Appendix 3: Aircraft Response Transfer Functions Referred to Aircraft Body Axes

A3.1 Longitudinal Response Transfer Functions in Terms of Dimensional Derivatives

The following longitudinal numerator polynomials describe the motion of the aircraft in response to elevator η input. To obtain the numerators describing the response to engine thrust input, it is simply necessary to replace the subscript η with τ.

Common Denominator Polynomial

$$\Delta(s) = as^4 + bs^3 + cs^2 + ds + e$$

$$a \quad mI_y\left(m - \mathring{Z}_{\dot w}\right)$$

$$b \quad I_y\left(\mathring{X}_u \mathring{Z}_{\dot w} - \mathring{X}_{\dot w} \mathring{Z}_u\right) - mI_y\left(\mathring{X}_u + \mathring{Z}_w\right) - mM_{\dot w}\left(\mathring{Z}_q + mU_e\right) - m\mathring{M}_q\left(m - \mathring{Z}_{\dot w}\right)$$

$$\begin{aligned} c \quad & I_y\left(\mathring{X}_u \mathring{Z}_w - \mathring{X}_w \mathring{Z}_u\right) + \left(\mathring{X}_u \mathring{M}_{\dot w} - \mathring{X}_{\dot w} \mathring{M}_u\right)\left(\mathring{Z}_q + mU_e\right) \\ & + \mathring{Z}_u\left(\mathring{X}_{\dot w} \mathring{M}_q - \mathring{X}_q \mathring{M}_{\dot w}\right) + \left(\mathring{X}_u \mathring{M}_q - \mathring{X}_q \mathring{M}_u\right)\left(m - \mathring{Z}_{\dot w}\right) \\ & + m\left(\mathring{M}_q \mathring{Z}_w - \mathring{M}_w \mathring{Z}_q\right) + mW_e\left(\mathring{M}_{\dot w} \mathring{Z}_u - \mathring{M}_u \mathring{Z}_{\dot w}\right) \\ & + m^2\left(\mathring{M}_{\dot w} g\sin\theta_e + W_e \mathring{M}_u - U_e \mathring{M}_w\right) \end{aligned}$$

$$\begin{aligned} d \quad & \left(\mathring{X}_u \mathring{M}_w - \mathring{X}_w \mathring{M}_u\right)\left(\mathring{Z}_q + mU_e\right) \\ & \quad + \left(\mathring{M}_u \mathring{Z}_w - \mathring{M}_w \mathring{Z}_u\right)\left(\mathring{X}_q - mW_e\right) + \mathring{M}_q\left(\mathring{X}_w \mathring{Z}_u - \mathring{X}_u \mathring{Z}_w\right) \\ & \quad + mg\cos\theta_e\left(\mathring{M}_{\dot w} \mathring{Z}_u + \mathring{M}_u\left(m - \mathring{Z}_{\dot w}\right)\right) + mg\sin\theta_e\left(\mathring{X}_{\dot w} \mathring{M}_u - \mathring{X}_u \mathring{M}_{\dot w} + m\mathring{M}_w\right) \end{aligned}$$

$$e \quad mg\sin\theta_e\left(\mathring{X}_w \mathring{M}_u - \mathring{X}_u \mathring{M}_w\right) + mg\cos\theta_e\left(\mathring{M}_w \mathring{Z}_u - \mathring{M}_u \mathring{Z}_w\right)$$

Numerator Polynomial	
$N_\eta^u(s) = as^3 + bs^2 + cs + d$	
a	$I_y\left(\mathring{X}_{\dot{w}}\mathring{Z}_\eta + \mathring{X}_\eta\left(m - \mathring{Z}_{\dot{w}}\right)\right)$
b	$\mathring{X}_\eta\left(-I_y\mathring{Z}_w - M_{\dot{w}}\left(\mathring{Z}_q + mU_e\right) - \mathring{M}_q\left(m - \mathring{Z}_{\dot{w}}\right)\right)$ $+ \mathring{Z}_\eta\left(I_y\mathring{X}_w - \mathring{X}_{\dot{w}}\mathring{M}_q + \mathring{M}_{\dot{w}}\left(\mathring{X}_q - mW_e\right)\right)$ $+ \mathring{M}_\eta\left(\left(\mathring{X}_q - mW_e\right)\left(m - \mathring{Z}_{\dot{w}}\right) + \mathring{X}_{\dot{w}}\left(\mathring{Z}_q + mU_e\right)\right)$
c	$\mathring{X}_\eta\left(\mathring{Z}_w\mathring{M}_q - \mathring{M}_w\left(\mathring{Z}_q + mU_e\right) + mg\sin\theta_e\mathring{M}_{\dot{w}}\right)$ $+ \mathring{Z}_\eta\left(\mathring{M}_w\left(\mathring{X}_q - mW_e\right) - \mathring{X}_w\mathring{M}_q - mg\cos\theta_e\mathring{M}_{\dot{w}}\right)$ $+ \mathring{M}_\eta\left(\mathring{X}_w\left(\mathring{Z}_q + mU_e\right) - \mathring{Z}_w\left(\mathring{X}_q - mW_e\right) - mg\cos\theta_e\left(m - \mathring{Z}_{\dot{w}}\right) - mg\sin\theta_e\mathring{X}_{\dot{w}}\right)$
d	$\mathring{X}_\eta\mathring{M}_w mg\sin\theta_e - \mathring{Z}_\eta\mathring{M}_w mg\cos\theta_e + \mathring{M}_\eta\left(\mathring{Z}_w mg\cos\theta_e - \mathring{X}_w mg\sin\theta_e\right)$

