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Aerodynamics and Numerical Simulation Methods

Integral Methods for Turbulent Boundary Layers

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Why integral boundary layer methods? (laminar, turbulent or otherwise!)

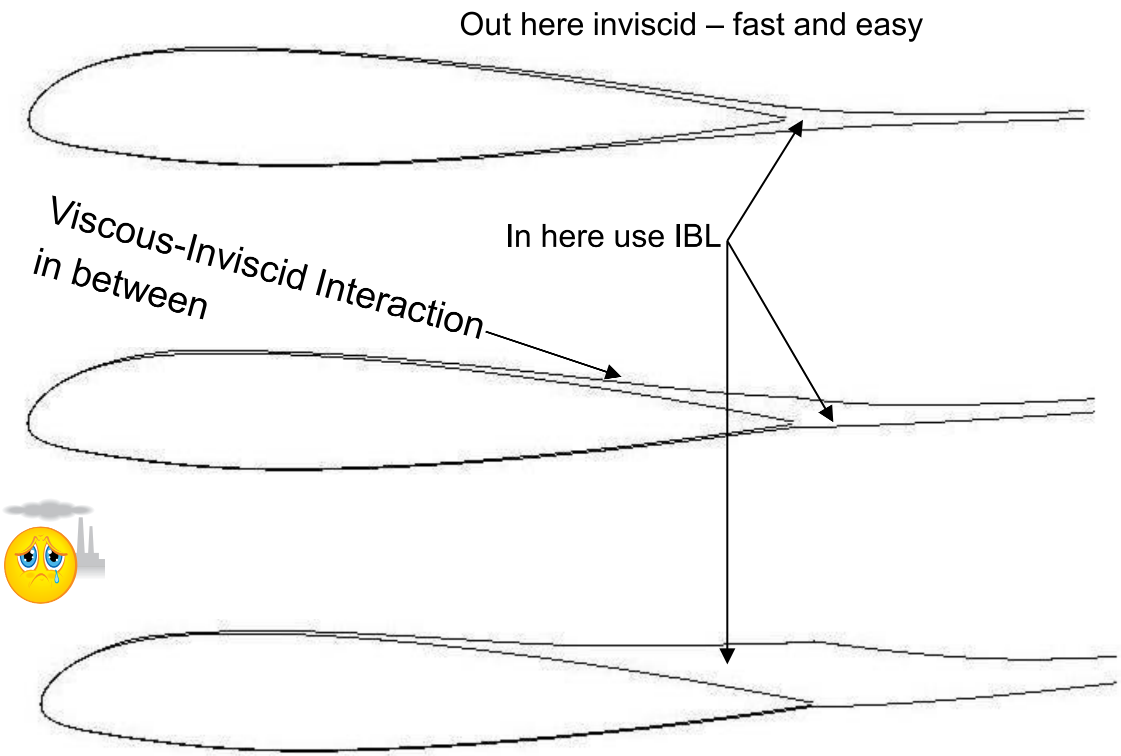
Engineers like

methods that are:

1) Fast 

2) Accurate 

3) Always applicable

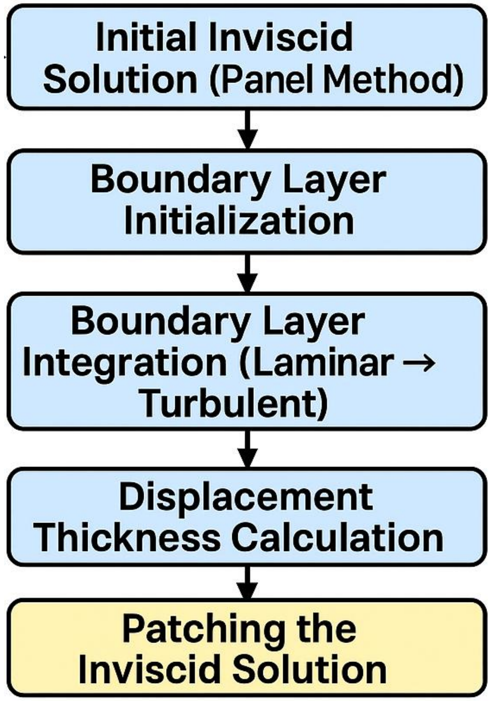


If you couple a simple inviscid panel method to an IBL

method, it is fast and accurate, but not always applicable

2/3 is not bad! **When it works** IBL beats full CFD easily…

Overview of Boundary Layer Patching Strategy in Xfoil



1. ***Initial Inviscid Solution (Panel Method)***
   This provides the surface pressure distribution and velocity field assuming no viscosity.

