boldsymbol{\Phi a}$ at the nodal coordinates

$$\boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{u}(x=0) \\ \boldsymbol{u}(x=L) \end{bmatrix} = \begin{bmatrix} \delta_{y1} \\ \phi_1 \\ \delta_{y2} \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \bar{\boldsymbol{\Phi}}\boldsymbol{a}$$

Beam Finite Element Formulation

Combining

$$\boldsymbol{\delta} = \bar{\boldsymbol{\Phi}}\boldsymbol{a} \qquad \text{and} \qquad \boldsymbol{u} = \boldsymbol{\Phi}\boldsymbol{a}$$

We find the $\boldsymbol{N}$ matrix

$$\boldsymbol{a} = \bar{\boldsymbol{\Phi}}^{-1}\boldsymbol{\delta} \quad \Rightarrow \quad \boldsymbol{u} = \boldsymbol{\Phi}\bar{\boldsymbol{\Phi}}^{-1}\boldsymbol{\delta} = \boldsymbol{N}\boldsymbol{\delta}$$

Beam Finite Element Formulation

$$\boldsymbol{N} = \begin{bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} & x - \frac{2x^2}{L} + \frac{x^3}{L^2} & \frac{3x^2}{L^2} - \frac{2x^3}{L^3} & -\frac{x^2}{L} + \frac{x^3}{L^2} \\ -\frac{6x}{L^2} + \frac{6x^2}{L^3} & 1 - \frac{4x}{L} + \frac{3x^2}{L^2} & \frac{6x}{L^2} - \frac{6x^2}{L^3} & -\frac{2x}{L} + \frac{3x^2}{L^2} \end{bmatrix}$$

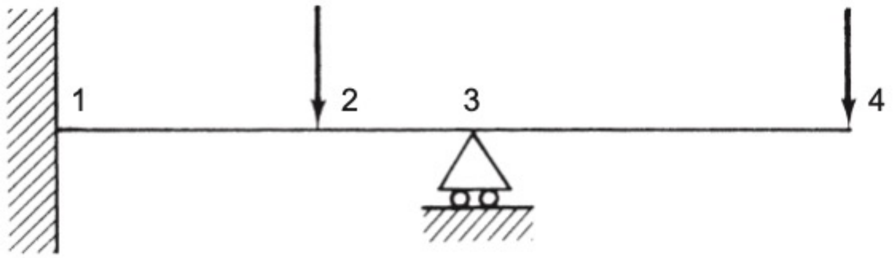
Note: As $\boldsymbol{N}$ was derived based on exact (within the limits of Euler-Bernoulli's beam theory) analytical relations, it will lead to equally exact results.

Does this fact have any implications?

Beam Finite Element Formulation

It does:

1. On how we mesh frames.
2. On the need for mesh refinement… it's not needed!



Beam Finite Element Formulation

Having determined the shape matrix, we can now calculate the stiffness matrix, $\boldsymbol{K}$, and the vector of equivalent forces, $\boldsymbol{f}$.

$$\boldsymbol{K} = \int_V \boldsymbol{B}^\mathsf{T} \boldsymbol{E} \boldsymbol{B} \, dV \qquad\qquad \boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T} \boldsymbol{p} \, dS + \int_V \boldsymbol{N}^\mathsf{T} \boldsymbol{b} \, dV$$

Recall $\boldsymbol{B} = \boldsymbol{D}\boldsymbol{N}$. To determine $\boldsymbol{K}$, we need a specialised version of $\boldsymbol{D}$ adapted for beam theory.

Governing Equations of Linear Elasticity: Beam Elements

Since $u_z = 0$, $\epsilon_{xy} = 0$, and there are no dependencies on z, $\boldsymbol{\varepsilon} = \boldsymbol{Du}$ simplifies as follows

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \cancel{\epsilon_{xy}} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \cancel{0} \\ 0 & \frac{\partial}{\partial y} & \cancel{0} \\ 0 & 0 & \cancel{\frac{\partial}{\partial z}} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \cancel{0} \\ 0 & \cancel{\frac{\partial}{\partial z}} & \cancel{\frac{\partial}{\partial y}} \\ \cancel{\frac{\partial}{\partial z}} & 0 & \cancel{\frac{\partial}{\partial x}} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ \cancel{u_z} \end{bmatrix} \Rightarrow \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Governing Equations of Linear Elasticity: Beam Elements

