CHAPTER

Normal Force, Shear Force, Bending Moment and Torsion

3

The purpose of a structure is to support the loads for which it has been designed. To accomplish this it must be able to transmit a load from one point to another, i.e. from the loading point to the supports. In Fig. 2.36, for example, the beam transmits the effects of the loads at B and C to the built-in end A. It achieves this by developing an *internal force* system and it is the distribution of these internal forces which must be determined before corresponding stress distributions and displacements can be found.

A knowledge of stress is essential in structural design where the cross-sectional area of a member must be such that stresses do not exceed values that would cause breakdown in the crystalline structure of the material of the member; in other words, a structural failure. In addition to stresses, strains, and thereby displacements, must be calculated to ensure that as well as strength a structural member possesses sufficient stiffness to prevent excessive distortions damaging surrounding portions of the complete structure.

In this chapter we shall examine the different types of load to which a structural member may be subjected and then determine corresponding internal force distributions.

3.1 Types of load

Structural members may be subjected to complex loading systems apparently comprised of several different types of load. However, no matter how complex such systems appear to be, they consist of a maximum of four basic load types: axial loads, shear loads, bending moments and torsion.

Axial load

Axial loads are applied along the longitudinal or centroidal axis of a structural member. If the action of the load is to increase the length of the member, the member is said to be in *tension* (Fig. 3.1(a)) and the applied load is *tensile.* A load that tends to shorten a member places the member in *compression* and is known as a *compressive* load (Fig. 3.1(b)). Members such as those shown in Fig. 3.1(a) and (b) are commonly found in pin-jointed frameworks where a member in tension is called a *tie* and one in compression a *strut* or *column.* More frequently, however, the name 'column' is associated with a vertical member carrying a compressive load, as illustrated in Fig. 3.1(c).

Shear load

Shear loads act perpendicularly to the axis of a structural member and have one of the forms shown in Fig. 3.2; in this case the members are *beams*. Figure 3.2(a) shows a *concentrated* shear load, W, applied to a cantilever beam. The shear load in Fig. 3.2(b) is *distributed* over a length of the beam and is of *intensity* w (force units) per unit length (see Section 1.7).

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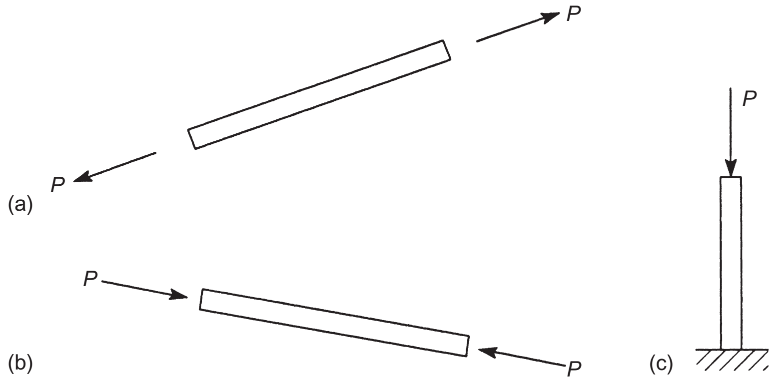

FIGURE 3.1

Axially loaded members.

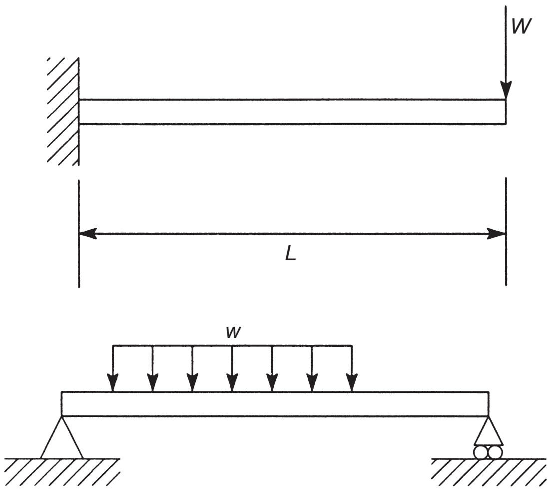

FIGURE 3.2

Shear loads applied to beams.

Bending moment

In practice it is difficult to apply a pure bending moment such as that shown in Fig. 3.3(a) to a beam. Generally, pure bending moments arise through the application of other types of load to adjacent structural members. For example, in Fig. 3.3(b), a vertical member BC is attached to the cantilever AB and carries a horizontal shear load, P (as far as BC is concerned). AB is therefore subjected to a pure moment, $M=Ph$, at B together with an axial load, P.

Torsion

A similar situation arises in the application of a pure torque, T (Fig. 3.4(a)), to a beam. A practical example of a torque applied to a cantilever beam is given in Fig. 3.4(b) where the horizontal member BC supports a vertical shear load at C. The cantilever AB is then subjected to a pure torque, $T=Wh$, plus a shear load, W.

All the loads illustrated in Figs 3.1–3.4 are applied to the various members by some external agency and are therefore *externally applied loads*. Each of these loads induces reactions in the support systems of the different beams; examples of the calculation of support reactions are given in Section 2.5. Since structures are in equilibrium under a force system of externally applied loads and support reactions, it follows that the support reactions are themselves externally applied loads.

Now consider the cantilever beam of Fig. 3.2(a). If we were to physically cut through the beam at some section ‘mm’ (Fig. 3.5(a)) the portion BC would no longer be able to support the load, W. The portion AB of the beam therefore performs the same function for the portion BC as does the wall for the complete beam. Thus at the section mm the portion AB applies a force W and a moment M to the portion BC at B, thereby maintaining its equilibrium (Fig. 3.5(b)); by the law of action and reaction (Newton’s Third Law of Motion),

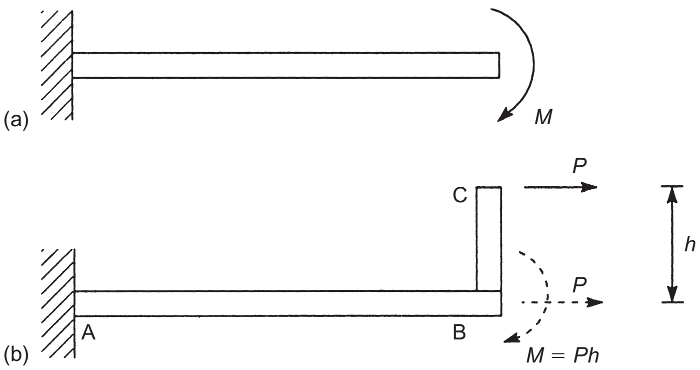

FIGURE 3.3

Moments applied to beams.

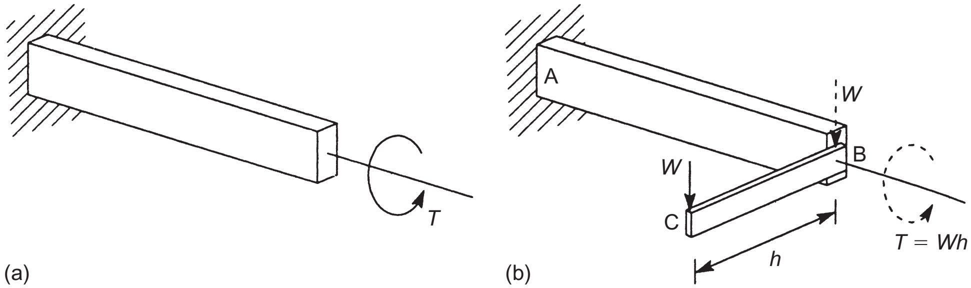

FIGURE 3.4

Torques applied to a beam.

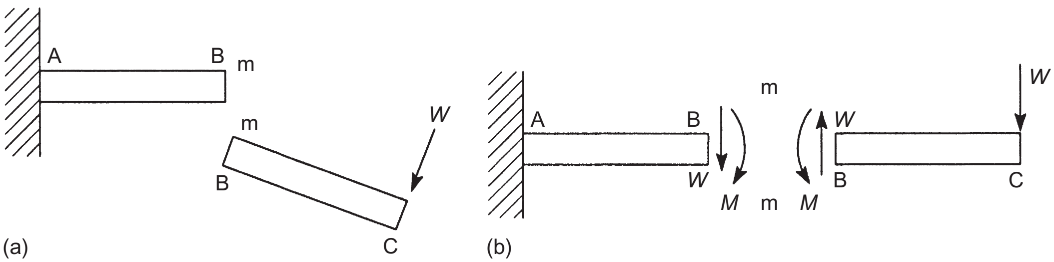

FIGURE 3.5

Internal force system generated by an external shear load.

BC exerts an equal force system on AB, but opposite in direction. The complete force systems acting on the two faces of the section mm are shown in Fig. 3.5(b).

Systems of forces such as those at the section mm are known as *internal forces.* Generally, they vary throughout the length of a structural member as can be seen from Fig. 3.5(b) where the internal moment, M, increases in magnitude as the built-in end is approached due to the increasing rotational effect of W. We note that applied loads of one type can induce internal forces of another. For example, in Fig. 3.5(b) the external shear load, W, produces both shear and bending at the section mm.

Internal forces are distributed throughout beam sections in the form of stresses. It follows that the resultant of each individual stress distribution must be the corresponding internal force; internal forces are therefore often known as *stress resultants.* However, before an individual stress distribution can be found it is necessary to determine the corresponding internal force. Also, in design problems, it is necessary to determine the position and value of maximum stress and displacement. Usually, the first step in the analysis of a structure is to calculate the distribution of each of the four basic internal force types throughout the component structural members. We shall therefore determine the distributions of the four internal force systems in a variety of structural members. First, however, we shall establish a notation and sign convention for each type of force.

3.2 Notation and sign convention

We shall be concerned initially with structural members having at least one longitudinal plane of symmetry. Normally this will be a vertical plane and will contain the externally applied loads. Later, however, we shall investigate the bending and shear of beams having unsymmetrical sections so that as far as possible the notation and sign convention we adopt now will be consistent with that required later.

The axes system we shall use is the right-handed system shown in Fig. 3.6 in which the x axis is along the longitudinal axis of the member and the y axis is vertically upwards. Externally applied loads W (concentrated) and w (distributed) are shown acting vertically downwards since this is usually the situation in practice. In fact, choosing a sign convention for these externally applied loads is not particularly important and can be rather confusing since they will generate support reactions, which are external loads themselves, in an opposite sense. An external axial load P is positive when tensile and a torque T is positive if applied in an anticlockwise sense when viewed in the direction xO. Later we shall be concerned with displacements in structural members and here the vertical displacement v is positive in the positive direction of the y axis.

We have seen that external loads generate internal force systems and for these it is essential to adopt a sign convention since, unless their directions and senses are known, it is impossible to calculate stress distributions.

Figure 3.7 shows a positive set of internal forces acting at two sections of a beam.

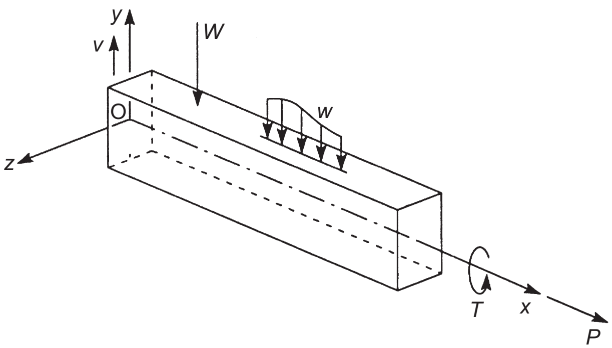

FIGURE 3.6

Notation and sign conventions for displacements and externally applied loads.

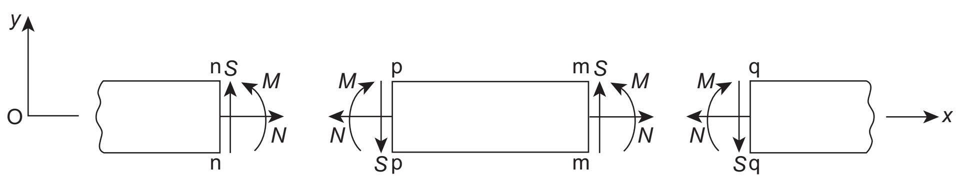

FIGURE 3.7

Positive internal force systems.

Note that the forces and moments acting on opposite faces of a section are identical and act in opposite directions since the internal equilibrium of the beam must be maintained. If this were not the case one part of the beam would part company with the other. A difficulty now arises in that a positive internal force, say the shear force S, acts upwards on one face of a section and downwards on the opposite face. We must therefore specify the face of the section we are considering. We can do this by giving signs to the different faces. In Fig. 3.7 we define a *positive face* as having an outward normal in the positive direction of the x axis (faces nn and mm) and a *negative face* as having an outward normal in the negative direction of the x axis (faces pp and qq). At nn and mm positive internal forces act in positive directions on positive faces while at pp and qq positive internal forces act in negative directions on negative faces.

A positive bending moment M, clockwise on the negative face pp and anticlockwise on the positive face mm, will cause the upper surface of the beam to become concave and the lower surface convex. This, for obvious reasons, is called a *sagging* bending moment. A negative bending moment will produce a convex upper surface and a concave lower one and is therefore termed a *hogging* bending moment.

The axial, or normal, force N is positive when tensile, i.e. it pulls away from either face of a section, and a positive internal torque T is anticlockwise on positive internal faces.

Generally the structural engineer will need to know peak values of these internal forces in a structural member. To determine these peak values *internal force diagrams* are constructed; the methods will be illustrated by examples.

3.3 Normal force

EXAMPLE 3.1

Construct a normal force diagram for the beam AB shown in Fig. 3.8(a).

The first step is to calculate the support reactions using the methods described in Section 2.5. In this case, since the beam is on a roller support at B, the horizontal load at B is reacted at A; clearly $R_{A,H}=10$ kN acting to the left.

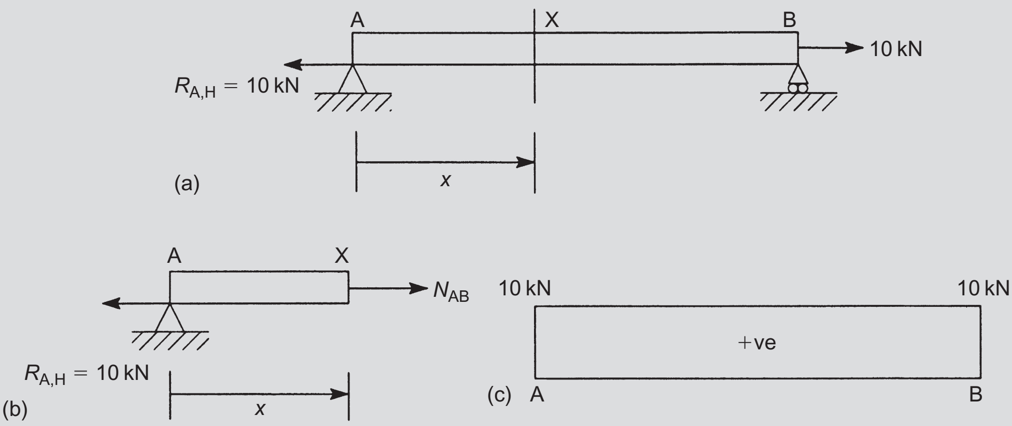

FIGURE 3.8

Normal force diagram for the beam of Ex. 3.1.

