ADVANCED STRUCTURES & MATERIALS

Finite Element Analysis Principles – Lecture 1

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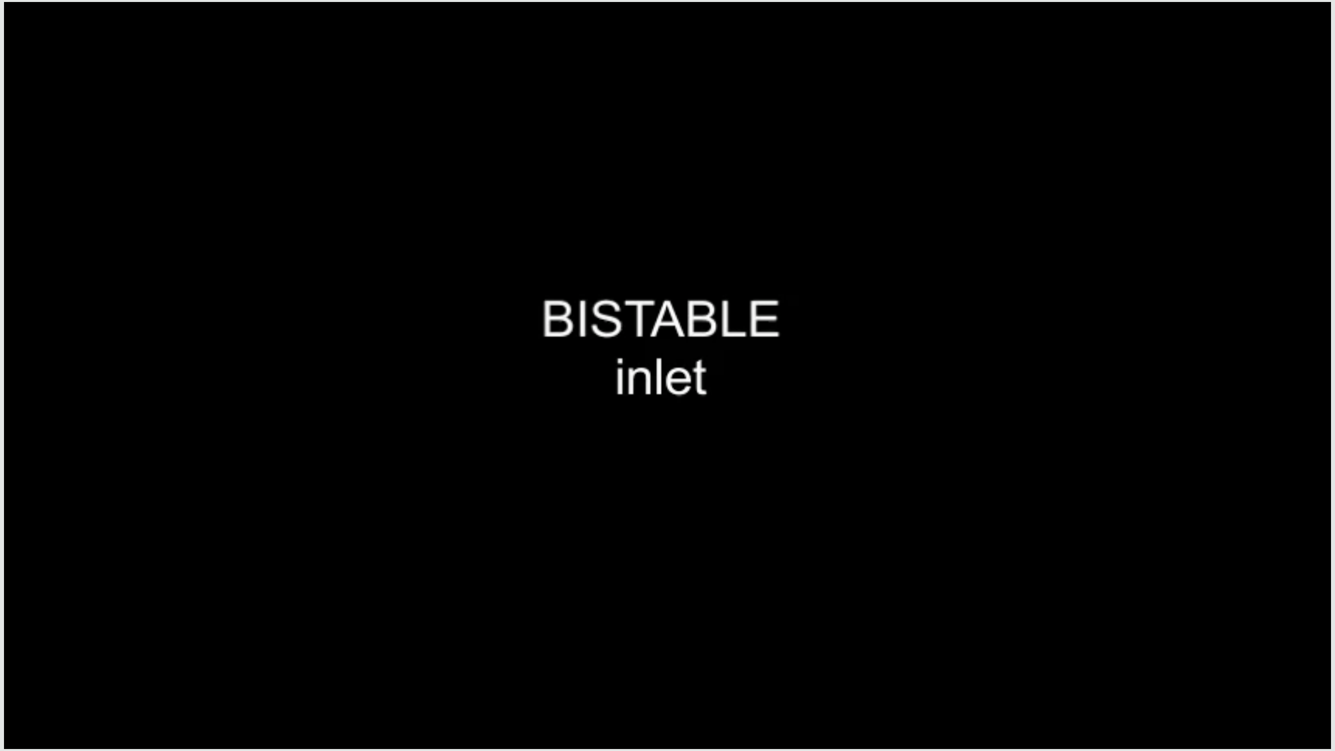
A LITTLE BIT ABOUT ME

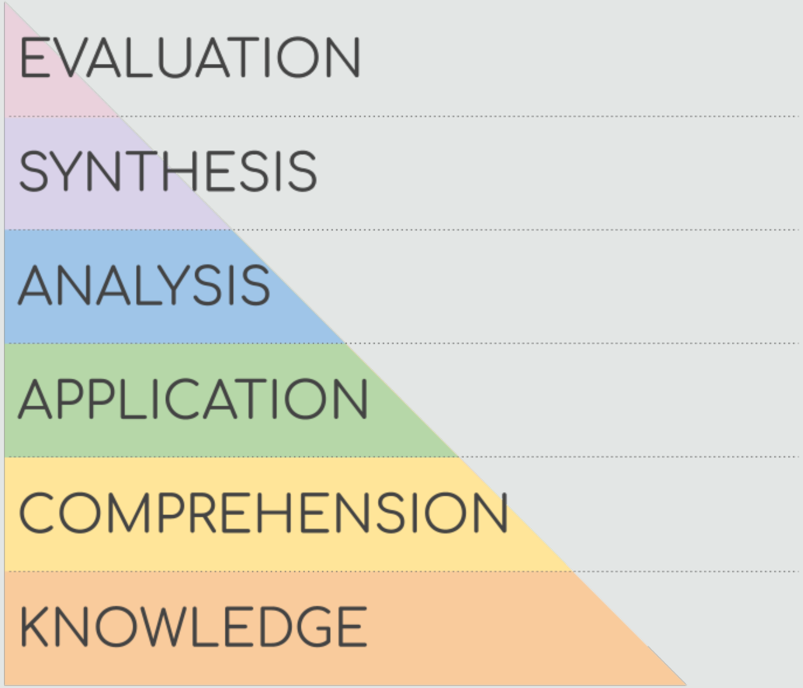
- Prof. of Nonlinear Structural Mechanics
- MSc in Aerospace Engineering – Università degli studi di Palermo, Italy
- PhD – University of Bristol
- CoSEM CDT co-director & taught programme director
- EPRSC research fellow (2015-2020)

Research Interests

- Well-behaved nonlinear structures
- Structural (in)stability
- Computational elasticity
- Adaptive and morphing structures
- Wind turbines
- Aeroelastic tailoring


BISTABLE
inlet

ABOUT THIS CLASS



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Applying

- Apply finite element analysis to solve simple problems in structural mechanics.
  - Example: Solve a beam bending problem using FEA software and interpret the results.

