

EQUATIONS OF MOTION 2

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6DOF Equations of Motion

Equations (8), (16), (17) and (18) form the 12 equations of motion for general atmospheric 6 degree-of-freedom flight.

They are all first-order differential equations that can be expressed in nonlinear state-space form as:

$$\dot{x} = f(x, \delta) \quad (19)$$

where $x = [U, V, W, p, q, r, \psi, \theta, \phi, x_E, y_E, z_E]'$ (state vector)

and δ is the vector of input parameters of interest. These are often aerodynamic or propulsion system inputs such as aileron, elevator, rudder deflection or thrust (upon which the formulation of the forces X , Y and Z will depend).

Coordinates and Forces

- The motions that are described are a consequence of the applied forces and moments from outside the body, being:
 - **aerodynamic** (including control forces and gusts)
 - **propulsive** (all additional forces due to propulsion)
 - **gravitational**
- The list above must be all inclusive, e.g. drag due to lowered undercarriage or lift, drag and pitching moment change due to flaps.
- Aerodynamic forces are typically modelled using **aerodynamic data tables** (aerodynamic coefficients and/or stability & control derivatives as function of angle-of-attack, Mach no., etc.)

Coordinates and Forces

- In eqn (8), the three general forces X, Y, Z in the directions of the coordinate axes include those due to aerodynamics, propulsion and gravity.
- When the CG is taken as the axes origin there are no gravity-induced moments so the three moments L, M, N have no gravity terms present.
- Sometimes it is convenient to use only a subset of equations.

For example, for static trim and stability studies, we may assume that the propulsive force is constant and that thrust $F = \text{drag}$; then the aircraft is in equilibrium w.r.t. the x -axis and we may assume that the X -force equation is not perturbed from equilibrium and hence omit this equation.

- Different forms of the equations may be used to address different problems.

Notation

- The steady velocities are U , V , W (upper case).
- For conventional flight conditions, U is relatively large and W and V are small, giving a small angle of attack, α , and small angle of sideslip, β .

Recall that: $\tan \alpha = \frac{W}{U}$ and

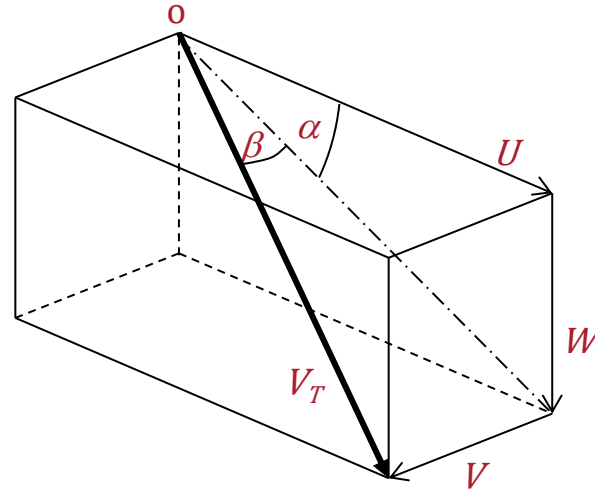
$$\sin \beta = \frac{V}{V_T}$$

where $V_T = \sqrt{U^2 + V^2 + W^2}$

Therefore, for small α and β :

$$\alpha \approx W/U$$

$$\beta \approx V/V_T.$$



- Thus, if preferred, we could write the three translational equations of motion in terms of α , β and V_T instead of U , V and W .

Notation

- Later, when we consider perturbations from the steady state condition, these short-term variations from steady values will be given the symbols u, v, w (lower case) for the U, V and W equations.
- The steady *angular* velocities are written p, q, r . This is the same when considering perturbations from steady state as we usually regard the steady state as having zero angular rate.
- We will often consider flight with body-axis forward speed U and vertical component W but with zero sideslip velocity, $V=0$. And if the perturbation is also longitudinal-only then we will see only U, W, u and w in the perturbation equations, and not V or v .
(If formulated in wind axes, W would also be zero as the x axis would be aligned with the unperturbed total vel.)