Generally the distribution of an internal force will change at a loading discontinuity. In this case there is no loading discontinuity at any section of the beam so that we can determine the complete distribution of the normal force by calculating the normal force at any section X, a distance x from A.

Consider the length AX of the beam as shown in Fig. 3.8(b) (equally we could consider the length XB). The internal normal force acting at X is N_{AB} which is shown acting in a positive (tensile) direction. The length AX of the beam is in equilibrium under the action of $R_{A,H}$ (=10 kN) and N_{AB}. Thus, from Section 2.4, for equilibrium in the x direction

$$N_{AB} - R_{A,H} = N_{AB} - 10 = 0$$

which gives

$$N_{AB} = +10 \text{ kN}$$

N_{AB} is positive and therefore acts in the assumed positive direction; the normal force diagram for the complete beam is then as shown in Fig. 3.8(c).

When the equilibrium of a portion of a structure is considered as in Fig. 3.8(b) we are using what is termed a *free body diagram.*

EXAMPLE 3.2

Draw a normal force diagram for the beam ABC shown in Fig. 3.9(a).

Again by considering the overall equilibrium of the beam we see that $R_{A,H}$ = 10 kN acting to the left (C is the roller support).

In this example there is a loading discontinuity at B so that the distribution of the normal force in AB will be different to that in BC. We must therefore determine the normal force at an arbitrary section X_1 between A and B, and then at an arbitrary section X_2 between B and C.

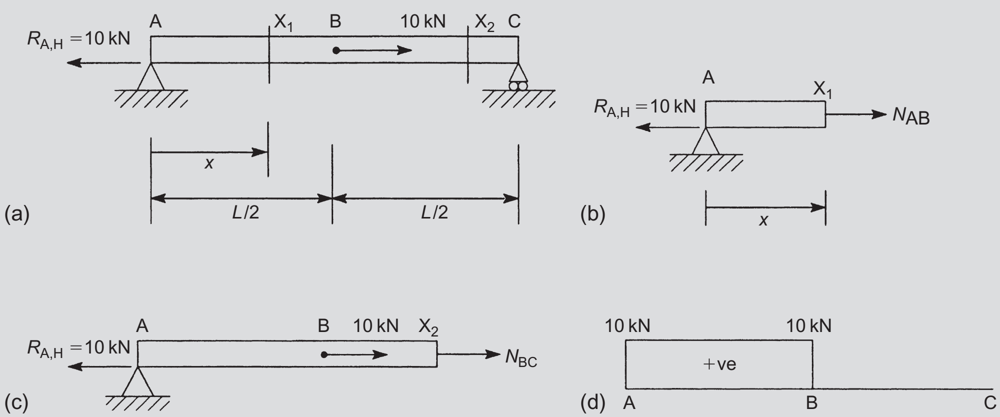

FIGURE 3.9

Normal force diagram for the beam of Ex. 3.2.

The free body diagram for the portion of the beam AX_1 is shown in Fig. 3.9(b). (Alternatively we could consider the portion X_1C). As before, we draw in a positive normal force, N_{AB}. Then, for equilibrium of AX_1 in the x direction

$$N_{AB} - 10 = 0$$

so that

$$N_{AB} = +10 \text{ kN (tension)}$$

Now consider the length ABX_2 of the beam; again we draw in a positive normal force, N_{BC}. Then for equilibrium of ABX_2 in the x direction

$$N_{BC} + 10 - 10 = 0$$

which gives

$$N_{BC} = 0$$

Note that we would have obtained the same result by considering the portion X_2C of the beam. Finally the complete normal force diagram for the beam is drawn as shown in Fig. 3.9(d).

EXAMPLE 3.3

Figure 3.10(a) shows a beam ABCD supporting three concentrated loads, two of which are inclined to the longitudinal axis of the beam. Construct the normal force diagram for the beam and determine the maximum value.

In this example we are only concerned with determining the normal force distribution in the beam, so that it is unnecessary to calculate the vertical reactions at the supports. Further, the horizontal components of the inclined loads can only be resisted at A since D is a roller support. Thus, considering the horizontal equilibrium of the beam

$$R_{A,H} + 6\cos 60° - 4\cos 60° = 0$$

which gives

$$R_{A,H} = -1 \text{ kN}$$

The negative sign of $R_{A,H}$ indicates that the reaction acts to the right and not to the left as originally assumed. However, rather than change the direction of $R_{A,H}$ in the diagram, it is simpler to retain the assumed direction and then insert the negative value as required.

Although there is an apparent loading discontinuity at B, the 2 kN load acts perpendicularly to the longitudinal axis of the beam and will therefore not affect the normal force. We may therefore consider the normal force at any section X_1 between A and C. The free body diagram for the portion AX_1 of the beam is shown in Fig. 3.10(b); again we draw in a positive normal force N_{AC}. For equilibrium of AX_1

$$N_{AC} - R_{A,H} = 0$$

so that

$$N_{AC} = R_{A,H} = -1 \text{ kN (compression)}$$

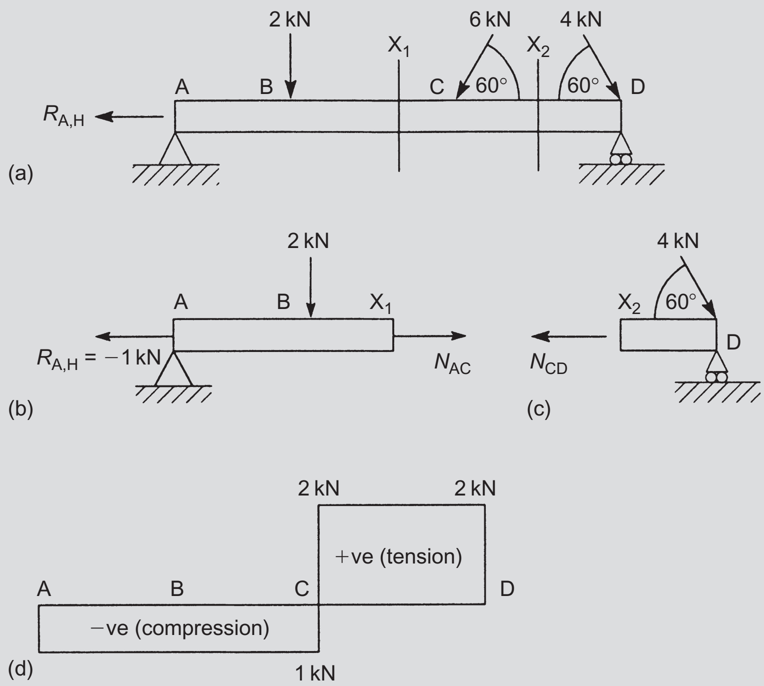

FIGURE 3.10

Normal force diagram for the beam of Ex. 3.3.

The horizontal component of the inclined load at C produces a loading discontinuity so that we now consider the normal force at any section X_2 between C and D. Here it is slightly simpler to consider the equilibrium of the length X_2D of the beam rather than the length AX_2. Thus, from Fig. 3.10(c)

$$N_{CD} - 4\cos 60° = 0$$

which gives

$$N_{CD} = +2\ \text{kN (tension)}$$

From the completed normal force diagram in Fig. 3.10(d) we see that the maximum normal force in the beam is 2 kN (tension) acting at all sections between C and D.

EXAMPLE 3.4

Construct the normal force diagram for the cranked cantilever beam shown in Fig. 3.11(a).

Note that in this example there will be two components of support reaction at the built-in end of the beam, $R_{A,H}$, and $R_{A,V}$ (there will also be a moment reaction but since we are concerned only with normal force this is irrelevant). However, if we consider the equilibrium of portions of the beam away from the built-in end it will not be necessary to calculate them. Note also that there is a loading discontinuity at B

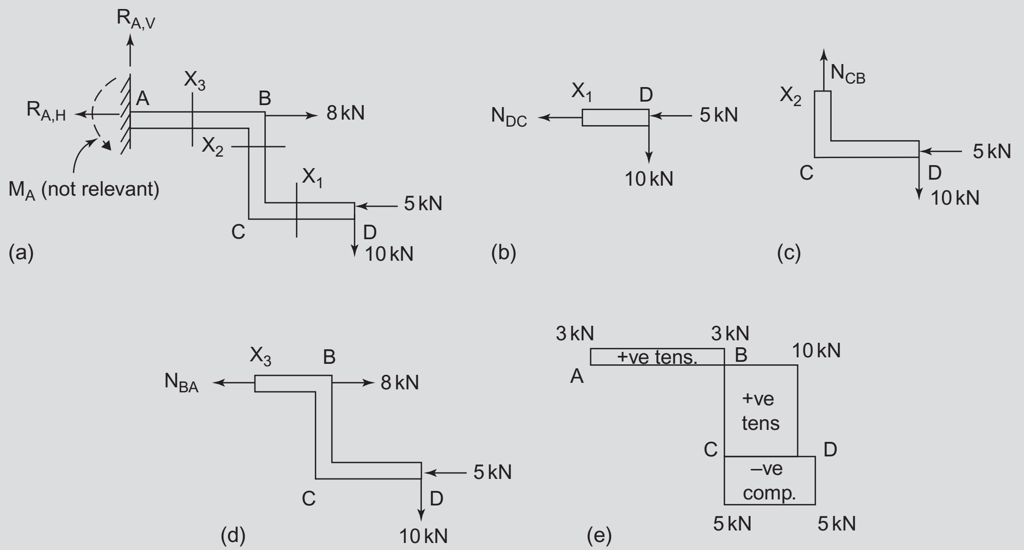

FIGURE 3.11

Normal force diagram for the beam of Ex. 3.4.

and structural discontinuities at C and B. Initially, therefore, we consider the normal force, N_{DC}, at the section X_1 as shown in Fig. 3.11(b).

For horizontal equilibrium of the length DX_1 of the beam

$$N_{DC} + 5 = 0$$

so that

$$N_{DC} = -5 \text{ kN (compression)}$$

The vertical 10 kN load acting at D will produce a normal force in CB. Then, considering the vertical equilibrium of the portion DCX_2 of the beam in Fig. 3.11(c)

$$N_{CB} - 10 = 0$$

which gives

$$N_{CB} = +10 \text{ kN (tension)}$$

Finally we consider the horizontal equilibrium of the portion $DCBX_3$ of the beam in Fig. 3.11(d).

$$N_{BA} - 8 + 5 = 0$$

from which

$$N_{BA} = +3 \text{ kN (tension)}$$

The normal force diagram for the complete beam is then as shown in Fig. 3.11(e). The normal force for the vertical portion CB may be drawn on either side of CB as is convenient.

EXAMPLE 3.5

The cranked cantilever ABC shown in Fig. 3.12(a) carries a uniformly distributed load of 10 kN/m of horizontal length. Construct the normal force diagram for the cantilever.

Consider any section X_1 of the cantilever between C and B as shown in Fig. 3.12(b).

Resolving forces horizontally for this length of beam gives

$$N_{BC} = 0$$

Now consider any section X_2 between B and A where X_2 is perpendicular to the axis of the beam as shown in Fig. 3.12(c).

Resolving forces parallel to the axis of the beam gives

$$N_{BA} - 10x \ \cos \ 60° - 10 \times 1 \times \cos \ 60° = 0$$

so that $N_{BA} = 5(x + 1)$

When $x = 0$, $N_{BA} = 5$ kN (tension) and when $x = 1.5$m, $N_{BA} = 12.5$ kN (tension).

The complete normal force distribution is shown in Fig. 3.12(d).

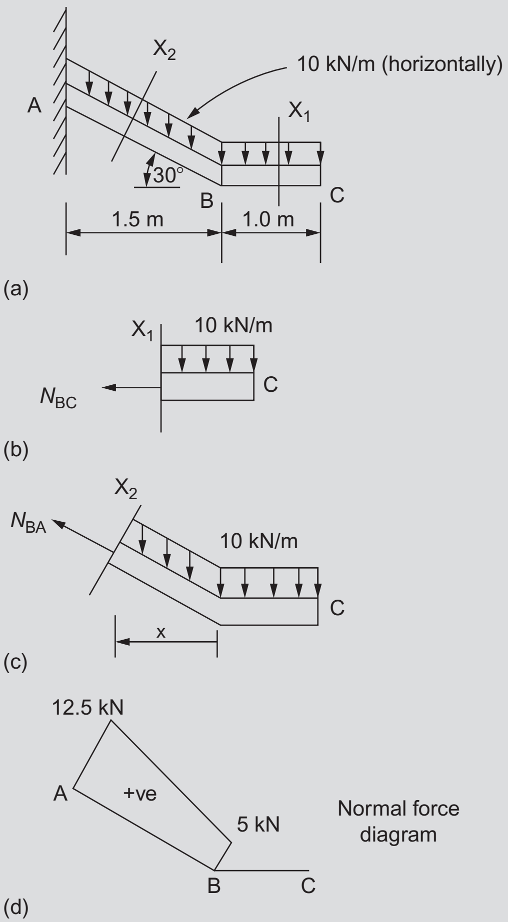

FIGURE 3.12

Normal force diagrams for the beam of Ex. 3.5.

3.4 Shear force and bending moment

It is convenient to consider shear force and bending moment distributions in beams simultaneously since, as we shall see in Section 3.5, they are directly related. Again the method of construction of shear force and bending moment diagrams will be illustrated by examples.

EXAMPLE 3.6

Cantilever beam with a concentrated load at the free end (Fig. 3.13(a)).

Generally, as in the case of normal force distributions, we require the variation in shear force and bending moment along the length of a beam. Again, loading discontinuities, such as concentrated loads and/or a sudden change in the intensity of a distributed load, cause discontinuities in the distribution of shear force and bending moment so that it is necessary to consider a series of sections, one between each loading discontinuity. In this example, however, there are no loading discontinuities between the built-in end A and the free end B so that we may consider a section X at any point between A and B.

For many beams the value of each support reaction must be calculated before the shear force and bending moment distributions can be obtained. In Fig. 3.13(a) a consideration of the overall equilibrium of the beam (see Section 2.5) gives a vertical reaction, W, and a moment reaction, WL, at the built-in end. However, if we consider the equilibrium of the length XB of the beam as shown in the free body diagram in Fig. 3.13(b), this calculation is unnecessary.

