Signals, Systems and Control

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1.3 The complex exponential

1.3.1 Describing signals with the complex exponential function

In this lecture we are starting to explore mathematical functions that can describe signals. A useful place to start this analysis is with the complex exponential:

$$f(t) = Ce^{at}$$

Where 'C' and 'a' can be real or complex numbers ('t' is time)

The complex exponential is useful because it can describe many common signals, including the ***basis functions*** used in the frequency and Laplace domains.

> *A basis function is an elemental function from which more complex functions in the same domain are constructed.*

Let us consider what happens to the complex exponential function as we change the nature of 'C' and 'a'.

'C' and 'a' are real

When 'C' is real it is an amplitude *scaling* term; then:

If 'a' is real and >0 then *f(t)* is a growing exponential

If 'a' is real and <0 then *f(t)* is a decaying exponential

If 'a' is real and =0 then *f(t)* is a constant signal

'C' is real, 'a' is purely imaginary

'a' is imaginary, so let us substitute a = jω. We now have:

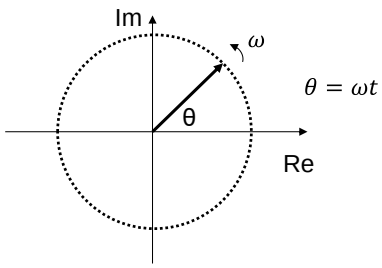
$$f(t) = Ce^{j\omega t}$$

and from Euler's identity we know that:

$$f(t) = Ce^{j\omega t} = Ccos(\omega t) + jCsin(\omega t)$$

So, what is the interpretation of this signal? It is a vector rotating on the complex plane with magnitude 'C' and rotational speed 'ω'. This is a *Phasor*.

This is not just abstract – the complex plane is often used to represent rotation in physical space, for example the cross section of an electrical machine (motor or generator). The phasor could also represent the energy in a lossless mass/spring system



$e^{j\omega t}$ Rotation CCW

$e^{-j\omega t}$ Rotation CW

Thinking more abstractly, ***encoding time as the phase*** in a polar coordinate system is central to being able to describe periodic functions, since the phase 'wraps' around every 360 degrees.

'C' is real, 'a' is complex (real and imaginary components)

How about when 'a' has real and imaginary parts?

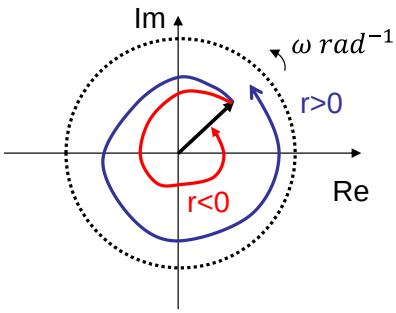
$$f(t) = Ce^{(r+j\omega)t}$$

rewrite as: $$f(t) = C(e^{rt}e^{j\omega t})$$

Exponential increase or decrease

Phasor with angular velocity ω

Depending on the sign of 'r' the signal is either a growing or a shrinking phasor – the signals will look like spirals on the complex plane:



'C' and 'a' are both complex with real and imaginary parts

First, we write the complex number 'C' in polar form: $C = |C|e^{j\theta}$

Then the function becomes: $f(t) = |C|e^{j\theta}e^{(r+j\omega)t}$

We can re-arrange this to $f(t) = |C|e^{rt}e^{j(\omega t+\theta)}$

Thus, this expression describes a spiral on the complex plane (increasing or decreasing) that can have a phase offset, Θ.

1.3.2 Interpretation of sines and cosines

In the previous section we described using time to encode the phase a function, thus creating periodic behaviour (time is linear, but phase wraps around). This then begs the question if a rotation in 2-dimensional space (i.e. on the complex plane) is fundamental to periodic functions what are common functions like sine/cosine describing?

