CHAPTER

# Bending of Beams

9

In Chapter 7 we saw that an axial load applied to a member produces a uniform direct stress across the cross section of the member (Fig. 7.2). A different situation arises when the applied loads cause a beam to bend which, if the loads are vertical, will take up a sagging or hogging shape (Section 3.2). This means that for loads which cause a beam to sag the upper surface of the beam must be shorter than the lower surface as the upper surface becomes concave and the lower one convex; the reverse is true for loads which cause hogging. The strains in the upper regions of the beam will, therefore, be different to those in the lower regions and since we have established that stress is directly proportional to strain (Eq. (7.7)) it follows that the stress will vary through the depth of the beam.

The truth of this can be demonstrated by a simple experiment. Take a reasonably long rectangular rubber eraser and draw three or four lines on its longer faces as shown in Fig. 9.1(a); the reason for this will become clear a little later. Now hold the eraser between the thumb and forefinger at each end and apply pressure as shown by the direction of the arrows in Fig. 9.1(b). The eraser bends into the shape shown and the lines on the side of the eraser *remain straight* but are now further apart at the top than at the bottom. Reference to Section 2.2 shows that a couple, or pure moment, has been applied to each end of the eraser and, in this case, has produced a hogging shape.

Since, in Fig. 9.1(b), the upper fibres have been stretched and the lower fibres compressed there will be fibres somewhere in between which are neither stretched nor compressed; the plane containing these fibres is called the *neutral plane*.

Now rotate the eraser so that its shorter sides are vertical and apply the same pressure with your fingers. The eraser again bends but now requires much less effort. It follows that the geometry and orientation of a

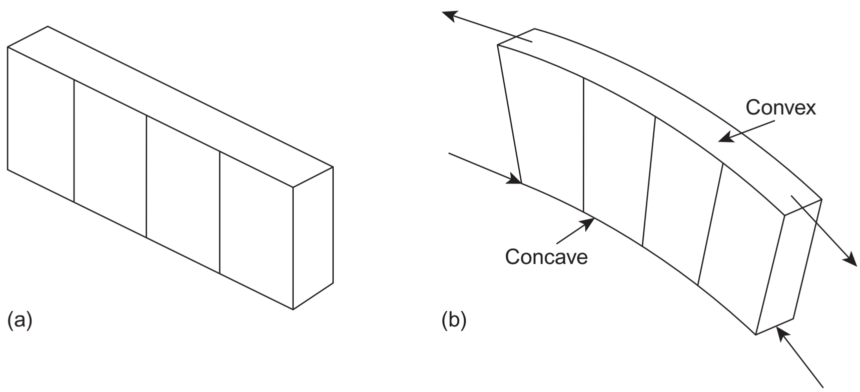



**FIGURE 9.1**

Bending of a rubber eraser.

Structural and Stress Analysis. https://doi.org/10.1016/B978-0-08-102586-4.00009-3

beam section must affect its *bending stiffness.* This is more readily demonstrated with a plastic ruler. When flat it requires hardly any effort to bend it but when held with its width vertical it becomes almost impossible to bend. What does happen is that the lower edge tends to move sideways (for a hogging moment) but this is due to a type of instability which we shall investigate later.

We have seen in Chapter 3 that bending moments in beams are produced by the action of either pure bending moments or shear loads. Reference to problem P.3.7 also shows that two symmetrically placed concentrated shear loads on a simply supported beam induce a state of pure bending, i.e. bending without shear, in the central portion of the beam. It is also possible, as we shall see in Section 9.2, to produce bending moments by applying loads parallel to but offset from the centroidal axis of a beam. Initially, however, we shall concentrate on beams subjected to pure bending moments and consider the corresponding internal stress distributions.

## 9.1 Symmetrical bending

Although symmetrical bending is a special case of the bending of beams of arbitrary cross section, we shall investigate the former first, so that the more complex general case may be more easily understood.

Symmetrical bending arises in beams which have either singly or doubly symmetrical cross sections; examples of both types are shown in Fig. 9.2.

Suppose that a length of beam, of rectangular cross section, say, is subjected to a pure, sagging bending moment, $M$, applied in a vertical plane. The length of beam will bend into the shape shown in Fig. 9.3(a) in which the upper surface is concave and the lower convex. It can be seen that the upper longitudinal fibres of the beam are compressed while the lower fibres are stretched. It follows that, as in the case of the eraser, between these two extremes there are fibres that remain unchanged in length.

Thus the direct stress varies through the depth of the beam from compression in the upper fibres to tension in the lower. Clearly the direct stress is zero for the fibres that do not change in length; we have called the plane containing these fibres the *neutral plane.* The line of intersection of the neutral plane and any cross section of the beam is termed the *neutral axis* (Fig. 9.3(b)).

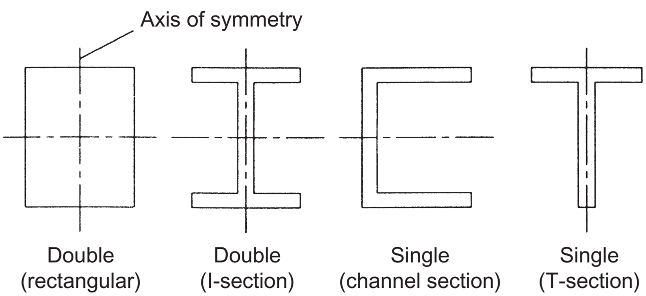



**FIGURE 9.2**

Symmetrical section beams.

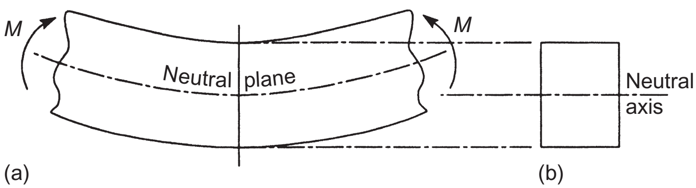



**FIGURE 9.3**

Beam subjected to a pure sagging bending moment.

The problem, therefore, is to determine the variation of direct stress through the depth of the beam, the values of the stresses and subsequently to find the corresponding beam deflection.

## Assumptions

The primary assumption made in determining the direct stress distribution produced by pure bending is that plane cross sections of the beam remain plane and normal to the longitudinal fibres of the beam after bending. Again, we saw this from the lines on the side of the eraser. We shall also assume that the material of the beam is linearly elastic, i.e. it obeys Hooke's law, and that the material of the beam is homogeneous. Cases of composite beams are considered in Chapter 12.

## Direct stress distribution

Consider a length of beam (Fig. 9.4(a)) that is subjected to a pure, sagging bending moment, $M$, applied in a vertical plane; the beam cross section has a vertical axis of symmetry as shown in Fig. 9.4(b). The bending moment will cause the length of beam to bend in a similar manner to that shown in Fig. 9.3(a) so that a neutral plane will exist which is, as yet, unknown distances $y_1$ and $y_2$ from the top and bottom of the beam, respectively. Coordinates of all points in the beam are referred to axes O$xyz$ (see Section 3.2) in which the origin O lies in the neutral plane of the beam. We shall now investigate the behaviour of an elemental length, $\delta x$, of the beam formed by parallel sections MIN and PGQ (Fig. 9.4(a)) and also the fibre ST of cross-sectional area $\delta A$ a distance $y$ above the neutral plane. Clearly, before bending takes place MP = IG = ST = NQ = $\delta x$.

The bending moment $M$ causes the length of beam to bend about a *centre of curvature* C as shown in Fig. 9.5(a). Since the element is small in length and a pure moment is applied we can take the curved shape of the beam to be circular with a *radius of curvature* $R$ measured to the neutral plane. This is a useful reference point since, as we have seen, strains and stresses are zero in the neutral plane.

The previously parallel plane sections MIN and PGQ remain plane as we have demonstrated but are now inclined at an angle $\delta\theta$ to each other. The length MP is now shorter than $\delta x$ as is ST while NQ is longer; IG, being in the neutral plane, is still of length $\delta x$. Since the fibre ST has changed in length it has suffered a strain $\varepsilon_x$ which is given by

$$\varepsilon_x = \frac{\text{change in length}}{\text{original length}} \quad \text{(see Eq. (7.4))}$$

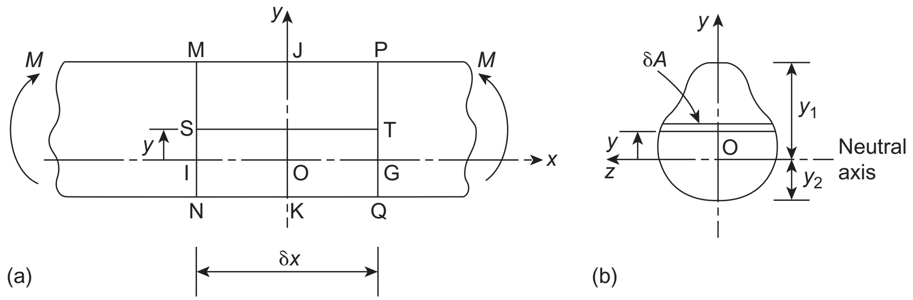



**FIGURE 9.4**

Bending of a symmetrical section beam.

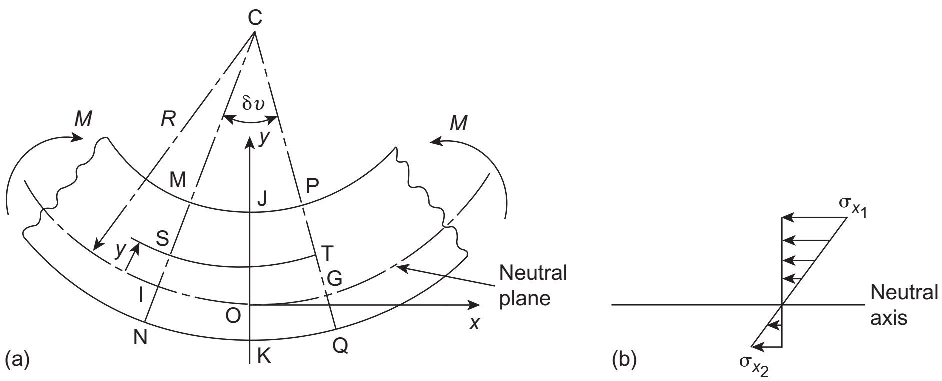



**FIGURE 9.5**

Length of beam subjected to a pure bending moment.

Then

$$\varepsilon_x = \frac{(R-y)\delta\theta - R\delta\theta}{\delta x}$$

i.e.

$$\varepsilon_x = \frac{(R-y)\delta\theta - R\delta\theta}{R\delta\theta}$$

so that

$$\varepsilon_x = -\frac{y}{R} \tag{9.1}$$

The negative sign in Eq. (9.1) indicates that fibres in the region where $y$ is positive will shorten when the bending moment is positive. Then, from Eq. (7.7), the direct stress $\sigma_x$ in the fibre ST is given by

$$\sigma_x = -E\frac{y}{R} \tag{9.2}$$

The direct or normal force on the cross section of the fibre ST is $\sigma_x \delta A$. However, since the direct stress in the beam section is due to a pure bending moment, in other words there is no axial load, the resultant normal force on the complete cross section of the beam must be zero. Then

$$\int_A \sigma_x \, \mathrm{d}A = 0 \tag{9.3}$$

where $A$ is the area of the beam cross section.

Substituting for $\sigma_x$ in Eq. (9.3) from Eq. (9.2) gives

$$-\frac{E}{R}\int_A y \, \mathrm{d}A = 0 \tag{9.4}$$

in which both $E$ and $R$ are constants for a beam of a given material subjected to a given bending moment. Thus

$$\int_A y \, \mathrm{d}A = 0 \tag{9.5}$$

Equation (9.5) states that the first moment of the area of the cross section of the beam with respect to the neutral axis, i.e. the $z$ axis, is equal to zero. Thus we see that *the neutral axis passes through the centroid of area of the cross section.* Since the $y$ axis in this case is also an axis of symmetry, it must also pass through the centroid of the cross section. Hence the origin, O, of the coordinate axes, coincides with the centroid of area of the cross section.

Equation (9.2) shows that for a sagging (i.e. positive) bending moment the direct stress in the beam section is negative (i.e. compressive) when $y$ is positive and positive (i.e. tensile) when $y$ is negative.

Consider now the elemental strip $\delta A$ in Fig. 9.4(b); this is, in fact, the cross section of the fibre ST. The strip is above the neutral axis so that there will be a *compressive* force acting on its cross section of $\sigma_x \delta A$ which is *numerically* equal to $(Ey/R)\delta A$ from Eq. (9.2). Note that this force will act at all sections along the length of ST. At S this force will exert a clockwise moment $(Ey/R)y\delta A$ about the neutral axis while at T the force will exert an identical anticlockwise moment about the neutral axis. Considering either end of ST we see that the moment resultant about the neutral axis of the stresses on all such fibres must be *equivalent* to the applied moment $M$, i.e.

$$M = \int_A E\frac{y^2}{R}\, \mathrm{d}A$$

or

$$M = \frac{E}{R}\int_A y^2\, \mathrm{d}A \tag{9.6}$$

The term $\int_A y^2\, \mathrm{d}A$ is known as the *second moment of area* of the cross section of the beam about the neutral axis and is given the symbol $I$. Rewriting Eq. (9.6) we have

$$M = \frac{EI}{R} \tag{9.7}$$

or, combining this expression with Eq. (9.2)

$$\frac{M}{I} = \frac{E}{R} = -\frac{\sigma_x}{y} \tag{9.8}$$

From Eq. (9.8) we see that

$$\sigma_x = -\frac{My}{I} \tag{9.9}$$

The direct stress, $\sigma_x$, at any point in the cross section of a beam is therefore directly proportional to the distance of the point from the neutral axis and so varies linearly through the depth of the beam as shown, for the section JK, in Fig. 9.5(b). Clearly, for a positive, or sagging, bending moment $\sigma_x$ is positive, i.e. tensile, when $y$ is negative and compressive (i.e. negative) when $y$ is positive. Thus in Fig. 9.5(b)

$$\sigma_{x,1} = \frac{My_1}{I}\ (\text{compression}) \quad \sigma_{x,2} = \frac{My_2}{I}\ (\text{tension}) \tag{9.10}$$

Furthermore, we see from Eq. (9.7) that the curvature, $1/R$, of the beam is given by

$$\frac{1}{R} = \frac{M}{EI} \tag{9.11}$$

and is therefore directly proportional to the applied bending moment and inversely proportional to the product $EI$ which is known as the *flexural rigidity* of the beam.

## Elastic section modulus

Equation (9.10) may be written in the form

$$\sigma_{x,1} = \frac{M}{Z_{e,1}} \quad \sigma_{x,2} = \frac{M}{Z_{e,2}} \tag{9.12}$$

in which the terms $Z_{e,1}(= I/y_1)$ and $Z_{e,2}(= I/y_2)$ are known as the *elastic section moduli* of the cross section. For a beam section having the $z$ axis as an axis of symmetry, say, $y_1 = y_2$ and $Z_{e,1} = Z_{e,2} = Z_e$. Then, numerically

$$\sigma_{x,1} = \sigma_{x,2} = \frac{M}{Z_e} \tag{9.13}$$

Expressing the extremes of direct stress in a beam section in this form is extremely useful in elastic design where, generally, a beam of a given material is required to support a given bending moment. The maximum allowable stress in the material of the beam is known and a minimum required value for the section modulus, $Z_e$, can be calculated. A suitable beam section may then be chosen from handbooks which list properties and dimensions, including section moduli, of standard structural shapes.

The selection of a beam cross section depends upon many factors; these include the type of loading and construction, the material of the beam and several others. However, for a beam subjected to bending and fabricated from material that has the same failure stress in compression as in tension, it is logical to choose a doubly symmetrical beam section having its centroid (and therefore its neutral axis) at mid-depth. Also it can be seen from Fig. 9.5(b) that the greatest values of direct stress occur at points furthest from the neutral axis so that the most efficient section is one in which most of the material is located as far as possible from the neutral axis. Such a section is the I-section shown in Fig. 9.2.

### EXAMPLE 9.1

A simply supported beam, 6 m long, is required to carry a uniformly distributed load of 10 kN/m. If the allowable direct stress in tension and compression is 155 N/mm$^2$, select a suitable cross section for the beam.

From Fig. 3.17(d) we see that the maximum bending moment in a simply supported beam of length $L$ carrying a uniformly distributed load of intensity $w$ is given by

$$M_{max} = \frac{wL^2}{8} \tag{i}$$

Therefore in this case

$$M_{max} = \frac{10 \times 6^2}{8} = 45 \text{ kNm}$$

The required section modulus of the beam is now obtained using Eq. (9.13), thus

$$Z_{e,min} = \frac{M_{max}}{\sigma_{x,max}} = \frac{45 \times 10^6}{155} = 290\,323 \text{ mm}^3$$

From tables of structural steel sections it can be seen that a Universal Beam, 254 mm×102 mm×28 kg/m, has a section modulus (about a centroidal axis parallel to its flanges) of 307 600 mm$^3$. This is the smallest beam section having a section modulus greater than that required and allows a margin for the increased load due to the self-weight of the beam. However, we must now check that the allowable stress is not exceeded due to self-weight. The total load intensity produced by the applied load and self-weight is

$$10+\frac{28\times 9.81}{10^3}=10.3\ \text{kN/m}$$

Hence, from Eq. (i)

$$M_{\max}=\frac{10.3\times 6^2}{8}=46.4\ \text{kNm}$$

Therefore from Eq. (9.13)

$$\sigma_{x,\max}=\frac{46.4\times 10^3\times 10^3}{307\,600}=150.8\ \text{N/mm}^2$$

The allowable stress is 155 $\text{N/mm}^2$ so that the Universal Beam, 254 mm×102 mm×28 kg/m, is satisfactory.