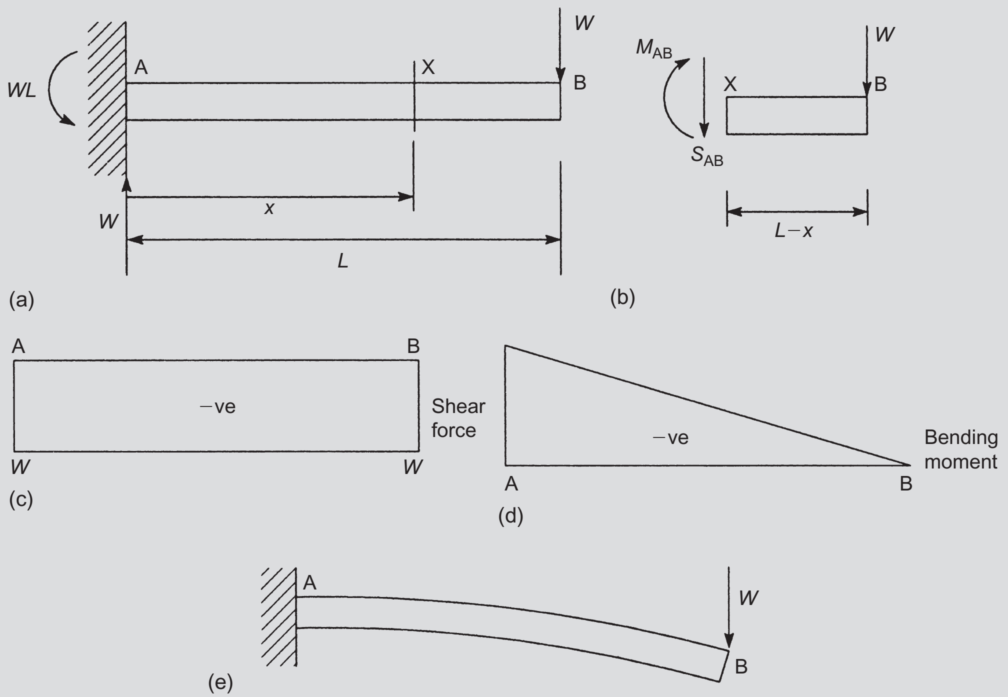

FIGURE 3.13

Shear force and bending moment diagrams for the beam of Ex. 3.6.

As in the case of normal force distributions we assign positive directions to the shear force, S_{AB}, and bending moment, M_{AB}, at the section X. Then, for vertical equilibrium of the length XB of the beam we have

$$S_{AB} + W = 0$$

which gives

$$S_{AB} = -W$$

The shear force is therefore constant along the length of the beam and the shear force diagram is rectangular in shape, as shown in Fig. 3.13(c).

The bending moment, M_{AB}, is now found by considering the moment equilibrium of the length XB of the beam about the section X. Alternatively we could take moments about B, but this would involve the moment of the shear force, S_{AB}, about B. This approach, although valid, is not good practice since it includes a previously calculated quantity; in some cases, however, this is unavoidable. Thus, taking moments about the section X we have

$$M_{AB} + W(L-x) = 0$$

so that

$$M_{AB} = -W(L-x) \tag{i}$$

Equation (i) shows that M_{AB} varies linearly along the length of the beam, is negative, i.e. hogging, at all sections and increases from zero at the free end ($x = L$) to $-WL$ at the built-in end where $x = 0$.

It is usual to draw the bending moment diagram on the tension side of a beam. This procedure is particularly useful in the design of reinforced concrete beams since it shows directly the surface of the beam near which the major steel reinforcement should be provided. Also, drawing the bending moment diagram on the tension side of a beam can give an indication of the deflected shape as illustrated in Exs 3.6–3.9. This is not always the case, however, as we shall see in Exs 3.10 and 3.11.

In this case the beam will bend as shown in Fig. 3.13(e), so that the upper surface of the beam is in tension and the lower one in compression; the bending moment diagram is therefore drawn on the upper surface as shown in Fig. 3.13(d). Note that negative (hogging) bending moments applied in a vertical plane will always result in the upper surface of a beam being in tension.

EXAMPLE 3.7

Cantilever beam carrying a uniformly distributed load of intensity w.

Again it is unnecessary to calculate the reactions at the built-in end of the cantilever; their values are, however, shown in Fig. 3.14(a). Note that for the purpose of calculating the moment reaction the uniformly distributed load may be replaced by a concentrated load ($= wL$) acting at a distance $L/2$ from A.

There is no loading discontinuity between A and B so that we may consider the shear force and bending moment at any section X between A and B. As before, we insert positive directions for the shear force, S_{AB}, and bending moment, M_{AB}, in the free body diagram of Fig. 3.14(b). Then, for vertical equilibrium of the length XB of the beam

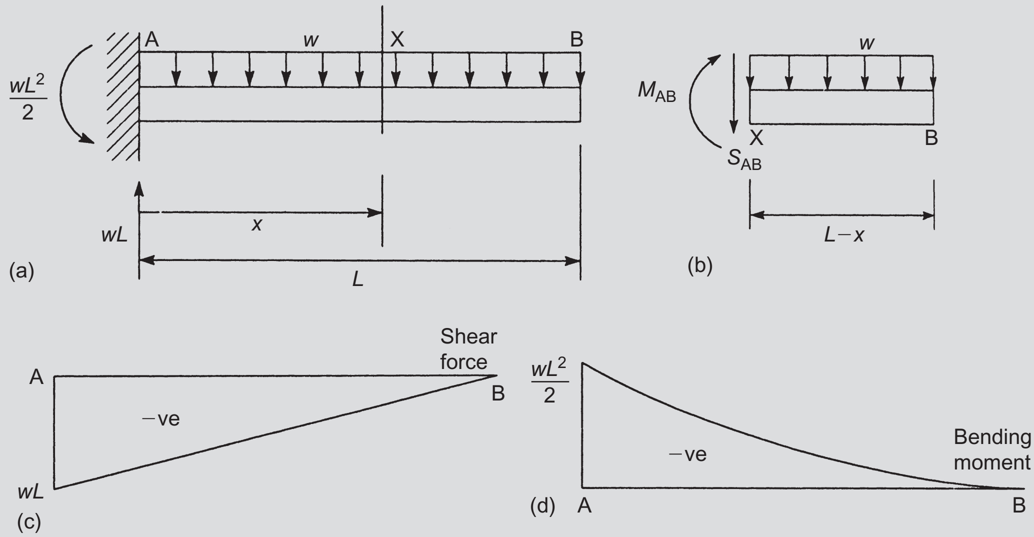

FIGURE 3.14

Shear force and bending moment diagrams for the beam of Ex. 3.7.

$$S_{AB} + w(L - x) = 0$$

so that

$$S_{AB} = -w(L - x) \tag{i}$$

Therefore S_{AB} varies linearly with x and varies from zero at B to $-wL$ at A (Fig. 3.14(c)).

Now consider the moment equilibrium of the length XB of the beam and take moments about X

$$M_{AB} + \frac{w}{2}(L - x)^2 = 0$$

which gives

$$M_{AB} = -\frac{w}{2}(L - x)^2 \tag{ii}$$

Note that the total load on the length XB of the beam is $w(L-x)$, which we may consider acting as a concentrated load at a distance $(L-x)/2$ from X. From Eq. (ii) we see that the bending moment, M_{AB}, is negative at all sections of the beam and varies parabolically as shown in Fig. 3.14(d) where the bending moment diagram is again drawn on the tension side of the beam. The actual shape of the bending moment diagram may be found by plotting values or, more conveniently, by examining Eq. (ii). Differentiating with respect to x we obtain

$$\frac{dM_{AB}}{dx} = w(L - x) \tag{iii}$$

so that when $x = L$, $dM_{AB}/dx = 0$ and the bending moment diagram is tangential to the datum line AB at B. Furthermore it can be seen from Eq. (iii) that the gradient (dM_{AB}/dx) of the bending moment diagram decreases as x increases, so that its shape is as shown in Fig. 3.14(d).

EXAMPLE 3.8

Simply supported beam carrying a central concentrated load.

In this example it is necessary to calculate the value of the support reactions, both of which are seen, from symmetry, to be $W/2$ (Fig. 3.15(a)). Also, there is a loading discontinuity at B, so that we must consider the shear force and bending moment first at an arbitrary section X_1 say, between A and B and then at an arbitrary section X_2 between B and C.

From the free body diagram in Fig. 3.15(b) in which both S_{AB} and M_{AB} are in positive directions we see, by considering the vertical equilibrium of the length AX_1 of the beam, that

$$S_{AB} + \frac{W}{2} = 0$$

which gives

$$S_{AB} = -\frac{W}{2}$$

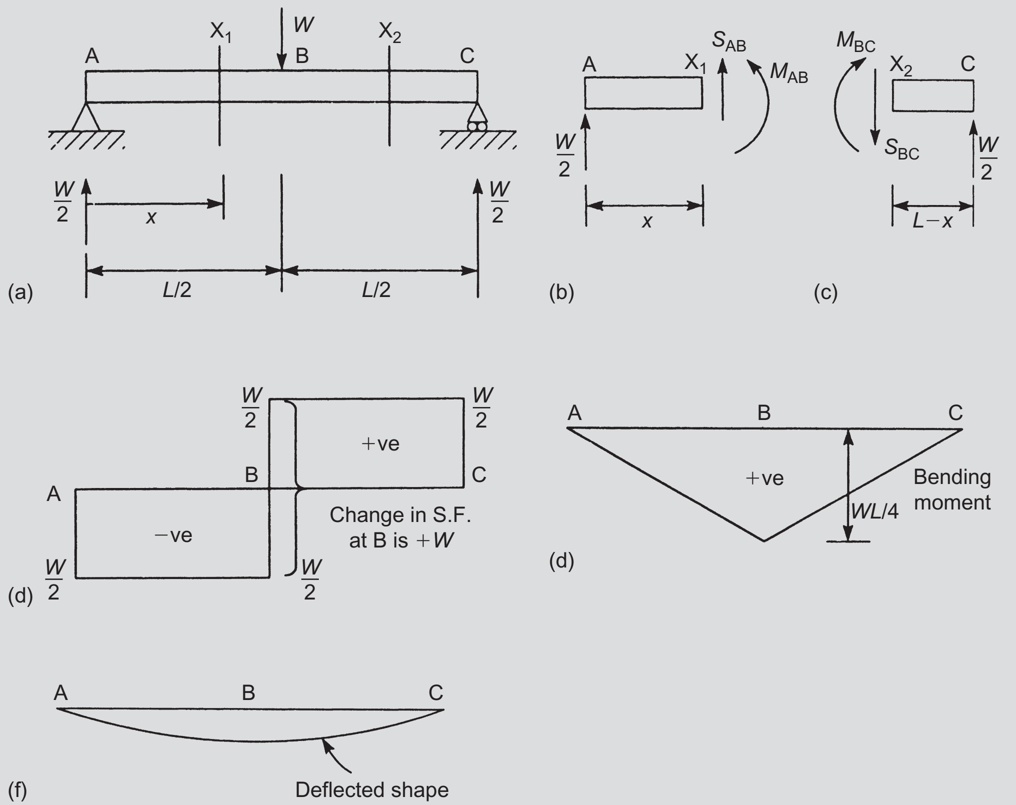

FIGURE 3.15

Shear force and bending moment diagrams for the beam of Ex. 3.8.

S_{AB} is therefore constant at all sections of the beam between A and B, in other words, from a section immediately to the right of A to a section immediately to the left of B.

Now consider the free body diagram of the length X_2C of the beam in Fig. 3.15(c). Note that, equally, we could have considered the length ABX_2, but this would have been slightly more complicated in terms of the number of loads acting. For vertical equilibrium of X_2C

$$S_{BC} - \frac{W}{2} = 0$$

from which

$$S_{BC} = +\frac{W}{2}$$

and we see that S_{BC} is constant at all sections of the beam between B and C so that the complete shear force diagram has the form shown in Fig. 3.15(d). Note that the *change* in shear force from that at a section immediately to the left of B to that at a section immediately to the right of B is $+W$. We shall consider the implications of this later in the chapter.

It would also appear from Fig. 3.15(d) that there are two different values of shear force at the same section B of the beam. This results from the assumption that W is concentrated at a point which, practically, is impossible since there would then be an infinite bearing pressure on the surface of the beam. In practice, the load W and the support reactions would be distributed over a small length of beam (Fig. 3.16(a)) so that the actual shear force distribution would be that shown in Fig. 3.16(b).

The distribution of the bending moment in AB is now found by considering the moment equilibrium about X_1 of the length AX_1 of the beam in Fig. 3.15(b). Thus

$$M_{AB} - \frac{W}{2}x = 0$$

or

$$M_{AB} = \frac{W}{2}x \tag{i}$$

Therefore M_{AB} varies linearly from zero at A ($x = 0$) to $+WL/4$ at B ($x = L/2$).

Now considering the length X_2C of the beam in Fig. 3.15(c) and taking moments about X_2.

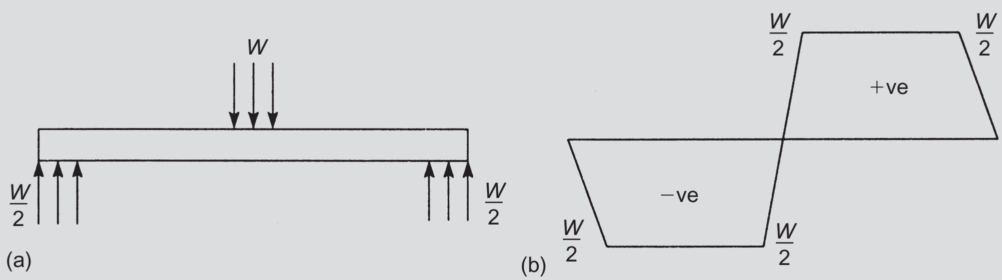

FIGURE 3.16

Shear force diagram in a practical situation.

$$M_{\mathrm{BC}} - \frac{W}{2}(L - x) = 0$$

which gives

$$M_{\mathrm{BC}} = +\frac{W}{2}(L - x) \tag{ii}$$

From Eq. (ii) we see that M_{BC} varies linearly from $+WL/4$ at B ($x = L/2$) to zero at C ($x = L$).

The complete bending moment diagram is shown in Fig. 3.15(e). Note that the bending moment is positive (sagging) at all sections of the beam so that the lower surface of the beam is in tension. In this example the deflected shape of the beam would be that shown in Fig. 3.15(f).

EXAMPLE 3.9

Simply supported beam carrying a uniformly distributed load.

The symmetry of the beam and its load may again be used to determine the support reactions which are each $wL/2$. Furthermore, there is no loading discontinuity between the ends A and B of the beam so that it is sufficient to consider the shear force and bending moment at just one section X, a distance x, say, from A; again we draw in positive directions for the shear force and bending moment at the section X in the free body diagram shown in Fig. 3.17(b).