Numerator Polynomial	
$N_\eta^w(s) = as^3 + bs^2 + cs + d$	
a	$mI_y\mathring{Z}_\eta$
b	$I_y\mathring{X}_\eta\mathring{Z}_u - \mathring{Z}_\eta\left(I_y\mathring{X}_u + m\mathring{M}_q\right) + m\mathring{M}_\eta\left(\mathring{Z}_q + mU_e\right)$
c	$\mathring{X}_\eta\left(\mathring{M}_u\left(\mathring{Z}_q + mU_e\right) - \mathring{Z}_u\mathring{M}_q\right) + \mathring{Z}_\eta\left(\mathring{X}_u\mathring{M}_q - \mathring{M}_u\left(\mathring{X}_q - mW_e\right)\right)$ $+ \mathring{M}_\eta\left(\mathring{Z}_u\left(\mathring{X}_q - mW_e\right) - \mathring{X}_u\left(\mathring{Z}_q + mU_e\right) - m^2 g\sin\theta_e\right)$
d	$-\mathring{X}_\eta\mathring{M}_u mg\sin\theta_e + \mathring{Z}_\eta\mathring{M}_u mg\cos\theta_e + \mathring{M}_\eta\left(\mathring{X}_u mg\sin\theta_e - \mathring{Z}_u mg\cos\theta_e\right)$

Numerator Polynomials	
$N_\eta^q(s) = s(as^2 + bs + c)$ and $N_\eta^\theta(s) = as^2 + bs + c$	
a	$m\mathring{Z}_\eta\mathring{M}_{\dot{w}} + m\mathring{M}_\eta\left(m - \mathring{Z}_{\dot{w}}\right)$
b	$\mathring{X}_\eta\left(\mathring{Z}_u\mathring{M}_{\dot{w}} + \mathring{M}_u\left(m - \mathring{Z}_{\dot{w}}\right)\right) + \mathring{Z}_\eta\left(m\mathring{M}_w - \mathring{X}_u\mathring{M}_{\dot{w}} + \mathring{M}_u\mathring{X}_{\dot{w}}\right)$ $+ \mathring{M}_\eta\left(-\mathring{X}_u\left(m - \mathring{Z}_{\dot{w}}\right) - \mathring{Z}_u\mathring{X}_{\dot{w}} - m\mathring{Z}_w\right)$
c	$\mathring{X}_\eta\left(\mathring{Z}_u\mathring{M}_w - \mathring{M}_u\mathring{Z}_w\right) + \mathring{Z}_\eta\left(\mathring{X}_w\mathring{M}_u - \mathring{M}_w\mathring{X}_u\right) + \mathring{M}_\eta\left(\mathring{X}_u\mathring{Z}_w - \mathring{Z}_u\mathring{X}_w\right)$

A3.2 Lateral-Directional Response Transfer Functions in Terms of Dimensional Derivatives

The following lateral-directional numerator polynomials describe the motion of the aircraft in response to aileron ξ input. To obtain the numerators describing the response to rudder input, it is simply necessary to replace the subscript ξ with ζ.