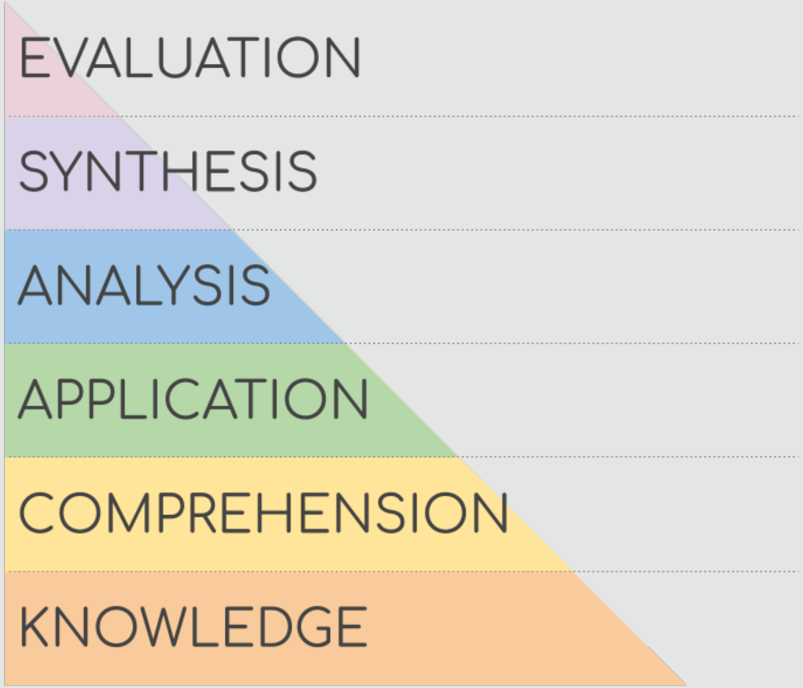
Understanding

- Explain the theoretical foundation of finite element methods, including the weak form of differential equations and variational principles.
  - Example: Describe how the finite element method approximates solutions for differential equations.

Knowledge

- Recall and define basic concepts of finite element analysis, such as nodes, elements, meshing, and stiffness matrices.
  - Example: Define terms such as “degree of freedom” and “shape function”.

Intended Learning Outcomes
based on Bloom’s taxonomy of educational goals



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Evaluating

- Critically evaluate the accuracy and reliability of FEA models by comparing them with analytical solutions or experiments.
  - Example: Judge the effectiveness of various FEA methods in predicting stresses in complex geometries.

Creating

- Design a finite element model for a complex engineering problem, including the selection of appropriate elements, boundary conditions, and material properties.
  - Example: Develop a finite element model for a mechanical structure and optimize it based on performance criteria.

Analysing

- Analyse the effect of different element types, mesh densities, and boundary conditions on FEA results.
  - Example: Compare how increasing mesh density affects the accuracy of stress distribution results.

Intended Learning Outcomes

based on Bloom’s taxonomy of educational goals

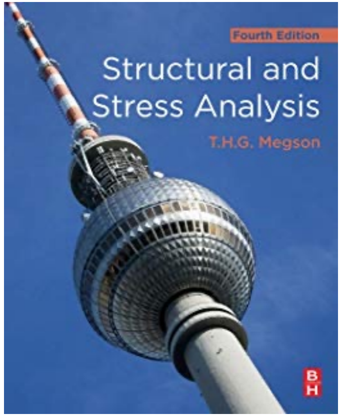
Prerequisites

- Linear Algebra
- Vector and Matrix algebra
- Calculus
- Newtonian mechanics: statics
- Solid mechanics: stresses and strains
- Structural mechanics: beam theory

Course content and textbooks

- Elements of Solid Mechanics
- Governing equations in strong and weak form
- Structural idealisation and displacement method
- 1D elements
- 2D elements (Jacobian and numerical integration)
- Modelling considerations and good practice

- **Finite Element Modelling for Stress Analysis** – R.D. Cook
- **Introduction To Finite Element Method** – J.N. Reddy
- The Finite Element Method – O.C. Zienkiewicz & R.L. Taylor
- A First Course in Finite Elements – J. Fish & T. Belytschko

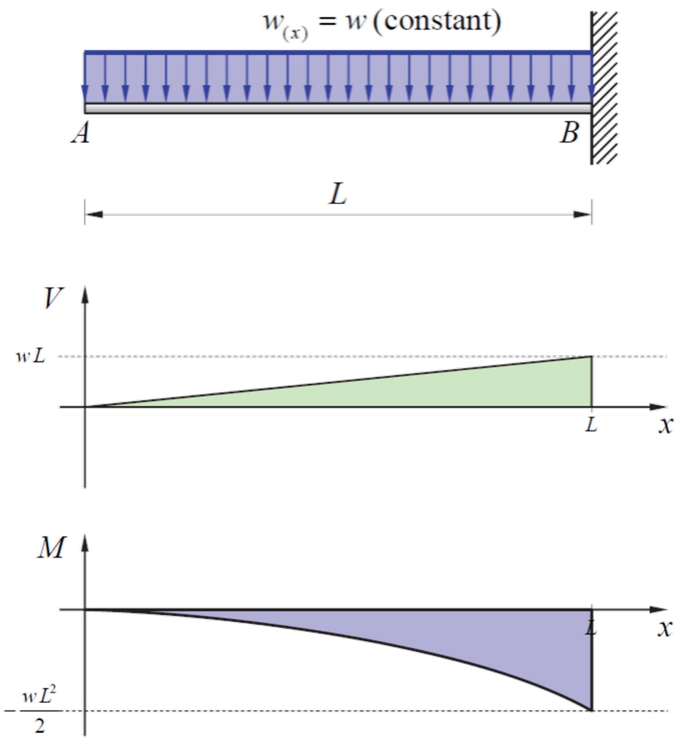


Structural and Stress Analysis
T.H.G. Megson
Chapter 17 (and 15)

PREAMBLE

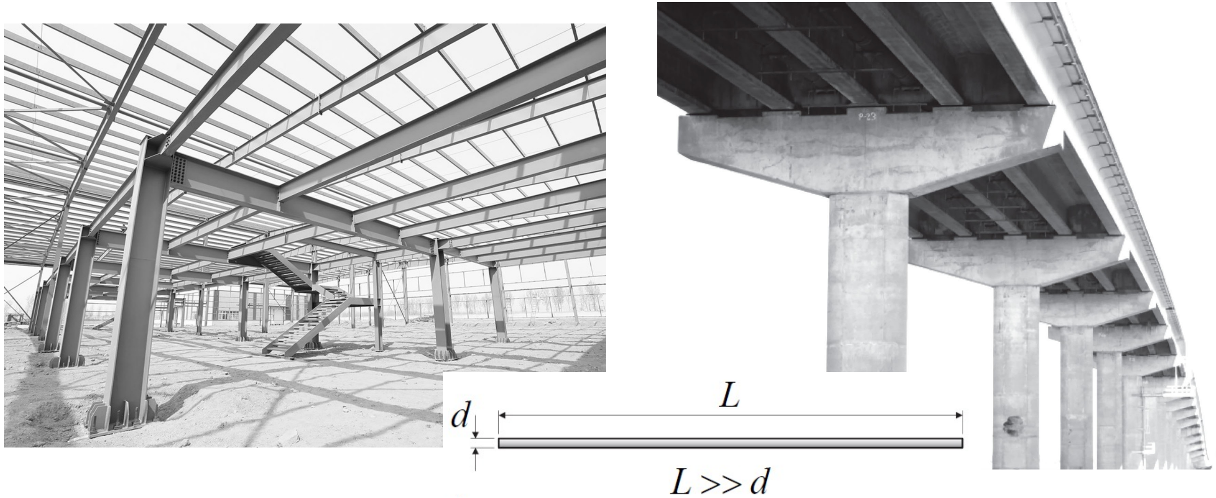
Your knowledge of structural mechanics so far