Notation

- Do not confuse the notation for dynamic pressure, \bar{q} ($\bar{q} = \frac{1}{2}\rho V_T^2$ or $\frac{1}{2}\rho U^2$) with pitch rate q .
- We shall also run up against the problem of distinguishing between L =lift and L =rolling moment.
- Moments and products of inertia:

$$I_{xx} = \sum \delta m (y^2 + z^2)$$

$$I_{xy} = \sum \delta m .xy$$

$$I_{yy} = \sum \delta m (x^2 + z^2)$$

$$I_{xz} = \sum \delta m .xz$$

$$I_{zz} = \sum \delta m (x^2 + y^2)$$

$$I_{yz} = \sum \delta m .yz$$

Learn the notation!

The Formal Equations

With the gravitational forces separated from the aerodynamic/propulsive forces, the translational equations can be written as follows (allowing for non-zero values of V , W and pitch and roll angles, θ and ϕ):

Fore/ Aft:

$$m(\dot{U} - rV + qW) = X - mg \sin \theta$$

Lateral

$$m(\dot{V} - pW + rU) = Y + mg \cos \theta \sin \phi$$

Transverse:

$$m(\dot{W} - qU + pV) = Z + mg \cos \theta \cos \phi$$

The Formal Equations

Similarly, the rotational equations are expressed in their complete form as:

Roll:

$$I_{xx}\dot{p} - (I_{yy} - I_{zz})qr - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) = L$$

Pitch:

$$I_{yy}\dot{q} - (I_{zz} - I_{xx})rp - I_{zx}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) = M$$

Yaw:

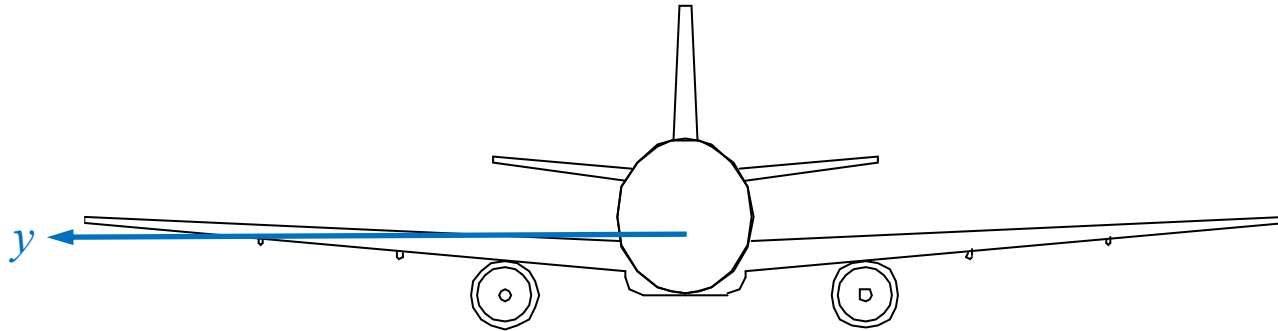
$$I_{zz}\dot{r} - (I_{xx} - I_{yy})pq - I_{xy}(p^2 - q^2) - I_{yz}(\dot{q} + rp) - I_{xz}(\dot{p} - qr) = N$$

(You are not expected to remember these – [see Cook for further reading](#)).