## EXAMPLE 9.2

The cross section of a beam has the dimensions shown in Fig. 9.6(a). If the beam is subjected to a sagging bending moment of 100 kNm applied in a vertical plane, determine the distribution of direct stress through the depth of the section.

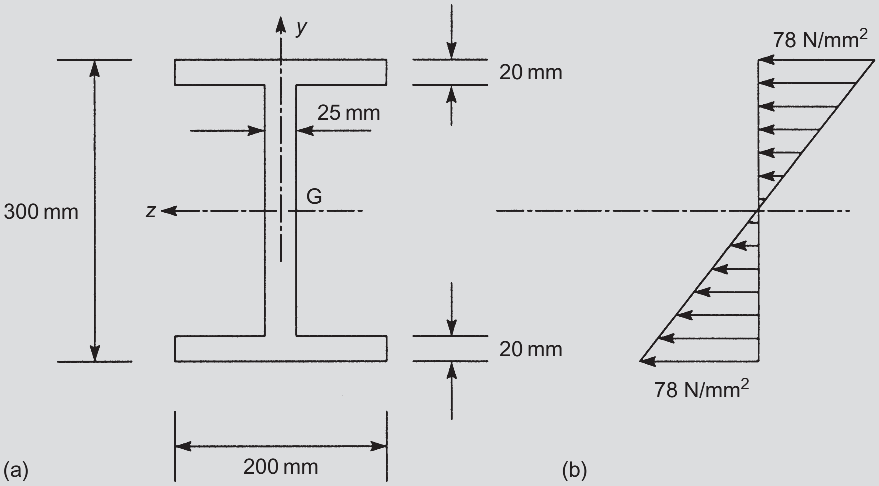



**FIGURE 9.6**

Direct stress distribution in beam of Ex. 9.2.

The cross section of the beam is doubly symmetrical so that the centroid, G, of the section, and therefore the origin of axes, coincides with the mid-point of the web. Furthermore, the bending moment is applied to the beam section in a vertical plane so that the $z$ axis becomes the neutral axis of the beam section; we therefore need to calculate the second moment of area, $I_z$, about this axis. Thus

$$I_z=\frac{200\times 300^3}{12}-\frac{175\times 260^3}{12}=193.7\times 10^6\ \text{mm}^4\quad \text{(see Section 9.6)}$$

From Eq. (9.9) the distribution of direct stress, $\sigma_x$, is given by

$$\sigma_x = -\frac{100 \times 10^6}{193.7 \times 10^6} y = -0.52y \tag{i}$$

The direct stress, therefore, varies linearly through the depth of the section from a value

$$-0.52 \times (+150) = -78 \text{ N/mm}^2 \text{ (compression)}$$

at the top of the beam to

$$-0.52 \times (-150) = +78 \text{ N/mm}^2 \text{ (tension)}$$

at the bottom as shown in Fig. 9.6(b).

## EXAMPLE 9.3

Now determine the distribution of direct stress in the beam of Ex. 9.2 if the bending moment is applied in a horizontal plane and in a clockwise sense about G$y$ when viewed in the direction G$y$.

In this case the beam will bend about the vertical $y$ axis which therefore becomes the neutral axis of the section. Thus Eq. (9.9) becomes

$$\sigma_x = -\frac{M}{I_y} z \tag{i}$$

where $I_y$ is the second moment of area of the beam section about the $y$ axis. Again from Section 9.6

$$I_y = 2 \times \frac{20 \times 200^3}{12} + \frac{260 \times 25^3}{12} = 27.0 \times 10^6 \text{ mm}^4$$

Hence, substituting for $M$ and $I_y$ in Eq. (i)

$$\sigma_x = -\frac{100 \times 10^6}{27.0 \times 10^6} z = -3.7z$$

We have not specified a sign convention for bending moments applied in a horizontal plane; clearly in this situation the sagging/hogging convention loses its meaning. However, a physical appreciation of the problem shows that the left-hand edges of the beam are in tension while the right-hand edges are in compression. Again the distribution is linear and varies from $3.7 \times (+100) = 370$ N/mm$^2$ (tension) at the left-hand edges of each flange to $3.7 \times (-100) = -370$ N/mm$^2$ (compression) at the right-hand edges.

We note that the maximum stresses in this example are very much greater than those in Ex. 9.2. This is due to the fact that the bulk of the material in the beam section is concentrated in the region of the neutral axis where the stresses are low. The use of an I-section in this manner would therefore be structurally inefficient.

## EXAMPLE 9.4

The beam section of Ex. 9.2 is subjected to a bending moment of 100 kN m applied in a plane parallel to the longitudinal axis of the beam but inclined at 30° to the left of vertical. The sense of the bending moment is clockwise when viewed from the left-hand edge of the beam section. Determine the distribution of direct stress.

The bending moment is first resolved into two components, $M_z$ in a vertical plane and $M_y$ in a horizontal plane. Equation (9.9) may then be written in two forms

$$\sigma_x = -\frac{M_z}{I_z}y \quad \sigma_x = -\frac{M_y}{I_y}z \tag{i}$$

The separate distributions can then be determined and superimposed. A more direct method is to combine the two equations (i) to give the total direct stress at any point $(y, z)$ in the section. Thus

$$\sigma_x = -\frac{M_z}{I_z}y - \frac{M_y}{I_y}z \tag{ii}$$

Now

$$\left.\begin{aligned} M_z &= 100\cos 30° = 86.6 \text{ kNm} \\ M_y &= 100\sin 30° = 50.0 \text{ kNm} \end{aligned}\right\} \tag{iii}$$

$M_z$ is, in this case, a negative bending moment producing tension in the upper half of the beam where $y$ is positive. Also $M_y$ produces tension in the left-hand half of the beam where $z$ is positive; we shall therefore call $M_y$ a negative bending moment. Substituting the values of $M_z$ and $M_y$ from Eq. (iii) but with the appropriate sign in Eq. (ii) together with the values of $I_z$ and $I_y$ from Exs 9.2 and 9.3 we obtain

$$\sigma_x = \frac{86.6 \times 10^6}{193.7 \times 10^6}y + \frac{50.0 \times 10^6}{27.0 \times 10^6}z \tag{iv}$$

or

$$\sigma_x = 0.45y + 1.85z \tag{v}$$

Equation (v) gives the value of direct stress at any point in the cross section of the beam and may also be used to determine the distribution over any desired portion. Thus on the upper edge of the top flange $y = +150$ mm, 100 mm $\geq z \geq -100$ mm, so that the direct stress varies linearly with $z$. At the top left-hand corner of the top flange

$$\sigma_x = 0.45 \times (+150) + 1.85 \times (+100) = +252.5 \text{ N/mm}^2 \text{ (tension)}$$

At the top right-hand corner

$$\sigma_x = 0.45 \times (+150) + 1.85 \times (-100) = -117.5 \text{ N/mm}^2 \text{ (compression)}$$

The distributions of direct stress over the outer edge of each flange and along the vertical axis of symmetry are shown in Fig. 9.7. Note that the neutral axis of the beam section does not in this case coincide

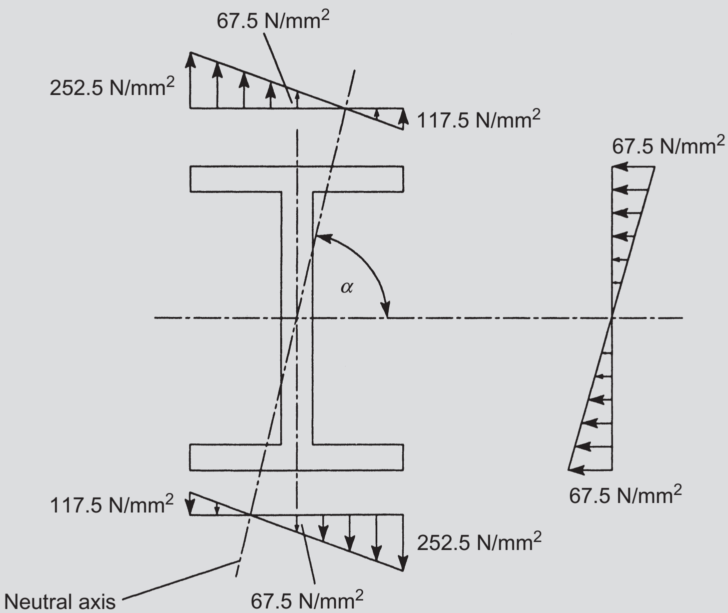



**FIGURE 9.7**

Direct stress distribution in beam of Ex. 9.4.

with either the $z$ or $y$ axes, although it still passes through the centroid of the section. Its inclination, $\alpha$, to the $z$ axis, say, can be found by setting $\sigma_x = 0$ in Eq. (v). Thus

$$0 = 0.45y + 1.85z$$

or

$$-\frac{y}{z} = \frac{1.85}{0.45} = 4.11 = \tan\alpha$$

which gives

$$\alpha = 76.3°$$

Note that $\alpha$ may be found in general terms from Eq. (ii) by again setting $\sigma_x = 0$. Hence

$$\frac{y}{z} = -\frac{M_y I_z}{M_z I_y} = \tan\alpha \tag{9.14}$$

or

$$\tan\alpha = \frac{M_y I_z}{M_z I_y}$$

since $y$ is positive and $z$ is negative for a positive value of $\alpha$.

## EXAMPLE 9.5

A beam, 6 m long, is simply supported at its left-hand end and at 1.5 m from its right-hand end. If the cross section of the beam is that shown in Fig. 9.8 and it carries a uniformly distributed load of 7.5 kN/m over its full length calculate the maximum tensile and compressive stresses in the beam.

The first step is to find the position of the centroid of area, G, of the beam section. This will lie on the vertical axis of symmetry and is a distance $\bar{y}$ from the top of the flange.

Taking moments of area about the top of the flange

$$(125 \times 25 + 125 \times 25)\bar{y} = 125 \times 25 \times 12.5 + 125 \times 25 \times 87.5$$

which gives

$$\bar{y} = 50 \text{ mm}$$

Since the loading is applied in the vertical plane of symmetry the direct stress distribution is given by Eq. (9.9) in which $I$ is the second moment of area of the beam cross section about the $z$ axis. Then, using the method described in Section 9.6

$$I_z = \frac{125 \times 25^3}{12} + 125 \times 25 \times 37.5^2 + \frac{25 \times 125^3}{12} + 25 \times 125 \times 37.5^2$$

which gives

$$I_z = 13.02 \times 10^6 \text{ mm}^4$$

It is clear from Ex. 3.10 that the maximum sagging bending moment in the beam will occur at a section between the support points and that the maximum hogging bending moment will occur at the right-hand support. Referring to Fig. 9.9 and taking moments about B

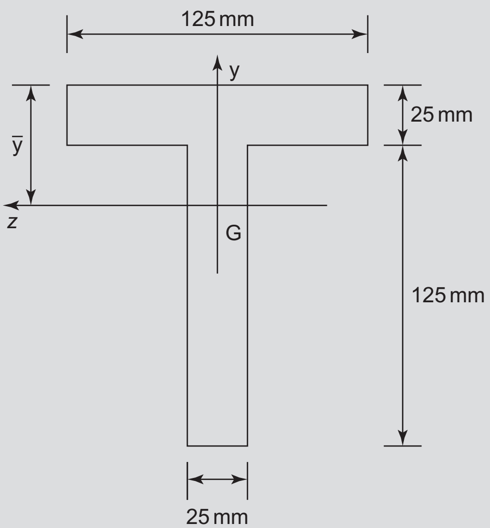



**FIGURE 9.8**

Beam section of Ex. 9.5.

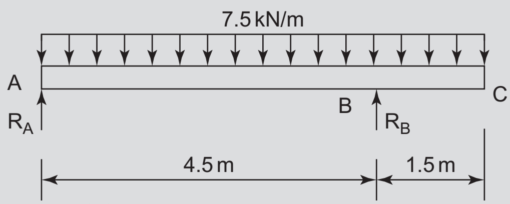



**FIGURE 9.9**

Beam of Ex. 9.5.

$$R_A \times 4.5 - 7.5 \times 6.0 \times 1.5 = 0$$

from which

$$R_A = 15\,\text{kN}$$

From Eq. (3.4) we see that the bending moment diagram will have a mathematical maximum between A and B when the shear force is zero. Then

$$S_{AB} = R_A - 7.5x = 15 - 7.5x = 0$$

so that $S_{AB} = 0$ when $x = 2$ m. The maximum sagging bending moment is therefore given by

$$M_z\,(\text{max.sagging}) = 15 \times 2.0 - \frac{7.5 \times 2.0^2}{2} = 15.0\ \text{kNm}$$

The maximum hogging bending moment occurs at B and is given by

$$M_z\,(\text{max.hogging}) = -\frac{7.5 \times 1.5^2}{2} = -8.45\ \text{kNm}$$

The maximum sagging bending moment will produce a direct compressive stress at the top of the flange and a direct tensile stress at the base of the leg. Then, from Eq. (9.9)

$$\sigma_x\,(\text{top of flange}) = -\frac{15 \times 10^6 \times 50}{13.02 \times 10^6} = -57.6\ \text{N/mm}^2$$

$$\sigma_x\,(\text{base of leg}) = -\frac{15 \times 10^6 \times (-100)}{13.02 \times 10^6} = +115.2\ \text{N/mm}^2$$

Similarly, due to the maximum hogging bending moment

$$\sigma_x\,(\text{top of flange}) = -\frac{(-8.45) \times 10^6 \times 50}{13.02 \times 10^6} = +32.5\ \text{N/mm}^2$$

$$\sigma_x\,(\text{base of leg}) = -\frac{(-8.45) \times 10^6 \times (-100)}{13.02 \times 10^6} = -64.9\ \text{N/mm}^2$$

Therefore the maximum tensile stress in the beam is 115.2 $\text{N/mm}^2$ occurring at a section 2 m from the left-hand support and the maximum compressive stress is 64.9 $\text{N/mm}^2$ occurring at the right-hand support.

## 9.2 Combined bending and axial load

In many practical situations beams and columns are subjected to combinations of axial loads and bending moments. For example, the column shown in Fig. 9.10 supports a beam seated on a bracket attached to the column. The loads on the beam produce a vertical load, $P$, on the bracket, the load being offset a distance $e$ from the neutral plane of the column. The action of $P$ on the column is therefore equivalent to an axial load, $P$, plus a bending moment, $Pe$. The direct stress at any point in the cross section of the column is therefore the algebraic sum of the direct stress due to the axial load and the direct stress due to bending.

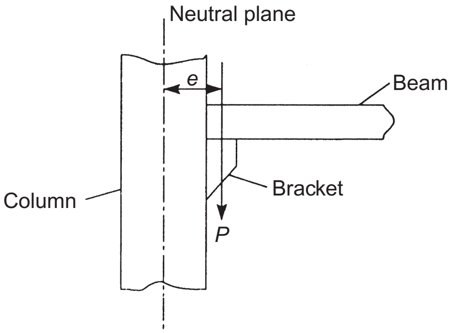



**FIGURE 9.10**

Combined bending and axial load on a column.

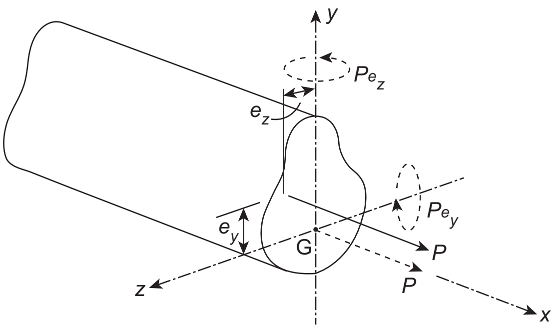



**FIGURE 9.11**

Combined bending and axial load on a beam section.

Consider now a length of beam having a vertical plane of symmetry and subjected to a tensile load, $P$, which is offset by positive distances $e_y$ and $e_z$ from the $z$ and $y$ axes, respectively (Fig. 9.11). It can be seen that $P$ is equivalent to an axial load $P$ plus bending moments $Pe_y$ and $Pe_z$ about the $z$ and $y$ axes, respectively. The moment $Pe_y$ is a negative or hogging bending moment while the moment $Pe_z$ induces tension in the region where $z$ is positive; $Pe_z$ is, therefore, also regarded as a negative moment. Thus at any point ($y$,$z$) the direct stress, $\sigma_x$, due to the combined force system, using Eqs (7.1) and (9.9), is

$$\sigma_x = \frac{P}{A} + \frac{Pe_y}{I_z}y + \frac{Pe_z}{I_y}z \tag{9.15}$$

Equation (9.15) gives the value of $\sigma_x$ at any point ($y$,$z$) in the beam section for any combination of signs of $P$, $e_z$, $e_y$.