- Beam theory
- Elements of solid mechanics for beam theory
- Only statically determinate problems



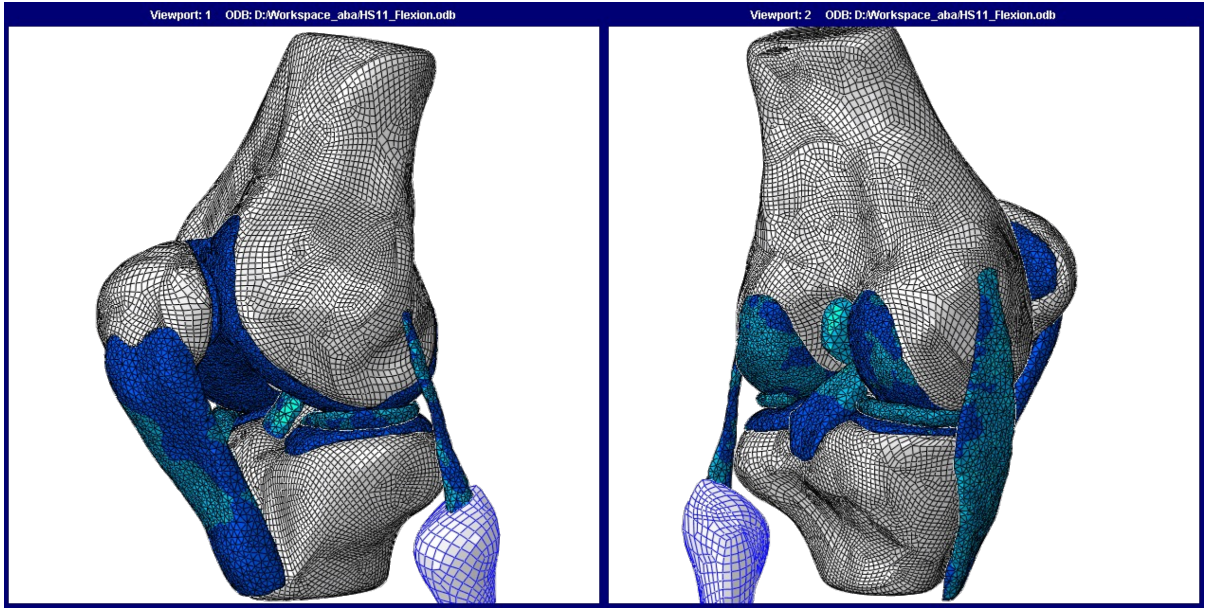
Is it sufficient? What is a beam?

Slender, 1D structural element(*) capable of carrying normal, transverse and torsional loads.

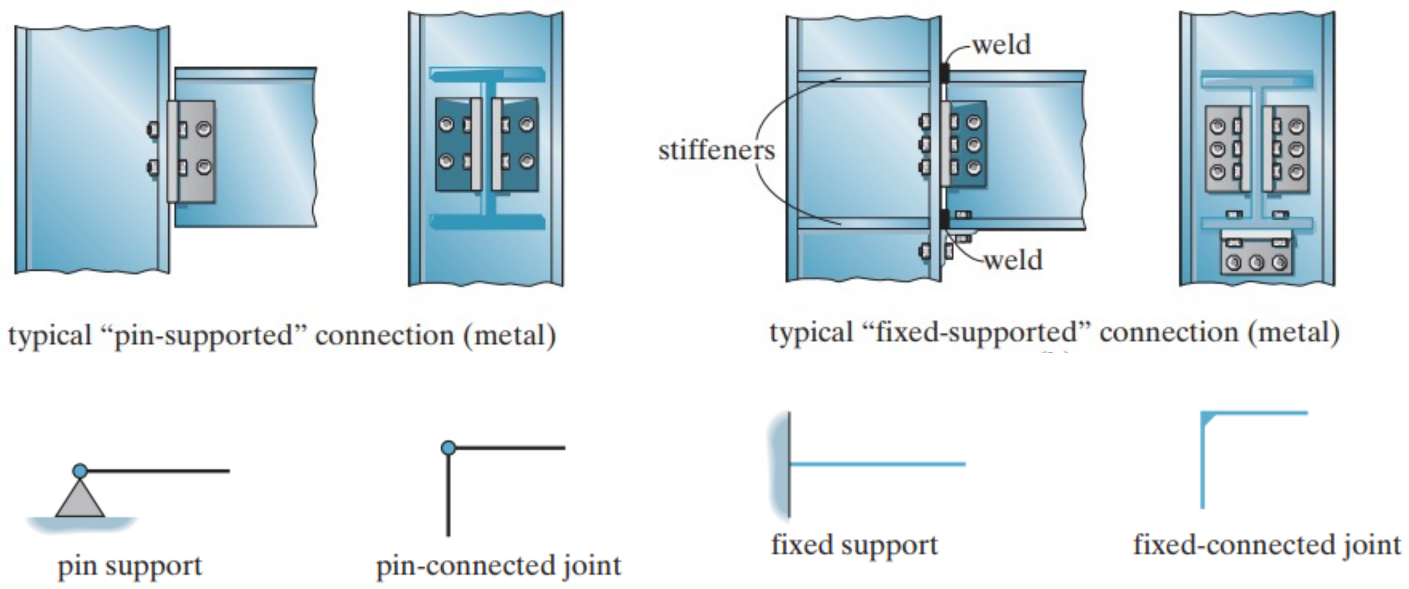


(*)characteristic length of the cross section, *d*, much smaller that axial length, *L*.

