CHAPTER

Stability

9

9.1 Introduction

Stability is referred to frequently in the foregoing chapters without a formal definition, so it is perhaps useful to revisit the subject in a little more detail in this chapter. Having established the implications of both static and dynamic stability in the context of aircraft response to controls, it is convenient to develop some simple analytical and graphical tools to help in the interpretation of aircraft stability.

9.1.1 A definition of stability

There are many different definitions of stability which are dependent on the kind of system to which they are applied. Fortunately, in the present context the aircraft model is linearised by limiting its motion to small perturbations. The definition of the stability of a linear system is the simplest and most commonly encountered, and it is adopted here for application to the aeroplane. The definition of stability of a linear system may be found in many texts in applied mathematics, in system analysis, and in control theory. A typical definition of the stability of a linear system with particular reference to the aeroplane may be stated as follows.

A system which is initially in a state of static equilibrium is said to be stable if, after a disturbance of finite amplitude and duration, the response ultimately becomes vanishingly small.

Stability is therefore concerned with the nature of the free motion of the system following a disturbance. When the system is linear the nature of the response, and hence its stability, is independent of the nature of the disturbing input. The small-perturbation equations of motion of an aircraft are linear since, by definition, the perturbations are small. Consequently, it is implied that the disturbing input must also be small to preserve that linearity. When, as is often the case, input disturbances which are not really small are applied to the linear small-perturbation equations of motion, some degradation in the interpretation of stability from the observed response must be anticipated. However, for most applications this does not give rise to major difficulties because the linearity of the aircraft model usually degrades relatively slowly with increasing perturbation amplitude. Thus it is considered reasonable to use linear system stability theory for general aircraft applications.

9.1.2 Non-linear systems

Many modern aircraft, especially combat aircraft, which depend on flight control systems for their normal flying qualities, can, under certain conditions, demonstrate substantial non-linearity in their behaviour. This may be due, for example, to large amplitude manoeuvring at the extremes of the flight envelope, where the aerodynamic properties of the airframe are decidedly non-linear. A rather more common source of non-linearity, often found in an otherwise nominally linear aeroplane and often overlooked, arises from the characteristics of common flight control system components. For example, control surface actuators all demonstrate static friction, hysteresis, amplitude and rate limiting to a greater or lesser extent. The non-linear response associated with these characteristics is not normally intrusive unless the demands on the actuator are limiting, such as might be the case in the fly-by-wire control system of a high-performance aircraft.

The mathematical models describing non-linear behaviour are much more difficult to create and the applicable stability criteria are rather more sophisticated; in any event, they are beyond the scope of the present discussion. Non-linear system theory, more popularly known as *chaotic system theory* today, is developing rapidly to provide the mathematical tools, understanding, and stability criteria for dealing with the problems posed by modern highly augmented aircraft.

9.1.3 Static and dynamic stability

Any discussion of stability must consider the total stability of the aeroplane at the flight condition of interest. However, it is usual and convenient to discuss static stability and dynamic stability separately since the related dependent characteristics can be identified explicitly in aircraft behaviour. In reality, static and dynamic stability are inseparable and must be considered as an entity. An introductory discussion of static and dynamic stability is presented in Chapter 3 (Section 3.1), and simple definitions of the two are re-iterated here. Static stability is commonly interpreted to describe the tendency of the aircraft to converge on the initial equilibrium condition following a small disturbance from trim. Dynamic stability describes the transient motion involved in recovery of equilibrium following the disturbance. It is very important that an aeroplane possesses both static and dynamic stability in order for it to be safe. However, the degree of stability is also very important since this determines the effectiveness of the controls of the aeroplane.

9.1.4 Control

By definition, a stable aeroplane is resistant to disturbance; in other words, it attempts to remain at its trimmed equilibrium flight condition. The “strength” of the resistance to disturbance is determined by the *degree* of stability possessed by the aeroplane. It follows, then, that a stable aeroplane is reluctant to respond when a disturbance is deliberately introduced as the result of pilot control action. Thus the degree of stability is critically important to aircraft handling. An aircraft which is very stable requires a greater pilot control action in order to manoeuvre about the trim state, and clearly, too much stability may limit its controllability and hence its manoeuvrability. On the other hand, too little stability in an otherwise stable aeroplane may give rise to an over responsive aeroplane with the resultant pilot tendency to over control. Therefore, too much stability can be as hazardous as too little, and it is essential to place upper and lower bounds on the acceptable degree of stability in order that the aeroplane shall remain completely controllable at all flight conditions.

As described in Chapter 3, the degree of static stability is governed by *cg* position, and this has a significant effect on controllability and the pilot workload. Interpretation of control characteristics as a function of degree of stability and, consequently *cg* position is summarised in Fig. 9.1.

