Signals, **Systems** and Control

Contents

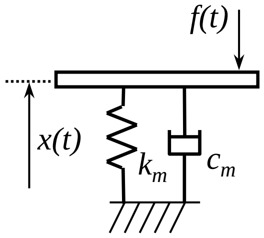
2.2 Lecture eight – Systems in the frequency domain

We ended the last lecture by stating that it was much easier to model systems in the frequency domain – the response of the systems is described by the Transfer Function (which is the ratio of output to input in the frequency domain) and thus the system output derived by simply multiplying the input by the Transfer Function. In this lecture we will show how to drive the transfer function of common simple systems and how we can develop the general technique to create a multi-domain modelling tool.

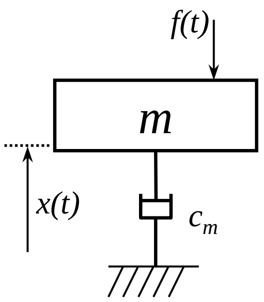
2.2.1 What do our systems look like?

We have described how our systems must conform to the LTI approximation in order to 'solve' them in the frequency domain, but what might our systems physically look like? Systems that conform to the LTI approximation tend to be electrical and mechanical – acoustic/aerodynamic/pneumatic less so because they often feature more significant non-linear effects. With care you can still use the frequency domain but you must be aware of the limitations.

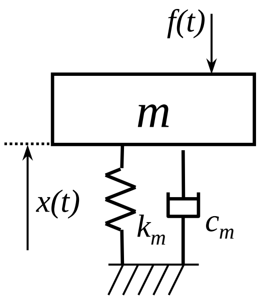
Consider these simple mechanical systems and their associated differential equations. In these systems the inputs and outputs are the forces (f(t)) and displacements (x(t)), and we start by writing down the differential equations found by summing forces.



$$f(t) = c_m \frac{dx}{dt} + k_m x$$

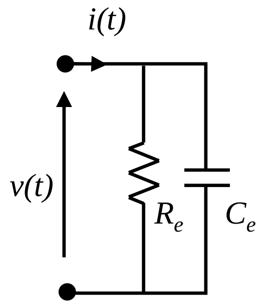


$$f(t) = m\frac{d^2x}{dt^2} + c_m \frac{dx}{dt}$$

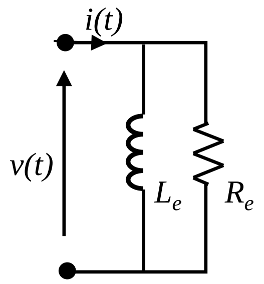


$$f(t) = m\frac{d^2x}{dt^2} + c_m \frac{dx}{dt} + k_m x$$

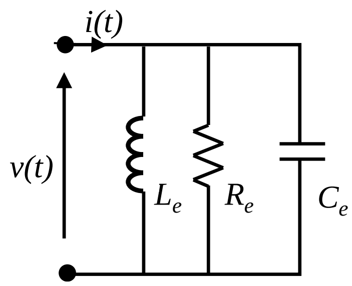
Similarly, we can look at electrical circuits, in this case summing currents. In this case our inputs/outputs are currents (i(t)) and voltages (v(t)).



$$i(t) = C_e\frac{dv}{dt} + \frac{1}{R_e}v$$



$$i(t) = \frac{1}{R_e}v + \frac{1}{L_e}\int vdt$$



$$i(t) = C_e\frac{dv}{dt} + \frac{1}{R_e}v + \frac{1}{L_e}\int vdt$$

It is useful to remember that our LTI system rules mean that there will be no constant terms in our differential expressions and we are at liberty to manipulate our expressions without worrying about the constant of integration.

2.2.2 Describing systems in the frequency domain

At the end of section 2.1 we have described how the differential equations describing the system can be used to derive the impulse response, and that we can take the Laplace of the impulse response to derive the transfer function. However, there are several steps required to go from the differential equations to the impulse response, hence there is still significant effort required.

The solution is to convert the differential equations describing the system directly into the frequency domain first and then these can then be rearranged algebraically to find the transfer function. We can do this because the Laplace is a *linear* operator and thus it doesn't matter the order in which we do our steps.