Note: *Product terms & cross inertias.*

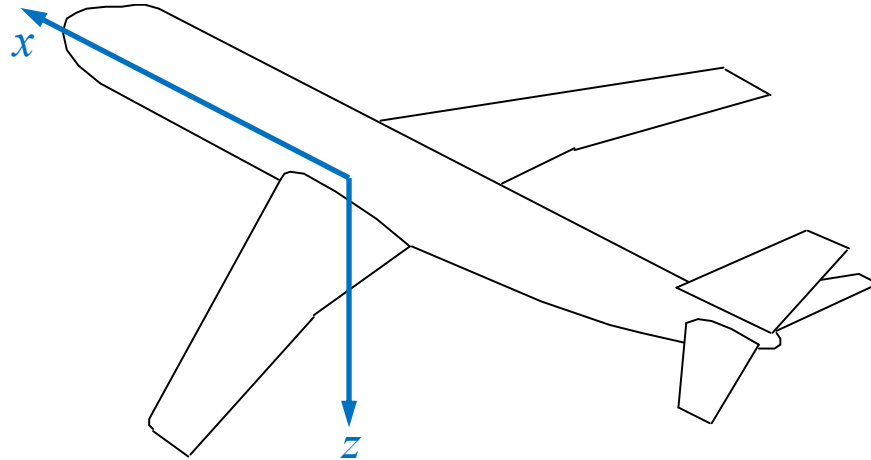
Reminder: the cross-inertias

- As mentioned previously, we usually adopt a simpler form of the equations on the previous slide because we assume the aircraft to have an inertial plane of symmetry. This results in I_{xy} and I_{zy} being zero, yielding eqn (16).
- An interpretation of this is that for any dimension x away from the origin (fore/aft) there are equal masses at $\pm y$ which cancel in the inertia calculations.



Reminder: the cross-inertias

- Similarly, for a choice of some fixed position z away from the origin (vertically) there will be equal masses at $\pm y$.
- However, I_{zx} is not zero for a typical aircraft geometry. The vertical tail (fin) configuration, for example, means that at aft positions x we do not have equal masses at $\pm z$ as there is no fin below the fuselage.



Alternative Form of the Equations

TABLE 2.4-1. The Flat-Earth, Body Axes 6-DOF Equations

Force Equations

$$\begin{aligned}\dot{U} &= RV - QW - g'_0 \sin \theta + \frac{F_x}{m} \\ \dot{V} &= -RU + PW + g'_0 \sin \phi \cos \theta + \frac{F_y}{m} \\ \dot{W} &= QU - PV + g'_0 \cos \phi \cos \theta + \frac{F_z}{m}\end{aligned}\tag{2.4-2}$$

Kinematic Equations

$$\begin{aligned}\dot{\phi} &= P + \tan \theta (Q \sin \phi + R \cos \phi) \\ \dot{\theta} &= Q \cos \phi - R \sin \phi \\ \dot{\psi} &= \frac{Q \sin \phi + R \cos \phi}{\cos \theta}\end{aligned}\tag{2.4-3}$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)

Alternative Form of the Equations

Moment Equations

$$\begin{aligned}\dot{P} &= (c_1 R + c_2 P)Q + c_3 \bar{L} + c_4 N \\ \dot{Q} &= c_5 PR - c_6(P^2 - R^2) + c_7 M \\ \dot{R} &= (c_8 P - c_2 R)Q + c_4 \bar{L} + c_9 N\end{aligned}\tag{2.4-4}$$

Navigation Equations

$$\begin{aligned}\dot{p}_N &= U \cos \theta \cos \psi + V(-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi) \\ &\quad + W(\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi) \\ \dot{p}_E &= U \cos \theta \sin \psi + V(\cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi) \\ &\quad + W(-\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi) \\ \dot{h} &= U \sin \theta - V \sin \phi \cos \theta - W \cos \phi \cos \theta\end{aligned}\tag{2.4-5}$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)

Alternative Form of the Equations – mass properties

$$\begin{aligned}\Gamma c_1 &= (J_y - J_z)J_z - J_{xz}^2, & \Gamma c_2 &= (J_x - J_y + J_z)J_{xz} \\ \Gamma c_3 &= J_z, & \Gamma c_4 &= J_{xz} \\ c_5 &= \frac{J_z - J_x}{J_y}, & c_6 &= \frac{J_{xz}}{J_y} \\ c_7 &= \frac{1}{J_y}, & \Gamma c_8 &= J_x(J_x - J_y) + J_{xz}^2, \\ \Gamma c_9 &= J_x, & & \end{aligned} \tag{2.4-6}$$

where

$$\Gamma = J_x J_z - J_{xz}^2 \quad [\text{as in (1.3-19b)}].$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)