## EXAMPLE 9.6

A beam has the cross section shown in Fig. 9.12(a). It is subjected to a normal tensile force, $P$, whose line of action passes through the centroid of the horizontal flange. Calculate the maximum allowable value of $P$ if the maximum direct stress is limited to $\pm 150\ \text{N/mm}^2$.

The first step in the solution of the problem is to determine the position of the centroid, G, of the section. Thus, taking moments of areas about the top edge of the flange we have

$$(200 \times 20 + 200 \times 20)\bar{y} = 200 \times 20 \times 10 + 200 \times 20 \times 120$$

from which

$$\bar{y} = 65\ \text{mm}$$

The second moment of area of the section about the $z$ axis is then obtained using the methods of Section 9.6 and is

$$I_z = \frac{200 \times 65^3}{3} - \frac{180 \times 45^3}{3} + \frac{20 \times 155^3}{3} = 37.7 \times 10^6\ \text{mm}^4$$

Since the line of action of the load intersects the $y$ axis, $e_z$ in Eq. (9.15) is zero so that

$$\sigma_x = \frac{P}{A} + \frac{Pe_y}{I_z}y \tag{i}$$

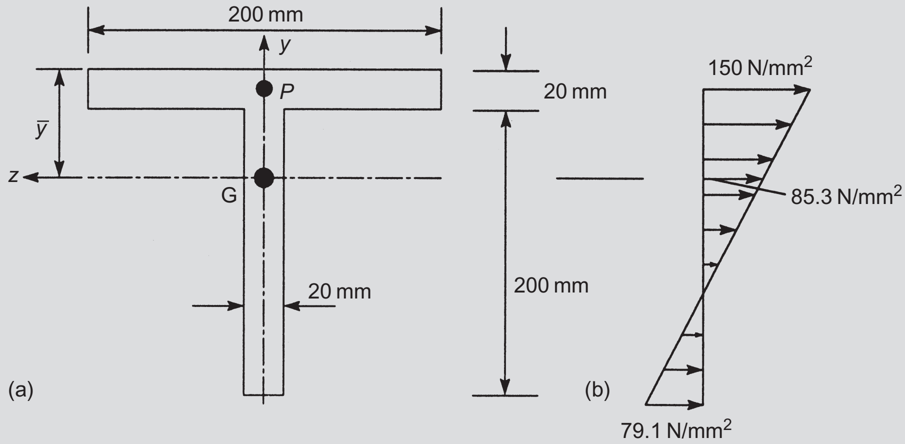



**FIGURE 9.12**

Direct stress distribution in beam section of Ex. 9.6.

Also $e_y = +55$ mm so that $Pe_y = +55\ P$ and Eq. (i) becomes

$$\sigma_x = P\left(\frac{1}{8000} + \frac{55}{37.7 \times 10^6}y\right)$$

or

$$\sigma_x = P(1.25 \times 10^{-4} + 1.46 \times 10^{-6}y) \tag{ii}$$

It can be seen from Eq. (ii) that $\sigma_x$ varies linearly through the depth of the beam from a tensile value at the top of the flange where $y$ is positive to either a tensile or compressive value at the bottom of the leg depending on whether the bracketed term is positive or negative. Therefore at the top of the flange

$$+150 = P[1.25 \times 10^{-4} + 1.46 \times 10^{-6} \times (+65)]$$

which gives the limiting value of $P$ as 682 kN.

At the bottom of the leg of the section $y = -155$ mm so that the right-hand side of Eq. (ii) becomes

$$P[1.25 \times 10^{-4} + 1.46 \times 10^{-6} \times (-155)] \equiv -1.01 \times 10^{-4}P$$

which is negative for a tensile value of $P$. Hence the resultant direct stress at the bottom of the leg is compressive so that for a limiting value of $P$

$$-150 = -1.01 \times 10^{-4}P$$

from which

$$P = 1485 \text{ kN}$$

Therefore, we see that the maximum allowable value of $P$ is 682 kN, giving the direct stress distribution shown in Fig. 9.12(b).

## Core of a rectangular section

In some structures, such as brick-built chimneys and gravity dams which are usually constructed from concrete, it is inadvisable for tension to be developed in any cross section. Clearly, from our previous discussion, it is possible for a compressive load that is offset from the neutral axis of a beam section to induce a resultant tensile stress in some regions of the cross section if the tensile stress due to bending in those regions is greater than the compressive stress produced by the axial load. Therefore, we require to impose limits on the eccentricity of such a load so that no tensile stresses are induced.

Consider the rectangular section shown in Fig. 9.13 subjected to an eccentric compressive load, $P$, applied parallel to the longitudinal axis in the positive $yz$ quadrant. Note that if $P$ were inclined at some angle to the longitudinal axis, then we need only consider the component of $P$ normal to the section since the in-plane component would induce only shear stresses. Since $P$ is a compressive load and therefore negative, Eq. (9.15) becomes

$$\sigma_x = -\frac{P}{A} - \frac{Pe_y}{I_z}y - \frac{Pe_z}{I_y}z \tag{9.16}$$

Note that both $Pe_y$ and $Pe_z$ are positive moments according to the sign convention we have adopted. In the region of the cross section where $z$ and $y$ are negative, tension will develop if

$$\left|\frac{Pe_y}{I_z}y + \frac{Pe_z}{I_y}z\right| > \left|\frac{P}{A}\right|$$

The limiting case arises when the direct stress is zero at the corner of the section, i.e. when $z = -b/2$ and $y = -d/2$. Therefore, substituting these values in Eq. (9.16) we have

$$0 = -\frac{P}{A} - \frac{Pe_y}{I_z}\left(-\frac{d}{2}\right) - \frac{Pe_z}{I_y}\left(-\frac{b}{2}\right)$$

or, since $A = bd$, $I_z = bd^3/12$, $I_y = db^3/12$ (see Section 9.6)

$$0 = -bd + 6be_y + 6de_z$$

which gives

$$be_y + de_z = \frac{bd}{6}$$

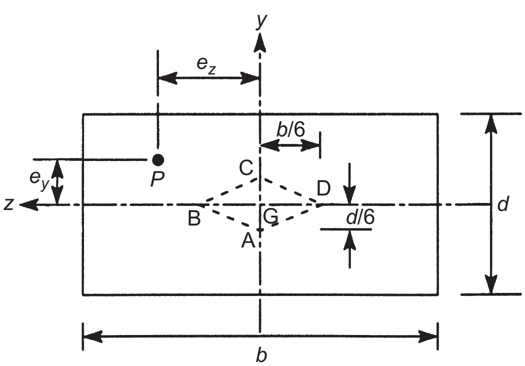



**FIGURE 9.13**

Core of a rectangular section.

Rearranging we obtain

$$e_y = -\frac{d}{b}e_z + \frac{d}{6} \tag{9.17}$$

Equation (9.17) defines the line BC in Fig. 9.13 which sets the limit for the eccentricity of $P$ from both the $z$ and $y$ axes. It follows that $P$ can be applied at any point in the region BCG for there to be no tension developed anywhere in the section.

Since the section is doubly symmetrical, a similar argument applies to the regions GAB, GCD and GDA; the rhombus ABCD is known as the *core of the section* and has diagonals BD $= b/3$ and AC $= d/3$. If the load $P$ were applied along the $z$ axis, then for zero tension anywhere in the section $P$ must be applied between B and D. The length BD $= b/3$ and is called the *middle third* of the section. Similarly, if the load were applied along the $y$ axis AC would be the middle third of the section. This is known as *the middle third rule*.

## Core of a circular section

Bending, produced by an eccentric load $P$, in a circular cross section always takes place about a diameter that is perpendicular to the radius on which $P$ acts. It is therefore logical to take this diameter and the radius on which $P$ acts as the coordinate axes of the section (Fig. 9.14).

Suppose that $P$ in Fig. 9.14 is a compressive load. The direct stress, $\sigma_x$, at any point $(z, y)$ is given by Eq. (9.15) in which $e_y = 0$. Hence

$$\sigma_x = -\frac{P}{A} - \frac{Pe_z}{I_y}z \tag{9.18}$$

Tension will occur in the region where $z$ is negative if

$$\left|\frac{Pe_z}{I_y}z\right| > \left|\frac{P}{A}\right|$$

The limiting case occurs when $\sigma_x = 0$ and $z = -R$; hence

$$0 = -\frac{P}{A} - \frac{Pe_z}{I_y}(-R)$$

Now $A = \pi R^2$ and $I_y = \pi R^4/4$ (see Section 9.6) so that

$$0 = -\frac{1}{\pi R^2} + \frac{4e_z}{\pi R^3}$$

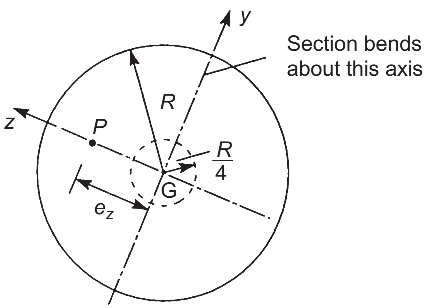



**FIGURE 9.14**

Core of a circular section beam.

from which

$$e_z = \frac{R}{4}$$

Thus the core of a circular section is a circle of radius $R/4$.

## EXAMPLE 9.7

A free-standing masonry wall is 7 m high, 0.6 m thick and has a density of 2000 kg/m$^3$. Calculate the maximum, uniform, horizontal wind pressure that can occur without tension developing at any point in the wall.

Consider a 1 m length of wall. The forces acting are the horizontal resultant, $P$, of the uniform wind pressure, $p$, and the weight, $W$, of the 1 m length of wall (Fig. 9.15).

Clearly the base section is the one that experiences the greatest compressive normal load due to self-weight and also the greatest bending moment due to wind pressure.

It is also the most critical section since the bending moment that causes tension is a function of the square of the height of the wall, whereas the weight causing compression is a linear function of wall height. From Fig. 9.13 it is clear that the resultant, $R$, of $P$ and $W$ must lie within the central 0.2 m of the base section, i.e. within the middle third of the section, for there to be no tension developed anywhere in the base cross section. The reason for this is that $R$ may be resolved into vertical and horizontal components at any point in its line of action. At the base of the wall the vertical component is then a compressive load parallel to the vertical axis of the wall (i.e. the same situation as in Fig. 9.13) and the horizontal component is a shear load which has no effect as far as tension in the wall is concerned. The limiting case arises when $R$ passes through m, one of the middle third points, in which case the direct stress at B is zero and the moment of $R$ (and therefore the sum of the moments of $P$ and $W$) about m is zero. Hence

$$3.5P = 0.1W \tag{i}$$

where

$$P = p \times 7 \times 1\,\mathrm{N} \text{ if } p \text{ is in } \mathrm{N/m^2}$$

and

$$W = 2000 \times 9.81 \times 0.6 \times 7\ \mathrm{N}$$

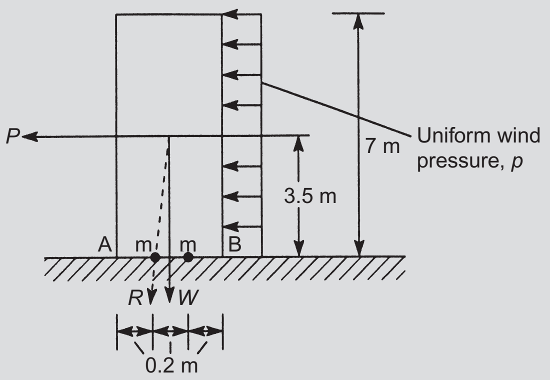



**FIGURE 9.15**

Masonry wall of Ex. 9.7.

Substituting for $P$ and $W$ in Eq. (i) and solving for $p$ gives

$$p = 336.3\ \mathrm{N/m^2}$$

In addition to the requirement of zero tension in the cross sections of gravity structures, they must also be investigated for tendencies to overturn and slide.

## EXAMPLE 9.8

Examine the tendency for the masonry wall of Ex. 9.7 to overturn due to the calculated wind pressure. Also, if the coefficient of friction between the base of the wall and the ground is 0.5, determine whether or not the wall will slide.

If the wall were to overturn, it would rotate about the point A at its base. Then for the point B to just leave the ground

$$P \times 3.5 = W \times 0.3.$$

But

$$P \times 3.5 = 336.3 \times 7 \times 1 \times 3.5 = 8239.4\,\mathrm{Nm}$$

and

$$W \times 0.3 = 2000 \times 9.81 \times 0.6 \times 7 \times 1 \times 0.3 = 24721.2\,\mathrm{Nm}$$

Clearly the overturning moment due to the wind is much less than the restraining moment due to its self-weight. The wall will therefore not overturn.

For the wall to slide $P > 0.5\ W$ but $P = 2354.1$ N and $0.5W = 41202$ N so that the wall will not slide.

## EXAMPLE 9.9

A chimney is constructed from brick and has a square cross section of side equal to 0.6 m as shown in Fig. 9.16. If the walls of the chimney are 100 mm thick and the density of the brickwork is 2000 kg/m$^3$, calculate the maximum allowable height of the chimney if the maximum wind pressure is 1.4 kN/m$^2$ applied in a direction perpendicular to a diagonal.

Suppose the height of the chimney is $h$ m. Then the weight of the chimney is given by

$$\text{Weight} = \left(0.6^2 - 0.4^2\right) h \times 2000 \times 9.81 = 3924h\,\mathrm{N}$$

The compressive stress on the base of the chimney due to its self-weight is then

$$\text{Compressive stress} = \frac{3924h}{\left(0.6^2 - 0.4^2\right)} = 19620h\,\mathrm{N/m^2}$$

It may be shown that the second moment of area of a square section about a diagonal is equal to the second moment of area about a centroidal axis parallel to one of its sides. Alternatively, the second moment of area may be calculated using Table A.1 in Appendix A. Then, using the first approach

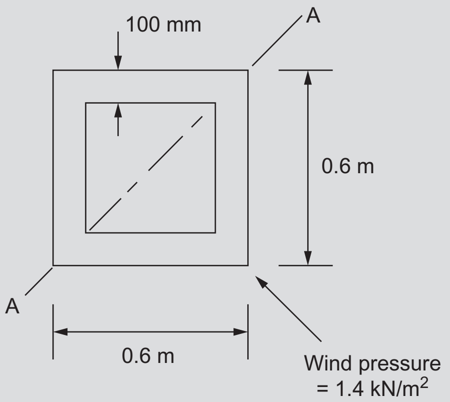



**FIGURE 9.16**

Chimney of Ex. 9.9.

$$I_{AA} = \frac{(0.6^4 - 0.4^4)}{12} = 8.67 \times 10^{-3}\,\text{m}^4$$

The maximum tensile stress due to bending occurs at C and is given by

$$\text{Max. tensile stress due to bending} = \frac{1.4 \times 10^3 \times (2 \times 0.6 \sin 45^\circ)(h^2/2) \times 0.6 \sin 45^\circ}{8.67 \times 10^{-3}}$$
$$= 29065.7\,h^2\ \text{N/m}^2$$

For zero tension at C, this must be numerically equal to the compressive stress produced by the chimney weight. That is

$$29065.7\,h^2 = 19620\,h$$

so that

$$h = 0.68\,\text{m}.$$

Note that in this example the middle third rule does not apply since the cross section of the chimney is hollow.

## EXAMPLE 9.10

An airship mooring mast is constructed from concrete, is 24 m high, and tapers from a diameter of 3 m at its base to 1.5 m at its top as shown in Fig. 9.17. The density of the concrete is 2000 kg/m$^3$. If the horizontal force on the top of the mast produced by the drag on the airship is $30p$, where $p$ is the wind pressure in kN/m$^2$, calculate the maximum allowable value of $p$ for there to be no tension developed at the base of the mast allowing fully for the effect of wind pressure on the mast itself.

For the value of pressure calculated, determine the stress distribution in the mast at mid-height. Comment on the result.

The forces on the mast produced by wind pressure may be calculated separately as follows.

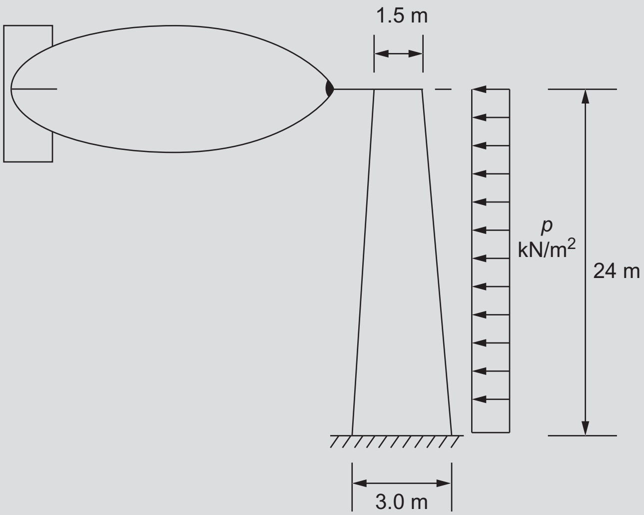



**FIGURE 9.17**

Airship mooring mast of Ex. 9.10.