Laplace transforms are normally performed by using tables of known transform pairs and in our cases the two that are most useful are:

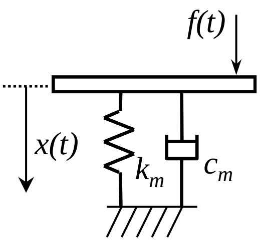
$$s \leftrightarrow \frac{d}{dt} \qquad \text{and for higher orders} \qquad s^n \leftrightarrow \frac{d^n}{dt^n}$$

$$\frac{1}{s} \leftrightarrow \int \quad dt \qquad \text{and for higher orders} \qquad \underbrace{\int \dots \int}_{n} \quad \leftrightarrow \frac{1}{s^n}$$

This approach is the same as solving ODE's using *differential operators.*

Examples

a) Spring with damping



step 1 derive the differential equation and take the Laplace:

$$f(t) = c_m \frac{dx}{dt} + k_m x \quad \xrightarrow{\mathcal{L}} \quad F(s) = c_m s X(s) + k_m X(s)$$

We can walk that through if you need more explanation:

- Our time domain signals *f(t)* and *x(t)* transform to *F(s)* and *X(s)* – capitalise the letter denoting the signal and replace the argument with 's'. It is common not to use the argument on the right hand side of these expressions (I didn't use '*x(t)*' in the time domain expression and you likely didn't bat an eyelid!)
- The differential *dx/dt* produces a 's' in the Laplace domain, so maps to *sX(s)*
- The constant parameter values stay the same.

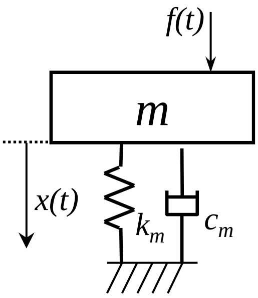
The next step is to factorise and rearrange in the Laplace domain:

$$\frac{F(s)}{X(s)} = c_m s + k_m$$

Now, by definition: ***Output = Transfer function x Input***, so **TF = output / input.** If the input to the system is the force (*F(s)*):

$$TF = \frac{output}{input} = \frac{X(s)}{F(s)} = \frac{1}{c_m s + k_m}$$

b) Mass with spring and damper



$$f(t) = m\frac{d^2x}{dt^2} + c_m\frac{dx}{dt} + k_m x$$

$$F(s) = ms^2X(s) + c_m sX(s) + k_m X(s)$$

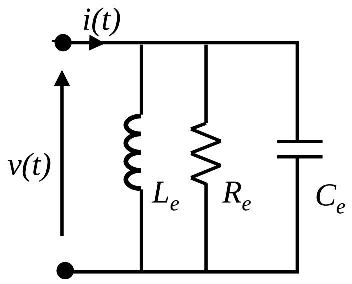
$$\frac{F(s)}{X(s)} = ms^2 + c_m s + k_m$$

$$TF = \frac{output}{input} = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + c_m s + k_m}$$

How about if the input was the displacement and the force was our output?

$$TF = \frac{output}{input} = \frac{F(s)}{X(s)} = ms^2 + c_m s + k_m$$

c) Parallel capacitor, resistor and inductor



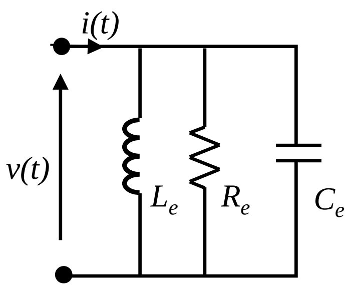
$$i(t) = C_e \frac{dv}{dt} + \frac{1}{R_e} v + \frac{1}{L_e} \int v dt$$

$$I(s) = C_e s V(s) + \frac{1}{R_e} V(s) + \frac{1}{L_e s} V(s)$$

$$\frac{I(s)}{V(s)} = C_e s + \frac{1}{R_e} + \frac{1}{L_e s}$$

2.2.3 Impedance modelling

If you have ever worked with electrical circuits you are probably thinking the method used in that last example is not how you would go about solving that circuit – you would likely use impedances:



You might start with Ohm's law:

$$\frac{I}{V} = \frac{1}{Z_p}$$

Where Z_p is the parallel impedance of the network

and where:

$$Z_p = \frac{1}{(\frac{1}{Z_L} + \frac{1}{Z_R} + \frac{1}{Z_C})}$$

You will (hopefully) recall that:

$$Z_L = j\omega L \qquad Z_R = R \qquad Z_c = \frac{1}{j\omega C}$$

so:

$$Z_p = \frac{1}{(\frac{1}{j\omega L_e} + \frac{1}{R_e} + j\omega C_e)}$$

Now if we let $s = j\omega$ *

$$\frac{I(s)}{V(s)} = \frac{1}{Z_p} = C_e s + \frac{1}{R_e} + \frac{1}{L_e s}$$

** Why did we remove the real part of S? Setting S =jw gives the 'steady state' response of a system e.g. the response to a continuous sine wave excitation.*

We have arrived at the same expression in a slightly different manner. When dealing with impedances we *started* with expressions in the frequency domain, skipping differential equations altogether. Once you are familiar with working with impedances solving more complex systems becomes easy – you just need to remember Ohm's Law and that series impedances add and for parallel impedances it is the reciprocal of the reciprocals added i.e.:

$$Z_{Total\_Series} = Z_1 + Z_2 + \cdots Z_3 \qquad Z_{Total\_Parallel} = \frac{1}{(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots \frac{1}{Z_n})}$$

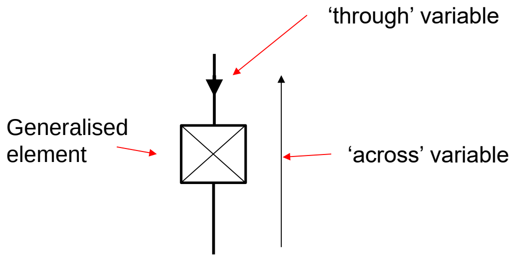
A TF is the ratio of two complex qualities, defined as output/input, but impedance is a little more constrained as it the ratio of voltage/current. However, even if our TF is not a voltage/current ratio we can still use impedances to derive the TF we want.

We can apply the same thinking to mechanical systems: The mechanical equivalent is called mechanical impedance.

2.2.4 Immittance

The general term for what we are looking at here is immittance – we will see where this term comes from shortly – and the concept and the associated system diagrams can be thought of as a semi-graphical technique to solve differential equations.

We define our system as made up of 'lumped elements', each with a through and across variable:



Impedance describes the ratio of *across/through*; **admittance** describes the ratio of *through/across,* or the inverse of impedance

You may now have spotted that immittance is a portmanteau of the impedance and admittance. The term was coined by H W Bode – a name we will come across in the next lecture. Find out more at: https://en.wikipedia.org/wiki/Hendrik_Wade_Bode

We can apply immittance theory to many domains where components conform to an LTI approximation. In the mechanical domain we refer to ***'mechanical impedance***' and '***mechanical admittance'.***

- The complex ratio of ***Volts to Amps*** is called ***impedance*** and measured in ***ohms***
- The complex ratio of ***Amps to Volts*** is called ***admittance*** and measured in ***siemens***
- The complex ratio of ***Force to Velocity*** is called ***mech. impedance*** and measured in ***ohms***
- The complex ratio of ***Velocity to Force*** is called ***mech. admittance*** (or '***mobility'***) and measured in ***siemens***