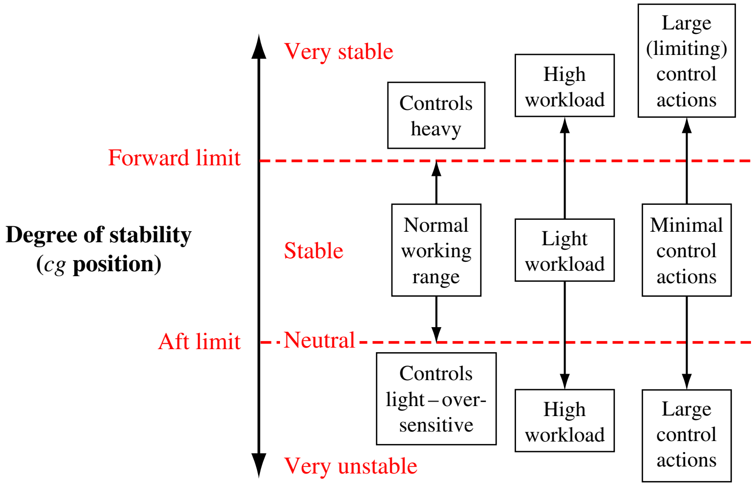

FIGURE 9.1 Stability and control.

In particular, the control action, interpreted as stick displacement and force, becomes larger at the extremes of stability, which has implications for pilot workload. It is also quite possible for very large pilot action to reach the limit of stick displacement or the limit of the pilot's ability to move the control against the force. For this reason, constraints are placed on the permitted *cg* operating range in the aircraft, as discussed in Chapter 3.

The control characteristics are also influenced by the dynamic stability properties, which are governed by *cg* position and also by certain aerodynamic properties of the airframe. This has implications for pilot workload if the dynamic characteristics of the aircraft are not within acceptable limits. However, the dynamic aspects of control are rather more concerned with the time dependency of the response, but in general the observations shown in Fig. 9.1 remain applicable. By reducing the total stability to static and dynamic components, which are further reduced to the individual dynamic modes, it becomes relatively easy to assign the appropriate degree of stability to each mode in order to achieve a safe controllable aeroplane in total. However, this may require the assistance of a command and stability augmentation system, and it may also require control force shaping by means of an artificial feel system.

9.2 The characteristic equation

It was shown in previous chapters that the denominator of every aircraft response transfer function defines the characteristic polynomial, the roots of which determine the stability modes of the aeroplane. Equating the characteristic polynomial to zero defines the classical characteristic equation, and thus far two such equations have been identified. Since only decoupled motion is considered, solution of the equations of motion of the aeroplane results in two fourth-order characteristic equations, one relating to longitudinal symmetric motion and one relating to lateral-directional asymmetric motion. In the event that the decoupled equations of motion provide an inadequate aircraft model, as is often the case for the helicopter, then a single characteristic equation, typically of eighth order, describes the stability characteristics of the aircraft in total.

For aircraft with significant stability augmentation, the flight control system introduces additional dynamics, resulting in a higher order characteristic equation. For advanced combat aircraft the longitudinal characteristic equation, for example, can be of order 30 or more! Interpretation of high order characteristic equations can be something of a challenge for the flight dynamicist.

The characteristic equation of a general system of order n may be expressed in the familiar format as a function of the Laplace operator s:

$$\Delta(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + a_{n-3}s^{n-3} + \cdots\cdots + a_1 s + a_0 = 0 \tag{9.1}$$

and the stability of the system may be determined by the n roots of this equation. Provided that the constant coefficients in equation (9.1) are real, the roots may be real, complex pairs, or a combination of the two. Thus the roots may be written in the general form as follows:

Single real roots—for example, $s = -\sigma_1$ with time solution $k_1 e^{-\sigma_1 t}$
Complex pairs of roots—for example, $s = -\sigma_2 \pm j\gamma_2$ with time solution $k_2 e^{-\sigma_2 t}\sin(\gamma_2 t + \phi_2)$ or, more familiarly, $s^2 + 2\sigma_2 s + (\sigma_2^2 + \gamma_2^2) = 0$

where σ is the real part, γ is the imaginary part, ϕ is the phase angle, and k is a gain constant. When all the roots have negative real parts, the transient component of the response to a disturbance decays to zero as $t \to \infty$ and the system is said to be stable. The system is unstable when any root has a positive real part, and neutrally stable when any root has a zero real part. Thus the stability and dynamic behaviour of any linear system is governed by the sum of the dynamics associated with each root of its characteristic equation. The interpretation of the stability and dynamics of a linear system is summarised in Appendix 6.

