Appendix 6: The Dynamics of a Linear Second Order System

The solution of the linearised small-perturbation equations of motion of an aircraft contains recognisable classical second order system terms. A review of the dynamics of a second order system is therefore useful as an aid to the correct interpretation of the solution of these equations.

Consider the classical mass-spring-damper system whose motion is described by the equation of motion

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \tag{A6.1}$$

where $x(t)$ is the displacement of the mass and $f(t)$ is the forcing function. The constants of the system comprise the mass m, the viscous damping c, and the spring stiffness k.

Classical unforced motion results when the forcing $f(t)$ is made zero and the mass is displaced by, say, A and then released. Equation (A6.1) may then be written as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{A6.2}$$

and the initial conditions are defined: $\dot{x}(0) = 0$ and $x(0) = A$. The time response of the motion of the mass may be found by solving equation (A6.2) subject to the constraints imposed by the initial conditions. This is readily achieved with the aid of the Laplace transform.

Thus,

$$\begin{aligned} \mathrm{L}\{m\ddot{x}(t) + c\dot{x}(t) + kx(t)\} &= m(s^2x(s) - sx(0) - \dot{x}(0)) + c(sx(s) - x(0)) + kx(s) \\ &= m(s^2x(s) - sA) + c(sx(s) - A) + kx(s) = 0 \end{aligned}$$

which after some rearrangement may be written as

$$x(s) = \frac{A(ms + c)}{(ms^2 + cs + k)} \tag{A6.3}$$

or, alternatively,

$$x(s) = \frac{A(s + 2\zeta\omega)}{(s^2 + 2\zeta\omega s + \omega^2)} \tag{A6.4}$$

where

$$\begin{aligned} 2\zeta\omega &= \frac{c}{m} \\ \omega^2 &= \frac{k}{m} \\ \zeta &= \text{system damping ratio} \\ \omega &= \text{system undamped natural frequency} \end{aligned} \tag{A6.5}$$

The time response $x(t)$ may be obtained by determining the inverse Laplace transform of equation (A6.4), and the form of the solution obviously depends on the magnitudes of the physical constants of the system: m, c, and k. The characteristic equation of the system is given by equating the denominator of equation (A6.3) or (A6.4) to zero:

$$ms^2 + cs + k = 0 \tag{A6.6}$$

or, equivalently,

$$s^2 + 2\zeta\omega s + \omega^2 = 0 \tag{A6.7}$$

To facilitate the determination of the inverse Laplace transform of equation (A6.4), the denominator is first factorised and the expression on the right-hand side is split into partial fractions. Whence

$$\begin{aligned} x(s) &= \frac{A(s + 2\zeta\omega)}{\left(s + \omega\left(\zeta + \sqrt{\zeta^2 - 1}\right)\right)\left(s + \omega\left(\zeta - \sqrt{\zeta^2 - 1}\right)\right)} \\ &= \frac{A}{2}\left(\frac{\left(1 + \dfrac{\zeta}{\sqrt{\zeta^2 - 1}}\right)}{\left(s + \omega\left(\zeta + \sqrt{\zeta^2 - 1}\right)\right)} + \frac{\left(1 - \dfrac{\zeta}{\sqrt{\zeta^2 - 1}}\right)}{\left(s + \omega\left(\zeta - \sqrt{\zeta^2 - 1}\right)\right)}\right) \end{aligned} \tag{A6.8}$$

With reference to the table of transform pairs given in Appendix 5, transform pair 2, the inverse Laplace transform of equation (A6.8) is readily obtained:

$$x(t) = \frac{Ae^{-\omega\zeta t}}{2}\left(\left(1 + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\right)e^{-\omega t\sqrt{\zeta^2 - 1}} + \left(1 - \frac{\zeta}{\sqrt{\zeta^2 - 1}}\right)e^{\omega t\sqrt{\zeta^2 - 1}}\right) \tag{A6.9}$$

Equation (A6.9) is the general solution describing the unforced motion of the mass, and the type of response depends on the value of the damping ratio.