Further, since

$$\begin{bmatrix} u_x(x,y) \\ u_y(x) \end{bmatrix} = \begin{bmatrix} -y\varphi(x) \\ u_y(x) \end{bmatrix} = \begin{bmatrix} 0 & -y \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_y \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 & -y \\ 1 & 0 \end{bmatrix} \boldsymbol{u}$$

$$\Downarrow$$

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} 0 & -y \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_y \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 & -y\frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} u_y \\ \varphi \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -y\frac{\partial}{\partial x} \\ 0 & 0 \end{bmatrix}}_{\text{new } \boldsymbol{D}} \boldsymbol{u}$$

Governing Equations of Linear Elasticity: Beam Elements

We also need $\boldsymbol{E}$…

In beam theory, the kinematic assumptions negate Poisson's effects, so

$$\begin{bmatrix}\sigma_{xx}\\\sigma_{yy}\\\sigma_{zz}\\\sigma_{xy}\\\sigma_{yz}\\\sigma_{zx}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\\epsilon_{yy}\\\epsilon_{zz}\\\epsilon_{xy}\\\epsilon_{yz}\\\epsilon_{zx}\end{bmatrix} \Rightarrow \sigma_{xx} = E\varepsilon_{xx}$$

Beam Element Stiffness Matrix

Having determined $\boldsymbol{N}$, $\boldsymbol{D}$, and $\boldsymbol{E}$, we can integrate* to find $\boldsymbol{K}$

$$\boldsymbol{K} = \begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} & -12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} & -6\dfrac{EI}{L^2} & 2\dfrac{EI}{L} \\ -12\dfrac{EI}{L^3} & -6\dfrac{EI}{L^2} & 12\dfrac{EI}{L^3} & -6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 2\dfrac{EI}{L} & -6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix}$$

*Will we always be able to integrate?

Accounting for both axial and flexural behaviour

$$\boldsymbol{K} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^2} & 4\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 0 & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix}$$

Equivalent Nodal Forces

Onto the vector of equivalent forces now…

$$\boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T} \boldsymbol{p} \, dS + \int_V \boldsymbol{N}^\mathsf{T} \boldsymbol{b} \, dV$$

Equivalent Nodal Forces

The vectors of distributed pressures and volume forces take the form

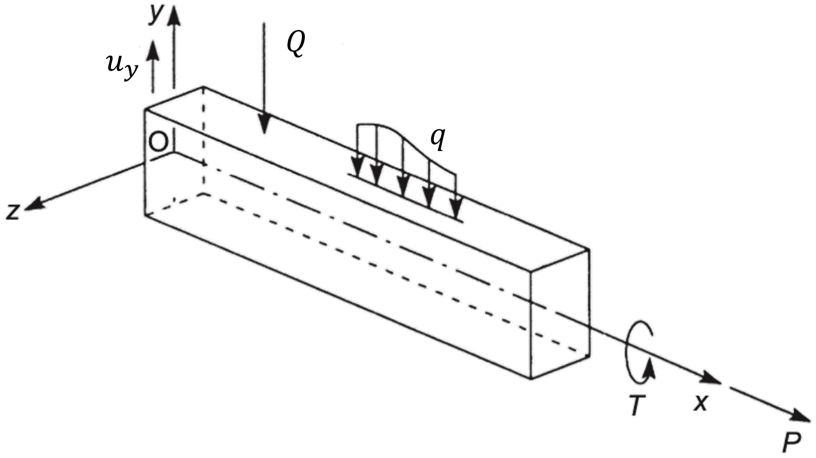
$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_x \\ b_y \\ 0 \end{bmatrix}$$

with the components along $\boldsymbol{z}$ being zero, as the problem is planar.

