Claudia Nicolai
claudia.nicolai@bristol.ac.uk

Aerodynamics and Numerical Simulation Methods

Introduction to Turbulent Boundary Layers: RECAP

University of BRISTOL

The Turbulent Boundary Layer Equations

- As with laminar boundary layers, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

- But now we represent the flow properties *u,v* etc. as a sum of mean plus turbulent fluctuation, i.e.

$$u = \hat{u} + u'$$

Reference: Frank White Page 360 (Reynolds time-averaging concept)

So, returning to the continuity equation, and time averaging:

$$\overline{\frac{\partial(\hat{u} + u')}{\partial x}} + \overline{\frac{\partial(\hat{v} + v')}{\partial y}} = 0$$

$$\Longrightarrow \quad \overline{\frac{\partial \hat{u}}{\partial x}} + \overline{\frac{\partial u'}{\partial x}} + \overline{\frac{\partial \hat{v}}{\partial y}} + \overline{\frac{\partial v'}{\partial y}} = 0$$

but, using our rules from earlier, the time average of the fluctuating differentials are zero, hence

$$\Longrightarrow \quad \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0$$

Similarly, we can show that the x-momentum equation becomes

$$\hat{u}\frac{\partial \hat{u}}{\partial x}+\hat{v}\frac{\partial \hat{u}}{\partial y}=-\frac{1}{\rho}\frac{d\hat{p}}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial \hat{u}}{\partial y}-\overline{u'v'}\right)-\frac{\partial \overline{u'^2}}{\partial x}$$

where the last two terms are called ***Reynolds Stresses***. We sometimes ignore the last one due to the thin layer assumption (=gradients in normal direction dominant).

Overall, the continuity and momentum equations are the same in turbulent flow as laminar, as long as we use time averaged values, and include this extra term involving *u'v'.*

We now have the RANS equations (Reynolds Averaged NS).

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}\quad \boldsymbol{+}\quad \begin{array}{l} u=\hat{u}+u' \quad v=\hat{v}+v' \\ p=\hat{p}+p' \end{array}$$

$\downarrow$ Conservative form

$$\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\underbrace{\frac{\partial(\hat{u}+u')^2}{\partial x}}_{\text{LHS - 1}}+\underbrace{\frac{\partial(\hat{u}+u')(\hat{v}+v')}{\partial y}}_{\text{LHS - 2}}=\underbrace{-\frac{1}{\rho}\frac{d(\hat{p}+p')}{dx}+\nu\frac{\partial^2(\hat{u}+u')}{\partial y^2}}_{\text{RHS}}$$

LHS – 1: $$\frac{\partial(\hat{u}+u')^2}{\partial x}=\frac{\partial\hat{u}^2}{\partial x}+\frac{\partial u'^2}{\partial x}+\frac{\partial 2\hat{u}u'}{\partial x}=\frac{\partial\hat{u}^2}{\partial x}+\frac{\partial u'^2}{\partial x}+2\hat{u}\frac{\partial u'}{\partial x}+2u'\frac{\partial\hat{u}}{\partial x}$$

LHS – 2: $$\frac{\partial(\hat{u}+u')(\hat{v}+v')}{\partial y}=\frac{\partial(\hat{u}\hat{v})}{\partial y}+\frac{\partial(\hat{u}v')}{\partial y}+\frac{\partial(u'\hat{v})}{\partial y}+\frac{\partial(u'v')}{\partial y}$$

Ref. to the notes on Blackboard for this derivation

$$\overline{\frac{\partial \hat{u}^2}{\partial x}} + \overline{\frac{\partial u'^2}{\partial x}} + \overline{2\hat{u}\frac{\partial u'}{\partial x}} + \overline{2u'\frac{\partial \hat{u}}{\partial x}} + \overline{\frac{\partial(\hat{u}\hat{v})}{\partial y}} + \overline{\frac{\partial(\hat{u}v')}{\partial y}} + \overline{\frac{\partial(u'\hat{v})}{\partial y}} + \overline{\frac{\partial(u'v')}{\partial y}} =$$

$$= -\frac{1}{\rho}\overline{\frac{d\hat{p}}{dx}} - \frac{1}{\rho}\overline{\frac{dp'}{dx}} + \nu\overline{\frac{\partial^2 \hat{u}}{\partial y^2}} + \nu\overline{\frac{\partial^2 u'}{\partial y^2}}$$