Limitations: geometry

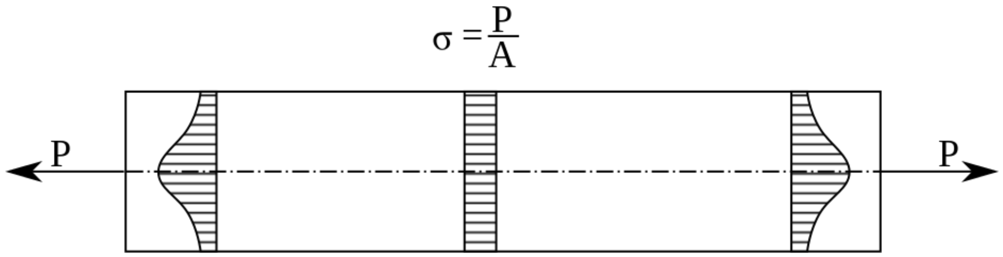


Limitations: representativeness

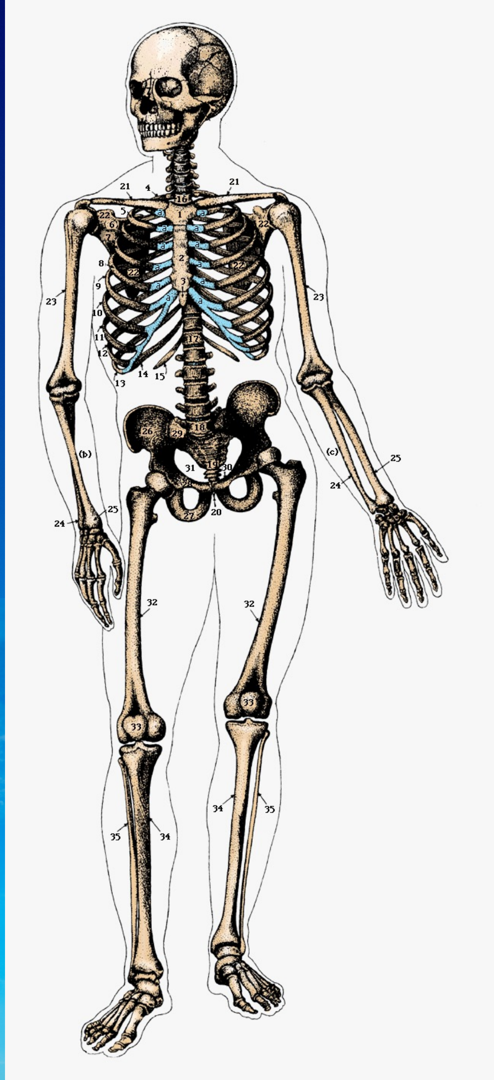


Saint-Venant's principle

The difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from load.



Can we break free from these limitations?



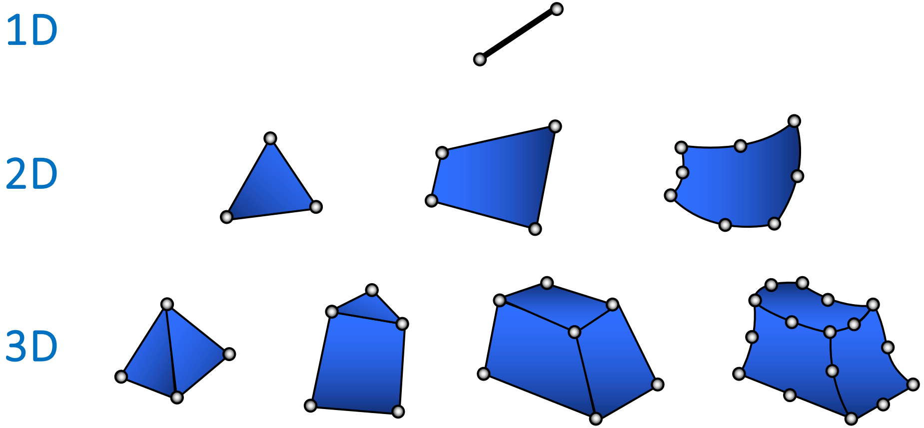

YES,
WE
CAN!
...with the Finite
Element Method

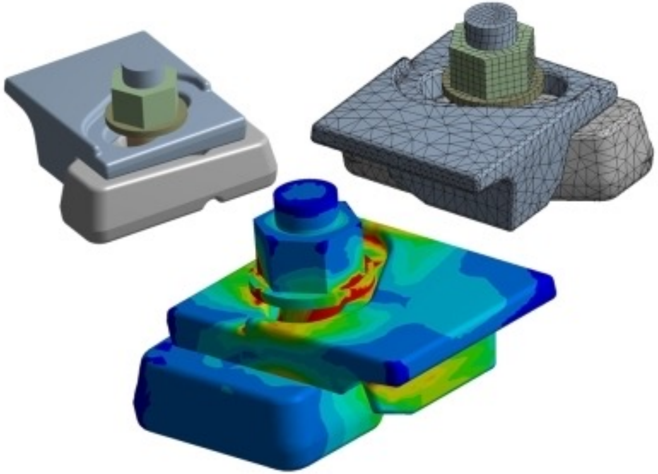
THE FINITE ELEMENT METHOD

The Finite Element Method (FEM)

- Partial differential equations (PDEs) describe nature.
- FEM is used for numerically solving PDEs in engineering and mathematical modelling.
- Originally developed for structural analysis, it’s now used for multiple physics.
- FEM solves initial boundary value problems.
- The method subdivides large systems into smaller, simpler parts called finite elements.
- This is done discretising continuum domains with a mesh of elements connected at nodes.
- The formulation leads to a system of algebraic equations, cast in matrix form, approximating the unknown function over the domain.
- A solution is approximated by minimising an error function through the calculus of variations.

What do Elements, Nodes, Meshes look like?





THE BROADER CONTEXT

Computer-Aided Engineering (CAE)

- Suite of computer methods to bring a product to market.
- Nowadays it encompasses **Computer-Aided Design (CAD)** and **Computer-Aided Manufacturing (CAM)**, using
  - Finite Element Analysis (FEA)
  - Computational Fluid Dynamics (CFD)
  - Multibody Dynamics
  - Optimisation
  - Etc.

Computer-Aided Engineering (CAE)

- CAE origins can be traced back to the development of FEM.
- FEM was first developed in the Structural Mechanics domain.
- In the early days, often called the Matrix Method of Analysis.
  - Name not *en vogue* anymore, because nowadays most numerical methods are cast in matrix form.

Computer-Aided Engineering (CAE)