Horizontal force on mast due to wind pressure on triangular projected area is given by

Horizontal force $= \frac{1}{2} \times 1.5 \times 24\ p = 18p$ kN acting at $24/3 = 8$ m from base.

Horizontal force on mast due to wind pressure on rectangular projected area is given by

Horizontal force $= 1.5 \times 24\ p = 36\ p$ acting 12 m from base.

Weight of mast = Average cross sectional area × height × density

i.e. Weight of mast $= \frac{\pi}{4} \times 2\ 25^2 \times 24 \times 2000 \times 9.81 \times 10^{-3} = 1872.3$ kN

The core of the base section has a radius $= 1.5/4 = 0.375$ m. Therefore, taking moments about a point on the core

$$30p \times 24 + 18p \times 8 + 36p \times 12 = 1872.3 \times 0.375$$

from which $p = 0.54$ kN/m$^2$.

The weight of the top half of the mast is given by

$$\text{Weight (top half)} = \frac{\pi}{4} \times 1.875^2 \times 12 \times 2000 \times 9.81 \times 10^{-3} = 650.1\,\text{kN}$$

$$\text{Wind pressure (triangular portion)} = \frac{1}{2} \times 0.75 \times 12p = 4.5p\,\text{kN}$$

Wind pressure (rectangular portion) $= 1.5 \times 12p = 18p$ kN

The compressive stress on the cross section of the mast at mid-height due to self-weight is

$$\text{Comp.stress (self-weight)} = \frac{650.1 \times 10^3}{(\pi/4) \times 2.25^2 \times 10^6} = 0.16\,\text{N/mm}^2$$

The bending moment at the mid-height of the mast is

$$\text{BM (mid-height)} = 30p \times 12 + 4.5p \times 4 + 18p \times 6 = 486p\,\text{kNm}$$

and since $p = 0.54$ kN/m$^2$

$$\text{BM(mid-height)} = 262.4\,\text{kNm}$$

The direct stress distribution due to wind pressure is then (see Eq. (9.9))

$$\sigma(\text{wind pressure}) = \frac{262.4 \times 10^6 y}{(\pi/64) \times 2.25^4 \times 10^{12}} = 2.09 \times 10^{-4} y\,\text{N/mm}^2$$

At the extremities of the cross section where $y = \pm\,1125$ mm, the direct stress due to wind pressure is then

$$\sigma(\text{wind pressure}) = \pm 2.09 \times 10^{-4} \times 1125 = \pm 0.24\,\text{N/mm}^2$$

Therefore, the direct stress distribution at the mid-height cross section varies linearly from $0.24 - 0.16 = +\,0.08$ N/mm$^2$ (tension) to $-0.24 - 0.16 = -0.4$ N/mm$^2$ (compression).

Although there is tension in the cross section of the mast, its value is extremely small and within the allowable limit for concrete.

## EXAMPLE 9.11

A brick-built chimney has a height of 20 m, an external diameter of 4 m, and walls 0.6 m thick. If the density of the brickwork is 2000 kg/m$^3$, determine the maximum wind pressure the chimney can withstand for there to be no tension developed anywhere in the cross section.

If the actual wind pressure is 1.2 kN/mm$^2$, determine the distribution of direct stress across the base of the chimney.

The self-weight of the chimney is $= \dfrac{\pi}{4}(4^2 - 2.8^2) \times 20 \times 2000 \times 9.81 \times 10^{-3}$

i.e. Weight $= 2514.8$ kN

The compressive stress due to the weight of the chimney on the chimney base is σ (self-weight)
$= \dfrac{2514.8 \times 10^3}{(\pi/4)\left(4^2 - 2.8^2\right) \times 10^6} = 0.39\,\text{N/mm}^2$

If the wind pressure is $p$ kN/mm$^2$ then the total wind force, based on projected area is given by

Wind force $= 4 \times 20p = 80p$ kN acting at mid-height.

The bending moment at the chimney base is then $80p \times 10$ kNm and the direct stress distribution is (see Eq. (9.9))

$$\sigma = \frac{80p \times 10 \times 10^6}{(\pi/64)\left(4^4 - 2.8^4\right) \times 10^{12}} y = 83.8p \times 10^6\, y$$

For no tension to be developed

$$83.8p \times 10^6 \times 2.0 \times 10^3 = 0.39,$$

which gives $p = 2.3$ kN/m$^2$

Note that since the chimney is hollow use of the core of a circular section is not applicable.

The wind force $P$ corresponding to a wind pressure of 1.2 kN/m$^2$ is given by

$$P = 1.2 \times 20 \times 4 = 96\,\text{kN}$$

The bending moment $M$ at the base of the chimney is then

$$M = 96 \times 10 = 960\,\text{kNm}$$

and from Eq. (9.9) the corresponding direct stress is

$$\sigma = \frac{960 \times 10^6}{(\pi/64)\left(4^4 - 2.8^4\right) \times 10^{12}} y = 100.5 \times 10^{-6} y$$

When $y = \pm 2.0$ m ($\pm$ 2000 mm)

$$\sigma = \pm 0.2\ \text{N/mm}^2$$

The direct stress therefore varies linearly across the section from

$$-0.39 + 0.2 = -0.19\ \text{N/mm}^2 \text{ to } -0.39 - 0.2 = -0.59\ \text{N/mm}^2.$$

## EXAMPLE 9.12

A masonry dam has the trapezoidal cross section shown in Fig. 9.18. The depth of the dam is 30 m, the width of the top of the dam is required to be 5 m to accommodate vehicle access, and it contains water to a depth of 27 m. If the density of the masonry is 2200 kg/m$^3$ and that of the water is 1000 kg/m$^3$, determine the minimum width of the base for there to be no tension developed in the base section.

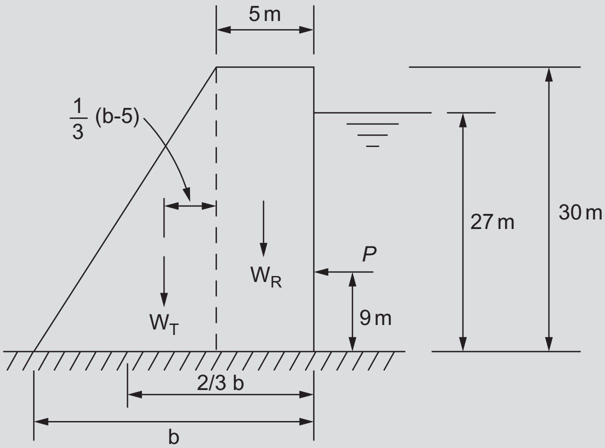



**FIGURE 9.18**

Masonry dam of Ex. 9.12.

Water pressure increases linearly with depth and is equal to $\rho h$ where $\rho$ is the density and $h$ the depth. The distribution is therefore triangular and its resultant will act at $2h/3$ from the water surface. Consider 1 m width of dam. The water pressure, $P$, is given by

$$P = 1000 \times 9.81 \times 10^{-3} \times 27^2/2 = 3575.7\ \text{kN}.$$

The weight of the 1 m length of dam is most easily calculated by dividing the cross section into a rectangle and a triangle. Then

$$\text{Weight}(1\,\text{m length}) = 2200 \times 9.81 \times 10^{-3} \times 1 \times 30\left[5 + \frac{(b-5)}{2}\right]$$

i.e.

$$\text{Weight (1 m length)} = 3237.3 + 323.7(b - 5) \text{ kN}$$

For there to be no tension developed in the base section, the resultant of the water pressure and the dam weight must lie within the middle third of the base section.

Then, taking moments about the middle third point

$$P \times 9 - 3237.3\left(\frac{2b}{3} - 2.5\right) - \frac{323.7}{3}(b-5)(b-10) = 0$$

Substituting for $P$ (= 3575.7 kN) and rearranging gives a quadratic equation in $b$

i.e.

$$b^2 + 5.0b - 323.3 = 0$$

Solving

$$b = 15.6 \text{ m (The negative solution has no significance)}$$

## EXAMPLE 9.13

An earth-retaining wall is constructed from brick and has the cross section shown in Fig. 9.19. If the density of the soil is 1500 kg/m$^3$ and that of the brickwork is 2100 kg/m$^3$, determine the maximum depth of soil the retaining wall can support for there to be no tension developed in the wall. Then, for this depth of soil, check the stability of the wall against overturning and also against sliding if the coefficient of friction between the base of the wall and the ground is 0.7.

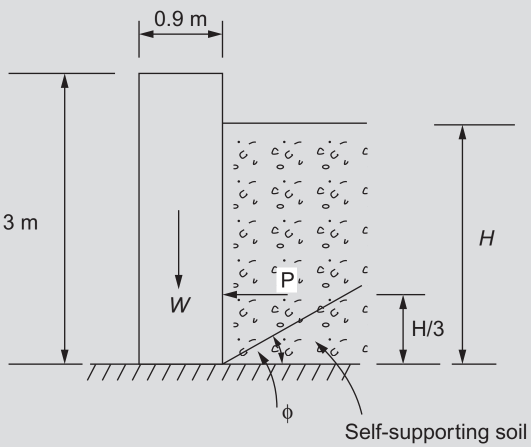



**FIGURE 9.19**

Earth retaining wall of Ex. 9.13.

Note that the Rankine theory for the soil pressure, $p$, at a depth $h$ is given by

$$p = wh\frac{(1 - \sin\phi)}{(1 + \sin\phi)} \text{ kN/mm}^2$$

where $w$ is the soil density and $\phi$ the angle of repose (see Fig. 9.19).

A reasonable assumption for $\phi$ is 30°. Then, the soil pressure at a depth $h$ is

$$p = 1500 \times 9.81 \times 10^{-3} h \frac{(1-0.5)}{(1+0.5)} = 4.91 h\,\mathrm{kN/m^2}$$

Since the soil pressure varies linearly with depth the distribution is triangular. The total soil pressure $P$ acting on a 1 m length of wall is then given by

$$P = \frac{1}{2} \times 4.91\,H^2 = 2.46\,H^2\,\mathrm{kN}$$

acting at $H/3$ from the base of the wall.

The weight, $W$, of 1 m length of wall is

$$W = 2100 \times 9.81 \times 10^{-3} \times 0.9 \times 3 \times 1 = 55.62\,\mathrm{kN}$$

For zero tension at B, the resultant of $P$ and $W$ must pass through the middle third point. Therefore, taking moments about the middle third point

$$P(H/3) - W \times 0.15 = 0$$

i.e.

$$2.46\,H^3 - 55.62 \times 0.15 = 0,$$

which gives

$$H = 2.2\ \mathrm{m}.$$

For this depth of soil $P = 2.46 \times 2.2^2 = 11.9$ kN and its moment about A is $11.9 \times 2.2/3 = 8.73$ kNm. The moment of $W$ about A is $55.62 \times 0.45 = 25.03$ kNm.

The wall will not therefore overturn.

The wall will slide if $P > 0.7W$, i.e. if $P > 0.7 \times 55.62$ (38.93 kN). Since $P = 11.9$ kN, the wall will not slide.

## 9.3 Anticlastic bending

In the rectangular beam section shown in Fig. 9.20(a) the direct stress distribution due to a positive bending moment applied in a vertical plane varies from compression in the upper half of the beam to tension in the lower half (Fig. 9.20(b)). However, due to the Poisson effect (see Section 7.8) the compressive stress produces a lateral elongation of the upper fibres of the beam section while the tensile stress produces a lateral contraction of the lower. The section does not therefore remain rectangular but distorts as shown in Fig. 9.20 (c); the effect is known as *anticlastic bending*.

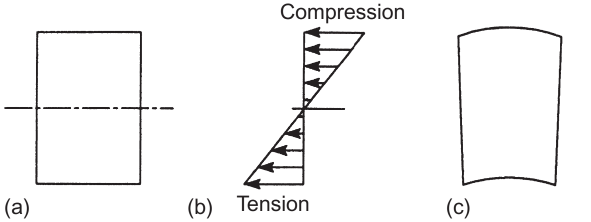



**FIGURE 9.20**

Anticlastic bending of a beam section.

Anticlastic bending is of interest in the analysis of thin-walled box beams in which the cross sections are maintained by stiffening ribs. The prevention of anticlastic distortion induces local variations in stress distributions in the webs and covers of the box beam and also in the stiffening ribs.

## 9.4 Strain energy in bending

A positive bending moment applied to a length of beam causes the upper longitudinal fibres to be compressed and the lower ones to stretch as shown in Fig. 9.5(a). The bending moment therefore does work on the length of beam and this work is absorbed by the beam as strain energy.

Suppose that the bending moment, $M$, in Fig. 9.5(a) is gradually applied so that when it reaches its final value the angle subtended at the centre of curvature by the element $\delta x$ is $\delta\theta$. From Fig. 9.5(a) we see that

$$R\,\delta\theta = \delta x$$

Substituting in Eq. (9.7) for $R$ we obtain

$$M = \frac{EI_z}{\delta x}\delta\theta \tag{9.19}$$

so that $\delta\theta$ is a linear function of $M$. It follows that the work done by the gradually applied moment $M$ is $M\,\delta\theta/2$ subject to the condition that the limit of proportionality is not exceeded. The strain energy, $\delta U$, of the elemental length of beam is therefore given by

$$\delta U = \frac{1}{2} M\,\delta\theta \tag{9.20}$$

or, substituting for $\delta\theta$ from Eq. (9.19) in Eq. (9.20)

$$\delta U = \frac{1}{2}\frac{M^2}{EI_z}\delta x$$

The total strain energy, $U$, due to bending in a beam of length $L$ is therefore

$$U = \int_L \frac{M^2}{2EI_z}\mathrm{d}x \tag{9.21}$$

## 9.5 Unsymmetrical bending

Frequently in civil engineering construction beam sections do not possess any axes of symmetry. Typical examples are shown in Fig. 9.21 where the angle section has legs of unequal length and the Z-section possesses anti- or skew symmetry about a horizontal axis through its centroid, but not symmetry. We shall now develop the theory of bending for beams of arbitrary cross section.

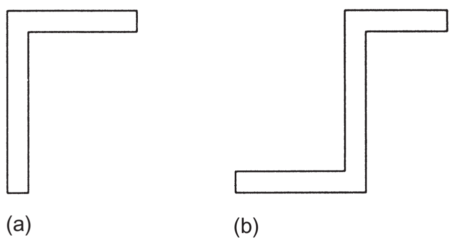


**FIGURE 9.21**

Unsymmetrical beam sections.

## Assumptions

We shall again assume, as in the case of symmetrical bending, that plane sections of the beam remain plane after bending and that the material of the beam is homogeneous and linearly elastic.

## Sign conventions and notation

Since we are now concerned with the general case of bending we may apply loading systems to a beam in any plane. However, no matter how complex these loading systems are, they can always be resolved into components in planes containing the three coordinate axes of the beam. We shall use an identical system of axes to that shown in Fig. 3.6, but our notation for loads must be extended and modified to allow for the general case.

As far as possible we shall adopt sign conventions and a notation which are consistent with those shown in Fig. 3.6. Thus, in Fig. 9.22, the externally applied shear load $W_y$ is parallel to the $y$ axis but vertically downwards, i.e. in the negative $y$ direction as before; similarly we take $W_z$ to act in the negative $z$ direction. The distributed loads $w_y(x)$ and $w_z(x)$ can be functions of $x$ and are also applied in the negative directions of the axes. The bending moment $M_z$ in the vertical $xy$ plane is, as before, a sagging (i.e. positive) moment and will produce compressive direct stresses in the positive $yz$ quadrant of the beam section. In the same way $M_y$ is positive when it produces compressive stresses in the positive $yz$ quadrant of the beam section. The applied torque $T$ is positive when anticlockwise when viewed in the direction $x$O and the displacements, $u$, $v$ and $w$ are positive in the positive directions of the $z$, $y$ and $x$ axes, respectively.

The positive directions and senses of the internal forces acting on the positive face (see Section 3.2) of a beam section are shown in Fig. 9.23 and agree, as far as the shear force and bending moment in the vertical $xy$ plane are concerned, with those in Fig. 3.7. The positive internal horizontal shear force $S_z$ is in the positive direction of the $z$ axis while the internal moment $M_y$ produces compression in the positive $yz$ quadrant of the beam section.