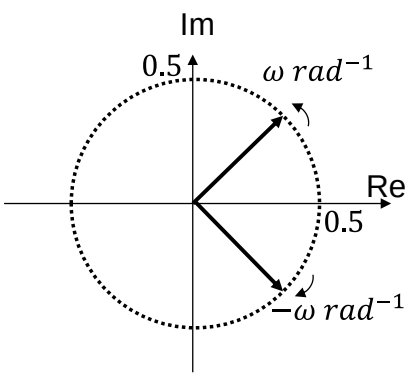
(you might like to stop and cogitate on this now)

This is easy to work out and an interesting result.

- *We know that:* $cos(\omega t) = Re(e^{j\omega t})$
- *And in general:* $Re(z) = \frac{1}{2}(z + \bar{z})$
- *Thus:*

$$cos(\omega t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

Hence cosine can be represented mathematically as the sum of two phasors of equal magnitude (1/2) rotating in opposite directions! Sine can be expressed in a similar way, from consideration of the expression for the imaginary part of a complex number.



You can think of Sine and Cosine as just incompletely specified rotations, and we will see later how this ambiguity manifests when we perform time/frequency transformations.

You may have realised some of these concepts from the first time you encountered the math; however, it is something I would like you all to have a grasp of – it is one of the 'threshold concepts' I would like you to come away with an appreciation of – so I will keep on trying to explain the importance. There is something special about a rotation in being able to elegantly describe periodic functions. If you try and describing a periodic function directly in a rectangular coordinate space (i.e. x,y) – they are inelegant (think of the power series expansions of sin and cos), or you can 'cheat' by twisting the maths of rotation (as above). A rotation in two-dimensional space has something fundamental about it.

$$sin(x) = Im\left(e^{jx}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n+1)!}$$

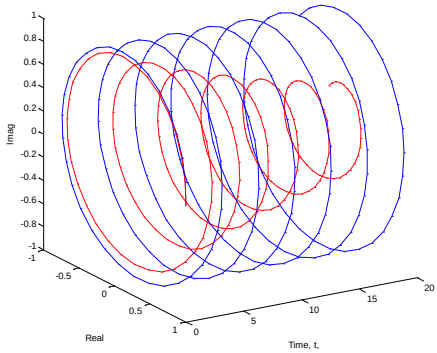
Sine can be represented elegantly in terms of a phasor, or through a cumbersome expansion

Finally for this section, if you think this is all very theoretical then you might be surprised that there are many real-word examples: The phases of our electricity supply act like the axes on our complex plane – each phase is a conductor providing a sinusoidal electric 'signal'. We have single phase supplied to our homes and a single phase induction motors (most domestic motors above ~500W) aren't very efficient because they can only produce a statically varying magnetic field – equivalent to producing a forward *and* a backward rotating magnetic field; the backward rotating field is wasted energy (single phase motors also cannot start without help). The three-phase supply found in many industrial or commercial premises have three sinusoidal electrical signals 120° offset, and allows motors to be considerably more efficient because they can produce a unique rotating magnetic field.

(You might note that we could also create a unique rotating field with two phases – one aligned with the real and imaginary axes of our motor and supplied with sine and cosine electrical signals, but three phases also have other useful properties since the sum of the three phases will be zero at any moment in time)

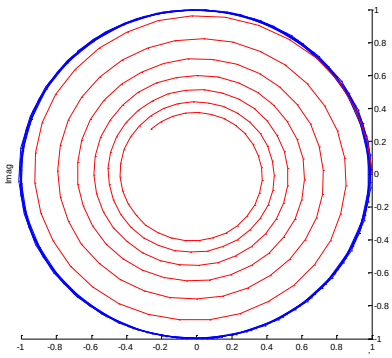
1.3.3 Visualising the complex exponential

