Topics for today

- Review of Navier-Stokes and continuity
- Boundary layer equations
  - Assumptions about the flow
  - Thin boundary layer assumptions
  - Boundary layer equations
  - Approaches to solve them

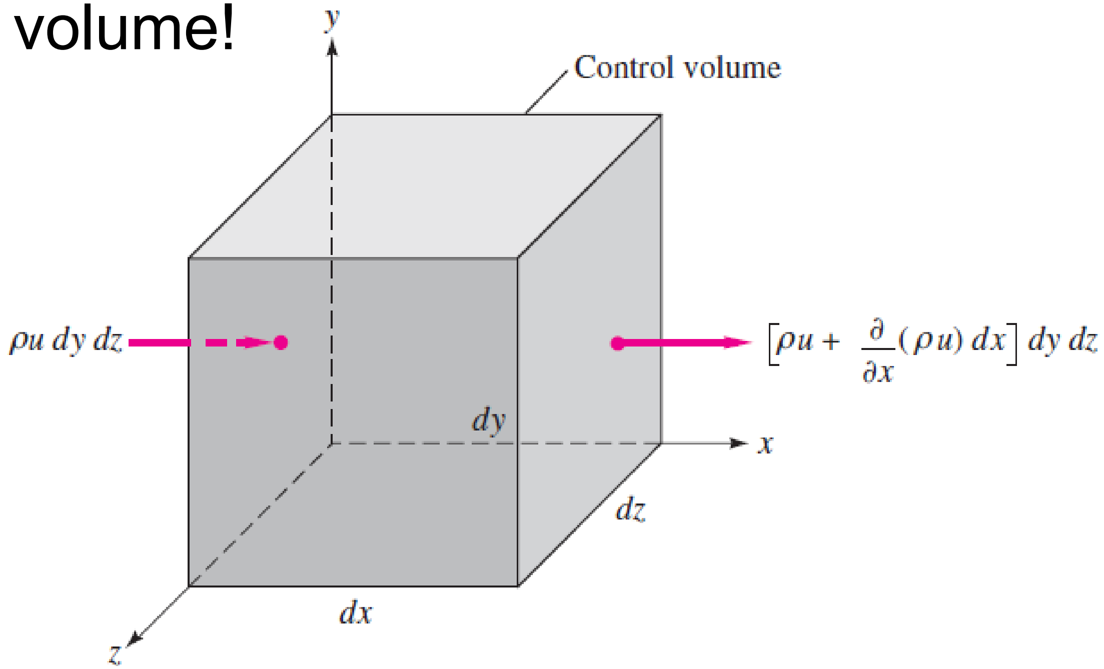
Aerodynamics and Numerical Simulation Methods

Navier-Stokes and Continuity equations

University of Bristol

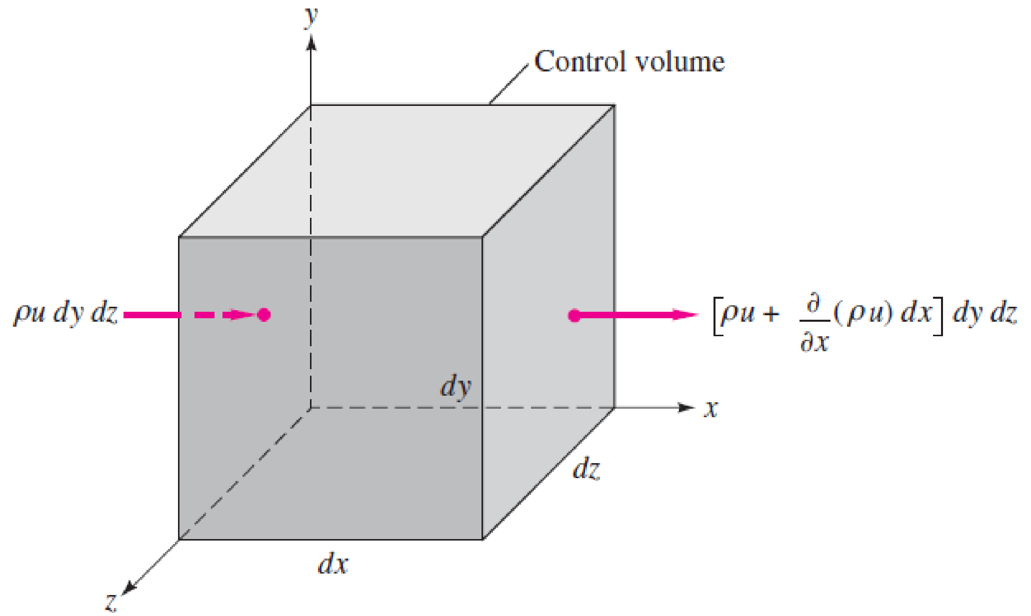
Conservation of mass

- Continuity relation (conservation of mass)
- What goes in, must come out, or accumulate in the control volume!