Differentiation and averaging are commutative

$$\frac{\partial \overline{\hat{u}^2}}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x} + \overline{2\hat{u}\frac{\partial u'}{\partial x}} + \overline{2u'\frac{\partial \hat{u}}{\partial x}} + \frac{\partial(\overline{\hat{u}\hat{v}})}{\partial y} + \frac{\partial(\overline{\hat{u}v'})}{\partial y} + \frac{\partial(\overline{u'\hat{v}})}{\partial y} + \frac{\partial(\overline{u'v'})}{\partial y} =$$

$$= -\frac{1}{\rho}\frac{d\overline{\hat{p}}}{dx} - \frac{1}{\rho}\frac{d\overline{p'}}{dx} + \nu\frac{\partial^2 \overline{\hat{u}}}{\partial y^2} + \nu\frac{\partial^2 \overline{u'}}{\partial y^2}$$

Properties seen in lecture 7

$$\frac{\partial \overline{\hat{u}^2}}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x} + 0 + 0 + \frac{\partial(\overline{\hat{u}\hat{v}})}{\partial y} + 0 + 0 + \frac{\partial(\overline{u'v'})}{\partial y} =$$

$$= -\frac{1}{\rho}\frac{d\overline{\hat{p}}}{dx} - 0 + \nu\frac{\overline{\partial^2 \hat{u}}}{\partial y^2} + 0$$

Ref. to the notes on Blackboard for this derivation

$$\frac{\partial \overline{\hat{u}^2}}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x} + \quad 0 \quad + \quad 0 \quad + \frac{\partial (\overline{\hat{u}\hat{v}})}{\partial y} + \quad 0 \quad + \quad 0 \quad + \frac{\partial (\overline{u'v'})}{\partial y} =$$

$$= -\frac{1}{\rho}\frac{d\overline{\hat{p}}}{dx} - \quad 0 \quad + \nu \frac{\partial^2 \hat{u}}{\partial y^2} + \quad 0$$

$$\hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{u}\frac{\partial \hat{v}}{\partial y} + \hat{v}\frac{\partial \hat{v}}{\partial y} = -\frac{1}{\rho}\frac{d\hat{p}}{dx} + \nu\frac{\partial^2 \hat{u}}{\partial y^2} - \frac{\partial (\overline{u'v'})}{\partial y} - \frac{\partial \overline{u'^2}}{\partial x}$$

$$\hat{u}\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\right) + \hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial \hat{v}}{\partial y} = -\frac{1}{\rho}\frac{d\hat{p}}{dx} + \nu\frac{\partial^2 \hat{u}}{\partial y^2} - \frac{\partial (\overline{u'v'})}{\partial y} - \frac{\partial \overline{u'^2}}{\partial x}$$

Continuity equation

$$0 \quad + \hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial \hat{v}}{\partial y} = -\frac{1}{\rho}\frac{d\hat{p}}{dx} + \nu\frac{\partial^2 \hat{u}}{\partial y^2} - \frac{\partial (\overline{u'v'})}{\partial y} - \frac{\partial \overline{u'^2}}{\partial x}$$

$$\hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{v}\frac{\partial \hat{u}}{\partial y} = -\frac{1}{\rho}\frac{d\hat{p}}{dx} + \frac{\partial}{\partial y}\left(\nu\frac{\partial \hat{u}}{\partial y} - \overline{u'v'}\right) - \frac{\partial \overline{u'^2}}{\partial x}$$

