CHAPTER

13 Deflection of Beams

In Chapters 9, 10 and 11 we investigated the *strength* of beams in terms of the stresses produced by the action of bending, shear and torsion, respectively. An associated problem is the determination of the deflections of beams caused by different loads for, in addition to strength, a beam must possess sufficient *stiffness* so that excessive deflections do not have an adverse effect on adjacent structural members. In many cases, maximum allowable deflections are specified by Codes of Practice in terms of the dimensions of the beam, particularly the span; typical values are quoted in Section 8.7. We also saw in Section 8.7 that beams may be designed using either elastic or plastic analysis. However, since beam deflections must always occur within the elastic limit of the material of a beam they are determined using elastic theory.

There are several different methods of obtaining deflections in beams, the choice depending upon the type of problem being solved. For example, the double integration method gives the complete shape of a beam whereas the moment-area method can only be used to determine the deflection at a particular beam section. The latter method, however, is also useful in the analysis of statically indeterminate beams.

Generally beam deflections are caused primarily by the bending action of applied loads. In some instances, however, where a beam's cross-sectional dimensions are not small compared with its length, deflections due to shear become significant and must be calculated. We shall consider beam deflections due to shear in addition to those produced by bending. We shall also include deflections due to unsymmetrical bending.

13.1 Differential equation of symmetrical bending

In Chapter 9 we developed an expression relating the curvature, $1/R$, of a beam to the applied bending moment, M, and flexural rigidity, EI, i.e.

$$\frac{1}{R}=\frac{M}{EI} \quad \text{(Eq. (9.11))}$$

For a beam of a given material and cross section, EI is constant so that the curvature is directly proportional to the bending moment. We have also shown that bending moments produced by shear loads vary along the length of a beam, which implies that the curvature of the beam also varies along its length; Eq. (9.11) therefore gives the curvature at a particular section of a beam.

Consider a beam having a vertical plane of symmetry and loaded such that at a section of the beam the deflection of the neutral plane, referred to arbitrary axes Oxy, is v and the slope of the tangent to the neutral plane at this section is dv/dx (Fig. 13.1). Also, if the applied loads produce a positive, i.e. sagging, bending moment at this section, then the upper surface of the beam is concave and the centre of curvature lies above the beam as shown. For the system of axes shown in Fig. 13.1, the sign convention usually adopted in mathematical theory gives a positive value for this curvature, i.e.

Structural and Stress Analysis. https://doi.org/10.1016/B978-0-08-102586-4.00013-5

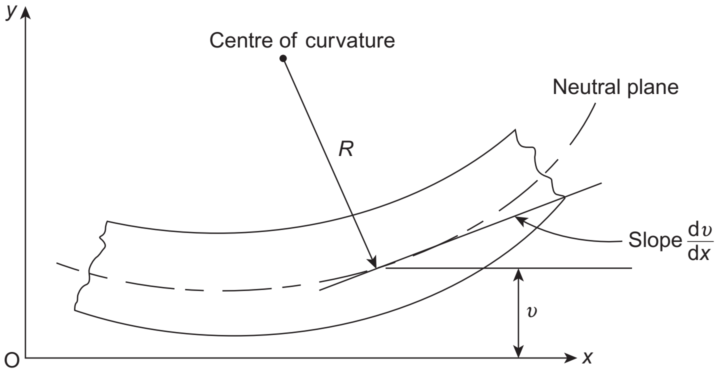

FIGURE 13.1

Deflection and curvature of a beam due to bending.

$$\frac{1}{R} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}} \tag{13.1}$$

For small deflections dv/dx is small so that $(dv/dx)^2$ is negligibly small compared with unity. Equation (13.1) then reduces to

$$\frac{1}{R} = \frac{d^2v}{dx^2} \tag{13.2}$$

whence, from Eq. (9.11)

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \tag{13.3}$$

Double integration of Eq. (13.3) then yields the equation of the deflection curve of the neutral plane of the beam.

In the majority of problems concerned with beam deflections the bending moment varies along the length of a beam and therefore M in Eq. (13.3) must be expressed as a function of x before integration can commence. Alternatively, it may be convenient in cases where the load is a known function of x to use the relationships of Eq. (3.8). Thus

$$\frac{d^3v}{dx^3} = -\frac{S}{EI} \tag{13.4}$$

$$\frac{d^4v}{dx^4} = -\frac{w}{EI} \tag{13.5}$$

We shall now illustrate the use of Eqs (13.3), (13.4) and (13.5) by considering some standard cases of beam deflection.

EXAMPLE 13.1

Determine the deflection curve and the deflection of the free end of the cantilever shown in Fig. 13.2(a); the flexural rigidity of the cantilever is EI.

The load W causes the cantilever to deflect such that its neutral plane takes up the curved shape shown in Fig. 13.2(b); the deflection at any section X is then v while that at its free end is v_{tip}. The axis system is chosen so that the origin coincides with the built-in end where the deflection is clearly zero.

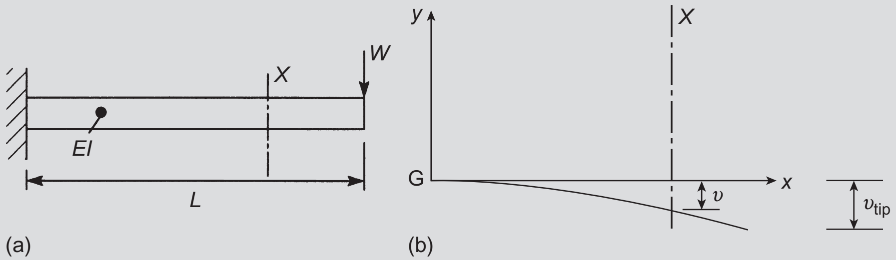

FIGURE 13.2

Deflection of a cantilever beam carrying a concentrated load at its free end (Ex. 13.1).

The bending moment, M, at the section X is, from Fig. 13.2(a)

$$M = -W(L-x) \quad \text{(i.e. hogging)} \tag{i}$$

Substituting for M in Eq. (13.3) we obtain

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\frac{W}{EI}(L-x)$$

or in more convenient form

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -W(L-x) \tag{ii}$$

Integrating Eq. (ii) with respect to x gives

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = -W\left(Lx - \frac{x^2}{2}\right) + C_1$$

where C_1 is a constant of integration which is obtained from the boundary condition that $\mathrm{d}v/\mathrm{d}x = 0$ at the built-in end where $x = 0$. Hence $C_1 = 0$ and

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = -W\left(Lx - \frac{x^2}{2}\right) \tag{iii}$$

Integrating Eq. (iii) we obtain

$$EIv = -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

in which C_2 is again a constant of integration. At the built-in end $v = 0$ when $x = 0$ so that $C_2 = 0$. Hence the equation of the deflection curve of the cantilever is

$$v = -\frac{W}{6EI}(3Lx^2 - x^3) \tag{iv}$$

The deflection, v_{tip}, at the free end is obtained by setting $x = L$ in Eq. (iv). Thus

$$v_{tip} = -\frac{WL^3}{3EI} \tag{v}$$

and is clearly negative and downwards.

EXAMPLE 13.2

Determine the deflection curve and the deflection of the free end of the cantilever shown in Fig. 13.3(a).

The bending moment, M, at any section X is given by

$$M = -\frac{w}{2}(L-x)^2 \tag{i}$$

Substituting for M in Eq. (13.3) and rearranging we have

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\frac{w}{2}(L-x)^2 = -\frac{w}{2}(L^2 - 2Lx + x^2) \tag{ii}$$

Integration of Eq. (ii) yields

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{w}{2}\left(L^2 x - Lx^2 + \frac{x^3}{3}\right) + C_1$$

When $x=0$ at the built-in end, $\mathrm{d}v/\mathrm{d}x = 0$ so that $C_1 = 0$ and

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{w}{2}\left(L^2 x - Lx^2 + \frac{x^3}{3}\right) \tag{iii}$$

Integrating Eq. (iii) we have

$$EIv = -\frac{w}{2}\left(L^2\frac{x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) + C_2$$

and since $v=0$ when $x=0$, $C_2 = 0$. The deflection curve of the beam therefore has the equation

$$v = -\frac{w}{24EI}(6L^2x^2 - 4Lx^3 + x^4) \tag{iv}$$

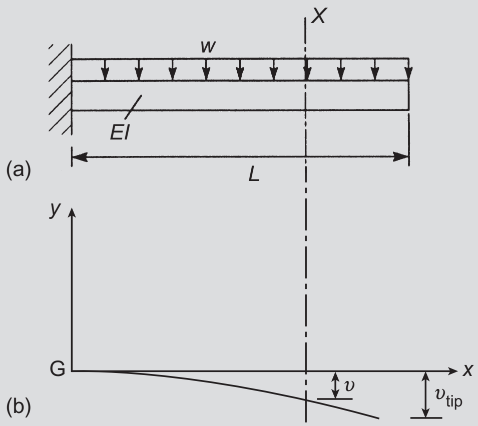

FIGURE 13.3

Deflection of a cantilever beam carrying a uniformly distributed load.

and the deflection at the free end where $x = L$ is

$$v_{\text{tip}} = -\frac{wL^4}{8EI} \tag{v}$$

which is again negative and downwards. The applied loading in this case may be easily expressed in mathematical form so that a solution can be obtained using Eq. (13.5), i.e.

$$\frac{d^4 v}{dx^4} = -\frac{w}{EI} \tag{vi}$$

in which $w = $ constant. Integrating Eq. (vi) we obtain

$$EI\frac{d^3 v}{dx^3} = -wx + C_1$$

We note from Eq. (13.4) that

$$\frac{d^3 v}{dx^3} = -\frac{S}{EI} \quad (\text{i.e.} - S = -wx + C_1)$$

When $x = 0$, $S = -wL$ so that

$$C_1 = wL$$

Alternatively we could have determined C_1 from the boundary condition that when $x = L$, $S = 0$. Hence

$$EI\frac{d^3 v}{dx^3} = -w(x - L) \tag{vii}$$

Integrating Eq. (vii) gives

$$EI\frac{d^2 v}{dx^2} = -w\left(\frac{x^2}{2} - Lx\right) + C_2$$

From Eq. (13.3) we see that

$$\frac{d^2 v}{dx^2} = \frac{M}{EI}$$

and when $x = 0$, $M = -wL^2/2$ (or when $x = L$, $M = 0$) so that

$$C_2 = -\frac{wL^2}{2}$$

and

$$EI\frac{d^2 v}{dx^2} = -\frac{w}{2}(x^2 - 2Lx + L^2)$$

which is identical to Eq. (ii). The solution then proceeds as before.

EXAMPLE 13.3

The cantilever beam shown in Fig. 13.4(a) carries a uniformly distributed load over part of its span. Calculate the deflection of the free end.

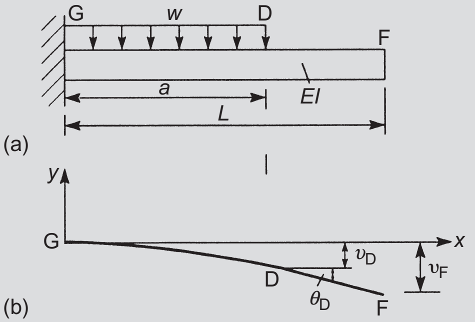

FIGURE 13.4

Cantilever beam of Ex. 13.3.

If we assume that the cantilever is weightless then the bending moment at all sections between D and F is zero. It follows that the length DF of the beam remains straight. The deflection at D can be deduced from Eq. (v) of Ex. 13.2 and is

$$v_D = -\frac{wa^4}{8EI}$$

Similarly the slope of the cantilever at D is found by substituting $x = a$ and $L = a$ in Eq. (iii) of Ex. 13.2; thus

$$\left(\frac{dv}{dx}\right)_D = \theta_D = -\frac{wa^3}{6EI}$$

The deflection, v_F, at the free end of the cantilever is then given by

$$v_F = -\frac{wa^4}{8EI} - (L - a)\frac{wa^3}{6EI}$$

which simplifies to

$$v_F = -\frac{wa^3}{24EI}(4L - a)$$

EXAMPLE 13.4

Determine the deflection curve and the mid-span deflection of the simply supported beam shown in Fig. 13.5(a).

The support reactions are each $wL/2$ and the bending moment, M, at any section X, a distance x from the left-hand support is

$$M = \frac{wL}{2}x - \frac{wx^2}{2} \qquad \text{(i)}$$

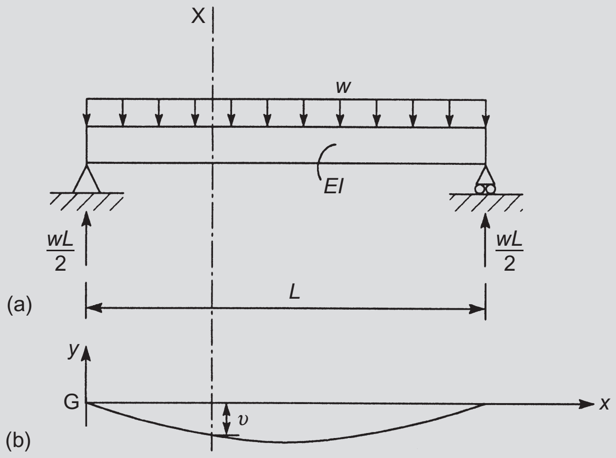

FIGURE 13.5

Deflection of a simply supported beam carrying a uniformly distributed load (Ex. 13.4).

Substituting for M in Eq. (13.3) we obtain

$$EI\frac{d^2\upsilon}{dx^2} = \frac{w}{2}(Lx - x^2) \tag{ii}$$

Integrating we have

$$EI\frac{d\upsilon}{dx} = \frac{w}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1$$

From symmetry it is clear that at the mid-span section the gradient $d\upsilon/dx = 0$. Hence

$$0 = \frac{w}{2}\left(\frac{L^3}{8} - \frac{L^3}{24}\right) + C_1$$

whence

$$C_1 = -\frac{wL^3}{24}$$

Therefore

$$EI\frac{d\upsilon}{dx} = \frac{w}{24}(6Lx^2 - 4x^3 - L^3) \tag{iii}$$

Integrating again gives

$$EI\upsilon = \frac{w}{24}(2Lx^3 - x^4 - L^3x) + C_2$$

Since $\upsilon = 0$ when $x = 0$ (or since $\upsilon = 0$ when $x = L$) it follows that $C_2 = 0$ and the deflected shape of the beam has the equation

$$\upsilon = \frac{w}{24EI}(2Lx^3 - x^4 - L^3x) \tag{iv}$$

The maximum deflection occurs at mid-span where $x = L/2$ and is

$$v_{\text{mid-span}} = -\frac{5wL^4}{384EI} \tag{v}$$

So far the constants of integration were determined immediately they arose. However, in some cases a relevant boundary condition, say a value of gradient, is not obtainable. The method is then to carry the unknown constant through the succeeding integration and use known values of deflection at two sections of the beam. Thus in the previous example Eq. (ii) is integrated twice to obtain

$$EIv = \frac{w}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_1 x + C_2$$

The relevant boundary conditions are $v = 0$ at $x = 0$ and $x = L$. The first of these gives $C_2 = 0$ while from the second we have $C_1 = -wL^3/24$. Thus the equation of the deflected shape of the beam is

$$v = \frac{w}{24EI}(2Lx^3 - x^4 - L^3 x)$$

as before.

EXAMPLE 13.5

Figure 13.6(a) shows a simply supported beam carrying a concentrated load W at mid-span. Determine the deflection curve of the beam and the maximum deflection.

The support reactions are each $W/2$ and the bending moment M at a section X a distance x ($\leq L/2$) from the left-hand support is

$$M = \frac{W}{2}x \tag{i}$$

From Eq. (13.3) we have

$$EI\frac{d^2v}{dx^2} = \frac{W}{2}x \tag{ii}$$

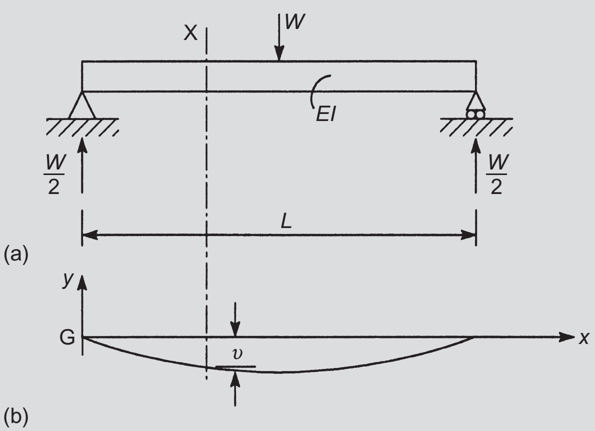

FIGURE 13.6

Deflection of a simply supported beam carrying a concentrated load at mid-span (Ex. 13.5).