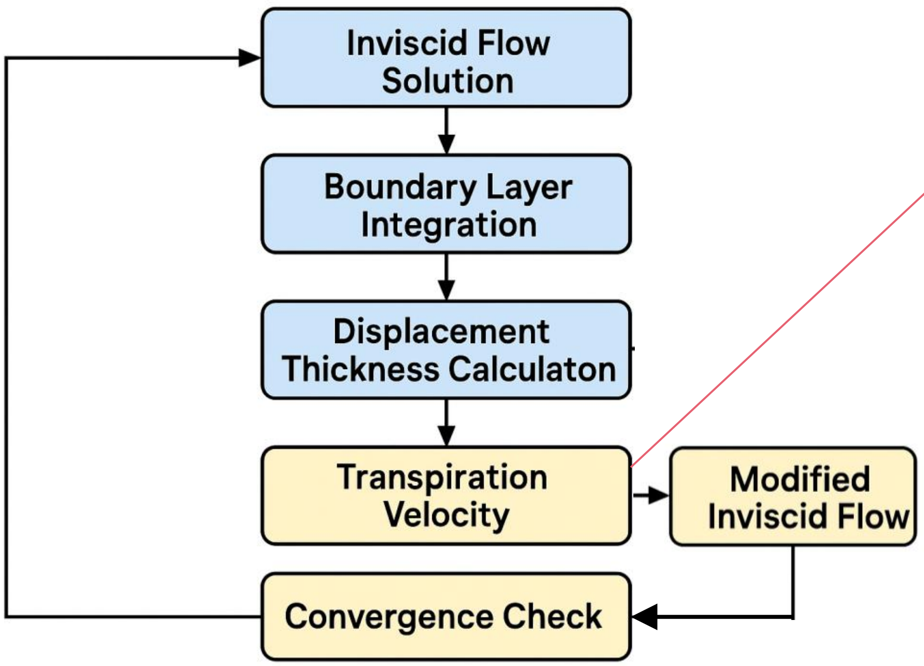
2. ***Boundary Layer Initialization and Integration***
   Using the inviscid surface velocities, XFOIL initializes the boundary layer calculation at the LE and begins integrating the boundary layer equations downstream.

3. ***Displacement Thickness Calculation***
   δ^*, θ modify the effective shape of the airfoil seen by the inviscid flow.

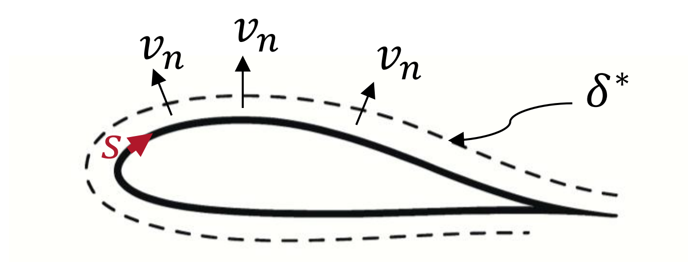
4. ***Patching the Inviscid Flow Solution***
   The inviscid flow is "patched" by modifying the airfoil geometry using δ^*. This is an iterative process.

Iterative coupling with transpiration velocity

The goal: **inviscid outer flow** and the **viscous boundary layer** are consistent.
This is done iteratively, and the **transpiration velocity** is the key mechanism.



$$v_n = \frac{d\delta^*}{ds}$$



- A **normal velocity** added at the surface of the airfoil in the inviscid solver.
- Mimics the effect of the boundary layer pushing the outer flow away from the surface. **New B.C. for inviscid solution**

More info at: https://ntrs.nasa.gov/api/citations/19880003084/downloads/19880003084.pdf

Finding Solutions

- As with laminar flows, we have two options, differential or integral. For the usual reasons, we will look at integral first
  - Quicker
  - Simpler to use
- Due to the greater complexity of the flow, formal analytical solutions do not generally exist, and we shall be considering mainly empirical methods
- In particular, we shall concentrate on Power Law methods

Power Law Methods

- Simplest approximation is to assume a velocity profile.
- From empirical data, it has been found that a *power law* gives a good approximation to a turbulent boundary layer under zero pressure gradient
  - i.e. scale the boundary layer with respect to local height, δ, through the boundary layer variable $\eta = y/\delta$
  - The velocity ratio U/U_e is a power of η

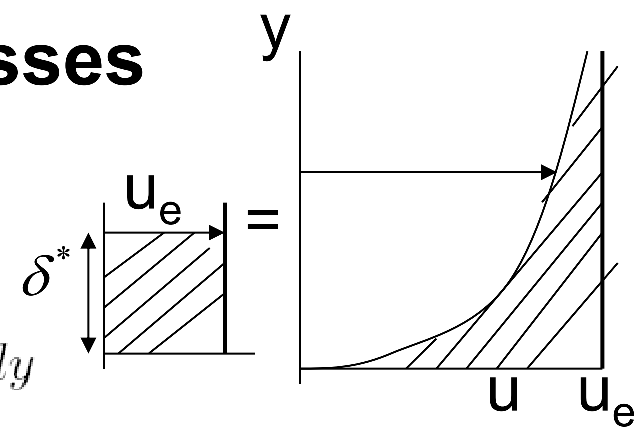
Power Law Methods

- Developed from pipe flow results
  - implicitly assumes velocity profile is *similar* (like Blasius)
  - Hence only really applicable to zero pressure gradient flows
  - Has *no* theoretical underpinnings, i.e. is entirely empirical (unlike Blasius)
- Most common form is $\frac{u}{u_e} = \eta^{\frac{1}{n}}$