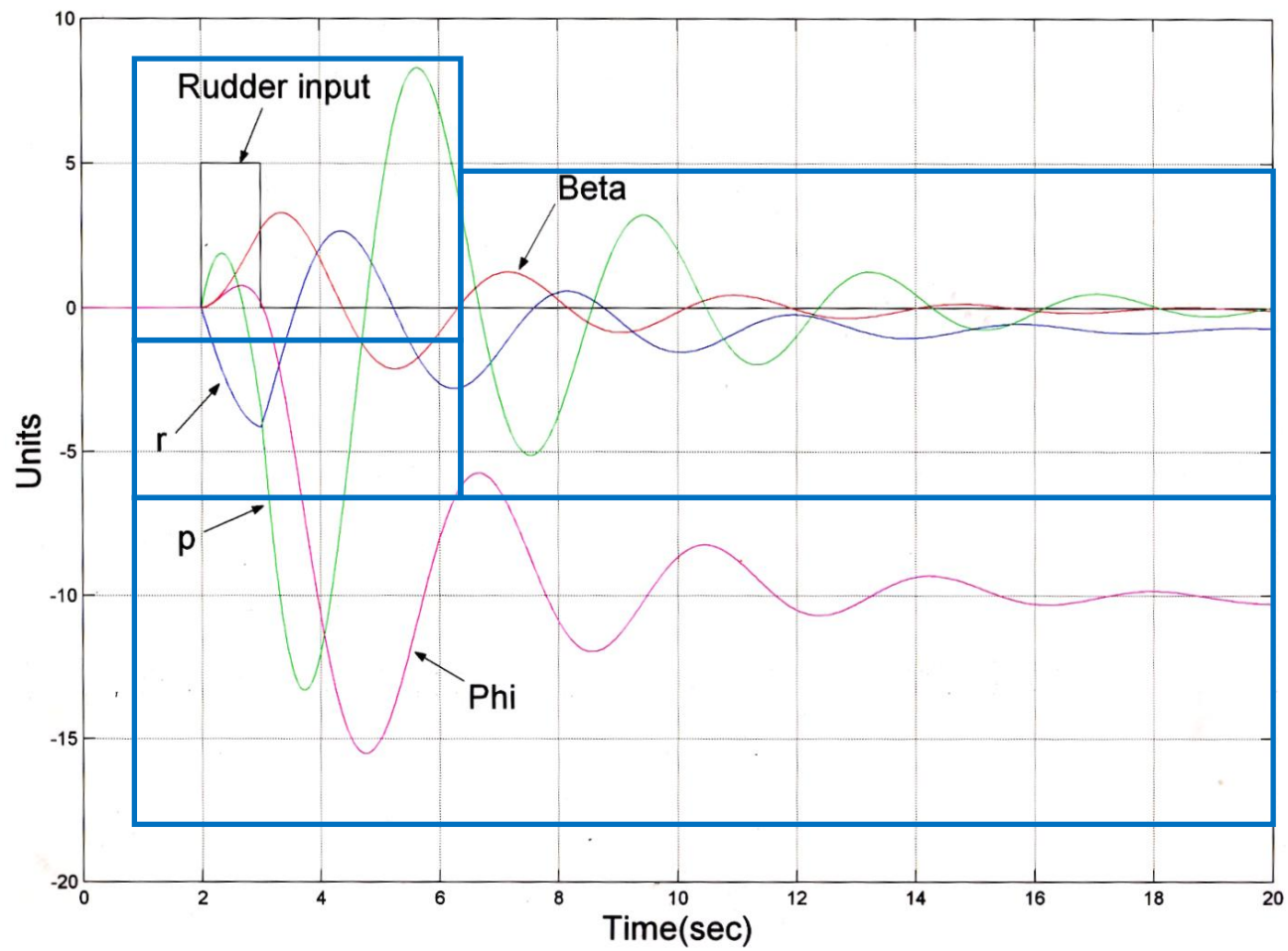
Alternative Form of the Equations

$$\begin{aligned}\dot{V}_T &= \frac{U\dot{U} + V\dot{V} + W\dot{W}}{V_T} \\ \dot{\beta} &= \frac{\dot{V}_T - V\dot{V}_T}{V_T^2 \cos \beta} \\ \dot{\alpha} &= \frac{U\dot{W} - W\dot{U}}{U^2 + W^2}.\end{aligned}\tag{2.4-8}$$