Claudia Nicolai

claudia.nicolai@bristol.ac.uk

Aerodynamics and Numerical Simulation Methods

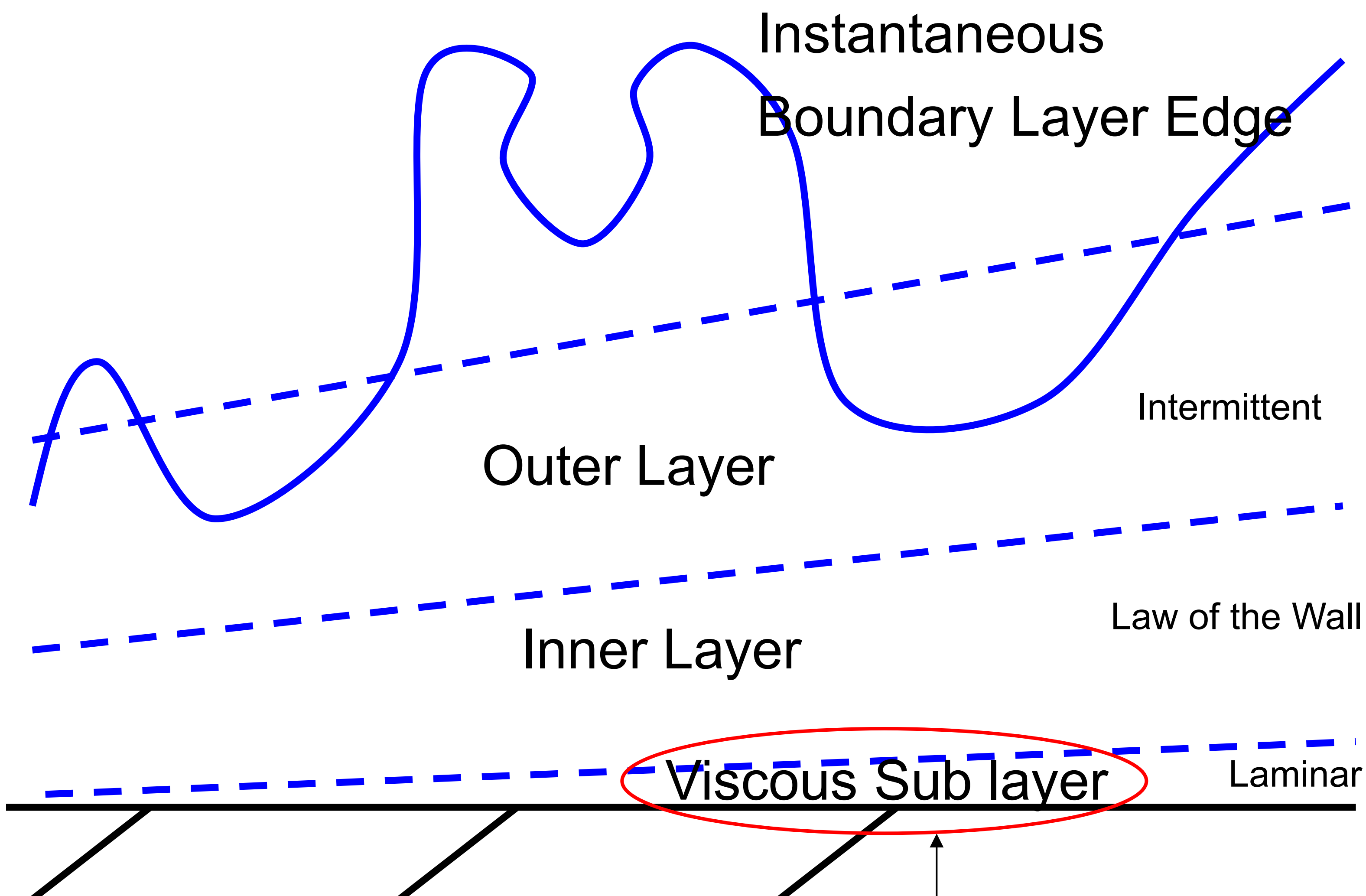
Velocity and Turbulence Profiles of Turbulent Boundary Layers

University of BRISTOL

Topics for today