Source: Frank White pg 232

Conservation of mass



$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

Since control volume is very small,

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} \approx \frac{\partial \rho}{\partial t} dx\, dy\, dz$$

Source: Frank White pg 232

Conservation of mass

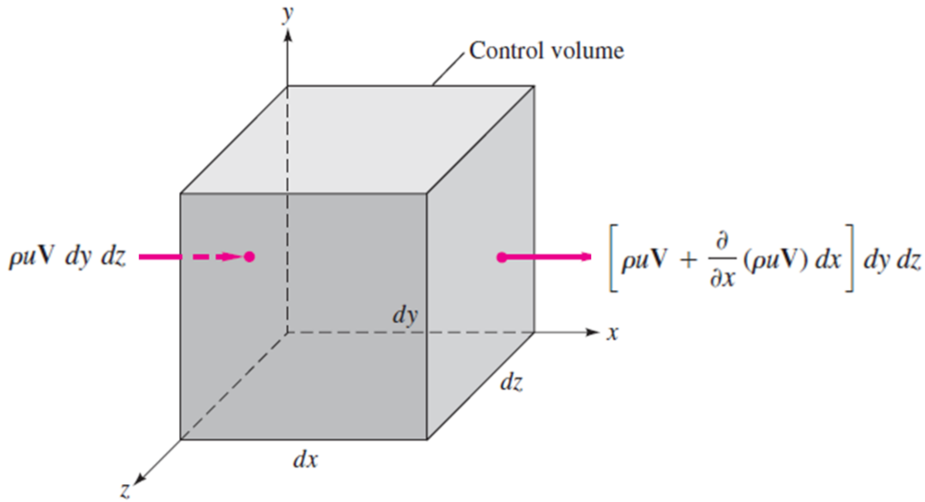
$$\underbrace{\frac{\partial \rho}{\partial t}\,dx\,dy\,dz}_{\text{Rate of change of mass in CV}} + \underbrace{\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz}_{\text{Net flux in x-direction}} + \underbrace{\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz}_{\text{Net flux in y-direction}} + \underbrace{\frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz}_{\text{Net flux in z-direction}} = 0$$

- dx, dy and dz cancels out, which leaves us with a partial differential equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

Source: Frank White pg 232

Conservation of momentum



$$\sum \mathbf{F} = \frac{\partial}{\partial t}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \sum (\dot{m}_i \mathbf{V}_i)_{out} - \sum (\dot{m}_i \mathbf{V}_i)_{in}$$

Additional term in F which describes all body and surface forces on the CV. Since CV is very small,

$$\frac{\partial}{\partial t}\int_{CV}(\mathbf{V}\rho\, d\mathcal{V}) \approx \frac{\partial}{\partial t}(\rho \mathbf{V})\, dx\, dy\, dz$$

Source: Frank White pg 238

Conservation of momentum

$$\underbrace{\sum \mathbf{F}}_{\text{Body and surface forces acting on the CV}} = dx\,dy\,dz\left[\underbrace{\frac{\partial}{\partial t}(\rho \mathbf{V})}_{\text{Rate of change of momentum}} + \underbrace{\frac{\partial}{\partial x}(\rho u \mathbf{V})}_{\text{Net flux in x-direction}} + \underbrace{\frac{\partial}{\partial y}(\rho \upsilon \mathbf{V})}_{\text{Net flux in y-direction}} + \underbrace{\frac{\partial}{\partial z}(\rho w \mathbf{V})}_{\text{Net flux in z-direction}}\right]$$

This is a vector relation. Rearranging terms on RHS (how?):

$$\frac{\partial}{\partial t}(\rho \mathbf{V}) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho \upsilon \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V})$$

$$= \underbrace{\cancel{\mathbf{V}\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V})\right]}}_{\text{Continuity}} + \underbrace{\rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + \upsilon\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Total acceleration, i.e. material derivative of the velocity}}$$

Source: Frank White pg 238

Conservation of momentum (Newton's law!)

$$\sum \mathbf{F} = \rho \frac{d\mathbf{V}}{dt}\, dx\, dy\, dz$$

Body and surface forces acting on the CV

Total acceleration

Gravity (body) force per unit volume

Pressure (surface) force per unit volume

Viscous (surface) force per unit volume

Density × Total acceleration

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

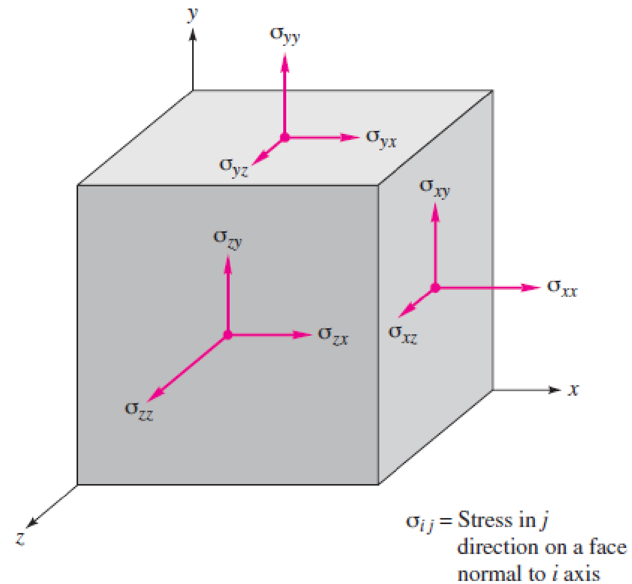
Source: Frank White pg 238

Conservation of momentum

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z \underbrace{- \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}}_{\text{Surface forces}} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$



