CHAPTER

Matrix Methods of Analysis

17

The methods described in Chapter 16 are basically methods of analysis which are suitable for use with a hand calculator. They also provide an insight into the physical behaviour of structures under different loading conditions and it is this fundamental knowledge which enables the structural engineer to design structures which are capable of fulfilling their required purpose. However, the more complex a structure the lengthier, and more tedious, hand methods of analysis become and the more the approximations which have to be made. It was this situation which led, in the late 1940s and early 1950s, to the development of *matrix methods* of analysis and, at the same time, to the emergence of high-speed, electronic, digital computers. Conveniently, matrix methods are ideally suited to expressing structural theory in a form suitable for numerical solution by computer.

The modern digital computer is capable of storing vast amounts of data and producing solutions for highly complex structural problems almost instantaneously. There is a wide range of program packages available which cover static and dynamic problems in all types of structure from skeletal to continuum. Unfortunately these packages are not foolproof and so it is essential for the structural engineer to be able to select the appropriate package and to check the validity of the results; without a knowledge of fundamental theory this is impossible.

In Section 16.1 we discussed the flexibility and stiffness methods of analysis of statically indeterminate structures and saw that the flexibility method involves releasing the structure, determining the displacements in the released structure and then finding the forces required to fulfil the compatibility of displacement condition in the complete structure. The method was applied to statically indeterminate beams, trusses, braced beams, portal frames and two-pinned arches in Sections 16.4–16.8. It is clear from the analysis of these types of structure that the greater the degree of indeterminacy the higher the number of simultaneous equations requiring solution; for large numbers of equations a computer approach then becomes necessary. Furthermore, the flexibility method requires judgements to be made in terms of the release selected, so that a more automatic procedure is desirable so long, of course, as the fundamental behaviour of the structure is understood.

In Section 16.9 we examined the slope–deflection method for the solution of statically indeterminate beams and frames; the slope–deflection equations also form the basis of the moment–distribution method described in Section 16.10. These equations are, in fact, force–displacement relationships as opposed to the displacement–force relationships of the flexibility method. The slope–deflection and moment–distribution methods are therefore *stiffness* or *displacement* methods.

The stiffness method basically requires that a structure, which has a degree of *kinematic indeterminacy* equal to n_k, is initially rendered determinate by imposing a system of n_k constraints. Thus, for example, in the slope–deflection analysis of a continuous beam (e.g. Ex. 16.15) the beam is initially fixed at each support and the fixed-end moments calculated. This generally gives rise to an unbalanced system of forces at each node. Then by allowing displacements to occur at each node we obtain a series of force–displacement states (Eqs (i)–(vi) in Ex. 16.15). The n_k equilibrium conditions at the nodes are then expressed in terms of the displacements, giving n_k equations (Eqs (vii)–(x) in Ex. 16.15), the solution of which gives the true

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values of the displacements at the nodes. The internal stress resultants follow from the known force–displacement relationships for each member of the structure (Eqs (i)–(vi) in Ex. 16.15) and the complete solution is then the sum of the determinate solution and the set of n_k indeterminate systems.

Again, as in the flexibility method, we see that the greater the degree of indeterminacy (kinematic in this case) the greater the number of equations requiring solution, so that a computer-based approach is necessary when the degree of indeterminacy is high. Generally this requires that the force–displacement relationships in a structure are expressed in matrix form. We therefore need to establish force–displacement relationships for structural members and to examine the way in which these individual force–displacement relationships are combined to produce a force–displacement relationship for the complete structure. Initially we shall investigate members that are subjected to axial force only.

17.1 Axially loaded members

Consider the axially loaded member, AB, shown in Fig. 17.1(a) and suppose that it is subjected to axial forces, F_A and F_B, and that the corresponding displacements are w_A and w_B; the member has a cross-sectional area, A, and Young's modulus, E. An elemental length, δx, of the member is subjected to forces and displacements as shown in Fig. 17.1(b) so that its change in length from its unloaded state is $w+\delta w - w = \delta w$. Thus, from Eq. (7.4), the strain, ε, in the element is given by

$$\varepsilon = \frac{dw}{dx}$$

Further, from Eq. (7.8)

$$\frac{F}{A} = E\frac{dw}{dx}$$

so that

$$dw = \frac{F}{AE}dx$$

Therefore the axial displacement at the section a distance x from A is given by

$$w = \int_0^x \frac{F}{AE}dx$$

which gives

$$w = \frac{F}{AE}x + C_1$$

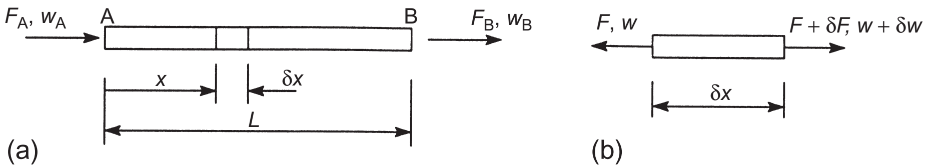

FIGURE 17.1

Axially loaded member.

in which C_1 is a constant of integration. When $x=0$, $w=w_A$ so that $C_1=w_A$ and the expression for w may be written as

$$w=\frac{F}{AE}x+w_A \tag{17.1}$$

In the absence of any loads applied between A and B, $F=F_B=-F_A$ and Eq. (17.1) may be written as

$$w=\frac{F_B}{AE}x+w_A \tag{17.2}$$

Thus, when $x=L$, $w=w_B$ so that from Eq. (17.2)

$$w_B=\frac{F_B}{AE}L+w_A$$

or

$$F_B=\frac{AE}{L}(w_B-w_A) \tag{17.3}$$

Furthermore, since $F_B=-F_A$ we have, from Eq. (17.3)

$$-F_A=\frac{AE}{L}(w_B-w_A)$$

or

$$F_A=-\frac{AE}{L}(w_B-w_A) \tag{17.4}$$

Equations (17.3) and (17.4) may be expressed in matrix form as follows

$$\begin{Bmatrix} F_A \\ F_B \end{Bmatrix}=\begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix}\begin{Bmatrix} w_A \\ w_B \end{Bmatrix}$$

or

$$\begin{Bmatrix} F_A \\ F_B \end{Bmatrix}=\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} w_A \\ w_B \end{Bmatrix} \tag{17.5}$$

Equation (17.5) may be written in the general form

$$\{F\}=[K_{AB}]\{w\} \tag{17.6}$$

in which $\{F\}$ and $\{w\}$ are generalized force and displacement matrices and $[K_{AB}]$ is the *stiffness matrix* of the member AB.

Suppose now that we have two axially loaded members, AB and BC, in line and connected at their common node B as shown in Fig. 17.2.

In Fig. 17.2 the force, F_B, comprises two components: $F_{B,AB}$ due to the change in length of AB, and $F_{B,BC}$ due to the change in length of BC. Thus, using the results of Eqs (17.3) and (17.4)

$$F_A=\frac{A_{AB}E_{AB}}{L_{AB}}(w_A-w_B) \tag{17.7}$$

$$F_B=F_{B,AB}+F_{B,BC}=\frac{A_{AB}E_{AB}}{L_{AB}}(w_B-w_A)+\frac{A_{BC}E_{BC}}{L_{BC}}(w_B-w_C) \tag{17.8}$$

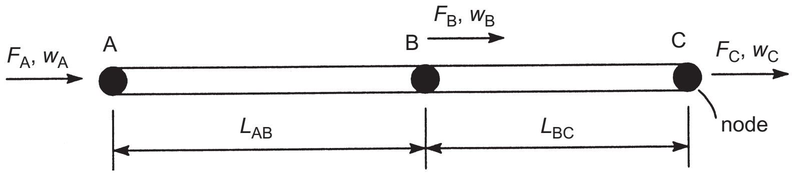

FIGURE 17.2

Two axially loaded members in line.

$$F_C = \frac{A_{BC}E_{BC}}{L_{BC}}(w_C - w_B) \tag{17.9}$$

in which A_{AB}, E_{AB} and L_{AB} are the cross-sectional area, Young's modulus and length of the member AB; similarly for the member BC. The term AE/L is a measure of the stiffness of a member, this we shall designate by k. Thus, Eqs (17.7)–(17.9) become

$$F_A = k_{AB}(w_A - w_B) \tag{17.10}$$

$$F_B = -k_{AB}w_A + (k_{AB} + k_{BC})w_B - k_{BC}w_C \tag{17.11}$$

$$F_C = k_{BC}(w_C - w_B) \tag{17.12}$$

Equations (17.10)–(17.12) are expressed in matrix form as

$$\begin{Bmatrix} F_A \\ F_B \\ F_C \end{Bmatrix} = \begin{bmatrix} k_{AB} & -k_{AB} & 0 \\ -k_{AB} & k_{AB} + k_{BC} & -k_{BC} \\ 0 & -k_{BC} & k_{BC} \end{bmatrix} \begin{Bmatrix} w_A \\ w_B \\ w_C \end{Bmatrix} \tag{17.13}$$

Note that in Eq. (17.13) the stiffness matrix is a symmetric matrix of order 3×3, which, as can be seen, connects *three* nodal forces to *three* nodal displacements. Also, in Eq. (17.5), the stiffness matrix is a 2×2 matrix connecting *two* nodal forces to *two* nodal displacements. We deduce, therefore, that a stiffness matrix for a structure in which n nodal forces relate to n nodal displacements will be a symmetric matrix of the order $n\times n$.

In more general terms the matrix in Eq. (17.13) may be written in the form

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{17.14}$$

in which the element k_{11} relates the force at node 1 to the displacement at node 1, k_{12} relates the force at node 1 to the displacement at node 2, and so on. Now, for the member connecting nodes 1 and 2

$$[k_{12}] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

and for the member connecting nodes 2 and 3

$$[k_{23}] = \begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix}$$

Therefore we may assemble a stiffness matrix for a complete structure, not by the procedure used in establishing Eqs (17.10)–(17.12) but by writing down the matrices for the individual members and then inserting them into the overall stiffness matrix such as that in Eq. (17.14). The element k_{22} appears in both $[K_{12}]$ and $[K_{23}]$ and will therefore receive contributions from both matrices. Hence, from Eq. (17.5)

$$[K_{AB}] = \begin{bmatrix} k_{AB} & -k_{AB} \\ -k_{AB} & k_{AB} \end{bmatrix}$$

and

$$[K_{BC}] = \begin{bmatrix} k_{BC} & -k_{BC} \\ -k_{BC} & k_{BC} \end{bmatrix}$$

Inserting these matrices into Eq. (17.14) we obtain

$$[K_{ABC}] = \begin{bmatrix} k_{AB} & -k_{AB} & 0 \\ -k_{AB} & k_{AB}+k_{BC} & -k_{BC} \\ 0 & -k_{BC} & k_{BC} \end{bmatrix}$$

as before. We see that only the k_{22} term (linking the force at node 2(B) to the displacement at node 2) receives contributions from both members AB and BC. This results from the fact that node 2(B) is directly connected to both nodes 1(A) and 3(C) while nodes 1 and 3 are connected directly to node 2. Nodes 1 and 3 are not directly connected so that the terms k_{13} and k_{31} are both zero, i.e. they are not affected by each other's displacement.

To summarize, the formation of the stiffness matrix for a complete structure is carried out as follows: terms of the form k_{ii} on the main diagonal consist of the sum of the stiffnesses of all the structural elements meeting at node i, while the off-diagonal terms of the form k_{ij} consist of the sum of the stiffnesses of all the elements connecting node i to node j.

Equation (17.13) may be solved for a specific case in which certain boundary conditions are specified. Thus, for example, the member AB may be fixed at A and loads F_B and F_C applied. Then $w_A = 0$ and F_A is a reaction force. Inversion of the resulting matrix enables w_B and w_C to be found.

In a practical situation a member subjected to an axial load could be part of a truss which would comprise several members set at various angles to one another. Therefore, to assemble a stiffness matrix for a complete structure, we need to refer axial forces and displacements to a common, or *global*, axis system.

Consider the member shown in Fig. 17.3. It is inclined at an angle θ to a global axis system denoted by xy. The member connects node i to node j, and has *member* or *local* axes $\bar{x}$, $\bar{y}$. Thus nodal forces and displacements referred to local axes are written as $\bar{F}$, $\bar{w}$, etc., so that, by comparison with Eq. (17.5), we see that

$$\begin{Bmatrix} \bar{F}_{x,i} \\ \bar{F}_{x,j} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{w}_i \\ \bar{w}_j \end{Bmatrix} \tag{17.15}$$

where the member stiffness matrix is written as $[\bar{K}_{ij}]$.

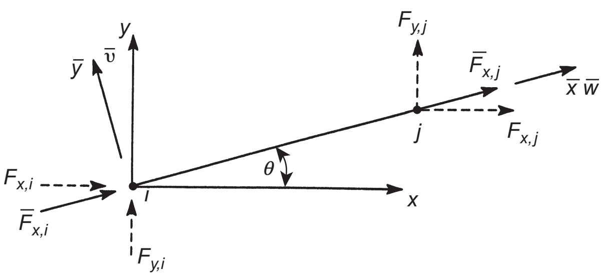

FIGURE 17.3

Local and global axes systems for an axially loaded member.

In Fig. 17.3 external forces $\overline{F}_{x,i}$ and $\overline{F}_{x,j}$ are applied to i and j. It should be noted that $\overline{F}_{y,i}$ and $\overline{F}_{y,j}$ do not exist since the member can only support axial forces. However, $\overline{F}_{x,i}$ and $\overline{F}_{x,j}$ have components $F_{x,i}$, $F_{y,i}$ and $F_{x,j}$, $F_{y,j}$ respectively, so that whereas only two force components appear for the member in local coordinates, four components are present when global coordinates are used. Therefore, if we are to transfer from local to global coordinates, Eq. (17.15) must be expanded to an order consistent with the use of global coordinates. Thus

$$\begin{Bmatrix} \overline{F}_{x,i} \\ \overline{F}_{y,i} \\ \overline{F}_{x,j} \\ \overline{F}_{y,j} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \overline{w}_i \\ \overline{v}_i \\ \overline{w}_j \\ \overline{v}_j \end{Bmatrix} \tag{17.16}$$

Expansion of Eq. (17.16) shows that the basic relationship between $\overline{F}_{x,i}$, $\overline{F}_{x,j}$ and $\overline{w}_i$, $\overline{w}_j$ as defined in Eq. (17.15) is unchanged.