9.3 The Routh-Hurwitz stability criterion

The development of a criterion for testing the stability of linear systems is generally attributed to Routh. Application of the criterion involves an analysis of the characteristic equation, and methods for interpreting and applying the criterion are very widely known and used, especially in control systems analysis. A similar analytical procedure for testing the stability of a system by analysis of the characteristic equation was developed simultaneously, and quite independently, by Hurwitz. As a result, both authors share the credit and the procedure is commonly known to control engineers as the Routh-Hurwitz criterion. The criterion provides an analytical means for testing the stability of a linear system of any order without having to obtain the roots of the characteristic equation.

With reference to the typical characteristic equation (9.1), if any coefficient is zero or if any coefficient is negative, then at least one root has a zero or positive real part, indicating the system to be unstable or, at best, neutrally stable. However, it is a necessary but not sufficient condition for stability that all coefficients in equation (9.1) be non-zero and of the same sign. When this condition exists the stability of the system described by the characteristic equation may be tested as follows.

An array, commonly known as the Routh array, is constructed from the coefficients of the characteristic equation arranged in descending powers of s as follows:

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots\cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots\cdots \\
s^{n-2} & u_1 & u_2 & u_3 & u_4 & \cdot \\
s^{n-3} & v_1 & v_2 & v_3 & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & & & & \\
s^1 & y & & & & \\
s^0 & z & & & &
\end{array}
\tag{9.2}
$$

The first row is written to include alternate coefficients starting with the highest power term; the second row includes the remaining alternate coefficients starting with the second highest-power term as indicated. The third row is constructed as follows:

$$u_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \quad u_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \quad u_3 = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

and so on, until all remaining u are zero. The fourth row is constructed similarly from coefficients in the two rows immediately above as follows:

$$v_1 = \frac{u_1 a_{n-3} - u_2 a_{n-1}}{u_1} \quad v_2 = \frac{u_1 a_{n-5} - u_3 a_{n-1}}{u_1} \quad v_3 = \frac{u_1 a_{n-7} - u_4 a_{n-1}}{u_1}$$

and so on, until all remaining v are zero. This process is repeated until all remaining rows of the array are completed. The array is triangular as indicated, and the last two rows comprise only one term each, y and z, respectively.

The Routh-Hurwitz criterion states,

The number of roots of the characteristic equation with positive real parts (unstable) is equal to the number of changes of sign of the coefficients in the first column of the array.

Thus, for the system to be stable, all the coefficients in the first column of the array must have the same sign.

EXAMPLE 9.1

The lateral-directional characteristic equation for the Douglas DC-8 aircraft in a low-altitude cruise flight condition, obtained from Teper (1969), is

$$\Delta(s) = s^4 + 1.326s^3 + 1.219s^2 + 1.096s - 0.015 = 0 \tag{9.3}$$

Inspection of this equation indicates an unstable aeroplane since the last coefficient has a negative sign. The number of unstable roots may be determined by constructing the array as described earlier:

$$\begin{array}{c|ccc} s^4 & 1 & 1.219 & -0.015 \\ s^3 & 1.326 & 1.096 & 0 \\ s^2 & 0.393 & -0.015 & 0 \\ s^1 & 1.045 & 0 & 0 \\ s^0 & -0.015 & 0 & 0 \end{array} \tag{9.4}$$

Working down the first column, there is one sign change, from 1.045 to -0.015, which indicates the characteristic equation has one unstable root. This is verified by obtaining the exact roots of the characteristic equation (9.2):

$$\begin{aligned} s &= -0.109 \pm 0.99j \\ s &= -1.21 \\ s &= +0.013 \end{aligned} \tag{9.5}$$

The complex pair of roots with negative real parts describes the stable dutch roll mode; the real root with negative real part describes the stable roll subsidence mode; and the real root with positive real part describes the unstable spiral mode. This is a typical solution for a classical aeroplane.

9.3.1 Special cases

Two special cases which may arise in the application of the Routh-Hurwitz criterion need to be considered, although they are unlikely to occur in aircraft applications. The first case occurs when, in the routine calculation of the array, a coefficient in the first column is zero. The second case occurs when, also in the routine calculation of the array, all coefficients in a row are zero. In either case no further progress is possible and an alternative procedure is required. The methods for dealing with these cases are best illustrated by example.

EXAMPLE 9.2

Consider the arbitrary characteristic equation

$$\Delta(s) = s^4 + s^3 + 6s^2 + 6s + 7 = 0 \tag{9.6}$$

the array for which is constructed in the usual way:

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 7 \\ s^3 & 1 & 6 & 0 \\ s^2 & \varepsilon & 7 & 0 \\ s^1 & \left(\dfrac{6\varepsilon - 7}{\varepsilon}\right) & 0 & 0 \\ s^0 & 7 & 0 & 0 \end{array} \tag{9.7}$$