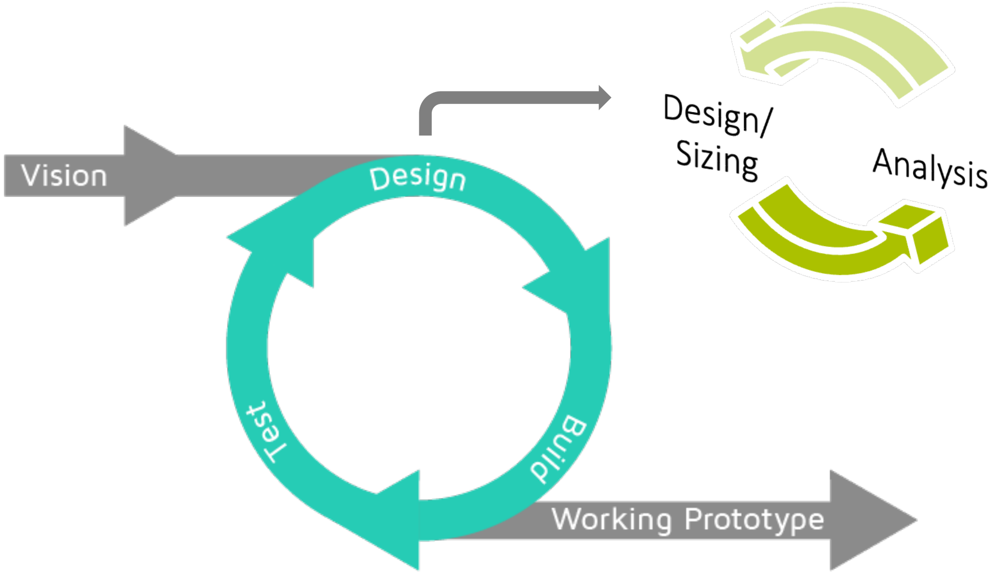
- Some (most?) of the impetus that led to modern FE software came from the aerospace sector.
- As modern computers became available, the Space Race accelerated their adoption.
- NASTRAN (**NA**SA **STR**uctural **AN**alysis) was written to support the design of more efficient vehicles such as the Space Shuttle.

On the topic of efficiency: the almighty aerospace driver …or why did the aerospace sector lead?

- Often measured by specific load-carrying capacity: load-to-mass ratio.
- For aircraft and spacecraft, minimising weight is crucial.
- Uncertainties and lack of modelling accuracy cannot be offset by using large reserve or safety factors.
- The shape of aerospace structures is determined by aerodynamic/aeroelastic demands, making it difficult to use familiar structural elements with predictable behaviour, e.g. beams and plates.
- Efficiency must be pursued with accurate analysis methods and enabling design methods.

FEA AND ENGINEERING DESIGN

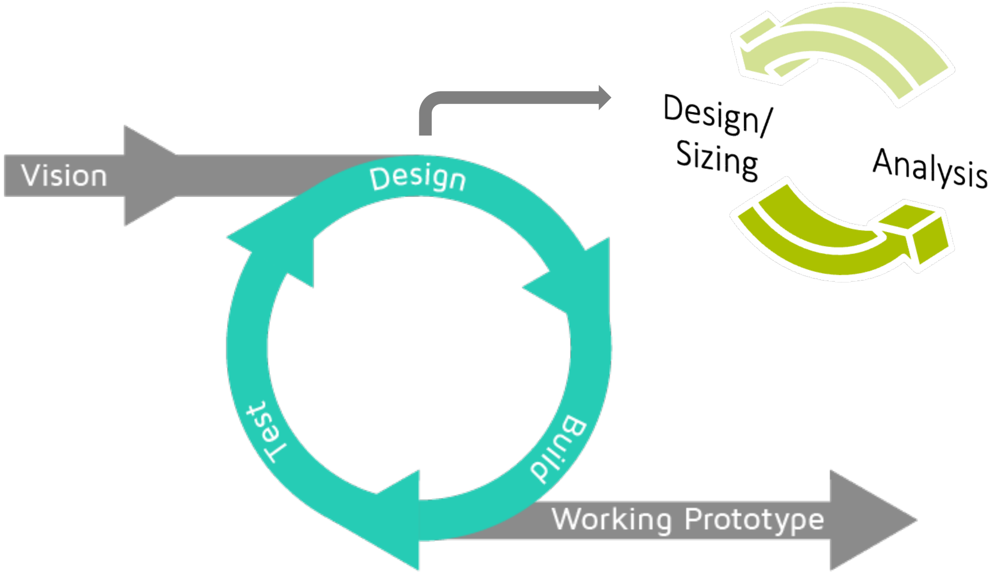
How is a human-made artefact born?



Design is an iterative process.

These days, most of the process is virtual, i.e. CAE-based.

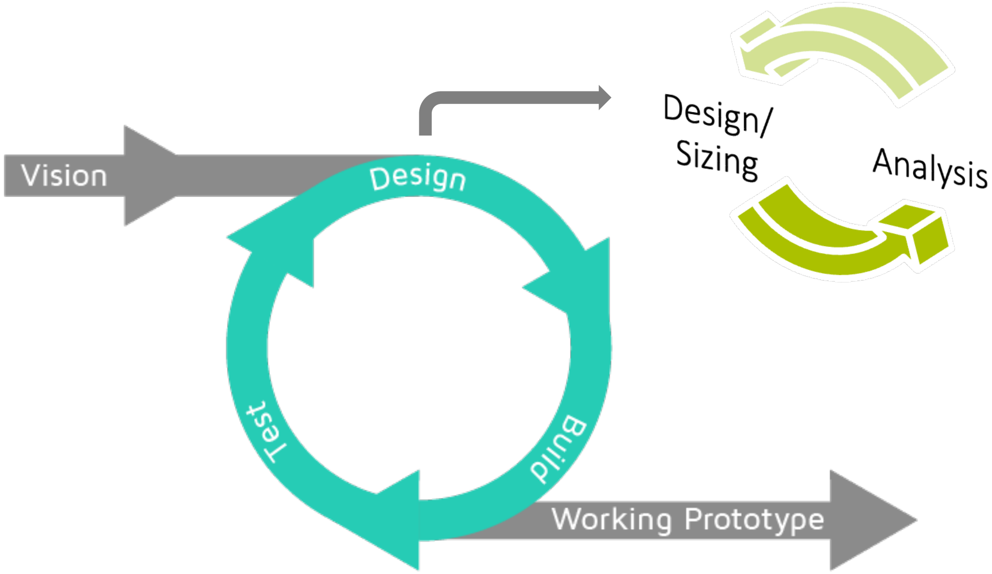
How is a human-made artefact born?