Considering the vertical equilibrium of the length AX of the beam gives

$$S_{\mathrm{AB}} - wx + w\frac{L}{2} = 0$$

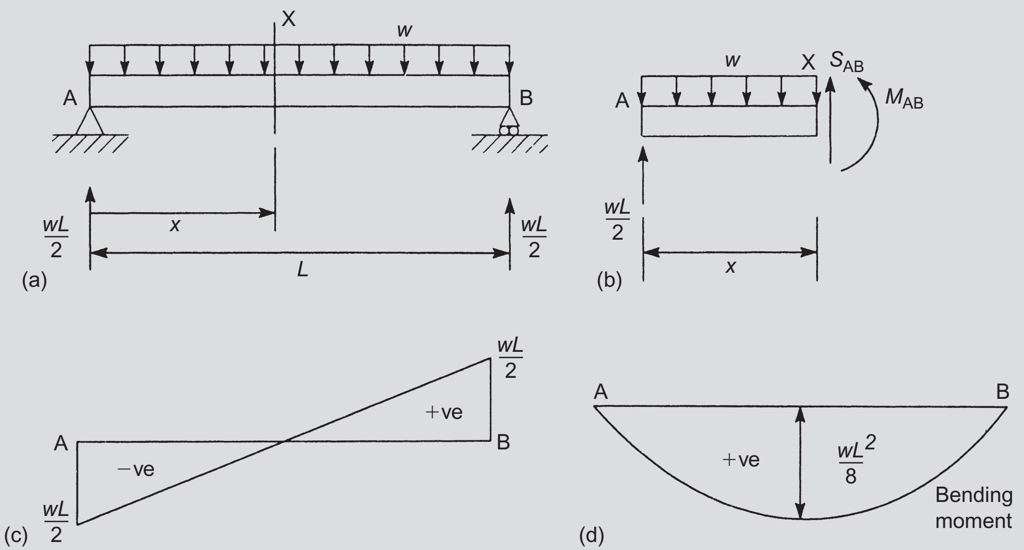

FIGURE 3.17

Shear force and bending moment diagrams for the beam of Ex. 3.9.

i.e.

$$S_{AB} = +w\left(x - \frac{L}{2}\right) \tag{i}$$

S_{AB} therefore varies linearly along the length of the beam from $-wL/2$ at A ($x = 0$) to $+wL/2$ at B ($x = L$). Note that $S_{AB} = 0$ at mid-span ($x = L/2$).

Now taking moments about X for the length AX of the beam in Fig. 3.17(b) we have

$$M_{AB} + \frac{wx^2}{2} - \frac{wL}{2}x = 0$$

from which

$$M_{AB} = +\frac{wx}{2}(L - x) \tag{ii}$$

Thus M_{AB} varies parabolically along the length of the beam and is positive (sagging) at all sections of the beam except at the supports ($x = 0$ and $x = L$) where it is zero.

Also, differentiating Eq. (ii) with respect to x gives

$$\frac{dM_{AB}}{dx} = w\left(\frac{L}{2} - x\right) \tag{iii}$$

From Eq. (iii) we see that $dM_{AB}/dx = 0$ at mid-span where $x = L/2$, so that the bending moment diagram has a turning value or mathematical maximum at this section. In this case this mathematical maximum is the maximum value of the bending moment in the beam and is, from Eq. (ii), $+wL^2/8$.

The bending moment diagram for the beam is shown in Fig. 3.17(d) where it is again drawn on the tension side of the beam; the deflected shape of the beam will be identical in form to the bending moment diagram.

Examples 3.6–3.9 may be regarded as 'standard' cases and it is useful to memorize the form that the shear force and bending moment diagrams take including the principal values.

EXAMPLE 3.10

Simply supported beam with cantilever overhang (Fig. 3.18(a)).

The support reactions are calculated using the methods described in Section 2.5. Thus, taking moments about B in Fig. 3.18(a) we have

$$R_A \times 2 - 2 \times 3 \times 0.5 + 1 \times 1 = 0$$

which gives

$$R_A = 1 \text{ kN}$$

From vertical equilibrium

$$R_B + R_A - 2 \times 3 - 1 = 0$$

so that

$$R_B = 6 \text{ kN}$$

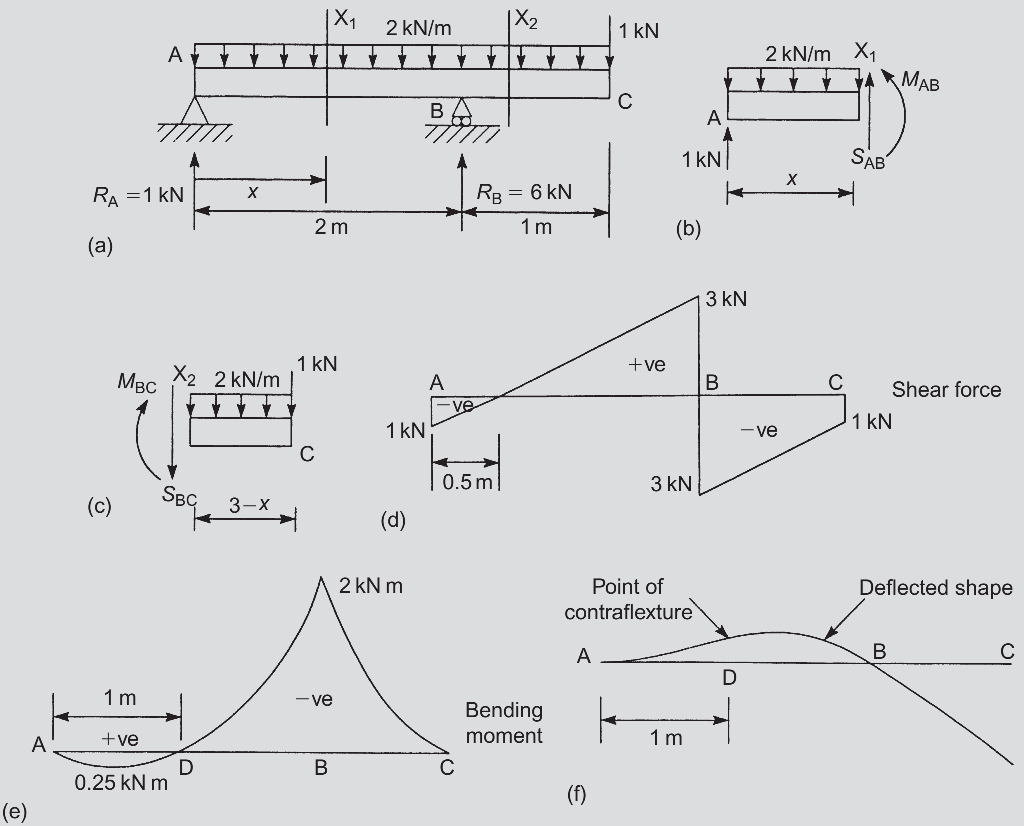

FIGURE 3.18

Shear force and bending moment diagrams for the beam of Ex. 3.10.

The support reaction at B produces a loading discontinuity at B so that we must consider the shear force and bending moment at two arbitrary sections of the beam, X_1 in AB and X_2 in BC. Free body diagrams are therefore drawn for the lengths AX_1 and X_2C of the beam and positive directions for the shear force and bending moment drawn in as shown in Fig. 3.18(b) and (c). Alternatively, we could have considered the lengths X_1BC and ABX_2, but this approach would have involved slightly more complicated solutions in terms of the number of loads applied.

Now from the vertical equilibrium of the length AX_1 of the beam in Fig. 3.18(b) we have

$$S_{AB} - 2x + 1 = 0$$

or

$$S_{AB} = 2x - 1 \tag{i}$$

The shear force therefore varies linearly in AB from -1 kN at A($x = 0$) to $+3$ kN at B($x = 2$ m). Note that $S_{AB} = 0$ at $x = 0.5$ m.

Consideration of the vertical equilibrium of the length X_2C of the beam in Fig. 3.18(c) gives

$$S_{BC} + 2(3 - x) + 1 = 0$$

from which

$$S_{BC} = 2x - 7 \tag{ii}$$

Equation (ii) shows that S_{BC} varies linearly in BC from -3 kN at B($x = 2$ m) to -1 kN at C($x = 3$ m).

The complete shear force diagram for the beam is shown in Fig. 3.18(d).

The bending moment, M_{AB}, is now obtained by considering the moment equilibrium of the length AX_1 of the beam about X_1 in Fig. 3.18(b). Hence

$$M_{AB} + 2x\frac{x}{2} - 1x = 0$$

so that

$$M_{AB} = x - x^2 \tag{iii}$$

which is a parabolic function of x. The distribution may be plotted by selecting a series of values of x and calculating the corresponding values of M_{AB}. However, this would not necessarily produce accurate estimates of either the magnitudes and positions of the maximum values of M_{AB} or, say, the positions of the zero values of M_{AB} which, as we shall see later, are important in beam design. A better approach is to examine Eq. (iii) as follows. Clearly when $x = 0$, $M_{AB} = 0$ as would be expected at the simple support at A. Also at B, where $x = 2$ m, $M_{AB} = -2$ kN so that although the support at B is a simple support and allows rotation of the beam, there is a moment at B; this is produced by the loads on the cantilever overhang BC. Rewriting Eq. (iii) in the form

$$M_{AB} = x(1 - x) \tag{iv}$$

we see immediately that $M_{AB} = 0$ at $x = 0$ (as demonstrated above) and that $M_{AB} = 0$ at $x = 1$ m, the point D in Fig. 3.18(e). We shall see later in Chapter 9 that at the point in the beam where the bending moment changes sign the curvature of the beam is zero; this point is known as a *point of contraflexure* or *point of inflection*. Now differentiating Eq. (iii) with respect to x we obtain

$$\frac{dM_{AB}}{dx} = 1 - 2x \tag{v}$$

and we see that $dM_{AB}/dx = 0$ at $x = 0.5$ m. In other words M_{AB} has a turning value or mathematical maximum at $x = 0.5$ m at which point $M_{AB} = 0.25$ kNm. Note that this is not the greatest value of bending moment in the span AB. Also it can be seen that for $0<x<0.5$ m, dM_{AB}/dx decreases with x while for $0.5 \text{ m}<x<2$ m, dM_{AB}/dx increases negatively with x.

Now we consider the moment equilibrium of the length X_2C of the beam in Fig. 3.18(c) about X_2

$$M_{BC} + \frac{2}{2}(3 - x)^2 + 1(3 - x) = 0$$

so that

$$M_{BC} = -12 + 7x - x^2 \tag{vi}$$

from which we see that dM_{BC}/dx is not zero at any point in BC and that as x increases dM_{BC}/dx decreases.

The complete bending moment diagram is therefore as shown in Fig. 3.18(e). Note that the value of zero shear force in AB coincides with the turning value of the bending moment.

In this particular example it is not possible to deduce the displaced shape of the beam from the bending moment diagram. Only three facts relating to the displaced shape can be stated with certainty; these are, the deflections at A and B are zero and there is a point of contraflexure at D, 1 m from A. However, using the method described in Section 13.2 gives the displaced shape shown in Fig. 3.18(f). Note that, although the beam is subjected to a sagging bending moment over the length AD, the actual deflection is upwards; clearly this could not have been deduced from the bending moment diagram.

EXAMPLE 3.11

Simply supported beam carrying a point moment.

From a consideration of the overall equilibrium of the beam (Fig. 3.19(a)) the support reactions are $R_A = M_0/L$ acting vertically upward and $R_C = M_0/L$ acting vertically downward. Note that R_A and R_C are independent of the point of application of M_0.

Although there is a loading discontinuity at B it is a point moment and will not affect the distribution of shear force. Thus, by considering the vertical equilibrium of either AX_1 in Fig. 3.19(b) or X_2C in Fig. 3.19(c) we see that

$$S_{AB} = S_{BC} = -\frac{M_0}{L} \tag{i}$$

The shear force is therefore constant along the length of the beam as shown in Fig. 3.19(d).

Now considering the moment equilibrium about X_1 of the length AX_1 of the beam in Fig. 3.19(b)

$$M_{AB} - \frac{M_0}{L}x = 0$$

or

$$M_{AB} = \frac{M_0}{L}x \tag{ii}$$

M_{AB} therefore increases linearly from zero at A ($x = 0$) to $+3M_0/4$ at B($x = 3L/4$). From Fig. 3.19(c) and taking moments about X_2 we have

$$M_{BC} + \frac{M_0}{L}(L - x) = 0$$

or

$$M_{BC} = \frac{M_0}{L}(x - L) \tag{iii}$$

M_{BC} therefore decreases linearly from $-M_0/4$ at B($x = 3L/4$) to zero at C($x = L$); the complete distribution of bending moment is shown in Fig. 3.19(e). The deflected form of the beam is shown in Fig. 3.19(f) where a point of contraflexure occurs at B, the section at which the bending moment changes sign.

In this example, as in Ex. 3.10, the exact form of the deflected shape cannot be deduced from the bending moment diagram without analysis. However, using the method of singularities described in Section 13.2, it may be shown that the deflection at B is negative and that the slope of the beam at C is positive, giving the displaced shape shown in Fig. 3.19(f).

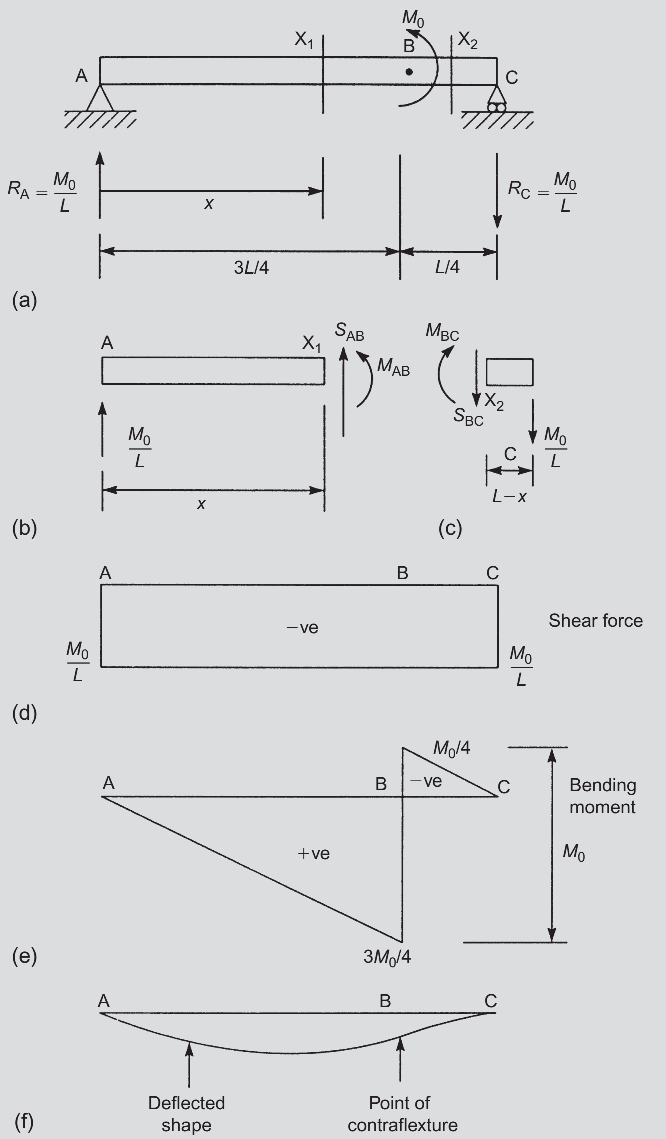

FIGURE 3.19

Shear force and bending moment diagrams for the beam of Ex. 3.11.

EXAMPLE 3.12

Construct shear force and bending moment diagrams for the truss shown in Fig. 3.20(a).

