Signals Theory

Contents

1.5 Discrete signals and sampling

1.5.1 Discrete signal nomenclature

In section 1.1 we briefly introduced discrete signals – sampled signals that only have a value defined at an instance in time. These are the signals we work with in digital systems. In this lecture we will explore them in greater detail.

Let's jump straight in and look at a sine wave in the continuous domain and its discrete time equivalent:

$$f(t) = \sin(\omega t) \qquad \text{and} \qquad f[n] = \sin(\Omega n)$$

Recall than 'n' is the sample number – an integer between 1 and the number of samples in our sequence. It is making 'n' an integer that creates the discrete sequence – it is not allowed to take a value of 1.5, 2.1 etc. 'f[n]' has square brackets to denote that is a discrete sequence, but 'sin()' has curved brackets because it is always a continuous function – it is only the input to the function that has been discretised.

So, the question is what is Ω? We can see that Ω is doing a job similar to ω, i.e. defining the frequency, but there is a little more to it…….

We know that for a sampled continuous waveform: *f[n] = f(nT),* where n is the sample number (an integer) and T is the sampling period in seconds.

Thus: $$f[n] = \sin(\Omega n) = f(nT) = \sin(\omega nT)$$

(If you are wondering about this step, it comes from substituting 'nT' in place of 't' in $f(t) = \sin(\omega t)$)

We can then say: *Ω = ωT*, which is 'radians per sample' (i.e. radians between samples).

Compare this to 'ω', 'radians per second' and you will see a key difference (and the point of this whole section!) – that in the discrete-time domain we can change 'time' by changing the sampling period.

Did you get this? You can look at the dimensions which can help explain:

ω has units of *rad / sec;*

T has units sec./sample;

ωT then has units *(rad.sec) / (sec.sample) = rad/sample.*

We have, in effect, normalised time to the frequency of the signal. We are going to refer to *Ω* as the ***Normalised frequency***

1.5.2 Exploring Ω and the influence of ω and T

In this section we will look at a few continuous sine-wave signals and how the signal frequency and the sampling period impact the discrete time sequence that results.

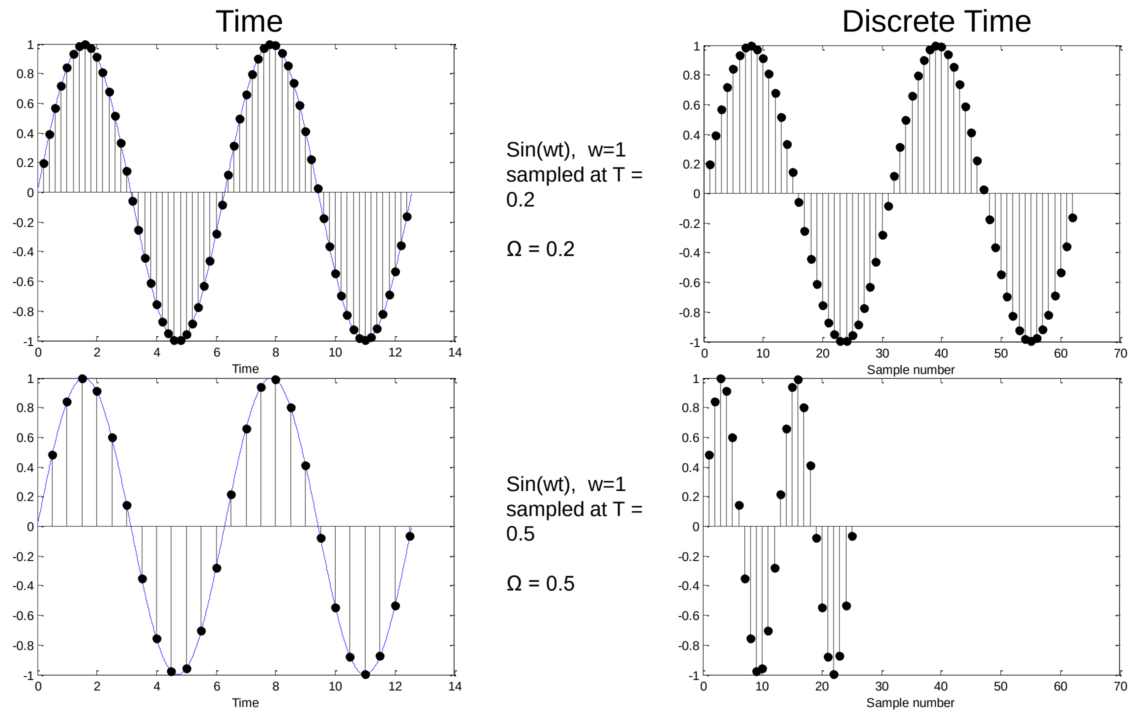


Fig 1. *Increasing the sample interval results in a higher normalised frequency in the discrete time domain*