The new state vector is

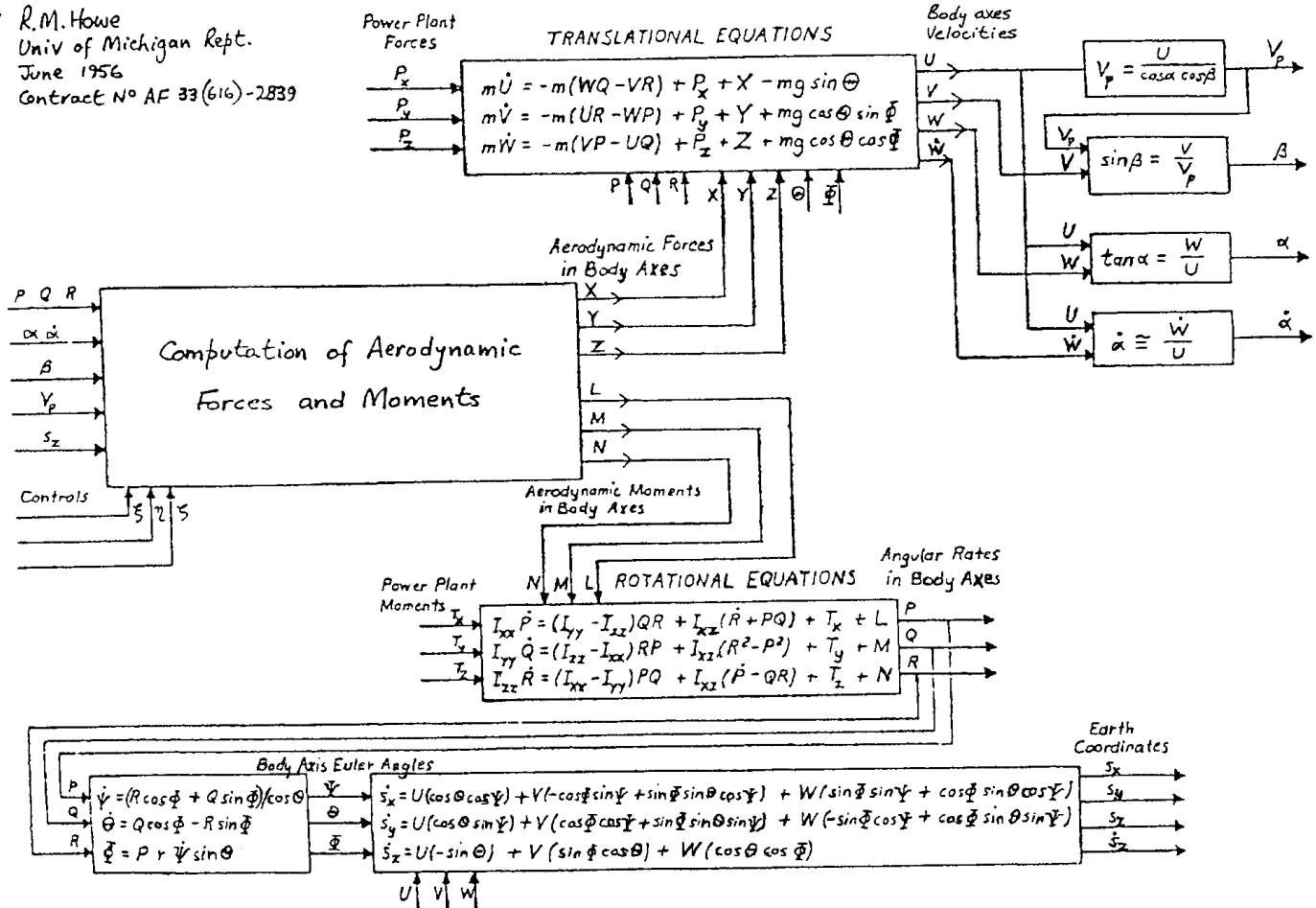
$$X^T = [V_T \quad \beta \quad \alpha \quad \phi \quad \theta \quad \psi \quad P \quad Q \quad R \quad p_N \quad p_E \quad h]. \tag{2.4-9}$$

