Chapter 0 Formulae

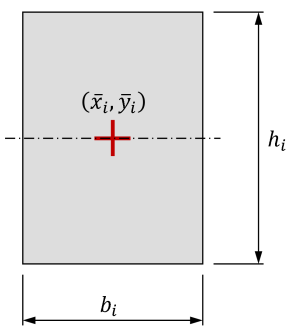
0.1 Solid Sections

0.1.1 Section properties

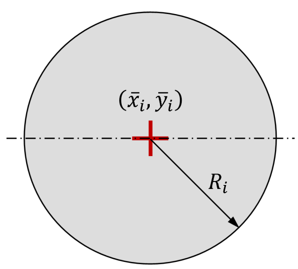
$$A = \int_A \mathrm{d}A = \sum_i (A_i)$$
$$Q_{XX} = \int_A Y \ \mathrm{d}A = \sum_i (A_i \ \bar{Y}_i)$$
$$Q_{YY} = \int_A X \ \mathrm{d}A = \sum_i (A_i \ \bar{X}_i)$$
$$\bar{X} = \frac{Q_{YY}}{A}$$
$$\bar{Y} = \frac{Q_{XX}}{A}$$

$$x = X - \bar{X}$$
$$y = Y - \bar{Y}$$

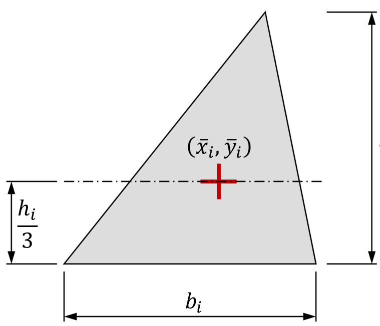
$$\bar{x}_i = \bar{X}_i - \bar{X}$$
$$\bar{y}_i = \bar{Y}_i - \bar{Y}$$



$$I_{\bar{x}_i\bar{x}_i} = \frac{b_i \ h_i{}^3}{12}$$



$$I_{\bar{x}_i\bar{x}_i} = \frac{\pi \ R_i{}^4}{4}$$



$$I_{\bar{x}_i\bar{x}_i} = \frac{b_i \ h_i{}^3}{36}$$

$$I_{xx} = \int_A y^2 \ \mathrm{d}A = \sum_i (I_{\bar{x}_i\bar{x}_i} + A_i \ \bar{y}_i{}^2)$$
$$I_{yy} = \int_A x^2 \ \mathrm{d}A = \sum_i (I_{\bar{y}_i\bar{y}_i} + A_i \ \bar{x}_i{}^2)$$
$$I_{xy} = \int_A x \ y \ \mathrm{d}A = \sum_i (I_{\bar{x}_i\bar{y}_i} + A_i \ \bar{x}_i \ \bar{y}_i)$$

0.1.2 Transformation of axes

For all equations in this section: $m = \cos\theta$ and $n = \sin\theta$.

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$x' = x \ \cos\theta + y \ \sin\theta$$
$$y' = y \ \cos\theta - x \ \sin\theta$$

$$\begin{Bmatrix} I_{x'x'} \\ I_{y'y'} \\ I_{x'y'} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2 \ m \ n \\ n^2 & m^2 & 2 \ m \ n \\ m \ n & -m \ n & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} I_{xx} \\ I_{yy} \\ I_{xy} \end{Bmatrix}$$

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \left(\frac{I_{xx} - I_{yy}}{2}\right) \cos 2\theta - (I_{xy}) \ \sin 2\theta$$
$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} - \left(\frac{I_{xx} - I_{yy}}{2}\right) \cos 2\theta + (I_{xy}) \ \sin 2\theta$$
$$I_{x'y'} = \left(\frac{I_{xx} - I_{yy}}{2}\right) \sin 2\theta + (I_{xy}) \ \cos 2\theta$$

$$\theta_\mathrm{p} = -\beta_\mathrm{p} = \frac{1}{2} \arctan\left(\frac{2 \ I_{xy}}{I_{yy} - I_{xx}}\right)$$
$$\beta_\mathrm{p} = -\theta_\mathrm{p} = \frac{1}{2} \arctan\left(\frac{2 \ I_{xy}}{I_{xx} - I_{yy}}\right)$$

Finding principal values from arbitrary orientation values:

$$I_{11} = \frac{I_{xx} + I_{yy}}{2} + \left(\frac{I_{xx} - I_{yy}}{2}\right) \cos 2\theta_{\mathrm{p}} - (I_{xy})\ \sin 2\theta_{\mathrm{p}}$$

$$I_{22} = \frac{I_{xx} + I_{yy}}{2} - \left(\frac{I_{xx} - I_{yy}}{2}\right) \cos 2\theta_{\mathrm{p}} + (I_{xy})\ \sin 2\theta_{\mathrm{p}}$$

$$I_{12} = 0$$

Finding arbitrary orientation values from principal values:

$$I_{xx} = \frac{I_{11} + I_{22}}{2} + \left(\frac{I_{11} - I_{22}}{2}\right) \cos 2\beta_{\mathrm{p}}$$