Integrating we obtain

$$EI\frac{dv}{dx} = \frac{W}{2}\frac{x^2}{2} + C_1$$

From symmetry the slope of the beam is zero at mid-span where $x = L/2$. Thus $C_1 = -WL^2/16$ and

$$EI\frac{dv}{dx} = \frac{W}{16}(4x^2 - L^2) \tag{iii}$$

Integrating Eq. (iii) we have

$$EIv = \frac{W}{16}\left(\frac{4x^3}{3} - L^2 x\right) + C_2$$

and when $x = 0$, $v = 0$ so that $C_2 = 0$. The equation of the deflection curve is, therefore

$$v = \frac{W}{48EI}(4x^3 - 3L^2 x) \tag{iv}$$

The maximum deflection occurs at mid-span and is

$$v_{\text{mid-span}} = -\frac{WL^3}{48EI} \tag{v}$$

Note that in this problem we could not use the boundary condition that $v = 0$ at $x = L$ to determine C_2 since Eq. (i) applies only for $0 \leq x \leq L/2$; it follows that Eqs (iii) and (iv) for slope and deflection apply only for $0 \leq x \leq L/2$ although the deflection curve is clearly symmetrical about mid-span.

EXAMPLE 13.6

The simply supported beam shown in Fig. 13.7(a) carries a concentrated load W at a distance a from the left-hand support. Determine the deflected shape of the beam, the deflection under the load and the maximum deflection.

Considering the moment and force equilibrium of the beam we have

$$R_A = \frac{W}{L}(L - a) \qquad R_B = \frac{Wa}{L}$$

At a section X_1, a distance x from the left-hand support where $x \leq a$, the bending moment is

$$M = R_A x \tag{i}$$

At the section X_2, where $x \geq a$

$$M = R_A x - W(x - a) \tag{ii}$$

Substituting both expressions for M in turn in Eq. (13.3) we obtain

$$EI\frac{d^2v}{dx^2} = R_A x \quad (x \leq a) \tag{iii}$$

and

$$EI\frac{d^2v}{dx^2} = R_A x - W(x - a) \quad (x \geq a) \tag{iv}$$

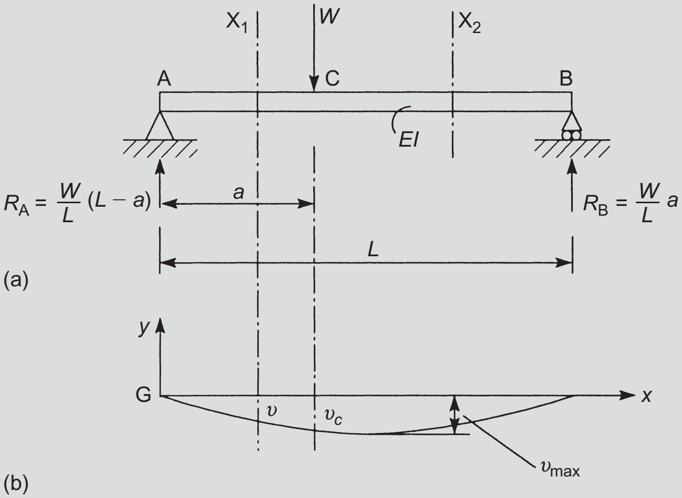

FIGURE 13.7

Deflection of a simply supported beam carrying a concentrated load not at mid-span (Ex. 13.6).

Integrating Eqs (iii) and (iv) we obtain

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = R_A\frac{x^2}{2} + C_1 \quad (x \le a) \tag{v}$$

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = R_A\frac{x^2}{2} - W\left(\frac{x^2}{2} - ax\right) + C'_1 \quad (x \ge a) \tag{vi}$$

and

$$EIv = R_A\frac{x^3}{6} + C_1 x + C_2 \quad (x \le a) \tag{vii}$$

$$EIv = R_A\frac{x^3}{6} - W\left(\frac{x^3}{6} - \frac{ax^2}{2}\right) + C'_1 x + C'_2 x \quad (x \ge a) \tag{viii}$$

in which C_1, C'_1, C_2, C'_2 are arbitrary constants. In using the boundary conditions to determine these constants, it must be remembered that Eqs (v) and (vii) apply only for $0 \le x \le a$ and Eqs (vi) and (viii) apply only for $a \le x \le L$. At the left-hand support $v = 0$ when $x = 0$, therefore, from Eq. (vii), $C_2 = 0$. It is not possible to determine C_1, C'_1 and C'_2 directly since the application of further known boundary conditions does not isolate any of these constants. However, since $v = 0$ when $x = L$ we have, from Eq. (viii)

$$0 = R_A\frac{L^3}{6} - W\left(\frac{L^3}{6} - \frac{aL^2}{2}\right) + C'_1 L + C'_2$$

which, after substituting $R_A = W(L - a)/L$, simplifies to

$$0 = \frac{WaL^2}{3} + C'_1 L + C'_2 \tag{ix}$$

Additional equations are obtained by considering the continuity which exists at the point of application of the load; at this section Eqs (v)–(viii) apply. Thus, from Eqs (v) and (vi)

$$R_A\frac{a^2}{2}+C_1=R_A\frac{a^2}{2}-W\left(\frac{a^2}{2}-a^2\right)+C'_1$$

which gives

$$C_1=\frac{Wa^2}{2}+C'_1 \tag{x}$$

Now equating values of deflection at $x=a$ we have, from Eqs (vii) and (viii)

$$R_A\frac{a^3}{6}+C_1a=R_A\frac{a^3}{6}-W\left(\frac{a^3}{6}-\frac{a^3}{2}\right)+C'_1a+C'_2$$

which yields

$$C_1a=\frac{Wa^3}{3}+C'_1a+C'_2 \tag{xi}$$

Solution of the simultaneous Eqs (ix), (x) and (xi) gives

$$C_1 \quad =-\frac{Wa}{6L}(a-2L)(a-L)$$

$$C'_1 \quad =-\frac{Wa}{6L}(a^2+2L^2)$$

$$C'_2 \quad =\frac{Wa^3}{6}$$

Equations (v)–(vii) then become respectively

$$EI\frac{dv}{dx}=-\frac{W(a-L)}{6L}\left[3x^2+a(a-2L)\right] \quad (x\le a) \tag{xii}$$

$$EI\frac{dv}{dx}=-\frac{Wa}{6L}(3x^2-6Lx+a^2+2L^2) \quad (x\ge a) \tag{xiii}$$

$$EIv=-\frac{W(a-L)}{6L}\left[x^3+a(a-2L)x\right] \quad (x\le a) \tag{xiv}$$

$$EIv=-\frac{Wa}{6L}\left[x^2-3Lx^2+(a^2+2L^2)x-a^2L\right] \quad (x\ge a) \tag{xv}$$

The deflection of the beam under the load is obtained by putting $x=a$ into either of Eq. (xiv) or (xv). Thus

$$v_C=-\frac{Wa^2(a-L)^2}{3EIL} \tag{xvi}$$

This is not, however, the maximum deflection of the beam. This will occur, if $a<L/2$, at some section between C and B. Its position may be found by equating dv/dx in Eq. (xiii) to zero. Hence

$$0=3x^2-6Lx+a^2+2L^2 \tag{xvii}$$

The solution of Eq. (xvii) is then substituted in Eq. (xv) and the maximum deflection follows.

For a central concentrated load $a=L/2$ and

$$v_C=-\frac{WL^3}{48EI}$$

as before.

EXAMPLE 13.7

Determine the deflection curve of the beam AB shown in Fig. 13.8 when it carries a distributed load that varies linearly in intensity from zero at the left-hand support to w_0 at the right-hand support.

To find the support reactions we first take moments about B. Thus

$$R_A L = \frac{1}{2} w_0 L \frac{L}{3}$$

which gives

$$R_A = \frac{w_0 L}{6}$$

Resolution of vertical forces then gives

$$R_B = \frac{w_0 L}{3}$$

The bending moment, M, at any section X, a distance x from A is

$$M = R_A x - \frac{1}{2}\left(w_0 \frac{x}{L}\right) x \frac{x}{3}$$

or

$$M = \frac{w_0}{6L}(L^2 x - x^3) \tag{i}$$

Substituting for M in Eq. (13.3) we obtain

$$EI \frac{d^2 v}{dx^2} = \frac{w_0}{6L}(L^2 x - x^3) \tag{ii}$$

which, when integrated, becomes

$$EI \frac{dv}{dx} = \frac{w_0}{6L}\left(L^2 \frac{x^2}{2} - \frac{x^4}{4}\right) + C_1 \tag{iii}$$

Integrating Eq. (iii) we have

$$EIv = \frac{w_0}{6L}\left(L^2 \frac{x^3}{6} - \frac{x^5}{20}\right) + C_1 x + C_2 \tag{iv}$$

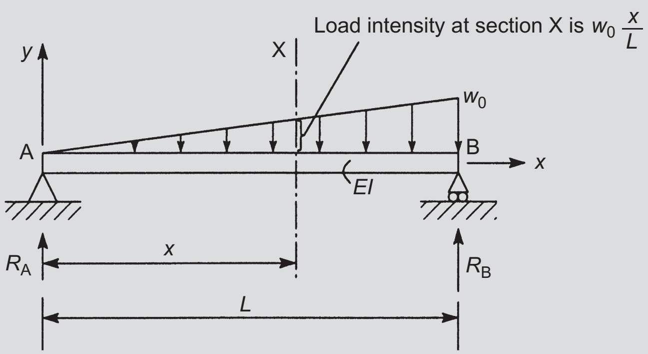

FIGURE 13.8

Deflection of a simply supported beam carrying a triangularly distributed load.

The deflection $v = 0$ at $x = 0$ and $x = L$. From the first of these conditions we obtain $C_2 = 0$, while from the second

$$0 = \frac{w_0}{6L}\left(\frac{L^5}{6} - \frac{L^5}{20}\right) + C_1 L$$

which gives

$$C_1 = -\frac{7w_0 L^4}{360}$$

The deflection curve then has the equation

$$v = -\frac{w_0}{360EIL}(3x^5 - 10L^2x^3 + 7L^4x) \tag{v}$$

An alternative method of solution is to use Eq. (13.5) and express the applied load in mathematical form. Thus

$$EI\frac{\mathrm{d}^4v}{\mathrm{d}x^4} = -w = -w_0\frac{x}{L} \tag{vi}$$

Integrating we obtain

$$EI\frac{\mathrm{d}^3v}{\mathrm{d}x^3} = -w_0\frac{x^2}{2L} + C_3$$

When $x = 0$ we see from Eq. (13.4) that

$$EI\frac{\mathrm{d}^3v}{\mathrm{d}x^3} = R_A = \frac{w_0 L}{6}$$

Hence

$$C_3 = \frac{w_0 L}{6}$$

and

$$EI\frac{\mathrm{d}^3v}{\mathrm{d}x^3} = -w_0\frac{x^2}{2L} + \frac{w_0 L}{6} \tag{vii}$$

Integrating Eq. (vii) we have

$$EI\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = -\frac{w_0x^3}{6L} + \frac{w_0 L}{6}x + C_4$$

Since the bending moment is zero at the supports we have

$$EI\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = 0 \quad \text{when } x = 0$$

Hence $C_4 = 0$ and

$$EI\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = -\frac{w_0}{6L}(x^3 - L^2x)$$

as before.

13.2 Singularity functions

A comparison of Exs 13.5 and 13.6 shows that the double integration method becomes extremely lengthy when even relatively small complications such as the lack of symmetry due to an offset load are introduced. Again the addition of a second concentrated load on the beam of Ex. 13.6 would result in a total of six equations for slope and deflection producing six arbitrary constants. Clearly the computation involved in determining these constants would be tedious, even though a simply supported beam carrying two concentrated loads is a comparatively simple practical case. An alternative approach is to introduce so-called *singularity* or *half-range* functions. Such functions were first applied to beam deflection problems by Macauley in 1919 and hence the method is frequently known as *Macauley's method.*

We now introduce a quantity $[x-a]$ and define it to be zero if $(x-a)<0$, i.e. $x<a$, and to be simply $(x-a)$ if $x>a$. The quantity $[x-a]$ is known as a singularity or half-range function and is defined to have a value only when the argument is positive in which case the square brackets behave in an identical manner to ordinary parentheses. Thus in Ex. 13.6 the bending moment at a section of the beam furthest from the origin for x may be written as

$$M = R_{\mathrm{A}}x - W[x-a]$$

This expression applies to both the regions AC and CB since $W[x-a]$ disappears for $x<a$. Equations (iii) and (iv) in Ex. 13.6 then become the single equation

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = R_{\mathrm{A}}x - W[x-a]$$

which on integration yields

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = R_{\mathrm{A}}\frac{x^2}{2} - \frac{W}{2}[x-a]^2 + C_1$$

and

$$EIv = R_{\mathrm{A}}\frac{x^3}{6} - \frac{W}{6}[x-a]^3 + C_1x + C_2$$

Note that the square brackets *must be retained* during the integration. The arbitrary constants C_1 and C_2 are found using the boundary conditions that $v=0$ when $x=0$ and $x=L$. From the first of these and remembering that $[x-a]^3$ is zero for $x<a$, we have $C_2=0$. From the second we have

$$0 = R_{\mathrm{A}}\frac{L^3}{6} - \frac{W}{6}[L-a]^3 + C_1L$$

in which $R_{\mathrm{A}} = W(L-a)/L$.

Substituting for R_{A} gives

$$C_1 = -\frac{Wa(L-a)}{6L}(2L-a)$$

Then

$$EIv = -\frac{W}{6L}\{-(L-a)x^3 + L[x-a]^3 + a(L-a)(2L-a)x\}$$

The deflection of the beam under the load is then

$$v_C = -\frac{Wa^2(L-a)^2}{3EIL}$$

as before.

EXAMPLE 13.8

Determine the position and magnitude of the maximum upward and downward deflections of the beam shown in Fig. 13.9.

A consideration of the overall equilibrium of the beam gives the support reactions; thus

$$R_A = \frac{3}{4}W\ (\text{upward}) \quad R_F = \frac{3}{4}W\ (\text{downward})$$

Using the method of singularity functions and taking the origin of axes at the left-hand support, we write down an expression for the bending moment, M, at any section X between D and F, *the region of the beam furthest from the origin.* Thus

$$M = R_A x - W[x-a] - W[x-2a] + 2W[x-3a] \tag{i}$$

Substituting for M in Eq. (13.3) we have

$$EI\frac{d^2v}{dx^2} = \frac{3}{4}Wx - W[x-a] - W[x-2a] + 2W[x-3a] \tag{ii}$$

Integrating Eq. (ii) and retaining the square brackets we obtain

$$EI\frac{dv}{dx} = \frac{3}{8}Wx^2 - \frac{W}{2}[x-a]^2 - \frac{W}{2}[x-2a]^2 + W[x-3a]^2 + C_1 \tag{iii}$$

and

$$EIv = \frac{1}{8}Wx^3 - \frac{W}{6}[x-a]^3 - \frac{W}{6}[x-2a]^3 + \frac{W}{3}[x-3a]^3 + C_1x + C_2 \tag{iv}$$

in which C_1 and C_2 are arbitrary constants. When $x=0$ (at A), $v=0$ and hence $C_2=0$. Note that the second, third and fourth terms on the right-hand side of Eq. (iv) disappear for $x<a$. Also $v=0$ at $x=4a$ (F) so that, from Eq. (iv), we have

$$0 = \frac{W}{8}64a^3 - \frac{W}{6}27a^3 - \frac{W}{6}8a^3 + \frac{W}{3}a^3 + 4aC_1$$

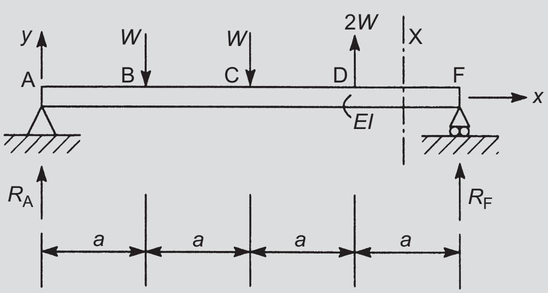

FIGURE 13.9

Macauley's method for the deflection of a simply supported beam (Ex. 13.8).

which gives

$$C_1 = -\frac{5}{8}Wa^2$$

Equations (iii) and (iv) now become

$$EI\frac{dv}{dx} = \frac{3}{8}Wx^2 - \frac{W}{2}[x-a]^2 - \frac{W}{2}[x-2a]^2 + W[x-3a]^2 - \frac{5}{8}Wa^2 \tag{v}$$

and

$$EIv = \frac{1}{8}Wx^3 - \frac{W}{6}[x-a]^3 - \frac{W}{6}[x-2a]^3 + \frac{W}{3}[x-3a]^3 - \frac{5}{8}Wa^2x \tag{vi}$$

respectively.