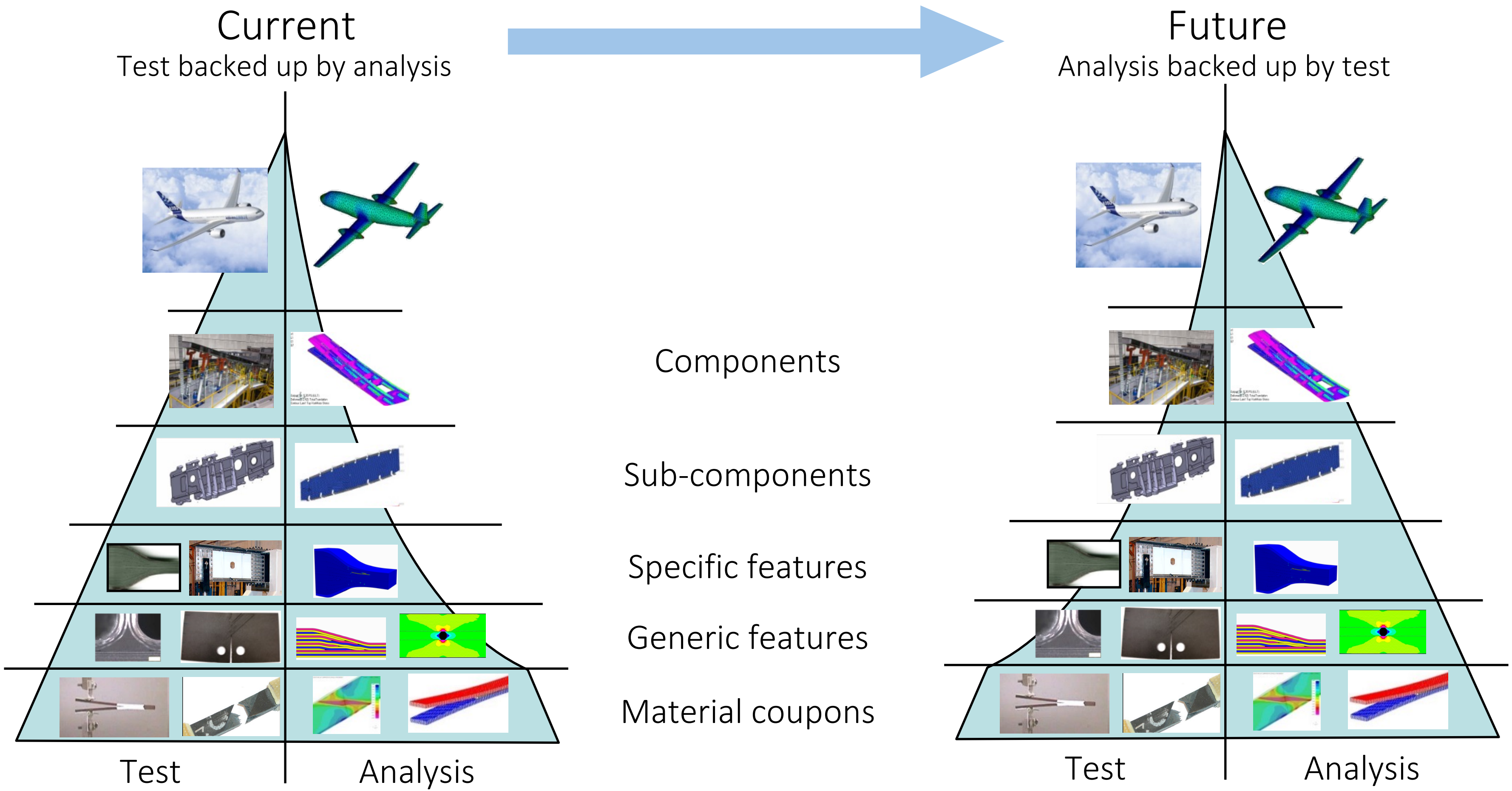
CAE-based design meets the need for:

- Increased precision
- Improved efficiency
- Handling complexity
- Faster process
- Cost reductions
- Integration of Disciplines

How is a human-made artefact born?



Accurate analysis and design can save (a lot of) time and money.


Current
Test backed up by analysis
Future
Analysis backed up by test
Components
Sub-components
Specific features
Generic features
Material coupons
Test
Analysis
Test
Analysis

The FEA (and generally CAE) workflow

1. PRE-PROCESSING
   - Idealisation, model generation
     - What are we interested in?
     - What features of the real system does my model need to capture?
     - What physics does my model need to capture?
     - Discretisation. Using what type of element?

2. PROCESSING
   - Number crunching by the analysis solver to compute the unknowns.

3. POST-PROCESSING
   - Computation of field variables depending on the unknowns and visualisation.

The FEA (and generally CAE) workflow

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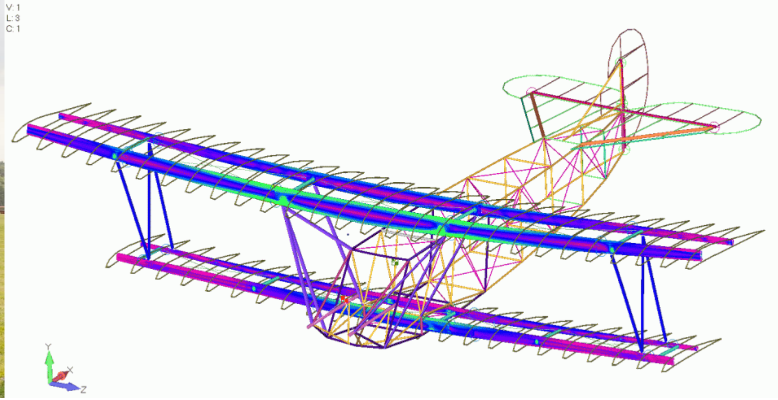
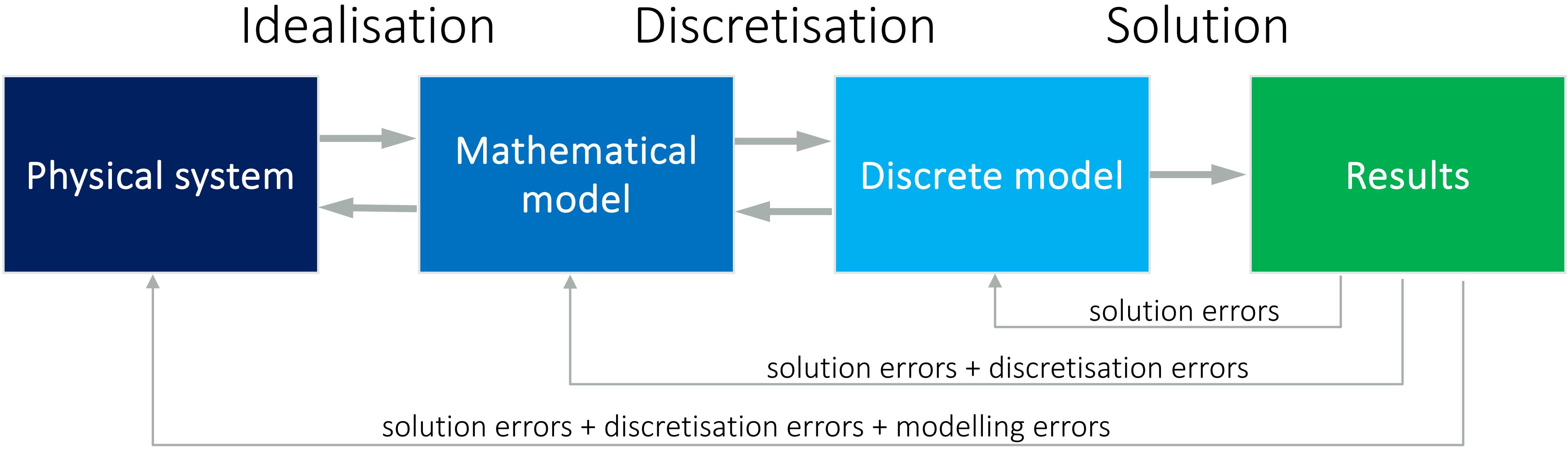


https://aerospaceengineeringblog.com/fundamentals-versus-finite-elements-a-case-study-on-the-fokker-d-vii-biplane/

Sources of error across the workflow



Solution + discretisation errors require **verification**.
Modelling errors require **validation**.

