6. d'Alembert's method

d'Alembert's method

The separation of variables method is one way of finding solutions of the wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

It is well suited to the case where we have **boundary conditions**. Then the spatial wavelength is determined by the boundary conditions. For example, what is the note played by the guitar string, the organ pipe or a percussive instrument?

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

The solution is said to be a **standing wave**.

General solution of the wave equation

This is not a viable method if the domain over which we solve the PDE is infinite. Of course, nothing is truly infinite. Really, what we mean by an infinite domain is a long domain in which the boundaries are far away and cannot influence the wave length.

For example, what is the shape of ripples if we drop a stone into a pond? How do stop-go waves propagate in motorway traffic? How do acoustic waves travel in the ocean? How do waves propagate along a long cable — in a cable stayed bridge, or bacterial flagellum?

The kind of solution we are looking for is a **travelling wave**.

Travelling wave solution of the wave equation

In order to find travelling wave solutions of the wave equation we note the following.

Theorem (d'Alembert's solution of the wave equation)

The function $u(x,t)$ given by

$$u(x,t) = \boxed{f(x-ct)} + \boxed{g(x+ct)}$$

is a solution of the wave equation for any functions f and g. It is made up of two travelling waves, with:

1. fixed shape f, moving to the right, at speed c
2. fixed shape g, moving to the left, at speed c

Travelling wave solution: proof

We just substitute the d'Alembert solution into the wave equation.

To do so we need to find the partial derivatives of $u(x, t)$ with respect to x and t.

If $u(x, t) = f(x - ct) + g(x + ct)$, then

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} f(x - ct) + \frac{\partial}{\partial t} g(x + ct)$$

make a change of variables $\xi = x - ct$, $\eta = x + ct$, and use the chain rule

$$\begin{aligned}
&= \frac{\partial}{\partial t} f(\xi) + \frac{\partial}{\partial t} g(\eta) \\
&= \frac{\partial \xi}{\partial t} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) + \frac{\partial \eta}{\partial t} \frac{\mathrm{d}}{\mathrm{d}\eta} g(\eta) \\
&= -cf'(\xi) + cg'(\eta) \\
&= -cf'(x - ct) + cg'(x + ct)
\end{aligned}$$

Travelling wave solution: proof (2)

We use the same technique to find all the partial derivatives of $u(x,t) = f(x-ct) + g(x+ct)$ with respect to x and t:

$$\begin{aligned} u_t &= -cf'(x-ct) + cg'(x+ct) \\ u_{tt} &= c^2 f''(x-ct) + c^2 g''(x+ct) \\ u_x &= f'(x-ct) + g'(x+ct) \\ u_{xx} &= f''(x-ct) + g''(x+ct) \end{aligned}$$

Hence

$$u_{tt} = c^2 f''(x-ct) + c^2 g''(x+ct) = c^2 \left[f''(x-ct) + g''(x+ct) \right] = c^2 u_{xx}$$

$\square$

Quick quiz #1

Why is $f(x-ct)$ a travelling wave?

Sketch graphs of $u = \mathrm{e}^{-(x-ct)^2}$ *against* x*, at the following (fixed) values of* t*:* $t = 0$, $t = 1/c$, $t = 2/c$, *and* $t = 3/c$. *How far does the wave travel between each of the graphs? What is the wave's speed? In which direction is the wave travelling?*

Summary

- The d'Alembert solution of the wave equation is, for any functions f and g,

$$u(x,t) = f(x-ct) + g(x+ct)$$

- It's made up of a pair of travelling waves
- The d'Alembert solution has two 'unknowns': the functions f and g
- We'll use the initial and boundary conditions to find f and g
- The process will be slightly different for
  - waves on infinite domains (dropping a stone in a pond)
  - waves on semi-infinite domains (wave propagation along a flagellum)

Walk-through example 1

Solving the wave equation on an infinite domain

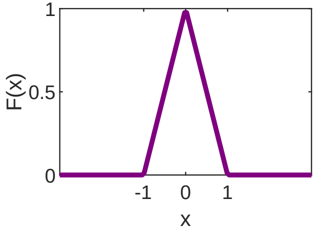
Find the solution $u(x,t)$ of the wave equation on an infinite domain