Reference: *Aircraft Control & Simulation*,
Stevens & Lewis, Wiley (1st edition)



Equations of motion in simulation model

Ref R.M. Howe
Univ of Michigan Rept.
June 1956
Contract N° AF 33(616)-2839



Coupled Motion

- In general, the motions are coupled, and, for example, an impulsive force (moment) applied in roll will eventually disturb the motions in all 6 degrees of freedom and there will be resultant non-zero values for all 6 motion variables.
- However, for many applications it is standard practice, and sufficient, to separate the equations into two de-coupled sets of three freedoms each to provide what is called:
 - the longitudinal equations in U, W, q
 - the lateral-directional equations in V, p, r .

Coupled Motion

In the form quoted above, the equations are not yet **linearised** and they show for example, products of the variables. However, after linearisation we shall find that the form of the full set of six can be displayed as follows:

$$\begin{array}{c|c}
 \begin{array}{c} \text{Translational} \\ \text{in upper} \end{array} & \text{equations} \\
 \hline
 \begin{array}{c} \text{Rotational} \\ \text{in lower} \end{array} & \text{equations}
 \end{array}
 \begin{array}{c}
 u \\
 v \\
 w \\
 \hline
 p \\
 q \\
 r
 \end{array}
 \text{ or }
 \begin{array}{c}
 x \\
 y \\
 z \\
 \hline
 \phi \\
 \theta \\
 \psi
 \end{array}
 =
 \begin{array}{c}
 \text{Fwd. force} \\
 \text{Lateral force} \\
 \text{Transv. force} \\
 \hline
 \text{Roll.mom.} \\
 \text{Pitch.mom.} \\
 \text{Yaw.mom.}
 \end{array}$$

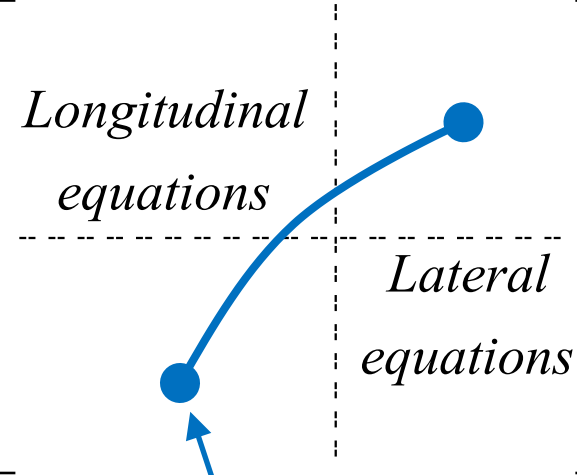
Coupled Motion

In this form, all four partitions of the LHS matrix will have non-zero terms. If we change the order of the forces and the order of the variables to allow for the lateral/longitudinal split, namely to display

- **longitudinal actions**: forces and motions which are within the plane of symmetry, and
- **lateral-directional actions**: forces which act out of the plane of symmetry and consequent motions of that plane away from its normal vertical position,

we can re-form the equations to display them as:

Coupled Motion



$$\begin{bmatrix} \text{Longitudinal} \\ \text{equations} \end{bmatrix} \quad \begin{bmatrix} u \\ w \\ q \\ v \\ p \\ r \end{bmatrix} = \begin{bmatrix} \text{Fwd. force} \\ \text{Transv. force} \\ \text{Pitch. mom.} \\ \text{Lat. force} \\ \text{Roll. mom.} \\ \text{Yaw. mom.} \end{bmatrix}$$

both of these partitions are nominally null

Next Session

Equations of Motion 3

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