Common Denominator Polynomial	
$\Delta(s) = s(as^4 + bs^3 + cs^2 + ds + e)$	
a	$m(I_x I_z - I_{xz}^2)$
b	$-\mathring{Y}_v(I_x I_z - I_{xz}^2) - m\left(I_x \mathring{N}_r + I_{xz}\mathring{L}_r\right) - m\left(I_z \mathring{L}_p + I_{xz}\mathring{N}_p\right)$
c	$\mathring{Y}_v\left(I_x \mathring{N}_r + I_{xz}\mathring{L}_r\right) + \mathring{Y}_v\left(I_z \mathring{L}_p + I_{xz}\mathring{N}_p\right) - \left(\mathring{Y}_p + mW_e\right)\left(I_z \mathring{L}_v + I_{xz}\mathring{N}_v\right)$ $- \left(\mathring{Y}_r - mU_e\right)\left(I_x \mathring{N}_v + I_{xz}\mathring{L}_v\right) + m\left(\mathring{L}_p \mathring{N}_r - \mathring{L}_r \mathring{N}_p\right)$
d	$\mathring{Y}_v\left(\mathring{L}_r \mathring{N}_p - \mathring{L}_p \mathring{N}_r\right) + \left(\mathring{Y}_p + mW_e\right)\left(\mathring{L}_v \mathring{N}_r - \mathring{L}_r \mathring{N}_v\right)$ $+ \left(\mathring{Y}_r - mU_e\right)\left(\mathring{L}_p \mathring{N}_v - \mathring{L}_v \mathring{N}_p\right)$ $- mg\cos\theta_e\left(I_z \mathring{L}_v + I_{xz}\mathring{N}_v\right) - mg\sin\theta_e\left(I_x \mathring{N}_v + I_{xz}\mathring{L}_v\right)$
e	$mg\cos\theta_e\left(\mathring{L}_v \mathring{N}_r - \mathring{L}_r \mathring{N}_v\right) + mg\sin\theta_e\left(\mathring{L}_p \mathring{N}_v - \mathring{L}_v \mathring{N}_p\right)$

Numerator Polynomial	
$N_\xi^v(s) = s(as^3 + bs^2 + cs + d\,)$	
a	$\mathring{Y}_\xi(I_x I_z - I_{xz}^2)$
b	$\mathring{Y}_\xi\left(-I_x \mathring{N}_r - I_z \mathring{L}_p - I_{xz}\left(\mathring{L}_r + \mathring{N}_p\right)\right) + \mathring{L}_\xi\left(I_z\left(\mathring{Y}_p + mW_e\right) + I_{xz}\left(\mathring{Y}_r - mU_e\right)\right)$ $+ \mathring{N}_\xi\left(I_x\left(\mathring{Y}_r - mU_e\right) + I_{xz}\left(\mathring{Y}_p + mW_e\right)\right)$
c	$\mathring{Y}_\xi\left(\mathring{L}_p \mathring{N}_r - \mathring{L}_r \mathring{N}_p\right)$ $+ \mathring{L}_\xi\left(\mathring{N}_p\left(\mathring{Y}_r - mU_e\right) - \mathring{N}_r\left(\mathring{Y}_p + mW_e\right) + mg\,(I_z\cos\theta_e + I_{xz}\sin\theta_e)\right)$ $+ \mathring{N}_\xi\left(\mathring{L}_r\left(\mathring{Y}_p + mW_e\right) - \mathring{L}_p\left(\mathring{Y}_r - mU_e\right) + mg\,(I_x\sin\theta_e + I_{xz}\cos\theta_e)\right)$
d	$\mathring{L}_\xi\left(\mathring{N}_p\, mg\sin\theta_e - \mathring{N}_r\, mg\cos\theta_e\right) + \mathring{N}_\xi\left(\mathring{L}_r\, mg\cos\theta_e - \mathring{L}_p\, mg\sin\theta_e\right)$

Numerator Polynomials	
$N_{\xi}^{p}(s) = s(as^3 + bs^2 + cs + d)$ and $N_{\xi}^{\phi}(s) = as^3 + bs^2 + cs + d$	
a	$m\left(I_z \mathring{L}_{\xi} + I_{xz} \mathring{N}_{\xi}\right)$
b	$\mathring{Y}_{\xi}\left(I_z \mathring{L}_v + I_{xz} \mathring{N}_v\right) + \mathring{L}_{\xi}\left(-I_z \mathring{Y}_v - m\mathring{N}_r\right) + \mathring{N}_{\xi}\left(m\mathring{L}_r - I_{xz}\mathring{Y}_v\right)$
c	$\mathring{Y}_{\xi}\left(\mathring{L}_r \mathring{N}_v - \mathring{L}_v \mathring{N}_r\right) + \mathring{L}_{\xi}\left(\mathring{N}_r \mathring{Y}_v - \mathring{N}_v \mathring{Y}_r + mU_e \mathring{N}_v\right)$ $+ \mathring{N}_{\xi}\left(\mathring{L}_v \mathring{Y}_r - \mathring{L}_r \mathring{Y}_v - mU_e \mathring{L}_v\right)$
d	$mg\sin\theta_e\left(\mathring{L}_v \mathring{N}_{\xi} - \mathring{L}_{\xi} \mathring{N}_v\right)$