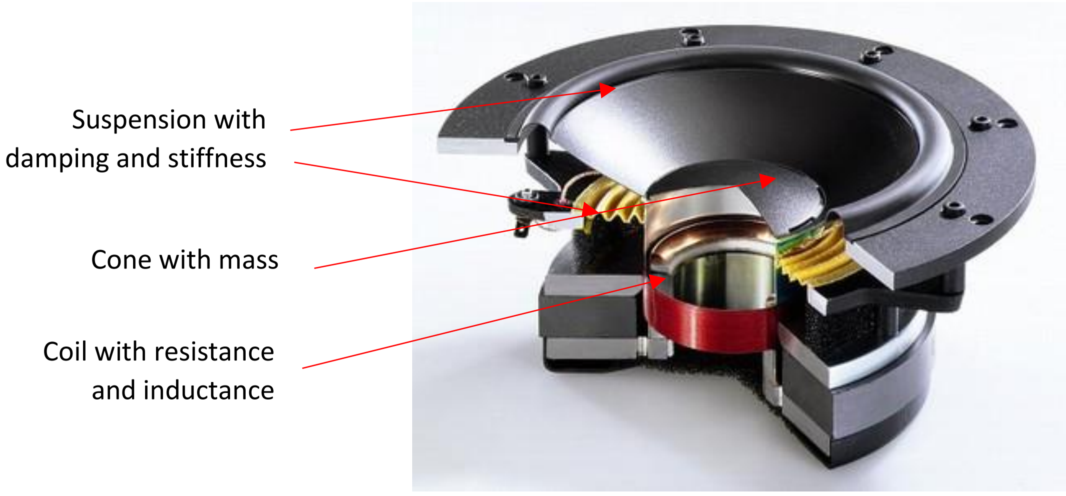
2.2.5 Electro-mechanical analogies

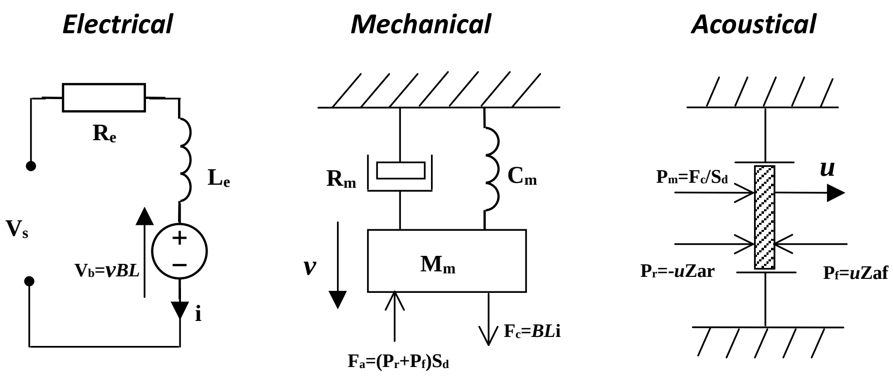
Having a unified mathematical description of systems in both the mechanical and electrical domains is incredibly useful. Firstly, we can use the well-developed tools of electrical circuit theory to help us solve mechanical systems; Secondly, we can model combined systems i.e. a system which has both mechanical and electrical components

We tend to view this as electrical techniques applied to mechanical systems, because although the maths is the same, the tools for circuit reduction and manipulation have already been developed for circuits. Circuit theory is just a ‘tool box’ for frequency domain problems; the frequency domain here is an approach to solve differential equations.

Loudspeaker example

Consider a loudspeaker – this device has dynamic components in the electrical, mechanical and acoustical domains (not shown)





R_e = Resistance.
L_e = Inductance.
V_b = Back emf.
i = Coil current.
V_s = Input signal.
BL = Tesla/metre product.

R_m = Mechanical loss.
C_m = Suspension Compliance.
M_m = Moving mass.
v = Cone velocity.
F_c = Force from voice coil.
F_a = Force reflected back through cone.
S_d = Cone surface area.

u = Volume velocity.
P_m = Pressure created by force from voice coil.
Z_{ar} = Radiation impedance acting on rear of cone.
Z_{af} = Radiation impedance acting on front of cone.

We can draw our loudspeaker as a schematic and use our electro-mechanical analogies to investigate this system, but we need to understand how we describe components across three domains in a way that is unified – this is where the analogies come in.