From Fig. 17.3 we see that

$$\overline{F}_{x,i} = F_{x,i} \cos\theta + F_{y,i} \sin\theta$$
$$\overline{F}_{y,i} = -F_{x,i} \sin\theta + F_{y,i} \cos\theta$$

and

$$\overline{F}_{x,j} = F_{x,j} \cos\theta + F_{y,j} \sin\theta$$
$$\overline{F}_{y,j} = -F_{x,j} \sin\theta + F_{y,j} \cos\theta$$

Writing λ for $\cos\theta$ and μ for $\sin\theta$ we express the above equations in matrix form as

$$\begin{Bmatrix} \overline{F}_{x,i} \\ \overline{F}_{y,i} \\ \overline{F}_{x,j} \\ \overline{F}_{y,j} \end{Bmatrix} = \begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{bmatrix} \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{Bmatrix} \tag{17.17}$$

or, in abbreviated form

$$\{\overline{F}\} = [T]\{F\} \tag{17.18}$$

where $[T]$ is known as the *transformation matrix*. A similar relationship exists between the sets of nodal displacements. Thus

$$\{\overline{\delta}\} = [T]\{\delta\} \tag{17.19}$$

in which $\{\overline{\delta}\}$ and $\{\delta\}$ are generalized displacements referred to the local and global axes, respectively. Substituting now for $\{\overline{F}\}$ and $\{\overline{\delta}\}$ in Eq. (17.16) from Eqs (17.18) and (17.19) we have

$$[T]\{F\} = [\overline{K}_{ij}][T]\{\delta\}$$

Hence

$$\{F\} = [T^{-1}][\overline{K}_{ij}][T]\{\delta\} \tag{17.20}$$

It may be shown that the inverse of the transformation matrix is its transpose, i.e.

$$[T^{-1}] = [T]^{\mathrm{T}}$$

Thus we rewrite Eq. (17.20) as

$$\{F\} = [T]^{\mathrm{T}}[\overline{K}_{ij}][T]\{\delta\} \tag{17.21}$$

The nodal force system referred to the global axes, $\{F\}$, is related to the corresponding nodal displacements by

$$\{F\} = [K_{ij}]\{\delta\} \tag{17.22}$$

in which $[K_{ij}]$ is the member stiffness matrix referred to global coordinates. Comparison of Eqs (17.21) and (17.22) shows that

$$\{K_{ij}\} = [T]^{\mathrm{T}}[\overline{K}_{ij}][T]$$

Substituting for $[T]$ from Eq. (17.17) and $[\overline{K}_{ij}]$ from Eq. (17.16) we obtain

$$[K_{ij}] = \frac{AE}{L}\begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & \mu^2 & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \tag{17.23}$$

Evaluating λ ($= \cos\theta$) and μ ($= \sin\theta$) for each member and substituting in Eq. (17.23) we obtain the stiffness matrix, referred to global axes, for each member of the framework.

EXAMPLE 17.1

Determine the horizontal and vertical components of the deflection of node 2 and the forces in the members of the truss shown in Fig. 17.4. The product AE is constant for all members.

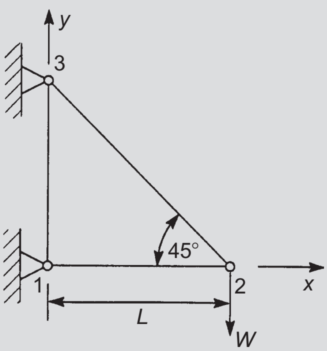

FIGURE 17.4

Truss of Ex. 17.1.

We see from Fig. 17.4 that the nodes 1 and 3 are pinned to the foundation and are therefore not displaced. Hence, referring to the global coordinate system shown,

$$w_1 = v_1 = w_3 = v_3 = 0$$

The external forces are applied at node 2 such that $F_{x,2} = 0$, $F_{y,2} = -W$; the nodal forces at 1 and 3 are then unknown reactions.

The first step in the solution is to assemble the stiffness matrix for the complete framework by writing down the member stiffness matrices referred to the global axes using Eq. (17.23). The direction cosines λ and μ take different values for each of the three members; therefore, remembering that the angle θ is measured anticlockwise from the positive direction of the x axis we have the following:

Member	θ (deg)	λ	μ
12	0	1	0
13	90	0	1
23	135	−0.707	0.707

The member stiffness matrices are therefore

$$[K_{12}]=\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [K_{13}]=\frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K_{23}]=\frac{AE}{1.414L}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \tag{i}$$

The complete stiffness matrix is now assembled using the method suggested in the discussion of Eq. (17.14). The matrix will be a 6×6 matrix since there are six nodal forces connected to six nodal displacements; thus

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.354 & -0.354 & -0.354 & 0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & -1 & 0.354 & -0.354 & -0.354 & 1.354 \end{bmatrix}\begin{Bmatrix} w_1=0 \\ v_1=0 \\ w_2 \\ v_2 \\ w_3=0 \\ v_3=0 \end{Bmatrix} \tag{ii}$$

If we now delete rows and columns in the stiffness matrix corresponding to zero displacements, we obtain the unknown nodal displacements w_2 and v_2 in terms of the applied loads $F_{x,2}$ $(=0)$ and $F_{y,2}$ $(=-W)$. Thus

$$\begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1.354 & -0.354 \\ -0.354 & 0.354 \end{bmatrix}\begin{Bmatrix} w_2 \\ v_2 \end{Bmatrix} \tag{iii}$$

Inverting Eq. (iii) gives

$$\begin{Bmatrix} w_2 \\ v_2 \end{Bmatrix} = \frac{L}{AE}\begin{bmatrix} 1 & 1 \\ 1 & 3.828 \end{bmatrix}\begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} \tag{iv}$$

from which

$$w_2 = \frac{L}{AE}(F_{x,2}+F_{y,2}) = \frac{-WL}{AE}$$

$$v_2 = \frac{L}{AE}(F_{x,2}+3.828F_{y,2}) = \frac{-3.828WL}{AE}$$

The reactions at nodes 1 and 3 are now obtained by substituting for w_2 and v_2 from Eq. (iv) into Eq. (ii). Hence

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,3} \\ F_{y,3} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -0.354 & 0.354 \\ 0.354 & -0.354 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3.828 \end{bmatrix} \begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix}$$

giving

$$\begin{aligned} F_{x,1} &= -F_{x,2} - F_{y,2} = W \\ F_{y,1} &= 0 \\ F_{x,3} &= F_{y,2} = -W \\ F_{y,3} &= W \end{aligned}$$

The internal forces in the members may be found from the axial displacements of the nodes. Thus, for a member ij, the internal force F_{ij} is given by

$$F_{ij} = \frac{AE}{L}(\overline{w}_j - \overline{w}_i) \tag{v}$$

But

$$\begin{aligned} \overline{w}_j &= \lambda w_j + \mu v_j \\ \overline{w}_i &= \lambda w_i + \mu v_i \end{aligned}$$

Hence

$$\overline{w}_j - \overline{w}_i = \lambda(w_j - w_i) + \mu(v_j - v_i)$$

Substituting in Eq. (v) and rewriting in matrix form,

$$F_{ij} = \frac{AE}{L}[\lambda \quad \mu] \begin{Bmatrix} w_j & w_i \\ v_j & v_i \end{Bmatrix} \tag{vi}$$

Thus, for the members of the framework

$$F_{12} = \frac{AE}{L}[1 \quad 0] \begin{Bmatrix} \dfrac{-WL}{AE} - 0 \\ \dfrac{-3.828WL}{AE} - 0 \end{Bmatrix} = -W \text{ (compression)}$$

$$F_{13} = \frac{AE}{L}[0 \quad 1] \begin{Bmatrix} 0 - 0 \\ 0 - 0 \end{Bmatrix} = 0 \text{ (obvious from inspection)}$$

$$F_{23} = \frac{AE}{1.414L}[-0.707 \quad 0.707] \begin{Bmatrix} 0 + \dfrac{WL}{AE} \\ 0 + \dfrac{3.828WL}{AE} \end{Bmatrix} = 1.414W \text{ (tension)}$$

The matrix method of solution for the statically determinate truss of Ex. 17.1 is completely general and therefore applicable to any structural problem. We observe from the solution that the question of statical determinacy of the truss did not arise. Statically indeterminate trusses are therefore solved in an identical manner with the stiffness matrix for each redundant member being included in the complete stiffness matrix as described above. Clearly, the greater the number of members the greater the size of the stiffness matrix, so that a computer-based approach is essential.

The procedure for the matrix analysis of space trusses is similar to that for plane trusses. The main difference lies in the transformation of the member stiffness matrices from local to global coordinates since, as we see from Fig. 17.5, axial nodal forces $\overline{F}_{x,i}$ and $\overline{F}_{x,j}$ have each, now, three global components $F_{x,i}$, $F_{y,i}$, $F_{z,i}$ and $F_{x,j}$, $F_{y,j}$, $F_{z,j}$, respectively. The member stiffness matrix referred to global coordinates is therefore of the order 6×6 so that $[K_{ij}]$ of Eq. (17.15) must be expanded to the same order to allow for this. Hence

$$[\overline{K}_{ij}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cccccc} \overline{w}_i & \overline{v}_i & \overline{\mu}_i & \overline{w}_j & \overline{v}_j & \overline{\mu}_j \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{17.24}$$

In Fig. 17.5 the member ij is of length L, cross-sectional area A and modulus of elasticity E. Global and local coordinate systems are designated as for the two-dimensional case. Further, we suppose that

$$\theta_{x\overline{x}} = \text{angle between } x \text{ and } \overline{x}$$
$$\theta_{x\overline{y}} = \text{angle between } x \text{ and } \overline{y}$$
$$\vdots$$
$$\theta_{z\overline{y}} = \text{angle between } z \text{ and } \overline{y}$$
$$\vdots$$

Therefore, nodal forces referred to the two systems of axes are related as follows

$$\left.\begin{aligned} \overline{F}_x &= F_x \cos\theta_{x\overline{x}} + F_y \cos\theta_{x\overline{y}} + F_z \cos\theta_{x\overline{z}} \\ \overline{F}_y &= F_x \cos\theta_{y\overline{x}} + F_y \cos\theta_{y\overline{y}} + F_z \cos\theta_{y\overline{z}} \\ \overline{F}_z &= F_x \cos\theta_{z\overline{x}} + F_y \cos\theta_{z\overline{y}} + F_z \cos\theta_{z\overline{z}} \end{aligned}\right\} \tag{17.25}$$

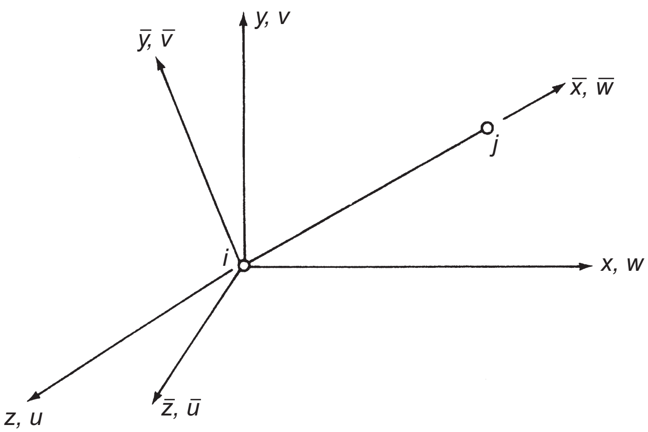

FIGURE 17.5

Local and global coordinate systems for a member in a space truss.

Writing

$$\left.\begin{array}{lll} \lambda_{\bar{x}} = \cos\theta_{x\bar{x}} & \lambda_{\bar{y}} = \cos\theta_{x\bar{y}} & \lambda_{\bar{z}} = \cos\theta_{x\bar{z}} \\ \mu_{\bar{x}} = \cos\theta_{y\bar{x}} & \mu_{\bar{y}} = \cos\theta_{y\bar{y}} & \mu_{\bar{z}} = \cos\theta_{y\bar{z}} \\ \nu_{\bar{x}} = \cos\theta_{z\bar{x}} & \nu_{\bar{y}} = \cos\theta_{z\bar{y}} & \nu_{\bar{z}} = \cos\theta_{z\bar{z}} \end{array}\right\} \tag{17.26}$$

we may express Eq. (17.25) for nodes i and j in matrix form as

$$\begin{Bmatrix} \overline{F}_{x,i} \\ \overline{F}_{y,i} \\ \overline{F}_{z,i} \\ \overline{F}_{x,j} \\ \overline{F}_{y,j} \\ \overline{F}_{z,j} \end{Bmatrix} = \begin{bmatrix} \lambda_{\bar{x}} & \mu_{\bar{x}} & \nu_{\bar{x}} & 0 & 0 & 0 \\ \lambda_{\bar{y}} & \mu_{\bar{y}} & \nu_{\bar{y}} & 0 & 0 & 0 \\ \lambda_{\bar{z}} & \mu_{\bar{z}} & \nu_{\bar{z}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\bar{x}} & \mu_{\bar{x}} & \nu_{\bar{x}} \\ 0 & 0 & 0 & \lambda_{\bar{y}} & \mu_{\bar{y}} & \nu_{\bar{y}} \\ 0 & 0 & 0 & \lambda_{\bar{z}} & \mu_{\bar{z}} & \nu_{\bar{z}} \end{bmatrix} \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{z,i} \\ F_{x,j} \\ F_{y,j} \\ F_{z,j} \end{Bmatrix} \tag{17.27}$$

or in abbreviated form

$$\{\overline{F}\} = [T]\{F\}$$

The derivation of $[K_{ij}]$ for a member of a space frame proceeds on identical lines to that for the plane frame member. Thus, as before

$$[K_{ij}] = [T]^{\mathrm{T}}[\overline{K}_{ij}]\,[T]$$

Substituting for $[T]$ and $[\overline{K}_{ij}]$ from Eqs (17.27) and (17.24) gives

$$[K_{ij}] = \frac{AE}{L}\begin{bmatrix} \lambda_{\bar{x}}^2 & \lambda_{\bar{x}}\mu_{\bar{x}} & \lambda_{\bar{x}}\nu_{\bar{x}} & -\lambda_{\bar{x}}^2 & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\lambda_{\bar{x}}\nu_{\bar{x}} \\ \lambda_{\bar{x}}\mu_{\bar{x}} & \mu_{\bar{x}}^2 & \mu_{\bar{x}}\nu_{\bar{x}} & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\mu_{\bar{x}}^2 & -\mu_{\bar{x}}\nu_{\bar{x}} \\ \lambda_{\bar{x}}\nu_{\bar{x}} & \mu_{\bar{x}}\nu_{\bar{x}} & \nu_{\bar{x}}^2 & -\lambda_{\bar{x}}\nu_{\bar{x}} & -\mu_{\bar{x}}\nu_{\bar{x}} & -\nu_{\bar{x}}^2 \\ -\lambda_{\bar{x}}^2 & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\lambda_{\bar{x}}\nu_{\bar{x}} & \lambda_{\bar{x}}^2 & \lambda_{\bar{x}}\mu_{\bar{x}} & \lambda_{\bar{x}}\nu_{\bar{x}} \\ -\lambda_{\bar{x}}\mu_{\bar{x}} & -\mu_{\bar{x}}^2 & -\mu_{\bar{x}}\nu_{\bar{x}} & \lambda_{\bar{x}}\mu_{\bar{x}} & \mu_{\bar{x}}^2 & \mu_{\bar{x}}\nu_{\bar{x}} \\ -\lambda_{\bar{x}}\nu_{\bar{x}} & -\mu_{\bar{x}}\nu_{\bar{x}} & -\nu_{\bar{x}}^2 & \lambda_{\bar{x}}\nu_{\bar{x}} & \mu_{\bar{x}}\nu_{\bar{x}} & \nu_{\bar{x}}^2 \end{bmatrix} \tag{17.28}$$

All the suffixes in Eq. (17.28) are $\bar{x}$ so that we may rewrite the equation in simpler form, namely

$$[K_{ij}] = \frac{AE}{L}\left[\begin{array}{ccc:ccc} \lambda^2 & & & & & \textbf{SYM} \\ \lambda\mu & \mu^2 & & & & \\ \lambda\nu & \mu\nu & \nu^2 & & & \\ \hdashline -\lambda^2 & -\lambda\mu & -\lambda\nu & \lambda^2 & & \\ -\lambda\mu & -\mu^2 & -\mu\nu & \lambda\mu & \mu^2 & \\ -\lambda\nu & -\mu\nu & -\nu^2 & \lambda\nu & \mu\nu & \nu^2 \end{array}\right] \tag{17.29}$$

where λ, μ and υ are the direction cosines between the x, y, z and $\bar{x}$ axes, respectively.