To determine the maximum upward and downward deflections we need to know in which bays $dv/dx = 0$ and thereby which terms in Eq. (v) disappear when the exact positions are being located. One method is to select a bay and determine the sign of the slope of the beam at the extremities of the bay. A change of sign will indicate that the slope is zero within the bay.

By inspection of Fig. 13.9 it seems likely that the maximum downward deflection will occur in BC. At B, using Eq. (v)

$$EI\frac{dv}{dx} = \frac{3}{8}Wa^2 - \frac{5}{8}Wa^2$$

which is clearly negative. At C

$$EI\frac{dv}{dx} = \frac{3}{8}W4a^2 - \frac{W}{2}a^2 - \frac{5}{8}Wa^2$$

which is positive. Therefore, the maximum downward deflection does occur in BC and its exact position is located by equating dv/dx to zero for any section in BC. Thus, from Eq. (v)

$$0 = \frac{3}{8}Wx^2 - \frac{W}{2}[x-a]^2 - \frac{5}{8}Wa^2$$

or, simplifying,

$$0 = x^2 - 8ax + 9a^2 \tag{vii}$$

Solution of Eq. (vii) gives

$$x = 1.35a$$

so that the maximum downward deflection is, from Eq. (vi)

$$EIv = \frac{1}{8}W(1.35a)^3 - \frac{W}{6}(0.35a)^3 - \frac{5}{8}Wa^2(1.35a)$$

i.e.

$$v_{\max}\,(\text{downward}) = -\frac{0.54Wa^3}{EI}$$

In a similar manner it can be shown that the maximum upward deflection lies between D and F at $x = 3.42a$ and that its magnitude is

$$v_{\max}\ (\text{upward}) = \frac{0.04Wa^3}{EI}$$

An alternative method of determining the position of maximum deflection is to select a possible bay, set $dv/dx = 0$ for that bay and solve the resulting equation in x. If the solution gives a value of x that lies within the bay, then the selection is correct, otherwise the procedure must be repeated for a second and possibly a third and a fourth bay. This method is quicker than the former if the correct bay is selected initially; if not, the equation corresponding to each selected bay must be completely solved, a procedure clearly longer than determining the sign of the slope at the extremities of the bay.

EXAMPLE 13.9

Determine the position and magnitude of the maximum deflection in the beam of Fig. 13.10.

Following the method of Ex. 13.8 we determine the support reactions and find the bending moment, M, at any section X in the bay furthest from the origin of the axes. Thus

$$M = R_A x - w\frac{L}{4}\left[x - \frac{5L}{8}\right] \tag{i}$$

Examining Eq. (i) we see that the singularity function $[x - 5L/8]$ does not become zero until $x \leq 5L/8$ although Eq. (i) is only valid for $x \geq 3L/4$. To obviate this difficulty we extend the distributed load to the support D while simultaneously restoring the status quo by applying an upward distributed load of the same intensity and length as the additional load (Fig. 13.11).

At the section X, a distance x from A, the bending moment is now given by

$$M = R_A x - \frac{w}{2}\left[x - \frac{L}{2}\right]^2 + \frac{w}{2}\left[x - \frac{3L}{4}\right]^2 \tag{ii}$$

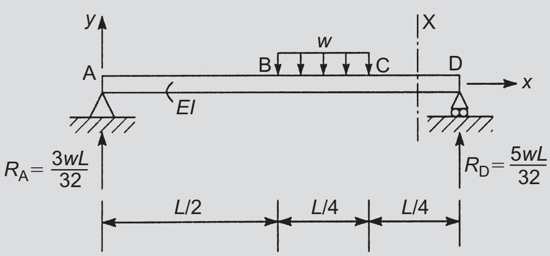

FIGURE 13.10

Deflection of a beam carrying a part span uniformly distributed load (Ex. 13.9).

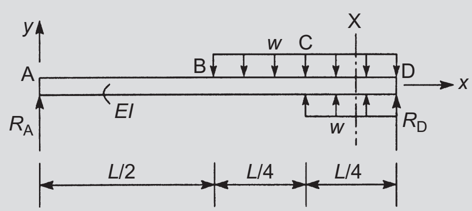

FIGURE 13.11

Method of solution for a part span uniformly distributed load.

Equation (ii) is now valid for all sections of the beam if the singularity functions are discarded as they become zero. Substituting Eq. (ii) into Eq. (13.3) we obtain

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{3}{32}wLx - \frac{w}{2}\left[x - \frac{L}{2}\right]^2 + \frac{w}{2}\left[x - \frac{3L}{4}\right]^2 \tag{iii}$$

Integrating Eq. (iii) gives

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3}{64}wLx^2 - \frac{w}{6}\left[x - \frac{L}{2}\right]^3 + \frac{w}{6}\left[x - \frac{3L}{4}\right]^3 + C_1 \tag{iv}$$

$$EIv = \frac{wLx^3}{64} - \frac{w}{24}\left[x - \frac{L}{2}\right]^4 + \frac{w}{24}\left[x - \frac{3L}{4}\right]^4 + C_1 x + C_2 \tag{v}$$

where C_1 and C_2 are arbitrary constants. The required boundary conditions are $v = 0$ when $x = 0$ and $x = L$. From the first of these we obtain $C_2 = 0$ while the second gives

$$0 = \frac{wL^4}{64} - \frac{w}{24}\left(\frac{L}{2}\right)^4 + \frac{w}{24}\left(\frac{L}{4}\right)^4 + C_1 L$$

from which

$$C_1 = -\frac{27wL^3}{2048}$$

Equations (iv) and (v) then become

$$EI\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3}{64}wLx^2 - \frac{w}{6}\left[x - \frac{L}{2}\right]^3 + \frac{w}{6}\left[x - \frac{3L}{4}\right]^3 - \frac{27wL^3}{2048} \tag{vi}$$

and

$$EIv = \frac{wLx^3}{64} - \frac{w}{24}\left[x - \frac{L}{2}\right]^4 + \frac{w}{24}\left[x - \frac{3L}{4}\right]^4 - \frac{27wL^3}{2048}x \tag{vii}$$

In this problem, the maximum deflection clearly occurs in the region BC of the beam. Thus equating the slope to zero for BC we have

$$0 = \frac{3}{64}wLx^2 - \frac{w}{6}\left[x - \frac{L}{2}\right]^3 - \frac{27wL^3}{2048}$$

which simplifies to

$$x^3 - 1.78Lx^2 + 0.75xL^2 - 0.046L^3 = 0 \tag{viii}$$

Solving Eq. (viii) by trial and error, we see that the slope is zero at $x \simeq 0.6L$. Hence from Eq. (vii) the maximum deflection is

$$v_{\max} = -\frac{4.53 \times 10^{-3} wL^4}{EI}$$

EXAMPLE 13.10

Determine the deflected shape of the beam shown in Fig. 13.12.

In this problem an external moment M_0 is applied to the beam at B. The support reactions are found in the normal way and are

$$R_A = -\frac{M_0}{L} \text{ (downwards)} \quad R_C = \frac{M_0}{L} \text{ (upwards)}$$

The bending moment at any section X between B and C is then given by

$$M = R_A x + M_0 \tag{i}$$

Equation (i) is valid only for the region BC and clearly does not contain a singularity function which would cause M_0 to vanish for $x \leq b$. We overcome this difficulty by writing

$$M = R_A x + M_0[x-b]^0 \quad (\text{Note}: [x-b]^0 = 1) \tag{ii}$$

Equation (ii) has the same value as Eq. (i) but is now applicable to all sections of the beam since $[x-b]^0$ disappears when $x \leq$ b. Substituting for M from Eq. (ii) in Eq. (13.3) we obtain

$$EI\frac{d^2v}{dx^2} = R_A x + M_0[x-b]^0 \tag{iii}$$

Integration of Eq. (iii) yields

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} + M_0[x-b] + C_1 \tag{iv}$$

and

$$EIv = R_A\frac{x^3}{6} + \frac{M_0}{2}[x-b]^2 + C_1x + C_2 \tag{v}$$

where C_1 and C_2 are arbitrary constants. The boundary conditions are $v = 0$ when $x = 0$ and $x = L$. From the first of these we have $C_2 = 0$ while the second gives

$$0 = -\frac{M_0}{L}\frac{L^3}{6} + \frac{M_0}{2}[L-b]^2 + C_1L$$

from which

$$C_1 = -\frac{M_0}{6L}(2L^2 - 6Lb + 3b^2)$$

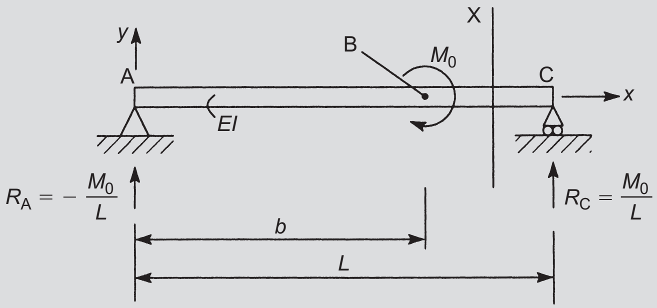

FIGURE 13.12

Deflection of a simply supported beam carrying a point moment (Ex. 13.10).

The equation of the deflection curve of the beam is then

$$v = \frac{M_0}{6EIL}\{x^3 + 3L[x-b]^2 - (2L^2 - 6Lb + 3b^2)x\} \tag{vi}$$

EXAMPLE 13.11

Determine the vertical deflection of the point D in the beam ABCD shown in Fig. 13.13 in terms of its flexural rigidity EI; state clearly its direction.

The support reactions R_A and R_C are obtained in the usual way and are -3.75 kN and 18.75 kN respectively. Note that R_A is a downward reaction.

The distributed load is now extended to the end D of the beam as shown in Fig. 13.14 and the status quo restored by applying an equal and upward distributed load between C and D.

The bending moment at the section X in the bay CD a distance x from A is then given by

$$M = R_A x - \frac{5}{2}[x-1]^2 + \frac{5}{2}[x-2]^2 + R_C[x-2] \tag{i}$$

Substituting for M in Eq.(13.3) we obtain

$$EI\frac{d^2v}{dx^2} = -3.75x - \frac{5}{2}[x-1]^2 + \frac{5}{2}[x-2]^2 + 18.75[x-2] \tag{ii}$$

Then

$$EI\frac{dv}{dx} = -3.75\frac{x^2}{2} - \frac{5}{6}[x-1]^3 + \frac{5}{6}[x-2]^3 + \frac{18.75}{2}[x-2]^2 + C_1 \tag{iii}$$

and

$$EIv = -3.75\frac{x^3}{6} - \frac{5}{24}[x-1]^4 + \frac{5}{24}[x-2]^4 + \frac{18.75}{6}[x-2]^3 + C_1x + C_2 \tag{iv}$$

The boundary conditions are that $v = 0$ when $x = 0$ and $x = 2$ m. From the first of these $C_2 = 0$ and from the second

$$0 = \frac{-3.75 \times 2^3}{6} - \frac{5}{24}[2-1]^4 + 2C_1$$

which gives $C_1 = 2.6$

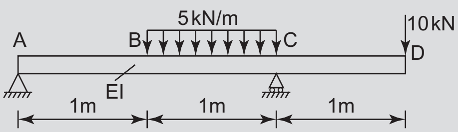

FIGURE 13.13

Beam of Ex. 13.11

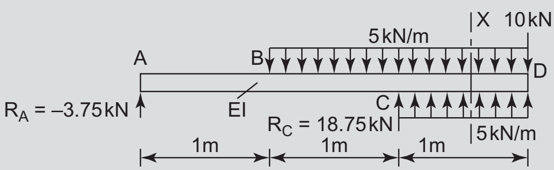

FIGURE 13.14

Solution of Ex. 13.11

Eq. (iv) therefore becomes

$$EIv = \frac{-3.75x^3}{6} - \frac{5}{24}[x-1]^4 + \frac{5}{24}[x-2]^4 + \frac{18.75}{6}[x-2]^3 + 2.6x \tag{v}$$

Then, at D where $x = 3$ m

$$EIv_D = -9.08$$

or

$$v_D = \frac{-9.08}{EI} \text{ (ie downwards)}$$

Note that the 10 kN load does not enter directly into the moment equation. It could be included by adding an imaginary extension of the beam past D which would result in an additional term $-10[x-3]$ in the expression for bending moment, Eq. (i). However it is clear that this term would always disappear when considering any section of the beam between A and D so that such an approach is unnecessary."

13.3 Moment-area method for symmetrical bending

The double integration method and the method of singularity functions are used when the complete deflection curve of a beam is required. However, if only the deflection of a particular point is required, the moment-area method is generally more suitable.

Consider the curvature–moment equation (Eq. (13.3)), i.e.

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

Integration of this equation between any two sections, say A and B, of a beam gives

$$\int_A^B \frac{d^2v}{dx^2}\,dx = \int_A^B \frac{M}{EI}\,dx \tag{13.6}$$

or

$$\left[\frac{dv}{dx}\right]_A^B = \int_A^B \frac{M}{EI}\,dx$$

which gives

$$\left(\frac{dv}{dx}\right)_B - \left(\frac{dv}{dx}\right)_A = \int_A^B \frac{M}{EI}\,dx \tag{13.7}$$

In qualitative terms Eq. (13.7) states that the change of slope between two sections A and B of a beam is numerically equal to the area of the *M/EI* diagram between those sections.

We now return to Eq. (13.3) and multiply both sides by x thereby retaining the equality. Thus

$$\frac{d^2v}{dx^2}x = \frac{M}{EI}x \tag{13.8}$$

Integrating Eq. (13.8) between two sections A and B of a beam we have

$$\int_A^B \frac{d^2 v}{dx^2} x \, dx = \int_A^B \frac{M}{EI} x \, dx \tag{13.9}$$

The left-hand side of Eq. (13.9) may be integrated by parts and gives

$$\left[x \frac{dv}{dx}\right]_A^B - \int_A^B \frac{dv}{dx} dx = \int_A^B \frac{M}{EI} x \, dx$$

or

$$\left[x \frac{dv}{dx}\right]_A^B - [v]_A^B = \int_A^B \frac{M}{EI} x \, dx$$

Hence, inserting the limits we have

$$x_B \left(\frac{dv}{dx}\right)_B - x_A \left(\frac{dv}{dx}\right)_A - (v_B - v_A) = \int_A^B \frac{M}{EI} x \, dx \tag{13.10}$$

in which x_B and x_A represent the x coordinate of each of the sections B and A, respectively, while $(dv/dx)_B$ and $(dv/dx)_A$ are the respective slopes; v_B and v_A are the corresponding deflections. The right-hand side of Eq. (13.10) represents the moment of the area of the M/EI diagram between the sections A and B *about* A.

Equations (13.7) and (13.10) may be used to determine values of slope and deflection at any section of a beam. We note that in both equations we are concerned with the geometry of the M/EI diagram. This will be identical in shape to the bending moment diagram unless there is a change of section. Furthermore, the form of the right-hand side of both Eqs (13.7) and (13.10) allows two alternative methods of solution. In cases where the geometry of the M/EI diagram is relatively simple, we can employ a *semi-graphical* approach based on the actual geometry of the M/EI diagram. Alternatively, in complex problems, the bending moment may be expressed as a function of x and a completely analytical solution obtained. Both methods are illustrated in the following examples.

EXAMPLE 13.12

Determine the slope and deflection of the free end of the cantilever beam shown in Fig. 13.15.