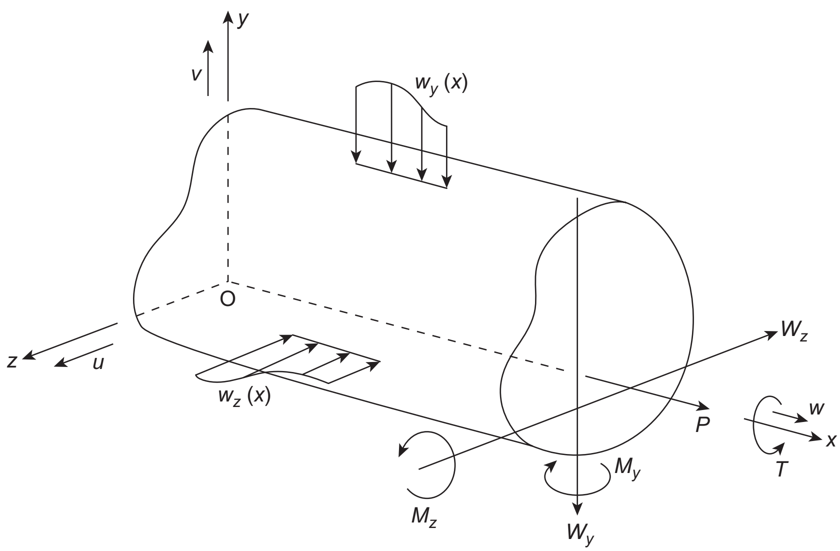



**FIGURE 9.22**

Sign conventions and notation.

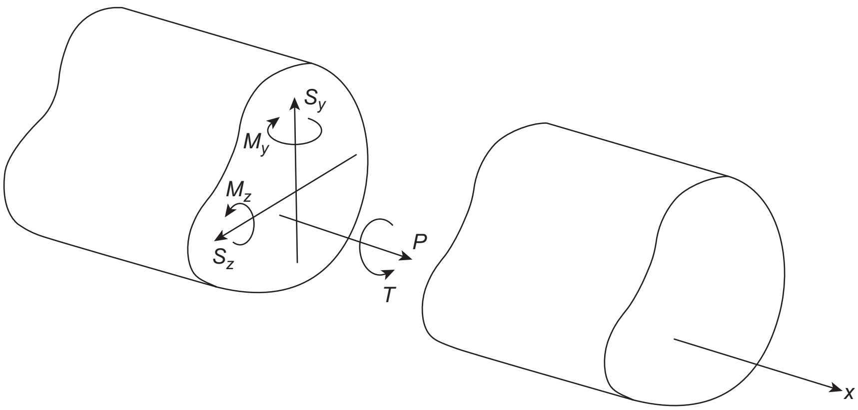



**FIGURE 9.23**

Internal force system.

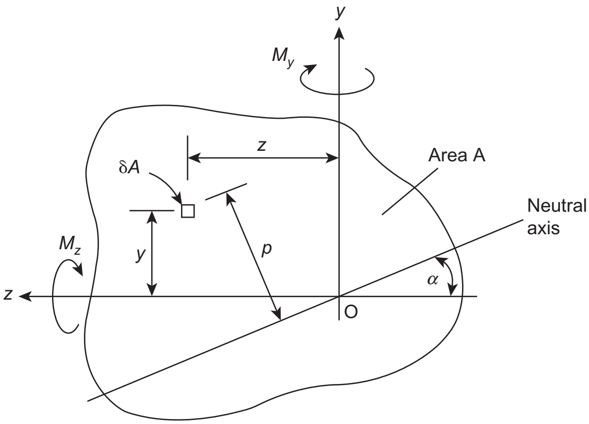



**FIGURE 9.24**

Bending of an unsymmetrical section beam.

## Direct stress distribution

Figure 9.24 shows the positive face of the cross section of a beam which is subjected to positive internal bending moments $M_z$ and $M_y$. Suppose that the origin O of the $y$ and $z$ axes lies on the neutral axis of the beam section; as yet the position of the neutral axis and its inclination to the $z$ axis are unknown.

We have seen in Section 9.1 that a beam bends about the neutral axis of its cross section so that the radius of curvature, $R$, of the beam is perpendicular to the neutral axis. Therefore, by direct comparison with Eq. (9.2) it can be seen that the direct stress, $\sigma_x$, on the element, $\delta A$, a perpendicular distance $p$ from the neutral axis, is given by

$$\sigma_x = -E\frac{p}{R} \tag{9.22}$$

The beam section is subjected to a pure bending moment so that the resultant direct load on the section is zero. Hence

$$\int_A \sigma_x \, \mathrm{d}A = 0$$

Replacing $\sigma_x$ in this equation from Eq. (9.22) we have

$$-\int_A E\frac{p}{R}\,\mathrm{d}A = 0$$

or, for a beam of a given material subjected to a given bending moment

$$\int_A p\,\mathrm{d}A = 0 \tag{9.23}$$

Qualitatively Eq. (9.23) states that the first moment of area of the beam section about the neutral axis is zero. It follows that in problems involving the pure bending of beams the neutral axis always passes through the centroid of the beam section. We shall therefore choose the centroid, G, of a section as the origin of axes.

From Fig. 9.20 we see that

$$p = z\sin\alpha + y\cos\alpha \tag{9.24}$$

so that from Eq. (9.22)

$$\sigma_x = -\frac{E}{R}(z\sin\alpha + y\cos\alpha) \tag{9.25}$$

The moment resultants of the direct stress distribution are equivalent to $M_z$ and $M_y$ so that

$$M_z = -\int_A \sigma_x y\,\mathrm{d}A \quad M_y = -\int_A \sigma_x z\,\mathrm{d}A \quad (\text{see Section 9.1}) \tag{9.26}$$

Substituting for $\sigma_x$ from Eq. (9.25) in Eq. (9.26), we obtain

$$\left.\begin{aligned} M_z &= \frac{E\sin\alpha}{R}\int_A zy\,\mathrm{d}A + \frac{E\cos\alpha}{R}\int_A y^2\,\mathrm{d}A \\ M_y &= \frac{E\sin\alpha}{R}\int_A z^2\,\mathrm{d}A + \frac{E\cos\alpha}{R}\int_A zy\,\mathrm{d}A \end{aligned}\right\} \tag{9.27}$$

In Eq. (9.27)

$$\int_A zy\,\mathrm{d}A = I_{zy} \quad \int_A y^2\,\mathrm{d}A = I_z \quad \int_A z^2\,\mathrm{d}A = I_y$$

where $I_{zy}$ is the product second moment of area of the beam section about the $z$ and $y$ axes, $I_z$ is the second moment of area about the $z$ axis and $I_y$ is the second moment of area about the $y$ axis. Equation (9.27) may therefore be rewritten as

$$\left.\begin{aligned} M_z &= \frac{E\sin\alpha}{R}I_{zy} + \frac{E\cos\alpha}{R}I_z \\ M_y &= \frac{E\sin\alpha}{R}I_y + \frac{E\cos\alpha}{R}I_{zy} \end{aligned}\right\} \tag{9.28}$$

Solving Eq. (9.28)

$$\frac{E\sin\alpha}{R} = \frac{M_y I_z - M_z I_{zy}}{I_z I_y - I_{zy}^2} \tag{9.29}$$

$$\frac{E\cos\alpha}{R} = \frac{M_z I_y - M_y I_{zy}}{I_z I_y - I_{zy}^2} \tag{9.30}$$

Now substituting these expressions in Eq. (9.25)

$$\sigma_x = -\left(\frac{M_y I_z - M_z I_{zy}}{I_z I_y - I_{zy}^2}\right) z - \left(\frac{M_z I_y - M_y I_{zy}}{I_z I_y - I_{zy}^2}\right) y \tag{9.31}$$

In the case where the beam section has either O$z$ or O$y$ (or both) as an axis of symmetry $I_{zy} = 0$ (see Section 9.6) and Eq. (9.31) reduces to

$$\sigma_x = -\frac{M_y}{I_y} z - \frac{M_z}{I_z} y \tag{9.32}$$

which is identical to Eq. (ii) in Ex. 9.4.

### Position of the neutral axis

We have established that the neutral axis of a beam section passes through the centroid of area of the section whether the section has an axis of symmetry or not. The inclination $\alpha$ of the neutral axis to the $z$ axis in Fig. 9.24 is obtained from Eq. (9.31) using the fact that the direct stress is zero at all points on the neutral axis. Then, for a point $(z_{\mathrm{NA}}, y_{\mathrm{NA}})$

$$0 = (M_z I_{zy} - M_y I_z) z_{\mathrm{NA}} + (M_y I_{zy} - M_z I_y) y_{\mathrm{NA}}$$

so that

$$\frac{y_{\mathrm{NA}}}{z_{\mathrm{NA}}} = -\frac{(M_z I_{zy} - M_y I_z)}{(M_y I_{zy} - M_z I_y)}$$

or, referring to Fig. 9.24

$$\tan\alpha = \frac{(M_z I_{zy} - M_y I_z)}{(M_y I_{zy} - M_z I_y)} \tag{9.33}$$

since $\alpha$ is positive when $y_{\mathrm{NA}}$ is positive and $z_{\mathrm{NA}}$ is negative. Again, for a beam having a cross section with either O$y$ or O$z$ as an axis of symmetry, $I_{zy} = 0$ and Eq. (9.33) reduces to

$$\tan\alpha = \frac{M_y I_z}{M_z I_y} \quad \text{(see Eq. (9.14) in Ex. 9.4)}$$

## 9.6 Calculation of section properties

It will be helpful at this stage to discuss the calculation of the various section properties required in the analysis of beams subjected to bending. Initially, however, two useful theorems are quoted.

### Parallel axes theorem

Consider the beam section shown in Fig. 9.25 and suppose that the second moment of area, $I_{\mathrm{G}}$, about an axis through its centroid G is known. The second moment of area, $I_{\mathrm{N}}$, about a parallel axis, NN, a distance $b$ from the centroidal axis is then given by

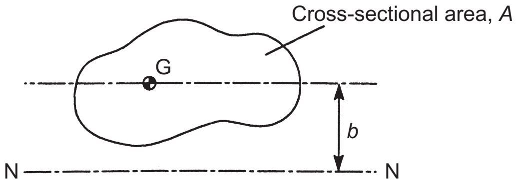



**FIGURE 9.25**

Parallel axes theorem.

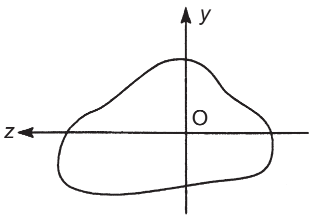



**FIGURE 9.26**

Theorem of perpendicular axes.

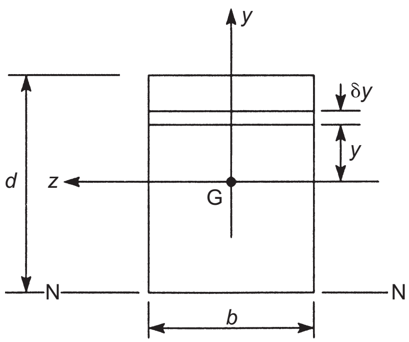



**FIGURE 9.27**

Second moments of area of a rectangular section.

$$I_N = I_G + Ab^2 \tag{9.34}$$

## Theorem of perpendicular axes

In Fig. 9.26 the second moments of area, $I_z$ and $I_y$, of the section about O$z$ and O$y$ are known. The second moment of area about an axis through O perpendicular to the plane of the section (i.e. a *polar second moment of area)* is then

$$I_o = I_z + I_y \tag{9.35}$$

## Second moments of area of standard sections

Many sections in use in civil engineering such as those illustrated in Fig. 9.2 may be regarded as comprising a number of rectangular shapes. The problem of determining the properties of such sections is simplified if the second moments of area of the rectangular components are known and use is made of the parallel axes theorem. Thus, for the rectangular section of Fig. 9.27

$$I_z = \int_A y^2\,\mathrm{d}A = \int_{-d/2}^{d/2} by^2\,\mathrm{d}y = b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

which gives

$$I_z = \frac{bd^3}{12} \tag{9.36}$$

Similarly

$$I_y = \frac{db^3}{12} \tag{9.37}$$

Frequently it is useful to know the second moment of area of a rectangular section about an axis which coincides with one of its edges. Thus in Fig. 9.27, and using the parallel axes theorem

$$I_{\mathrm{N}} = \frac{bd^3}{12} + bd\left(-\frac{d}{2}\right)^2 = \frac{bd^3}{3} \tag{9.38}$$

## EXAMPLE 9.14

Determine the second moments of area $I_z$ and $I_y$ of the I-section shown in Fig. 9.28.

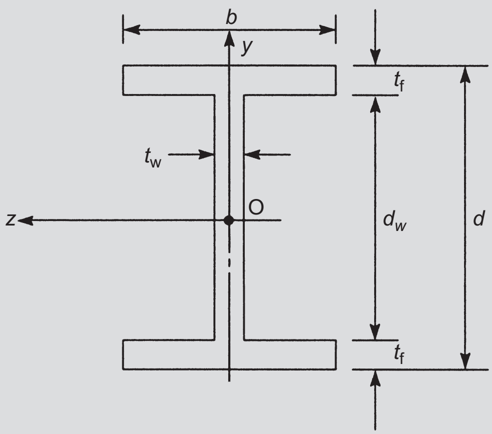



**FIGURE 9.28**

Second moments of area of an I-section.

Using Eq. (9.36)

$$I_z = \frac{bd^3}{12} - \frac{(b - t_w)d_w^3}{12}$$

Alternatively, using the parallel axes theorem in conjunction with Eq. (9.36)

$$I_z = 2\left[\frac{bt_f^3}{12} + bt_f\left(\frac{d_w + t_f}{2}\right)^2\right] + \frac{t_w d_w^3}{12}$$

The equivalence of these two expressions for $I_z$ is most easily demonstrated by a numerical example. Also, from Eq. (9.37)

$$I_y = 2\frac{t_f b^3}{12} + \frac{d_w t_w^3}{12}$$

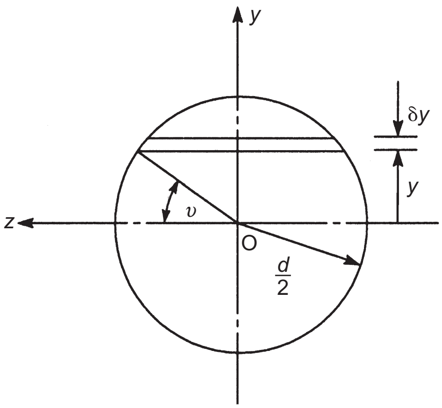



**FIGURE 9.29**

Second moments of area of a circular section.

It is also useful to determine the second moment of area, about a diameter, of a circular section. In Fig. 9.29 where the $z$ and $y$ axes pass through the centroid of the section

$$I_z = \int_A y^2 \, \mathrm{d}A = \int_{-d/2}^{d/2} 2\left(\frac{d}{2}\cos\theta\right) y^2 \, \mathrm{d}y \tag{9.39}$$

Integration of Eq. (9.39) is simplified if an angular variable, $\theta$, is used. Thus

$$I_z = \int_{-\pi/2}^{\pi/2} d\cos\theta\left(\frac{d}{2}\sin\theta\right)^2 \frac{d}{2}\cos\theta \, \mathrm{d}\theta$$

i.e.

$$I_z = \frac{d^4}{8}\int_{-\pi/2}^{\pi/2} \cos^2\theta \sin^2\theta \, \mathrm{d}\theta$$

which gives

$$I_z = \frac{\pi d^4}{64} \tag{9.40}$$

Clearly from symmetry

$$I_y = \frac{\pi d^4}{64} \tag{9.41}$$

Using the theorem of perpendicular axes, the polar second moment of area, $I_o$, is given by

$$I_o = I_z + I_y = \frac{\pi d^4}{32} \tag{9.42}$$

## Product second moment of area

The product second moment of area, $I_{zy}$, of a beam section with respect to $z$ and $y$ axes is defined by

$$I_{zy} = \int_A zy \, \mathrm{d}A \tag{9.43}$$

Thus each element of area in the cross section is multiplied by the product of its coordinates and the integration is taken over the complete area. Although second moments of area are always positive since elements of area are multiplied by the square of one of their coordinates, it is possible for $I_{zy}$ to be negative if the section lies predominantly in the second and fourth quadrants of the axes system. Such a situation would arise in the case of the Z-section of Fig. 9.30(a) where the product second moment of area of each flange is clearly negative.

A special case arises when one (or both) of the coordinate axes is an axis of symmetry so that for any element of area, $\delta A$, having the product of its coordinates positive, there is an identical element for which the product of its coordinates is negative (Fig. 9.30(b)).

Summation (i.e. integration) over the entire section of the product second moment of area of all such pairs of elements results in a zero value for $I_{zy}$.

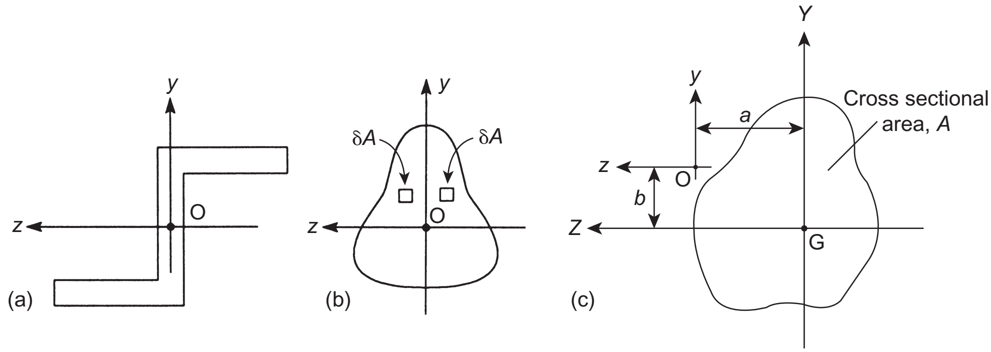



**FIGURE 9.30**

Product second moment of area.