Let's use MATLAB to plot $f(t) = Ce^{j\omega t}$, and $f(t) = Ce^{(r+j\omega)t}$

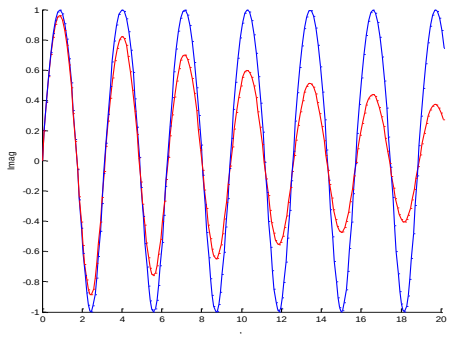


Plotting in 3 axes (t, Im, Re) we can see that $f(t) = Ce^{j\omega t}$ forms a helix of constant magnitude (blue trace)

$f(t) = Ce^{(r+j\omega)t}$ forms a helix decreasing in magnitude over time (red trace)



Looking on the complex plane we see the expected circular trace and the decreasing spiral



And looking on the time/ imaginary axis we see sinusoidal and decaying sinusoidal shapes

**Look at these plots in MALAB – use the plot tools to rotate the graphs to see these images*

1.3.4 Analytic signals

A signal that has both real and imaginary components defined as functions of time is known as an 'analytic signal'. These signals occupy three dimensions – two on the complex plane and time.

But why do we make life more complex? *(pun intended)*

- *As we have seen sinusoids are quite difficult to define on their own, but when thought of as the projection of a phasor along one axis they are simple – that might indicate something more fundamental about a phasor (i.e. rotation) than a sinusoid, and that can make the math 'work'.*
- *In practice real-world systems can conform to this math: consider the energy in a mass/spring oscillator – the energy in the mass follows the real part; the energy in the spring follows the imaginary part and the energy in the system follows the analytical signal.*

So analytic signals can be thought of as representing real phenomena, but also mathematically they are arguably more elegant. That all said, we don't use the analytic representation very often, and this maybe the first time you have encountered them - but there are important consequences that need to be borne in mind when we get to the transforms between time and frequency domains.

The real and imaginary parts of an analytic signal are linked by the 'Hilbert transform', and this transform can be used to reconstruct an analytic signal from a real-valued signal. For example, the Hilbert transform of a sine is a cosine.

1.3.5 An introduction to Fourier and Laplace

Fourier is credited with the idea that time domain waveforms can be represented by summations of sinusoids (his *basis functions*).

A signal made up of summed individual sinusoids is called a 'Fourier Series'. Fourier series are often used to describe periodic functions – you will likely have come across the idea that a square waveform is made up of various individual frequency components added together.

Since a sinusoid represents a single frequency component, then we can start to imagine how we map between time and frequency domains (time domain – t, frequency domain – ω or f).

> *However, when we look at the mathematical definition of the Fourier transform later you will see that the basis function are analytic signals, rather than sinusoids.*

Fourier representations work well for periodic signals that continue indefinitely or signals which we can define as constant over some time. Thus, Fourier analysis in its various forms is very useful for signals.

Laplace went one stage further than Fourier and represented his time domain functions as summations of sinusoidal *basis functions* which could grow or shrink in amplitude. The Laplace domain is the ***complex frequency domain***.

But, as before the maths uses analytic representations!

The physical significance of basis functions that grow or shrink over time is that the complex frequency domain can model systems that gain or lose energy over time – e.g. the free response of a mass/spring/damper.

This idea is central to the control theory you will study next term – stable systems conserve or lose 'energy' over time, unstable systems gain energy.

1.3.6 Visualising transformations into the frequency domain

The signals below show how an analytical signal made up of a single phasor maps to a single frequency of particular in the frequency domain. The lower signal is the sum of the two previous. **Note**: this is simplified and a stepping-stone to the full description we will build up to later.