$$I_{yy} = \frac{I_{11} + I_{22}}{2} - \left(\frac{I_{11} - I_{22}}{2}\right) \cos 2\beta_{\mathrm{p}}$$

$$I_{xy} = \left(\frac{I_{11} - I_{22}}{2}\right) \sin 2\beta_{\mathrm{p}}$$

0.1.3 Mohr's circle

$$C = \frac{I_{xx} + I_{yy}}{2}$$

$$R_{\mathrm{Mohr}} = \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + {I_{xy}}^2}$$

$$I_{11} = C + R_{\mathrm{Mohr}}$$

$$I_{22} = C - R_{\mathrm{Mohr}}$$

$$\theta_{\mathrm{p}} = \frac{1}{2}\ \arctan\left(\frac{I_{xy}}{C - I_{xx}}\right)$$

0.1.4 Direct stresses

$$\sigma_z = -\left(\frac{M_y\ I_{xx} + M_x\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2}\right)\ x\ +\ \left(\frac{M_x\ I_{yy} + M_y\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2}\right)\ y$$

$$\alpha\ =\ \arctan\left(\frac{M_y\ I_{xx} + M_x\ I_{xy}}{M_x\ I_{yy} + M_y\ I_{xy}}\right)$$

0.1.5 Shear stresses

$$\tau_{(y_1)} = \frac{S_Y}{I_{xx}\ b_{(y_1)}} \int_{y_1}^{h} y\ \mathrm{d}A$$

0.2 Thin-walled Sections

$$A = \int_0^b t_{(s)} \; \mathrm{d}s = \sum_i \int_0^{b_i} {}^i t_{(s)} \; \mathrm{d}s$$

$$Q_{XX} = \int_0^b Y_{(s)} \; t_{(s)} \; \mathrm{d}s = \sum_i \int_0^{b_i} {}^i Y_{(s)} \; {}^i t_{(s)} \; \mathrm{d}s$$
$$Q_{YY} = \int_0^b X_{(s)} \; t_{(s)} \; \mathrm{d}s = \sum_i \int_0^{b_i} {}^i X_{(s)} \; {}^i t_{(s)} \; \mathrm{d}s$$

$$\bar{X} = \frac{Q_{YY}}{A}$$
$$\bar{Y} = \frac{Q_{XX}}{A}$$

$$x_{(s)} = X_{(s)} - \bar{X}$$
$$y_{(s)} = Y_{(s)} - \bar{Y}$$

$${}^i x_{(s)} = {}^i X_{(s)} - \bar{X}$$
$${}^i y_{(s)} = {}^i Y_{(s)} - \bar{Y}$$

$$I_{xx} = \int_0^b {y_{(s)}}^2 \; t_{(s)} \; \mathrm{d}s = \sum_i \int_0^{b_i} {}^i {y_{(s)}}^2 \; {}^i t_{(s)} \; \mathrm{d}s$$
$$I_{yy} = \int_0^b {x_{(s)}}^2 \; t_{(s)} \; \mathrm{d}s = \sum_i \int_0^{b_i} {}^i {x_{(s)}}^2 \; {}^i t_{(s)} \; \mathrm{d}s$$
$$I_{xy} = \int_0^b x_{(s)} \; y_{(s)} \; t_{(s)} \; \mathrm{d}s = \sum_i \int_0^{b_i} {}^i x_{(s)} \; {}^i y_{(s)} \; {}^i t_{(s)} \; \mathrm{d}s$$

0.3 Open Thin-walled Sections

0.3.1 Shear flow

$$q_{(s)} = \tau_{zs_{(s)}}\ t_{(s)}$$

$$C_x = \left(\frac{S_x\ I_{xx} - S_y\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2} \right) \qquad C_y = \left(\frac{S_y\ I_{yy} - S_x\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2} \right)$$

$$q_{(s)} = q_0\ -\ C_x \int_0^s\ x_{(s)}\ t_{(s)}\,\mathrm{d}s\ -\ C_y \int_0^s\ y_{(s)}\ t_{(s)}\,\mathrm{d}s$$

$${}^{i}q_{(s)} = q_{[i-1]}\ -\ C_x \int_0^s\ {}^{i}x_{(s)}\ {}^{i}t_{(s)}\,\mathrm{d}s\ -\ C_y \int_0^s\ {}^{i}y_{(s)}\ {}^{i}t_{(s)}\,\mathrm{d}s$$

0.3.2 Shear centre

$$S_Y\ e_X - S_X\ e_Y = \int_0^b\ q_{(s)}\ r_{(s)}\ \ \mathrm{d}s = \sum_i \int_0^{b_i}\ {}^{i}q_{(s)}\ {}^{i}r_{(s)}\ \ \mathrm{d}s$$