The complete stiffness matrix for a space frame is assembled from the member stiffness matrices in a similar manner to that for the plane frame and the solution completed as before.

17.2 Stiffness matrix for a uniform beam

Our discussion so far has been restricted to structures comprising members capable of resisting axial loads only. Many structures, however, consist of beam assemblies in which the individual members resist shear and

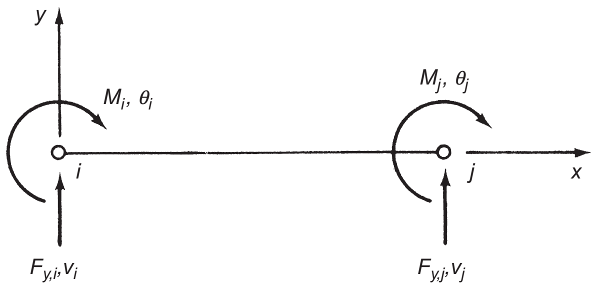

FIGURE 17.6

Forces and moments on a beam element.

bending forces, in addition to axial loads. We shall now derive the stiffness matrix for a uniform beam and consider the solution of rigid jointed frameworks formed by an assembly of beams, or beam elements as they are sometimes called.

Figure 17.6 shows a uniform beam ij of flexural rigidity EI and length L subjected to nodal forces $F_{y,i}$, $F_{y,j}$ and nodal moments M_i, M_j in the xy plane. The beam suffers nodal displacements and rotations υ_i, υ_j and θ_i, θ_j. We do not include axial forces here since their effects have already been determined in our investigation of trusses.

The stiffness matrix [K$_{ij}$] may be written down directly from the beam slope–deflection equations (16.27). Note that in Fig. 17.6 θi and θj are opposite in sign to θ_A and θ_B in Fig. 16.32. Then

$$M_i = -\frac{6EI}{L^2}\upsilon_i + \frac{4EI}{L}\theta_i + \frac{6EI}{L^2}\upsilon_j + \frac{2EI}{L}\theta_j \tag{17.28}$$

and

$$M_j = -\frac{6EI}{L^2}\upsilon_i + \frac{2EI}{L}\theta_i + \frac{6EI}{L^2}\upsilon_j + \frac{4EI}{L}\theta_j \tag{17.29}$$

Also

$$-F_{y,i} = F_{y,j} = -\frac{12EI}{L^3}\upsilon_i + \frac{6EI}{L^2}\theta_i + \frac{12EI}{L^3}\upsilon_j + \frac{6EI}{L^2}\theta_j \tag{17.30}$$

Expressing Eqs (17.28), (17.29) and (17.30) in matrix form yields

$$\begin{Bmatrix} F_{y,i} \\ M_i \\ F_{y,j} \\ M_j \end{Bmatrix} = EI \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ -6/L^2 & 4/L & 6/L^2 & 2/L \\ -12/L^3 & 6/L^2 & 12/L^3 & 6/L^2 \\ -6/L^2 & 2/L & 6/L^2 & 4/L \end{bmatrix} \begin{Bmatrix} \upsilon_i \\ \theta_i \\ \upsilon_j \\ \theta_j \end{Bmatrix} \tag{17.31}$$

which is of the form

$$\{F\} = [Kij]\{\delta\}$$

where $[K_{ij}]$ is the stiffness matrix for the beam.

It is possible to write Eq. (17.31) in an alternative form such that the elements of $[K_{ij}]$ are pure numbers. Thus

$$\begin{Bmatrix} F_{y,i} \\ M_i/L \\ F_{y,j} \\ M_j/L \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix} \begin{Bmatrix} \upsilon_i \\ \theta_i L \\ \upsilon_j \\ \theta_j L \end{Bmatrix}$$

This form of Eq. (17.31) is particularly useful in numerical calculations for an assemblage of beams in which EI/L^3 is constant.

Equation (17.31) is derived for a beam whose axis is aligned with the x axis so that the stiffness matrix defined by Eq. (17.31) is actually $[\overline{K}_{ij}]$ the stiffness matrix referred to a local coordinate system. If the beam is positioned in the xy plane with its axis arbitrarily inclined to the x axis then the x and y axes form a global coordinate system and it becomes necessary to transform Eq. (17.31) to allow for this. The procedure is similar to that for the truss member of Section 17.1 in that $[\overline{K}_{ij}]$ must be expanded to allow for the fact that nodal displacements $\overline{w}_i$ and $\overline{w}_j$, which are irrelevant for the beam in local coordinates, have components w_i, v_i and w_j, v_j in global coordinates. Thus

$$
[\overline{K}_{ij}] = EI \begin{array}{c} \begin{array}{cccccc} w_i & v_i & \theta_i & w_j & v_j & \theta_j \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12/L^3 & -6/L^2 & 0 & -12/L^3 & -6/L^2 \\ 0 & -6/L^2 & 4/L & 0 & 6/L^2 & 2/L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12/L^3 & 6/L^2 & 0 & 12/L^3 & 6/L^2 \\ 0 & -6/L^2 & 2/L & 0 & 6/L^2 & 4/L \end{bmatrix} \end{array} \tag{17.32}
$$

We may deduce the transformation matrix $[T]$ from Eq. (17.17) if we remember that although w and v transform in exactly the same way as in the case of a truss member the rotations θ remain the same in either local or global coordinates.

Hence

$$
[T] = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17.33}
$$

where λ and μ have previously been defined. Thus since

$$
[K_{ij}] = [T]^{\mathrm{T}}[\overline{K}_{ij}][T]
$$

we have, from Eqs (17.32) and (17.33)

$$
[K_{ij}] = EI \begin{bmatrix} 12\mu^2/L^3 & & & & & \text{SYM} \\ -12\lambda\mu/L^3 & 12\lambda^2/L^3 & & & & \\ 6\mu/L^2 & -6\lambda/L^2 & 4/L & & & \\ -12\mu^2/L^3 & 12\lambda\mu/L^3 & -6\mu/L^2 & 12\mu^2/L^3 & & \\ 12\lambda\mu/L^3 & -12\lambda^2/L^3 & 6\lambda/L^2 & -12\lambda\mu/L^3 & 12\lambda^2/L^3 & \\ 6\mu/L^2 & -6\lambda/L^2 & 2/L & -6\mu/L^2 & 6\lambda/L^2 & 4/L \end{bmatrix} \tag{17.34}
$$

Again the stiffness matrix for the complete structure is assembled from the member stiffness matrices, the boundary conditions are applied and the resulting set of equations solved for the unknown nodal displacements and forces.

The internal shear forces and bending moments in a beam may be obtained in terms of the calculated nodal displacements. Thus, for a beam joining nodes i and j we shall have obtained the unknown values of υ_i, θ_i and υ_j, θ_j. The nodal forces $F_{y,i}$ and M_i are then obtained from Eq. (17.31) if the beam is aligned with the x axis. Hence

$$\left.\begin{aligned} F_{y,i} &= EI\left(\frac{12}{L^3}\upsilon_i - \frac{6}{L^2}\theta_i - \frac{12}{L^3}\upsilon_j - \frac{6}{L^2}\theta_j\right) \\ M_i &= EI\left(-\frac{6}{L^2}\upsilon_i + \frac{4}{L}\theta_i + \frac{6}{L^2}\upsilon_j + \frac{2}{L}\theta_j\right) \end{aligned}\right\} \tag{17.35}$$

Similar expressions are obtained for the forces at node j. From Fig. 17.6 we see that the shear force S_y and bending moment M in the beam are given by

$$\left.\begin{aligned} S_y &= F_{y,i} \\ M &= F_{y,i}x + M_i \end{aligned}\right\} \tag{17.36}$$

Substituting Eq. (17.35) into Eq. (17.36) and expressing in matrix form yields

$$\begin{Bmatrix} S_y \\ M \end{Bmatrix} = EI\begin{bmatrix} \dfrac{12}{L^3} & -\dfrac{6}{L^2} & -\dfrac{12}{L^3} & -\dfrac{6}{L^2} \\ \dfrac{12}{L^3}x - \dfrac{6}{L^2} & -\dfrac{6}{L^2}x + \dfrac{4}{L} & -\dfrac{12}{L^3}x + \dfrac{6}{L^2} & -\dfrac{6}{L^2}x + \dfrac{2}{L} \end{bmatrix}\begin{Bmatrix} \upsilon_i \\ \theta_i \\ \upsilon_j \\ \theta_j \end{Bmatrix} \tag{17.37}$$

The matrix analysis of the beam in Fig. 17.6 is based on the condition that no external forces are applied between the nodes. Obviously in a practical case a beam supports a variety of loads along its length and therefore such beams must be idealized into a number of *beam-elements* for which the above condition holds. The idealization is accomplished by merely specifying nodes at points along the beam such that any element lying between adjacent nodes carries, at the most, a uniform shear and a linearly varying bending moment. For example, the beam of Fig. 17.7 would be idealized into beam-elements 1–2, 2–3 and 3–4 for which the unknown nodal displacements are υ_2, θ_2, θ_3, υ_4 and θ_4 ($\upsilon_1 = \theta_1 = \upsilon_3 = 0$).

Beams supporting distributed loads require special treatment in that the distributed load is replaced by a series of statically equivalent point loads at a selected number of nodes. Clearly the greater the number of nodes chosen, the more accurate but more complicated and therefore time consuming will be the analysis. Figure 17.8 shows a typical idealization of a beam supporting a uniformly distributed load. The method of idealization may be found in specialist texts on matrix analysis.

Many simple beam problems may be idealized into a combination of two beam-elements and three nodes. A few examples of such beams are shown in Fig. 17.9. If we therefore assemble a stiffness matrix for the general case of a two beam-element system we may use it to solve a variety of problems simply by inserting the appropriate loading and support conditions. Consider the assemblage of two beam-elements

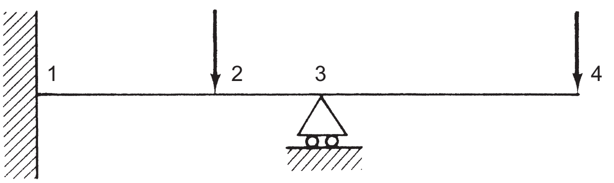

FIGURE 17.7

Idealization of a beam into beam-elements.

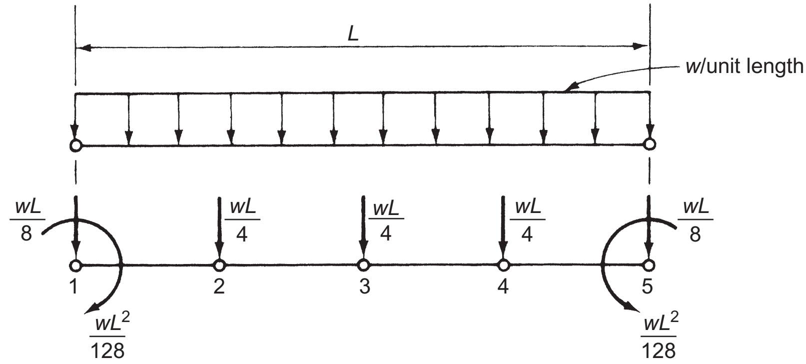

FIGURE 17.8

Idealization of a beam supporting a uniformly distributed load.

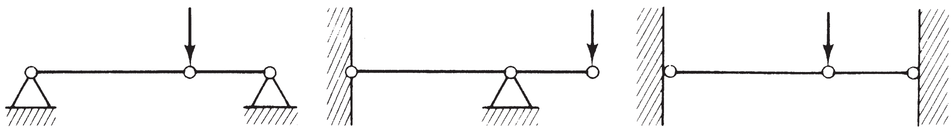

FIGURE 17.9

Idealization of beams into beam-elements.

shown in Fig. 17.10. The stiffness matrices for the beam-elements 1–2 and 2–3 are obtained from Eq. (17.31); thus

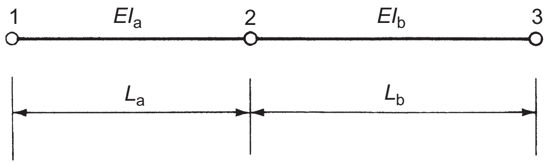

FIGURE 17.10

Assemblage of two beam-elements.

$$
[K_{12}] = EI_a \begin{array}{c} \begin{array}{cccc} \quad v_1 & \qquad \theta_1 & \qquad v_2 & \qquad \theta_2 \end{array} \\ \left[\begin{array}{cc} \underbrace{\begin{bmatrix} 12/L_a^3 & -6/L_a^2 \\ -6/L_a^2 & 4/L_a \end{bmatrix}}_{k_{11}} & \underbrace{\begin{bmatrix} -12/L_a^3 & -6/L_a^2 \\ 6/L_a^2 & 2/L_a \end{bmatrix}}_{k_{12}} \\ \underbrace{\begin{bmatrix} -12/L_a^3 & 6/L_a^2 \\ -6/L_a^2 & 2/L_a \end{bmatrix}}_{k_{21}} & \underbrace{\begin{bmatrix} 12/L_a^3 & 6/L_a^2 \\ 6/L_a^2 & 4/L_a \end{bmatrix}}_{k_{22}} \end{array}\right] \end{array} \qquad (17.38)
$$

$$
[K_{23}] = EI_{\mathrm{b}} \begin{array}{c} \begin{array}{cccc} \quad v_2 \quad & \quad \theta_2 \quad & \quad v_3 \quad & \quad \theta_3 \quad \end{array} \\ \left[\begin{array}{cc} \left[\begin{array}{ccc} 12/L_{\mathrm{b}}^3 & & -6/L_{\mathrm{b}}^2 \\ & k_{22} & \\ -6/L_{\mathrm{b}}^2 & & 4/L_{\mathrm{b}} \end{array} \right] & \left[\begin{array}{ccc} -12/L_{\mathrm{b}}^3 & & -6/L_{\mathrm{b}}^2 \\ & k_{23} & \\ 6/L_{\mathrm{b}}^2 & & 2/L_{\mathrm{b}} \end{array} \right] \\ \left[\begin{array}{ccc} -12/L_{\mathrm{b}}^3 & & 6/L_{\mathrm{b}}^2 \\ & k_{32} & \\ -6/L_{\mathrm{b}}^2 & & 2/L_{\mathrm{b}} \end{array} \right] & \left[\begin{array}{ccc} 12/L_{\mathrm{b}}^3 & & 6/L_{\mathrm{b}}^2 \\ & k_{33} & \\ 6/L_{\mathrm{b}}^2 & & 4/L_{\mathrm{b}} \end{array} \right] \end{array} \right] \end{array} \tag{17.39}
$$