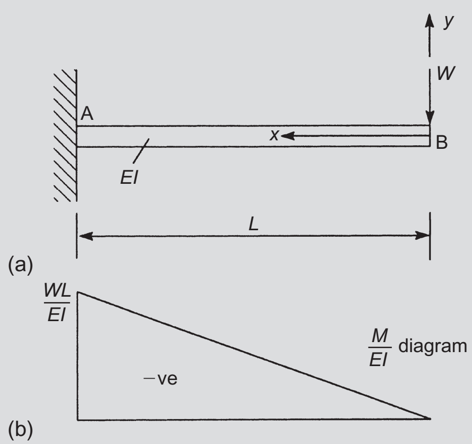

FIGURE 13.15

Moment-area method for the deflection of a cantilever (Ex. 13.12).

We choose the origin of the axes at the free end B of the cantilever. Equation (13.7) then becomes

$$\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{A}} - \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{B}} = \int_{\mathrm{A}}^{\mathrm{B}} \frac{M}{EI}\,\mathrm{d}x$$

or, since $(\mathrm{d}v/\mathrm{d}x)_{\mathrm{A}} = 0$

$$-\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{B}} = \int_0^L \frac{M}{EI}\,\mathrm{d}x \tag{i}$$

Generally at this stage we decide which approach is most suitable; however, both semi-graphical and analytical methods are illustrated here. Using the geometry of Fig. 13.15(b) we have

$$-\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{B}} = \frac{1}{2}L\left(\frac{-WL}{EI}\right)$$

which gives

$$\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{B}} = \frac{WL^2}{2EI}$$

(compare with the value given by Eq. (iii) of Ex. 13.1. Note the change in sign due to the different origin for x).

Alternatively, since the bending moment at any section a distance x from B is $-Wx$ we have, from Eq. (i)

$$-\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{B}} = \int_0^L -\frac{Wx}{EI}\,\mathrm{d}x$$

which again gives

$$\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{B}} = \frac{WL^2}{2EI}$$

With the origin for x at B, Eq. (13.10) becomes

$$x_{\mathrm{A}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{A}} - x_{\mathrm{B}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{B}} - (v_{\mathrm{A}} - v_{\mathrm{B}}) = \int_{\mathrm{B}}^{\mathrm{A}} \frac{M}{EI}x\,\mathrm{d}x \tag{ii}$$

Since $(\mathrm{d}v/\mathrm{d}x)_{\mathrm{A}} = 0$ and $x_{\mathrm{B}} = 0$ and $v_{\mathrm{A}} = 0$, Eq. (ii) reduces to

$$v_{\mathrm{B}} = \int_0^L \frac{M}{EI}x\,\mathrm{d}x \tag{iii}$$

Again we can now decide whether to proceed semi-graphically or analytically. Using the former approach and taking the moment of the area of the M/EI diagram about B, we have

$$v_{\mathrm{B}} = \frac{1}{2}L\left(\frac{-WL}{EI}\right)\frac{2}{3}L$$

which gives

$$v_B = -\frac{WL^3}{3EI} \quad \text{(compare with Eq. (v) of Ex. 13.1)}$$

Alternatively we have

$$v_B = \int_0^L \frac{(-Wx)}{EI} x \, dx = -\int_0^L \frac{Wx^2}{EI} \, dx$$

which gives

$$v_B = -\frac{WL^3}{3EI}$$

as before.

Note that if the built-in end had been selected as the origin for x, we could not have determined v_B directly since the term $x_B(dv/dx)_B$ in Eq. (ii) would not have vanished. The solution for v_B would then have consisted of two parts, first the determination of $(dv/dx)_B$ and then the calculation of v_B.

EXAMPLE 13.13

Determine the maximum deflection in the simply supported beam shown in Fig. 13.16(a).

From symmetry we deduce that the beam reactions are each $wL/2$; the M/EI diagram has the geometry shown in Fig. 13.16(b).

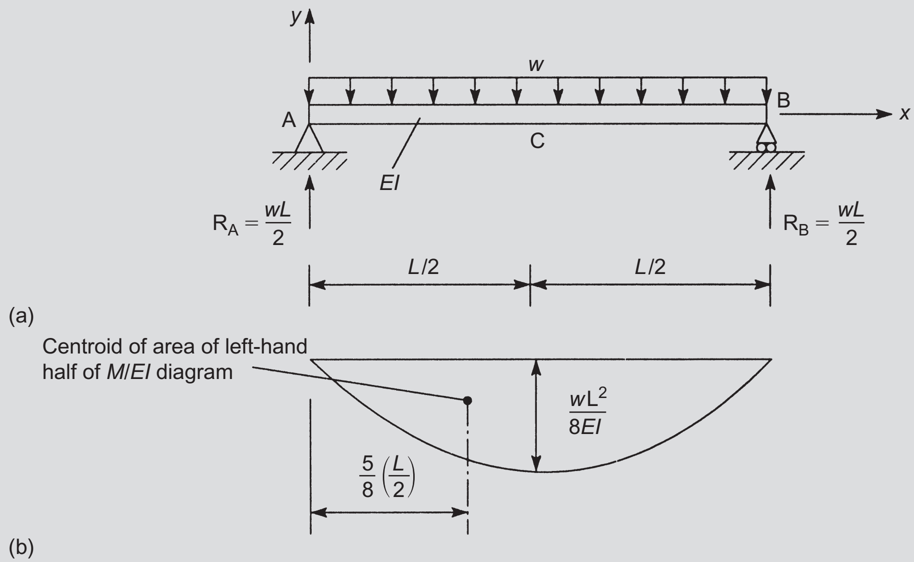

FIGURE 13.16

Moment-area method for a simply supported beam carrying a uniformly distributed load.

If we take the origin of axes to be at A and consider the half-span AC, Eq. (13.10) becomes

$$x_{\mathrm{C}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{C}}-x_{\mathrm{A}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{A}}-(v_{\mathrm{C}}-v_{\mathrm{A}})=\int_{\mathrm{A}}^{\mathrm{C}}\frac{M}{EI}x\,\mathrm{d}x \tag{i}$$

In this problem $(\mathrm{d}v/\mathrm{d}x)_{\mathrm{C}}=0$, $x_{\mathrm{A}}=0$ and $v_{\mathrm{A}}=0$; hence Eq. (i) reduces to

$$v_{\mathrm{C}}=-\int_0^{L/2}\frac{M}{EI}x\,\mathrm{d}x \tag{ii}$$

Using the geometry of the *M/EI* diagram, i.e. the semi-graphical approach, and taking the moment of the area of the *M/EI* diagram between A and C about A we have from Eq. (ii)

$$v_{\mathrm{C}}=-\frac{2}{3}\frac{wL^2}{8EI}\frac{L}{2}\frac{5}{8}\left(\frac{L}{2}\right)$$

which gives

$$v_{\mathrm{C}}=-\frac{5wL^4}{384EI}\text{(see Eq. (v) in Ex. 13.4)}$$

For the completely analytical approach we express the bending moment M as a function of x; thus

$$M=\frac{wL}{2}x-\frac{wx^2}{2}$$

or

$$M=\frac{w}{2}(Lx-x^2)$$

Substituting for M in Eq. (ii) we have

$$v_{\mathrm{C}}=-\int_0^{L/2}\frac{w}{2EI}(Lx^2-x^3)\,\mathrm{d}x$$

which gives

$$v_{\mathrm{C}}=-\frac{w}{2EI}\left[\frac{Lx^3}{3}-\frac{x^4}{4}\right]_0^{L/2}$$

Then

$$v_{\mathrm{C}}=-\frac{5wL^4}{384EI}$$

EXAMPLE 13.14

Figure 13.17(a) shows a cantilever beam of length L carrying a concentrated load W at its free end. The section of the beam changes midway along its length so that the second moment of area of its cross section is reduced by half. Determine the deflection of the free end.

In this problem the bending moment and M/EI diagrams have different geometrical shapes. Choosing the origin of axes at C, Eq. (13.10) becomes

$$x_{\mathrm{A}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{A}} - x_{\mathrm{C}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{\mathrm{C}} - (v_{\mathrm{A}} - v_{\mathrm{C}}) = \int_{\mathrm{C}}^{\mathrm{A}} \frac{M}{EI} x\,\mathrm{d}x \tag{i}$$

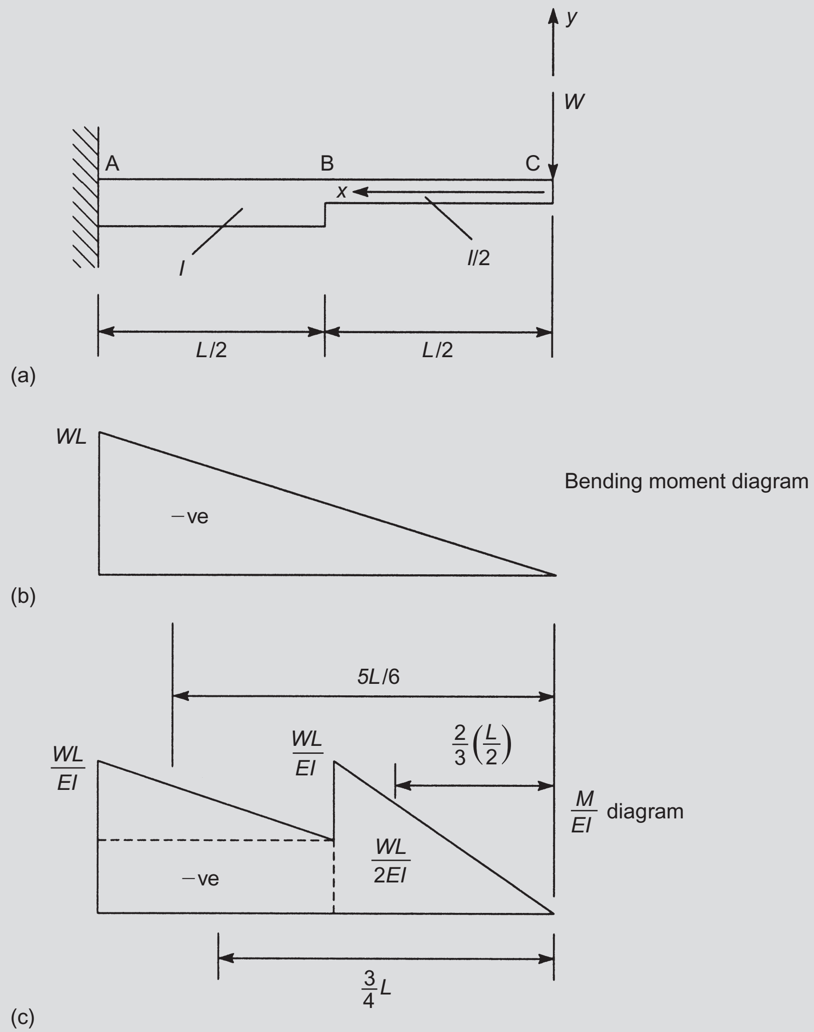

FIGURE 13.17

Deflection of a cantilever of varying section.

in which $(\mathrm{d}\upsilon/\mathrm{d}x)_A = 0$, $x_C = 0$, $\upsilon_A = 0$. Hence

$$\upsilon_C = \int_0^L \frac{M}{EI} x \, \mathrm{d}x \tag{ii}$$

From the geometry of the *M/EI* diagram (Fig. 13.17(c)) and taking moments of areas about C we have

$$\upsilon_C = \left[\left(\frac{-W}{2EI}\right)\frac{L}{2}\frac{3L}{4} + \frac{1}{2}\left(\frac{-WL}{2EI}\right)\frac{L}{2}\frac{5L}{6} + \frac{1}{2}\left(\frac{-WL}{EI}\right)\frac{L}{2}\frac{2L}{3\,2}\right]$$

which gives

$$\upsilon_C = -\frac{3WL^3}{8EI}$$

Analytically we have

$$\upsilon_C = \left[\int_0^{L/2} \frac{-Wx^2}{EI/2}\,\mathrm{d}x + \int_{L/2}^{L} \frac{-Wx^2}{EI}\,\mathrm{d}x\right]$$

or

$$\upsilon_C = -\frac{W}{EI}\left\{\left[\frac{2x^3}{3}\right]_0^{L/2} + \left[\frac{x^3}{3}\right]_{L/2}^{L}\right\}$$

Hence

$$\upsilon_C = -\frac{3WL^3}{8EI}$$

as before.

EXAMPLE 13.15

The cantilever beam shown in Fig. 13.18 tapers along its length so that the second moment of area of its cross section varies linearly from its value I_0 at the free end to $2I_0$ at the built-in end. Determine the deflection at the free end when the cantilever carries a concentrated load W.

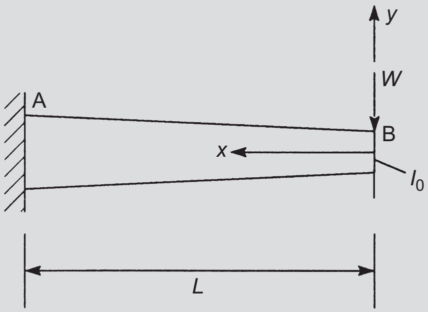

FIGURE 13.18

Deflection of a cantilever of tapering section.

Choosing the origin of axes at the free end B we have, from Eq. (13.10)

$$x_A\left(\frac{dv}{dx}\right)_A - x_B\left(\frac{dv}{dx}\right)_B - (v_A - v_B) = \int_B^A \frac{M}{EI_X}x\,dx \tag{i}$$

in which I_x, the second moment of area at any section X, is given by

$$I_X = I_0\left(1+\frac{x}{L}\right)$$

Also $(dv/dx)_A = 0$, $x_B = 0$, $v_A = 0$ so that Eq. (i) reduces to

$$v_B = \int_0^L \frac{Mx}{EI_0\left(1+\frac{x}{L}\right)}dx \tag{ii}$$

The geometry of the M/EI diagram in this case will be complicated so that the analytical approach is most suitable. Therefore since $M = -Wx$, Eq. (ii) becomes

$$v_B = -\int_0^L \frac{Wx^2}{EI_0\left(1+\frac{x}{L}\right)}dx$$

or

$$v_B = -\frac{WL}{EI_0}\int_0^L \frac{x^2}{L+x}dx \tag{iii}$$

Rearranging Eq. (iii) we have

$$v_B = -\frac{WL}{EI_0}\left[\int_0^L (x-L)\,dx + \int_0^L \frac{L^2}{L+x}dx\right]$$

which may be written in the form

$$v_B = -\frac{WL}{EI_0}\left[\int_0^L (x-L)\,dx + L^2\int_0^L \frac{dx}{L(1+x/L)}\right]$$

Hence

$$v_B = -\frac{WL}{EI_0}\left[\left(\frac{x^2}{2} - Lx\right) + L^2\log_e(1+x/L)\right]_0^L$$

so that

$$v_B = -\frac{WL^3}{EI_0}\left(-\frac{1}{2} + \log_e 2\right)$$

i.e.

$$v_B = -\frac{0.19\,WL^3}{EI_0}$$

13.4 Deflections due to unsymmetrical bending

We noted in Chapter 9 that a beam bends about its neutral axis whose inclination to arbitrary centroidal axes is determined from Eq. (9.33). Beam deflections, therefore, are always perpendicular in direction to the neutral axis.