0.4 Closed Thin-walled Sections

0.4.1 Pure torsion

$$T = \bar{q}\ \oint r_{(s)}\,\mathrm{d}s \qquad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{4\ \bar{A}^2} \oint \frac{1}{G\ t_{(s)}}\,\mathrm{d}s$$

$$T = \bar{q}\ \left(2\ \bar{A}\right) \qquad \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2\ \bar{A}} \oint \frac{\bar{q}}{G\ t_{(s)}}\,\mathrm{d}s$$

0.4.2 Torsional constant

$$T = \frac{\mathrm{d}\theta}{\mathrm{d}z}\ G\ J$$

$$J = \frac{4\ \bar{A}^2}{\oint \frac{\mathrm{d}s}{t}}$$

0.4.3 Open shear flow

$$C_x = \left(\frac{S_x\ I_{xx} - S_y\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2} \right) \qquad C_y = \left(\frac{S_y\ I_{yy} - S_x\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2} \right)$$

$$q_{(s)}^{\text{open}} = q_0\ -\ C_x \int_0^s\ x_{(s)}\ t_{(s)}\ \mathrm{d}s\ -\ C_y \int_0^s\ y_{(s)}\ t_{(s)}\ \mathrm{d}s$$

$$^{i}q_{(s)}^{\text{open}} = q_{[i-1]}\ -\ C_x \int_0^s\ {}^{i}x_{(s)}\ {}^{i}t_{(s)}\,\mathrm{d}s\ -\ C_y \int_0^s\ {}^{i}y_{(s)}\ {}^{i}t_{(s)}\,\mathrm{d}s$$

0.4.4 Closed-cell constant

$$\bar{q} = -\frac{\oint q_{(s)}^{\text{open}}\,\mathrm{d}s}{\oint \mathrm{d}s}$$

0.4.5 Closed shear flow

$$q_{(s)}^{\text{closed}} = q_{(s)}^{\text{open}} + \bar{q}$$

0.4.6 Angle of twist

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2\bar{A}} \oint \frac{q_{(s)}^{\text{closed}}}{G\ t_{(s)}}\,\mathrm{d}s$$

0.4.7 Shear centre

$$S_Y\ e_X - S_X\ e_Y = \oint\ q_{(s)}^{\text{closed}}\ r_{(s)}\ \mathrm{d}s = \sum_i \int_0^{b_i}\ {}^{i}q_{(s)}^{\text{closed}}\ {}^{i}r_{(s)}\ \mathrm{d}s$$

0.5 Idealised Sections

0.5.1 Effective boom areas

Shear load S_X:

$$B_k = J_k + \sum_{\text{connected booms } l} \frac{b_{kl}\ t_{kl}}{6} \left(2 + \frac{x_l}{x_k} \right)$$

Shear load S_Y:

$$B_k = J_k + \sum_{\text{connected booms } l} \frac{b_{kl}\ t_{kl}}{6} \left(2 + \frac{y_l}{y_k} \right)$$

where:

- J_k is the cross-sectional area of any 'joint' (or local reinforcement) at boom k ;
- l indexes all neighbouring booms connected to boom k ;
- b_{kl} is the arc-length of the skin connecting k and l ;
- t_{kl} is the thickness of the skin connecting k and l .

0.5.2 Idealised shear flow

$$C_x = \left(\frac{S_x\ I_{xx} - S_y\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2} \right) \qquad C_y = \left(\frac{S_y\ I_{yy} - S_x\ I_{xy}}{I_{xx}\ I_{yy} - {I_{xy}}^2} \right)$$

$$q_i^{\text{open}} = -\ C_x \sum_k x_k\ B_k\ -\ C_y \sum_k y_k\ B_k$$

$$q_i^{\text{closed}} = q_i^{\text{open}} + \bar{q}$$

0.5.3 Multi-cell sections

Rate of twist of each cell $j = 1...n$:

$$\sum_{i \in j} \left[({q_i}^{\text{open}} + {}^*\bar{q}_i) \ \frac{b_i}{t_i} \right] = \left(2\ \bar{A}_j\ G\right) \frac{\mathrm{d}\theta}{\mathrm{d}z}$$

where:

$$^*\bar{q}_i = \begin{cases} \bar{q}_j & \text{if wall } i \text{ is exclusive to cell } j \\ \bar{q}_j - \bar{q}_{[j\pm1]} & \text{if wall } i \text{ is shared with cell } [j \pm 1] \end{cases}$$

Balance of torques about the origin of (X, Y):

$$T + S_Y\ e_X - S_X\ e_Y = \sum_{\text{all walls } i} ({q_i}^{\text{open}}\ b_i\ r_i) + \sum_{\text{all cells } j} \left(2\ \bar{A}_j\ \bar{q}_j\right)$$

where r_i is the orthogonal distance function $r_{(s)}$ for wall i with respect to the origin of (X, Y).

If $r_{(s)}$ is not constant within wall i then the 'subtended area' $\hat{A}_i$ can be used instead:

$$T + S_Y\ e_X - S_X\ e_Y = \sum_{\text{all walls } i} \left(2\ \hat{A}_i\ {q_i}^{\text{open}}\right) + \sum_{\text{all cells } j} \left(2\ \bar{A}_j\ \bar{q}_j\right)$$