The complete stiffness matrix is formed by superimposing $[K_{12}]$ and $[K_{23}]$ as described in Ex. 17.1. Hence

$$
[K] = E \begin{bmatrix} \dfrac{12I_{\mathrm{a}}}{L_{\mathrm{a}}^3} & -\dfrac{6I_{\mathrm{a}}}{L_{\mathrm{a}}^2} & -\dfrac{12I_{\mathrm{a}}}{L_{\mathrm{a}}^3} & -\dfrac{6I_{\mathrm{a}}}{L_{\mathrm{a}}^2} & 0 & 0 \\ -\dfrac{6I_{\mathrm{a}}}{L_{\mathrm{a}}^2} & \dfrac{4I_{\mathrm{a}}}{L_{\mathrm{a}}} & \dfrac{6I_{\mathrm{a}}}{L_{\mathrm{a}}^2} & \dfrac{2I_{\mathrm{a}}}{L_{\mathrm{a}}^2} & 0 & 0 \\ -\dfrac{12I_{\mathrm{a}}}{L_{\mathrm{a}}^3} & \dfrac{6I_{\mathrm{a}}}{L_{\mathrm{a}}^2} & 12\left(\dfrac{I_{\mathrm{a}}}{L_{\mathrm{a}}^3} + \dfrac{I_{\mathrm{b}}}{L_{\mathrm{b}}^3}\right) & 6\left(\dfrac{I_{\mathrm{a}}}{L_{\mathrm{a}}^2} - \dfrac{I_{\mathrm{b}}}{L_{\mathrm{b}}^2}\right) & -\dfrac{12I_{\mathrm{b}}}{L_{\mathrm{b}}^3} & -\dfrac{6I_{\mathrm{b}}}{L_{\mathrm{b}}^2} \\ -\dfrac{6I_{\mathrm{a}}}{L_{\mathrm{a}}^2} & \dfrac{2I_{\mathrm{a}}}{L_{\mathrm{a}}} & 6\left(\dfrac{I_{\mathrm{a}}}{L_{\mathrm{a}}^2} - \dfrac{I_{\mathrm{b}}}{L_{\mathrm{b}}^2}\right) & 4\left(\dfrac{I_{\mathrm{a}}}{L_{\mathrm{a}}} + \dfrac{I_{\mathrm{b}}}{L_{\mathrm{b}}}\right) & \dfrac{6I_{\mathrm{b}}}{L_{\mathrm{b}}^2} & \dfrac{2I_{\mathrm{b}}}{L_{\mathrm{b}}} \\ 0 & 0 & -\dfrac{12I_{\mathrm{b}}}{L_{\mathrm{b}}^3} & \dfrac{6I_{\mathrm{b}}}{L_{\mathrm{b}}^2} & \dfrac{12I_{\mathrm{b}}}{L_{\mathrm{b}}^3} & \dfrac{6I_{\mathrm{b}}}{L_{\mathrm{b}}^2} \\ 0 & 0 & -\dfrac{6I_{\mathrm{b}}}{L_{\mathrm{b}}^2} & \dfrac{2I_{\mathrm{b}}}{L_{\mathrm{b}}} & \dfrac{6I_{\mathrm{b}}}{L_{\mathrm{b}}^2} & \dfrac{4I_{\mathrm{b}}}{L_{\mathrm{b}}} \end{bmatrix} \tag{17.40}
$$

EXAMPLE 17.2

Determine the unknown nodal displacements and forces in the beam shown in Fig. 17.11. The beam is of uniform section throughout.

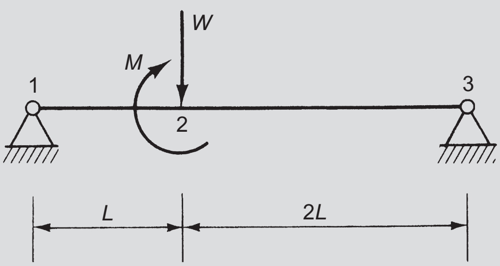

FIGURE 17.11

Beam of Ex. 17.2.

The beam may be idealized into two beam-elements, 1–2 and 2–3. From Fig. 17.11 we see that $v_1 = v_3 = 0$, $F_{y,2} = -W$, $M_2 = +M$. Therefore, eliminating rows and columns corresponding to zero displacements from Eq. (17.40), we obtain

$$\begin{Bmatrix} F_{y,2}=-W \\ M_2=M \\ M_1=0 \\ M_3=0 \end{Bmatrix} = EI \begin{bmatrix} 27/2L^3 & 9/2L^2 & 6/L^2 & -3/2L^2 \\ 9/2L^2 & 6/L & 2/L & 1/L \\ 6/L^2 & 2/L & 4/L & 0 \\ -3/2L^2 & 1/L & 0 & 2/L \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_1 \\ \theta_3 \end{Bmatrix} \tag{i}$$

Equation (i) may be written such that the elements of [K] are pure numbers

$$\begin{Bmatrix} F_{y,2}=-W \\ M_2/L=M/L \\ M_1/L=0 \\ M_3/L=0 \end{Bmatrix} = \frac{EI}{2L^3} \begin{bmatrix} 27 & 9 & 12 & -3 \\ 9 & 12 & 4 & 2 \\ 12 & 4 & 8 & 0 \\ -3 & 2 & 0 & 4 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 L \\ \theta_1 L \\ \theta_3 L \end{Bmatrix} \tag{ii}$$

Expanding Eq. (ii) by matrix multiplication we have

$$\begin{Bmatrix} -W \\ M/L \end{Bmatrix} = \frac{EI}{2L^3}\left(\begin{bmatrix} 27 & 9 \\ 9 & 12 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 L \end{Bmatrix} + \begin{bmatrix} 12 & -3 \\ 4 & 2 \end{bmatrix} \begin{Bmatrix} \theta_1 L \\ \theta_3 L \end{Bmatrix} \right) \tag{iii}$$

and

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \frac{EI}{2L^3}\left(\begin{bmatrix} 12 & 4 \\ -3 & 2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 L \end{Bmatrix} + \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \theta_1 L \\ \theta_3 L \end{Bmatrix} \right) \tag{iv}$$

Equation (iv) gives

$$\begin{Bmatrix} \theta_1 L \\ \theta_3 L \end{Bmatrix} = \begin{bmatrix} -\dfrac{3}{2} & -\dfrac{1}{2} \\ -\dfrac{3}{4} & -\dfrac{1}{2} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 L \end{Bmatrix} \tag{v}$$

Substituting Eq. (v) in Eq. (iii) we obtain

$$\begin{Bmatrix} v_2 \\ \theta_2 L \end{Bmatrix} = \frac{L^3}{9EI} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{Bmatrix} -W \\ M/L \end{Bmatrix} \tag{vi}$$

from which the unknown displacements at node 2 are

$$v_2 = -\frac{4}{9}\frac{WL^3}{EI} - \frac{2}{9}\frac{ML^2}{EI}$$

$$\theta_2 = \frac{2}{9}\frac{WL^2}{EI} + \frac{1}{3}\frac{ML}{EI}$$

In addition, from Eq. (v) we find that

$$\theta_1 = \frac{5}{9}\frac{WL^2}{EI} + \frac{1}{6}\frac{ML}{EI}$$

$$\theta_3 = -\frac{4}{9}\frac{WL^2}{EI} - \frac{1}{3}\frac{ML}{EI}$$

It should be noted that the solution has been obtained by inverting two 2×2 matrices rather than the 4×4 matrix of Eq. (ii). This simplification has been brought about by the fact that $M_1 = M_3 = 0$.

The internal shear forces and bending moments can now be found using Eq. (17.37). For the beam-element 1–2 we have

$$S_{y,12} = EI\left(\frac{12}{L^3}v_1 - \frac{6}{L^2}\theta_1 - \frac{12}{L^3}v_2 - \frac{6}{L^2}\theta_2\right)$$

or

$$S_{y,12} = \frac{2}{3}W - \frac{1}{3}\frac{M}{L}$$

and

$$M_{12} = EI\left[\left(\frac{12}{L^3}x - \frac{6}{L^2}\right)v_1 + \left(-\frac{6}{L^2}x + \frac{4}{L}\right)\theta_1 + \left(-\frac{12}{L^3}x + \frac{6}{L^2}\right)v_2 + \left(-\frac{6}{L^2}x + \frac{2}{L}\right)\theta_2\right]$$

which reduces to

$$M_{12} = \left(\frac{2}{3}W - \frac{1}{3}\frac{M}{L}\right)x$$

EXAMPLE 17.3

The beam shown in Fig. 17.12 is simply supported at nodes 1 and 2 and is attached to a roller support at node 3, which prevents horizontal and rotational displacements. If the flexural rigidity of the beam is EI and it carries a vertical load W at node 3, find the rotation of the beam at nodes 1 and 2, the vertical displacement at node 3, and hence the reactions at the supports. Finally sketch the bending moment diagram for the beam showing the principal values.

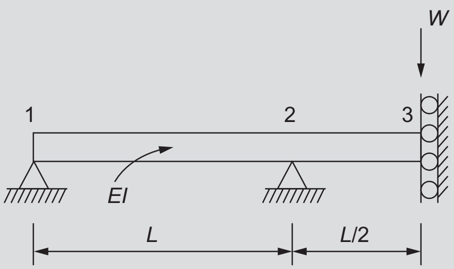

FIGURE 17.12

Beam of Ex. 17.3.

From Eq. (17.31), the stiffness matrices for the beam are

$$[K_{12}] = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \tag{i}$$

$$[K_{23}] = \frac{EI}{L^3}\begin{bmatrix} 96 & -24L & -96 & -24L \\ -24L & 8L^2 & 24L & 4L^2 \\ -96 & 24L & 96 & 24L \\ -24L & 4L^2 & 24L & 8L^2 \end{bmatrix} \tag{ii}$$

Combining Eqs (i) and (ii) and referring to Eq. (17.31) for the complete beam

$$\begin{bmatrix} F_{y,1} \\ M_1 = 0 \\ F_{y,2} \\ M_2 = 0 \\ F_{y,3} = -W \\ M_3 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L & 0 & 0 \\ -6L & 4L^2 & 6L & 2L^2 & 0 & 0 \\ -12 & 6L & 108 & -18L & -96 & -24L \\ -6L & 2L^2 & -18L & 12L^2 & 24L & 4L^2 \\ 0 & 0 & -96 & 24L & 96 & 24L \\ 0 & 0 & -24L & 4L^2 & 24L & 8L^2 \end{bmatrix}\begin{bmatrix} v_1 = 0 \\ \theta_1 \\ v_2 = 0 \\ \theta_2 \\ v_3 \\ \theta_3 = 0 \end{bmatrix} \quad \text{(iii)}$$

Then

$$M_1 = 0 = 4L^2\theta_1 + 2L^2\theta_2$$

so that

$$\theta_1 = -\frac{\theta_2}{2} \quad \text{(iv)}$$

Also

$$M_2 = 0 = 2L^2\theta_1 + 12L^2\theta_2 + 24Lv_3,$$

which gives

$$v_3 = -\frac{11}{24}\theta_2 L \quad \text{(v)}$$

Further

$$F_{y,3} = -W = \frac{EI}{L^3}(24L\theta_2 + 96v_3) \quad \text{(vi)}$$

Substituting for v_3 from Eq. (v) in Eq. (vi) gives

$$\theta_2 = \frac{WL^2}{20EI} = 0.05\frac{WL^2}{EI}$$

Then, from Eq. (iv)

$$\theta_1 = -0.025\frac{WL^2}{EI}$$

and, from Eq. (v)

$$v_3 = -0.023\frac{WL^3}{EI}$$

From Eqs (iii)

$$F_{y,1} = \frac{EI}{L^3}(-6L\theta_1 - 6L\theta_2) = -0.15W$$

$$F_{y,2} = \frac{EI}{L^3}(6L\theta_1 - 18L\theta_2 - 96v_3) = 1.15W$$

and

$$M_3 = \frac{EI}{L^3}\left(4L^2\theta_2 + 24Lv_3\right) = -0.35WL$$

The bending moment diagram is shown in Fig. 17.13.

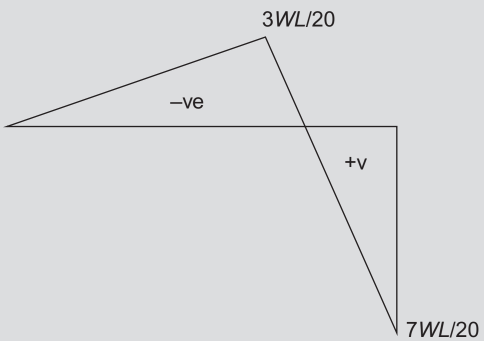

FIGURE 17.13

Bending moment diagram for the beam of Ex. 17.3.

EXAMPLE 17.4

The rigidly jointed frame shown in Fig. 17.14 is fixed at the joint 1 and pinned at the joint 4; all members have the same flexural rigidity *EI*. Use the matrix displacement method to find the displacements at the joints and hence calculate the support reactions at the joint 1.

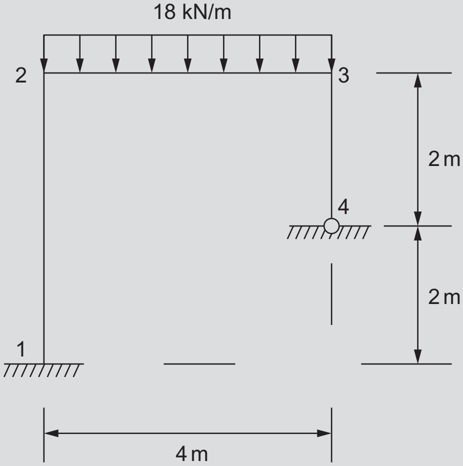

FIG. 17.14

Frame of Ex. 17.4.