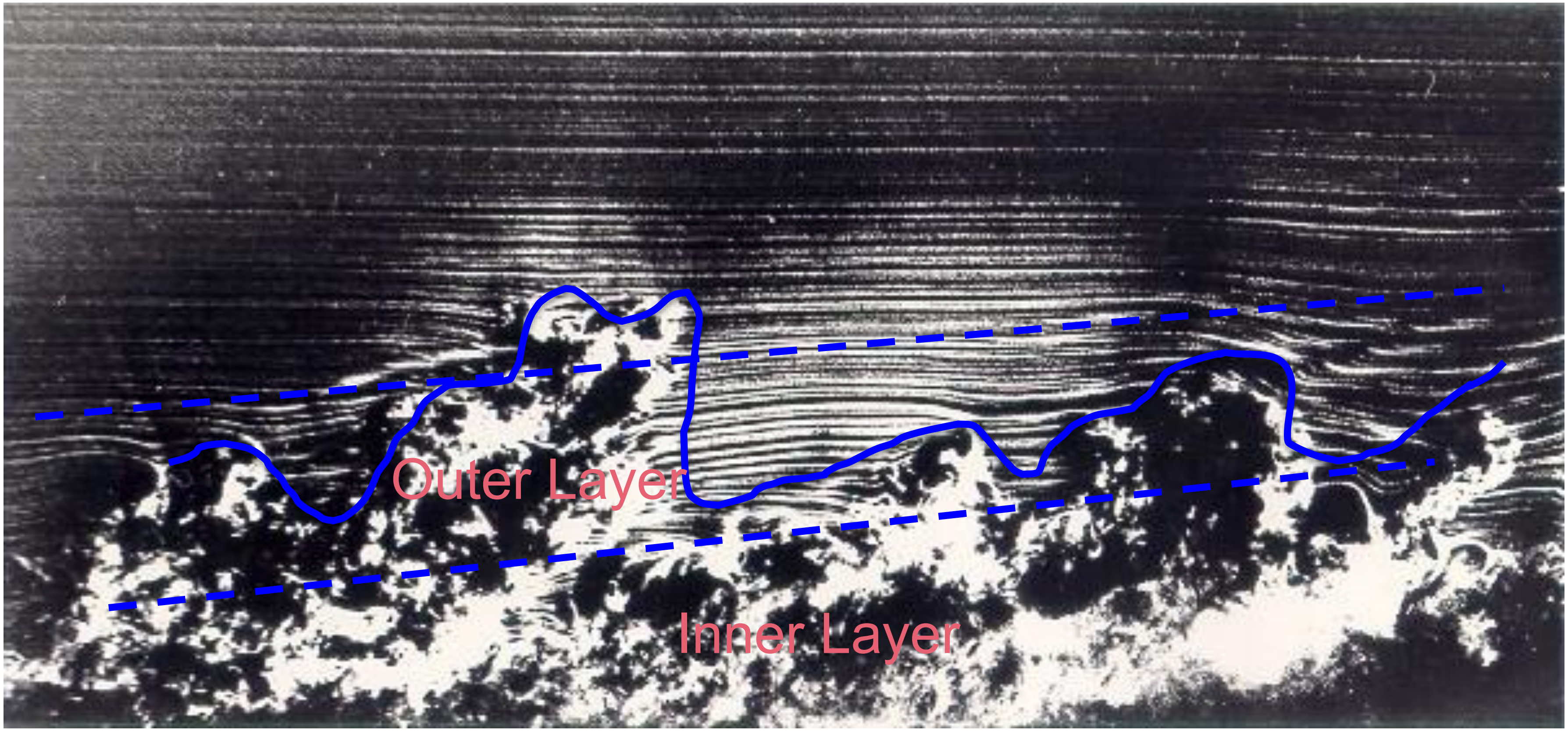
- Turbulent boundary layer
  - Velocity profile
  - Structure
- Reynolds stresses
- Understanding the turbulent flow motions with quadrant analysis
- Coherent structures in the turbulent boundary layer