First, we establish whether to describe each domain as impedances or admittances. To explore this, we can consider the differential equations of mechanical and electrical systems and consider which variables map to which:

Impedance analogy

First consider this equation for a mechanical systems comprising mass, compliance and damping:

$$f = M\frac{dv}{dt} + Cv + K\int vdt$$

We can compare it to this expression for an electrical system comprising inductance, resistance and capacitance. Note we have denoted voltage as 'e' to distinguish it from velocity (one of the problems of multi-domain modelling is you run into conflict with symbols!)

$$e = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt$$

Drawing equivalence between these expressions is known as the 'impedance analogy' and results in:

***Variables**:*

Force, f = voltage, e

Velocity, v = current, i

Parameters:

Mass, M = inductance, L

Damping, C = resistance, R

Compliance, 1/K = Capacitance, C

The challenge for our loudspeaker is that the transducer is electromagnetic – i.e. a current in the electrical domain produces a force in the mechanical domain, hence we can't draw our system using the impedance analogy because the impedance analogy maps a voltage to a force.

Mobility analogy

This time we will take our differential expression describing a mechanical system and compare it to a different electrical expression where we have swapped 'e' and 'i':

$$f = M\frac{dv}{dt} + Cv + K\int vdt$$

$$i = C\frac{de}{dt} + \frac{1}{R}e + \frac{1}{L}\int e\,dt$$

Drawing equivalence between these expressions results in the mobility analogy which maps:

Variables;

Force, f = current, i

Velocity, v = voltage, e

Parameters;

Mass, M = Capacitance, C

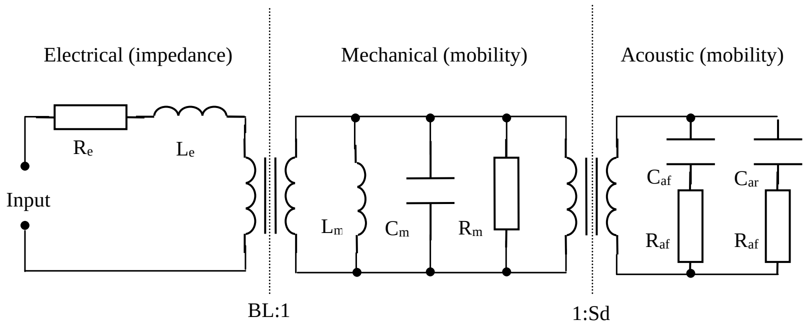
Damping, C = conductance, 1/R

Compliance, 1/K = Inductance, L

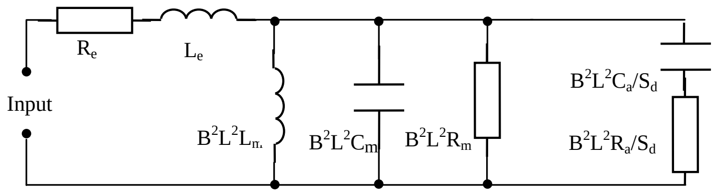
This now works with our electro-magnetic transducer but you will notice that we have 'flipped' the parameters in one domain – we have impedance in one domain and admittance in the other.

When we use the mobility analogy, by convention we deal with admittance in the mechanical domain – because most people are more familiar with electrical impedance and the physical components, so it is harder to think in admittance in the electrical domain.

We can now draw out an equivalent electric circuit for our loudspeaker:



In our explanation we will skip over describing the acoustic domain, but you will notice how the transducers (electromagnetic transduction between elec/mech and a piston between mech and acoustic) can be represented by transformers – a simple linear scaling between domains. This can be drawn as a simplified electrical circuit that can be readily analysed and solved:



For this unit I'm not going to ask you to learn electromechanical analogies like the loudspeaker example, but you should be aware that they exist and understand that electrical circuits are really just a method to solve differential equations, and that these techniques can be applied to other systems where the components conform to the LTI approximation.

2.2.6 Test yourself

1) Which types of physical system can be represented well in the frequency domain?
2) Which types of physical systems cannot always be represented in the frequency domain and why?
3) What is the first step in creating the TF for a system?
4) What are the Laplace transforms of the integral and differential operators?
5) What is the relationship between admittance and impedance?
6) Who was Bode?
7) What is the ratio of force to voltage known as?
8) In the loudspeaker example, what determines if we use the impedance or mobility analogy to model the mechanical components?