From Table 16.6, the fixing moments at the joints 2 and 3 in the member 23 produced by the uniformly distributed load are

$$M_2 = -M_3 = \frac{18 \times 4^2}{12} = 24\,\text{kNm}$$

In addition, there will be equal vertical reactions at 2 and 3 given by

$$F_{y,2} = F_{y,3} = \frac{18 \times 4}{2} = 36\,\text{kN}$$

The member stiffness matrices are assembled using Eqs (17.34) where, for member 12, $\mu = 1, \lambda = 0$; for member 23, $\mu = 0$, $\lambda = 1$ and for member 34, $\mu = -1$, $\lambda = 0$. Then

$$[k_{12}] = EI \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12/64 & 0 & 6/16 & -12/64 & 0 & 6/16 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6/16 & 0 & 4/4 & -6/16 & 0 & 2/4 \\ -12/64 & 0 & -6/16 & 12/64 & 0 & -6/16 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6/16 & 0 & 2/4 & -6/16 & 0 & 4/4 \end{bmatrix} \end{array}$$

$$[k_{23}] = EI \begin{array}{c} \begin{array}{cccccc} u_2 & v_2 & \theta_2 & u_3 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12/64 & -6/16 & 0 & -12/64 & -6/16 \\ 0 & -6/16 & 4/4 & 0 & 6/16 & 2/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12/64 & 6/16 & 0 & 12/64 & 6/16 \\ 0 & -6/16 & 2/4 & 0 & 6/16 & 4/4 \end{bmatrix} \end{array}$$

$$[k_{34}] = EI \begin{array}{c} \begin{array}{cccccc} u_3 & v_3 & \theta_3 & u_4 & v_4 & \theta_4 \end{array} \\ \begin{bmatrix} 12/8 & 0 & -6/4 & -12/8 & 0 & -6/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6/4 & 0 & 4/2 & 6/4 & 0 & 2/2 \\ -12/8 & 0 & 6/4 & 12/8 & 0 & 6/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6/4 & 0 & 2/2 & 6/4 & 0 & 4/2 \end{bmatrix} \end{array}$$

The stiffness matrix for the complete frame is then obtained as described in Section 17.2. This matrix would be of the order 12×12 but may be reduced in size by using the boundary conditions that $u_1 = v_1 = \theta_1 = v_2 = v_3 = u_4 = v_4 = \theta_4 = 0$.

Note that the moment at 4 is zero since 4 is a pinned joint. Also, since the axial stiffness of the frame members is assumed to be high, $u_2 = u_3$. The stiffness matrix for the frame is then

$$\begin{Bmatrix} F_{x,2} \\ M_2 \\ F_{x,3} \\ M_3 \end{Bmatrix} = EI \begin{bmatrix} 12/64 & -6/16 & 0 & 0 \\ -6/16 & 2 & 0 & 2/4 \\ 0 & 0 & 12/8 & -6/4 \\ 0 & 2/4 & -6/4 & 3 \end{bmatrix} \begin{Bmatrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix} \tag{i}$$

Equations (i) may be simplified further since $F_{x,2} + F_{x,3} = 0$ and $u_2 = u_3$. Eqs (i) then reduce to a 3×3 matrix, i.e.

$$\begin{Bmatrix} 0 \\ 24 \\ -24 \end{Bmatrix} = EI \begin{bmatrix} 27/16 & -3/8 & -6/4 \\ -3/8 & 2 & 0.5 \\ -3/2 & 0.5 & 3 \end{bmatrix} \begin{Bmatrix} u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \tag{ii}$$

Solution of Eqs (ii) gives

$$u_2 = -\frac{10.9}{EI}, \quad \theta_2 = \frac{13.9}{EI}, \quad \theta_3 = -\frac{15.7}{EI}$$

These displacements are global displacements and must be transferred to local displacements for each member. Thus, u_2 (global) will be $-v_2$ (local) for the member 12, i.e. v_2 (local for member 12) $= 10.9/EI$. Note that θ is the same in both global and local systems. Then from Eq. (16.28)

$$M_{12}(\text{at } 1) = \frac{2EI}{4}\left[0 + \frac{13.9}{EI} + \frac{3}{4}\left(0 + \frac{10.9}{EI}\right)\right]$$

from which

$$M_{12} \text{ (at 1)} = 11.04 \text{ kNm}$$

Note that there is no fixing moment to be added to M_{12} (at 1) as there would be to M_{23} (at 2, $= -24$ kNm).

$$S_{12}(\text{at }1) = \frac{6EI}{4^2}\left[0 + \frac{13.9}{EI} + \frac{2}{4}\left(0 + \frac{10.9}{EI}\right)\right],$$

which gives

$$S_{12}\ (\text{at }1)\ =\ 7.3\ \text{kN}.$$

The horizontal reaction at 1 is

$$F_{y,1} + 36 = EI\left(\frac{-6}{16} \times \frac{13.9}{EI} + \frac{6}{16} \times \frac{15.7}{EI}\right)$$

i.e.

$$\text{Horizontal reaction at }1\ =\ 36.8\ \text{kN}.$$

17.3 Finite element method for continuum structures

In the previous sections we have discussed the matrix method of solution of structures composed of elements connected only at nodal points. For skeletal structures consisting of arrangements of beams these nodal points fall naturally at joints and at positions of concentrated loading. Continuum structures, such as flat plates, aircraft skins, shells, etc., do not possess such natural subdivisions and must therefore be artificially idealized into a number of elements before matrix methods can be used. These *finite elements*, as they are known, may be two- or three-dimensional but the most commonly used are two-dimensional triangular and quadrilateral shaped elements. The idealization may be carried out in any number of different ways depending on such factors as the type of problem, the accuracy of the solution required and the time and money available. For example, a *coarse* idealization involving a small number of large elements would provide a comparatively rapid but very approximate solution while a *fine* idealization of small elements would produce more accurate results but would take longer and consequently cost more. Frequently, *graded meshes* are used in which small elements are placed in regions where high stress concentrations are expected, e.g. around cut-outs and loading points. The principle is illustrated in Fig. 17.15 where a graded system of triangular elements is used to examine the stress concentration around a circular hole in a flat plate.

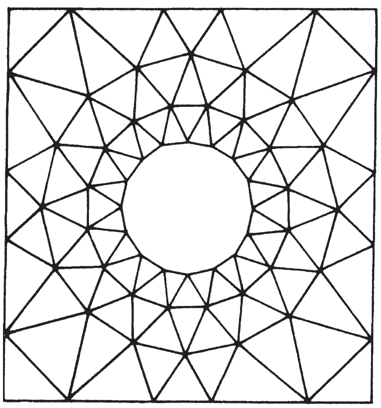

FIGURE 17.15

Finite element idealization of a flat plate with a central hole.

Although the elements are connected at an infinite number of points around their boundaries it is assumed that they are only interconnected at their corners or nodes. Thus, compatibility of displacement is only ensured at the nodal points. However, in the finite element method a displacement pattern is chosen for each element which may satisfy some, if not all, of the compatibility requirements along the sides of adjacent elements.

Since we are employing matrix methods of solution we are concerned initially with the determination of nodal forces and displacements. Thus, the system of loads on the structure must be replaced by an equivalent system of nodal forces. Where these loads are concentrated the elements are chosen such that a node occurs at the point of application of the load. In the case of distributed loads, equivalent nodal concentrated loads must be calculated.

The solution procedure is identical in outline to that described in the previous sections for skeletal structures; the differences lie in the idealization of the structure into finite elements and the calculation of the stiffness matrix for each element. The latter procedure, which in general terms is applicable to all finite elements, may be specified in a number of distinct steps. We shall illustrate the method by establishing the stiffness matrix for the simple one-dimensional beam-element of Fig. 17.6 for which we have already derived the stiffness matrix using slope–deflection.

Stiffness matrix for a beam-element

The first step is to choose a suitable coordinate and node numbering system for the element and define its nodal displacement vector $\{\delta^e\}$ and nodal load vector $\{F^e\}$. Use is made here of the superscript e to denote element vectors since, in general, a finite element possesses more than two nodes. Again we are not concerned with axial or shear displacements so that for the beam-element of Fig. 17.6 we have

$$\{\delta^e\} = \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} \quad \{F^e\} = \begin{Bmatrix} F_{y,i} \\ M_i \\ F_{y,j} \\ M_j \end{Bmatrix}$$

Since each of these vectors contains four terms the element stiffness matrix $[K^e]$ will be of order 4×4.

In the second step we select a displacement function which uniquely defines the displacement of all points in the beam-element in terms of the nodal displacements. This displacement function may be taken as a polynomial which must include four arbitrary constants corresponding to the four nodal degrees of freedom of the element. Thus

$$v(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{17.41}$$

Equation (17.41) is of the same form as that derived from elementary bending theory for a beam subjected to concentrated loads and moments and may be written in matrix form as

$$\{v(x)\} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

or in abbreviated form as

$$\{v(x)\} = [f(x)]\{\alpha\} \tag{17.42}$$

The rotation θ at any section of the beam-element is given by $\partial v/\partial x$; therefore

$$\theta = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2 \tag{17.43}$$

From Eqs (17.41) and (17.43) we can write down expressions for the nodal displacements v_i, θ_i and v_j, θ_j at $x = 0$ and $x = L$, respectively. Hence

$$\left.\begin{aligned} v_i &= \alpha_1 \\ \theta_i &= \alpha_2 \\ v_j &= \alpha_1 + \alpha_2 L + \alpha_3 L^2 + \alpha_4 L^3 \\ \theta_j &= \alpha_2 + 2\alpha_3 L + 3\alpha_4 L^2 \end{aligned}\right\} \tag{17.44}$$

Writing Eq. (17.44) in matrix form gives

$$\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \tag{17.45}$$

or

$$\{\delta^e\} = [A]\{\alpha\} \tag{17.46}$$

The third step follows directly from Eqs (17.45) and (17.42) in that we express the displacement at any point in the beam-element in terms of the nodal displacements. Using Eq. (17.46) we obtain

$$\{\alpha\} = [A^{-1}]\{\delta^e\} \tag{17.47}$$

Substituting in Eq. (17.42) gives

$$\{v(x)\} = [f(x)][A^{-1}]\{\delta^e\} \tag{17.48}$$

where $[A^{-1}]$ is obtained by inverting $[A]$ in Eq. (17.45) and may be shown to be given by

$$[A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^3 & 1/L^2 & -2/L^3 & 1/L^2 \end{bmatrix} \tag{17.49}$$

In step four we relate the strain $\{\varepsilon(x)\}$ at any point x in the element to the displacement $\{v(x)\}$ and hence to the nodal displacements $\{\delta^e\}$. Since we are concerned here with bending deformations only we may represent the strain by the curvature $\partial^2 v/\partial x^2$. Hence from Eq. (17.41)

$$\frac{\partial^2 v}{\partial x^2} = 2\alpha_3 + 6\alpha_4 x \tag{17.50}$$

or in matrix form

$$\{\varepsilon\} = [0 \quad 0 \quad 2 \quad 6x] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \tag{17.51}$$

which we write as

$$\{\varepsilon\} = [C]\{\alpha\} \tag{17.52}$$

Substituting for $\{\alpha\}$ in Eq. (17.52) from Eq. (17.47) we have

$$\{\varepsilon\} = [C][A^{-1}]\{\delta^{e}\} \tag{17.53}$$

Step five relates the internal stresses in the element to the strain $\{\varepsilon\}$ and hence, using Eq. (17.53), to the nodal displacements $\{\delta^{e}\}$. In our beam-element the stress distribution at any section depends entirely on the value of the bending moment M at that section. Thus we may represent a 'state of stress' $\{\sigma\}$ at any section by the bending moment M, which, from simple beam theory, is given by

$$M = -EI\frac{\partial^2 v}{\partial x^2}$$

or

$$\{\sigma\} = [EI]\{\varepsilon\} \tag{17.54}$$

which we write as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{17.55}$$

The matrix $[D]$ in Eq. (17.55) is the 'elasticity' matrix relating 'stress' and 'strain'. In this case $[D]$ consists of a single term, the flexural rigidity EI of the beam. Generally, however, $[D]$ is of a higher order. If we now substitute for $\{\varepsilon\}$ in Eq. (17.55) from Eq. (17.53) we obtain the 'stress' in terms of the nodal displacements, i.e.

$$\{\sigma\} = [D][C][A^{-1}]\{\delta^{e}\} \tag{17.56}$$

The element stiffness matrix is finally obtained in step six in which we replace the internal 'stresses' $\{\sigma\}$ by a statically equivalent nodal load system $\{F^{e}\}$, thereby relating nodal loads to nodal displacements (from Eq. (17.56)) and defining the element stiffness matrix $[K^{e}]$. This is achieved by employing the principle of the stationary value of the total potential energy of the beam (see Section 15.3) which comprises the internal strain energy U and the potential energy V of the nodal loads. Thus

$$U + V = \frac{1}{2}\int_{\text{vol}} \{\varepsilon\}^{\mathrm{T}}\{\sigma\}\,\mathrm{d(vol)} - \{\delta^{e}\}^{\mathrm{T}}\{F^{e}\} \tag{17.57}$$

Substituting in Eq. (17.57) for $\{\varepsilon\}$ from Eq. (17.53) and $\{\sigma\}$ from Eq. (17.56) we have

$$U + V = \frac{1}{2}\int_{\text{vol}} \{\delta^{e}\}^{\mathrm{T}}[A^{-1}]^{\mathrm{T}}[C]^{\mathrm{T}}[D][C][A^{-1}]\{\delta^{e}\}\mathrm{d(vol)} - \{\delta^{e}\}^{\mathrm{T}}\{F^{e}\} \tag{17.58}$$

The total potential energy of the beam has a stationary value with respect to the nodal displacements $\{\delta^{e}\}^{\mathrm{T}}$; hence, from Eq. (17.58)

$$\frac{\partial(U+V)}{\partial\{\delta^{e}\}^{\mathrm{T}}} = \int_{\text{vol}} [A^{-1}]^{\mathrm{T}}[C]^{\mathrm{T}}[D][C][A^{-1}]\{\delta^{e}\}\,\mathrm{d(vol)} - \{F^{e}\} = 0 \tag{17.59}$$

whence

$$\{F^e\} = \left[\int_{\text{vol}} [C]^{\mathrm{T}}[A^{-1}]^{\mathrm{T}}[D][C][A^{-1}]\,\mathrm{d(vol)}\right]\{\delta^e\} \tag{17.60}$$

or writing $[C][A^{-1}]$ as $[B]$ we obtain

$$\{F^e\} = \left[\int_{\text{vol}} [B]^{\mathrm{T}}[D][B]\,\mathrm{d(vol)}\right]\{\delta^e\} \tag{17.61}$$

from which the element stiffness matrix is clearly

$$\{K^e\} = \left[\int_{\text{vol}} [B]^{\mathrm{T}}[D][B]\,\mathrm{d(vol)}\right] \tag{17.62}$$

From Eqs (17.49) and (17.51) we have

$$[B] = [C][A^{-1}] = [0 \quad 0 \quad 2 \quad 6x]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^3 & 1/L^2 & -2/L^3 & 1/L^2 \end{bmatrix}$$

or

$$[B]^{\mathrm{T}} = \begin{bmatrix} -\dfrac{6}{L^2} + \dfrac{12x}{L^3} \\ -\dfrac{4}{L} + \dfrac{6x}{L^2} \\ \dfrac{6}{L^2} - \dfrac{12x}{L^3} \\ -\dfrac{2}{L} + \dfrac{6x}{L^2} \end{bmatrix} \tag{17.63}$$

Hence

$$[K^e] = \int_0^L \begin{bmatrix} -\dfrac{6}{L^2} + \dfrac{12x}{L^3} \\ -\dfrac{4}{L} + \dfrac{6x}{L^2} \\ \dfrac{6}{L^2} - \dfrac{12x}{L^3} \\ -\dfrac{2}{L} + \dfrac{6x}{L^2} \end{bmatrix} [EI]\left[-\frac{6}{L^2} + \frac{12x}{L^3} \quad -\frac{4}{L} + \frac{6x}{L^2} \quad \frac{6}{L^2} \quad -\frac{12x}{L^3} \quad -\frac{2}{L} + \frac{6x}{L^2}\right]\mathrm{d}x$$

which gives

$$[K^e] = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \tag{17.64}$$

Equation (17.64) is identical to the stiffness matrix (see Eq. (17.31)) for the uniform beam of Fig. 17.6.