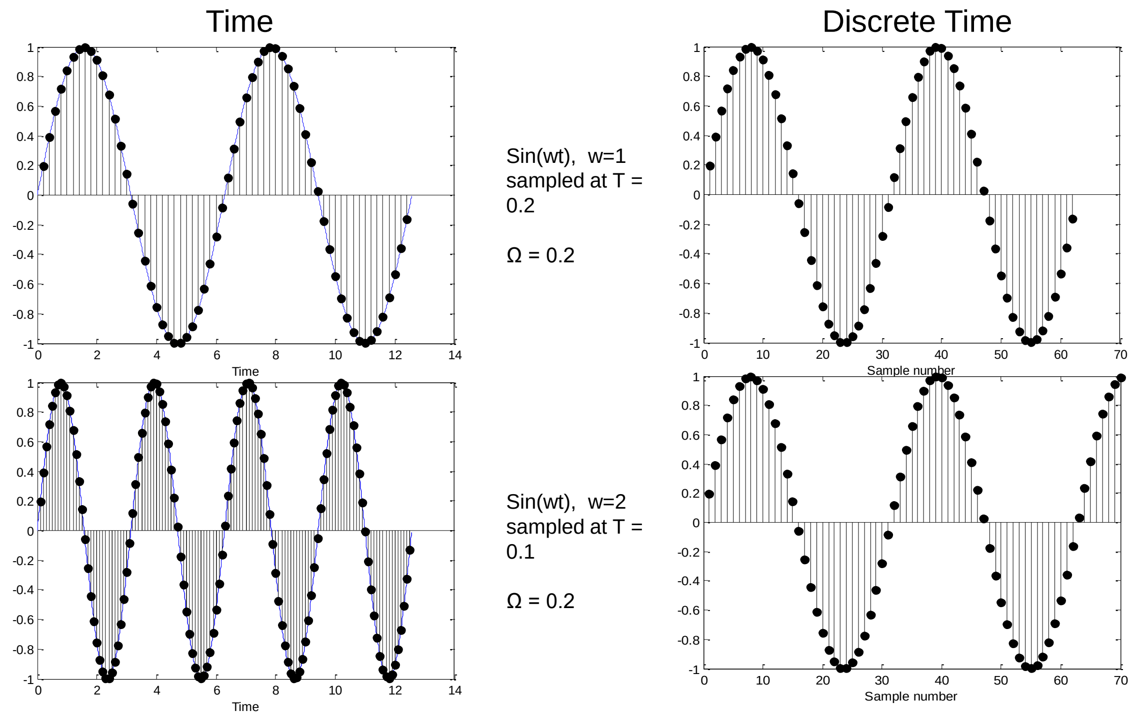


Fig 2. *Waveforms with the same normalised frequency can look different in the time domain*

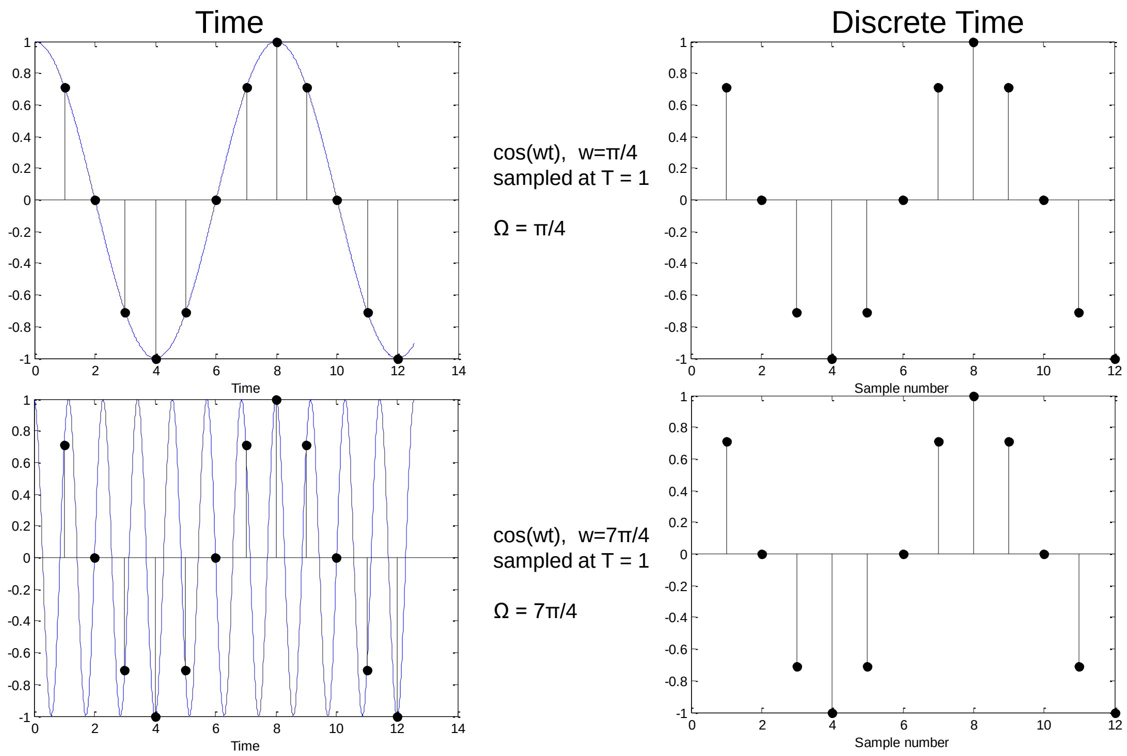


Fig 3. *Another important aspect of sampling – certain differing Ω can give the same discrete time series! More on this later.*

1.5.3 Discrete time complex exponential

The complex exponentials that we described in the continuous time domain can be described in the discrete time domain:

$$f[n] = Ce^{\beta n} \qquad \textit{Where 'C' and 'β' are complex numbers}$$

When 'β' has an imaginary part the result is a periodic signal

$$f[n] = Ce^{(\varepsilon + j\Omega)n}$$

The discrete time complex exponential is analogous to the continuous time version save the substitution of Ω for discrete frequency, (remembering Ω is the radians per sample period).