Overview of structure of a TBL



v=0 and perturbations in v, called v'=0 too

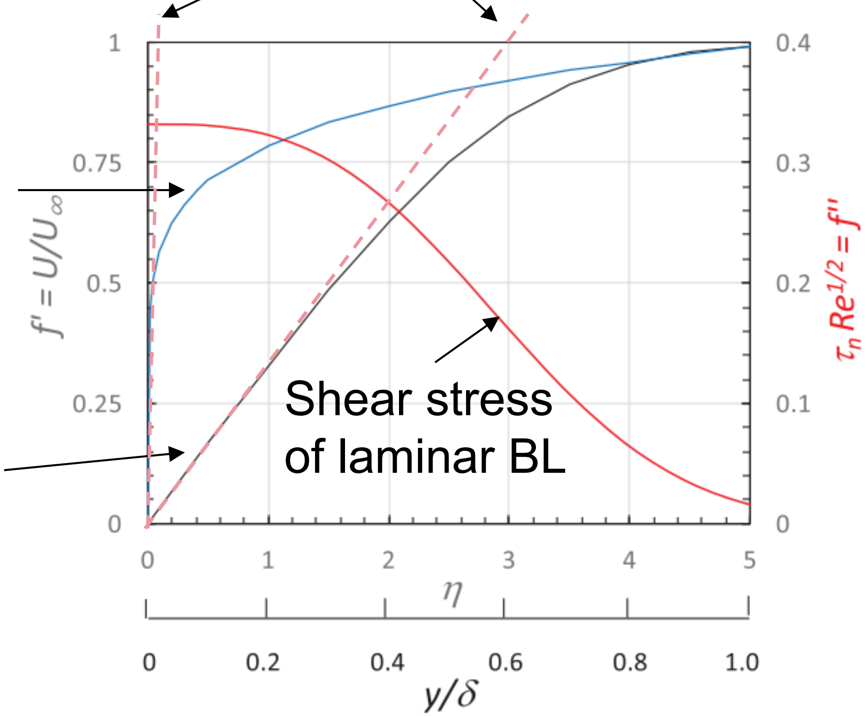
Overview of structure of a TBL



An Album of Fluid Motion *Milton Van Dyke*

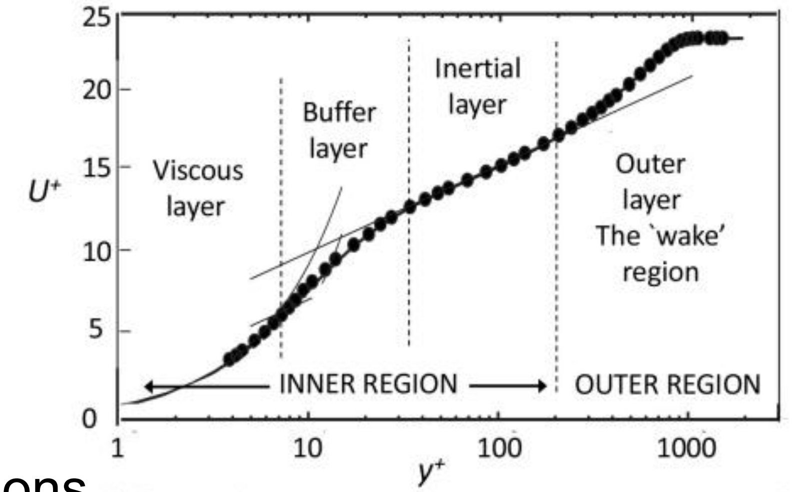
Velocity profile of a TBL

- High velocity gradient, hence high wall shear stress $\tau_w = \mu \frac{\partial u}{\partial y}_{y=0}$
- Scales approximately with a power law as: $\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$
- Limitations of the power law:
  - Infinite velocity gradient not physical
  - Does not account for Re



Reference: Turbulent Flows: an Introduction Chapter 8.3

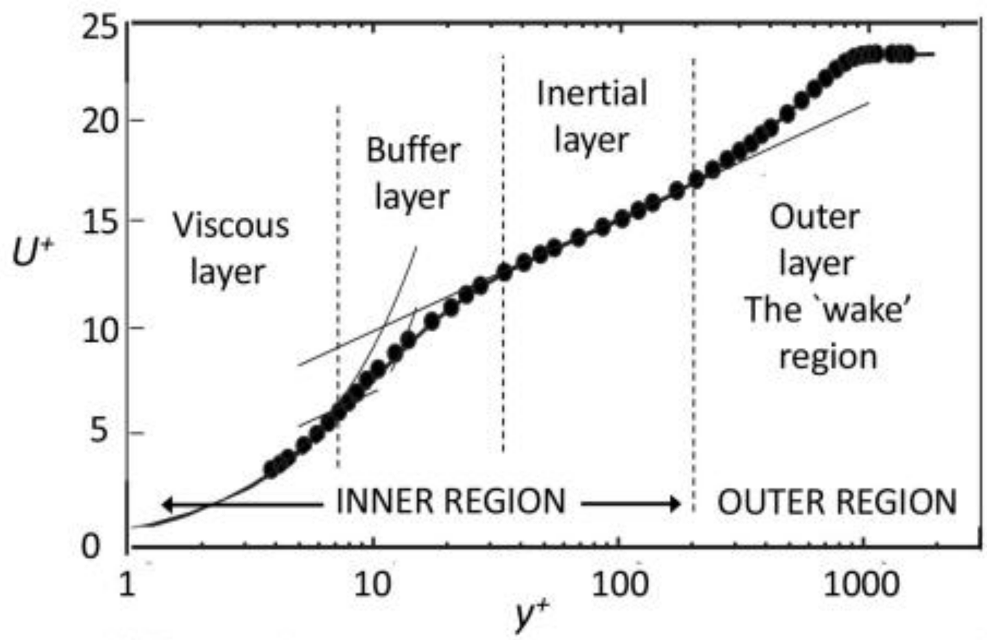
Velocity profile of a TBL



- TBL can be divided into:
- **Inner region** which has 3 subregions
  - Scales with *friction velocity* ($u_\tau = \sqrt{\frac{\tau_w}{\rho}}$) and *viscous length scale* ($\delta_\nu = \nu / u_\tau$)
  - The *wall unit* is a local Re: $y^+ = y/\delta_\nu = y u_\tau / \nu$
  - Law of the wall similarity relation: $\frac{U}{u_\tau} = f_i\left(\frac{y u_\tau}{\nu}\right)$
- **Outer region** also known as the wake region
  - Scales with u_τ and δ
  - Law of the wake (or defect law): $\frac{U_\infty - u}{u_\tau} = f_o\left(\frac{y}{\delta}, \frac{u_\tau}{U_\infty}\right)$