Finally, in step seven, we relate the internal 'stresses', $\{\sigma\}$, in the element to the nodal displacements $\{\delta^e\}$. In fact, this has been achieved to some extent in Eq. (17.56), namely

$$\{\sigma\} = [D][C][A^{-1}]\{\delta^e\}$$

or, from the above

$$\{\sigma\} = [D][B]\{\delta^e\} \tag{17.65}$$

Equation (17.65) is usually written

$$\{\sigma\} = [H]\{\delta^e\} \tag{17.66}$$

in which $[H] = [D][B]$ is the stress–displacement matrix. For this particular beam-element $[D] = EI$ and $[B]$ is defined in Eq. (17.63). Thus

$$[H] = EI\left[-\frac{6}{L^2}+\frac{12}{L^3}x \quad -\frac{4}{L}+\frac{6}{L^2}x \quad \frac{6}{L^2}-\frac{12}{L^3}x \quad -\frac{2}{L}+\frac{6}{L^2}x\right] \tag{17.67}$$

Stiffness matrix for a triangular finite element

Triangular finite elements are used in the solution of plane stress and plane strain problems. Their advantage over other shaped elements lies in their ability to represent irregular shapes and boundaries with relative simplicity.

In the derivation of the stiffness matrix we shall adopt the step by step procedure of the previous example. Initially, therefore, we choose a suitable coordinate and node numbering system for the element and define its nodal displacement and nodal force vectors. Figure 17.16 shows a triangular element referred to axes Oxy and having nodes i, j and k lettered anticlockwise. It may be shown that the inverse of the $[A]$ matrix for a triangular element contains terms giving the actual area of the element; this area is positive if the above node lettering or numbering system is adopted. The element is to be used for plane elasticity problems and has therefore two degrees of freedom per node, giving a total of six degrees of freedom for the element, which will result in a 6×6 element stiffness matrix $[K^e]$. The nodal forces and displacements are shown and the complete displacement and force vectors are

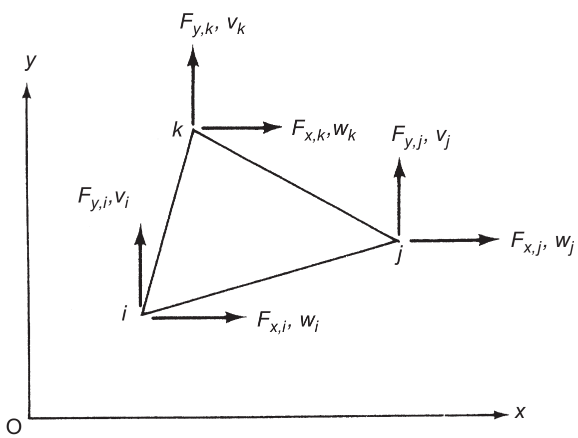

FIGURE 17.16

Triangular element for plane elasticity problems.

$$\{\delta^e\} = \begin{Bmatrix} w_i \\ v_i \\ w_j \\ v_j \\ w_k \\ v_k \end{Bmatrix} \quad \{F^e\} = \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \\ F_{x,k} \\ F_{y,k} \end{Bmatrix} \tag{17.68}$$

We now select a displacement function which must satisfy the boundary conditions of the element, i.e. the condition that each node possesses two degrees of freedom. Generally, for computational purposes, a polynomial is preferable to, say, a trigonometric series since the terms in a polynomial can be calculated much more rapidly by a digital computer. Furthermore, the total number of degrees of freedom is six, so that only six coefficients in the polynomial can be obtained. Suppose that the displacement function is

$$\left.\begin{aligned} w(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned}\right\} \tag{17.69}$$

The constant terms, α_1 and α_4, are required to represent any in-plane rigid body motion, i.e. motion without strain, while the linear terms enable states of constant strain to be specified; Eq. (17.69) ensures compatibility of displacement along the edges of adjacent elements. Writing Eq. (17.69) in matrix form gives

$$\begin{Bmatrix} w(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \tag{17.70}$$

Comparing Eq. (17.70) with Eq. (17.42) we see that it is of the form

$$\begin{Bmatrix} w(x,y) \\ v(x,y) \end{Bmatrix} = [f(x,y)]\{\alpha\} \tag{17.71}$$

Substituting values of displacement and coordinates at each node in Eq. (17.71) we have, for node i

$$\begin{Bmatrix} w_i \\ v_i \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \end{bmatrix} \{\alpha\}$$

Similar expressions are obtained for nodes j and k so that for the complete element we obtain

$$\begin{Bmatrix} w_i \\ v_i \\ w_j \\ v_j \\ w_k \\ v_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 1 & x_j & x_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \tag{17.72}$$

From Eq. (17.68) and by comparison with Eqs (17.45) and (17.46) we see that Eq. (17.72) takes the form

$$\{\delta^e\} = [A]\{\alpha\}$$

Hence (step 3) we obtain

$$\{\alpha\} = [A^{-1}]\{\delta^e\} \text{ (compare with Eq. (17.47))}$$

The inversion of $[A]$, defined in Eq. (17.72), may be achieved algebraically as illustrated in Ex. 17.5. Alternatively, the inversion may be carried out numerically for a particular element by computer. Substituting for $\{\alpha\}$ from the above into Eq. (17.71) gives

$$\begin{Bmatrix} w(x,y) \\ v(x,y) \end{Bmatrix} = [f(x,y)][A^{-1}]\{\delta^e\} \text{ (compare with Eq. (17.48))} \tag{17.73}$$

The strains in the element are

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{17.74}$$

Direct and shear strains may be defined in the form

$$\varepsilon_x = \frac{\partial w}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x} \tag{17.75}$$

Substituting for w and v in Eq. (17.75) from Eq. (17.69) gives

$$\begin{aligned} \varepsilon_x &= \alpha_2 \\ \varepsilon_y &= \alpha_6 \\ \gamma_{xy} &= \alpha_3 + \alpha_5 \end{aligned}$$

or in matrix form

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \tag{17.76}$$

which is of the form

$$\{\varepsilon\} = [C]\{\alpha\} \text{ (see Eps (17.51) and (17.52))}$$

Substituting for $\{\alpha\}$ $(= [A^{-1}]\{\delta^e\})$ we obtain

$$\{\varepsilon\} = [C][A^{-1}]\{\delta^e\} \text{ (compare with Eq. (17.53))}$$

or

$$\{\varepsilon\} = [B]\{\delta^e\} \text{ (see Eq. (17.63))}$$

where $[C]$ is defined in Eq. (17.76).

In step five we relate the internal stresses $\{\sigma\}$ to the strain $\{\varepsilon\}$ and hence, using step four, to the nodal displacements $\{\delta^e\}$. For plane stress problems

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{17.77}$$

and

$$\left.\begin{aligned}\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_x}{E}\\ \varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E}\\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy}\end{aligned}\right\}\text{(see Chapter 7)}$$

Thus, in matrix form

$$\{\varepsilon\} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} \tag{17.78}$$

It may be shown that

$$\{\sigma\} = \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \gamma_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \tag{17.79}$$

which has the form of Eq. (17.55), i.e.

$$\{\sigma\} = [D]\{\varepsilon\}$$

Substituting for $\{\varepsilon\}$ in terms of the nodal displacements $\{\delta^e\}$ we obtain

$$\{\sigma\} = [D][B]\{\delta^e\}\ \text{(see Eq. (17.56))}$$

In the case of plane strain the elasticity matrix $[D]$ takes a different form to that defined in Eq. (17.79). For this type of problem

$$\begin{aligned}\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}\\ \varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_z}{E}\\ \varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} = 0\\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy}\end{aligned}$$

Eliminating σ_z and solving for σ_x, σ_y and τ_{xy} gives

$$\{\sigma\} = \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E(1-\nu)}{(1-\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \tag{17.80}$$

which again takes the form

$$\{\sigma\} = [D]\{\varepsilon\}$$

Step six, in which the internal stresses $\{\sigma\}$ are replaced by the statically equivalent nodal forces $\{F^e\}$ proceeds, in an identical manner to that described for the beam-element. Thus

$$[F^e] = \left[\int_{\text{vol}} [B]^{\text{T}}[D][B]\,\text{d(vol)}\right]\{\delta^e\}$$

as in Eq. (17.61), whence

$$[K^e] = \left[\int_{\text{vol}} [B]^{\text{T}}[D][B]\,\text{d(vol)}\right]$$

In this expression $[B] = [C][A^{-1}]$ where $[A]$ is defined in Eq. (17.72) and $[C]$ in Eq. (17.76). The elasticity matrix $[D]$ is defined in Eq. (17.79) for plane stress problems or in Eq. (17.80) for plane strain problems. We note that the $[C]$, $[A]$ (therefore $[B]$) and $[D]$ matrices contain only constant terms and may therefore be taken outside the integration in the expression for $[K^e]$, leaving only $\int \text{d(vol)}$ which is simply the area, A, of the triangle times its thickness t. Thus

$$[K^e] = [[B]^{\text{T}}[D][B]At] \tag{17.81}$$

Finally the element stresses follow from Eq. (17.66), i.e.

$$\{\sigma\} = [H]\{\delta^e\}$$

where $[H] = [D][B]$ and $[D]$ and $[B]$ have previously been defined. It is usually found convenient to plot the stresses at the centroid of the element.

Of all the finite elements in use the triangular element is probably the most versatile. It may be used to solve a variety of problems ranging from two-dimensional flat plate structures to three-dimensional folded plates and shells. For three-dimensional applications the element stiffness matrix $[K^e]$ is transformed from an in-plane xy coordinate system to a three-dimensional system of global coordinates by the use of a transformation matrix similar to those developed for the matrix analysis of skeletal structures. In addition to the above, triangular elements may be adapted for use in plate flexure problems and for the analysis of bodies of revolution.

EXAMPLE 17.5

A constant strain triangular element has corners 1(0, 0), 2(4, 0) and 3(2, 2) referred to a Cartesian Oxy axes system and is 1 unit thick. If the elasticity matrix $[D]$ has elements $D_{11} = D_{22} = a$, $D_{12} = D_{21} = b$, $D_{13} = D_{23} = D_{31} = D_{32} = 0$ and $D_{33} = c$, derive the stiffness matrix for the element.

From Eq. (17.69)

$$w_1 = \alpha_1 + \alpha_2(0) + \alpha_3(0)$$

i.e.

$$w_1 = \alpha_1 \tag{i}$$

$$w_2 = \alpha_1 + \alpha_2(4) + \alpha_3(0)$$

i.e.

$$w_2 = \alpha_1 + 4\alpha_2 \tag{ii}$$

$$w_3 = \alpha_1 + \alpha_2(2) + \alpha_3(2)$$

i.e.

$$w_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \tag{iii}$$

From Eq. (i)

$$\alpha_1 = w_1 \tag{iv}$$

and from Eqs (ii) and (iv)

$$\alpha_2 = \frac{w_2 - w_1}{4} \tag{v}$$

Then, from Eqs (iii)–(v)

$$\alpha_3 = \frac{2w_3 - w_1 - w_2}{4} \tag{vi}$$

Substituting for α_1, α_2 and α_3 in the first of Eq. (17.69) gives

$$w = w_1 + \left(\frac{w_2 - w_1}{4}\right)x + \left(\frac{2w_3 - w_1 - w_2}{4}\right)y$$

or

$$w = \left(1 - \frac{x}{4} - \frac{y}{4}\right)w_1 + \left(\frac{x}{4} - \frac{y}{4}\right)w_2 + \frac{y}{2}w_3 \tag{vii}$$

Similarly

$$v = \left(1 - \frac{x}{4} - \frac{y}{4}\right)v_1 + \left(\frac{x}{4} - \frac{y}{4}\right)v_2 + \frac{y}{2}v_3 \tag{viii}$$

Now from Eq. (17.75)

$$\begin{aligned}
\varepsilon_x &= \frac{\partial w}{\partial x} = -\frac{w_1}{4} + \frac{w_2}{4} \\
\varepsilon_y &= \frac{\partial v}{\partial y} = -\frac{v_1}{4} - \frac{v_2}{4} + \frac{v_3}{2}
\end{aligned}$$

and

$$\gamma_{xy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x} = -\frac{w_1}{4} - \frac{w_2}{4} - \frac{v_1}{4} + \frac{v_2}{4}$$

Hence

$$[B]\{\delta^e\} = \begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -1 & -1 & 1 & 2 & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ v_1 \\ w_2 \\ v_2 \\ w_3 \\ v_3 \end{Bmatrix} \tag{ix}$$

Also

$$[D] = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

Hence

$$[D][B] = \frac{1}{4}\begin{bmatrix} -a & -b & a & -b & 0 & 2b \\ -b & -a & b & -a & 0 & 2a \\ -c & -c & -c & c & 2c & 0 \end{bmatrix}$$

and

$$[B]^{\mathrm{T}}[D][B] = \frac{1}{16}\begin{bmatrix} a+b & b+c & -a+c & b-c & -2c & -2b \\ b+c & a+c & -b+c & a-c & -2c & -2a \\ -a+c & -b+c & a+c & -b-c & -2c & 2b \\ b-c & a-c & -b-c & a+c & 2c & -2a \\ -2c & -2c & -2c & 2c & 4c & 0 \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

Then, from Eq. (17.81)

$$[K^{\mathrm{e}}] = \frac{1}{4}\begin{bmatrix} a+c & b+c & -a+c & b-c & -2c & -2b \\ b+c & a+c & -b+c & a-c & -2c & -2a \\ -a+c & -b+c & a+c & -b-c & -2c & 2b \\ b-c & a-c & -b-c & a+c & 2c & -2a \\ -2c & -2c & -2c & 2c & 4c & 0 \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

Stiffness matrix for a quadrilateral element

Quadrilateral elements are frequently used in combination with triangular elements to build up particular geometrical shapes.