Normal progress cannot be made beyond the third row since the first coefficient is zero. To proceed, the zero is replaced with a small positive number, denoted ε. The array can be completed as at (9.7) and as $\varepsilon \to 0$, so the first coefficient in the fourth row tends to a large negative value. The signs of the coefficients in the first column are then easily determined:

$$\begin{array}{c|c} s^4 & + \\ s^3 & + \\ s^2 & + \\ s^1 & - \\ s^0 & + \end{array} \tag{9.8}$$

There are two changes of sign, from row 3 to row 4 and from row 4 to row 5. Therefore, the characteristic equation (9.6) has two roots with positive real parts, which is verified by the exact solution,

$$\begin{aligned} s &= -0.6454 \pm 0.9965j \\ s &= +0.1454 \pm 2.224j \end{aligned} \tag{9.9}$$

EXAMPLE 9.3

To illustrate the required procedure when all coefficients in a row of the array are zero, consider the arbitrary characteristic equation

$$\Delta(s) = s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 6 = 0 \tag{9.10}$$

If the array is constructed in the usual way,

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 6 \\ s^3 & 0 & 0 & \end{array} \tag{9.11}$$

no further progress is possible because the third row comprises all zeros. To proceed, the zero row, the third row in this example, is replaced by an auxiliary function derived from the preceding non-zero row. Thus the function is created from the row commencing with the coefficient of s to the power of four as follows:

$$2s^4 + 8s^2 + 6 = 0 \text{ or, equivalently, } s^4 + 4s^2 + 3 = 0 \tag{9.12}$$

Only terms in alternate powers of s are included in the auxiliary function (9.12), commencing with the highest power term determined from the row of the array from which it is derived. The auxiliary function is differentiated with respect to s, and the resulting polynomial is used to replace the zero row. Equation (9.12) is differentiated to obtain

$$4s^3 + 8s = 0 \text{ or, equivalently, } s^3 + 2s = 0 \tag{9.13}$$

Substituting equation (9.13) into the third row of array (9.11), the array may then be completed in the usual way:

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 2 & 8 & 6 \\ s^3 & 1 & 2 & 0 \\ s^2 & 4 & 6 & 0 \\ s^1 & 0.5 & 0 & 0 \\ s^0 & 6 & 0 & 0 \end{array} \tag{9.14}$$

Inspection of the first column of (9.14) indicates that all roots of the characteristic equation (9.10) have negative real parts. However, the fact that, in the derivation of the array, one row comprises zero coefficients suggests that something is different. The exact solution of equation (9.10) confirms this suspicion:

$$\begin{aligned} s &= 0 \pm 1.732j \\ s &= 0 \pm 1.0j \\ s &= -2.0 \end{aligned} \tag{9.15}$$

Clearly the system is neutrally stable since the two pairs of complex roots both have zero real parts.

9.4 The stability quartic

Since both the longitudinal and lateral-directional characteristic equations derived from the aircraft small perturbation equations of motion are fourth order, considerable emphasis has always been placed on the solution of a fourth order polynomial, sometimes referred to as the *stability quartic*. A general quartic equation applicable to either longitudinal or lateral-directional motion may be written as

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0 \tag{9.16}$$

When all of the coefficients in equation (9.16) are positive, as is often the case, no conclusions may be drawn concerning stability unless the roots are found or the Routh-Hurwitz array is constructed. Constructing the Routh-Hurwitz array as described in Section 9.3, we obtain

$$\begin{array}{c|ccc} s^4 & A & C & E \\ s^3 & B & D & \\ s^2 & \left(\dfrac{BC-AD}{B}\right) & E & \\ s^1 & \left(\dfrac{D(BC-AD)-B^2E}{BC-AD}\right) & & \\ s^0 & E & & \end{array} \tag{9.17}$$

Assuming that all of the coefficients in the characteristic equation (9.16) are positive and that B and C are large compared with D and E, as is usually the case, then the coefficients in the first column of (9.17) are also positive with the possible exception of the coefficient in the fourth row. Writing

$$R = D(BC - AD) - B^2E \tag{9.18}$$

R is called *Routh's discriminant* and, since $(BC - AD)$ is positive, the outstanding condition for stability is

$$R > 0$$

For most classical aircraft operating within the constraints of small perturbation motion, the only coefficient in the characteristic equation (9.16) likely to be negative is E. Thus, typically, the necessary and sufficient conditions for an aeroplane to be stable are

$$R > 0 \quad \text{and} \quad E > 0$$

When an aeroplane is unstable some conclusions about the nature of the instability can be made simply by observing the values of R and E.

9.4.1 Interpretation of conditional instability

Case 1: When $R < 0$ and $E > 0$.
Observation of the signs of the coefficients in the first column of array (9.17) indicates that two roots of the characteristic equation (9.16) have positive real parts. For longitudinal motion this implies a pair of complex roots; in most cases this means an unstable phugoid mode since its stability margin is usually the smallest. For lateral-directional motion the implication is that either the two real roots or the pair of complex roots have positive real parts. This means either that the spiral and roll subsidence modes are unstable or that the dutch roll mode is unstable. Within the limitations of small-perturbation modeling, an unstable roll subsidence mode is not possible. Therefore, the instability must be determined by the pair of complex roots describing the dutch roll mode.