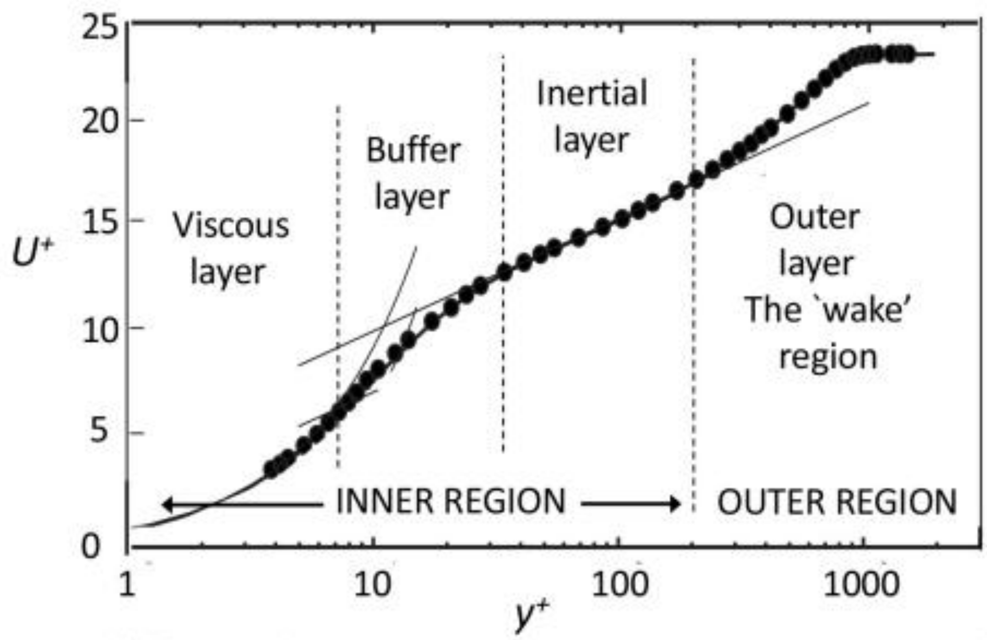
Inner region of a TBL

- *Viscous sublayer* extends to around 5 wall units ($y^+ = 5$)
- Can be described as: $U^+ = y^+$, where $U^+ = \frac{u}{u_\tau}$ and $y^+ = \frac{yu_t}{\nu}$
- Turbulent stress is negligible compared to viscous stress, which is constant within the viscous sublayer, i.e. $\tau = \mu \frac{\partial u}{\partial y} = \rho u_\tau^2 = \tau_w$



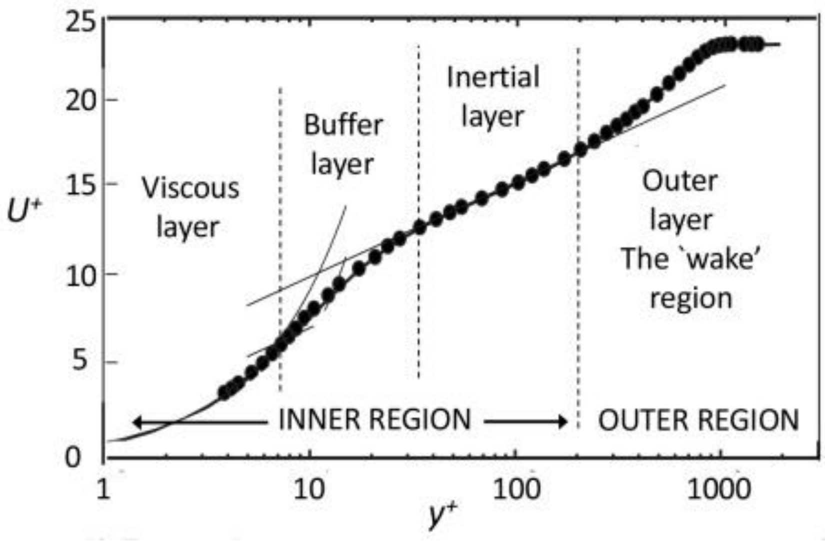
Inner region of a TBL

- *Buffer layer* is where turbulent fluctuations (i.e. turbulent stress) become increasingly important
- Both viscous and turbulent stress are important here
- In wall units, typically extends from $5 < y^+ < 30$



Inner region of a TBL

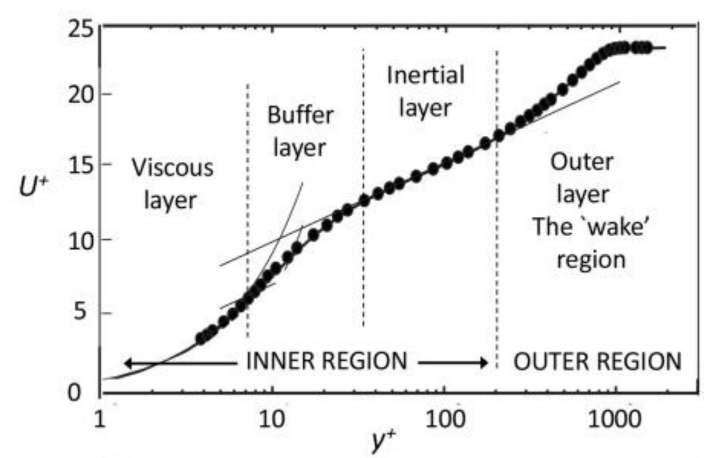
- *Inertial layer* is where viscosity is now irrelevant ($y \gg \nu/u_\tau$)
- y is still relatively small (relative to δ), typically up to around $\frac{y}{\delta} = 0.2$, so δ is irrelevant and wall scaling remains appropriate
- Log-law profile: $U^+ = \frac{1}{\kappa}\ln y^+ + B$
- Also commonly known as (logarithmic) law of the wall