Figure 17.17 shows a quadrilateral element referred to axes O*xy* and having corner nodes, *i, j, k* and *l*; the nodal forces and displacements are also shown and the displacement and force vectors are

$$\{\delta^{\mathrm{e}}\} = \begin{Bmatrix} w_i \\ v_i \\ w_j \\ v_j \\ w_k \\ v_k \\ w_l \\ v_l \end{Bmatrix} \quad \{F^{\mathrm{e}}\} = \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \\ F_{x,k} \\ F_{y,k} \\ F_{x,l} \\ F_{y,l} \end{Bmatrix} \tag{17.82}$$

As in the case of the triangular element we select a displacement function which satisfies the total of eight degrees of freedom of the nodes of the element; again this displacement function will be in the form of a polynomial with a maximum of eight coefficients. Thus

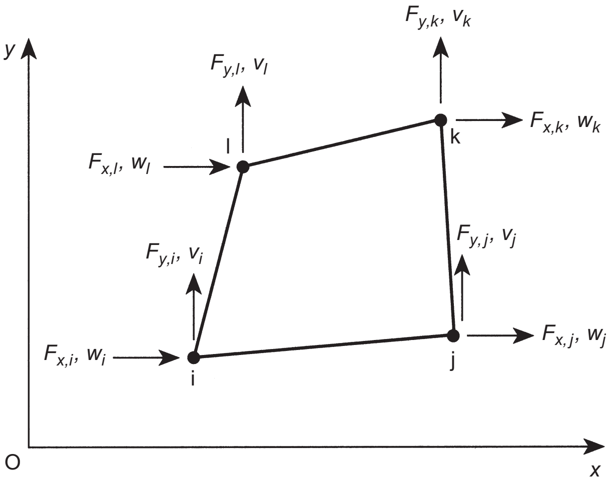

FIGURE 17.17

Quadrilateral element subjected to nodal in-plane forces and displacements.

$$\left.\begin{aligned} w(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \\ v(x,y) &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy \end{aligned}\right\} \tag{17.83}$$

The constant terms, α_1 and α_5, are required, as before, to represent the in-plane rigid body motion of the element while the two pairs of linear terms enable states of constant strain to be represented throughout the element. Further, the inclusion of the xy terms results in both the $w(x,y)$ and $v(x,y)$ displacements having the same algebraic form so that the element behaves in exactly the same way in the x direction as it does in the y direction.

Writing Eq. (17.83) in matrix form gives

$$\begin{Bmatrix} w(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \tag{17.84}$$

or

$$\begin{Bmatrix} w(x,y) \\ v(x,y) \end{Bmatrix} = [f(x,y)]\{\alpha\} \tag{17.85}$$

Now substituting the coordinates and values of displacement at each node we obtain

$$\begin{Bmatrix} w_i \\ v_i \\ w_j \\ v_j \\ w_k \\ v_k \\ w_l \\ v_l \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i & x_i y_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_i & y_i & x_i y_i \\ 1 & x_j & y_j & x_j y_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_j & y_j & x_j y_j \\ 1 & x_k & y_k & x_k y_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_k & y_k & x_k y_k \\ 1 & x_l & y_l & x_l y_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_l & y_l & x_l y_l \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \tag{17.86}$$

which is of the form

$$\{\delta^{e}\} = [A]\{\alpha\}$$

Then

$$\{\alpha\} = [A^{-1}]\{\delta^{e}\} \tag{17.87}$$

The inversion of $[A]$ is illustrated in Ex. 17.4 but, as in the case of the triangular element, is most easily carried out by means of a computer. The remaining analysis is identical to that for the triangular element except that the $\{\varepsilon\}$–$\{\alpha\}$ relationship (see Eq. (17.76)) becomes

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \tag{17.88}$$

EXAMPLE 17.6

A rectangular element used in a plane stress analysis has corners whose coordinates (in metres), referred to an Oxy axes system, are 1(−2, −1), 2(2, −1), 3(2, 1) and 4(−2, 1); the displacements (also in metres) of the corners were

$$w_1 = 0.001 \qquad w_2 = 0.003 \qquad w_3 = -0.003 \qquad w_4 = 0$$
$$v_1 = -0.004 \qquad v_2 = -0.002 \qquad v_3 = 0.001 \qquad v_4 = 0.001$$

If Young's modulus $E = 200\,000$ N/mm^2 and Poisson's ratio $\upsilon = 0.3$, calculate the stresses at the centre of the element.

From the first of Eq. (17.83)

$$w_1 = \alpha_1 - 2\alpha_2 - \alpha_3 + 2\alpha_4 = 0.001 \tag{i}$$

$$w_2 = \alpha_1 + 2\alpha_2 - \alpha_3 - 2\alpha_4 = 0.003 \tag{ii}$$

$$w_3 = \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = -0.003 \tag{iii}$$

$$w_4 = \alpha_1 - 2\alpha_2 + \alpha_3 - 2\alpha_4 = 0 \tag{iv}$$

Subtracting Eq. (ii) from Eq. (i)

$$\alpha_2 - \alpha_4 = 0.0005 \tag{v}$$

Now subtracting Eq. (iv) from Eq. (iii)

$$\alpha_2 + \alpha_4 = -0.00075 \tag{vi}$$

Then subtracting Eq. (vi) from Eq. (v)

$$\alpha_4 = -0.000625 \tag{vii}$$

whence, from either of Eqs (v) or (vi)

$$\alpha_2 = -0.000125 \tag{viii}$$

Adding Eqs (i) and (ii)

$$\alpha_1 - \alpha_3 = 0.002 \tag{ix}$$

Adding Eqs (iii) and (iv)

$$\alpha_1 + \alpha_3 = -0.0015 \tag{x}$$

Then adding Eqs (ix) and (x)

$$\alpha_1 = 0.00025 \tag{xi}$$

and, from either of Eqs (ix) or (x)

$$\alpha_3 = -0.00175 \tag{xii}$$

The second of Eq. (17.83) is used to determine α_5, α_6, α_7 and α_8 in an identical manner to the above. Thus

$$\alpha_5 = -0.001$$
$$\alpha_6 = 0.00025$$
$$\alpha_7 = 0.002$$
$$\alpha_8 = -0.00025$$

Now substituting for $\alpha_1, \alpha_2, \ldots, \alpha_8$ in Eq. (17.83)

$$w_i = 0.00025 - 0.000125x - 0.00175y - 0.000625xy$$

and

$$v_i = -0.001 + 0.00025x + 0.002y - 0.00025xy$$

Then, from Eq. (17.75)

$$\varepsilon_x = \frac{\partial w}{\partial x} = -0.000125 - 0.000625y$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0.002 - 0.00025x$$

$$\gamma_{xy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x} = -0.0015 - 0.000625x - 0.00025y$$

Therefore, at the centre of the element ($x = 0$, $y = 0$)

$$\varepsilon_x = -0.000125$$
$$\varepsilon_y = 0.002$$
$$\gamma_{xy} = -0.0015$$

so that, from Eq. (17.79)

$$\sigma_x = \frac{E}{1 - \nu^2}(\varepsilon_x + \nu\varepsilon_y) = \frac{200000}{1 - 0.3^2}(-0.000125 + (0.3 \times 0.002))$$

i.e.

$$\sigma_x = 104.4\,\mathrm{N/mm^2}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = \frac{200\,000}{1-0.3^2}(0.002 + (0.3 \times 0.000125))$$

i.e.

$$\sigma_y = 431.3\,\mathrm{N/mm^2}$$

and

$$\tau_{xy} = \frac{E}{1-\nu^2} \times \frac{1}{2}(1-\nu)\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

Thus

$$\tau_{xy} = \frac{200\,000}{2(1+0.3)} \times (-0.0015)$$

i.e.

$$\tau_{xy} = -11.5\,\mathrm{N/mm^2}$$

The application of the finite element method to three-dimensional solid bodies is a straightforward extension of the analysis of two-dimensional structures. The basic three-dimensional elements are the tetrahedron and the rectangular prism, both shown in Fig. 17.18. The tetrahedron has four nodes each possessing three degrees of freedom, a total of 12 for the element, while the prism has 8 nodes and therefore a total of 24 degrees of freedom. Displacement functions for each element require polynomials in x, y and z; for the tetrahedron the displacement function is of the first degree with 12 constant coefficients, while that for the prism may be of a higher order to accommodate the 24 degrees of freedom. A development in the solution of three-dimensional problems has been the introduction of curvilinear coordinates. This enables the tetrahedron and prism to be distorted into arbitrary shapes that are better suited for fitting actual boundaries.

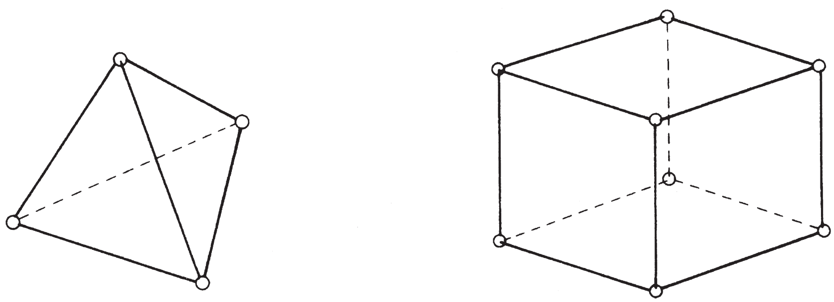

FIGURE 17.18

Tetrahedron and rectangular prism finite elements for three-dimensional problems.

New elements and new applications of the finite element method are still being developed, some of which lie outside the field of structural analysis. These fields include soil mechanics, heat transfer, fluid and seepage flow, magnetism and electricity.

PROBLEMS

P.17.1 Assemble the stiffness matrix for the pin-jointed frame shown in Fig. P.17.1 and hence determine the horizontal and vertical components of the displacement of the joint 2. The product AE is constant for all members.

Ans. Horizontal: $-WL/AE$, Vertical: $-3.83WL/AE$.

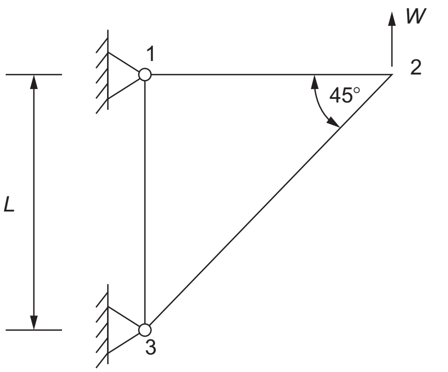

FIG. P.17.1

P.17.2 The truss shown in Fig. P.17.2 has members of cross sectional area 60 mm^2 and Young's modulus 210000 N/mm^2. Obtain the stiffness matrix for the truss and hence calculate the horizontal and vertical displacements at node 2.

Ans. 15.19 mm (to the right), 3.98 mm (downwards).

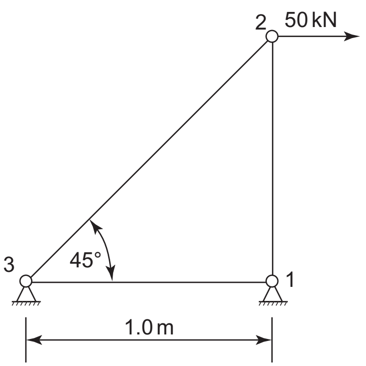

FIGURE P.17.2

P.17.3 Figure P.17.3 shows a square symmetrical pin-jointed truss 1234, pinned to rigid supports at 2 and 4 and loaded with a vertical load at 1. The axial rigidity EA is the same for all members.

Use the stiffness method to find the displacements at nodes 1 and 3 and hence solve for all the internal member forces and support reactions.

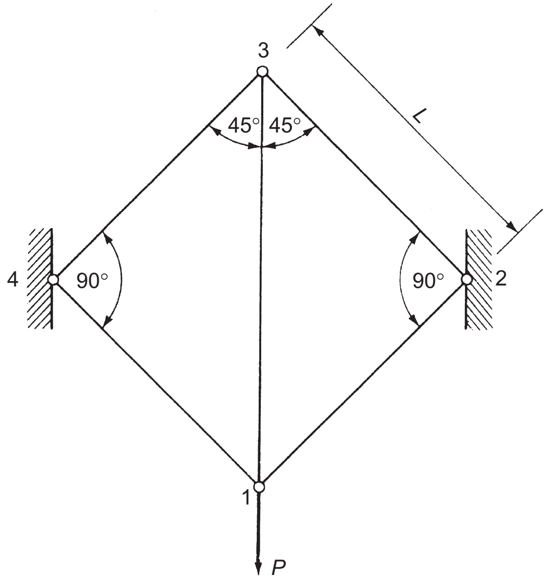

FIGURE P.17.3

Ans. $v_1 = -PL/\sqrt{2}AE$, $v_3 = -0.293PL/AE$, $F_{12} = P/2 = F_{14}$,
$F_{23} = -0.207P = F_{43}$, $F_{13} = 0.293P$, $F_{x,2} = -F_{x,4} = 0.207P$,
$F_{y,2} = F_{y,4} = P/2$.

P.17.4 Use the stiffness method to find the ratio H/P for which the displacement of node 4 of the plane pin-jointed frame shown loaded in Fig. P.17.4 is zero, and for that case give the displacements of nodes 2 and 3.

All members have equal axial rigidity EA.

Ans. $H/P = 0.449$, $v_2 = -4Pl/(9 + 2\sqrt{3})\ AE$, $v_3 = -6Pl/(9 + 2\sqrt{3})AE$.

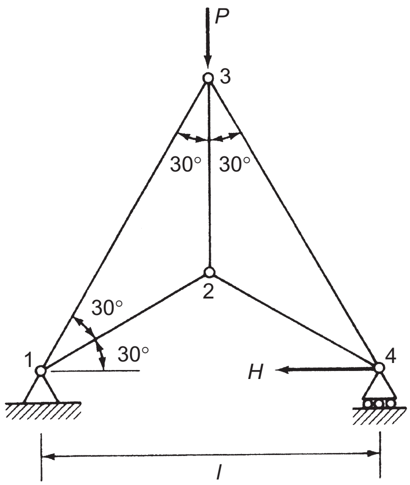

FIGURE P.17.4

P.17.5 Form the matrices required to solve completely the plane truss shown in Fig. P.17.5 and determine the force in member 24. All members have equal axial rigidity.

Ans. $F_{24} = 0$.

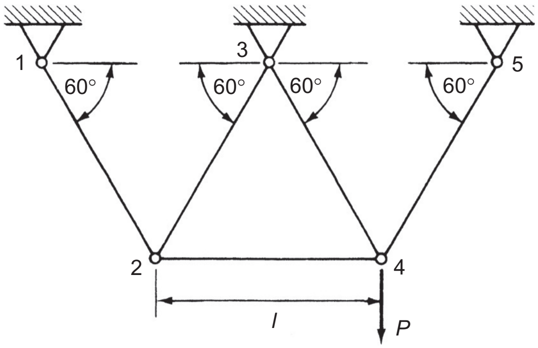

FIGURE P.17.5

P.17.6 The symmetrical plane rigid jointed frame 1234567, shown in Fig. P.17.6, is fixed to rigid supports at 1 and 5 and supported by rollers inclined at 45° to the horizontal at nodes 3 and 7. It carries a vertical point load P at node 4 and a uniformly distributed load w per unit length on the span 26. Assuming the same flexural rigidity EI for all members, set up the stiffness equations which, when solved, give the nodal displacements of the frame.

Explain how the member forces can be obtained.