The support at E is a roller support so that only a vertical reaction, $R_{E,V}$, can occur there. Considering the horizontal equilibrium of the truss

$$R_{A,H} - 1 = 0$$

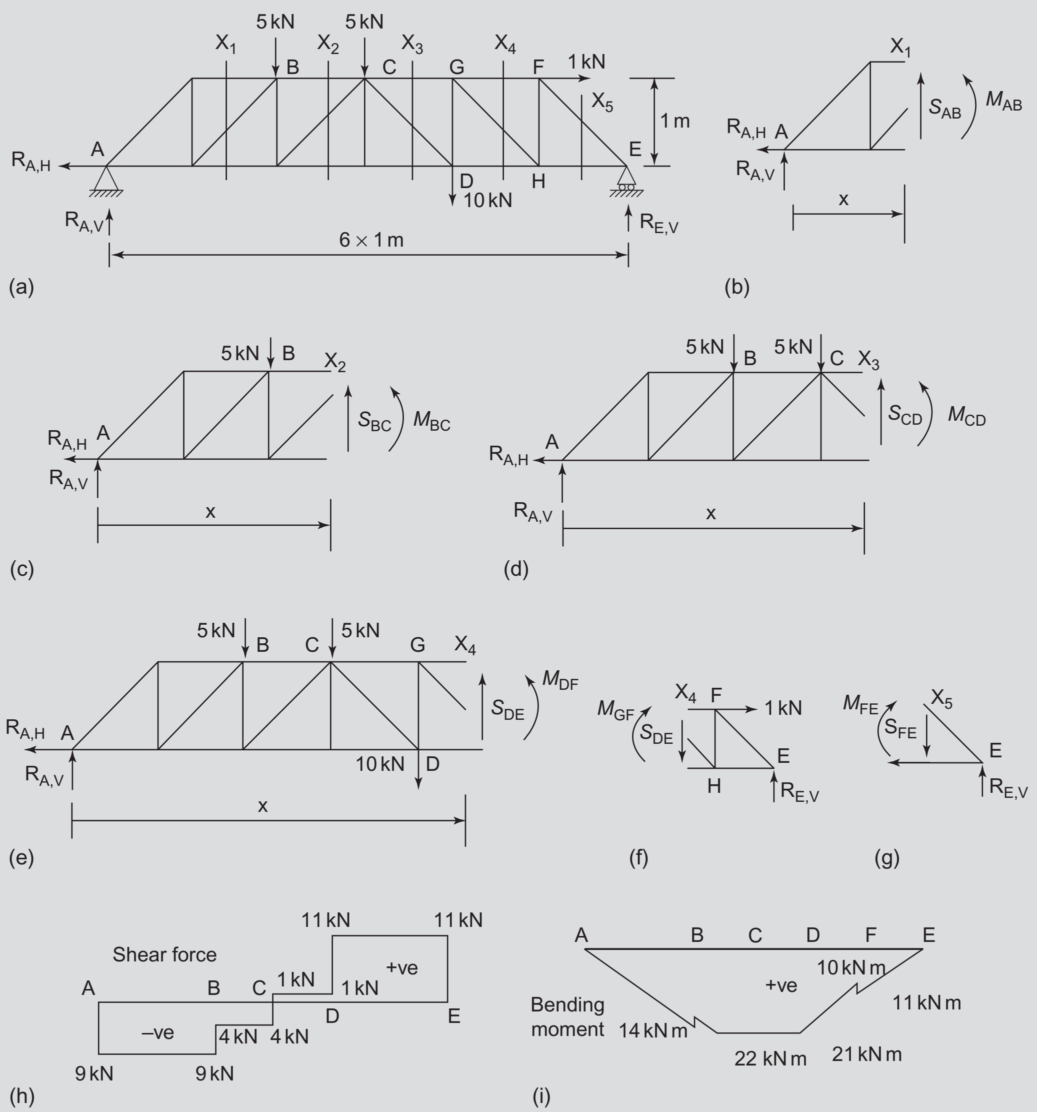

FIGURE 3.20

Shear force and bending moment diagrams for the truss of Ex. 3.12.

so that

$$R_{A,H} = 1 \text{ kN}$$

Now taking moments about E

$$R_{A,V} \times 6 - 5 \times 4 - 5 \times 3 - 10 \times 2 + 1 \times 1 = 0$$

which gives

$$R_{A,V} = 9 \text{ kN}$$

The vertical equilibrium of the truss gives

$$R_{E,V} + R_{A,V} - 5 - 5 - 10 = 0$$

from which

$$R_{E,V} = 11 \text{ kN}$$

With regard to vertical forces there are loading discontinuities at B, C and D; the horizontal load at F will not contribute to the shear force at any section of the truss. Initially, therefore, we consider a length, x, of the truss as shown in Fig. 3.20(b) and insert a positive shear force, S_{AB}, and a positive bending moment, M_{AB}, at the section X_1. Then, for vertical equilibrium of the length of truss

$$S_{AB} + R_{A,V} = 0$$

or

$$S_{AB} + 9 = 0$$

so that

$$S_{AB} = -9 \text{ kN}$$

Similarly, from Fig. 3.20(c)

$$S_{BC} + R_{A,V} - 5 = 0$$

which gives

$$S_{BC} = -4 \text{ kN}$$

Then, from Fig. 3.20(d)

$$S_{CD} + R_{A,V} - 5 - 5 = 0$$

from which

$$S_{CD} = +1 \text{ kN}$$

and from Fig. 3.20(e)

$$S_{DE} + R_{A,V} - 5 - 5 - 10 = 0$$

which gives

$$S_{DE} = +11 \text{ kN}$$

Alternatively, and slightly simpler, we could have considered the equilibrium of the portion of the truss to the right of the section X_4 as in Fig. 3.20(f). Then

$$S_{DE} - R_{E,V} = 0$$

which gives $S_{DE} = +11$ kN as before.

The complete shear force diagram is then as shown in Fig. 3.20(h).

With regard to the bending moment distribution there are loading discontinuities at B, C, D and also at F which is caused by the application of the horizontal 1 kN load. We must therefore consider sections of the truss between A and B, between B and C, between C and D, between D and F and between F and E.

Now considering the length of truss in Fig. 3.20(b) and taking moments about the section X_1 (thereby eliminating S_{AB})

$$M_{AB} - R_{A,V}x = 0$$

from which

$$M_{AB} = 9x \tag{i}$$

Eq. (i) shows that M_{AB} varies linearly from zero at A to $9\times2 = +18$ kNm at B.

Similarly, from Fig. 3.20(c) and taking moments about the section X_2

$$M_{BC} - R_{A,V}x + 5(x-2) = 0$$

so that

$$M_{BC} = 10 + 4x \tag{ii}$$

Therefore, from Eq. (ii), M_{BC} varies linearly from +18 kNm at B($x = 2$ m) to +22 kNm at C($x = 3$ m).

Now, from Fig. 3.20(d) and taking moments about the section X_3

$$M_{CD} - R_{A,V}x + 5(x-2) + 5(x-3) = 0$$

which gives

$$M_{CD} = 25 - x \tag{iii}$$

Eq. (iii) shows that M_{CD} varies linearly from +22 kNm at C($x = 3$ m) to +21 kNm at D($x = 4$ m).

The bending moment distribution in DF may be found by considering the equilibrium of the portion of the frame to the left of X_4 as shown in Fig. 3.20(e). Then, taking moments about X_4

$$M_{DF} - R_{A,V}x + 5(x-2) + 5(x-3) + 10(x-4) = 0$$

from which

$$M_{DF} = 65 - 11x \tag{iv}$$

Therefore, M_{DF} varies linearly from +21 kNm at D($x = 4$ m) to +10 kNm at F($x = 5$ m).

Now considering the length of truss to the right of the section X_5 and taking moments about the section X_5

$$M_{FE} - R_{E,V}(6-x) = 0$$

which gives

$$M_{FE} = 66 - 11x \tag{v}$$

so that M_{FE} varies linearly from +11 kNm at F ($x = 5$ m) to zero at E ($x = 6$ m); the complete bending moment distribution is then as shown in Fig. 3.20(i).

The discontinuity at F is due to the moment at F produced by the horizontal load at F which, together with the horizontal support reaction, $R_{A,H}$, may be regarded as forming a couple of magnitude $1\times1 = 1$ kNm acting at the truss section at F. (from the concept of the transmissibility of a force, $R_{A,H}$ may be regarded as acting at any point in its line of action). The situation is then similar to that in Ex. 3.11 where a point moment, M_0, is applied to the beam at B.

3.5 Load, shear force and bending moment relationships

It is clear from Exs 3.6–3.11 that load, shear force and bending moment are related. Thus, for example, uniformly distributed loads produce linearly varying shear forces and maximum values of bending moment coincide with zero shear force. We shall now examine these relationships mathematically.

The length of beam shown in Fig. 3.21(a) carries a general system of loading comprising concentrated loads and a distributed load $w(x)$. An elemental length δx of the beam is subjected to the force and moment system shown in Fig. 3.21(b); since δx is very small the distributed load may be regarded as constant over the length δx. For vertical equilibrium of the element

$$S + w(x)\delta x - (S + \delta S) = 0$$

so that

$$+w(x)\delta x - \delta S = 0$$

Thus, in the limit as $\delta x \rightarrow 0$

$$\frac{\mathrm{d}S}{\mathrm{d}x} = +w(x) \tag{3.1}$$

From Eq. (3.1) we see that the rate of change of shear force at a section of a beam, in other words the gradient of the shear force diagram, is equal to the value of the load intensity at that section. In Fig. 3.14(c), for example, the shear force changes linearly from $-wL$ at A to zero at B so that the gradient of the shear force diagram at any section of the beam is $+wL/L = +w$ where w is the load intensity. Equation (3.1) also applies at beam sections subjected to concentrated loads. In Fig. 3.15(a) the load intensity at B, theoretically, is infinite, as is the gradient of the shear force diagram at B (Fig. 3.15(d)). In practice the shear force diagram would have a finite gradient at this section as illustrated in Fig. 3.16.

Now integrating Eq. (3.1) with respect to x we obtain

$$S = +\int w(x)\,\mathrm{d}x + C_1 \tag{3.2}$$

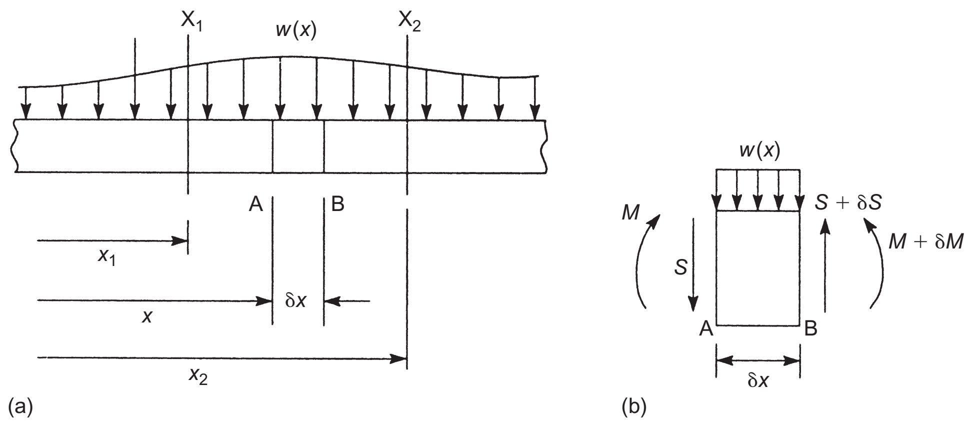

FIGURE 3.21

Load, shear force and bending moment relationships.

in which C_1 is a constant of integration which may be determined in a particular case from the loading boundary conditions.

If, for example, $w(x)$ is a uniformly distributed load of intensity w, i.e., it is not a function of x, Eq. (3.2) becomes

$$S = +wx + C_1$$

which is the equation of a straight line of gradient $+w$ as demonstrated for the cantilever beam of Fig. 3.14 in the previous paragraph. Furthermore, for this particular example, $S = 0$ at $x = L$ so that $C_1 = -wL$ and $S = -w(L-x)$ as before.

In the case of a beam carrying only concentrated loads then, in the bays between the loads, $w(x)=0$ and Eq. (3.2) reduces to

$$S = C_1$$

so that the shear force is constant over the unloaded length of beam (see Figs 3.13 and 3.15).

Suppose now that Eq. (3.1) is integrated over the length of beam between the sections X_1 and X_2. Then

$$\int_{x_1}^{x_2} \frac{dS}{dx} dx = +\int_{x_1}^{x_2} w(x) dx$$

which gives

$$S_2 - S_1 = \int_{x_1}^{x_2} w(x) dx \tag{3.3}$$

where S_1 and S_2 are the shear forces at the sections X_1 and X_2 respectively. Equation (3.3) shows that the *change* in shear force between two sections of a beam is equal to the area under the load distribution curve over that length of beam.

The argument may be applied to the case of a concentrated load W which may be regarded as a uniformly distributed load acting over an extremely small elemental length of beam, say δx. The area under the load distribution curve would then be $w\delta x$ ($=W$) and the change in shear force from the section x to the section $x+\delta x$ would be $+W$. In other words, the change in shear force from a section immediately to the left of a concentrated load to a section immediately to the right is equal to the value of the load, as noted in Ex. 3.8.

Now consider the rotational equilibrium of the element δx in Fig. 3.21(b) about B. Thus

$$M - S\delta x - w(x)\delta x\frac{\delta x}{2} - (M + \delta M) = 0$$

The term involving the square of δx is a second-order term and may be neglected. Hence

$$-S\delta x - \delta M = 0$$

or, in the limit as $\delta x \to 0$

$$\frac{dM}{dx} = -S \tag{3.4}$$

Equation (3.4) establishes for the general case what may be observed in particular in the shear force and bending moment diagrams of Exs 3.6–3.11, i.e. the gradient of the bending moment diagram at a beam section is equal to minus the value of the shear force at that section. For example, in Fig. 3.18(e) the bending moment in AB is a mathematical maximum at the section where the shear force is zero.

Integrating Eq. (3.4) with respect to x we have

$$M = -\int S\, dx + C_2 \tag{3.5}$$

in which C_2 is a constant of integration. Substituting for S in Eq. (3.5) from Eq. (3.2) gives

$$M = -\int \left[+ \int w(x)\,\mathrm{d}x + C_1 \right] \mathrm{d}x + C_2$$

or

$$M = -\int\int w(x)\,\mathrm{d}x + C_1 x + C_2 \tag{3.6}$$

If $w(x)$ is a uniformly distributed load of intensity w, Eq. (3.6) becomes

$$M = -w\frac{x^2}{2} - C_1 x + C_2$$

which shows that the equation of the bending moment diagram on a length of beam carrying a uniformly distributed load is parabolic.

In the case of a beam carrying concentrated loads only, then, between the loads, $w(x) = 0$ and Eq. (3.6) reduces to

$$M = -C_1 x + C_2$$

which shows that the bending moment varies linearly between the loads and has a gradient $-C_1$.

The constants C_1 and C_2 in Eq. (3.6) may be found, for a given beam, from the loading boundary conditions. Thus, for the cantilever beam of Fig. 3.14, we have already shown that $C_1 = -wL$ so that $M = -wx^2/2 + wLx + C_2$. Also, when $x = L$, $M = 0$ which gives $C_2 = -wL^2/2$ and hence $M = -wx^2/2 + wLx - wL^2/2$ as before.