Suppose that at some section of a beam, the deflection normal to the neutral axis (and therefore an absolute deflection) is ζ. Then, as shown in Fig. 13.19, the centroid G is displaced to G'. The components of ζ, u and v, are given by

$$u = \zeta \sin \alpha \quad v = \zeta \cos \alpha \tag{13.11}$$

The centre of curvature of the beam lies in a longitudinal plane perpendicular to the neutral axis of the beam and passing through the centroid of any section. Hence for a radius of curvature R, we see, by direct comparison with Eq. (13.2) that

$$\frac{1}{R} = \frac{d^2\zeta}{dx^2} \tag{13.12}$$

or, substituting from Eq. (13.11)

$$\frac{\sin \alpha}{R} = \frac{d^2 u}{dx^2} \quad \frac{\cos \alpha}{R} = \frac{d^2 v}{dx^2} \tag{13.13}$$

We observe from the derivation of Eq. (9.31) that

$$\frac{E \sin \alpha}{R} = \frac{M_y I_z - M_z I_{zy}}{I_z I_y - I_{zy}^2}$$

$$\frac{E \cos \alpha}{R} = \frac{M_z I_y - M_y I_{zy}}{I_z I_y - I_{zy}^2}$$

Therefore, from Eq. (13.13)

$$\frac{d^2 u}{dx^2} = \frac{M_y I_z - M_z I_{zy}}{E(I_z I_y - I_{zy}^2)} \tag{13.14}$$

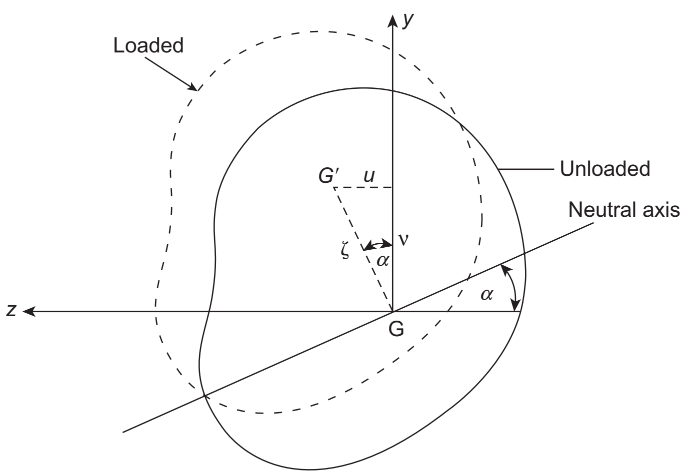

FIGURE 13.19

Deflection of a beam of unsymmetrical cross section.

$$\frac{d^2 v}{dx^2} = \frac{M_z I_y - M_y I_{zy}}{E(I_z I_y - I_{zy}^2)} \tag{13.15}$$

EXAMPLE 13.16

Determine the horizontal and vertical components of the deflection of the free end of the cantilever shown in Fig. 13.20. The second moments of area of its unsymmetrical section are I_z, I_y and I_{zy}.

The bending moments at any section of the beam due to the applied load W are

$$M_z = -W(L-x), \quad M_y = 0$$

Then Eq. (13.14) reduces to

$$\frac{d^2 u}{dx^2} = \frac{W(L-x)I_{zy}}{E(I_z I_y - I_{zy}^2)} \tag{i}$$

Integrating with respect to x

$$\frac{du}{dx} = \frac{W I_{zy}}{E(I_z I_y - I_{zy}^2)}\left(Lx - \frac{x^2}{2} + C_1\right)$$

When $x = 0$, $(du/dx) = 0$ so that $C_1 = 0$ and

$$\frac{du}{dx} = \frac{W I_{zy}}{E(I_z I_y - I_{zy}^2)}\left(Lx - \frac{x^2}{2}\right) \tag{ii}$$

Integrating Eq. (ii) with respect to x

$$u = \frac{W I_{zy}}{E(I_z I_y - I_{zy}^2)}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_2\right)$$

When $x = 0$, $u = 0$ so that $C_2 = 0$. Therefore

$$u = \frac{W I_{zy}}{6E(I_z I_y - I_{zy}^2)}(3Lx^2 - x^3) \tag{iii}$$

At the free end of the cantilever where $x = L$

$$u_{\text{fe}} = \frac{W I_{zy} L^3}{3E(I_z I_y - I_{zy}^2)} \tag{iv}$$

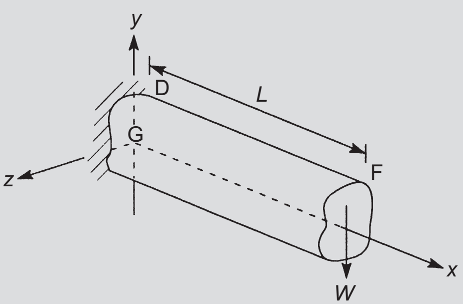

FIGURE 13.20

Deflection of a cantilever of unsymmetrical cross section carrying a concentrated load at its free end (Ex. 13.16).

The deflected shape of the beam in the xy plane is found in an identical manner from Eq. (13.15) and is

$$v = -\frac{WI_y}{6E(I_zI_y - I_{zy}^2)}(3Lx^2 - x^3) \tag{v}$$

from which the deflection at the free end is

$$v_{\text{fe}} = -\frac{WI_yL^3}{3E(I_zI_y - I_{zy}^2)} \tag{vi}$$

The absolute deflection, δ_{fe}, at the free end is given by

$$\delta_{\text{fe}} = (u_{\text{fe}}^2 + v_{\text{fe}}^2)^{\frac{1}{2}} \tag{vii}$$

and its direction is at $\tan^{-1}(u_{\text{fe}}/v_{\text{fe}})$ to the vertical.

Note that if either Gz or Gy is an axis of symmetry $I_{zy} = 0$ and Eqs. (iv) and (vi) reduce to

$$u_{\text{fe}} = 0 \quad v_{\text{fe}} = -\frac{WL^3}{3EI_z} \quad \text{(compare with Eq. (v) of Ex. 13.1)}$$

EXAMPLE 13.17

Determine the deflection of the free end of the cantilever beam shown in Fig. 13.21. The second moments of area of its cross section about a horizontal and vertical system of centroidal axes are I_z, I_y and I_{zy}.

The method of solution is identical to that in Ex. 13.16 except that the bending moments M_z and M_y are given by

$$M_z = -w(L-x)^2/2 \quad M_y = 0$$

The values of the components of the deflection at the free end of the cantilever are

$$u_{\text{fe}} = \frac{wI_{zy}L^4}{8E(I_zI_y - I_{zy}^2)} \quad v_{\text{fe}} = -\frac{wI_yL^4}{8E(I_zI_y - I_{zy}^2)}$$

Again, if either Gz or Gy is an axis of symmetry, $I_{zy} = 0$ and these expressions reduce to

$$u_{\text{fe}} = 0, \quad v_{\text{fe}} = -\frac{wL^4}{8EI_z} \quad \text{(compare with Eq. (v) of Ex 13.2)}$$

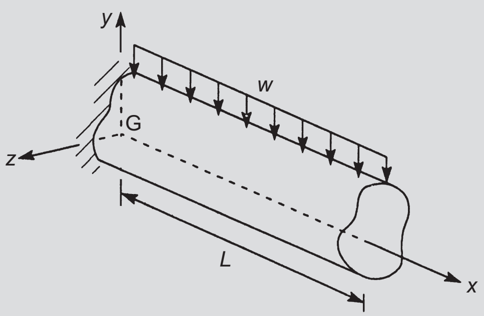

FIGURE 13.21

Deflection of a cantilever of unsymmetrical cross section carrying a uniformly distributed load (Ex. 13.17).

EXAMPLE 13.18

Determine the vertical and horizontal components of the displacement midway between the supports B and C of the thin-walled beam shown in Fig. 13.22(a). Young's modulus for the material of the beam is E and its cross section is shown in Fig. 13.22(b).

The centroid of the beam section coincides with the centre of the web 34. The second moments of area are calculated using the methods described in Section 9.6 and are:

$$I_z = 3.25a^3t, \quad I_y = 1.67a^3t, \quad I_{zy} = 1.75a^3t$$

Since only a vertical load is applied there will only be vertical support reactions at B and C. Therefore, taking moments about B

$$R_C \times 2L + WL = 0$$

so that

$$R_C = -W/2 \text{ (ie downwards)}$$

Taking the origin for x at C the bending moments at any section between B and C are given by

$$M_z = R_C x = -Wx/2, \; M_y = 0$$

Substituting these values in Eq. (13.14)

$$\frac{d^2u}{dx^2} = -\frac{(-Wx/2)I_{zy}}{E(I_z I_y - I_{zy}{}^2)}$$

Integrating with respect to x

$$\frac{du}{dx} = \frac{WI_{zy}}{2E(I_z I_y - I_{zy}{}^2)}\left(\frac{x^2}{2} + C_1\right)$$

and

$$u = \frac{WI_{zy}}{2E(I_z I_y - I_{zy}{}^2)}\left(\frac{x^3}{6} + C_1 x + C_2\right) \tag{i}$$

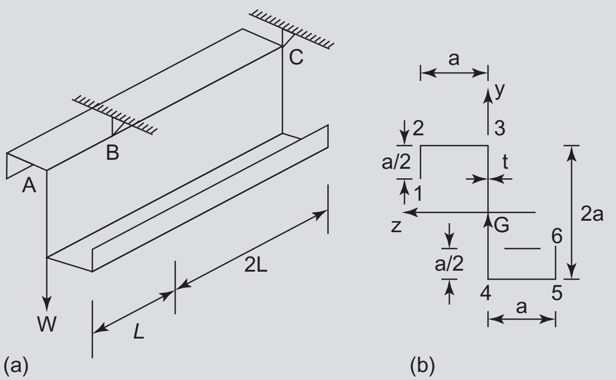

FIGURE 13.22

Beam of Ex. 13.18

When $x=0$, $u=0$ so that $C_2=0$. Also, when $x=2L$, $u=0$ so that, from Eq. (i)

$$\frac{8L^3}{6}+2LC_1=0$$

which gives

$$C_1=-2L^2/3$$

Eq. (i) may then be written

$$u=\frac{WI_{zy}}{12E(I_zI_y-I_{zy}{}^2)}(x^3-4L^2x) \tag{ii}$$

At the mid-point of BC, $x=L$ so that

$$u\,(\text{mid-point of BC})=-\frac{WI_{zy}L^3}{4E(I_zI_y-I_{zy}{}^2)}$$

Substituting the values of I_z etc gives

$$u\,(\text{mid-point of BC})=-\frac{0.186\,WL^3}{Ea^3t}\quad(\text{ie to the right})$$

Similarly

$$v\,(\text{mid-point of BC})=+\frac{0.177\,WL^3}{Ea^3t}\quad(\text{ie upwards})$$

Note that in this particular example the vertical displacement of the mid-point of BC may be obtained directly by replacing I_{zy} in Eq. (ii) by I_y and making allowance for the change in sign of the term involving M_z in Eq. (13.15).

13.5 Deflection due to shear

So far in this chapter we have been concerned with deflections produced by the bending action of shear loads. These shear loads however, as we saw in Chapter 10, induce shear stress distributions throughout beam sections which in turn produce shear strains and therefore shear deflections. Generally, shear deflections are small compared with bending deflections, but in some cases of deep beams they can be comparable. In the following we shall use strain energy to derive an expression for the deflection due to shear in a beam having a cross section which is at least singly symmetrical.

In Chapter 10 we showed that the strain energy U of a piece of material subjected to a uniform shear stress τ is given by

$$U=\frac{\tau^2}{2G}\times\text{volume}\quad(\text{Eq. (10.20)})$$

However, we also showed in Chapter 10 that shear stress distributions are not uniform throughout beam sections. We therefore write Eq. (10.20) as

$$U=\frac{\beta}{2G}\times\left(\frac{S}{A}\right)^2\times\text{volume} \tag{13.16}$$

in which S is the applied shear force, A is the cross-sectional area of the beam section and β is a constant which depends upon the distribution of shear stress through the beam section; β is known as the *form factor*.

To determine β we consider an element $b_0\,\delta y$ in an elemental length δx of a beam subjected to a vertical shear load S_y (Fig. 13.23); we shall suppose that the beam section has a vertical axis of symmetry. The shear stress τ is constant across the width, b_0, of the element (see Section 10.2). The strain energy, δU, of the element $b_0\,\delta y\,\delta x$, from Eq. (10.20) is

$$\delta U = \frac{\tau^2}{2G} \times b_0\,\delta y\,\delta x \tag{13.17}$$

Therefore the total strain energy U in the elemental length of beam is given by

$$U = \frac{\delta x}{2G}\int_{y_1}^{y_2} \tau^2 b_0\,\mathrm{d}y \tag{13.18}$$

Alternatively U for the elemental length of beam is obtained using Eq. (13.16); thus

$$U = \frac{\beta}{2G} \times \left(\frac{S_y}{A}\right)^2 \times A\delta x \tag{13.19}$$

Equating Eqs (13.19) and (13.18) we have

$$\frac{\beta}{2G} \times \left(\frac{S_y}{A}\right)^2 \times A\delta x = \frac{\delta x}{2G}\int_{y_1}^{y_2} \tau^2 b_0\,\mathrm{d}y$$

whence

$$\beta = \frac{A}{S_y^2}\int_{y_1}^{y_2} \tau^2 b_0\,\mathrm{d}y \tag{13.20}$$

The shear stress distribution in a beam having a singly or doubly symmetrical cross section and subjected to a vertical shear force, S_y, is given by Eq. (10.4), i.e.

$$\tau = -\frac{S_y A'\bar{y}}{b_0 I_z}$$

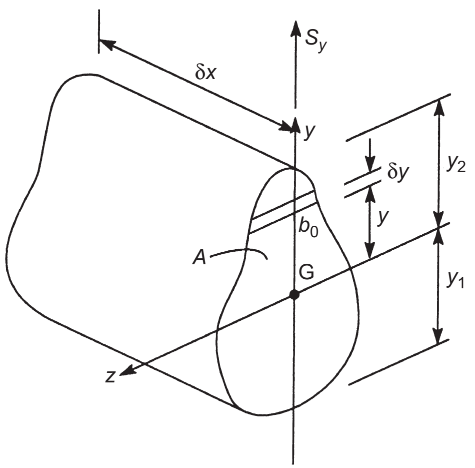

FIGURE 13.23

Determination of form factor β.

Substituting this expression for τ in Eq. (13.20) we obtain

$$\beta = \frac{A}{S_y^2}\int_{y_1}^{y_2}\left(\frac{S_y A'\bar{y}}{b_0 I_z}\right)^2 b_0\,\mathrm{d}y$$

which gives

$$\beta = \frac{A}{I_z^2}\int_{y_1}^{y_2}\frac{(A'\bar{y})_2}{b_0}\mathrm{d}y \tag{13.21}$$

Suppose now that δv_s is the deflection due to shear in the elemental length of beam of Fig. 13.23. The work done by the shear force S_y (assuming it to be constant over the length δx and gradually applied) is then

$$\frac{1}{2}S_y\,\delta v_s$$

which is equal to the strain energy stored. Hence

$$\frac{1}{2}S_y\,\delta v_s = \frac{\beta}{2G}\times\left(\frac{S_y}{A}\right)^2\times A\,\delta x$$

which gives

$$\delta v_s = \frac{\beta}{G}\left(\frac{S_y}{A}\right)\delta x$$

The total deflection due to shear in a beam of length L subjected to a vertical shear force S_y is then

$$v_s = \frac{\beta}{G}\int_L\left(\frac{S_y}{A}\right)\mathrm{d}x \tag{13.22}$$

EXAMPLE 13.19

A cantilever beam of length L has a rectangular cross section of breadth B and depth D and carries a vertical concentrated load, W, at its free end. Determine the deflection of the free end, including the effects of both bending and shear. The flexural rigidity of the cantilever is EI and its shear modulus G.

Using Eq. (13.21) we obtain the form factor β for the cross section of the beam directly. Thus

$$\beta = \frac{BD}{(BD^3/12)^2}\int_{-D/2}^{D/2}\frac{1}{B}\left[B\left(\frac{D}{2}-y\right)\frac{1}{2}\left(\frac{D}{2}+y\right)\right]^2\mathrm{d}y \quad \text{(see Ex. 10.1)}$$

which simplifies to

$$\beta = \frac{36}{D^5}\int_{-D/2}^{D/2}\left(\frac{D^4}{16}-\frac{D^2y^2}{2}+y^4\right)\mathrm{d}y$$

Integrating we obtain

$$\beta = \frac{36}{D^5}\left[\frac{D^4 y}{16}-\frac{D^2y^3}{6}+\frac{y^5}{5}\right]_{-D/2}^{D/2}$$

which gives

$$\beta = \frac{6}{5}$$

Note that the dimensions of the cross section do not feature in the expression for β. The form factor for any rectangular cross section is therefore 6/5 or 1.2.

Let us suppose that υ_s is the vertical deflection of the free end of the cantilever due to shear. Hence, from Eq. (13.22) we have

$$\upsilon_s = \frac{6}{5G}\int_0^L \left(\frac{-W}{BD}\right) \mathrm{d}x$$

so that

$$\upsilon_s = -\frac{6WL}{5GBD} \tag{i}$$

The vertical deflection due to bending of the free end of a cantilever carrying a concentrated load has previously been determined in Ex. 13.1 and is $-WL^3/3EI$. The total deflection, υ_T, produced by bending and shear is then

$$\upsilon_T = -\frac{WL^3}{3EI} - \frac{6WL}{5GBD} \tag{ii}$$

Rewriting Eq. (ii) we obtain

$$\upsilon_T = -\frac{WL^3}{3EI}\left[1 + \frac{3}{10}\frac{E}{G}\left(\frac{D}{L}\right)^2\right] \tag{iii}$$

For many materials (3E/10 G) is approximately unity so that the contribution of shear to the total deflection is $(D/L)^2$ of the bending deflection. Clearly this term only becomes significant for short, deep beams.