Outer region of a TBL

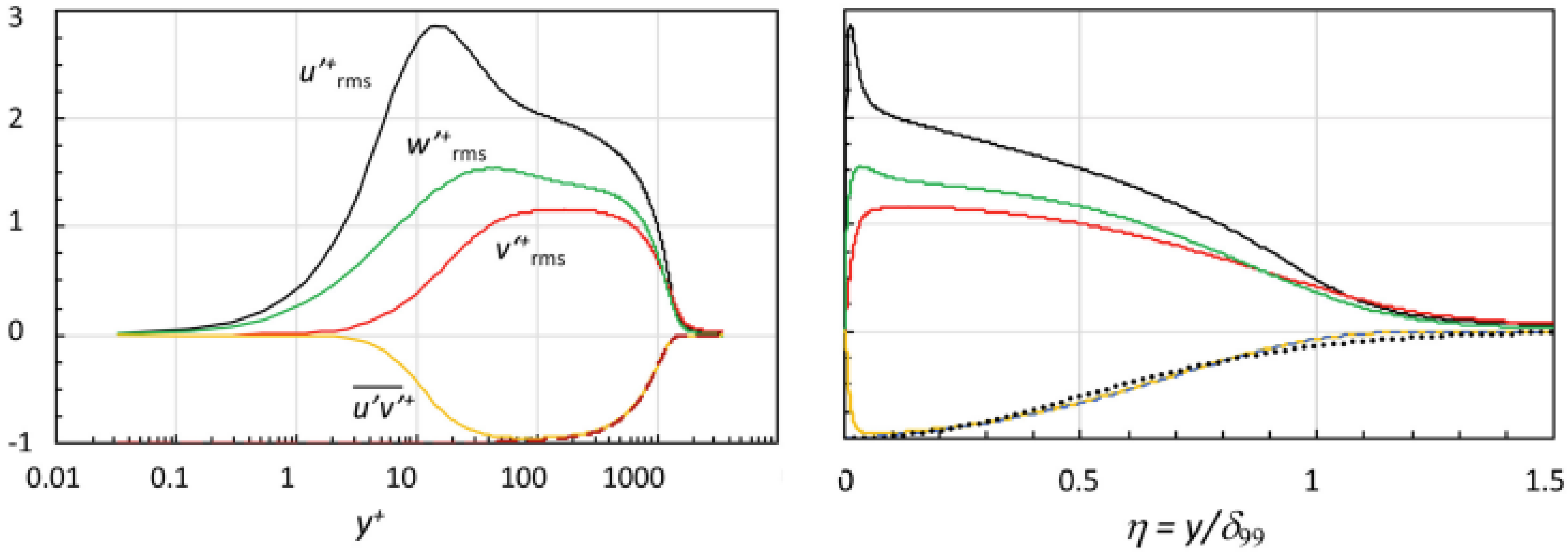
- Highly intermittent, turbulent-non turbulent interface (TNTI)
- Velocity profile over the entire boundary layer can be represented by the sum of two similarity functions, (1) *law of the wall* and (2) *law of the wake:*
- $U^+ = \frac{1}{\kappa}\ln y^+ + B + \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right)$
- where Π is the wake strength and w is the wake function
- Many different wake functions
- Common approximation:

$$w\left(\frac{y}{\delta}\right) = 2\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$



Reynolds stresses

- Recall that near the wall, viscosity dominates, and Reynolds stresses goes to zero
- Far away from the wall, we expect Reynolds stresses to go to zero too (why?)