ELEMENTS OF SOLID MECHANICS FOR THE FINITE ELEMENT METHOD

Assuming infinitesimal strains and linearity

Solid Mechanics – Linear Elasticity

- Solid mechanics is governed by **15 PDEs in 15 unknowns** (functions)
  - Three basic arguments: **Equilibrium**, **Compatibility**, **Constitutive**.
  - Deformable continua have **infinite degrees of freedom** (d.o.f):
    - 15 unknowns per material point: 3 **displacements**, 6 **strain** components, 6 **stress** components
- The determination of deformations, stress and strain states in any body subjected to, e.g., mechanical or thermal loads, and constraints, passes through solving this set of PDEs.
- Information on the structure's geometry, active loads, constraints, material properties, and so on, must be known.

Compatibility Equations

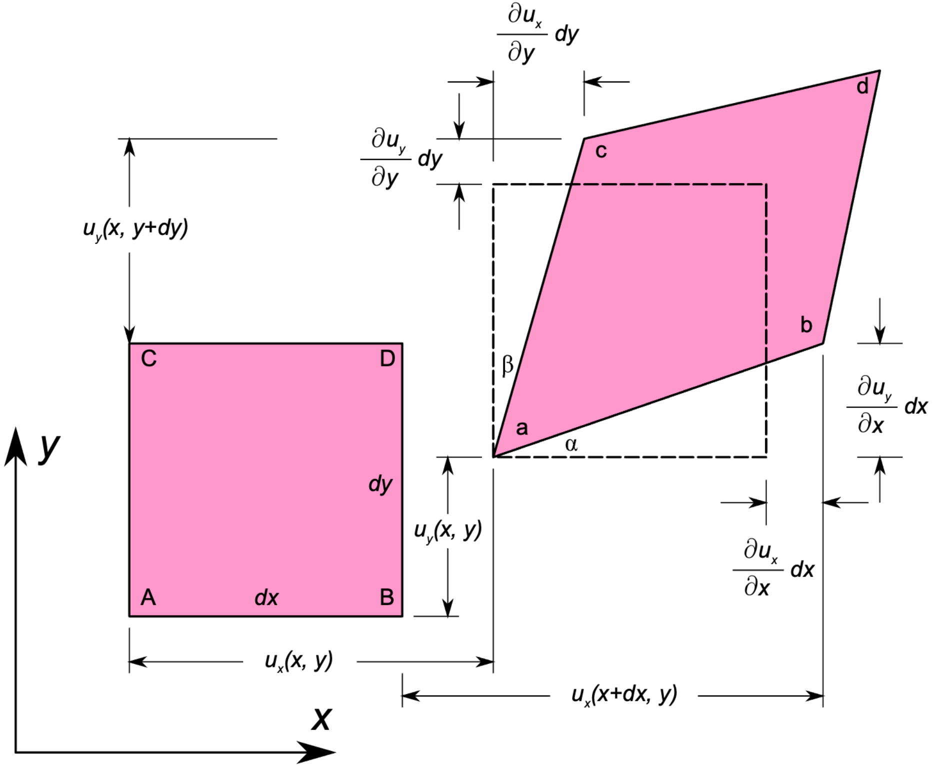
- These are the conditions that must be met by the displacement and strain components to ensure continuity (no material gaps or overlaps) and that the boundary constraints are respected.
- Any displacement field satisfying the compatibility equations defines a compatible structure configuration.

Compatibility Equations

Upon deformation, material points within the solid undergo displacements

$$u_x = u_x(x, y, z)$$
$$u_y = u_y(x, y, z)$$
$$u_z = u_z(x, y, z)$$

Leading to the development of strains.



2D_geometric_strain.png: Sanpazderivative work: Mircalla22, Public domain, via Wikimedia Commons

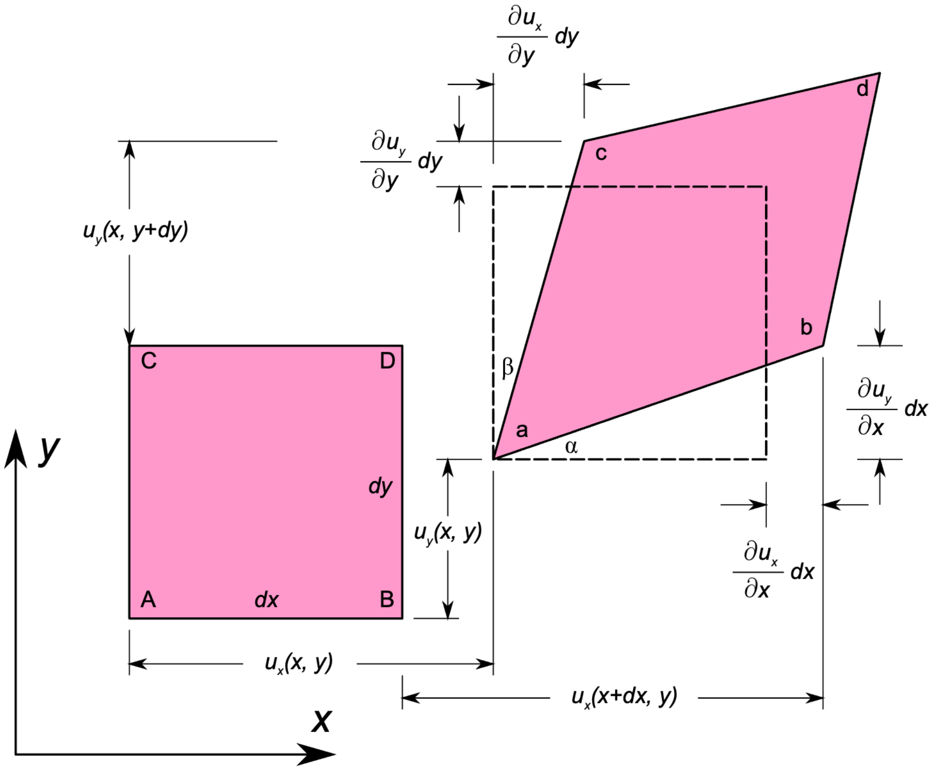
Compatibility Equations

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x},$$
$$\epsilon_{yy} = \frac{\partial u_y}{\partial y},$$
$$\epsilon_{zz} = \frac{\partial u_z}{\partial z},$$
$$\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x},$$
$$\epsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y},$$
$$\epsilon_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