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty, \quad t > 0$$

subject to the initial conditions

$$u(x,0) = F(x) = \begin{cases} 1+x & x \in [-1,0) \\ 1-x & x \in [0,1] \\ 0 & \textit{otherwise} \end{cases}$$

$$u_t(x,0) = 0, \quad \textit{for all } x \in \mathbb{R}$$



and (implicit) boundary conditions 'at infinity'

$$u(x,t) \to 0 \qquad \textit{as } x \to \pm\infty \textit{ for all } t > 0$$

Step 1: state the d'Alembert solution

The d'Alembert (travelling wave) solution of the wave equation is

$$u(x,t) = f(x - ct) + g(x + ct)$$

- No need to prove this every time you use it
- It only works for infinite (or semi-infinite) domains
- For finite domains (e.g. $0 \leqslant x \leqslant L$) with two boundary conditions you have to use separation of variables instead

Step 2: use the initial conditions

First we take the zero-velocity initial condition

$$0 = u_t(x, 0) = \left[-cf'(x - ct) + cg'(x + ct)\right]_{t=0},$$

which implies that, for all x,

$$-f'(x) + g'(x) = 0.$$

Integrate this expression with respect to x to get

$$-f(x) + g(x) = K$$

for some constant K.

Step 2: use the initial conditions

Now, applying the initial displacement condition we get

$$F(x) = u(x, 0) = [f(x - ct) + g(x + ct)]_{t=0} = f(x) + g(x),$$

for all x, where $F(x)$ is the known initial displacement function.

Hence we have two simultaneous equations, true for all x, for the two unknown functions f and g:

$$-f(x) + g(x) = K, \tag{1}$$

$$f(x) + g(x) = F(x). \tag{2}$$

Step 3: solve for f and g

To solve the two simultaneous equations (1) and (2) for the unknown functions f and g, first add them together, which gives

$$2g(x) = F(x) + K \qquad \Rightarrow \qquad g(x) = \frac{1}{2}F(x) + \frac{K}{2}$$

Then rearrange (1) to find $f(x)$

$$f(x) = g(x) - K = \frac{1}{2}F(x) + \frac{K}{2} - K = \frac{1}{2}F(x) - \frac{K}{2}$$

So now we know the two functions f and g

$$\boxed{f(x) = \frac{1}{2}F(x) - \frac{K}{2}} \qquad \boxed{g(x) = \frac{1}{2}F(x) + \frac{K}{2}}$$

Step 4: recombine to get general solution

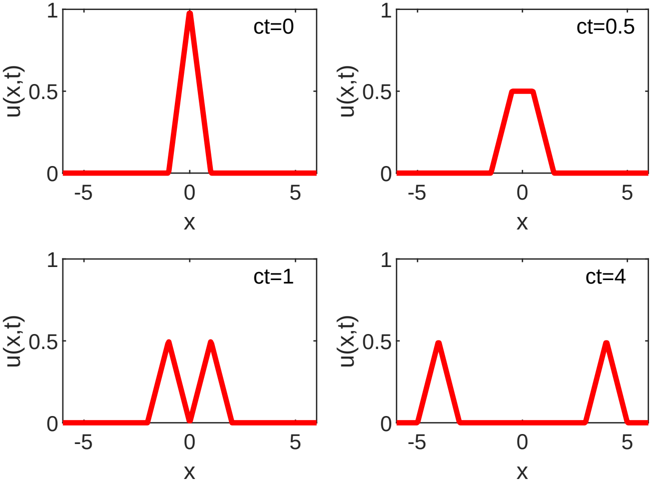
Substituting the expressions for f and g into the d'Alembert solution gets us $u(x,t)$:

$$\begin{aligned} u(x,t) &= f(x-ct) + g(x+ct) \\ &= \frac{1}{2}F(x-ct) - \frac{K}{2} + \frac{1}{2}F(x+ct) + \frac{K}{2} \end{aligned}$$