We have shown previously that the parallel axes theorem may be used to calculate second moments of area of beam sections comprising geometrically simple components. The theorem can be extended to the calculation of product second moments of area. Let us suppose that we wish to calculate the product second moment of area, $I_{zy}$, of the section shown in Fig. 9.30(c) about axes $zy$ when $I_{ZY}$ about its own, say centroidal, axes system GZY is known. From Eq. (9.43)

$$I_{zy} = \int_A zy \, \mathrm{d}A$$

or

$$I_{zy} = \int_A (Z-a)(Y-b) \, \mathrm{d}A$$

which, on expanding, gives

$$I_{zy} = \int_A ZY \, \mathrm{d}A - b\int_A Z \, \mathrm{d}A - a\int_A Y \, \mathrm{d}A + ab\int_A dA$$

If $Z$ and $Y$ are centroidal axes then $\int_A Z \, \mathrm{d}A = \int_A Y \, \mathrm{d}A = 0$. Hence

$$I_{zy} = I_{ZY} + abA \tag{9.44}$$

It can be seen from Eq. (9.44) that if either G$Z$ or G$Y$ is an axis of symmetry, i.e. $I_{ZY} = 0$, then

$$I_{zy} = abA \tag{9.45}$$

Thus for a section component having an axis of symmetry that is parallel to either of the section reference axes the product second moment of area is the product of the coordinates of its centroid multiplied by its area.

A table of the properties of a range of beam sections is given in Appendix A.

## EXAMPLE 9.15

A beam having the cross section shown in Fig. 9.31 is subjected to a hogging bending moment of 1500 Nm in a vertical plane. Calculate the maximum direct stress due to bending stating the point at which it acts.

The position of the centroid, G, of the section may be found by taking moments of areas about some convenient point. Thus

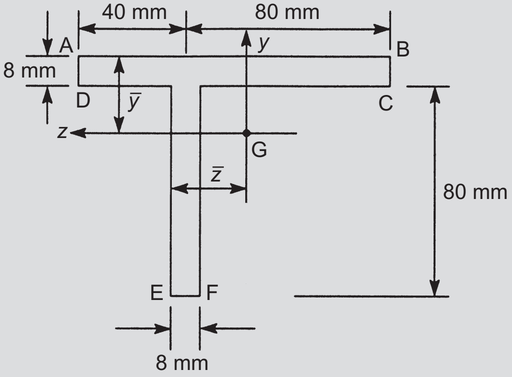



**FIGURE 9.31**

Beam section of Ex. 9.15.

$$(120 \times 8 + 80 \times 8)\bar{y} = 120 \times 8 \times 4 + 80 \times 8 \times 48$$

which gives

$$\bar{y} = 21.6 \text{ mm}$$

and

$$(120 \times 8 + 80 \times 8)\bar{z} = 80 \times 8 \times 4 + 120 \times 8 \times 24$$

giving

$$\bar{z} = 16 \text{ mm}$$

The second moments of area referred to axes G$zy$ are now calculated.

$$I_z = \frac{120 \times (8)^3}{12} + 120 \times 8 \times (17.6)^2 + \frac{8 \times (80)^3}{12} + 80 \times 8 \times (26.4)^2$$
$$= 1.09 \times 10^6 \text{ mm}^4$$
$$I_y = \frac{8 \times (120)^3}{12} + 120 \times 8 \times (8)^2 + \frac{80 \times (8)^3}{12} + 80 \times 8 \times (12)^2$$
$$= 1.31 \times 10^6 \text{ mm}^4$$
$$I_{zy} = 120 \times 8 \times (-8) \times (+17.6) + 80 \times 8 \times (+12) \times (-26.4)$$
$$= -0.34 \times 10^6 \text{ mm}^4$$

Since $M_z = -1500$ Nm and $M_y = 0$ we have from Eq. (9.31)

$$\sigma_x = -\frac{1500 \times 10^3 \times (-0.34 \times 10^6)z + 1500 \times 10^3 \times (1.31 \times 10^6)y}{1.09 \times 10^6 \times 1.31 \times 10^6 - (-0.34 \times 10^6)^2}$$

i.e.

$$\sigma_x = 0.39z + 1.5y \qquad \text{(i)}$$

Note that the denominator in both the terms in Eq. (9.31) is the same.

Inspection of Eq. (i) shows that $\sigma_x$ is a maximum at F where $z = 8$ mm, $y = -66.4$ mm. Hence

$$\sigma_{x,\max} = -96.5 \text{ N/mm}^2 \text{ (compressive)}$$

## Approximations for thin-walled sections

Modern civil engineering structures frequently take the form of thin-walled cellular box beams which combine the advantages of comparatively low weight and high strength, particularly in torsion. Other forms of thin-walled structure consist of 'open' section beams such as a plate girder which is constructed from thin plates stiffened against instability. In addition to these there are the cold-formed sections which we discussed in Chapter 1.

There is no clearly defined line separating 'thick' and 'thin-walled' sections; the approximations allowed in the analysis of thin-walled sections become increasingly inaccurate the 'thicker' a section becomes. However, as a guide, it is generally accepted that the approximations are reasonably accurate for sections for which the ratio

$$\frac{t_{max}}{b} \leq 0.1$$

where $t_{max}$ is the maximum thickness in the section and $b$ is a typical cross-sectional dimension.

In the calculation of the properties of thin-walled sections we shall assume that the thickness, $t$, of the section is small compared with its cross-sectional dimensions so that squares and higher powers of $t$ are neglected. The section profile may then be represented by the mid-line of its wall. Stresses are then calculated at points on the mid-line and assumed to be constant across the thickness.

### EXAMPLE 9.16

Calculate the second moment of area, $I_z$, of the channel section shown in Fig. 9.32(a).

The centroid of the section is located midway between the flanges; its horizontal position is not needed since only $I_z$ is required. Thus

$$I_z = 2\left(\frac{bt^3}{12} + bth^2\right) + t\frac{[2(h - t/2)]^3}{12}$$

which, on expanding, becomes

$$I_z = 2\left(\frac{bt^3}{12} + bth^2\right) + \frac{t}{12}\left[(2)^3\left(h^3 - \frac{3h^2t}{2} + \frac{3ht^2}{4} - \frac{t^3}{8}\right)\right]$$

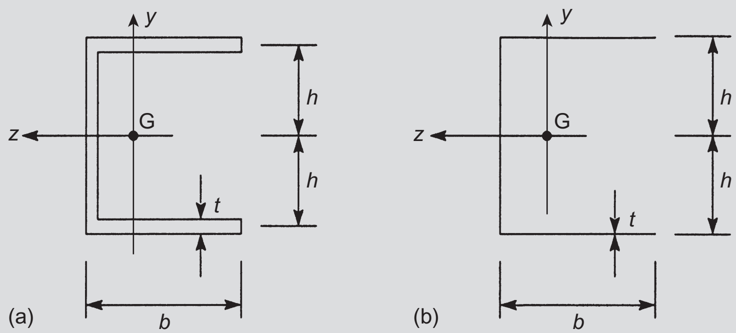



**FIGURE 9.32**

Calculation of the second moment of area of a thin-walled channel section.

Neglecting powers of $t^2$ and upwards we obtain

$$I_z = 2bth^2 + t\frac{(2h)^3}{12}$$

It is unnecessary for such calculations to be carried out in full since the final result may be obtained almost directly by regarding the section as being represented by a single line as shown in Fig. 9.32(b).

## EXAMPLE 9.17

A thin-walled beam has the cross section shown in Fig. 9.33. Determine the direct stress distribution produced by a hogging bending moment $M_z$.

The beam cross section is antisymmetrical so that its centroid is at the mid-point of the vertical web. Furthermore, $M_y=0$ so that Eq. (9.31) reduces to

$$\sigma_x = \frac{M_z I_{zy} z - M_z I_y y}{I_z I_y - I_{zy}^2} \tag{i}$$

But $M_z$ is a hogging bending moment and therefore negative. Eq. (i) must then be rewritten as

$$\sigma_x = \frac{-M_z I_{zy} z + M_z I_y y}{I_z I_y - I_{zy}^2} \tag{ii}$$

The section properties are calculated using the previously specified approximations for thin-walled sections; thus

$$I_z = 2\frac{ht}{2}\left(\frac{h}{2}\right)^2 + \frac{th^3}{12} = \frac{h^3 t}{3}$$

$$I_y = 2\frac{t}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3 t}{12}$$

$$I_{zy} = \frac{ht}{2}\left(\frac{h}{4}\right)\left(\frac{h}{2}\right) + \frac{ht}{2}\left(-\frac{h}{4}\right)\left(-\frac{h}{2}\right) = \frac{h^3 t}{8}$$

Substituting these values in Eq. (ii) we obtain

$$\sigma_x = \frac{M_z}{h^3 t}(6.86y - 10.3z) \tag{iii}$$

On the top flange $y = +h/2$, $h/2 \geq z \geq 0$ and the distribution of direct stress is given by

$$\sigma_x = \frac{M_z}{h^3 t}(3.43h - 10.3z)$$

which is linear. Hence

$$\sigma_{x,\mathrm{A}} = -\frac{1.72M_z}{h^2 t}(\text{compressive})$$

$$\sigma_{x,\mathrm{B}} = +\frac{3.43M_z}{h^2 t}(\text{tensile})$$

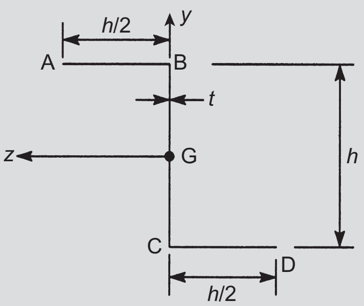



**FIGURE 9.33**

Beam section of Ex. 9.17.

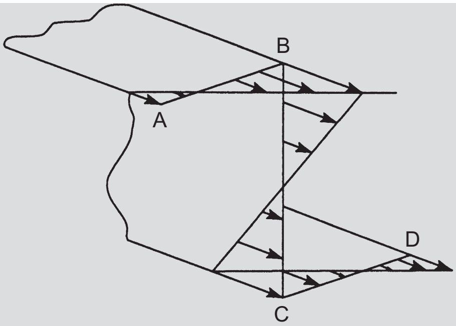



**FIGURE 9.34**

Distribution of direct stress in beam section of Ex. 9.17.

In the web $-h/2 \leq y \leq h/2$ and $z = 0$ so that Eq. (iii) reduces to

$$\sigma_x = \frac{6.86 M_z}{h^3 t} y$$

Again the distribution is linear and varies from

$$\sigma_{x,\mathrm{B}} = +\frac{3.43 M_z}{h^2 t} (\text{tensile})$$

to

$$\sigma_{x,\mathrm{C}} = -\frac{3.43 M_z}{h^2 t} (\text{compressive})$$

The distribution in the lower flange may be deduced from antisymmetry. The complete distribution is as shown in Fig. 9.34.

## Second moments of area of inclined and curved thin-walled sections

Thin-walled sections frequently have inclined or curved walls which complicate the calculation of section properties. Consider the inclined thin section of Fig. 9.35. The second moment of area of an element $\delta s$ about a horizontal axis through its centroid G is equal to $t \delta s y^2$. Therefore the total second moment of area of the section about G$z$, $I_z$, is given by

$$I_z = \int_{-a/2}^{a/2} t y^2 \, \mathrm{d}s = \int_{-a/2}^{a/2} t (s \sin \beta)^2 \, \mathrm{d}s$$

i.e.

$$I_z = \frac{a^3 t \sin^2 \beta}{12}$$

Similarly

$$I_y = \frac{a^3 t \cos^2 \beta}{12}$$

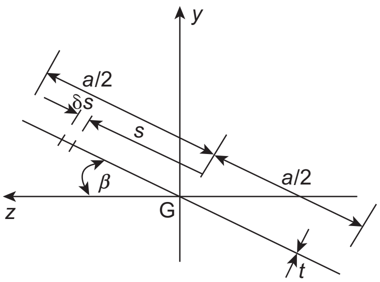



**FIGURE 9.35**

Second moments of area of an inclined thin-walled section.

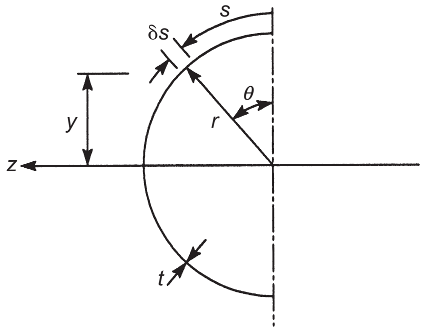



**FIGURE 9.36**

Second moment of area of a semicircular thin-walled section.

The product second moment of area of the section about G$zy$ is

$$I_{zy} = \int_{-a/2}^{a/2} tzy\,\mathrm{d}s = \int_{-a/2}^{a/2} t(s\cos\beta)(s\sin\beta)\,\mathrm{d}s$$

i.e.

$$I_{zy} = \frac{a^3 t \sin 2\beta}{24}$$

Properties of thin-walled curved sections are found in a similar manner. Thus $I_z$ for the semicircular section of Fig. 9.36 is

$$I_z = \int_0^{\pi r} ty^2\,\mathrm{d}s$$

Expressing $y$ and $s$ in terms of a single variable $\theta$ simplifies the integration; hence

$$I_z = \int_0^{\pi} t(r\cos\theta)^2 r\,\mathrm{d}\theta$$

from which

$$I_z = \frac{\pi r^3 t}{2}$$

## EXAMPLE 9.18

Calculate the second moments of area of the thin-walled beam section shown in Fig. 9.37. Note that the position of the centroid of area of the circular portion of the beam section is given.

The first step is to find the position of the centroid of area of the beam section. Therefore, taking moments of area about the flange 12

$$(2rt + 2rt + \pi rt/2)\bar{z} = 2rtr + \pi rt \times 2.45r/2$$

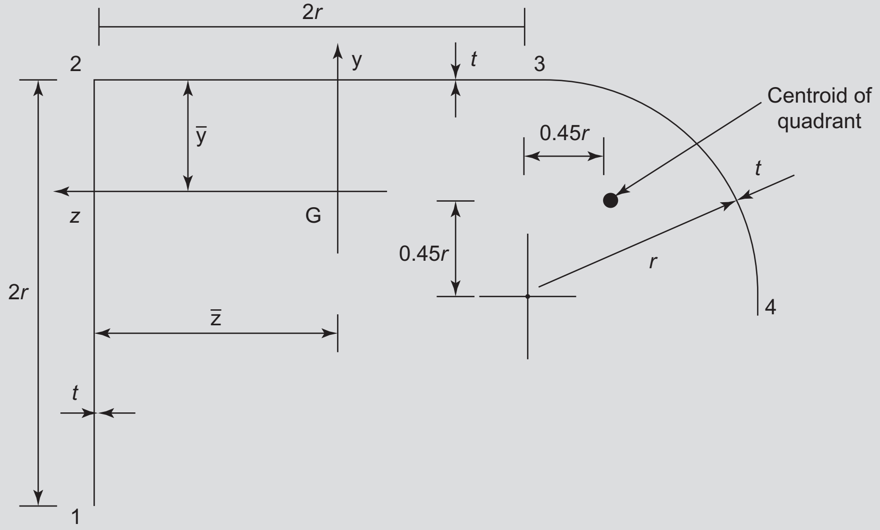



**FIGURE 9.37**

Beam section of Ex. 9.18.

which gives

$$\bar{z} = 1.05r$$

Now taking moments of area about the flange 23

$$(2rt + 2rt + \pi rt/2)\bar{y} = 2rtr + \pi rt \times 0.55r/2$$

from which

$$\bar{y} = 0.51r$$

The second moments of area of the quadrant portion of the section about axes $ZY$ through its own centroid may be found by referring to Fig. 9.38. Then

$$I_Z = \int_0^{\pi/r2} tY^2\, ds = \int_0^{\pi/2} t(r\cos\theta - 0.45r)^2 r\, d\theta$$

which gives

$$I_Z = 0.7tr^3 = I_Y$$

$$I_{ZY} = \int_0^{\pi r/2} tZY\, ds = \int_0^{\pi/2} t(0.45r - r\sin\theta)(r\cos\theta - 0.45r)r\, d\theta$$

from which

$$I_{ZY} = 0.58tr^3$$

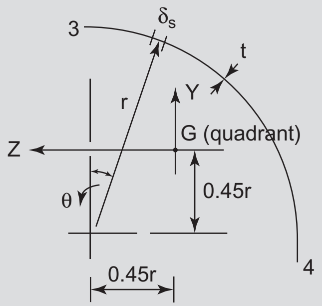



**FIGURE 9.38**

Quadrant of beam section of Ex. 9.18.