13.6 Statically indeterminate beams

The beams we have considered so far have been supported in such a way that the support reactions could be determined using the equations of statical equilibrium; such beams are therefore *statically determinate.* However, many practical cases arise in which additional supports are provided so that there are a greater number of unknowns than the possible number of independent equations of equilibrium; the support systems of such beams are therefore *statically indeterminate.* Simple examples are shown in Fig. 13.24 where, in Fig. 13.24(a), the cantilever does not, theoretically, require the additional support at its free end and in Fig. 13.24(b) any one of the three supports is again, theoretically, *redundant.* A beam such as that shown in Fig. 13.24(b) is known as a *continuous beam* since it has more than one span and is continuous over one or more supports.

We shall now use the results of the previous work in this chapter to investigate methods of solving statically indeterminate beam systems. Having determined the reactions, diagrams of shear force and bending moment follow in the normal manner.

The examples given below are relatively simple cases of statically indeterminate beams. We shall investigate more complex cases in Chapter 16.

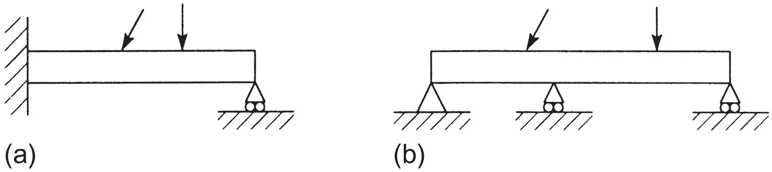

FIGURE 13.24

Examples of statically indeterminate beams.

Method of superposition

In Section 3.7 we discussed the principle of superposition and saw that the combined effect of a number of forces on a structural system may be found by the addition of their separate effects. The principle may be applied to the determination of support reactions in relatively simple statically indeterminate beams. We shall illustrate the method by examples.

EXAMPLE 13.20

The cantilever AB shown in Fig. 13.25(a) carries a uniformly distributed load and is provided with an additional support at its free end. Determine the reaction at the additional support.

Suppose that the reaction at the support B is R_B. Using the principle of superposition we can represent the combined effect of the distributed load and the reaction R_B as the sum of the two loads acting separately as shown in Fig. 13.25(b) and (c). Also, since the vertical deflection of B in Fig. 13.25(a) is zero, it follows that the vertical downward deflection of B in Fig. 13.25(b) must be numerically equal to the vertically upward deflection of B in Fig. 13.25(c). Therefore using the results of Exs (13.1) and (13.2) we have

$$\left|\frac{R_B L^3}{3EI}\right| = \left|\frac{wL^4}{8EI}\right|$$

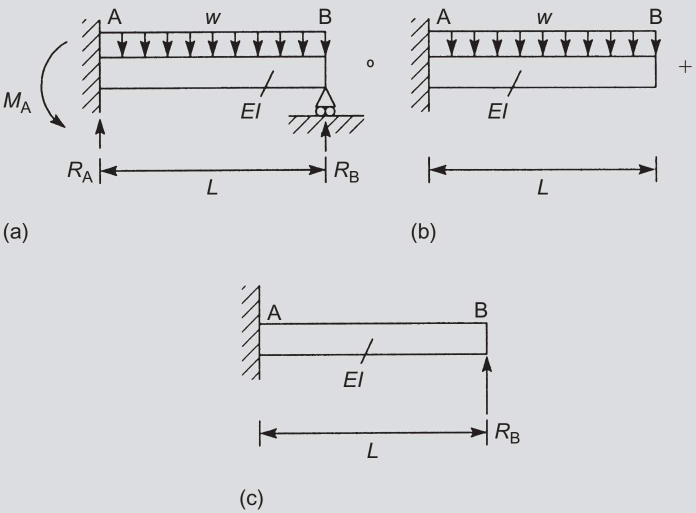

FIGURE 13.25

Propped cantilever of Ex. 13.20.

whence

$$R_B = \frac{3}{8}wL$$

It is now possible to determine the reactions R_A and M_A at the built-in end using the equations of simple statics. Taking moments about A for the beam in Fig. 13.25(a) we have

$$M_A = \frac{wL^2}{2} - R_B L = \frac{wL^2}{2} - \frac{3}{8}wL^2 = \frac{1}{8}wL^2$$

Resolving vertically

$$R_A = wL - R_B = wL - \frac{3}{8}wL = \frac{5}{8}wL$$

In the solution of Ex. 13.20 we selected R_B as the *redundancy;* in fact, any one of the three support reactions, M_A, R_A or R_B, could have been chosen. Let us suppose that M_A is taken to be the redundant reaction. We now represent the combined loading of Fig. 13.25(a) as the sum of the separate loading systems shown in Fig. 13.26(a) and (b) and work in terms of the rotations of the beam at A due to the distributed load and the applied moment, M_A. Clearly, since there is no rotation at the built-in end of a cantilever, the rotations produced separately in Fig. 13.26(a) and (b) must be numerically equal but opposite in direction. Using the method of Section 13.1 it may be shown that

$$\theta_A(\text{due to } w) = \frac{wL^3}{24EI} \quad \text{(clockwise)}$$

and

$$\theta_A(\text{due to } M_A) = \frac{M_A L}{3EI} \quad \text{(anticlockwise)}$$

Since

$$\left|\theta_A(M_A)\right| = \left|\theta_A(w)\right|$$

we have

$$M_A = \frac{wL^2}{8}$$

as before.

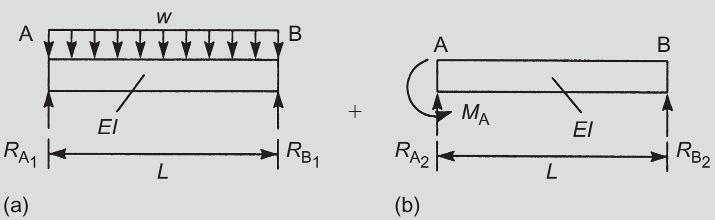

FIGURE 13.26

Alternative solution of Ex. 13.21.

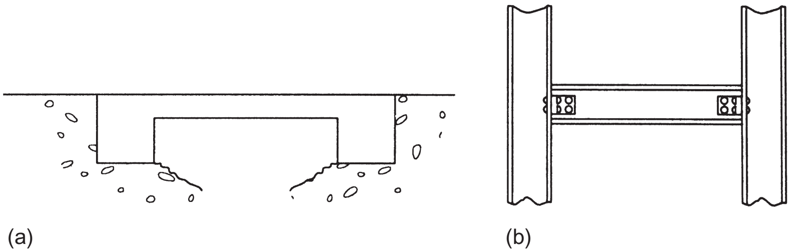

FIGURE 13.27

Practical examples of fixed beams.

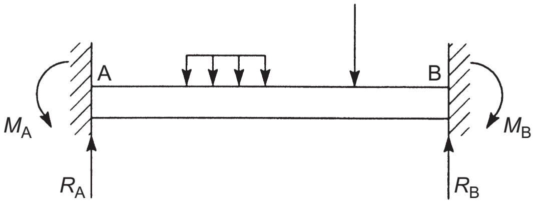

FIGURE 13.28

Support reactions in a fixed beam.

Built-in or fixed-end beams

In practice single-span beams may not be free to rotate about their supports but are connected to them in a manner that prevents rotation. Thus a reinforced concrete beam may be cast integrally with its supports as shown in Fig. 13.27(a) or a steel beam may be bolted at its ends to steel columns (Fig. 13.27(b)). Clearly neither of the beams of Fig. 13.27(a) or (b) can be regarded as simply supported.

Consider the fixed beam of Fig. 13.28. Any system of vertical loads induces reactions of force and moment, the latter arising from the constraint against rotation provided by the supports. There are then four unknown reactions and only two possible equations of statical equilibrium; the beam is therefore statically indeterminate and has two redundancies. A solution is obtained by considering known values of slope and deflection at particular beam sections.

EXAMPLE 13.21

Figure 13.29(a) shows a fixed beam carrying a central concentrated load, W. Determine the value of the fixed-end moments, M_A and M_B.

Since the ends A and B of the beam are prevented from rotating, moments M_A and M_B are induced in the supports; these are termed fixed-end moments. From symmetry we see that $M_A = M_B$ and $R_A = R_B = W/2$.

The beam AB in Fig. 13.29(a) may be regarded as a simply supported beam carrying a central concentrated load with moments M_A and M_B applied at the supports. The bending moment diagrams corresponding to these two loading cases are shown in Fig. 13.29(b) and (c) and are known as the *free bending moment diagram* and the *fixed-end moment diagram*, respectively. Clearly the concentrated load produces sagging (positive) bending moments, while the fixed-end moments induce hogging (negative) bending moments. The resultant or final bending moment diagram is constructed by superimposing the free and fixed-end moment diagrams as shown in Fig. 13.29(d).

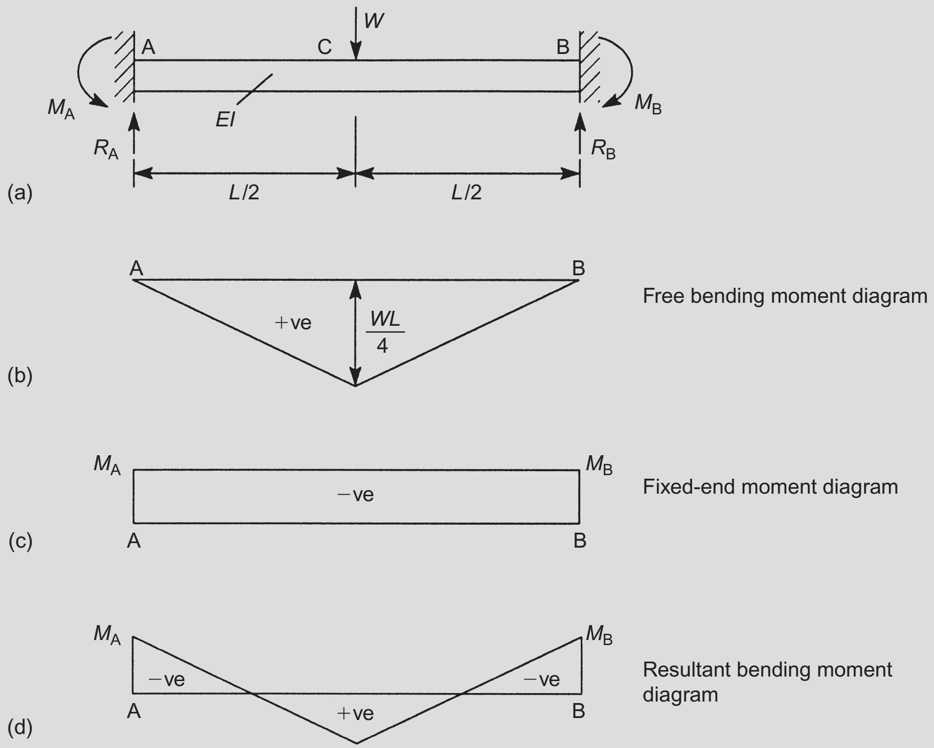

FIGURE 13.29

Bending moment diagram for a fixed beam (Ex. 13.21).

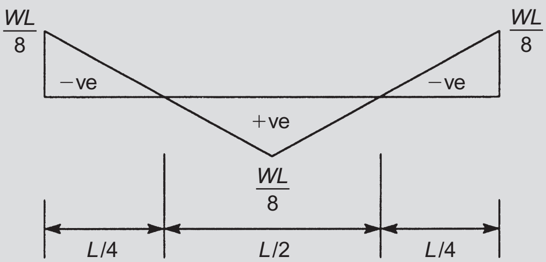

FIGURE 13.30

Complete bending moment diagram for fixed beam of Ex. 13.21.

The moment-area method is now used to determine the fixed-end moments, M_A and M_B. From Eq. (13.7) the change in slope between any two sections of a beam is equal to the area of the M/EI diagram between those sections. Therefore, the net area of the bending moment diagram of Fig. 13.29(d) must be zero since the change of slope between the ends of the beam is zero. It follows that the area of the free bending moment diagram is numerically equal to the area of the fixed-end moment diagram; thus

$$M_A L = \frac{1}{2}\frac{WL}{4}L$$

which gives

$$M_A = M_B = \frac{WL}{8}$$

and the resultant bending moment diagram has principal values as shown in Fig. 13.30. Note that the maximum positive bending moment is equal in magnitude to the maximum negative bending moment and that points of contraflexure (i.e. where the bending moment changes sign) occur at the quarter-span points.

Having determined the support reactions, the deflected shape of the beam may be found by any of the methods described in the previous part of this chapter.

EXAMPLE 13.22

Determine the fixed-end moments and the fixed-end reactions for the beam shown in Fig. 13.31(a).

The resultant bending moment diagram is shown in Fig. 13.31(b) where the line AB represents the datum from which values of bending moment are measured. Again the net area of the resultant bending moment diagram is zero since the change in slope between the ends of the beam is zero. Hence

$$\frac{1}{2}(M_A + M_B)L = \frac{1}{2}L\frac{Wab}{L}$$

which gives

$$M_A + M_B = \frac{Wab}{L} \qquad \text{(i)}$$

We require a further equation to solve for M_A and M_B. This we obtain using Eq. 13.10 and taking the origin for x at A; hence we have

$$x_B\left(\frac{dv}{dx}\right)_B - x_A\left(\frac{dv}{dx}\right)_A - (v_B - v_A) = \int_A^B \frac{M}{EI}x\,dx \qquad \text{(ii)}$$

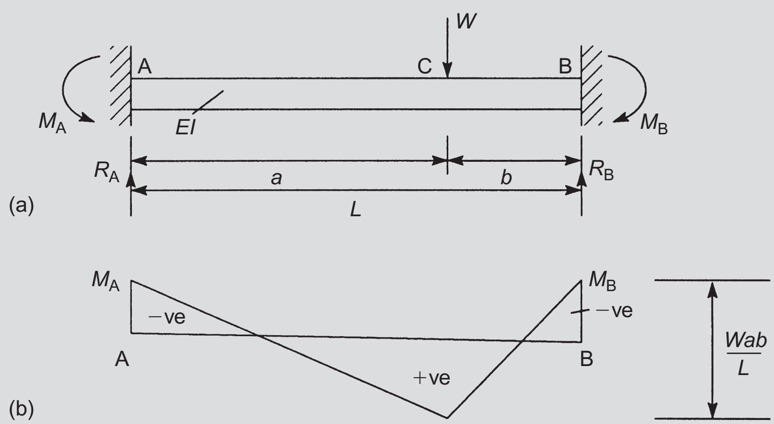

FIGURE 13.31

Fixed beam of Ex. 13.22.

In Eq. (ii) $(\mathrm{d}v/\mathrm{d}x)_{\mathrm{B}} = (\mathrm{d}v/\mathrm{d}x)_{\mathrm{A}} = 0$ and $v_{\mathrm{B}} = v_{\mathrm{A}} = 0$ so that

$$0 = \int_{\mathrm{A}}^{\mathrm{B}} \frac{M}{EI} x \, \mathrm{d}x \tag{iii}$$

and the moment of the area of the M/EI diagram between A and B about A is zero. Since EI is constant for the beam, we need only consider the bending moment diagram. Therefore from Fig. 13.31(b)

$$M_{\mathrm{A}} L \frac{L}{2} + (M_{\mathrm{B}} - M_{\mathrm{A}}) \frac{L}{2} \frac{2}{3} L = \frac{1}{2} a \frac{Wab}{L} \frac{2a}{3} + \frac{1}{2} b \frac{Wab}{L} \left(a + \frac{1}{3} b \right)$$

Simplifying, we obtain

$$M_{\mathrm{A}} + 2M_{\mathrm{B}} = \frac{Wab}{L^2}(2a + b) \tag{iv}$$

Solving Eqs (i) and (iv) simultaneously we obtain

$$M_{\mathrm{A}} = \frac{Wab^2}{L^2} \quad M_{\mathrm{B}} = \frac{Wa^2 b}{L^2} \tag{v}$$

We can now use statics to obtain R_{A} and R_{B}; hence, taking moments about B

$$R_{\mathrm{A}} L - M_{\mathrm{A}} + M_{\mathrm{B}} - Wb = 0$$

Substituting for M_{A} and M_{B} from Eq. (v) we have

$$R_{\mathrm{A}} L = \frac{Wab^2}{L^2} - \frac{Wa^2 b}{L^2} + Wb$$

whence

$$R_{\mathrm{A}} = \frac{Wb^2}{L^3}(3a + b)$$

Similarly

$$R_{\mathrm{B}} = \frac{Wa^2}{L^3}(a + 3b)$$

EXAMPLE 13.23

The fixed beam shown in Fig. 13.32(a) carries a uniformly distributed load of intensity w. Determine the support reactions.