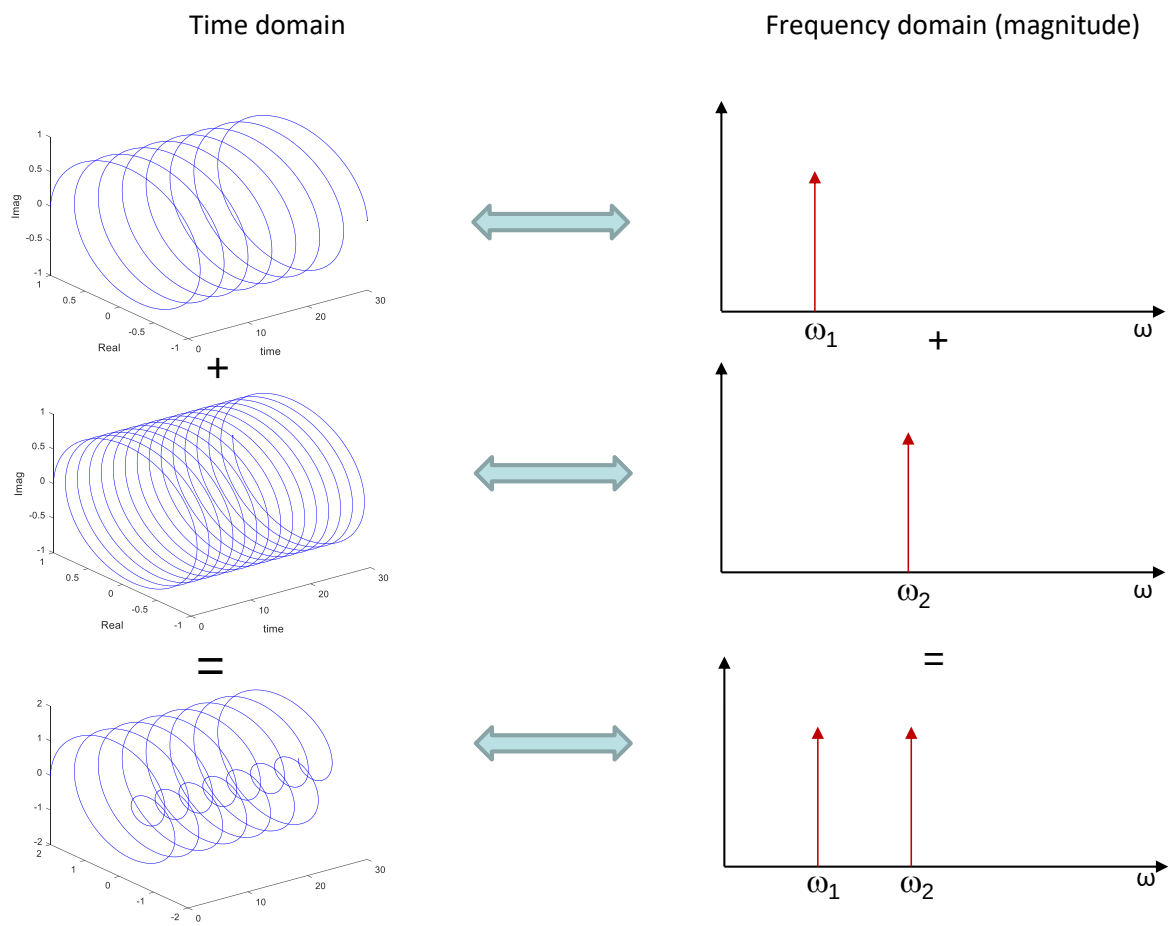


Fig 1. A simple mapping of our analytical signals to the frequency domain (the lower signal is the sum of the first two signals). Note the transformation is reversable.

The transform is reversible, we can go back and forth between domains. Both the time and frequency domain is continuous, but note how the continuous signal in the time domain maps to a scaled impulse in the frequency domain.

So, what happens if we transform non-analytical, singled-valued, signals into the frequency domain?