Now integrating Eq. (3.4) over the length of beam between the sections X_1 and X_2 (Fig. 3.21(a))

$$\int_{x_1}^{x_2} \frac{\mathrm{d}M}{\mathrm{d}x}\mathrm{d}x = -\int_{x_1}^{x_2} S\,\mathrm{d}x$$

which gives

$$M_2 - M_1 = -\int_{x_1}^{x_2} S\,\mathrm{d}x \tag{3.7}$$

where M_1 and M_2 are the bending moments at the sections X_1 and X_2, respectively. Equation (3.7) shows that the *change* in bending moment between two sections of a beam is equal to minus the area of the shear force diagram between those sections. Again, using the cantilever beam of Fig. 3.14 as an example, we see that the change in bending moment from A to B is $wL^2/2$ and that the area of the shear force diagram between A and B is $-wL^2/2$.

Finally, from Eqs (3.1) and (3.4)

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -\frac{\mathrm{d}S}{\mathrm{d}x} = -w(x) \tag{3.8}$$

The relationships established above may be used to construct shear force and bending moment diagrams for some beams more readily than when the methods illustrated in Exs 3.6–3.11 are employed. In addition they may be used to provide simpler solutions in some beam problems.

EXAMPLE 3.13

Construct shear force and bending moment diagrams for the beam shown in Fig. 3.22(a).

Initially the support reactions are calculated using the methods described in Section 2.5. Then, for moment equilibrium of the beam about E

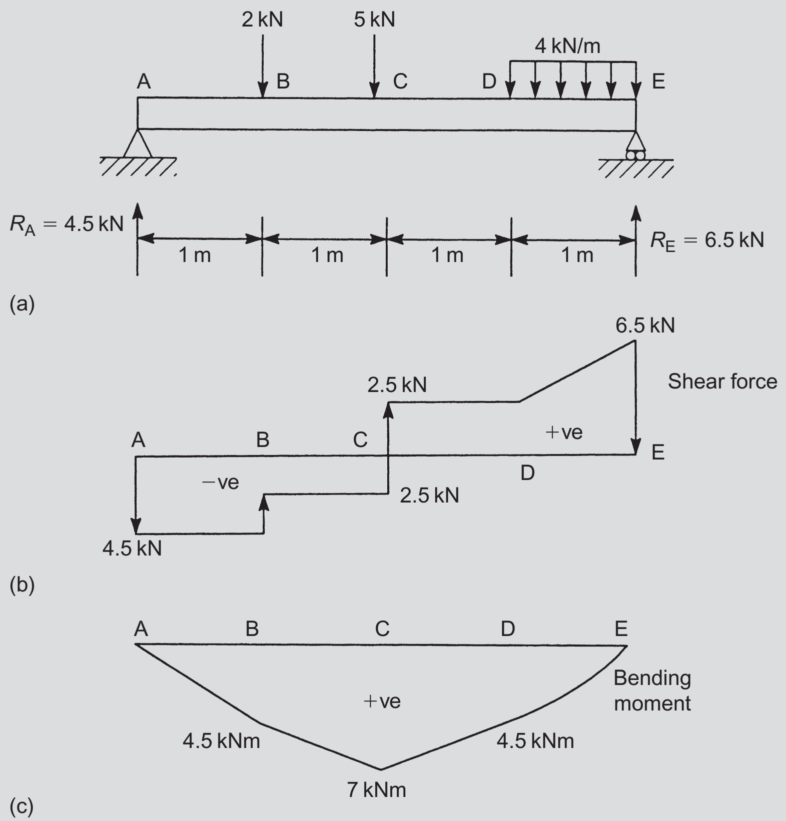

FIGURE 3.22

Shear force and bending moment diagrams for the beam of Ex. 3.13.

$$R_A \times 4 - 2 \times 3 - 5 \times 2 - 4 \times 1 \times 0.5 = 0$$

from which

$$R_{\mathrm{A}} = 4.5 \text{ kN}$$

Now considering the vertical equilibrium of the beam

$$R_{\mathrm{E}} + R_{\mathrm{A}} - 2 - 5 - 4 \times 1 = 0$$

so that

$$R_{\mathrm{E}} = 6.5 \text{ kN}$$

In constructing the shear force diagram we can make use of the facts that, as established above, the shear force is constant over unloaded bays of the beam, varies linearly when the loading is uniformly distributed and changes positively as a vertically downward concentrated load is crossed in the positive x direction by the value of the load. Thus in Fig. 3.22(b) the shear force increases negatively by 4.5 kN as we move from the left of A to the right of A, is constant between A and B, changes positively by 2 kN as we move from the left of B to the right of B, and so on. Note that between D and E the shear force changes linearly from +2.5 kN at D to +6.5 kN at a section immediately to the left of E, in other words it changes by +4 kN, the total value of the downward-acting uniformly distributed load.

The bending moment diagram may also be constructed using the above relationships, namely, the bending moment varies linearly over unloaded lengths of beam and parabolically over lengths of beam carrying a uniformly distributed load. Also, the change in bending moment between two sections of a beam is equal to minus the area of the shear force diagram between those sections. Thus in Fig. 3.22(a) we know that the bending moment at the pinned support at A is zero and that it varies linearly in the bay AB. The bending moment at B is then equal to minus the area of the shear force diagram between A and B, i.e. $-(-4.5 \times 1) = 4.5$ kNm. This represents, in fact, the change in bending moment from the zero value at A to the value at B. At C the area of the shear force diagram to the right or left of C is 7 kNm (note that the bending moment at E is also zero), and so on. In the bay DE the shape of the parabolic curve representing the distribution of bending moment over the length of the uniformly distributed load may be found using part of Eq. (3.8), i.e.

$$\frac{d^2M}{dx^2} = -w(x)$$

For a vertically downward uniformly distributed load this expression becomes

$$\frac{d^2M}{dx^2} = -w$$

which from mathematical theory shows that the curve representing the variation in bending moment is convex in the positive direction of bending moment. This may be observed in the bending moment diagrams in Fig. 3.14(d), 3.17(d) and 3.18(e). In this example the bending moment diagram for the complete beam is shown in Fig. 3.22(c) and is again drawn on the tension side of the beam.

EXAMPLE 3.14

A precast concrete beam of length L is to be lifted from the casting bed and transported so that the maximum bending moment is as small as possible. If the beam is lifted by two slings placed symmetrically, show that each sling should be $0.21L$ from the adjacent end.

The external load on the beam is comprised solely of its own weight, which is uniformly distributed along its length. The problem is therefore resolved into that of a simply supported beam carrying a uniformly distributed load in which the supports are positioned at some distance a from each end (Fig. 3.23(a)).

The shear force and bending moment diagrams may be constructed in terms of a using the methods described above and would take the forms shown in Fig. 3.23(b) and (c). Examination of the bending moment diagram shows that there are two possible positions for the maximum bending moment. First at B and C where the bending moment is hogging and has equal values from symmetry; second at the mid-span point where the bending moment has a turning value and is sagging if the supports at B and C are spaced a sufficient distance apart. Suppose that B and C are positioned such that the value of the hogging bending moment at B and C is numerically equal to the sagging bending moment at the mid-span point. If now B and C are moved further apart the mid-span moment will increase while the moment

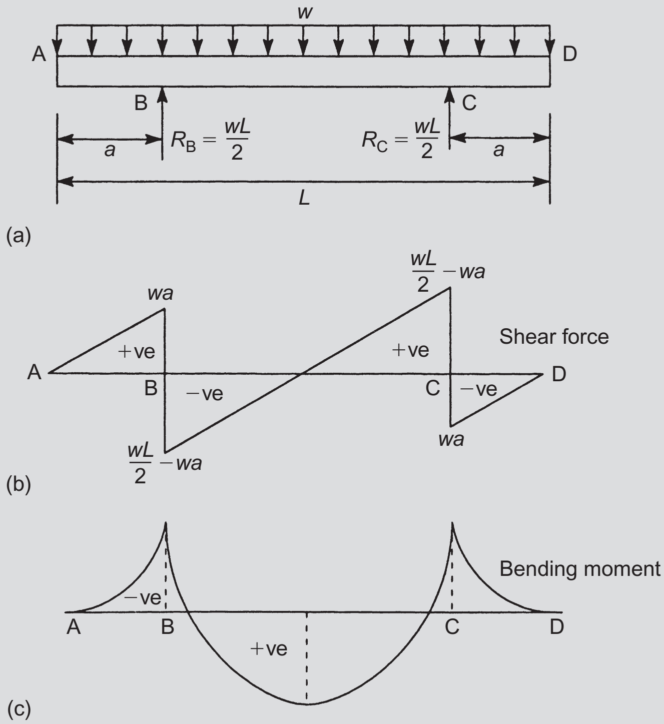

FIGURE 3.23

Determination of optimum position for supports in the precast concrete beam of Ex. 3.14.

at B and C decreases. Conversely, if B and C are brought closer together, the hogging moment at B and C increases while the mid-span moment decreases. It follows that the maximum bending moment will be as small as possible when the hogging moment at B and C is numerically equal to the sagging moment at mid-span.

The solution will be simplified if use is made of the relationship in Eq. (3.7). Thus, when the supports are in the optimum position, the change in bending moment from A to B (negative) is equal to minus half the change in the bending moment from B to the mid-span point (positive). It follows that the area of the shear force diagram between A and B is equal to minus half of that between B and the mid-span point. Then

$$+\frac{1}{2}awa = -\frac{1}{2}\left[-\frac{1}{2}\left(\frac{L}{2}-a\right)w\left(\frac{L}{2}-a\right)\right]$$

which reduces to

$$a^2 + La - \frac{L^2}{4} = 0$$

the solution of which gives

$$a = 0.21L \quad \text{(the negative solution has no practical significance)}$$

3.6 Torsion

The distribution of torque along a structural member may be obtained by considering the equilibrium in free body diagrams of lengths of member in a similar manner to that used for the determination of shear force distributions in Exs 3.6–3.11.

EXAMPLE 3.15

Construct a torsion diagram for the beam shown in Fig. 3.24(a).

There is a loading discontinuity at B so that we must consider the torque at separate sections X_1 and X_2 in AB and BC, respectively. Thus, in the free body diagrams shown in Fig. 3.24(b) and (c) we insert positive internal torques.

From Fig. 3.24(b)

$$T_{AB} - 10 + 8 = 0$$

so that

$$T_{AB} = +2 \text{ kN m}$$

From Fig. 3.24(c)

$$T_{BC} + 8 = 0$$

from which

$$T_{BC} = -8 \text{ kN m}$$

The complete torsion diagram is shown in Fig. 3.24(d).

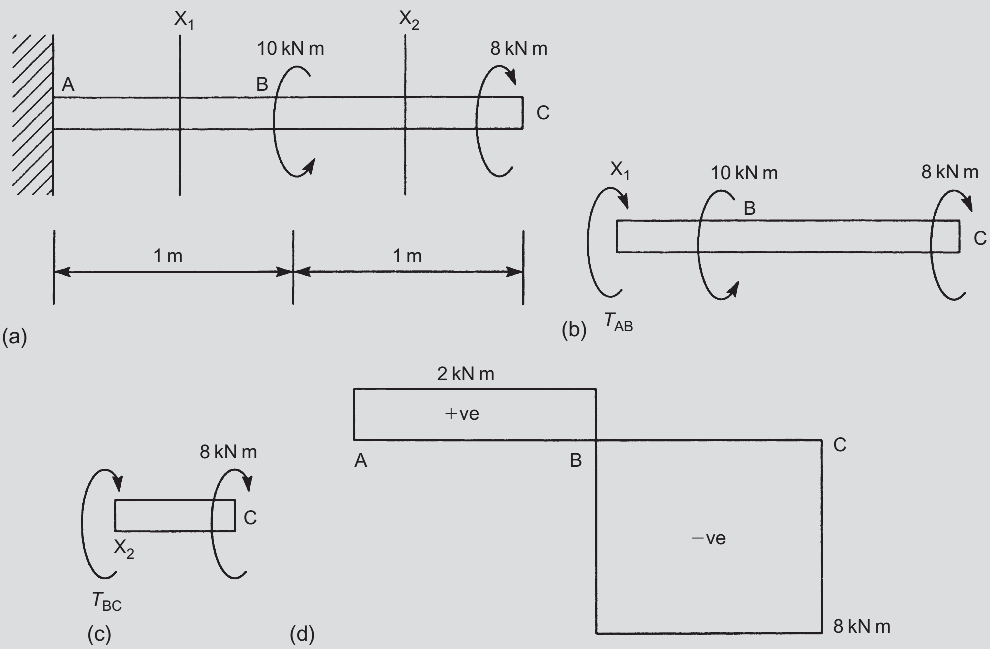

FIGURE 3.24

Torsion diagram for a cantilever beam.

EXAMPLE 3.16

The structural member ABC shown in Fig. 3.25 carries a distributed torque of 2 kNm/m together with a concentrated torque of 10 kNm at mid-span. The supports at A and C prevent rotation of the member in planes perpendicular to its axis. Construct a torsion diagram for the member and determine the maximum value of torque.

From the rotational equilibrium of the member about its longitudinal axis and its symmetry about the mid-span section at B, we see that the reactive torques T_A and T_C are each -9 kNm, i.e. clockwise when viewed in the direction CBA. In general, as we shall see in Chapter 11, reaction torques at supports form a statically indeterminate system.

In this particular problem there is a loading discontinuity at B so that we must consider the internal torques at two arbitrary sections X_1 and X_2 as shown in Fig. 3.26(a).

From the free body diagram in Fig. 3.26(b)

$$T_{AB} + 2x - 9 = 0$$

which gives

$$T_{AB} = 9 - 2x \quad \text{(i)}$$

From Eq. (i) we see that T_{AB} varies linearly from +9 kNm at A ($x = 0$) to +5 kNm at a section immediately to the left of B ($x = 2$ m). Furthermore, from Fig. 3.26(c)

$$T_{BC} - 2(4 - x) + 9 = 0$$

so that

$$T_{BC} = -2x - 1 \quad \text{(ii)}$$

from which we see that T_{BC} varies linearly from -5 kNm at a section immediately to the right of B ($x = 2$ m) to -9 kNm at C ($x = 4$ m). The resulting torsion diagram is shown in Fig. 3.26(d).

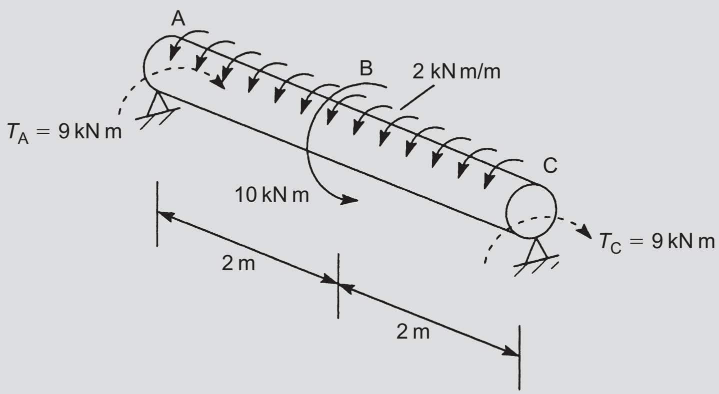

FIGURE 3.25

Beam of Ex. 3.16.