- When $\zeta = 0$, equation (A6.9) reduces to

$$x(t) = \frac{A}{2}(e^{-j\omega t} + e^{j\omega t}) = A \cos \omega t \tag{A6.10}$$

  which describes undamped harmonic motion or, alternatively, a neutrally stable system.
- When $0 < \zeta < 1$, equation (A6.9) may be modified by writing

$$\omega_n = \omega\sqrt{1 - \zeta^2}$$

where ω_n is the damped natural frequency. Thus the solution is given by

$$x(t) = \frac{Ae^{-\omega\zeta t}}{2}\left(\left(1 + \frac{j\zeta}{\sqrt{1-\zeta^2}}\right)e^{-j\omega_n t} + \left(1 - \frac{j\zeta}{\sqrt{1-\zeta^2}}\right)e^{j\omega_n t}\right)$$
$$= Ae^{-\omega\zeta t}\left(\cos\omega_n t - \frac{\omega\zeta}{\omega_n}\sin\omega_n t\right) \tag{A6.11}$$

which describes damped harmonic motion.

- When $\zeta = 1$, the coefficients of the exponential terms in equation (A6.9) become infinite. However, by expressing the exponentials as series and by letting $\zeta \to 1$, it may be shown that the damped natural frequency ω_n tends to zero and the solution is given by

$$x(t) = Ae^{-\omega t}(1 - \omega t) \tag{A6.12}$$

- When $\zeta > 1$, the solution is given by equation (A6.9) directly and so is a function of a number of exponential terms. The motion thus described is non-oscillatory and is exponentially convergent.

Typical response time histories for a range of values of damping ratio are shown in Fig. A6.1.

It is important to note that the type of response is governed entirely by the damping ratio and the undamped natural frequency, which in turn determine the roots of the characteristic equation, (A6.6) or (A6.7). Thus the dynamic properties of the system may be directly attributed to its physical properties. Consequently, the type of unforced response may be ascertained simply by inspection of the characteristic equation. A summary of these observations for a stable system are given in the following table.

Summary of a Stable System		
Damping Ratio	**Roots of Characteristic Equation**	**Type of Response**
$\zeta = 0$	$(s + j\omega)(s - j\omega) = 0$ Complex with zero real part	Undamped sinusoidal oscillation with frequency ω
$0 < \zeta < 1$	$(s + \omega\zeta + j\omega_n)(s + \omega\zeta - j\omega_n) = 0$ Complex with non-zero real part	Damped sinusoidal oscillation with frequency $\omega_n = \omega\sqrt{1-\zeta^2}$
$\zeta = 1$	$(s + \omega)^2 = 0$ Repeated real roots	Exponential convergence of form $e^{-\omega t}(1 - \omega t)$
$\zeta > 1$	$(s + r_1)(s + r_2) = 0$ Real roots, where $r_1 = \omega\left(\zeta + \sqrt{\zeta^2 - 1}\right)$ $r_2 = \omega\left(\zeta - \sqrt{\zeta^2 - 1}\right)$	Exponential convergence of general form $k_1e^{-r_1 t} + k_2e^{-r_2 t}$

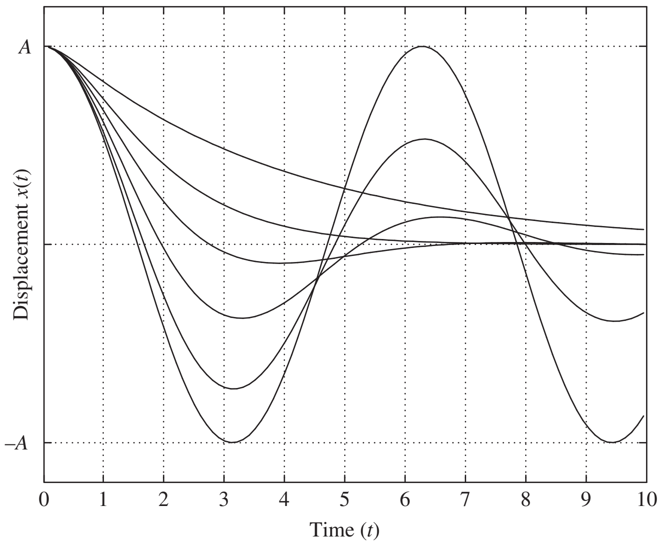

FIGURE A6.1 Typical second-order system responses.

The classical mass-spring-damper system is always stable. However, for rather more general systems which demonstrate similar properties this may not necessarily be so. For a more general interpretation, including unstable systems in which $\zeta < 0$, it is sufficient only to note that the solutions are similar except that the motion they describe is divergent rather than convergent. Aeroplanes typically demonstrate both stable and unstable characteristics, both of which are conveniently described by this simple linear second order model.