- Surface forces are due to stresses on CV faces (hydrostatic pressure + viscous stresses)
- For a Newtonian fluid, the viscous stresses are proportional to strain rates and viscosity!
- Need to replace with velocity terms

Source: Frank White pg 238

Conservation of momentum

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \underbrace{\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Viscous stress terms can be replaced with velocity terms for incompressible flow

Hence, differential momentum equation simplifies to the incompressible flow Navier-Stokes equation:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\frac{dw}{dt}$$

Source: Frank White pg 238

Final remark

- We have just derived the continuity and Navier Stokes equations
- It is recommended you read through the full derivation (any textbook dealing with viscous flow) and practice the derivations yourself
- Do expect to be tested on how to work with PDEs, using the continuity and Navier Stokes Eqns to derive other equations, etc

Aerodynamics and Numerical Simulation Methods

Boundary Layer Equations

University of BRISTOL

Topics for today

- ✓ Review of Navier-Stokes and continuity

- Boundary layer equations
  - Assumptions about the flow
  - Thin boundary layer assumptions
  - Boundary layer equations
  - Approaches to solve them

Boundary Layer Equations

- Before looking at the boundary layer equations specifically, we assume:
  - 2D flow
  - Steady flow
  - Incompressible Flow
  - Viscous flow
  - Negligible gravity and buoyancy

Derivation of 2D Boundary Layer Equations

- Exercise: Which are the terms that can dropped?

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho \frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho \frac{dw}{dt}$$

Recall: Total derivative of u, v, w

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Source: Frank White page 465

Given these assumptions, the continuity and Navier Stokes Eqns. become:

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

x-momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

y-momentum

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Kinematic viscosity

Source: Frank White page 465

Effect of Incompressible Flow:

- By definition, ρ is constant, μ, ν are also constant
  - Have one less variable -> need one less Eqn.
- In fact, this assumption de-couples the energy equation
  - we don't need to solve it unless heat transfer is important
  - This makes life a lot simpler, but *remember, only applies to incompressible flows*

Thin Layer Assumption

- If Re is large, the boundary layer thickness δ is very small compared to any chordwise length. ie.

$$\frac{\delta}{x} << 1$$

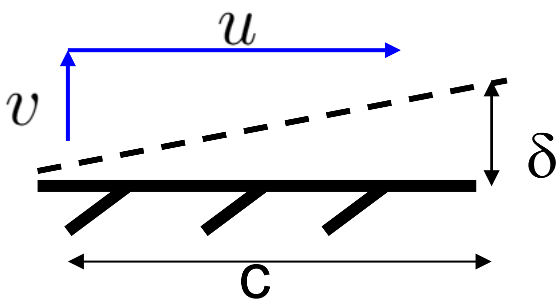
- In turn this means that

$$\frac{\partial}{\partial y} >> \frac{\partial}{\partial x}$$

- What effect does this have on the equations?

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- It may initially seem that the first term may be ignored. However:



$$\frac{v}{u_e} = \left(\frac{\delta}{c}\right) \ll 1$$

- Hence first term is a small differential of a large number, second is a large differential of small number, and hence are of similar magnitude.
- Neither can be neglected

x-momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Kinematic viscosity

- partial derivative w.r.t x is small, so second derivative is neglected:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

y-momentum: $$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Kinematic viscosity

- first term: v small, d/dx small -> first term neglected
- 2nd term: v small, dv/dy also small, neglect
- For similar reasons to above, can also neglect 2nd derivatives on RHS. Hence

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}\approx 0$$

- This means pressure is approximately constant through a boundary layer. If we assume this, do not need to solve y-momentum equation.

Solving the Boundary Layer Equations

- Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- x-Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

- We also need to assume small curvature:

$$\frac{\delta}{R} << 1$$

- This is generally the case for aerodynamic shapes

Integral vs. Differential

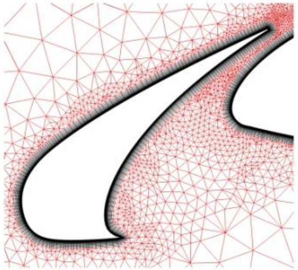
- Equations governing the flow in a steady, incompressible, 2D boundary layer are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

- These equations can be solved:
  - Numerically, on high-speed computers
  - In integral fashion – semi-analytical

Integral vs. Differential



- Differential methods solve the differential equations:
  - need high speed computers
  - defined grids
- Integral methods solve analytical integrals of flow properties along the boundary layer
  - Involve more assumptions
  - Less versatile, BUT
  - Much simpler, can be done by hand/in simple processes
  - Easier to calibrate accurately