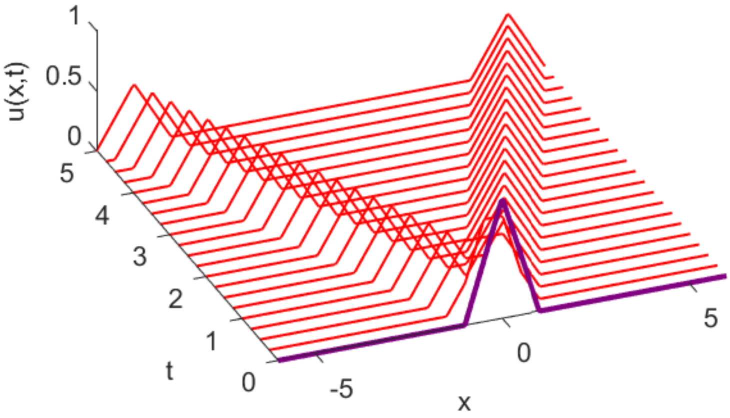
$$\boxed{u(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct)}$$

where $F(x) = \begin{cases} 1+x & -1 \leqslant x < 0 \\ 1-x & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

Plots of $u(x,t)$ for $ct = 0,\ 0.5,\ 1,\ 4$:



Step 5: plot the solution profile



Find the general solution of the wave equation on an infinite domain

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty, \quad t \geqslant 0,$$

subject to the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = x\,\mathrm{e}^{-x^2}, \qquad \text{for all } x \in \mathbb{R}$$

Sketch the solution profile at a (fixed) time $t > 0$.

d'Alembert method with a semi-infinite domain

Suppose we wish to solve the wave equation on a semi-infinite domain

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \infty, \quad t > 0,$$

This time, the process is slightly more involved.

- Again we will use the d'Alembert solution $u(x,t) = f(x-ct) + g(x+ct)$
- First, apply the initial conditions to find the functions f and g, as before. *However, the solution we get will only be valid for $x > ct$.*
- To find the solution for $0 < x < ct$ we will need to apply the boundary condition (known u or u_x at $x = 0$ for all $t > 0$)
- Then patch the two pieces together

Worked example 6.2

Solving the wave equation on a semi-infinite domain

Find the solution of the wave equation on a semi-infinite domain

$$\boxed{u_{tt} = c^2 u_{xx}} \qquad \boxed{0 < x < \infty, \quad t > 0}$$

subject to the initial conditions

$$\boxed{u(x,0) = 0, \quad u_t(x,0) = 0, \qquad \text{for all } x > 0}$$

and boundary conditions

$$\boxed{u(0,t) = \sin(\omega t), \qquad \text{for all } t > 0}$$

and (implicitly)

$$\boxed{u(x,t) \to 0 \text{ as } x \to \infty, \qquad \text{for all } t > 0}$$

Homework #6

Use the d'Alembert method to find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{for } -\infty < x < \infty \text{ and } t > 0,$$

with initial displacement and velocity given by

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \frac{1}{1+x^2}, \quad \text{for all } -\infty < x < \infty.$$

Sketch graphs of the solution $u(x,t)$ *as a function of* x*, at fixed times* $t = 0$*,* $t = 2/c$ *and* $t = 4/c$*.*

Summary

- The D'Alembert solution of the wave equation $u_{tt} = c^2 u_{xx}$ is

$$u(x,t) = f(x - ct) + g(x + ct)$$

  for arbitrary functions f and g
- It represents a pair of travelling waves, of fixed shapes f and g, moving to the right and left respectively, with fixed speed c
- On an infinite domain $-\infty < x < \infty$, initial conditions $u(x,0) = F(x)$ and $u_t(x,t) = G(x)$ determine the functions f and g and hence specify the solution
- For a semi-infinite domain $0 \leqslant x < \infty$, this solution works for $x > ct$. For $0 < x < ct$, we have to use the boundary condition $u(0,t) = A(t)$ or $u_x(0,t) = B(t)$