From symmetry, $M_{\mathrm{A}} = M_{\mathrm{B}}$ and $R_{\mathrm{A}} = R_{\mathrm{B}}$. Again the net area of the bending moment diagram must be zero since the change of slope between the ends of the beam is zero (Eq. (13.7)). Hence

$$M_{\mathrm{A}} L = \frac{2}{3} \frac{wL^2}{8} L$$

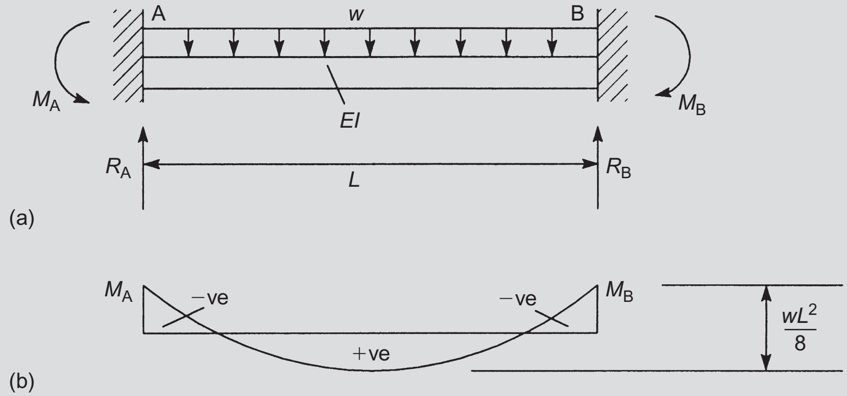

FIGURE 13.32

Fixed beam carrying a uniformly distributed load (Ex. 13.23).

so that

$$M_A = M_B = \frac{wL^2}{12}$$

From statics

$$R_A = R_B = \frac{wL}{2}$$

EXAMPLE 13.24

The fixed beam of Fig. 13.33 carries a uniformly distributed load over part of its span. Determine the values of the fixed-end moments.

Consider a small element δx of the distributed load. We can use the results of Ex. 13.22 to write down the fixed-end moments produced by this elemental load since it may be regarded, in the limit as $\delta x \rightarrow 0$, as a concentrated load. Therefore from Eq. (v) of Ex. 13.22 we have

$$\delta M_A = w\,\delta x \frac{x(L-x)^2}{L^2}$$

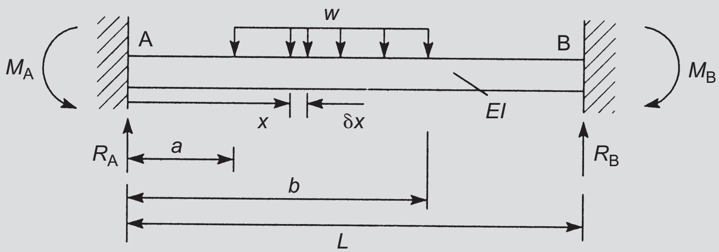

FIGURE 13.33

Fixed beam with part span uniformly distributed load (Ex. 13.24).

The total moment at A, M_A, due to all such elemental loads is then

$$M_A = \int_a^b \frac{w}{L^2} x(L-x)^2 \, dx$$

which gives

$$M_A = \frac{w}{L^2}\left[\frac{L^2}{2}(b^2-a^2) - \frac{2}{3}L(b^3-a^3) + \frac{1}{4}(b^4-a^4)\right] \qquad \text{(i)}$$

Similarly

$$M_B = \frac{wb^3}{L^2}\left(\frac{L}{3} - \frac{b}{4}\right) \qquad \text{(ii)}$$

If the load covers the complete span, $a=0$, $b=L$ and Eqs (i) and (ii) reduce to

$$M_A = M_B = \frac{wL^2}{12} \quad \text{(as in Ex. 13.21.)}$$

Fixed beam with a sinking support

In most practical situations the ends of a fixed beam would not remain perfectly aligned indefinitely. Since the ends of such a beam are prevented from rotating, a deflection of one end of the beam relative to the other induces fixed-end moments as shown in Fig. 13.34(a). These are in the same sense and for the relative displacement shown produce a total anticlockwise moment equal to $M_A + M_B$ on the beam. This moment is equilibrated by a clockwise couple formed by the force reactions at the supports. The resultant bending moment diagram is shown in Fig. 13.34(b) and, as in previous examples, its net area is zero since there is no change of slope between the ends of the beam and EI is constant (see Eq. (13.7)). This condition is satisfied by $M_A = M_B$.

Let us now assume an origin for x at A; Eq. (13.10) becomes

$$x_B\left(\frac{dv}{dx}\right)_B - x_A\left(\frac{dv}{dx}\right)_A - (v_B - v_A) = \int_A^B \frac{M}{EI} x \, dx \qquad \text{(i)}$$

in which $(dv/dx)_A = (dv/dx)_B = 0$, $v_A = 0$ and $v_B = -\delta$. Hence Eq. (i) reduces to

$$\delta = \int_0^L \frac{M}{EI} x \, dx$$

Using the semi-graphical approach and taking moments of areas about A we have

$$\delta = -\frac{1}{2}\frac{L}{2}\frac{M_A}{EI}\frac{L}{6} + \frac{1}{2}\frac{L}{2}\frac{M_A}{EI}\frac{5}{6}L$$

which gives

$$M_A = \frac{6EI\delta}{L^2} \quad \text{(hogging)}$$

It follows that

$$M_B = \frac{6EI\delta}{L^2} \quad \text{(sagging)}$$

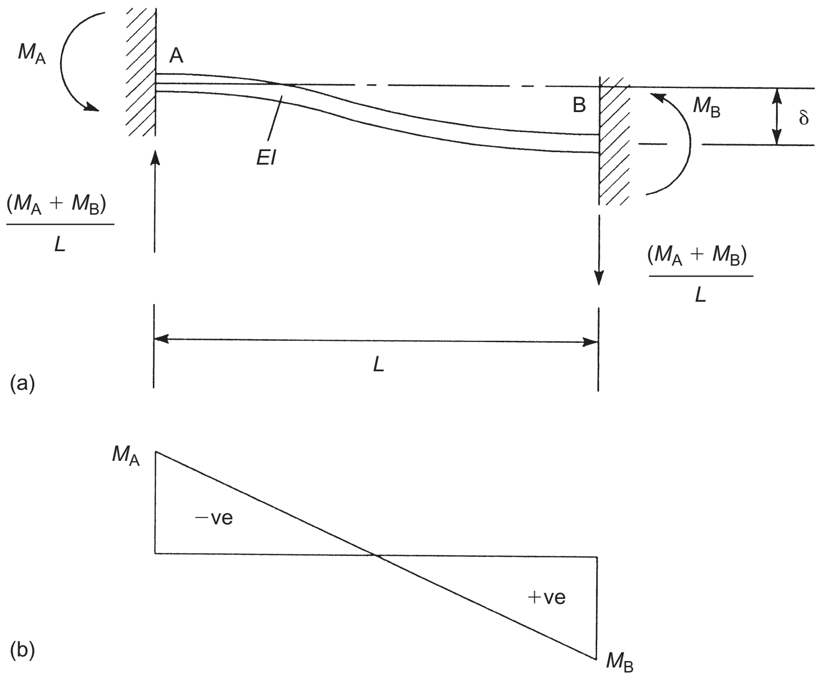

FIGURE 13.34

Fixed beam with a sinking support.

EXAMPLE 13.25

The balcony shown in Fig. 13.35 comprises a concrete slab 1.5 m wide and 100 mm thick. The slab is supported at intervals of 3 m by steel joists having a self-weight of 17 kg/m and a second moment of area about a horizontal axis of 800×10^4 mm^4. Each joist is built-in at its wall end and supported at its outer end by a hollow circular section steel column of external diameter 80 mm and having walls 5 mm thick. The connection between a joist and the top of a column allows rotation of the joist at its outer end. If the density of concrete is 2000 kg/m^3 and the balcony must be designed for an imposed load of 2 kN/m^2, calculate the maximum bending moment in a joist allowing for the effect of the compressive load in the supporting column.

$$\text{Take } E \text{ for steel as } 200000\ \text{N/mm}^2.$$

$$\text{Weight of the concrete slab/joist} = 2000 \times 9.81 \times 1.5 \times 3.0 \times 0.1 \times 10^{-3} = 8.83\ \text{kN}.$$

$$\text{Weight of a joist} = 17 \times 9.81 \times 1.5 \times 10^{-3} = 0.25\ \text{kN}.$$

$$\text{Imposed load/joist} = 2 \times 1.5 \times 3.0 = 9.0\ \text{kN}.$$

$$\text{Total load/joist} = 8.83 + 0.25 + 9.0 = 18.08\ \text{kN}.$$

$$\text{Then the uniformly distributed load/joist} = \frac{18.08}{1.5} = 12.05\ \text{kN/m}.$$

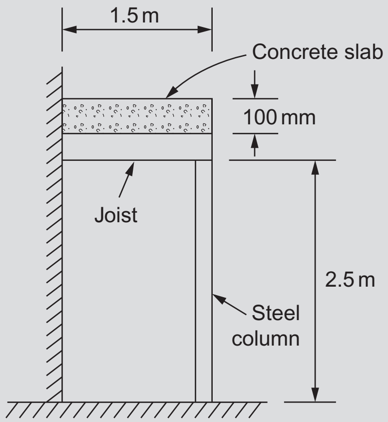

FIGURE 13.35

Balcony of Ex. 13.25.

From Ex. 13.20, the vertical reaction between a column and a joist is given by

$$\text{Vertical reaction} = \frac{3}{8} \times 12.05 \times 1.5 = 6.78\,\text{kN}$$

$$\text{The shortening of the column is then} = \frac{6.78 \times 10^3 \times 2.5 \times 10^3}{(\pi/4)(80^2 - 70^2) \times 200000} = 0.072\,\text{mm}$$

From Table 16.6, this shortening will induce an anticlockwise moment at the built-in end of a joist of

$$\text{Moment} = \frac{3 \times 200000 \times 800 \times 10^4 \times 0.072}{(1.5 \times 10^3)^2} = 0.154 \times 10^6\,\text{Nmm}$$

i.e.

$$\text{Moment} = 0.154\ \text{kNm}.$$

The bending moment at the built-in end due to the applied load and self-weight is, from Ex. 13.20

$$\text{Moment} = \frac{1}{8} \times 12.05 \times 1.5^2 = 3.39\,\text{kNm},$$

which is also anticlockwise. The total bending moment at the built-in end of a joist is then $0.154 + 3.39 = 3.544$ kNm, which will be the maximum value.

The effect of building in the ends of a beam is to increase both its strength and its stiffness. For example, the maximum bending moment in a simply supported beam carrying a central concentrated load W is $WL/4$ but it is $WL/8$ if the ends are built-in. A comparison of the maximum deflections shows a respective reduction from $WL^3/48EI$ to $WL^3/192EI$. It would therefore appear desirable for all beams to have their ends built-in if possible. However, in practice this is rarely done since, as we have seen, settlement of one of the supports induces additional bending moments in a beam. It is also clear that such moments can be induced during erection unless the supports are perfectly aligned. Furthermore, temperature changes can induce large stresses while live loads, which produce vibrations and fluctuating bending moments, can have adverse effects on the fixity of the supports.

One method of eliminating these difficulties is to employ a double cantilever construction. We have seen that points of contraflexure (i.e. zero bending moment) occur at sections along a fixed beam. Thus if hinges were positioned at these points the bending moment diagram and deflection curve would be unchanged but settlement of a support or temperature changes would have little or no effect on the beam.

PROBLEMS

P.13.1 The beam shown in Fig. P.13.1 is simply supported symmetrically at two points 2 m from each end and carries a uniformly distributed load of 5 kN/m together with two concentrated loads of 2 kN each at its free ends. Calculate the deflection at the mid-span point and at its free ends using the method of double integration. $EI = 43 \times 10^{12}$ Nmm^2.

Ans. 3.6 mm (downwards), 2.0 mm (upwards).

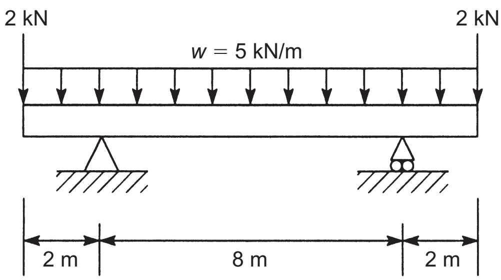

FIGURE P.13.1

P.13.2 A beam AB of length L (Fig. P.13.2) is freely supported at A and at a point C which is at a distance KL from the end B. If a uniformly distributed load of intensity w per unit length acts on AC, find the value of K which will cause the upward deflection of B to equal the downward deflection midway between A and C.

Ans. 0.24.

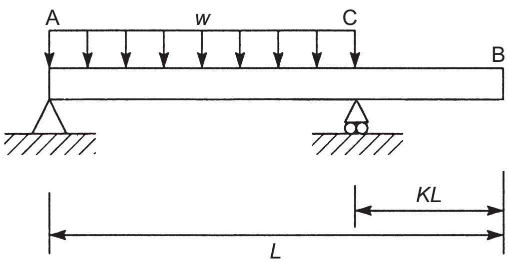

FIGURE P.13.2

P.13.3 A uniform beam is simply supported over a span of 6 m. It carries a trapezoidally distributed load with intensity varying from 30 kN/m at the left-hand support to 90 kN/m at the right-hand support. Find the equation of the deflection curve and hence the deflection at the mid-span point. The second moment of area of the cross section of the beam is 120×10^6 mm^4 and Young's modulus $E = 206\,000$ N/mm^2.

Ans. 41 mm (downwards).

P.13.4 A cantilever of length L and having a flexural rigidity EI carries a distributed load that varies in intensity from w per unit length at the built-in end to zero at the free end. Find the deflection of the free end.

Ans. $wL^4/30EI$ (downwards).

P.13.5 Determine the position and magnitude of the maximum deflection of the simply supported beam shown in Fig. P.13.5 in terms of its flexural rigidity EI.

Ans. $38.8/EI$ m downwards at 2.9 m from left-hand support.

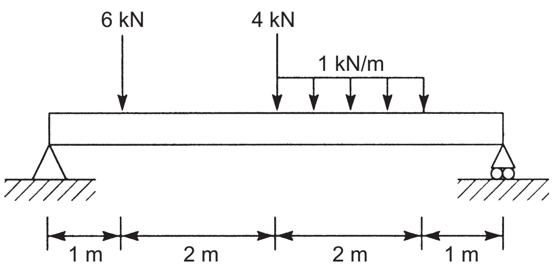

FIGURE P.13.5

P.13.6 Calculate the position and magnitude (in terms of EI) of the maximum deflection in the beam shown in Fig. P.13.6.

Ans. $1309.2/EI$ m downwards at 13.3 m from left-hand support.

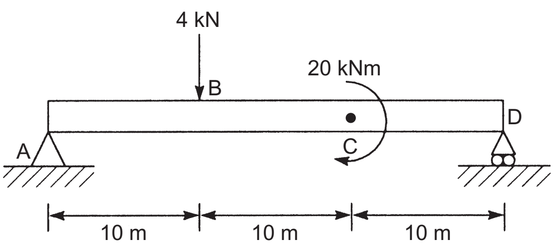

FIGURE P.13.6

P.13.7 Determine the equation of the deflection curve of the beam shown in Fig. P.13.7. The flexural rigidity of the beam is EI.

Ans.

$$v = -\frac{1}{EI}\left\{\frac{125}{6}x^3 - 50[x-1]^2 + \frac{50}{12}[x-2]^4 - \frac{50}{12}[x-4]^4 - \frac{525}{6}[x-4]^3 + 237.5x\right\}.$$

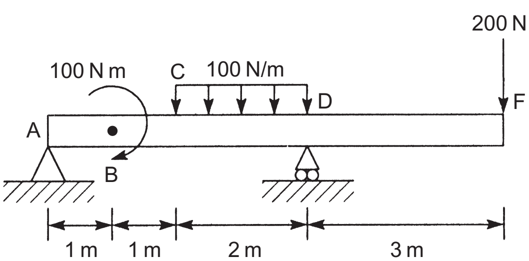

FIGURE P.13.7

P.13.8 The beam shown in Fig. P.13.8 has its central portion reinforced so that its flexural rigidity is twice that of the outer portions. Use the moment-area method to determine the central deflection.