Case 2: When $R < 0$ and $E < 0$.
For this case, the signs of the coefficients in the first column of array (9.17) indicate that only one root of the characteristic equation (9.16) has a positive real part. Clearly, the "unstable" root can only be a real root. For longitudinal motion this may be interpreted to mean that the phugoid mode has changed such that it is no longer oscillatory and is therefore described by a pair of real roots, one of which has a positive real part. The "stable" real root typically describes an exponential heave characteristic, whereas the "unstable" root describes an exponentially divergent speed mode. For lateral-directional motion the interpretation is similar, and in this case the only "unstable" real root must be that describing the spiral mode. This, of course, is a commonly encountered condition in lateral-directional dynamics.

Case 3: When $R > 0$ and $E < 0$.
As with the previous case, the signs of the coefficients in the first column of array (9.17) indicate that only one root of the characteristic equation (9.16) has a positive real part. Again, the "unstable" root can only be a real root. Interpretation of the stability characteristics corresponding with this particular condition is exactly the same as described in *Case 2*.

When all coefficients in the characteristic equation (9.16) are positive and R is negative, the instability can only be described by a pair of complex roots, the interpretation of which is described in *Case 1*. Since the unstable motion is oscillatory, the condition $R > 0$ is sometimes referred to as the criterion for dynamic stability. Alternatively, the most common unstable condition arises when the coefficients in the characteristic equation (9.16) are positive with the exception of E. In this case the instability can only be described by a single real root, the interpretation of which is described in *Case 3*. Now, the instability is clearly identified as a longitudinal speed divergence or as the divergent lateral-directional spiral mode, both of which are dynamic characteristics. However, the aerodynamic contribution to E is substantially dependent on static stability effects, and when $E < 0$ the cause is usually static instability. Consequently, the condition $E > 0$ is sometimes referred to as the criterion for static stability. This simple analysis emphasises the role of the characteristic equation in describing the *total stability* of the aeroplane and reinforces the reason that, in reality, static and dynamic stability are inseparable and should not be considered without referencing one to the other.

9.4.2 Interpretation of the coefficient *E*

Assuming the longitudinal equations of motion to be referred to a system of aircraft wind axes, the coefficient E in the longitudinal characteristic equation may be obtained directly from Appendix 3 as

$$E = mg\left(\overset{\circ}{M}_w \overset{\circ}{Z}_u - \overset{\circ}{M}_u \overset{\circ}{Z}_w\right) \tag{9.19}$$

and the longitudinal static stability criterion may be expressed in terms of dimensionless derivatives as

$$M_w Z_u > M_u Z_w \tag{9.20}$$

For most aeroplanes the derivatives in equation (9.20) have negative values so that the terms on either side of the inequality are usually both positive. M_w is a measure of the controls-fixed longitudinal static stability margin; Z_u is largely dependent on lift coefficient; Z_w is dominated by lift curve slope; and M_u only assumes significant values at a high Mach number. Thus, provided the aeroplane possesses a sufficient margin of controls-fixed longitudinal static stability, M_w is sufficiently large to ensure that inequality (9.20) is satisfied. At higher Mach numbers, when M_u becomes larger, the inequality is generally maintained because the associated aerodynamic changes also cause M_w to increase.

The coefficient E in the lateral-directional characteristic equation may similarly be obtained directly from Appendix 3 as

$$E = mg\left(\overset{\circ}{L}_v \overset{\circ}{N}_r - \overset{\circ}{L}_r \overset{\circ}{N}_v\right) \tag{9.21}$$

and the lateral-directional static stability criterion may be expressed in terms of dimensionless derivatives as

$$L_v N_r > L_r N_v \tag{9.22}$$

For most aeroplanes the derivatives L_v and N_r are both negative, the derivative L_r is usually positive, and the derivative N_v is always positive. Thus the terms on either side of inequality (9.22) are usually both positive. Satisfaction of the inequality is usually determined by the relative magnitudes of the derivatives L_v and N_v. Now, L_v and N_v are the derivatives describing the lateral and directional controls-fixed static stability of the aeroplane, respectively, as discussed in Sections 3.4.1 and 3.4.2. The magnitude of the derivative L_v is determined by the *lateral dihedral effect*, and the magnitude of the derivative N_v is determined by the *directional weathercock effect*. Inequality (9.22) also determines the condition for a stable spiral mode as described in Section 7.3.2 and, once again, the inseparability of static and dynamic stability is illustrated.