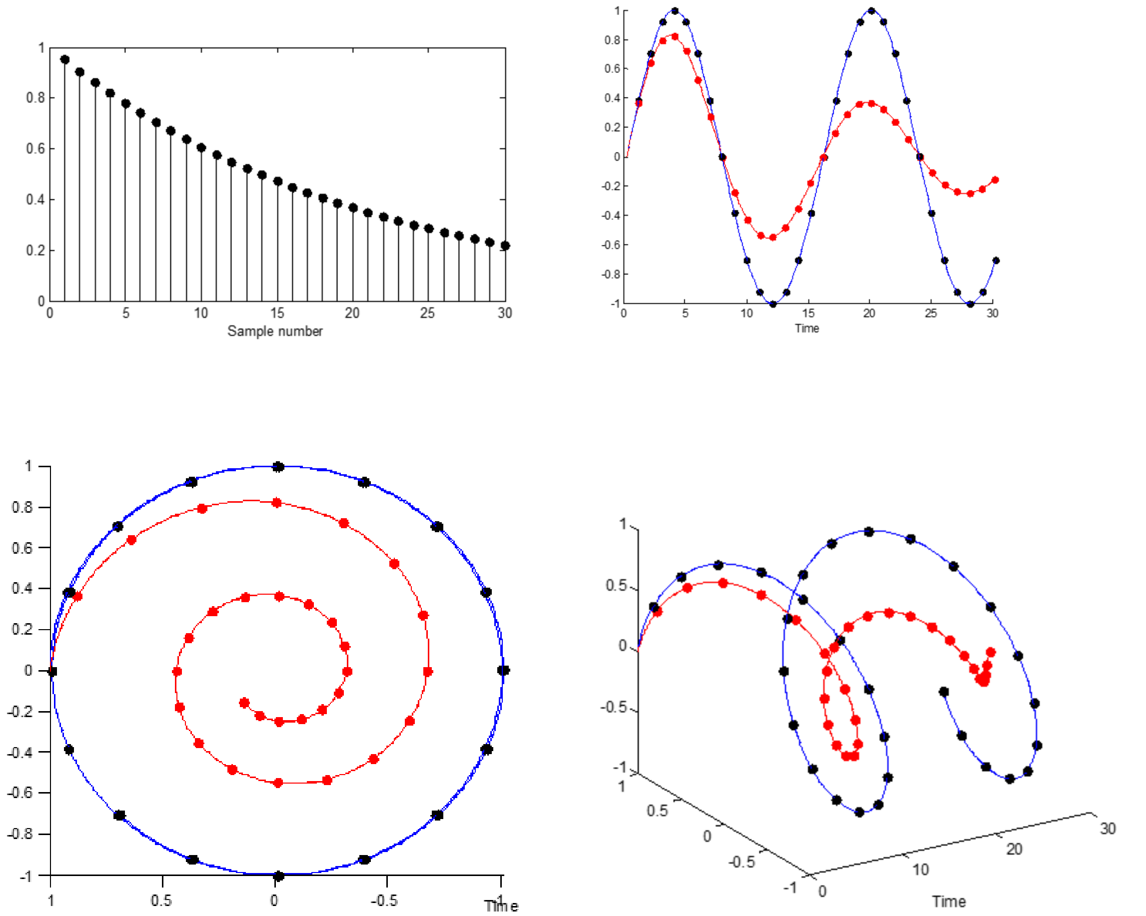


Fig. 4. *Sampled versions of exponential functions, sinusoids, decaying/growing sinusoids, phase shifted sinusoids can all be reproduced*

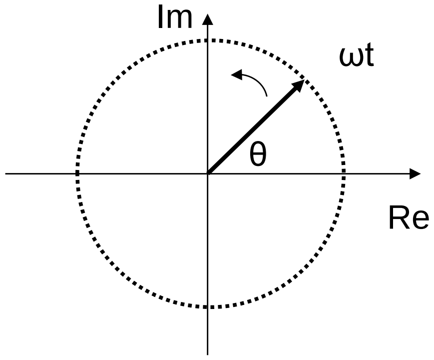
Note: in the images of the complex functions are shown as points superimposed on top of the continuous time equivalent and with a time axis instead of sample number – just a limitation on trying to draw these in MATLAB.

1.5.4 Sampling on the complex plane

The majority of digital signals will be produced by sampling a continuous time signal – for instance when any physical phenomena is recorded. To try and explain the behaviour of sampling we are going to consider what is happening on the complex plane.

Remember, we will refer to Ω (radians per sample period) as the 'normalised frequency' – because that is one interpretation, i.e. the angular velocity of the phasor normalised to the sample period.

Let's consider the signal we are going to sample. It is a phasor of some constant magnitude rotating on the complex phase with constant angular velocity ω.

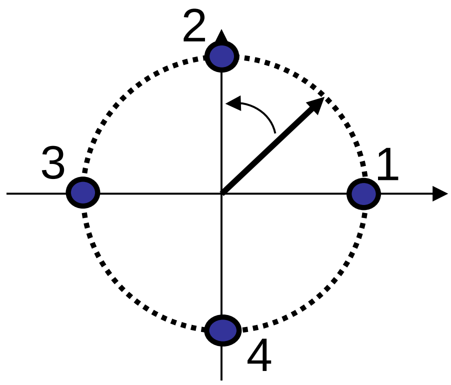


If the normalised frequency is 2π/4, i.e. the phasor moved 2π/4 radians in-between samples we get the following pattern – where a blue dot indicates where a sample is taken, the number indicates the sample number.

At each sample the value is represented by a complex number.

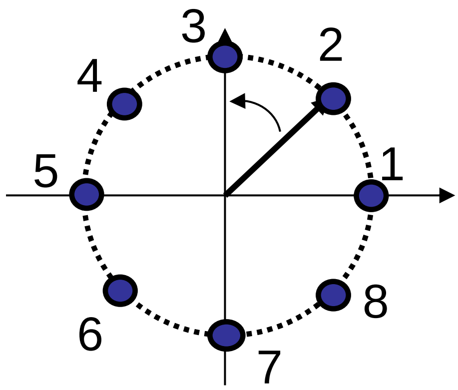
This signal would be:

f[n] = [(1+j0), (0+j), (-1+j0), (0-J)]



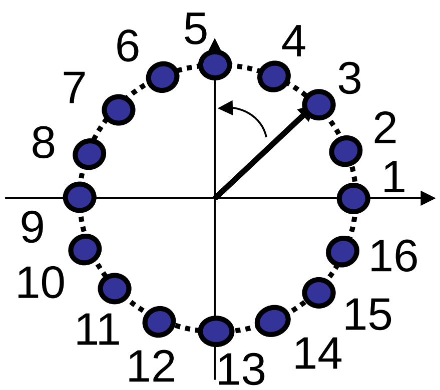
Ω = 2π/4

If we halve the normalised frequency – corresponding to reducing the radians per sample we see that the number of times the phasor is sampled during a rotation is increased.



Ω = π/4

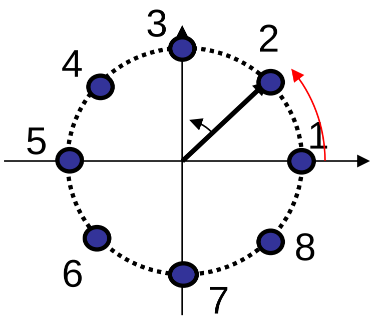
We can continue this and reduce the normalised frequency even further. In these examples I have used integer fractions of the 2π for the normalised frequency for clarity, resulting in a repeating sequence – i.e. we return to sample no. 1 each time, but this does not have to be the case.



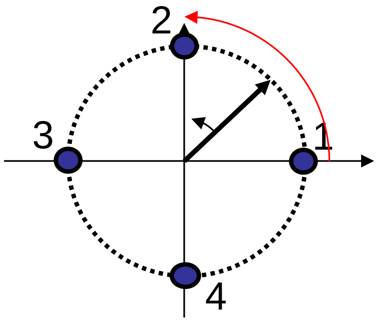
Ω = π/8

1.5.5 Increasing Ω

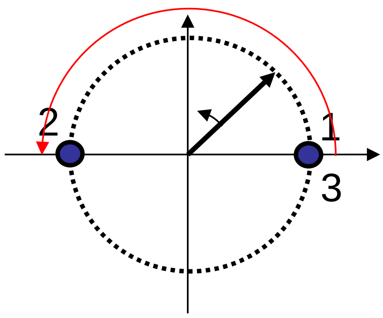
In the previous section we looked at the case where our normalised frequency was decreasing, but what happens when we start to increase it?