Now, from Fig. 9.37

$$I_z = \frac{t(2r^3)}{12} + 2rt(0.49r)^2 + 2rt(0.51r)^2 + 0.7tr^3 + \frac{\pi rt}{2}(0.04r)^2$$

so that

$$I_z = 2.87tr^3$$

$$I_y = 2rt(1.05r)^2 + \frac{t(2r)^3}{12} + 2tr(0.05r)^2 + 0.7tr^3 + \frac{\pi rt}{2}(1.4r)^2$$

which gives

$$I_y = 6.65tr^3$$

$$I_{zy} = 2rt(1.05r)(-0.49r) + 2rt(0.05r)(0.51r) + 0.58tr^3 + \frac{\pi rt}{2}(-1.4r)(0.04r)$$

from which

$$I_{zy} = -0.31tr^3$$

## 9.7 Principal axes and principal second moments of area

In any beam section there is a set of axes, neither of which need necessarily be an axis of symmetry, for which the product second moment of area is zero. Such axes are known as *principal axes* and the second moments of area about these axes are termed principal second moments of area.

Consider the arbitrary beam section shown in Fig. 9.39. Suppose that the second moments of area $I_z$, $I_y$ and the product second moment of area, $I_{zy}$, about arbitrary axes O$zy$ are known. By definition

$$I_z = \int_A y^2\,dA \quad I_y = \int_A z^2\,dA \quad I_{zy} = \int_A zy\,dA \tag{9.46}$$

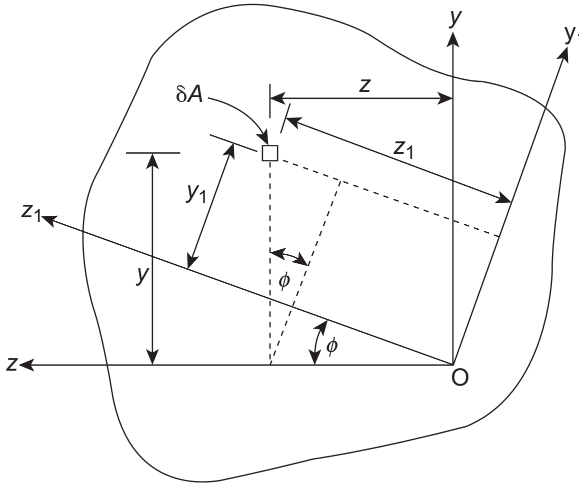



**FIGURE 9.39**

Principal axes in a beam of arbitrary section.

The corresponding second moments of area about axes $Oz_1y_1$ are

$$I_{z(1)} = \int_A y_1^2\, dA \quad I_{y(1)} = \int_A z_1^2\, dA \quad I_{z(1)y(1)} = \int_A z_1 y_1\, dA \tag{9.47}$$

From Fig. 9.39

$$z_1 = z\cos\phi + y\sin\phi \quad y_1 = y\cos\phi - z\sin\phi$$

Substituting for $y_1$ in the first of Eq. (9.47)

$$I_{z(1)} = \int_A (y\cos\phi - z\sin\phi)^2\, dA$$

Expanding, we obtain

$$I_{z(1)} = \cos^2\phi \int_A y^2\, dA + \sin^2\phi \int_A z^2 dA - 2\cos\phi\sin\phi \int_A zy\, dA$$

which gives, using Eq. (9.46)

$$I_{z(1)} = I_z\cos^2\phi + I_y\sin^2\phi - I_{zy}\sin 2\phi \tag{9.48}$$

Similarly

$$I_{y(1)} = I_y\cos^2\phi + I_z\sin^2\phi - I_{zy}\sin 2\phi \tag{9.49}$$

and

$$I_{z(1),y(1)} = \left(\frac{I_z - I_y}{2}\right)\sin 2\phi + I_{zy}\cos 2\phi \tag{9.50}$$

Equations (9.48)–(9.50) give the second moments of area and product second moment of area about axes inclined at an angle $\phi$ to the $x$ axis. In the special case where $Oz_1y_1$ are principal axes, $Oz_p\, y_p$, then $I_{z(p),y(p)} = 0$, $\phi = \phi_p$ and Eqs (9.48) and (9.49) become

$$I_{z(\mathrm{p})} = I_z \cos^2\phi_\mathrm{p} + I_y \sin^2\phi_\mathrm{p} - I_{zy}\sin 2\phi_\mathrm{p} \tag{9.51}$$

and

$$I_{y(\mathrm{p})} = I_y \cos^2\phi_\mathrm{p} + I_z \sin^2\phi_\mathrm{p} + I_{zy}\sin 2\phi_\mathrm{p} \tag{9.52}$$

respectively. Furthermore, since $I_{z(1),y(1)} = I_{z(\mathrm{p}),y(\mathrm{p})} = 0$, Eq. (9.50) gives

$$\tan 2\phi_\mathrm{p} = \frac{2I_{zy}}{I_y - I_z} \tag{9.53}$$

The angle $\phi_\mathrm{p}$ may be eliminated from Eqs (9.51) and (9.52) by first determining $\cos 2\phi_\mathrm{p}$ and $\sin 2\phi_\mathrm{p}$ using Eq. (9.53). Thus

$$\cos 2\phi_\mathrm{p} = \frac{(I_y - I_z)/2}{\sqrt{[(I_y - I_z)/2]^2 + I_{xy}^2}} \quad \sin 2\phi_\mathrm{p} = \frac{I_{zy}}{\sqrt{[(I_y - I_z)/2]^2 + I_{zy}^2}}$$

Rewriting Eq. (9.51) in terms of $\cos 2\phi_\mathrm{p}$ and $\sin 2\phi_\mathrm{p}$ we have

$$I_{z(\mathrm{p})} = \frac{I_z}{2}(1 + \cos 2\phi_\mathrm{p}) + \frac{I_y}{2}(1 - \cos 2\phi_\mathrm{p}) - I_{zy}\sin 2\phi_\mathrm{p}$$

Substituting for $\cos 2\phi_p$ and $\sin 2\phi_p$ from the above we obtain

$$I_{z(\mathrm{p})} = \frac{I_z + I_y}{2} - \frac{1}{2}\sqrt{(I_z - I_y)^2 + 4I_{zy}^2} \tag{9.54}$$

Similarly

$$I_{y(\mathrm{p})} = \frac{I_z + I_y}{2} + \frac{1}{2}\sqrt{(I_z - I_y)^2 + 4I_{zy}^2} \tag{9.55}$$

Note that the solution of Eq. (9.53) gives two values for the inclination of the principal axes, $\phi_\mathrm{p}$ and $\phi_\mathrm{p} + \pi/2$, corresponding to the axes $Oz_\mathrm{p}$ and $Oy_\mathrm{p}$.

The results of Eqs (9.48)–(9.55) may be represented graphically by Mohr's circle, a powerful method of solution for this type of problem. We shall discuss Mohr's circle in detail in Chapter 14 in connection with the analysis of complex stress and strain.

Principal axes may be used to provide an apparently simpler solution to the problem of unsymmetrical bending. Referring components of bending moment and section properties to principal axes having their origin at the centroid of a beam section, we see that Eq. (9.31) or Eq. (9.32) reduces to

$$\sigma_x = -\frac{M_{y(\mathrm{p})}}{I_{y(\mathrm{p})}} z_\mathrm{p} - \frac{M_{z(\mathrm{p})}}{I_{z(\mathrm{p})}} y_\mathrm{p} \tag{9.56}$$

However, it must be appreciated that before $I_{z(\mathrm{p})}$ and $I_{y(\mathrm{p})}$ can be determined $I_z$, $I_y$ and $I_{zy}$ must be known together with $\phi_\mathrm{p}$. Furthermore, the coordinates $(z, y)$ of a point in the beam section must be transferred to the principal axes as must the components, $M_z$ and $M_y$, of bending moment. Thus unless the position of the principal axes is obvious by inspection, the amount of computation required by the above method is far greater than direct use of Eq. (9.31) and an arbitrary, but convenient, set of centroidal axes.

## 9.8 Effect of shear forces on the theory of bending

So far our analysis has been based on the assumption that plane sections remain plane after bending. This assumption is only strictly true if the bending moments are produced by pure bending action rather than by shear loads, as is very often the case in practice. The presence of shear loads induces shear stresses in the cross section of a beam which, as shown by elasticity theory, cause the cross section to deform into the shape of a shallow inverted 's'. However, shear stresses in beams, the cross sectional dimensions of which are small in relation to their length, are comparatively low in value so that the assumption of plane sections remaining plane after bending may be used with reasonable accuracy.

## 9.9 Load, shear force and bending moment relationships, general case

In Section 3.5 we derived load, shear force and bending moment relationships for loads applied in the vertical plane of a beam whose cross section was at least singly symmetrical. These relationships are summarized in Eq. (3.8) and may be extended to the more general case in which loads are applied in both the horizontal *(xz)* and vertical *(yx)* planes of a beam of arbitrary cross section. Thus for loads applied in a horizontal plane Eq. (3.8) become

$$\frac{\partial^2 M_y}{\partial x^2} = -\frac{\partial S_x}{\partial x} = -w_z(x) \tag{9.57}$$

and for loads applied in a vertical plane Eq. (3.8) become

$$\frac{\partial^2 M_z}{\partial x^2} = -\frac{\partial S_y}{\partial x} = -w_y(x) \tag{9.58}$$

In Chapter 18 we shall return to the topic of beams subjected to bending but, instead of considering loads which produce stresses within the elastic range of the material of the beam, we shall investigate the behaviour of beams under loads which cause collapse.

## PROBLEMS

**P.9.1** A girder 10 m long has the cross section shown in Fig. P.9.1(a) and is simply supported over a span of 6 m (see Fig. P.9.1(b)). If the maximum direct stress in the girder is limited to 150 N/mm$^2$, determine the maximum permissible uniformly distributed load that may be applied to the girder.

*Ans.* 84.3 kN/m.

**P.9.2** A 230 mm$\times$300 mm timber cantilever of rectangular cross section projects 2.5 m from a wall and carries a load of 13 300 N at its free end. Calculate the maximum direct stress in the beam due to bending.

*Ans.*9.6 N/mm$^2$.

**P.9.3** A floor carries a uniformly distributed load of 16 kN/m$^2$ and is supported by joists 300 mm deep and 110 mm wide; the joists in turn are simply supported over a span of 4 m. If the maximum stress in the joists is not to exceed 7 N/mm$^2$, determine the distance apart, centre to centre, at which the joists must be spaced.

*Ans.* 0.36 m.

**P.9.4** A wooden mast 15 m high tapers linearly from 250 mm diameter at the base to 100 mm at the top. At what point will the mast break under a horizontal load applied at the top? If the maximum

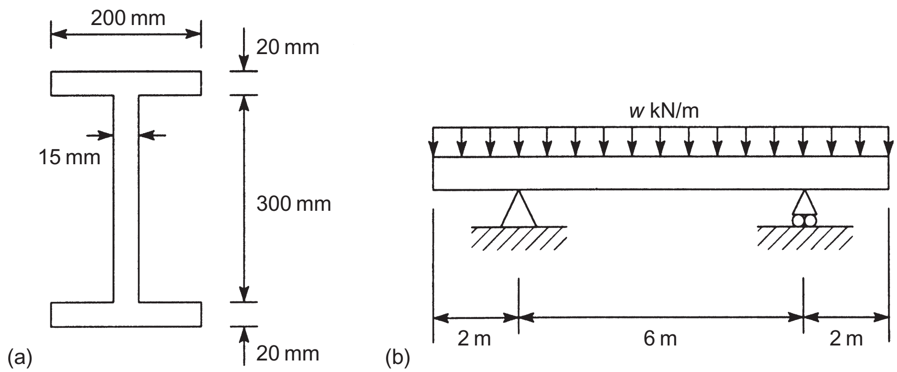



**FIGURE P.9.1**

permissible stress in the wood is 35 N/mm$^2$, calculate the magnitude of the load that will cause failure.

*Ans.* 5 m from the top, 2320 N.

**P.9.5** A main beam in a steel framed structure is 5 m long and simply supported at each end. The beam carries two cross-beams at distances of 1.5 and 3.5 m from one end, each of which transmits a load of 20 kN to the main beam. Design the main beam using an allowable stress of 155 N/mm$^2$; make adequate allowance for the effect of self-weight.

*Ans.* Universal Beam, 254 mm$\times$102 mm$\times$22 kg/m.

**P.9.6** A cantilever beam of length 2.5 m has the cross section shown in Fig. P.9.6 and carries a vertically downward concentrated load of 25 kN at its free end. If the maximum allowable direct stress in the beam is $\pm$165 N/mm$^2$ calculate the maximum intensity of uniformly distributed load the beam can carry over its complete length. What are the values of the elastic section moduli of the beam cross section about its horizontal $z$ axis?

*Ans.* 9.8 kN/m, 563835 mm$^3$, 1000925 mm$^3$.

**P.9.7** A beam has the singly symmetrical cross section shown in Fig. P.9.7 and is simply supported over a span of 2 m. If the direct stress is limited to $\pm$155 N/mm$^2$ and it carries a distributed load which varies in intensity from zero at the left-hand support to $w_0$ at the right-hand support calculate the maximum allowable value of $w_0$.

*Ans.* 308.6 kN/m.

**P.9.8** A steel pipe has an outside diameter of 300mm, walls 25 mm thick, and is used as an overflow pipe built horizontally out from a wall as shown in Fig. P.9.8(a). The pipe is 3m long and is unsupported apart from its built-in end. To stiffen the pipe, a steel strip 25mm wide and 300 mm deep is welded along the top of the pipe as shown in Fig. P.9.8(b). If the density of the steel is 7750 kg/m$^3$ and that of water is 1000 kg/m$^3$, determine the maximum stress in the pipe when running full of water.

*Ans.* 5.9 N/mm$^2$ (tension) at the top of the stiffener at the built-in end.

**P.9.9** A short column, whose cross section is shown in Fig. P.9.9 is subjected to a compressive load, $P$, at the centroid of one of its flanges. Find the value of $P$ such that the maximum compressive stress does not exceed 150 N/mm$^2$.

*Ans.* 846.4 kN.

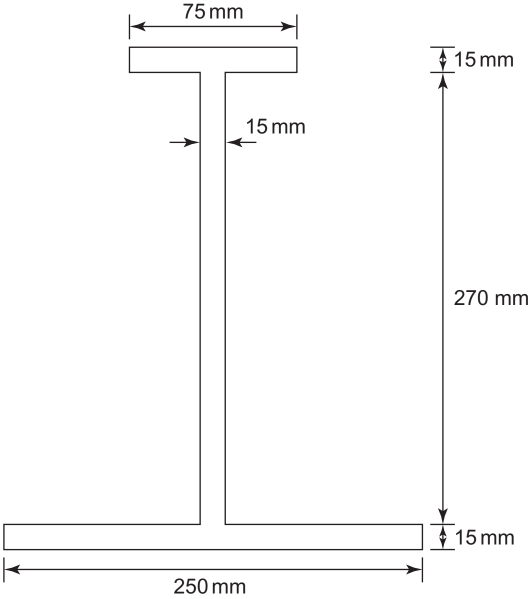



**FIGURE P.9.6**

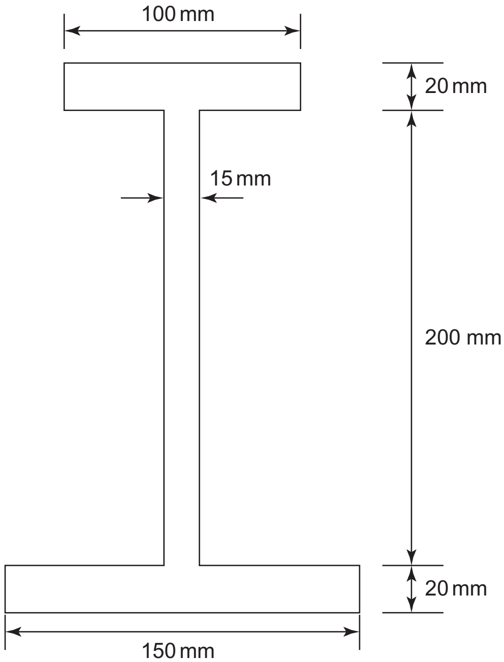
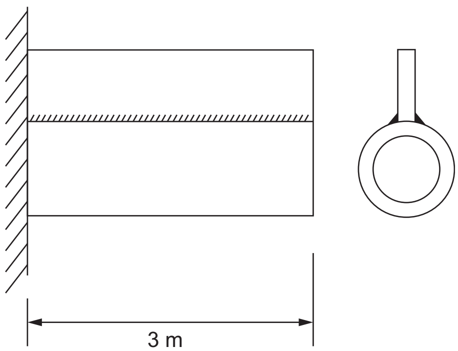



**FIGURE P.9.7**




(a) (b)

**FIGURE P.9.8**

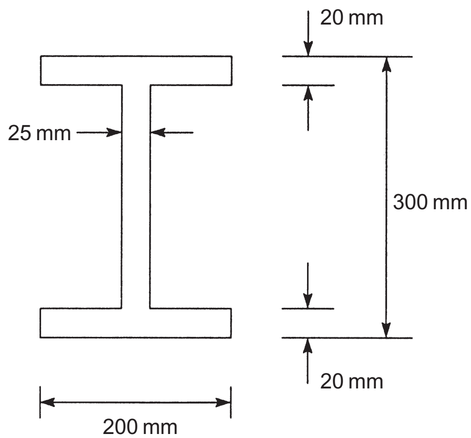



**FIGURE P.9.9**

**P.9.10** A compressive force, $P$, is applied eccentrically to a column whose cross section is shown in Fig. P.9.10. Find the maximum eccentricity, $e$, of $P$ if there is to be no tension anywhere in the cross section of the column. For this value of $e$ calculate the maximum compressive stress in the column when $P=450$ kN.