Equivalent Nodal Forces

By integrating these quantities along the perimeter, Γ, and area, A, of each cross section along the beam axis, we obtain the vector of distributed forces per unit length

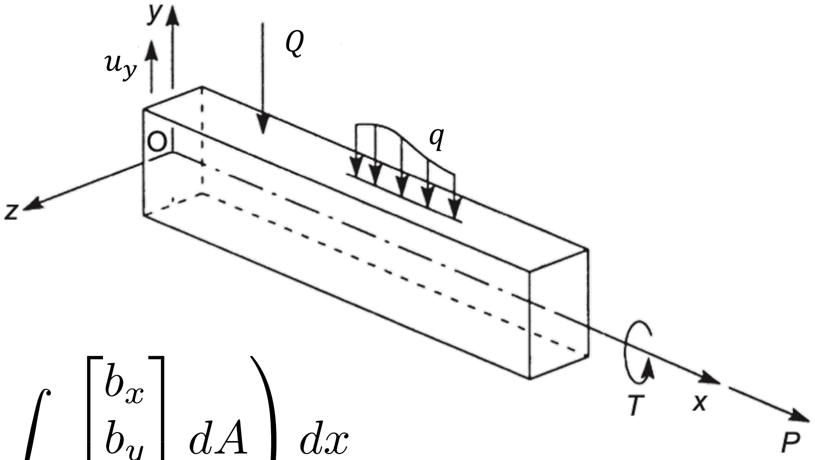
$$\boldsymbol{q} = \begin{bmatrix} n_x \\ q_y \\ 0 \end{bmatrix} = \int_\Gamma \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} d\Gamma + \int_A \begin{bmatrix} b_x \\ b_y \\ 0 \end{bmatrix} dA$$



Equivalent Nodal Forces

Plugging this into

$$\boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T} \boldsymbol{p}\, dS + \int_V \boldsymbol{N}^\mathsf{T} \boldsymbol{b}\, dV$$



we obtain

$$\boldsymbol{f} = \int_L \boldsymbol{N}^\mathsf{T} \boldsymbol{q}\, dx = \int_L \boldsymbol{N}^\mathsf{T} \left(\int_\Gamma \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} d\Gamma + \int_A \begin{bmatrix} b_x \\ b_y \\ 0 \end{bmatrix} dA \right) dx$$

Equivalent Nodal Forces

where, this time,

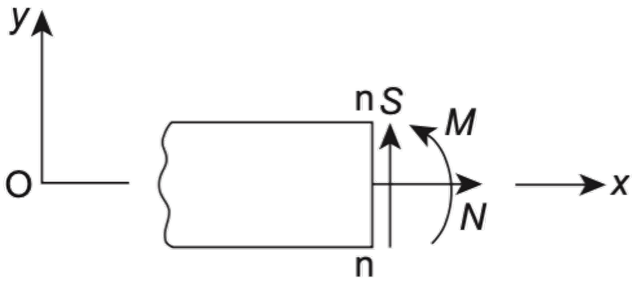
$$\boldsymbol{N} = \begin{bmatrix} 1 - \frac{x}{L} & 0 & 0 & \frac{x}{L} & 0 & 0 \\ 0 & 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} & x - \frac{2x^2}{L} + \frac{x^3}{L^2} & 0 & \frac{3x^2}{L^2} - \frac{2x^3}{L^3} & -\frac{x^2}{L} + \frac{x^3}{L^2} \\ 0 & -\frac{6x}{L^2} + \frac{6x^2}{L^3} & 1 - \frac{4x}{L} + \frac{3x^2}{L^2} & 0 & \frac{6x}{L^2} - \frac{6x^2}{L^3} & -\frac{2x}{L} + \frac{3x^2}{L^2} \end{bmatrix}$$

as we are considering axial and flexural behaviours simultaneously.