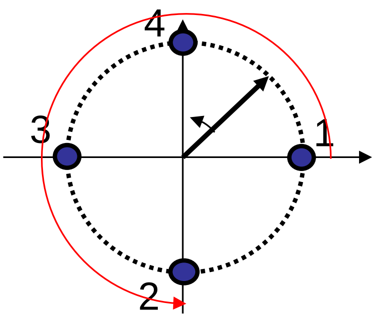
$\Omega = \pi/4$



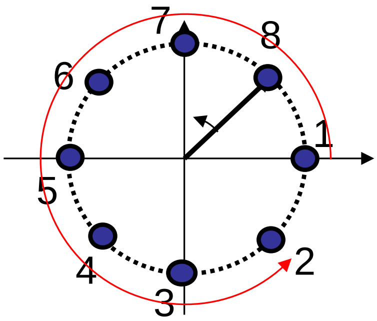
$\Omega = 2\pi/4$



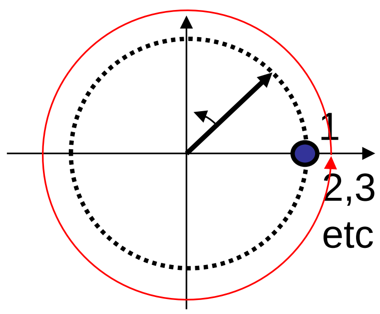
$\Omega = 4\pi/4$



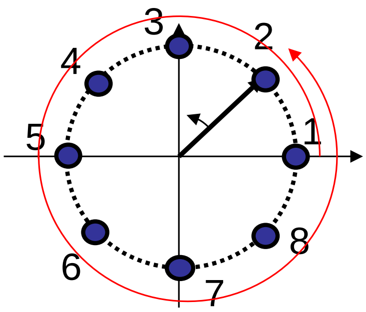
$\Omega = 6\pi/4$



$\Omega = 7\pi/4$



$\Omega = 8\pi/4$

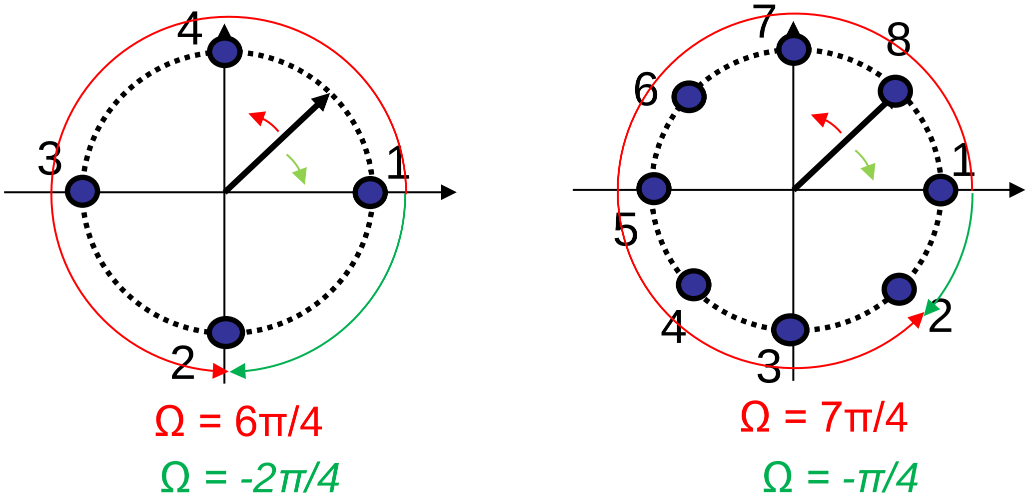


$\Omega = 9\pi/4$

As we increase the normalised frequency – increasing the size of the step between samples – we eventually get to a point where the sampling 'wraps around' and we produce the same discrete time sequence, this happens at $\Omega+2\pi$, $\Omega+4\pi$ etc

1.5.6 Negative Ω

We can also look at what happens when Ω is negative, i.e. the rotation is backwards. Just taking two examples from our previous section:



In this case we see that a backward rotating phasor produces the same discrete time sequence when sampled at Ω-2π, Ω-4π etc.

1.5.7 Unique sampling bandwidth

First of all, let’s define ‘bandwidth’ for those not familiar. It describes a ‘block’ or range of frequencies in the frequency domain. If this block of frequencies extends down to zero frequency (often termed ‘DC’) then it is described as ‘baseband’. Bandwidth is inextricably linked with information rate – with high rates of information (data) implying high frequencies, and a wider bandwidth.

From the discussion in sections 1.4.5 and 1.4.6, we can explore the bandwidth or range of normalised frequencies for which a unique sampled sequence is created – that is we can look at the sequences that are produced when we sample at different normalised frequencies and see where we get repeats.

In the figure below ignore the y axis – it is arbitrary. The discrete time sequence is encoded as the shade of the coloured block i.e. all deep red parts represent the same sequence, as do all the light blues etc.

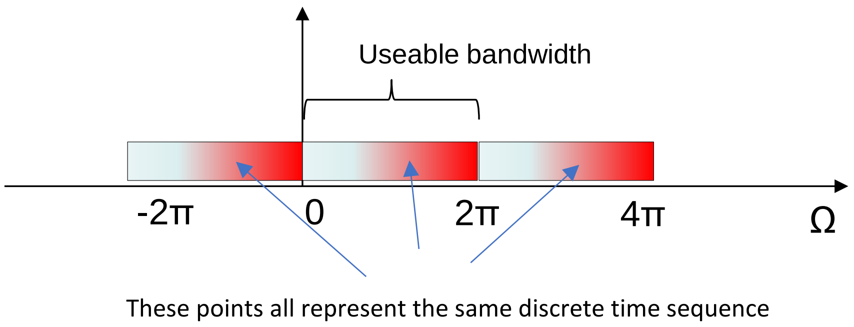


Fig. 5. *When a signal is sampled, it only produces a unique sequence over a particular bandwidth, which is then repeated for other Ω*

We see that for values of Ω in the range $0 < \Omega < 2\pi$ we get unique discrete time sequences (the useable bandwidth), but for values of $\Omega < 0$ and $\Omega > 2\pi$ we get a repetition of the sequence already seen in the $0 < \Omega < 2\pi$ range .