Ans. $3WL^3/256EI$ (downwards).

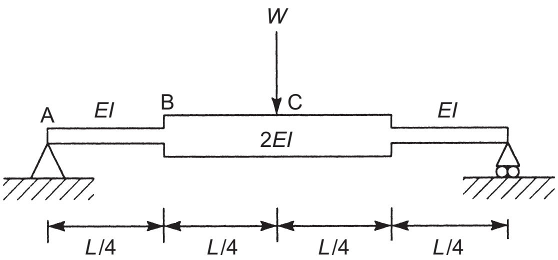

FIGURE P.13.8

P.13.9 A simply supported beam of flexural rigidity EI carries a triangularly distributed load as shown in Fig. P.13.9. Determine the deflection of the mid-point of the beam.

Ans. $w_0L^4/120EI$ (downwards).

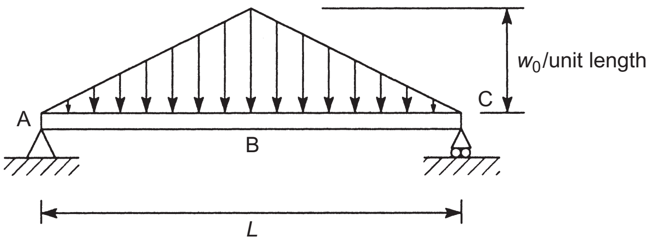

FIGURE P.13.9

P.13.10 The simply supported beam shown in Fig. P.13.10 has its outer regions reinforced so that their flexural rigidity may be regarded as infinite compared with the central region. Determine the central deflection.

Ans. $7WL^3/384EI$ (downwards).

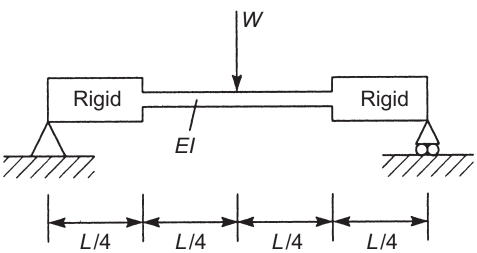

FIGURE P.13.10

P.13.11 Calculate the horizontal and vertical components of the deflection at the centre of the simply supported span AB of the thick Z-section beam shown in Fig. P.13.11. Take $E = 200\,000$ N/mm^2.

Ans. $u = 2.45$ mm (to right), $v = 1.78$ mm (upwards).

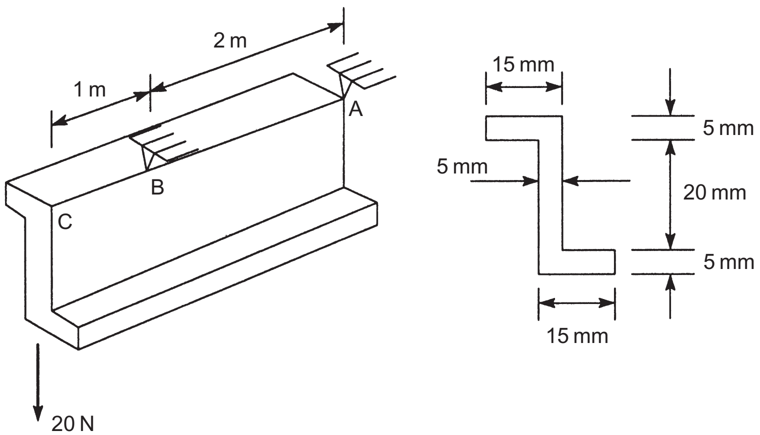

FIGURE P.13.11

P.13.12 A cantilever beam of length 5 m has the cross section shown in Fig. P.13.12. If the beam carries a vertically downward uniformly distributed load of intensity 10 kN/m calculate the magnitude and direction of the deflection of the free end of the beam. Young's modulus $E=15000$ N/mm^2.

Ans. 4.6 mm at 24.3° to the right of vertical.

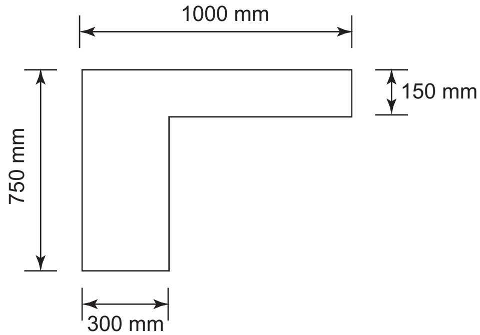

FIGURE P.13.12

P.13.13 The simply supported beam shown in Fig. P.13.13 supports a uniformly distributed load of 10 N/mm in the plane of its horizontal flange. The properties of its cross section referred to horizontal and vertical axes through its centroid are $I_z=1.67\times10^6$ mm^4, $I_y=0.95\times10^6$ mm^4 and $I_{zy}=-0.74\times10^6$ mm^4. Determine the magnitude and direction of the deflection at the mid-span section of the beam. Take $E=70\ 000$ N/mm^2.

Ans. 52.3 mm at 23.9° below horizontal.

P.13.14 A uniform cantilever of arbitrary cross section and length L has section properties I_z, I_y and I_{zy} with respect to the centroidal axes shown (Fig. P.13.14). It is loaded in the vertical plane by a tip load W. The tip of the beam is hinged to a horizontal link which constrains it to move in the vertical direction only (provided that the actual deflections are small). Assuming that the link is rigid and that there are no twisting effects, calculate the force in the link and the deflection of the tip of the beam.

Ans. WI_{zy}/I_z (compression if I_{zy} is positive), $WL^3/3EI_z$ (downwards).

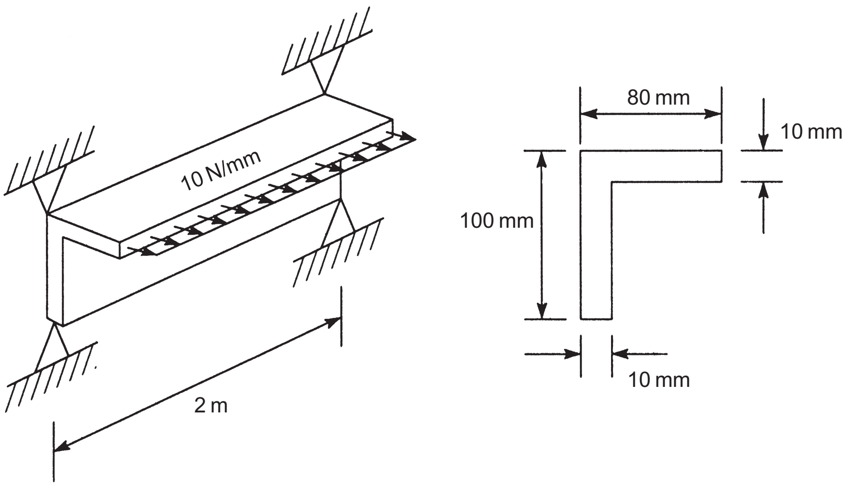

FIGURE P.13.13

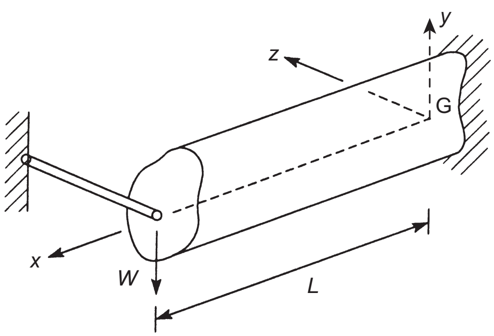

FIGURE P.13.14

P.13.15 A thin-walled beam has the cross section shown in Fig. P.13.15 and is simply supported over a span of 2 m. If the beam carries a horizontal uniformly distributed load of 10 kN/m applied in the plane of its flange together with a vertical uniformly distributed load of 20 kN/m applied in the plane of its leg, both loads being over the complete span, calculate the magnitude and direction of the deflection of the mid-span point. Take $E = 200000$ N/mm^2.

Ans. 24.4 mm at 64.8° to the right of vertical.

P.13.16 A thin-walled beam is simply supported at each end and supports a uniformly distributed load of intensity w per unit length in the plane of its lower horizontal flange (see Fig. P.13.16). Calculate the horizontal and vertical components of the deflection of the mid-span point. Take $E = 200$ 000 N/mm^2.

Ans. $u = -9.1$ mm, $v = 5.2$ mm.

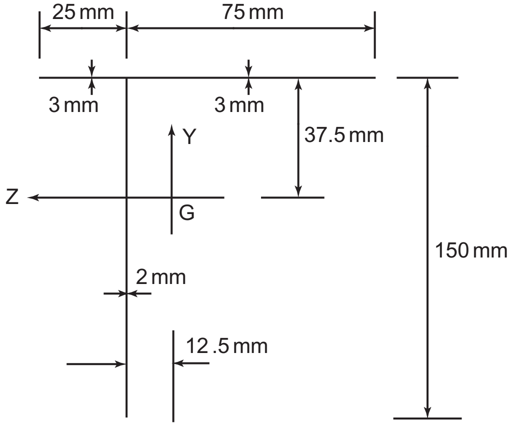

FIGURE P.13.15

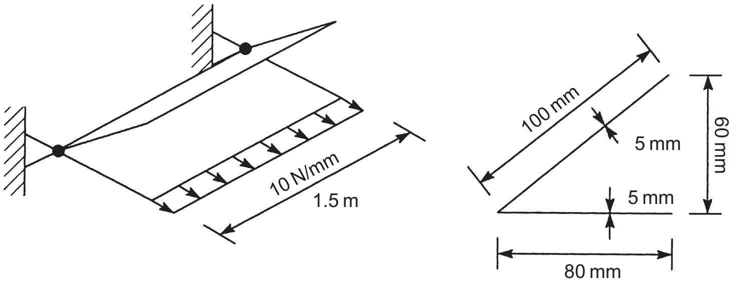

FIGURE P.13.16

P.13.17 A uniform beam of arbitrary unsymmetrical cross section and length $2L$ is built-in at one end and is simply supported in the vertical direction at a point half-way along its length. This support, however, allows the beam to deflect freely in the horizontal z direction (Fig. P.13.17). Determine the vertical reaction at the support.

Ans. 5 W/2.

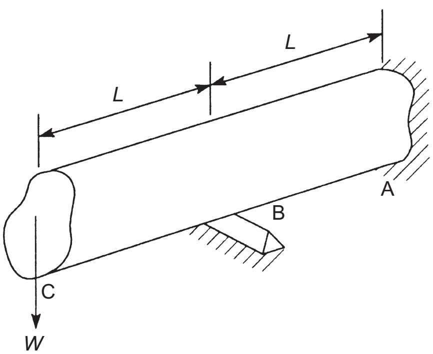

FIGURE P.13.17

P.13.18 A cantilever of length $3L$ has section second moments of area I_z, I_y and I_{zy} referred to horizontal and vertical axes through the centroid of its cross section. If the cantilever carries a vertically downward load W at its free end and is pinned to a support which prevents both vertical and horizontal movement at a distance $2L$ from the built-in end, calculate the magnitude of the vertical reaction at the support. Show also that the horizontal reaction is zero.

Ans. $7W/4$.

P.13.19 A beam of length 1.5 m has a rectangular cross section of width 50 mm and depth 200 mm. The beam is simply supported over a span of 1.0 m with an overhang of 0.5 m and carries a vertically downward load of 200 kN at the free end of the overhang. Calculate the deflection of the beam midway between the supports allowing for the effects of both bending and shear. Take $E = 200000$ N/mm^2 and $G = 70000$ N/mm^2. What percentage of the total deflection is due to shear?

Ans. 1.03 mm upwards, 8.7%.

P.13.20 Calculate the deflection due to shear at the mid-span point of a simply supported rectangular section beam of length L which carries a vertically downward load W at mid-span. The beam has a cross section of breadth B and depth D; the shear modulus is G.

Ans. $3WL/10GBD$ (downwards).

P.13.21 Determine the deflection due to shear at the free end of a cantilever of length L and rectangular cross section $B \times D$ which supports a uniformly distributed load of intensity w. The shear modulus is G.

Ans. $3wL^2/5GBD$ (downwards).

P.13.22 A cantilever of length L has a solid circular cross section of diameter D and carries a vertically downward load W at its free end. The modulus of rigidity of the cantilever is G. Calculate the shear stress distribution across a section of the cantilever and hence determine the deflection due to shear at its free end.

Ans. $\tau = 16W(1-4y^2/D^2)/3\pi D^2$, $40WL/9\pi GD^2$ (downwards).

P.13.23 Show that the deflection due to shear in a rectangular section beam supporting a vertical shear load S_y is 20% greater for a shear stress distribution given by the expression

$$\tau = -\frac{S_y A' \bar{y}}{b_o I_z}$$

than for a distribution assumed to be uniform.

A rectangular section cantilever beam 200 mm wide by 400 mm deep and 2 m long carries a vertically downward load of 500 kN at a distance of 1 m from its free end. Calculate the deflection at the free end taking into account both shear and bending effects. Take $E = 200\,000$ N/mm^2 and $G = 70\,000$ N/mm^2.

Ans. 2.06 mm (downwards).

P.13.24 Determine the form factor β for the beam section shown in Fig. P.13.24.

Ans. 1.88.

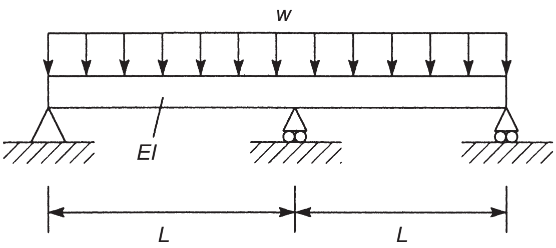

FIGURE P.13.24

P.13.25 A cantilever beam of length L has a solid circular cross section of diameter D and carries a vertically downward load W at its free end. Calculate the distribution of shear stress in a cross section of the beam and hence the form factor β. What is the deflection due to shear at the free end of the cantilever? The shear modulus is G and note that $\int_{-\pi/2}^{\pi/2} \cos^6\theta\,d\theta = 5\pi/16$

Ans. $\beta = 10/9$, $40WL/9\pi GD^2$.

P.13.26 The beam shown in Fig. P.13.26 is simply supported at each end and is provided with an additional support at mid-span. If the beam carries a uniformly distributed load of intensity w and has a flexural rigidity EI, use the principle of superposition to determine the reactions in the supports.

Ans. $5wL/4$ (central support), $3wL/8$ (outside supports).

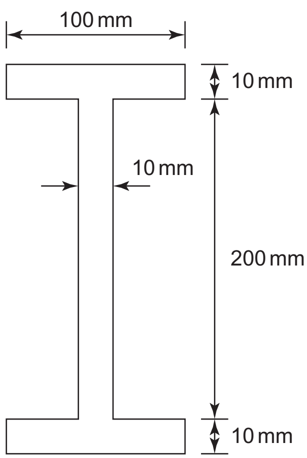

FIGURE P.13.26

P.13.27 A built-in beam ACB of span L carries a concentrated load W at C a distance a from A and b from B. If the flexural rigidity of the beam is EI, use the principle of superposition to determine the support reactions.

Ans. $R_A = Wb^2(L+2a)/L^3$, $R_B = Wa^2(L+2b)/L^3$, $M_A = Wab^2/L^2$, $M_B = Wa^2b/L^2$.

P.13.28 A beam has a second moment of area I for the central half of its span and $I/2$ for the outer quarters. If the beam carries a central concentrated load W, find the deflection at mid-span if the beam is simply supported and also the fixed-end moments when both ends of the beam are built-in.

Ans. $3WL^3/128EI$, $5WL/48$.

P.13.29 A cantilever beam projects 1.5 m from its support and carries a uniformly distributed load of 16 kN/m over its whole length together with a load of 30 kN at 0.75 m from the support. The outer end rests on a prop which compresses 0.12 mm for every kN of compressive load. If the value of EI for the beam is 2000 kNm^2, determine the reaction in the prop.

Ans. 23.4 kN.

P.13.30. If the attachment between a joist and a column in Ex. 13.25 prevents rotation of the joist calculate the bending moment in the joist at the built-in end.

Ans. -3.98 kNm (hogging).