9.5 Graphical interpretation of stability

Today, the foregoing analysis of stability is of limited practical value since all of the critical information is normally obtained in the process of solving the equations of motion exactly and directly using suitable computer software tools, as described elsewhere. However, its greatest value is in the understanding and interpretation of stability it provides. Of much greater practical value are the graphical tools much favoured by the control engineer for the interpretation of stability on the *s-plane*.

9.5.1 Root mapping on the *s*-plane

The roots of the characteristic equation are either real or complex pairs, as stated in Section 9.2. The possible forms of the roots may be mapped onto the *s*-plane as shown in Fig. 9.2. Since the roots describe various dynamic and stability characteristics possessed by the system to which they relate, the location of the roots on the *s*-plane conveys the same information in a highly accessible

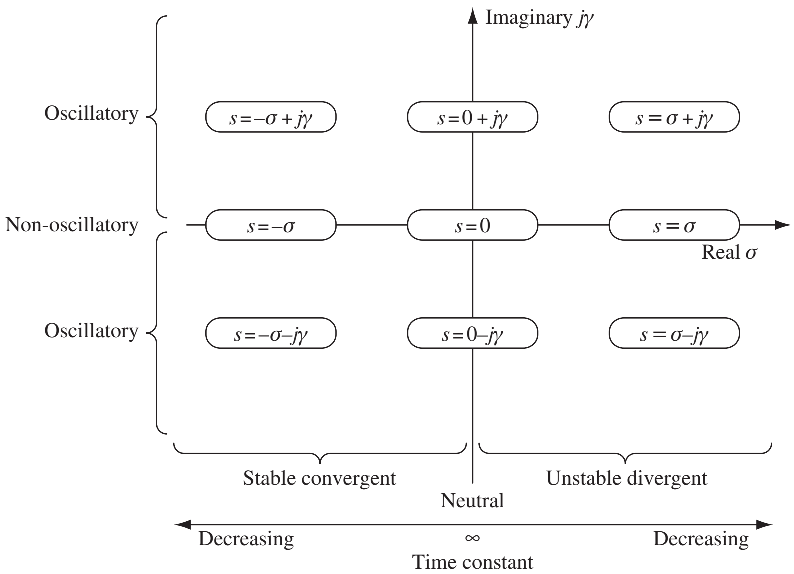

FIGURE 9.2 Roots on the *s*-plane.

form. "Stable" roots have negative real parts and lie on the left half of the s-plane; "unstable" roots have positive real parts and lie on the right half of the s-plane; roots describing neutral stability have zero real parts and lie on the imaginary axis. Complex roots lie in the upper half of the s-plane; their conjugates lie in the lower half, and, since their locations are mirrored in the real axis, it is usual to show the upper half of the plane only. Complex roots describe oscillatory motion, so all roots lying in the plane and not on the real axis describe such characteristics. Roots lying on the real axis describe non-oscillatory motions, the time constants of which are given by $T = 1/\sigma$. A root lying at the origin is therefore neutrally stable and has an infinite time constant. As real roots move away from the origin, their time constants decrease—in the stable sense on the left half of the plane and in the unstable sense on the right half of the plane.

Consider the interpretation of a complex pair of roots on the s-plane in rather greater detail. As stated in Section 9.2, the typical pair of complex roots may be written as

$$(s + \sigma + j\gamma)(s + \sigma - j\gamma) = s^2 + 2\sigma s + (\sigma^2 + \gamma^2) = 0 \tag{9.23}$$

which is equivalent to the familiar expression

$$s^2 + 2\zeta\omega s + \omega^2 = 0 \tag{9.24}$$

Whence

$$\begin{aligned} \zeta\omega &= \sigma \\ \omega^2 &= \sigma^2 + \gamma^2 \\ \zeta &= \cos\phi = \frac{\sigma}{\sqrt{\sigma^2 + \gamma^2}} \end{aligned} \tag{9.25}$$

where ϕ is referred to as the *damping angle*. This information is readily interpreted on the s-plane as shown in Fig. 9.3. The complex roots of equation (9.23) are plotted at p, with only the upper half of

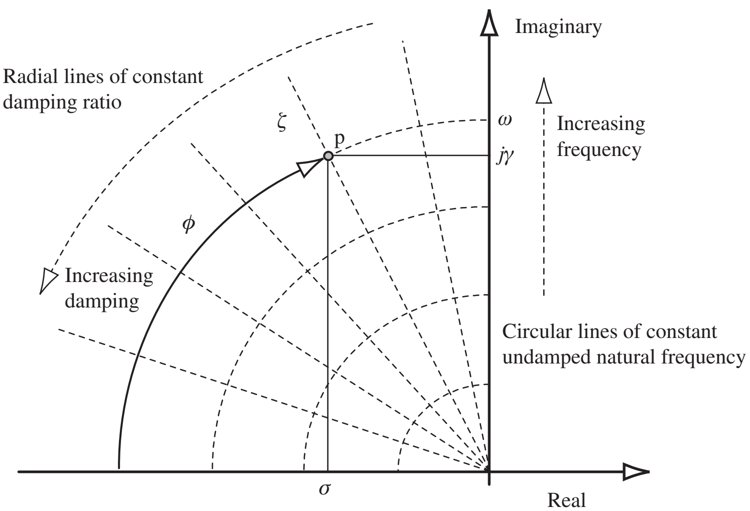

FIGURE 9.3 Typical complex roots on the s-plane.