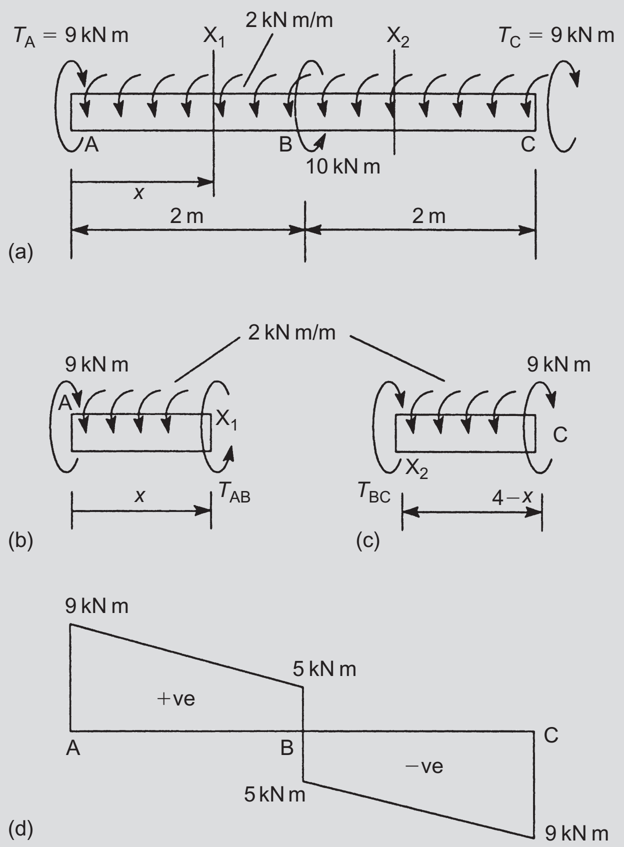

FIGURE 3.26

Torsion diagram for the beam of Ex. 3.16.

3.7 Principle of superposition

An extremely useful principle in the analysis of linearly elastic structures (see Chapter 8) is that of superposition. The principle states that if the displacements at all points in an elastic body are proportional to the forces producing them, that is the body is linearly elastic, the effect (i.e. stresses and displacements) on such a body of a number of forces acting simultaneously is the sum of the effects of the forces applied separately.

This principle can sometimes simplify the construction of shear force and bending moment diagrams.

EXAMPLE 3.17

Construct the bending moment diagram for the beam shown in Fig. 3.27(a).

Figures 3.27(b), (c) and (d) show the bending moment diagrams for the cantilever when each of the three loading systems acts separately. The bending moment diagram for the beam when the loads act simultaneously is obtained by adding the ordinates of the separate diagrams and is shown in Fig. 3.27(e).

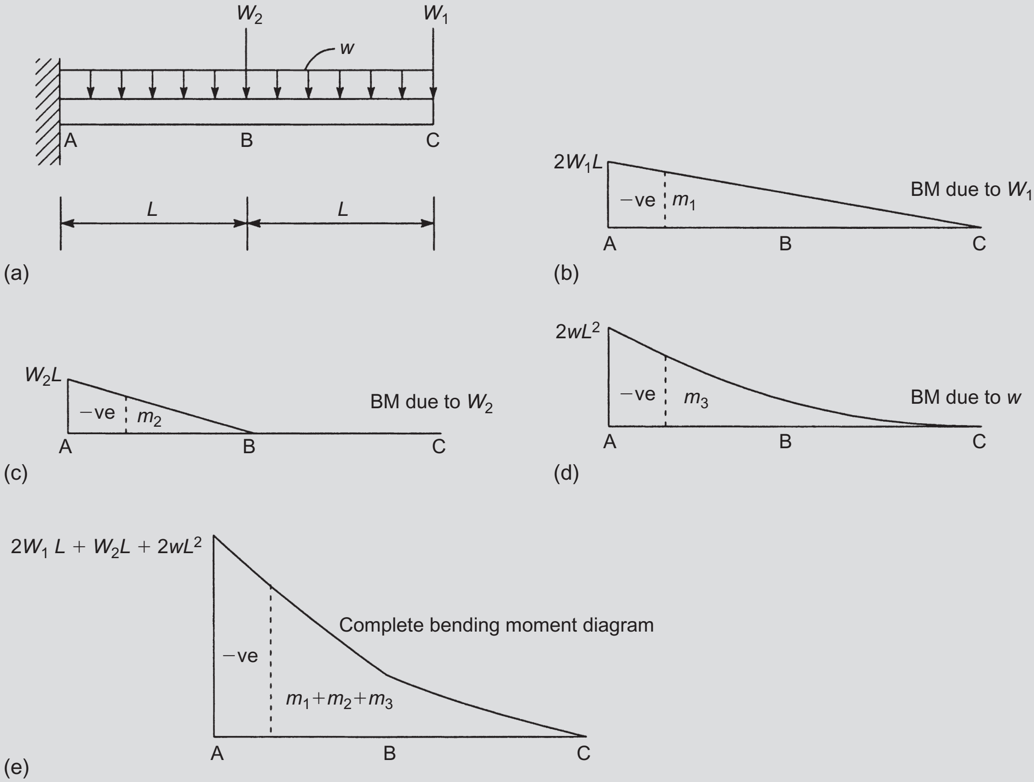

FIGURE 3.27

Bending moment (BM) diagram using the principle of superposition.

PROBLEMS

P.3.1 A transmitting mast of height 40 m and weight 4.5 kN/m length is stayed by three groups of four cables attached to the mast at heights of 15, 25 and 35 m. If each cable is anchored to the ground at a distance of 20 m from the base of the mast and tensioned to a force of 15 kN, draw a diagram of the compressive force in the mast.

Ans. Max. force = 314.9 kN.

P.3.2 Construct the normal force, shear force and bending moment diagrams for the beam shown in Fig. P.3.2.

Ans.

$N_{AB} = 9.2$ kN, $N_{BC} = 9.2$ kN, $N_{CD} = 5.7$ kN, $N_{DE} = 0$.
$S_{AB} = -6.9$ kN, $S_{BC} = -3.9$ kN, $S_{CD} = +2.2$ kN, $S_{DE} = +7.9$ kN.
$M_B = 27.6$ kNm, $M_C = 51$ kNm, $M_D = 40$ kNm.

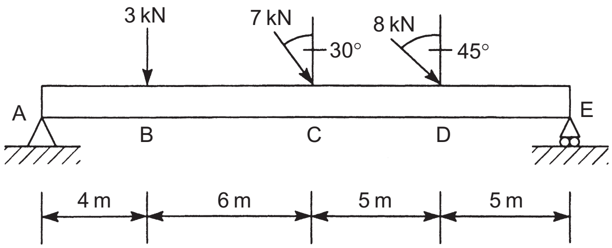

FIGURE P.3.2

P.3.3 Draw dimensioned sketches of the diagrams of normal force, shear force and bending moment for the beam shown in Fig. P.3.3.

Ans.

$N_{AB} = N_{BC} = N_{CD} = 0$, $N_{DE} = -6$ kN.
$S_A = 0$, S_B (in AB) $= +10$ kN, S_B (in BC) $= -10$ kN.
$S_C = -4$ kN, S_D (in CD) $= -4$ kN, $S_{DE} = +4$ kN.

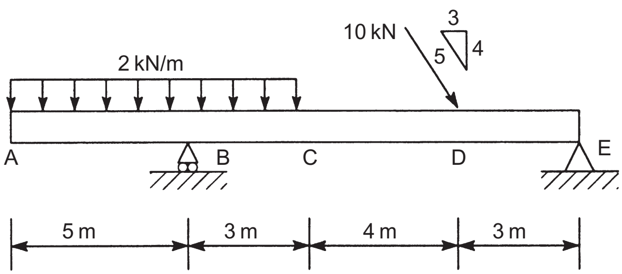

FIGURE P.3.3

$M_B = -25$ kNm, $M_C = -4$ kNm, $M_D = 12$ kNm.

P.3.4 Draw normal force, shear force and bending moment diagrams for the beam ABC shown in Fig. P.3.4. Insert all the principal values.

Ans.

$N_{AB} = +20$ kN, $N_{BC} = +10$ kN.
$S_A = +56.6$ kN, S_B (AB) $= +39.1$ kN, S_B (BC) $= +24.1$ kN, $S_C = +15$ kN.
$M_A = -181.0$ kNm, M_B (AB) $= -61.4$ kNm, M_B (BC) $= -43.4$ kNm,
$M_C = -18.0$ kNm.

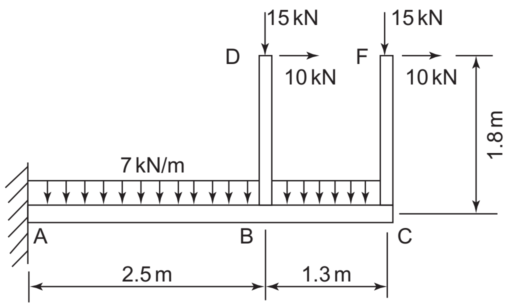

FIGURE P.3.4

P.3.5 Draw diagrams of normal force, shear force and bending moment for the cranked cantilever beam shown in Fig. P.3.5. Insert all principal values.

Ans.

$N_{AB} = 0$, $N_{BC} = +2$ kN, $N_{CD} = 0$.
$S_{AB} = +14$ kN, $S_{BC} = +3.46$ kN, S_C (CD) $= +4$ kN, $S_D = 0$.
$M_A = -20$ kNm, $M_B = -6$ kNm, $M_C = -2$ kNm, $M_D = 0$.

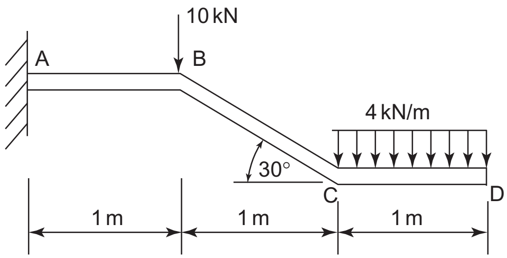

FIGURE P.3.5

P.3.6 Draw diagrams of normal force, shear force and bending moment for the beam ABCD shown in Fig. P.3.6 inserting all principal values. Also calculate the magnitude of the horizontal load required at D to reduce the bending moment at A to zero.

Ans.

$N_{DC} = -5$ kN, $N_{CBA} = -3.54$ kN,
$S_{DC} = 0$, $S_{CB} = +3.54$ kN, $S_A = +5.54$ kN.
$M_{DC} = 0$, $M_B = -3.54$ kNm, $M_A = -8.08$ kNm.
11.43 kN.

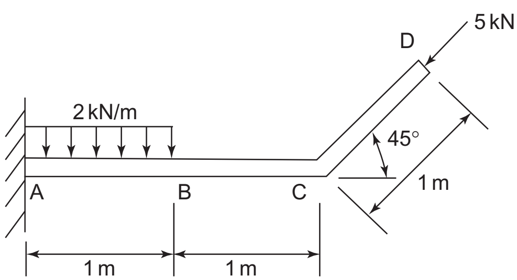

FIGURE P.3.6

P.3.7 Draw shear force and bending moment diagrams for the beam shown in Fig. P.3.7.

Ans.

$S_{AB} = -W$, $S_{BC} = 0$, $S_{CD} = +W$.
$M_B = M_C = WL/4$.

Note zero shear and constant bending moment in central span.

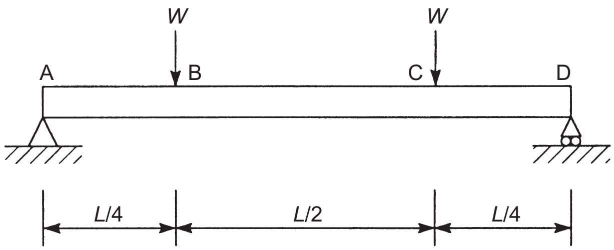

FIGURE P.3.7

P.3.8 The cantilever AB shown in Fig. P.3.8 carries a uniformly distributed load of 5 kN/m and a concentrated load of 15 kN at its free end. Construct the shear force and bending moment diagrams for the beam.

Ans.

$S_B = -15$ kN, $S_C = -65$ kN.
$M_B = 0$, $M_A = -400$ kNm.

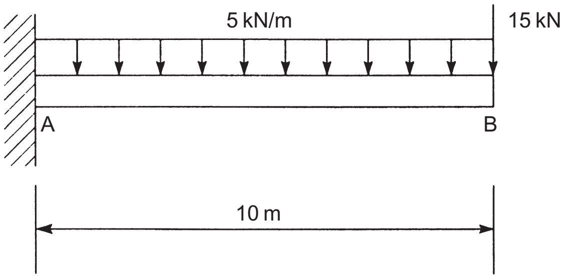

FIGURE P.3.8

P.3.9 Sketch the bending moment and shear force diagrams for the simply supported beam shown in Fig. P.3.9 and insert the principal values.

Ans.

S_B (in AB) = +5 kN, S_B (in BC) = −3.75 kN, S_C (in BC) = +6.25 kN.
$S_{CD} = -5$ kN, $M_B = -12.5$ kNm, $M_C = -25$ kNm.

Turning value of bending moment of −5.5 kNm in BC, 3.75 m from B.

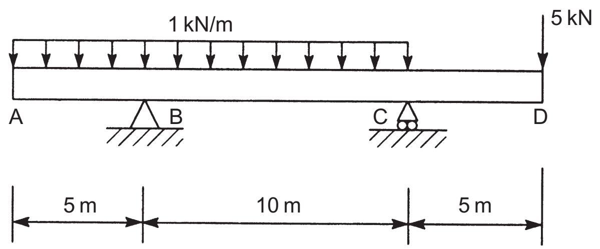

FIGURE P.3.9

P.3.10 Draw the shear force and bending moment diagrams for the beam shown in Fig. P.3.10 indicating the principal values.

Ans.

$S_{AB} = -5.6$ kN, S_B (in BC) $= +4.4$ kN, S_C (in BC) $= +7.4$ kN, S_C (in CD) $= -1.5$ kN.
$M_B = 16.8$ kNm, $M_C = -1.125$ kNm.

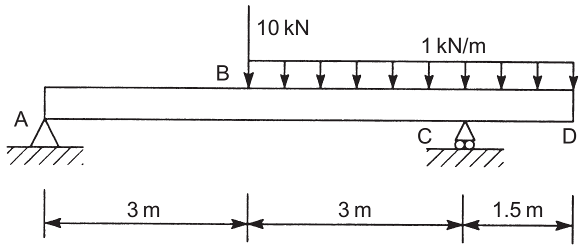

FIGURE P.3.10

P.3.11 Find the value of w in the beam shown in Fig. P.3.11 for which the maximum sagging bending moment occurs at a point 10/3 m from the left-hand support and determine the value of this moment.

Ans. $w = 1.2$ kN/m, 6.7 kNm.

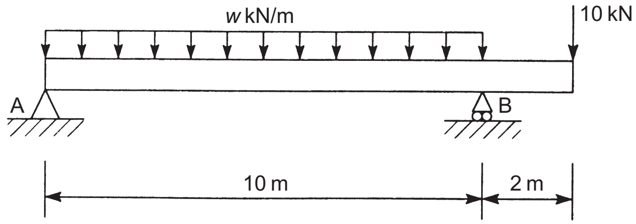

FIGURE P.3.11

P.3.12 Find the value of n for the beam shown in Fig. P.3.12 such that the maximum sagging bending moment occurs at $L/3$ from the right-hand support. Using this value of n determine the position of the point of contraflexure in the beam.