Equivalent Nodal Forces

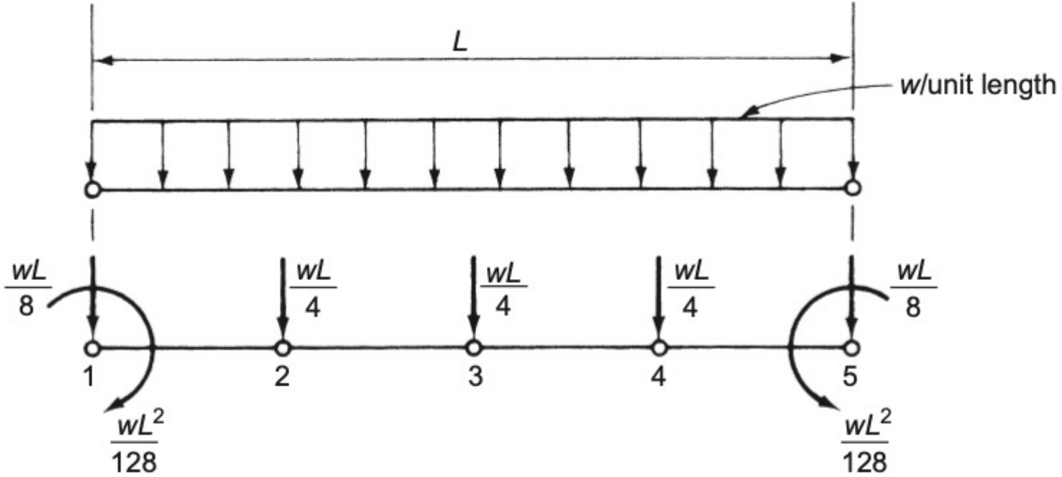
Given the chosen Lagrangian parameters

$$\boldsymbol{f} = \int_L \boldsymbol{N}^\mathsf{T} \boldsymbol{q}\, dx = \begin{bmatrix} N_1 \\ S_1 \\ M_1 \\ N_2 \\ S_2 \\ M_2 \end{bmatrix}$$



Equivalent Nodal Forces

An example…



Equivalent Nodal Forces

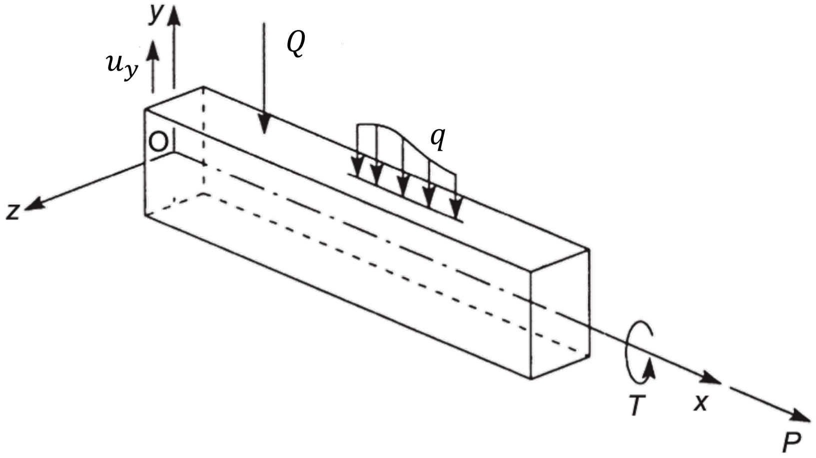
Point forces and moments can be included in $\boldsymbol{q}$ with Dirac functions, but they can more simply be added *ad hoc* on the node on which they are applied. Hence,

$$\boldsymbol{f} = \int_L \boldsymbol{N}^\mathsf{T} \boldsymbol{q}\, dx + \boldsymbol{f}_{\text{discrete}} = \begin{bmatrix} N_1 \\ S_1 \\ M_1 \\ N_2 \\ S_2 \\ M_2 \end{bmatrix} + \begin{bmatrix} N_1 \\ S_1 \\ M_1 \\ N_2 \\ S_2 \\ M_2 \end{bmatrix}_{\text{discrete}}$$

Equivalent Nodal Forces

Note that we have implicitly assumed that n_x and q_y act on the neutral axis and shear centre, respectively.

Conversely, they would generate distributed flexural and torsional moments per unit length, which we have not considered for the sake of simplicity.



Local to Global Transformation

At this point, the results obtained must be transformed into the global reference system.

The procedure is similar to that adopted for bar elements.

In this specific case, the number of Lagrangian parameters in the two reference systems is the same, which is six.

The transformation matrix $\boldsymbol{T}$ will therefore be a 6×6 matrix.

Since rotations lie on an axis perpendicular to the plane of the structure, it will maintain these displacements unchanged.

Local to Global Transformation

$$\boldsymbol{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$