*Ans.* 64.2 mm, 160.7 N/mm$^2$.

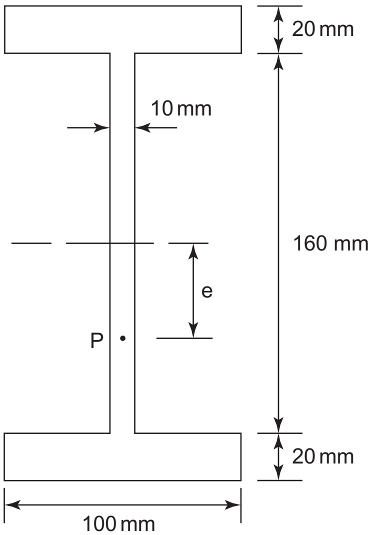



**FIGURE P.9.10**

**P.9.11** The cantilever beam shown in Fig. P.9.11(a) has the hollow cross section shown in Fig. 9.11(b). If the 130 kN load is applied at the centroid of the end cross section calculate the maximum value of compressive stress in the cross section of the beam.

*Ans.* 132.1 N/mm$^2$.

**P.9.12** A column 3 m high has the cross section shown in Fig. P.9.12 and is subjected to an axial load of 200 kN together with a horizontal load, $W$, applied in the direction of the web. If the maximum direct stress in the column is limited to 155 N/mm$^2$ calculate the maximum allowable value of $W$. If the 200 kN load is moved in the direction of $W$ to the outside edge of a flange but remains in the vertical plane of symmetry calculate the corresponding maximum allowable value of $W$.

*Ans.* 20.5 kN, 12.5 kN.

**P.9.13** A vertical chimney built in brickwork has a uniform square cross section as shown in Fig. P.9.13(a) and is built to a height of 15 m. The brickwork has a density of 2000 kg/m$^3$ and the wind pressure is equivalent to a uniform horizontal pressure of 750 N/m$^2$ acting over one face. Calculate the stress at each of the points A and B at the base of the chimney.

*Ans.* (A) 0.02 N/mm$^2$ (compression), (B) 0.60 N/mm$^2$ (compression).

**P.9.14** A cantilever beam of length 2 m has the cross section shown in Fig. P.9.14. If the beam carries a uniformly distributed load of 5 kN/m together with a compressive axial load of 100 kN applied at its free end, calculate the maximum direct stress in the cross section of the beam.

*Ans.* 121.5 N/mm$^2$ (compression) at the built-in end and at the bottom of the leg.

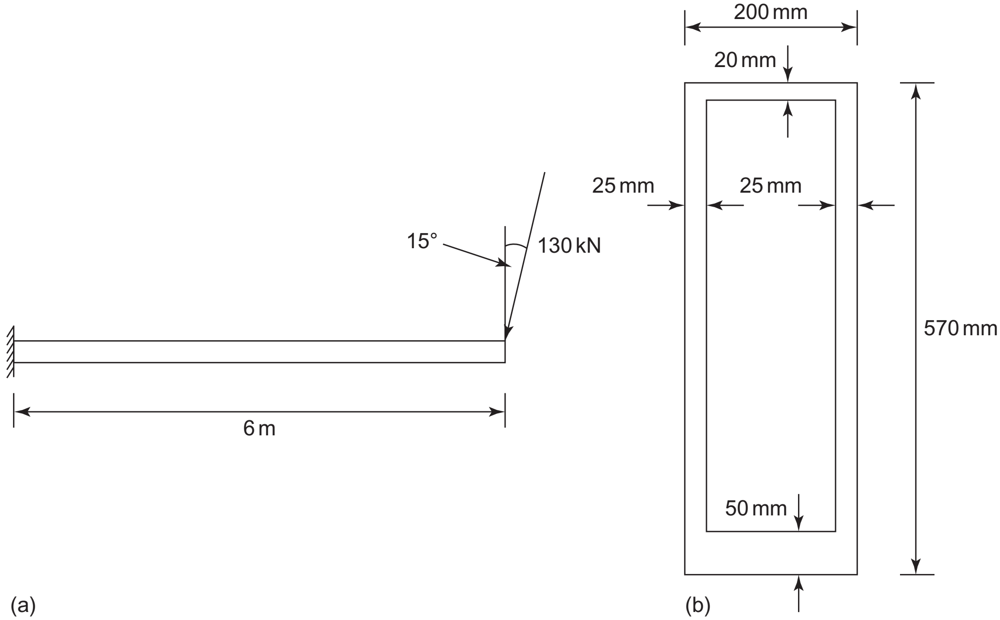



**FIGURE P.9.11**

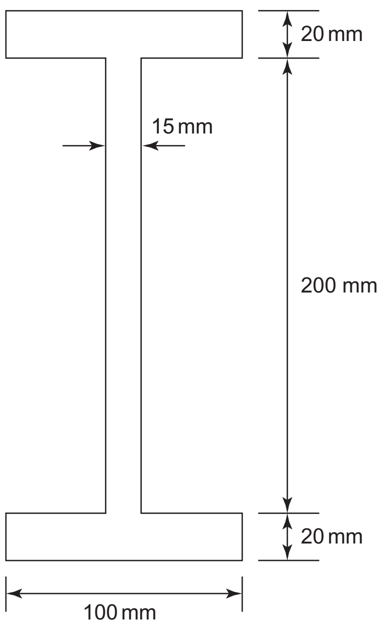



**FIGURE P.9.12**

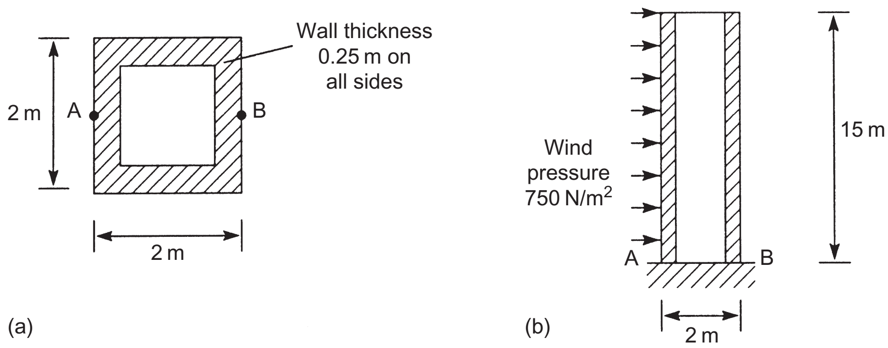



**FIGURE P.9.13**

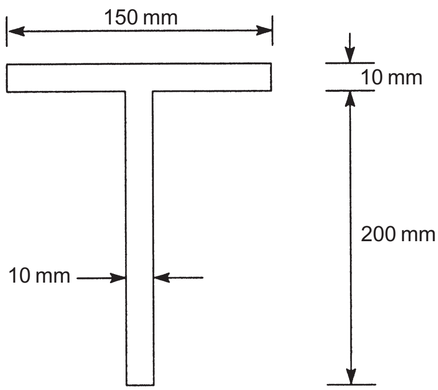



**FIGURE P.9.14**

**P.9.15** A concrete dam has a trapezoidal cross section, is 16 m high, and contains water to a depth of 15 m; the top of the dam is 2 m wide. If the density of the concrete is 2000 kg/m$^3$ and that of water is 1000 kg/m$^3$, calculate the minimum width of the base if no tension is to be developed in the base section.

*Ans.* 9.5 m.

**P.9.16** A free-standing earth retaining wall is built from brick, is 4 m high, and 1.2 m wide. If the density of the brickwork is 2000 kg/m$^3$ and that of the soil is 1500 kg/m$^3$, determine the maximum depth of soil the wall can support for there to be no tension developed in the wall. Assume an angle of repose for the soil of 30°.

For the depth of soil calculated, check the stability of the wall against overturning and also against sliding if the coefficient of friction between the base of the wall and the ground is 0.7.

*Ans.* 2.8 m. The wall will not overturn or slide.

**P.9.17** The section of a thick beam has the dimensions shown in Fig. P.9.17. Calculate the section properties $I_z$, $I_y$ and $I_{zy}$ referred to horizontal and vertical axes through the centroid of the section. Determine also the direct stress at the point A due to a bending moment $M_y = 55$ Nm.

*Ans.* –114 N/mm$^2$ (compression).

**P.9.18** A beam possessing the thick section shown in Fig. P.9.18 is subjected to a bending moment of 12 kNm applied in a plane inclined at 30° to the left of vertical and in a sense such that its

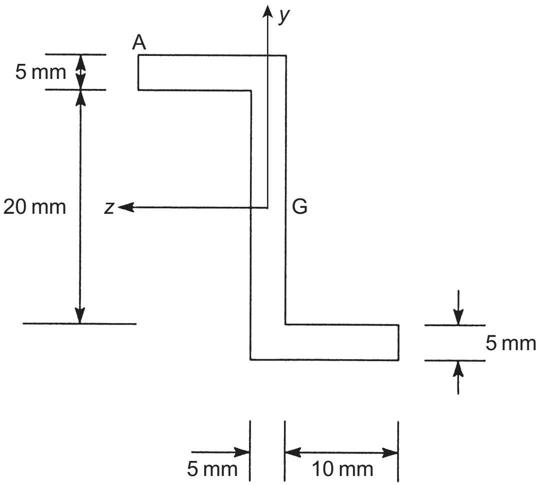



**FIGURE P.9.17**

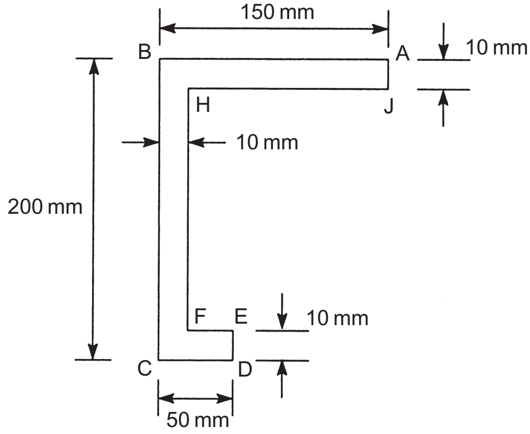



**FIGURE P.9.18**

components $M_z$ and $M_y$ are negative and positive, respectively. Calculate the magnitude and position of the maximum direct stress in the beam cross section.

*Ans.* 156.2 N/mm$^2$ (compression) at D.

**P.9.19** The cross section of a beam/floor slab arrangement is shown in Fig. P.9.19. The complete section is simply supported over a span of 10 m and, in addition to its self-weight, carries a concentrated load of 25 kN acting vertically downwards at mid-span. If the density of concrete is 2000 kg/m$^3$, calculate the maximum direct stress at the point A in its cross section.

*Ans.* 5.4 N/mm$^2$ (tension).

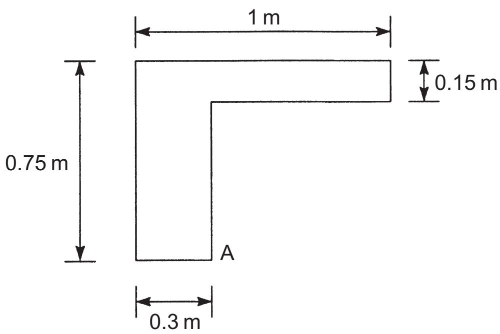



**FIGURE P.9.19**

**P.9.20** A precast concrete beam has the cross section shown in Fig. P.9.20 and carries a vertically downward uniformly distributed load of 100 kN/m over a simply supported span of 4 m. Calculate the maximum direct stress in the cross section of the beam, indicating clearly the point at which it acts.

*Ans.* –27.4 N/mm$^2$ (compression) at B.

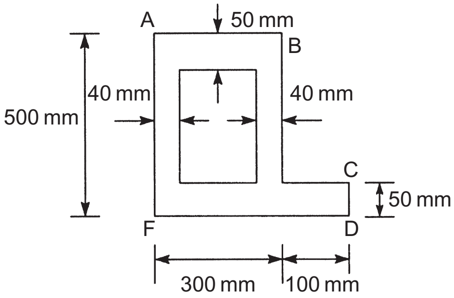



FIGURE P.9.20

**P.9.21** A thin-walled, cantilever beam of unsymmetrical cross section supports shear loads at its free end as shown in Fig. P.9.21. Calculate the value of direct stress at the extremity of the lower flange (point A) at a section half-way along the beam if the position of the shear loads is such that no twisting of the beam occurs.

*Ans.* 194.7 N/mm$^2$ (tension).

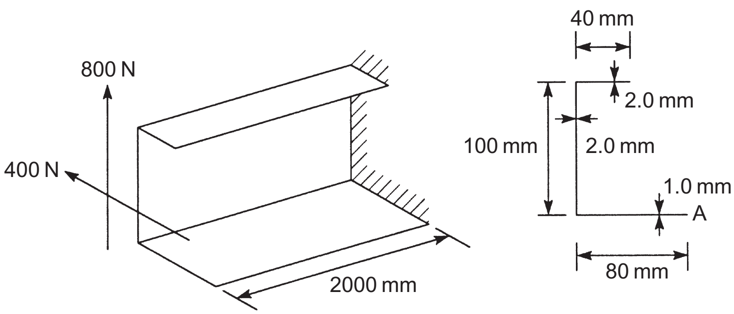



FIGURE P.9.21

**P.9.22** A thin-walled cantilever with walls of constant thickness $t$ has the cross section shown in Fig. P.9.22. The cantilever is loaded by a vertical force $P$ at the tip and a horizontal force $2P$ at the mid-section. Determine the direct stress at the points A and B in the cross section at the built-in end.

*Ans.* (A) $-1.85\ PL/td^2$, (B) $0.1\ PL/td^2$.

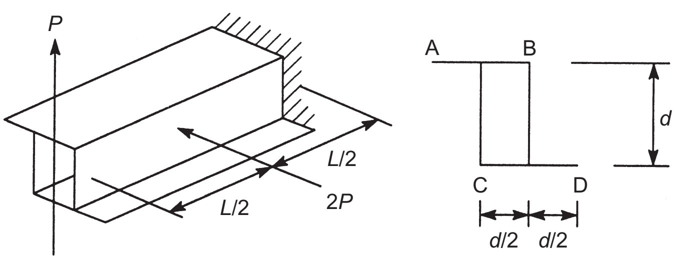



FIGURE P.9.22

**P.9.23** A tall building has the cross section shown in Fig. P.9.23 and is subjected to horizontal wind loads which are resisted by an unsymmetrical arrangement of concrete shear walls. If the height of the building is 60 m determine the maximum direct stress in the cross section produced by a uniform wind pressure of 750 N/m$^2$ acting as shown. Specify the point at which the maximum direct stress acts and assume, for the purposes of calculation, that the shear walls, all of thickness 200 mm, are thin.

*Ans.* 3.4 N/mm$^2$ (tension) at H.

**P.9.24** A cold-formed, thin-walled beam section of constant thickness has the profile shown in Fig. P.9.24. Calculate the position of the neutral axis and the maximum direct stress for a bending moment of 3.5 kNm applied about the horizontal axis G$z$.

*Ans.* $\alpha$=51.9°, ±101.0 N/mm$^2$.

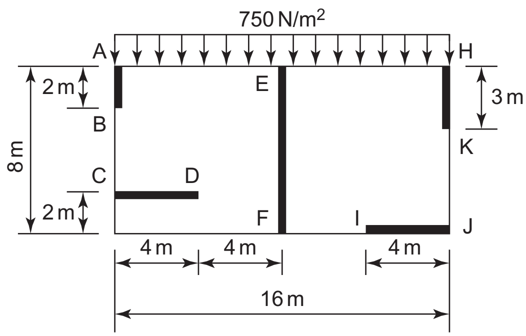



**FIGURE P.9.23**

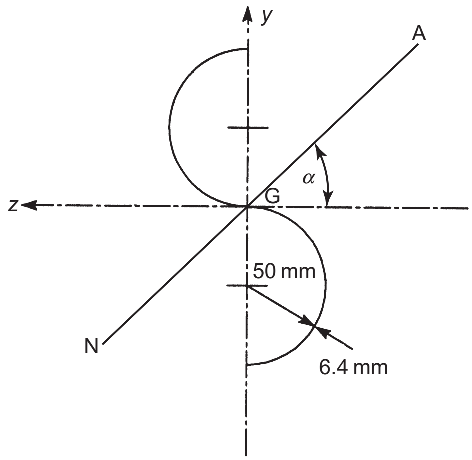



**FIGURE P.9.24**