the s-plane being shown since the lower half containing the complex conjugate root is a *mirror image* in the real axis. With reference to equations (9.24) and (9.25), it is evident that undamped natural frequency is given by the magnitude of the line joining the origin and the point p. Thus lines of constant frequency are circles concentric with the origin provided that both axes have the same scales. Care should be exercised when the scales are dissimilar, which is often the case, as the lines of constant frequency then become ellipses. Thus, clearly, roots indicating low frequency dynamics are near the origin and vice versa.

Whenever possible, it is good practice to draw s-plane plots and *root locus plots* on axes having the same scales to facilitate the easy interpretation of frequency. With reference to equations (9.25), it is evident that radial lines drawn through the origin are lines of constant damping. The imaginary axis then becomes a line of zero damping and the real axis becomes a line of *critical damping* where the damping ratio is unity and the roots become real. The upper left quadrant of the s-plane shown in Fig. 9.3 contains the stable region of positive damping ratio in the range $0 \le \zeta \le 1$ and is therefore the region of critical interest in most practical applications. Thus roots indicating stable well damped dynamics are seen towards the left of the region and *vice versa*. Thus, information about the dynamic behaviour of a system is instantly available on inspection of the roots of its characteristic equation on the *s-plane*. The interpretation of the stability of an aeroplane on the s-plane becomes especially useful for the assessment of stability augmentation systems on the *root locus plot* described in Chapter 11.

EXAMPLE 9.4

The Boeing B-747 is typical of a large classical transport aircraft, and the following characteristics were obtained from Heffley and Jewell (1972). The flight case chosen is representative of typical cruising flight at Mach 0.65 at an altitude of 20,000 ft. The longitudinal characteristic equation is

$$\Delta(s) = s^4 + 1.1955s^3 + 1.5960s^2 + 0.0106s + 0.00676 \tag{9.26}$$

with roots

$$\begin{aligned} s &= -0.001725 \pm 0.0653j \\ s &= -0.596 \pm 1.1101j \end{aligned} \tag{9.27}$$

describing stability mode characteristics

$$\begin{aligned} \omega_p &= 0.065 \text{ rad/s} \quad \zeta_p = 0.0264 \\ \omega_s &= 1.260 \text{ rad/s} \quad \zeta_s = 0.4730 \end{aligned} \tag{9.28}$$

The corresponding lateral-directional characteristic equation is

$$\Delta(s) = s^4 + 1.0999s^3 + 1.3175s^2 + 1.0594s + 0.01129 \tag{9.29}$$

with roots

$$\begin{aligned} s &= -0.0108 \\ s &= -0.9130 \\ s &= -0.0881 \pm 1.0664j \end{aligned} \tag{9.30}$$

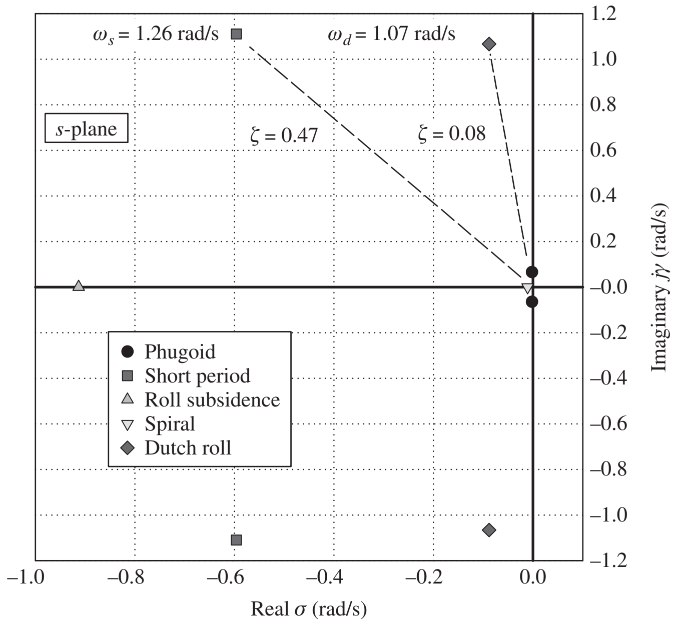

FIGURE 9.4 Boeing B-747 stability modes on the *s*-plane.