Ans. $n = 4/3$, $L/3$ from left-hand support.

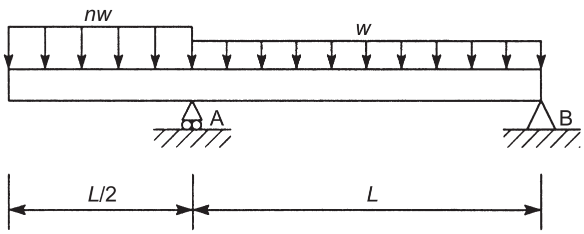

FIGURE P.3.12

P.3.13 Sketch the shear force and bending moment diagrams for the simply supported beam shown in Fig. P.3.13 and determine the positions of maximum bending moment and point of contraflexure. Calculate the value of the maximum moment.

Ans.

$S_A = -45$ kN, S_B (in AB) $= +55$ kN, $S_{BC} = -20$ kN.
$M_{max} = 202.5$ kNm at 9 m from A, $M_B = -100$ kNm.

Point of contraflexure is 18 m from A.

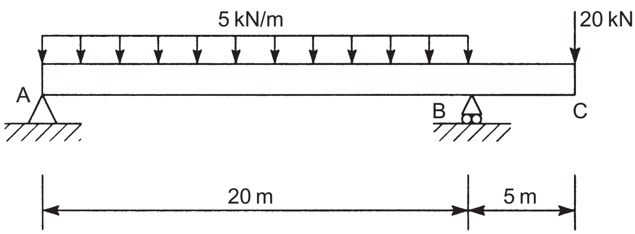

FIGURE P.3.13

P.3.14 Determine the position of maximum bending moment in a simply supported beam, 8 m span, which carries a load of 100 kN uniformly distributed over its complete length and, in addition, a load of 120 kN uniformly distributed over 2.5 m to the right from a point 2 m from the left support. Calculate the value of maximum bending moment and the value of bending moment at mid-span.

Ans.

$M_{max} = 294$ kN m at 3.6 m from left-hand support.
M (mid-span) $= 289$ kNm.

P.3.15 A simply supported beam AB has a span of 6 m and carries a distributed load which varies linearly in intensity from zero at A to 2 kN/m at B. Sketch the shear force and bending moment diagrams for the beam and insert the principal values.

Ans.

$S_{AB} = -2+x^2/6$, $S_A = -2$ kN, $S_B = +4$ kN.
$M_{AB} = 2x-x^3/18$, $M_{max} = 4.62$ kNm at 3.46 m from A.

P.3.16 A precast concrete beam of length L is to be lifted by a single sling and has one end resting on the ground. Show that the optimum position for the sling is 0.29 m from the nearest end.

P.3.17 Construct shear force and bending moment diagrams for the framework shown in Fig. P.3.17.

Ans.

$S_{AB} = -60$ kN, $S_{BC} = -10$ kN, $S_{CD} = +140$ kN.
$M_B = 480$ kNm, $M_C = 560$ kNm.

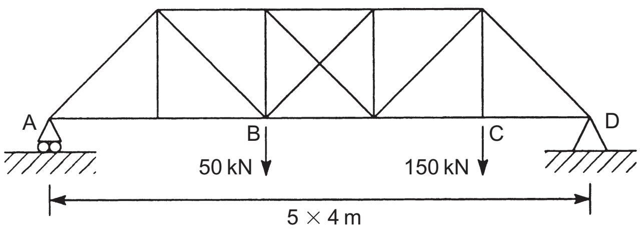

FIGURE P.3.17

P.3.18 Draw shear force and bending moment diagrams for the framework shown in Fig. P.3.18.

Ans.

$S_{AB} = +5$ kN, $S_{BC} = +15$ kN, $S_{CD} = +30$ kN, $S_{DE} = -12$ kN, $S_{EF} = -7$ kN, $S_{FG} = -5$ kN, $S_{GH} = 0$.
$M_B = -10$ kNm, $M_C = -40$ kNm, $M_D = -100$ kNm, $M_E = -76$ kNm,
$M_F = -20$ kNm, $M_G = M_H = 0$.

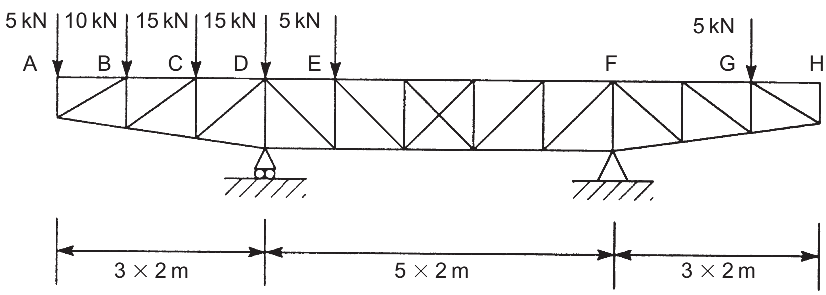

FIGURE P.3.18

P.3.19 Draw shear force and bending moment diagrams for the truss shown in Fig. P.3.19.

Ans.

$S_{AC} = -4$ kN, $S_{CD} = +1$ kN, $S_{DE} = +5$ kN.
$M_A = +1$ kNm, $M_C = +5$ kNm, $M_D = +3$ kNm, $M_E = -2$ kNm.

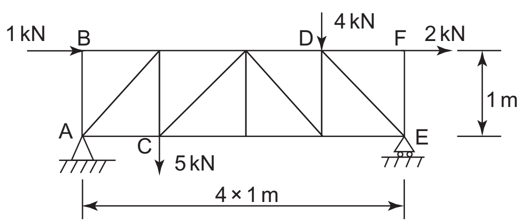

FIGURE P.3.19

P.3.20. A ship's lifeboat is supported by a pair of steel davits as shown in the lowering position in Fig. P.3.20. If the total weight of the lifeboat and its full complement of passengers is 15 kN, calculate the distribution of shear force and bending moment in the davit.

Ans. $S_{CB} = -7.5 \sin\theta$, $S_C = 0$, $S_D = -7.5$ kN, $S_B = 0 = S_{BA}$.
$M_{CB} = 11.25(\cos\theta - 1)$, $M_C = 0$, $M_D = -11.25$ kNm,
$M_B = M_{BA} = -22.5$ kNm.

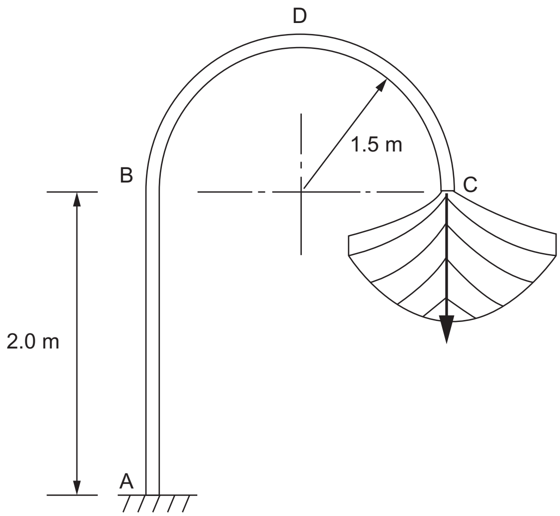

FIGURE P.3.20

P.3.21. A single line railway track over a road is supported by two trusses as shown in elevation in Fig. P.3.21(a). The load on the bridge produced by a train may be regarded as comprising two uniformly distributed loads, one representing the engine and one the line of freight cars as shown in Fig. P.3.21(b). With the front of the engine at mid-span, determine the distribution of shear force and bending moment in one of the trusses.

Ans. $S_D = 312.5$ kN, $S_C = 262.5$ kN, $S_B = S_A = -137.5$ kN,
$S_{CB} = 0$, 4.625 m from D.
$M_A = M_D = 0$, $M_C = 575$ kNm, $M_B = 825$ kNm, $M_{max} = 919.5$ kNm at 4.625 m from D.

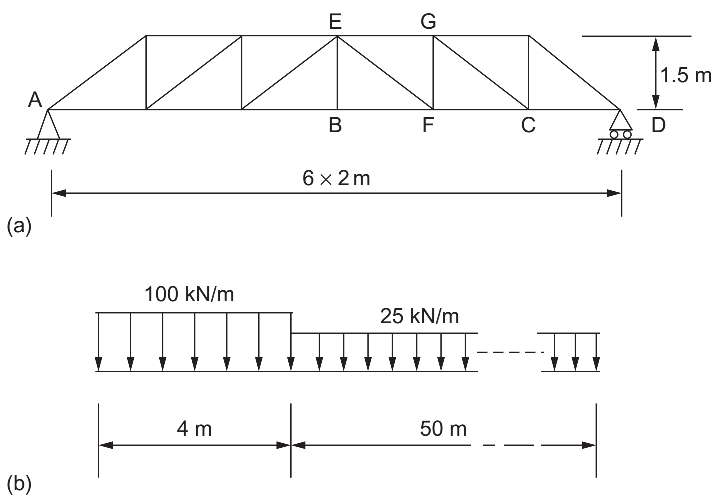

FIGURE P.3.21

P.3.22 The cranked cantilever ABC shown in Fig. P.3.22 carries a load of 3 kN at its free end. Draw shear force, bending moment and torsion diagrams for the complete beam.

Ans.

$S_{CB} = -3$ kN, $S_{BA} = -3$ kN
$M_C = 0$, M_B (in CB) $= -6$ kNm, M_B (in BA) $= 0$, $M_A = -9$ kNm.
$T_{CB} = 0$, $T_{BA} = 6$ kNm.

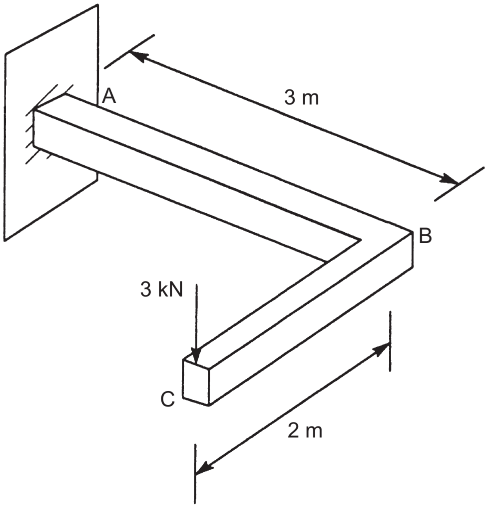

FIGURE P.3.22

P.3.23 Construct a torsion diagram for the beam shown in Fig. P.3.23.

Ans. $T_{CB} = -300$ N m, $T_{BA} = -400$ N m.

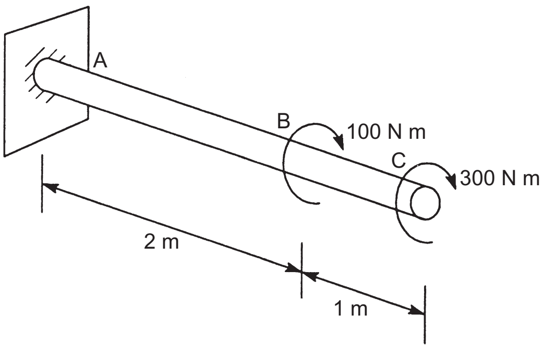

FIGURE P.3.23

P.3.24 The beam ABC shown in Fig. P.3.24 carries a distributed torque of 1 N m/mm over its outer half BC and a concentrated torque of 500 N m at B. Sketch the torsion diagram for the beam inserting the principal values.

Ans. $T_C = 0$, T_B (in BC) $= 1000$ N m, T_B (in AB) $= 1500$ N m.

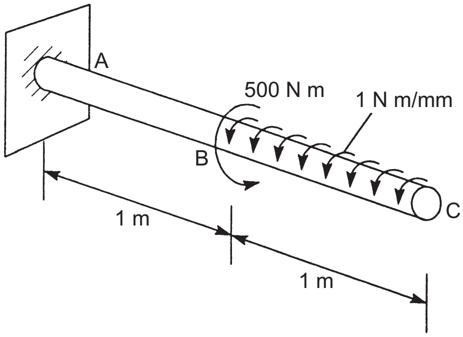

FIGURE P.3.24

P.3.25 The cylindrical bar ABCD shown in Fig. P.3.25 is supported symmetrically at B and C by supports that prevent rotation of the bar about its longitudinal axis. The bar carries a uniformly distributed torque of 2 N m/mm together with concentrated torques of 400 N m at each end. Draw the torsion diagram for the bar and determine the maximum value of torque.

Ans.

$T_{DC} = 400+2x$, $T_{CB} = 2x-2000$, $T_{BA} = 2x-4400$ (T in N m when x is in mm).
$T_{max} = 1400$ N m at C and B.

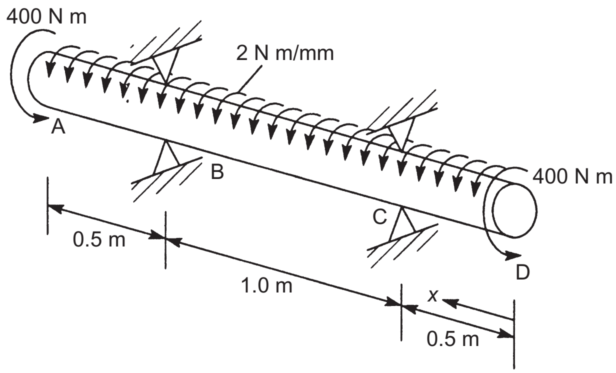

FIGURE P.3.25

P.3.26. A balcony consists of a concrete slab supported on two cantilever steel joists CD and EF welded to a Universal Beam AB as shown in Fig. P.3.26. The concrete slab is 3 m long, 1.5 m wide, and 150 mm thick. If the density of concrete is 22 kN/m^3 and the self-weight of the joists is 17 kg/m, determine the distribution of torque along the beam AB. The weight of the balcony rail may be represented by two concentrated loads each equal to 0.2 kN applied at D and F.

Ans. $T_{AC} = 6.1$ kNm, $T_{CE} = 0$, $T_{EB} = -6.1$ kNm.

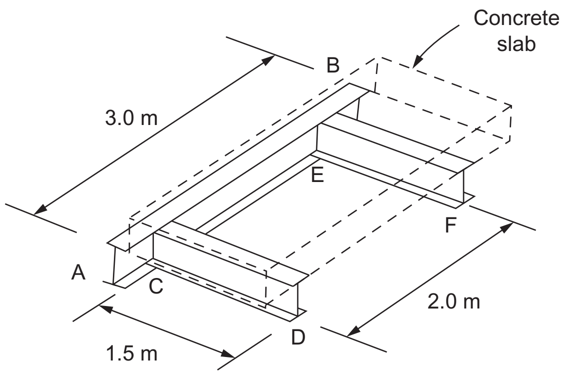

FIGURE P.3.26

P.3.27. Verify the solution of P.3.7 using the principle of superposition.