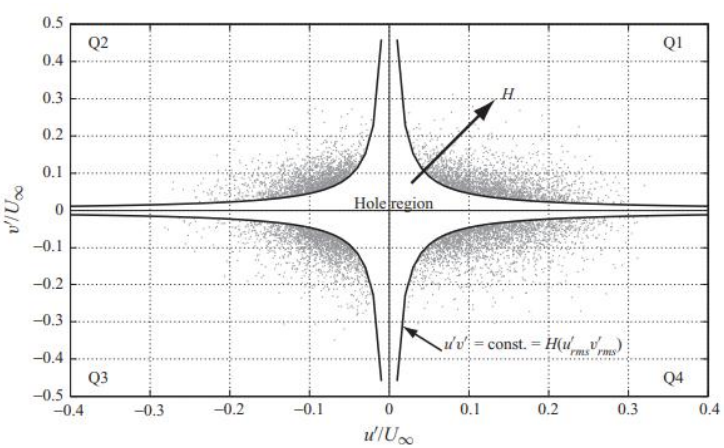


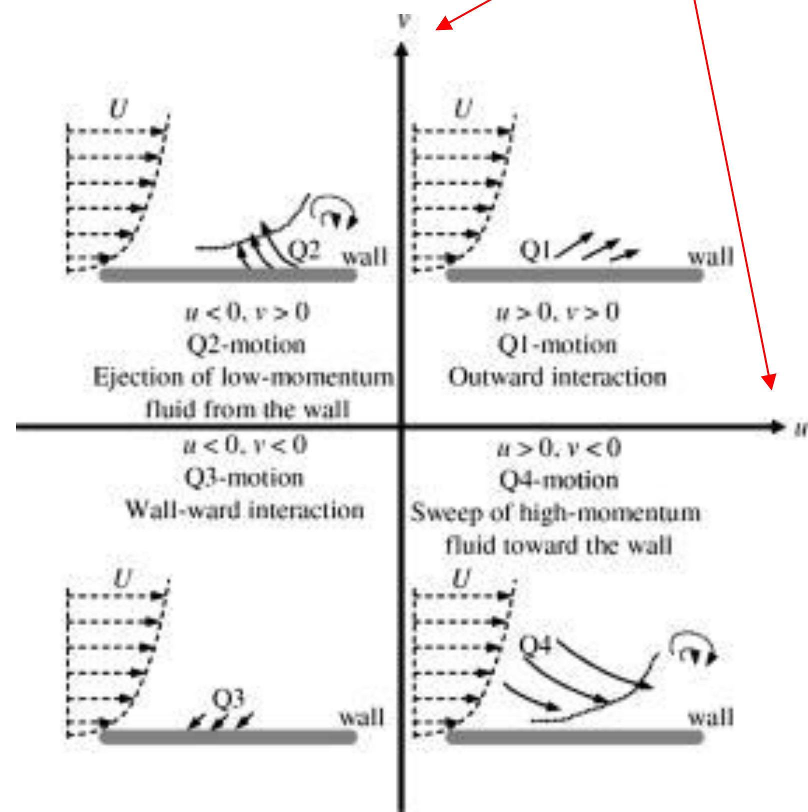
Understanding the turbulent flow motions

- Quadrant decomposition of the Reynolds shear stress (i.e. velocity fluctuation plots) by conditionally sampling the fluctuating components of velocity into quadrants

- Ejection (Q2) and sweep (Q4) events dominate vertical transport of momentum

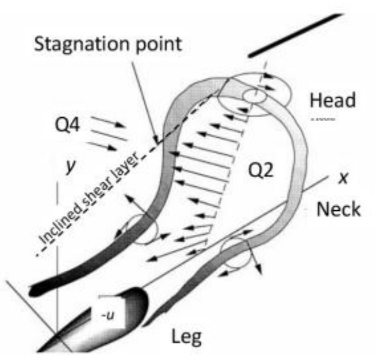
u' and v'



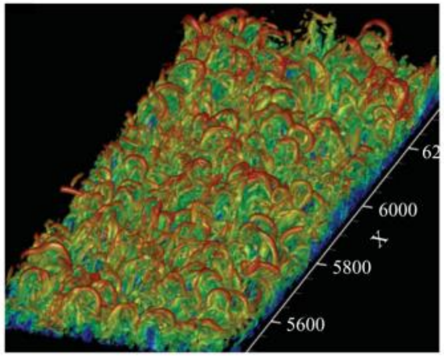


Structures of a turbulent boundary layer

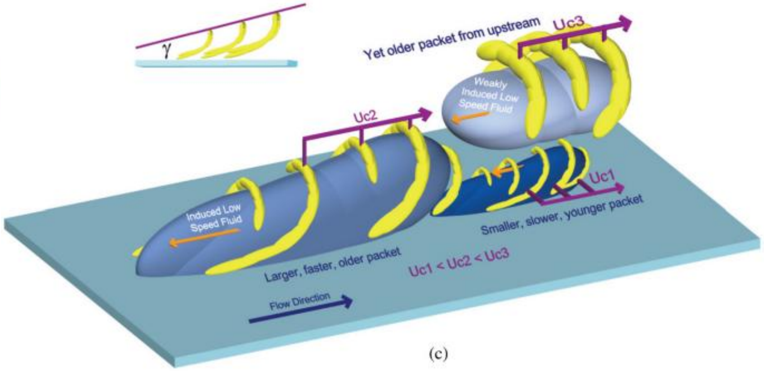
- Low-speed streaks very near the wall
- Ejection of low-speed fluid (Q2 events) outward from the wall
- Sweeps of high-speed fluid (Q4 events) towards the wall
- Vortical structures, typically in the form of hairpin packets
- Large-scale motion (LSM) $O(\delta)$ and very-large-scale motions (VLSM) $O(10\delta)$ are observed in the outer region



(a)



(b)



(c)