Note: This is n and not η

- Where
  - $5 \times 10^5 < Re < 10^7, n = 7$
  - $10^6 < Re < 10^8, n = 9$

Power Law Methods - Thicknesses



Consider usual integral variables:

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$

As we have incompressible flow, $\rho = \rho_e$, and further

$$\eta = \frac{y}{\delta} \Rightarrow \frac{d\eta}{dy} = \frac{1}{\delta} \Rightarrow \boxed{dy = \delta d\eta}$$

we can write this as

Displacement thickness

$$\delta^* = \delta \int_0^1 1 - \frac{u}{u_e} d\eta$$

$$\Rightarrow \frac{\delta^*}{\delta} = \int_0^1 1 - \eta^{\frac{1}{n}} d\eta = \left[\eta - \frac{n}{n+1}\eta^{\frac{n+1}{n}}\right]_0^1 = (1 - \frac{n}{n+1})$$

Momentum thickness

$$\theta = \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy$$

$$\Rightarrow \frac{\theta}{\delta} = \int_0^1 \eta^{\frac{1}{n}} - \eta^{\frac{2}{n}} d\eta = \left[\frac{n}{n+1}\eta^{\frac{n+1}{n}} - \frac{n}{n+2}\eta^{\frac{n+2}{n}}\right]_0^1$$

$$\Rightarrow \frac{\theta}{\delta} = \frac{n}{n+1} - \frac{n}{n+2} = \frac{n}{(n+1)(n+2)}$$

however, we have a problem with skin friction:

$$c_f = \frac{\tau_{wall}}{\frac{1}{2}\rho u_e^2}$$

$$\tau_{wall} = \mu \frac{\partial u}{\partial y} = \mu \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{\mu}{\delta} u_e \frac{1}{n} \eta^{\frac{1-n}{n}} = \frac{\mu u_e}{n\delta} \frac{1}{\eta^{\frac{n-1}{n}}}$$

for any value of n > 1, the second term on the rhs above tends to 1/η, and hence as η is zero at the wall, skin friction is infinite!

Obviously this is a problem with our analysis, not real, and hence skin friction is approximated through empirical relations (both of type $c_f = K_\delta \, \mathrm{Re}_\delta^{\frac{-2}{n+1}}$):

$$c_f = \frac{0.0468}{Re_\delta^{0.25}} \text{for n} = 7, \qquad c_f = \frac{0.0290}{Re_\delta^{0.2}} \text{for n} = 9$$

Use of the Momentum Integral Equation (MIE)

- The derivation of the MIE made no assumptions about the nature of the flow, so the MIE can be used for turbulent flows

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\cancel{\frac{du_e}{dx}}(H+2) = \frac{c_f}{2}$$

- Consider zero pressure gradient (i.e. no velocity gradients). MIE becomes

$$\frac{d\theta}{dx} = \frac{c_f}{2}$$

- We can solve for integral properties in the following manner:

Assume for now n = 7. Use the empirical Blasius Correction for skin friction:

$$c_f = \frac{0.0468}{\mathrm{Re}_\delta^{0.25}} = \frac{0.0468\nu^{0.25}}{\delta^{0.25} u_e^{0.25}}$$

so

$$2\frac{d\theta}{dx} = \frac{0.0468\nu_e^{0.25}}{u_e^{0.25}\delta^{0.25}}$$

3 slides ago we showed that $\dfrac{\theta}{\delta} = \dfrac{n}{(n+1)(n+2)}$

for n = 7, this gives δ=10.29θ. Substituting this for δ:

$$2\frac{d\theta}{dx} = \frac{0.0261\nu_e^{0.25}}{u_e^{0.25}\theta^{0.25}}$$

$$\Rightarrow 2\theta^{0.25} d\theta = \frac{0.0261\nu_e^{0.25}}{u_e^{0.25}} dx$$

integrate:

$$\Rightarrow 1.6\theta^{1.25} = \frac{0.0261\nu_e^{0.25}}{u_e^{0.25}} x$$

divide both sides by 1.6 and raise to the power of 0.8:

$$\theta = \frac{0.0372\nu_e^{0.2}}{u_e^{0.2}} \frac{x}{x^{0.2}}$$

$$\Rightarrow \frac{\theta}{x} = \frac{0.0372}{Re_x^{0.2}}$$

Now, we know one integral property as a function of x, and also as a function of δ: $\frac{\theta}{\delta} = \frac{n}{n+1} - \frac{n}{n+2} = \frac{n}{(n+1)(n+2)}$

All the others follow from relations derived previously

$$\frac{\delta}{x} = \frac{\theta}{x}\frac{\delta}{\theta}, \frac{\delta^*}{x} = \frac{\delta}{x}\frac{\delta^*}{\delta} \qquad \mathrm{Re}_\delta = \mathrm{Re}_x \frac{\delta}{x}$$

Turbulent boundary layer grows proportional to $x^{0.8}$ (from above $=x/x^{0.2}$) compared to $x^{0.5}$ in a laminar one – so turbulent boundary layers are **thicker**.