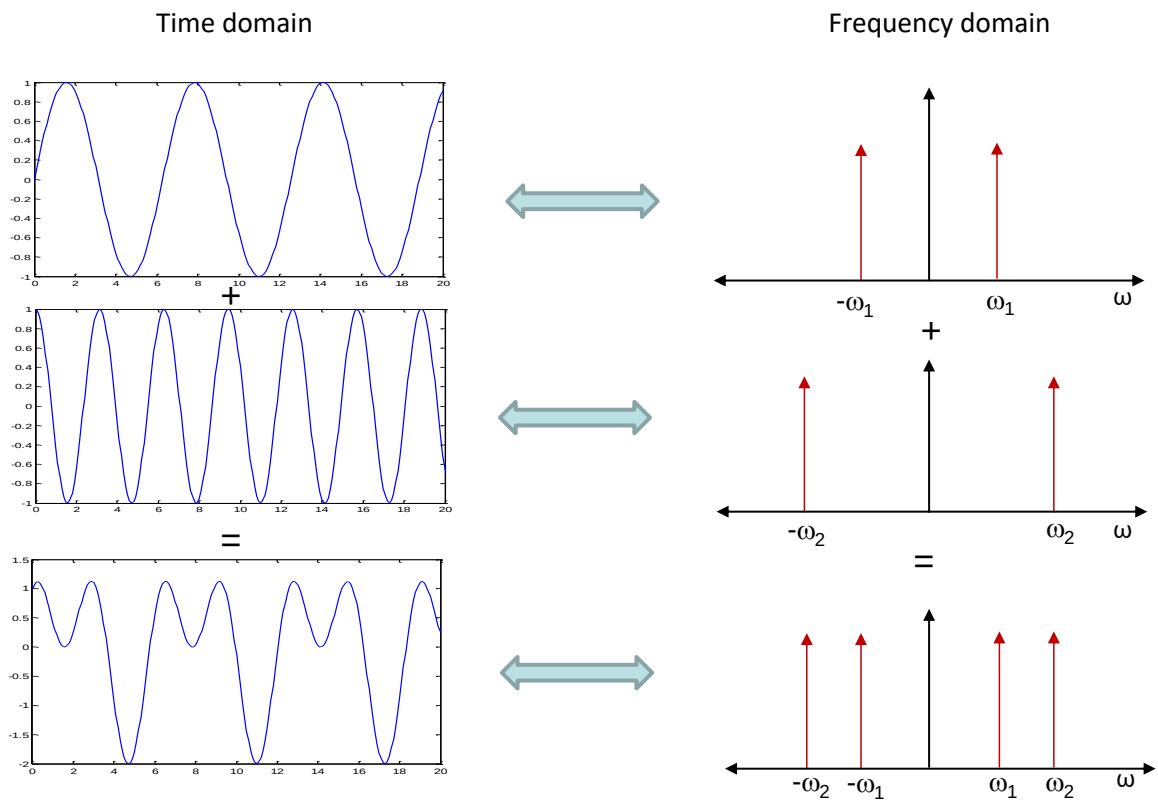


Fig 2. When we consider a single-valued signal we do not know which way the phasor is rotating – this means we end up with a half magnitude component rotating in one direction and a half magnitude component rotating in the other direction.

When we transform a non-analytical, single-valued signal into the frequency domain we see it maps to two frequency components – one forward (positive frequency) and one backward (negative frequency) component. Again, the transformation is reversable. This is expected based on our previous consideration of what a sine or cosine is.

These examples are to get you thinking – there is a simplification in the frequency domain that we will expand upon in later lectures, (specifically we have only shown magnitude and not phase in the frequency domain).

1.3.7 Test Yourself Section 1.3

The answers to all of these questions should be in the notes. Check if you can answer them before moving on

1) What is a basis function?
2) What is a phasor?
3) How are forward and backward rotating phasors distinguished mathematically?
4) Which common functions can be described as two counter-rotating phasors?
5) What is the key difference between the basis functions used by Fourier and Laplace?
6) What is the name given to a signal where for each instance in time, the signal is defined by a complex number?
7) If a real-valued, sinusoid time domain waveform is mapped to the frequency domain how does the frequency domain representation differ to an analytical representation of the same time domain waveform?

Stretch question: Can you think of a way to use the transformations to recreate the analytical representation of a signal for which you are given in single-valued form? (Answer below)

Answer: We can take the Fourier transform of a single valued signal, then in the frequency domain delete the negative frequency components before performing the inverse Fourier transform – the result is the analytical version on the original signal!