describing stability mode characteristics

$$\begin{aligned} T_s &= 92.6\,\text{s} \\ T_r &= 1.10\,\text{s} \\ \omega_d &= 1.070\ \text{rad/s} \quad \zeta_d = 0.082 \end{aligned} \tag{9.31}$$

The longitudinal roots given by equation (9.27) and the lateral-directional roots given by equation (9.30) are mapped on to the *s*-plane as shown in Fig. 9.4. The plot is absolutely typical for a large number of aeroplanes and shows the *stability modes*, represented by their corresponding roots, on regions of the *s*-plane normally associated with the modes. For example, the *slow modes*, the phugoid and the spiral, are clustered around the origin, whereas the *faster modes* are further out in the plane. Since the vast majority of aeroplanes have longitudinal and lateral-directional control bandwidths of less than 10 rad/s, the scales of the *s*-plane plot normally lie in the range -10 rad/s < real < 0 rad/s and -10 rad/s < imaginary < 10 rad/s. Clearly, the control bandwidth of the B-747 at the chosen flight condition is a little over 1 rad/s, as might be expected for such a large aeroplane. The important observation to be made from this illustration is the relative locations of the stability mode roots on the *s*-plane since they are quite typical of many aeroplanes.

References

Heffley, R. K., & Jewell, W. F. (1972). *Aircraft handling qualities data.* NASA Contractor Report, NASA CR-2144. Washington, DC: National Aeronautics and Space Administration.

Teper, G. L. (1969). *Aircraft stability and control data.* Systems Technology, STI Technical Report 176-1. Hawthorne, CA: Systems Technology, Inc.

PROBLEMS

9.1 By applying the Routh-Hurwitz stability criterion to the longitudinal characteristic equation, show that the minimum condition for stability, assuming a conventional aircraft, is $E > 0$ and $R > 0$, where the Routh discriminant R is given by

$$R = (BC - AD)D - B^2E$$

Using these conditions, test the longitudinal stability of the aircraft whose dimensional characteristics equation is

$$s^4 + 5.08s^3 + 13.2s^2 + 0.72s + 0.52 = 0$$

Verify your findings by obtaining the approximate solution of the equation. Describe in detail the characteristic longitudinal modes of the aircraft.

(CU 1982)

9.2 The Republic F-105B Thunderchief aircraft, data for which are given in Teper (1969), has a wing span of 10.6 m and is flying at a speed of 518 kts at an altitude where the lateral relative density parameter is $\mu_2 = 221.46$. The dimensionless controls-fixed lateral-directional stability quartic is

$$\lambda^4 + 29.3\lambda^3 + 1052.7\lambda^2 + 14913.5\lambda - 1154.6 = 0$$

(a) Using the Routh-Hurwitz stability criterion, test the lateral-directional stability of the aircraft.

(b) Given that the time constant of the spiral mode is $T_s = 115$ s and the time constant of the roll subsidence mode is $T_r = 0.5$ s, calculate the characteristics of the remaining mode. Determine the time to half or double amplitude of the non-oscillatory modes and hence describe the physical characteristics of the lateral-directional modes of motion of the aircraft.

(CU 1982)

9.3 The longitudinal characteristic equation for an aircraft may be written

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0$$

It may be assumed that it describes the usual short-period pitching oscillation and phugoid. State the Routh-Hurwitz stability criterion and thus show that for a conventional aircraft the conditions for stability are, $B > 0$, $D > 0$, $E > 0$, and $R > 0$, where Routh's discriminant R is given by

$$R = BCD - AD^2 - B^2E$$

Describe the significance of the coefficient E on the stability of the aircraft by considering the form of the roots of the quartic when E is positive, negative, or zero.

(CU 1985)

9.4 Explain the Routh-Hurwitz stability criterion as it might apply to the following typical aircraft stability quartic:

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0$$

What is Routh's discriminant R? Explain the special significance of R and of the coefficient E in the context of the lateral-directional stability characteristics of an aircraft.

(a) The coefficients of the lateral-directional stability quartic of an aircraft are
$A = 1 \quad B = 9.42 \quad C = 9.48 + N_v \quad D = 10.29 + 8.4N_v$
$E = 2.24 - 0.39N_v$

Find the range of values of N_v for which the aircraft will be both statically and dynamically stable. What do the limits on N_v mean in terms of the dynamic stability characteristics of the aircraft, and on what do they depend?

(CU 1987)

9.5 An unstable fly-by-wire combat aircraft has the longitudinal characteristic equation

$$s^4 + 36.87s^3 - 4.73s^2 + 1.09s - 0.13 = 0$$

(a) Test its stability using the Routh-Hurwitz criterion.

(b) The roots of the characteristic equation are $s = 0.0035 \pm 0.1697j$, $s = 0.122$, and $s = -37.0$. Describe the longitudinal stability modes of the aircraft.

(CU 1989)