Numerator Polynomials	
$N_{\xi}^{r}(s) = s(as^3 + bs^2 + cs + d)$ and $N_{\xi}^{\psi}(s) = as^3 + bs^2 + cs + d$	
a	$m\left(I_x \mathring{N}_{\xi} + I_{xz} \mathring{L}_{\xi}\right)$
b	$\mathring{Y}_{\xi}\left(I_x \mathring{N}_v + I_{xz} \mathring{L}_v\right) + \mathring{L}_{\xi}\left(m\mathring{N}_p - I_{xz}\mathring{Y}_v\right) - \mathring{N}_{\xi}\left(I_x \mathring{Y}_v + m\mathring{L}_p\right)$
c	$\mathring{Y}_{\xi}\left(\mathring{L}_v \mathring{N}_p - \mathring{L}_p \mathring{N}_v\right) + \mathring{L}_{\xi}\left(\mathring{N}_v \mathring{Y}_p - \mathring{N}_p \mathring{Y}_v + mW_e \mathring{N}_v\right)$ $+ \mathring{N}_{\xi}\left(\mathring{L}_p \mathring{Y}_v - \mathring{L}_v \mathring{Y}_p - mW_e \mathring{L}_v\right)$
d	$mg\cos\theta_e\left(\mathring{L}_{\xi} \mathring{N}_v - \mathring{L}_v \mathring{N}_{\xi}\right)$

A3.3 Longitudinal Response Transfer Functions in Terms of Concise Derivatives

Again, the longitudinal numerator polynomials describe the motion of the aircraft in response to elevator η input. To obtain the numerators describing the response to engine thrust input, it is simply necessary to replace the subscript η with τ.

Common Denominator Polynomial	
$\Delta(s) = as^4 + bs^3 + cs^2 + ds + e$	
a	1
b	$-(m_q + x_u + z_w)$
c	$(m_q z_w - m_w z_q) + (m_q x_u - m_u x_q) + (x_u z_w - x_w z_u) - m_\theta$
d	$(m_\theta x_u - m_u x_\theta) + (m_\theta z_w - m_w z_\theta) + m_q(x_w z_u - x_u z_w) + x_q(m_u z_w - m_w z_u) + z_q(m_w x_u - m_u x_w)$
e	$m_\theta(x_w z_u - x_u z_w) + x_\theta(m_u z_w - m_w z_u) + z_\theta(m_w x_u - m_u x_w)$

Numerator Polynomial	
$N_{\eta}^{u}(s) = as^3 + bs^2 + cs + d$	
a	x_{η}
b	$m_{\eta}x_q - x_{\eta}(m_q + z_w) + z_{\eta}x_w$
c	$m_{\eta}(x_w z_q - x_q z_w + x_{\theta}) + x_{\eta}(m_q z_w - m_w z_q - m_{\theta}) + z_{\eta}(m_w x_q - m_q x_w)$
d	$m_{\eta}(x_w z_{\theta} - x_{\theta} z_w) + x_{\eta}(m_{\theta} z_w - m_w z_{\theta}) + z_{\eta}(m_w x_{\theta} - m_{\theta} x_w)$

Numerator Polynomial	
$N_{\eta}^{w}(s) = as^3 + bs^2 + cs + d$	
a	z_{η}
b	$m_{\eta}z_q + x_{\eta}z_u - z_{\eta}(m_q + x_u)$
c	$m_{\eta}(x_q z_u - x_u z_q + z_{\theta}) + x_{\eta}(m_u z_q - m_q z_u) + z_{\eta}(m_q x_u - m_u x_q - m_{\theta})$
d	$m_{\eta}(x_{\theta} z_u - x_u z_{\theta}) + x_{\eta}(m_u z_{\theta} - m_{\theta} z_u) + z_{\eta}(m_{\theta} x_u - m_u x_{\theta})$

Numerator Polynomials	
$N_{\eta}^{q}(s) = s(as^2 + bs + c)$ and $N_{\eta}^{\theta}(s) = as^2 + bs + c$	
a	m_{η}
b	$-m_{\eta}(x_u + z_w) + x_{\eta}m_u + z_{\eta}m_w$
c	$m_{\eta}(x_u z_w - x_w z_u) + x_{\eta}(m_w z_u - m_u z_w) + z_{\eta}(m_u x_w - m_w x_u)$

A3.4 Lateral-Directional Response Transfer Functions in Terms of Concise Derivatives

As before, the lateral-directional numerator polynomials describe the motion of the aircraft in response to aileron ξ input. To obtain the numerators describing the response to rudder input, it is simply necessary to replace the subscript ξ with ζ.