+ boundary conditions defined on the constrained surface, S_c:

$$u = u_{bc}$$

(*Geometric* or *Essential BCs*)



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Compatibility Equations

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y};$$

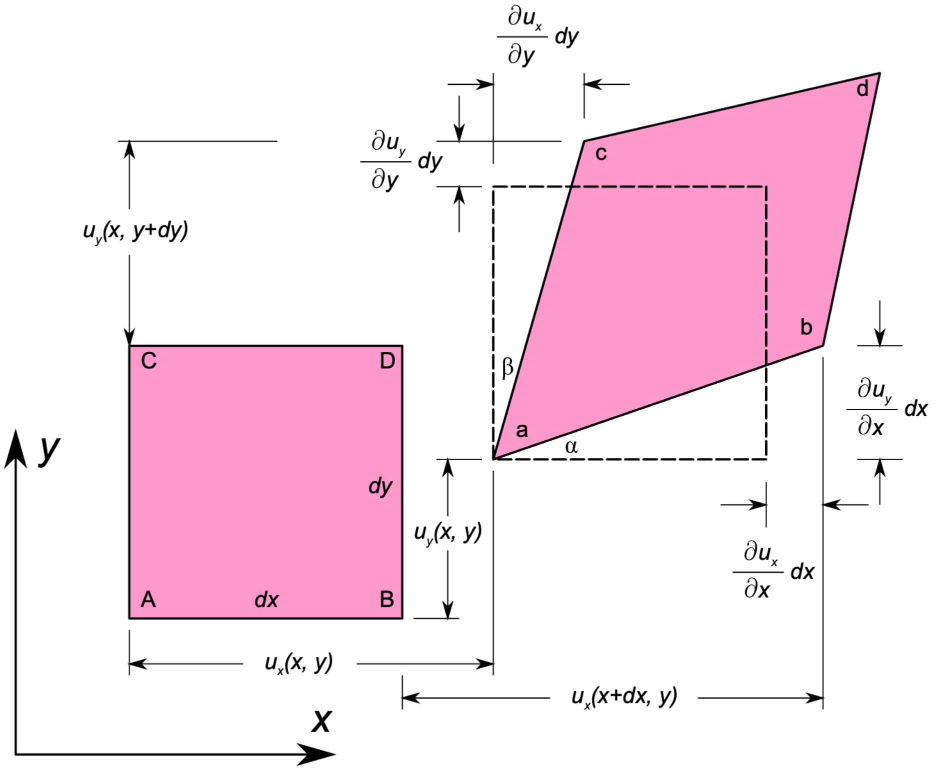
$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = \frac{\partial^2 \epsilon_{yz}}{\partial y \partial z};$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = \frac{\partial^2 \epsilon_{zx}}{\partial z \partial x};$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} = \frac{1}{2} \frac{\partial}{\partial x} \left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} \right);$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} = \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z} \right);$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} = \frac{1}{2} \frac{\partial}{\partial z} \left(\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z} \right).$$



2D_geometric_strain.png: Sanpazderivative work: Mircalla22, Public domain, via Wikimedia Commons

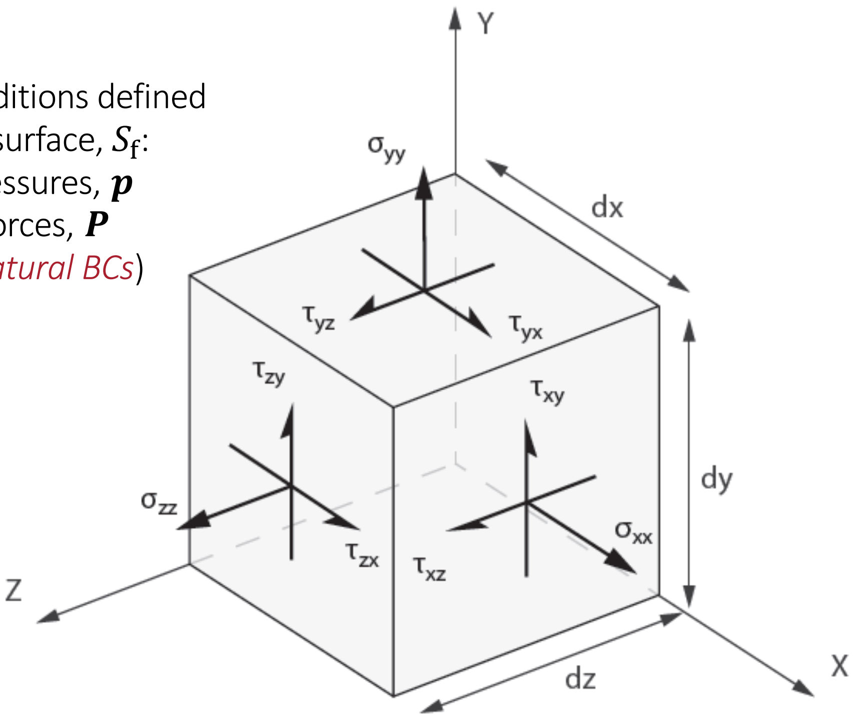
Equilibrium Equations

- These are the conditions that stresses must meet for the body to be in equilibrium under the action of active and boundary forces.
- Any generalized stress field satisfying the equilibrium equations defines a set of internal forces in equilibrium with external forces.

Equilibrium Equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

+ boundary conditions defined on the free surface, S_f:
applied pressures, $\boldsymbol{p}$
applied forces, $\boldsymbol{P}$
(*Force* or *Natural BCs*)



with b_x, b_y, b_z, being components of body forces per unit volume.

Constitutive Equation

- These are stress-strain relationships that describe the material's resistance to deformation under applied loads.
- For isotropic materials Hooke's law relates stresses and strains in terms of Young's modulus, E, and Poisson's ratio, ν.

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right)$$
$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right)$$
$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right)$$
$$\epsilon_{xy} = 2\frac{1+\nu}{E}\sigma_{xy}$$
$$\epsilon_{yz} = 2\frac{1+\nu}{E}\sigma_{yz}$$
$$\epsilon_{zx} = 2\frac{1+\nu}{E}\sigma_{zx}$$

Navier’s Equations

- Substituting the strain-displacement equations into the constitutive equations, and the resultant equations into the equilibrium equation, we obtain **Navier’s equations**.
- Strains and stresses are eliminated from the formulation, leaving the displacements as the only unknowns to be solved for. → **Displacement method** → **Basis for FEM formulation**.