In summary:

- When a signal is complex valued (analytical), as long as it is sampled more than once per cycle (of phasor rotation), the sample can represent that signal uniquely.
- When a signal is sampled less than once per cycle, the sampled signal appears identical to another in the range $0<\Omega<2\pi$
- When sampling, if signals outside the range $0<\Omega<2\pi$ are present, then they will reflect artefacts back into the range $0<\Omega<2\pi$ causing '***aliasing***' – signals at one frequency appearing at another.

Remember: Ω changes with both the frequency of the signal being sampled and the sampling period. In this section we have used diagrammatic explanations that can look as if the phasor (signal we are sampling) is of constant frequency and the sampling period is changing; in most real systems the sampling period is constant and it is the frequency of the sampled signal that changes.

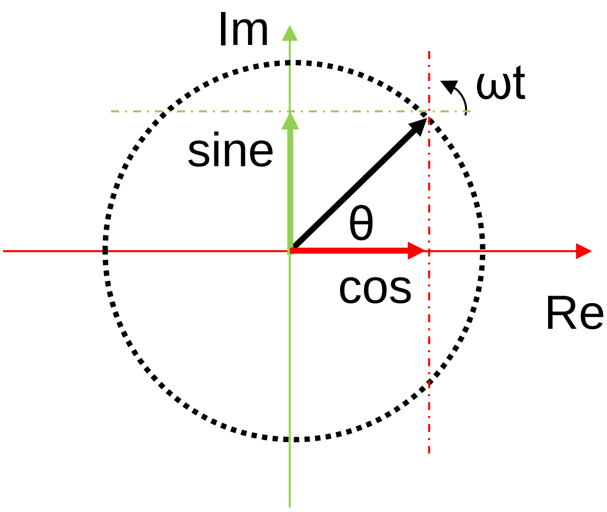
Look again at the last figure in section 1.4.2 – this illustrates two differing frequency signals producing the same discrete time sequence. This is an example of ***aliasing***.

If you are really getting this you will now be asking "this section is dealing with signals on the complex plane i.e. they are complex valued, but that figure I just referred you to is for a signal with real-only values!". This is the focus of the next section……..

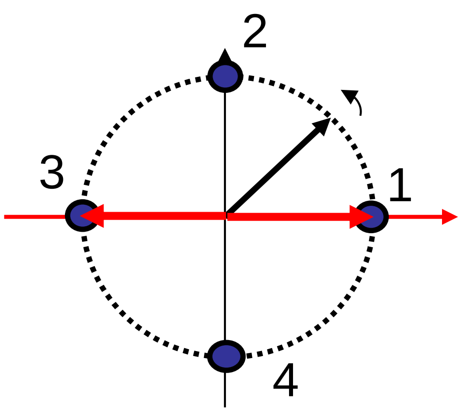
1.5.8 Sampling real valued signals

In the vast majority of applications, we are not sampling rotations – complex valued, analytical signals – we are sampling single real-valued signals with frequency components made up of sines and cosines.

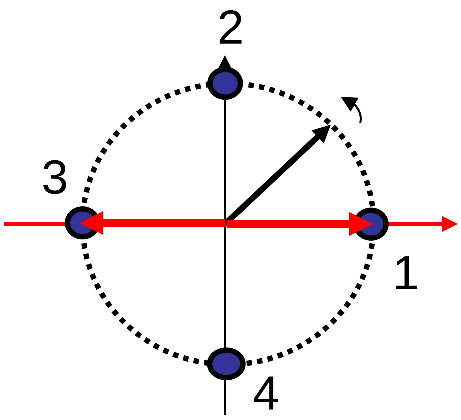
We have already seen that sine and cosine are projections of rotations on the Im and Re axis of the complex planes, respectively. When we sample a real valued signal, we are just capturing the data sequence associated with this projection along one axis.



When we sample real-valued signals we still see the same sequence repetition that at Ω +2π, and Ω - 2π that we saw for complex valued signals:

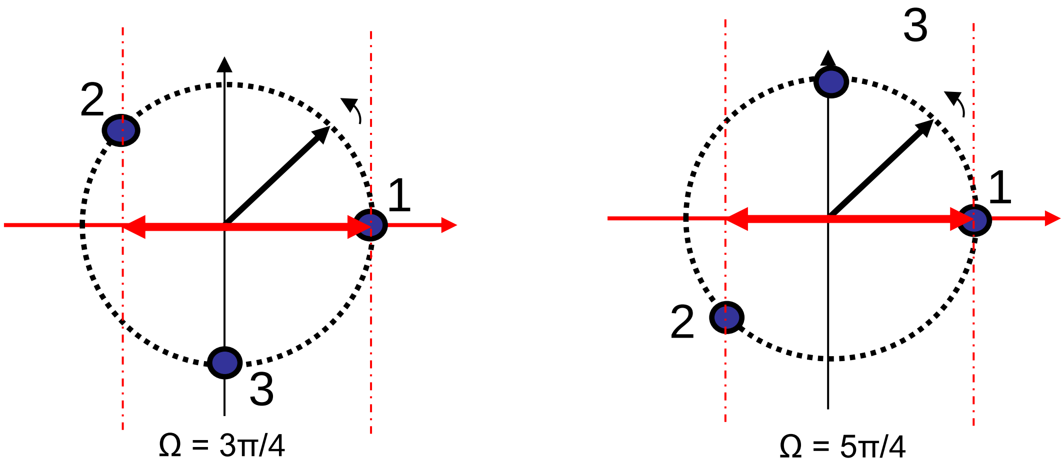


Ω = 2π/4

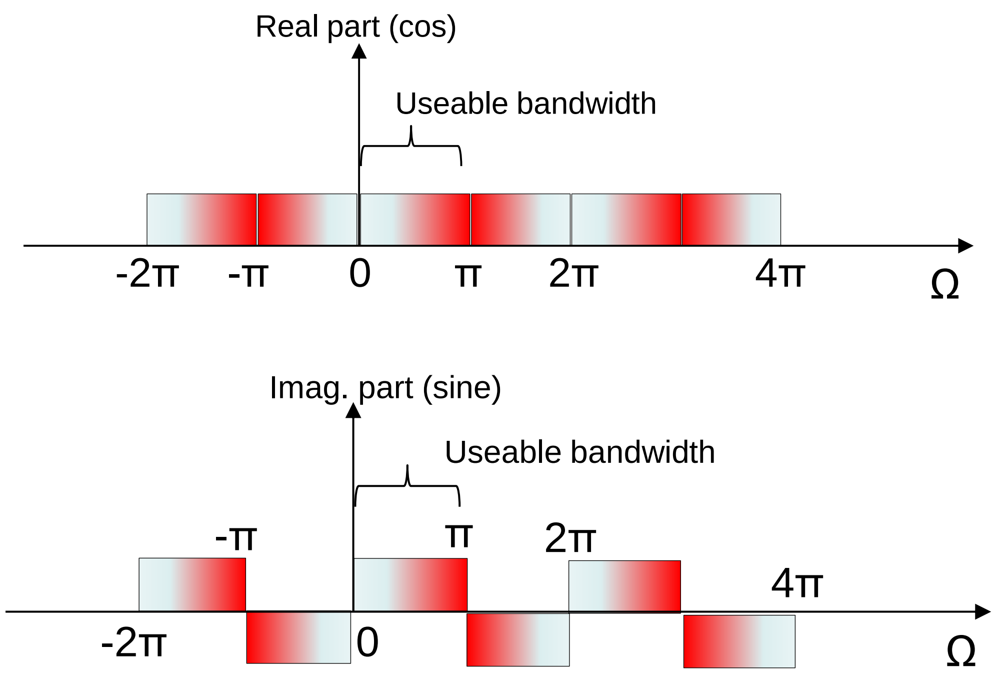


Ω = -6π/4 & 10π/4

However, there is another phenomenon - waveform symmetry causes a 'reflection' in the unique bandwidth at Ω = π. This can be seen below where sample 2 has the same projection along the Re axis when Ω = 3π/4 and when Ω = 5π/4, and is thus indistinguishable:



The diagrams below give a visualisation of the repetition and reflections effects from sampling real valued signals. In these diagrams the y axis tells you about the polarity of the sample.

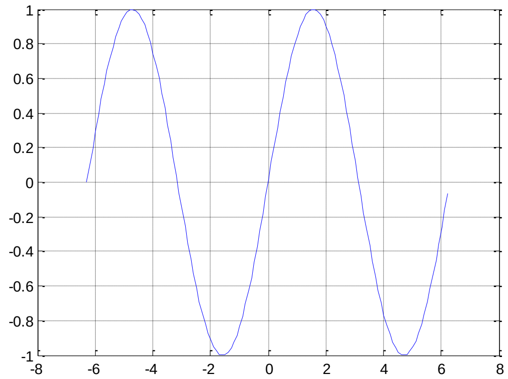


In summary:

- The case for real valued signals is more involved as the symmetry of cosine and sine causes reflection/inversions at Ω = π/2 intervals.
- When a signal has real values only, as long as it is sampled more than **twice per cycle**, the sample can represent that signal uniquely, however when a signal is sampled less than twice per cycle, the sampled signal appears identical to another in the range 0<Ω<π
- Signals outside the range 0<Ω<π will reflect artefacts back into the range 0<Ω<π causing 'aliasing'

1.5.9 Waveform symmetry and examples

In the last section we mentioned symmetry of waveforms. This is a useful concept, particularly with the *integral transforms* we will consider in this lecture series as it can reduce the range over which we integrate (in some cases).

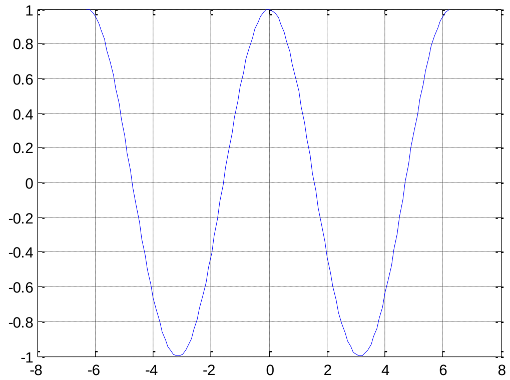


Sine is an odd function:

- sin(x) = -sin(-x)

Sine has quarter wave symmetry

It has even symmetry around π/2

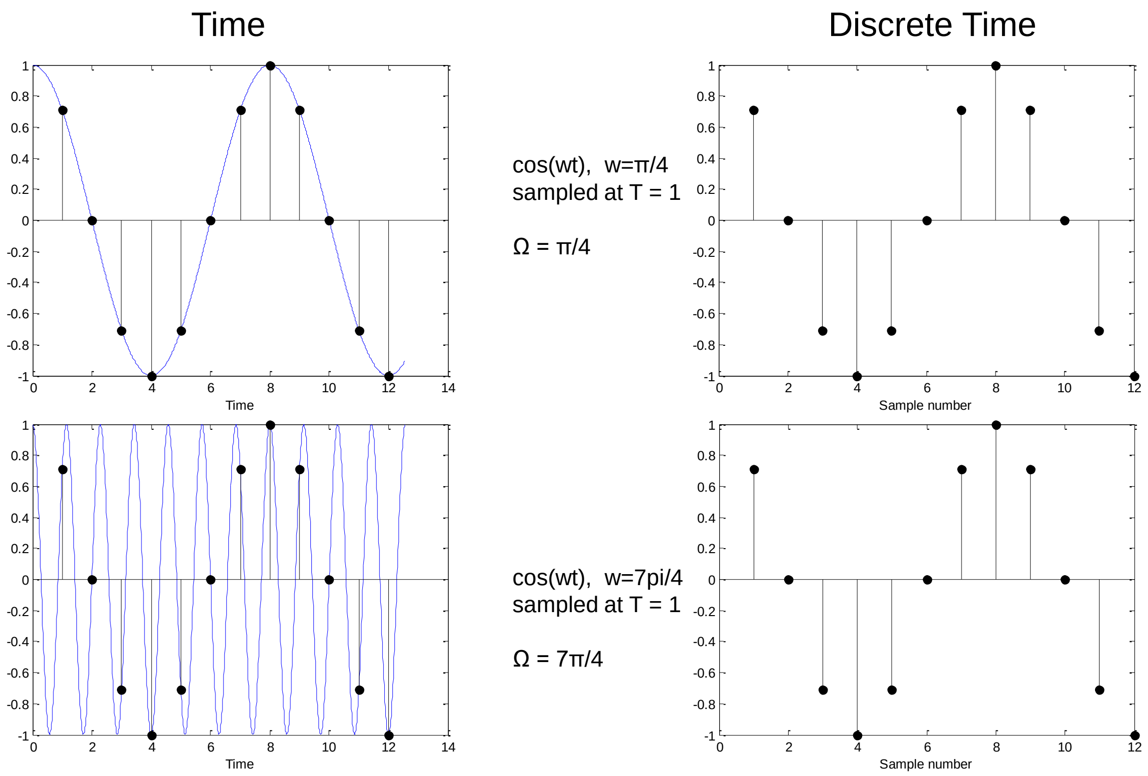


Cosine is an even function:

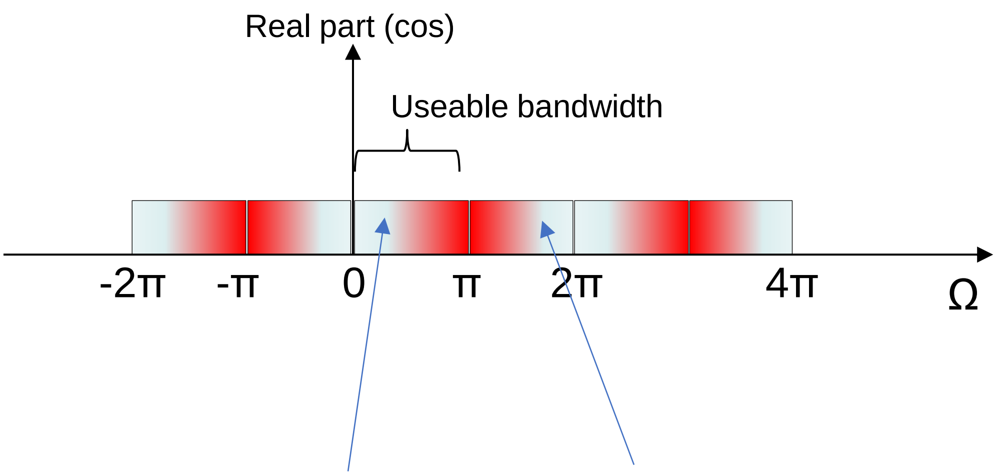
cos(x) = cos(-x)

Cosine has quarter wave symmetry

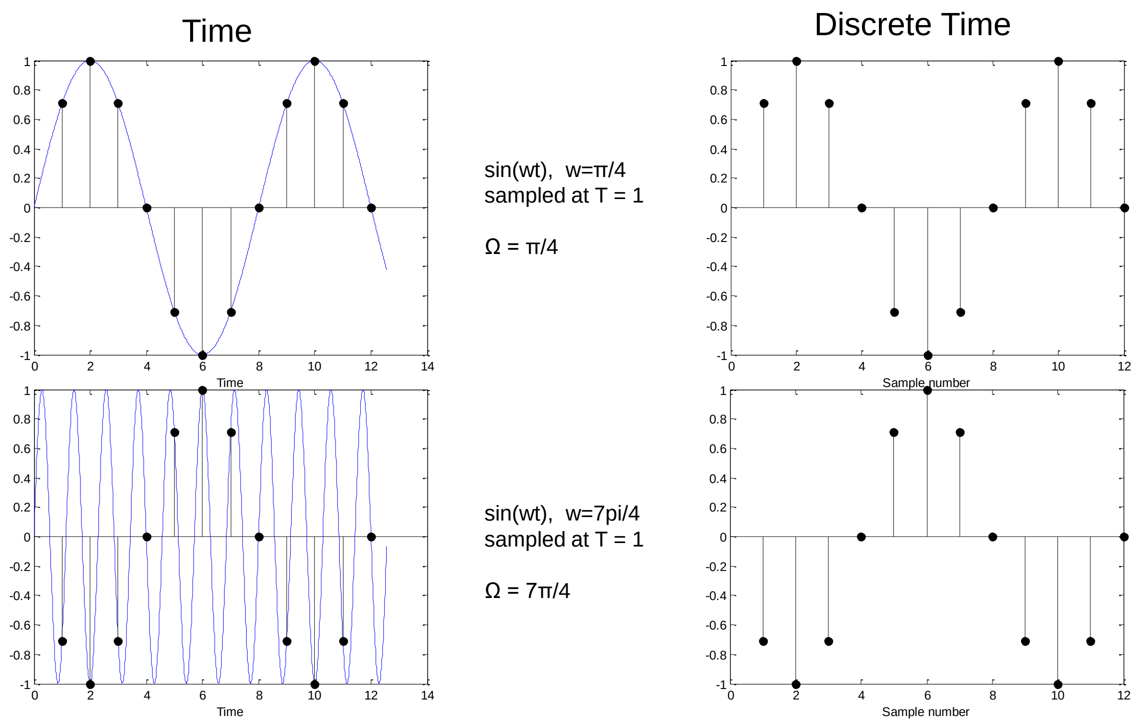
It has odd symmetry around π/2



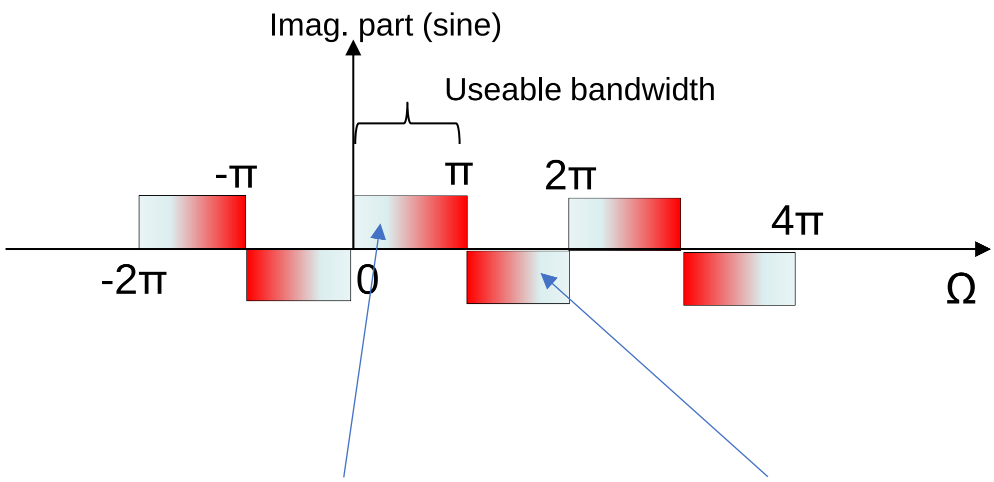
As we have seen before, f[n] = cos(nπ/4) produces the same discrete time sequence as f[n] = cos(n7π/4).



When sampling a cosine, Ω = π/4, is the same as Ω = 7π/4



In comparison, f[n] = sin(nπ/4) produces the same discrete time sequence as f[n] = -sin(n7π/4).