Common Denominator Polynomial	
$\Delta(s) = as^5 + bs^4 + cs^3 + ds^2 + es + f$	
a	1
b	$-(l_p + n_r + y_v)$
c	$(l_p n_r - l_r n_p) + (n_r y_v - n_v y_r) + (l_p y_v - l_v y_p) - (l_{\phi} + n_{\psi})$
d	$(l_p n_{\psi} - l_{\psi} n_p) + (l_{\phi} n_r - l_r n_{\phi}) + l_v(n_r y_p - n_p y_r - y_{\phi}) + n_v(l_p y_r - l_r y_p - y_{\psi}) + y_v(l_r n_p - l_p n_r + l_{\phi} + n_{\psi})$
e	$(l_{\phi} n_{\psi} - l_{\psi} n_{\phi}) + l_v((n_r y_{\phi} - n_{\phi} y_r) + (n_{\psi} y_p - n_p y_{\psi})) + n_v((l_{\phi} y_r - l_r y_{\phi}) + (l_p y_{\psi} - l_{\psi} y_p)) + y_v((l_r n_{\phi} - l_{\phi} n_r) + (l_{\psi} n_p - l_p n_{\psi}))$
f	$l_v(n_{\psi} y_{\phi} - n_{\phi} y_{\psi}) + n_v(l_{\phi} y_{\psi} - l_{\psi} y_{\phi}) + y_v(l_{\psi} n_{\phi} - l_{\phi} n_{\psi})$

Numerator Polynomial	
$N_{\xi}^{v}(s) = as^4 + bs^4 + cs^2 + ds + e$	
a	y_{ξ}
b	$l_{\xi}y_p + n_{\xi}y_r - y_{\xi}(l_p + n_r)$
c	$l_{\xi}(n_p y_r - n_r y_p + y_{\phi}) + n_{\xi}(l_r y_p - l_p y_r + y_{\psi}) + y_{\xi}(l_p n_r - l_r n_p - l_{\phi} - n_{\psi})$
d	$l_{\xi}(n_{\phi}y_r - n_r y_{\phi} + n_p y_{\psi} - n_{\psi}y_p) + n_{\xi}(l_r y_{\phi} - l_{\phi}y_r + l_{\psi}y_p - l_p y_{\psi}) + y_{\xi}(l_{\phi}n_r - l_r n_{\phi} + l_p n_{\psi} - l_{\psi}n_p)$
e	$l_{\xi}(n_{\phi}y_{\psi} - n_{\psi}y_{\phi}) + n_{\xi}(l_{\psi}y_{\phi} - l_{\phi}y_{\psi}) + y_{\xi}(l_{\phi}n_{\psi} - l_{\psi}n_{\phi})$

Numerator Polynomials	
$N_{\xi}^{p}(s) = s(as^3 + bs^2 + cs + d)$ and $N_{\xi}^{\phi}(s) = as^3 + bs^2 + cs + d$	
a	l_{ξ}
b	$-l_{\xi}(n_r + y_v) + n_{\xi}l_r + y_{\xi}l_v$
c	$l_{\xi}(n_r y_v - n_v y_r - n_{\psi}) + n_{\xi}(l_v y_r - l_r y_v + l_{\psi}) + y_{\xi}(l_r n_v - l_v n_r)$
d	$l_{\xi}(n_{\psi}y_v - n_v y_{\psi}) + n_{\xi}(l_v y_{\psi} - l_{\psi}y_v) + y_{\xi}(l_{\psi}n_v - l_v n_{\psi})$

Numerator Polynomials	
$N_{\xi}^{r}(s) = s(as^3 + bs^2 + cs + d)$ and $N_{\xi}^{\psi}(s) = as^3 + bs^2 + cs + d$	
a	n_{ξ}
b	$l_{\xi}n_p - n_{\xi}(l_p + y_v) + y_{\xi}n_v$
c	$l_{\xi}(n_v y_p - n_p y_v + n_{\phi}) + n_{\xi}(l_p y_v - l_v y_p - l_{\phi}) + y_{\xi}(l_v n_p - l_p n_v)$
d	$l_{\xi}(n_v y_{\phi} - n_{\phi}y_v) + n_{\xi}(l_{\phi}y_v - l_v y_{\phi}) + y_{\xi}(l_v n_{\phi} - l_{\phi}n_v)$