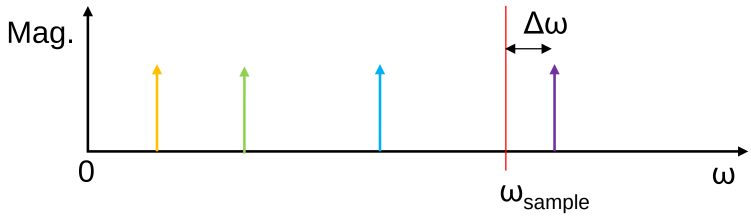
When sampling a sine, Ω = π/4, produces the inverse of Ω = 7π/4

1.5.10 Aliasing illustrated

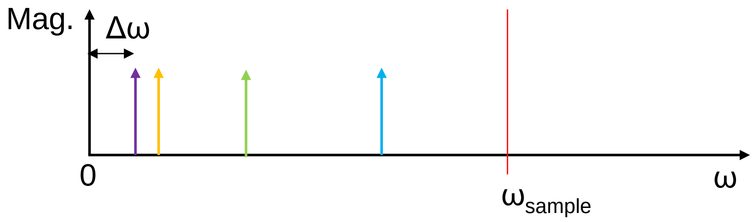
Complex signals

In the case of complex signals, we know we will get aliasing if we don't sample more than once per cycle. Consider the case below – we have a frequency present in our signal $\Delta\omega$ above the sampling frequency (the purple one). After sampling this will appear as if it is $\Delta\omega$ above zero frequency.

Actual frequency components of a signal



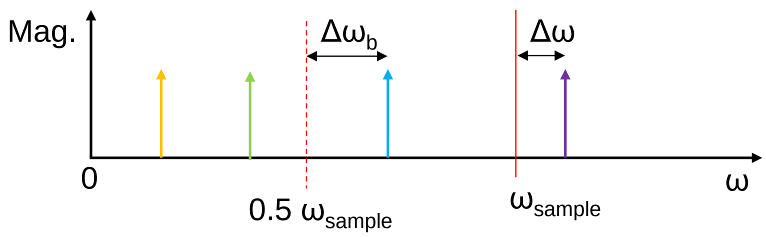
How they appear after sampling



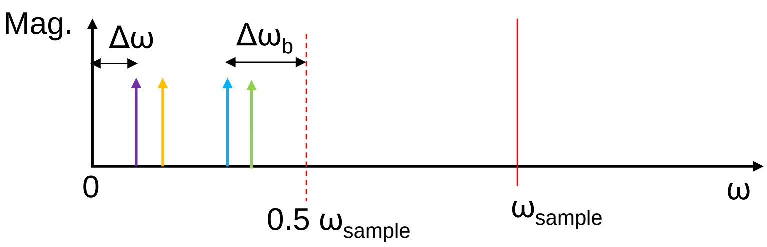
Single valued signals

Now let's consider the same frequency components but with single-valued signals. We know we need to sample more than twice per cycle to capture a frequency component uniquely. In this case we still get repetition from the component $\Delta\omega$ above the sapling frequency, but we also get reflection from the component $\Delta\omega_b$ above $0.5\omega_{sample}$.

Actual frequency components of a signal

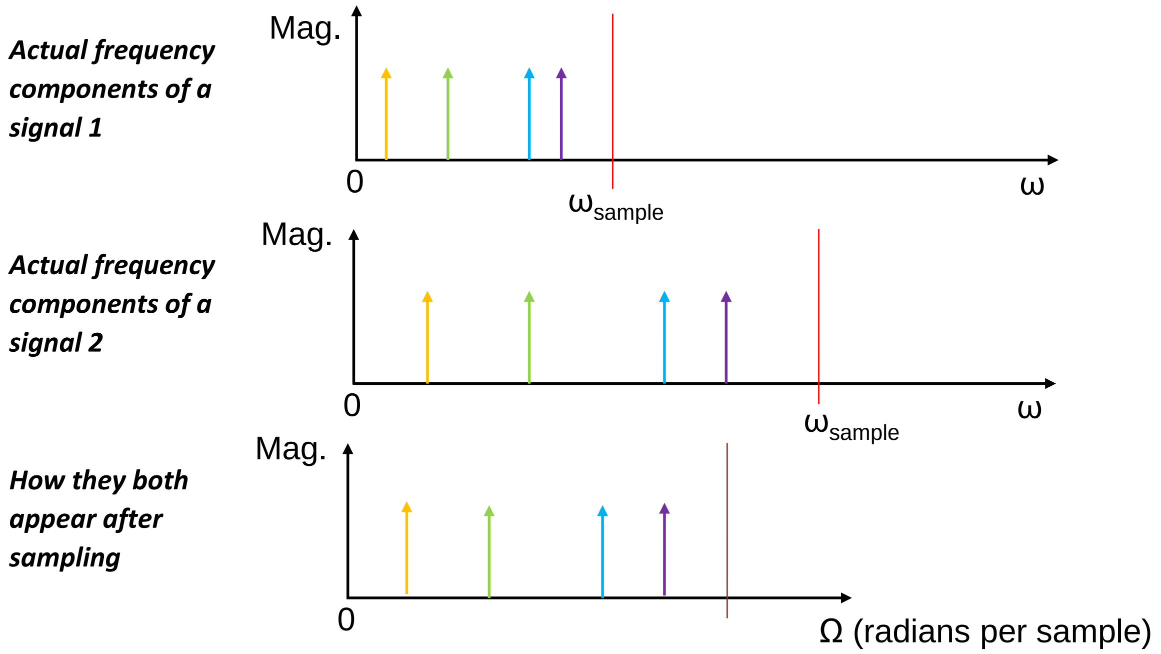


How they appear after sampling



Normalised frequency

Finally, for this section on sampling we can use these diagrams to show the effect of normalising to the sampling frequency. Both of the upper frequency domain signals map to the same normalised frequency domain signal, shown at the bottom.



1.5.11 Test Yourself Section 1.4

1) What is the difference between functions given square brackets [ ] compared to round ( )?
2) In this lecture series what does Ω represent and what are its units?
3) Under what conditions will signals of differing frequencies produce the same sequence when sampled?
4) What is the minimum number of times per period must an analytical (complex valued) signal be sampled for the sequence to be unique?
5) What range of values of Ω produce unique sequences?
6) What happens when Ω is outside of the range that produces unique sequences? What name is given to this phenomenon?
7) If a real-valued, sinusoid time domain waveform is sampled, how does the unique range for Ω change?