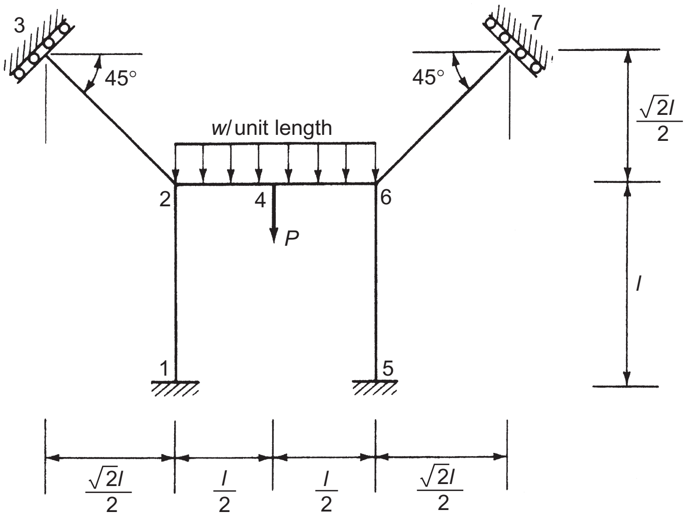

FIGURE P.17.6

P.17.7 The frame shown in Fig. P.17.7 has the planes xz and yz as planes of symmetry. The nodal coordinates of one quarter of the frame are given in Table P.17.7(i).

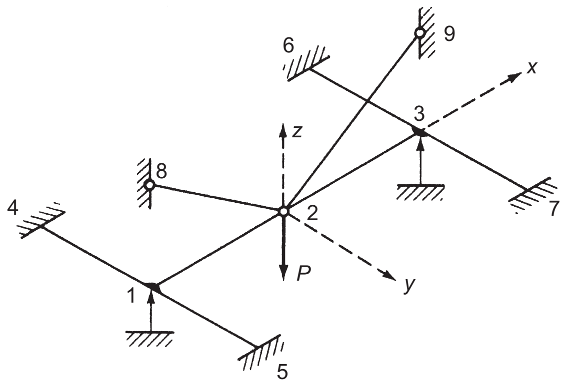

FIGURE P.17.7

In this structure the deformation of each member is due to a single effect, this being axial, bending or torsional. The mode of deformation of each member is given in Table P.17.7(ii), together with the relevant rigidity.

Use the *direct stiffness* method to find all the displacements and hence calculate the forces in all the members. For member 123 plot the shear force and bending moment diagrams.

Briefly outline the sequence of operations in a typical computer program suitable for linear frame analysis.

Ans. $F_{29} = F_{28} = \sqrt{2}P/6$ (tension), $M_3 = -M_1 = PL/9$ (hogging),

Table P.17.7(i)

Node	x	y	z
2	0	0	0
3	L	0	0
7	L	$0.8L$	0
9	L	0	L

Table P.17.7(ii)

	Effect		
Member	Axial	Bending	Torsional
23	–	EI	–
37	–	–	$GJ = 0.8EI$
29	$EA = 6\sqrt{2}\frac{EI}{L^2}$	–	–

$M_2 = 2PL/9$ (sagging), $F_{y,3} = -F_{y,2} = P/3$.

Twisting moment in 37, $PL/18$ (anticlockwise).

P.17.8 Given that the force–displacement (stiffness) relationship for the beam element shown in Fig. P.17.8(a) may be expressed in the following form:

$$\begin{Bmatrix} F_{y,1} \\ M_1/L \\ F_{y,2} \\ M_2/L \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 L \\ v_2 \\ \theta_2 L \end{Bmatrix}$$

obtain the force–displacement (stiffness) relationship for the variable section beam (Fig. P.17.8(b)), composed of elements 12, 23 and 34.

Such a beam is loaded and supported symmetrically as shown in Fig. P.17.8(c). Both ends are rigidly fixed and the ties FB, CH have a cross-sectional area a_1 and the ties EB, CG a cross-sectional

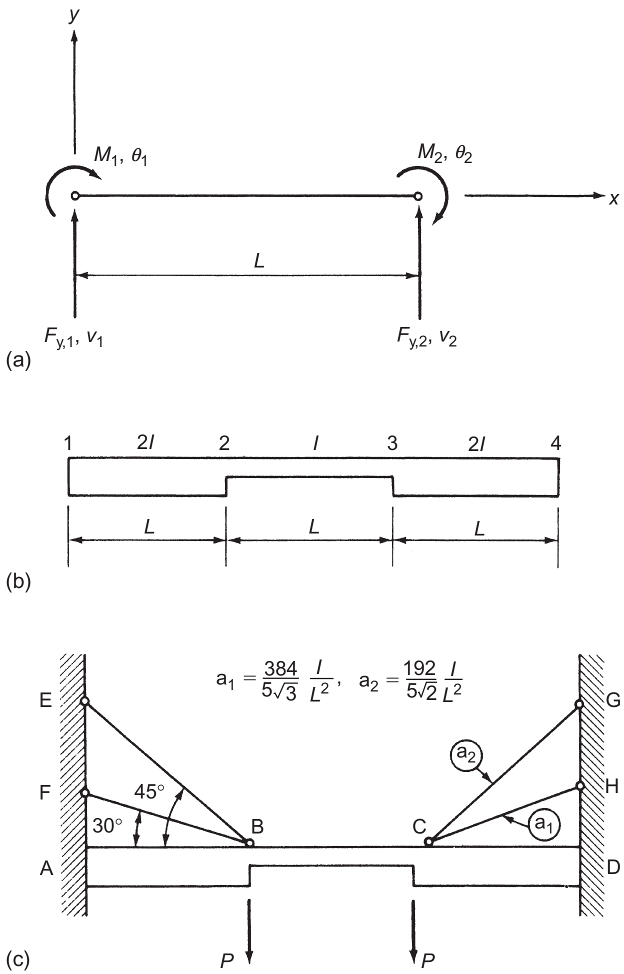

FIGURE P.17.8

area a_2. Calculate the deflections under the loads, the forces in the ties and all other information necessary for sketching the bending moment and shear force diagrams for the beam.

Neglect axial effects in the beam. The ties are made from the same material as the beam.

Ans. $v_B = v_C = -5PL^3/144EI$, $\theta_B = -\theta_C = PL^2/24EI$, $F_{BF} = 2P/3$,
$F_{BE} = \sqrt{2}P/3$, $F_{y,A} = P/3$, $M_A = -PL/4$.

P.17.9 The symmetrical rigid jointed grillage shown in Fig. P.17.9 is encastré at 6, 7, 8 and 9 and rests on simple supports at 1, 2, 4 and 5. It is loaded with a vertical point load P at 3.

Use the stiffness method to find the displacements of the structure and hence calculate the support reactions and the forces in all the members. Plot the bending moment diagram for 123. All members have the same section properties and $GJ = 0.8EI$.

Ans. $F_{y,1} = F_{y,5} = -P/16$
$F_{y,2} = F_{y,4} = 9P/16$
$M_{21} = M_{45} = -Pl/16$ (hogging)
$M_{23} = M_{43} = -Pl/12$ (hogging)

Twisting moment in 62, 82, 74 and 94 is $Pl/96$.

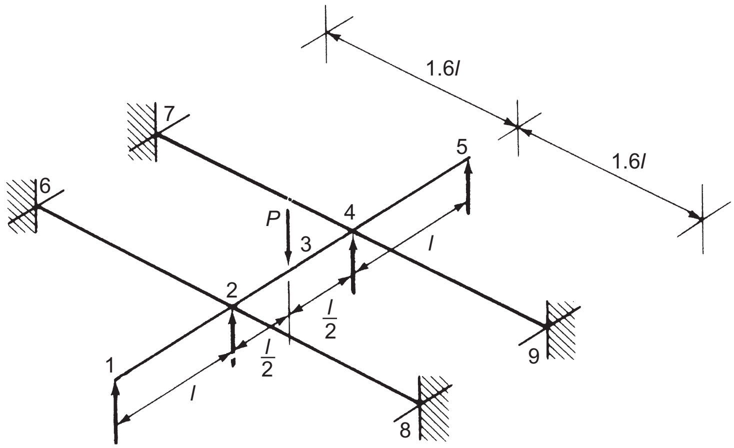

FIGURE P.17.9

P.17.10 Use the matrix displacement method to determine the rotation of the beam shown in Fig. P.17.10 at the points B and C and hence determine the values of bending moment at A, B, and C.

Ans. At B: $-24/EI$. At C: $12/EI$.
$M_A = -106$ kNm, $M_B = -58$ kNm, $M_C = -40$ kNm.

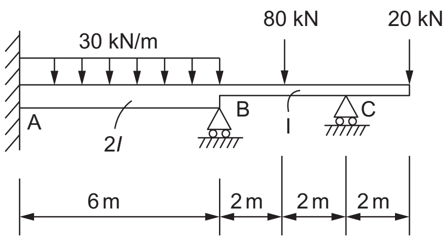

FIG. P.17.10

P.17.11 Find, using the matrix displacement method, the support reactions in the rigidly jointed frame shown in Fig. P.17.11.

Ans. Vertical reaction at A = 32.2 kN, at C = 6.2 kN.
Vertical reaction at D = 73.7 kN. Moment reaction at D = 3.8 kNm.

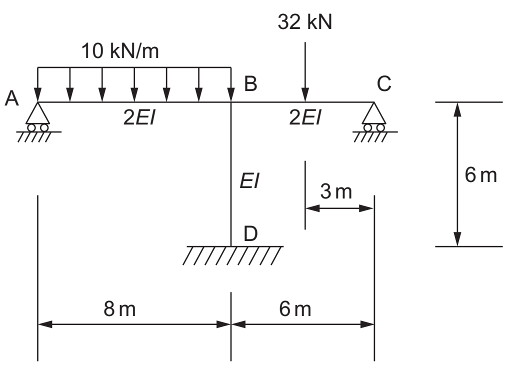

FIG. P.17.11

P.17.12 Find, using the matrix displacement method the joint displacements in the frame shown in Fig. P.17.12. The relative flexural rigidity of each member is shown where $EI = 10^4$ kNm^2.

Ans. Rotation at B $= 2.60 \times 10^{-3}$ rad, at C $= 5.05 \times 10^{-3}$ rad.

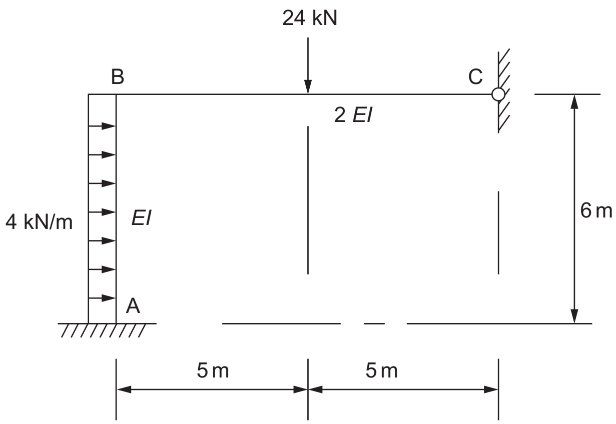

FIG. P.17.12

P.17.13 It is required to formulate the stiffness of a triangular element 123 with coordinates (0, 0) (*a*, 0) and (0, *a)* respectively, to be used for 'plane stress' problems.

a. Form the [*B*] matrix.
b. Obtain the stiffness matrix [K^e].

Why, in general, is a finite element solution not an exact solution?

P.17.14 It is required to form the stiffness matrix of a triangular element 123 for use in stress analysis problems. The coordinates of the element are (1, 1), (2, 1) and (2, 2) respectively.

a. Assume a suitable displacement field explaining the reasons for your choice.
b. Form the [*B*] matrix.
c. Form the matrix which gives, when multiplied by the element nodal displacements, the stresses in the element. Assume a general [*D*] matrix.

P.17.15 It is required to form the stiffness matrix for a rectangular element of side $2a \times 2b$ and thickness *t* for use in 'plane stress' problems.

a. Assume a suitable displacement field.
b. Form the [*C*] matrix.
c. Obtain $\int_{\text{vol}} [C]^{\text{T}}[D]\,[C]\,\text{d}V$.

Note that the stiffness matrix may be expressed as

$$[K^e] = [A^{-1}]^{\text{T}}\left[\int_{\text{vol}} [C]^{\text{T}}[D][C]\,\text{d}V\right][A^{-1}]$$

P.17.16 A square element 1234, whose corners have coordinates *x*, *y* (in m) of $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$, respectively, was used in a plane stress finite element analysis. The following nodal displacements (mm) were obtained:

$w_1 = 0.1$	$w_2 = 0.3$	$w_3 = 0.6$	$w_4 = 0.1$
$v_1 = 0.1$	$v_2 = 0.3$	$v_3 = 0.7$	$v_4 = 0.5$

If Young's modulus $E = 200\ 000$ N/mm^2 and Poisson's ratio $\nu = 0.3$, calculate the stresses at the centre of the element.

Ans. $\sigma_x = 51.65$ N/mm^2, $\sigma_y = 55.49$ N/mm^2, $\tau_{xy} = 13.46$ N/mm^2.

P.17.17 A triangular element with corners 1, 2 and 3, whose x, y coordinates in metres are (2.0, 3.0), (3.0, 3.0) and (2.5, 4.0), respectively, was used in a plane stress finite element analysis. The following nodal displacements (mm) were obtained.

$$w_1 = 0.04, \quad v_1 = 0.08, \quad w_2 = 0.10, \quad v_2 = 0.12, \quad w_3 = 0.20, \quad v_3 = 0.18$$

Calculate the stresses in the element if Young's modulus is 200 000 N/mm^2 and Poisson's ratio is 0.3.

Ans. $\sigma_x = 25.4$ N/mm^2, $\sigma_y = 28.5$ N/mm^2, $\tau_{xy} = 13.1$ N/mm^2.

P.17.18 A rectangular element 1234 has corners whose x, y coordinates in metres are, respectively, (−2, −1), (2, −1), (2, 1) and (−2, 1). The element was used in a plane stress finite element analysis and the following displacements (mm) were obtained.

	1	2	3	4
w	0.001	0.003	−0.003	0.0
v	−0.004	−0.002	0.001	0.001

If the stiffness of the element was derived assuming a linear variation of displacements, Young's modulus is 200 000 N/mm^2 and Poisson's ratio is 0.3, calculate the stresses at the centre of the element.

Ans. $\sigma_x = 104.4$ N/mm^2, $\sigma_y = 431.3$ N/mm^2, $\tau_{xy} = -115.4$ N/mm^2.

P.17.19 Derive the stiffness matrix of a constant strain, triangular finite element 123 of thickness t and coordinates (0, 0), (2, 0) and (0, 3), respectively, to be used for plane stress problems. The elements of the elasticity matrix **[D]** are as follows.

$$D_{11} = D_{22} = a, \quad D_{12} = b, \quad D_{13} = D_{23} = 0, \quad D_{33} = c$$

where a, b and c are material constants.

Ans. See Solutions Manual.

P.17.20 A constant strain triangular element has corners 1(0,0), 2(4,0) and 3(2,2) and is 1 unit thick. If the elasticity matrix $[D]$ has elements $D_{11} = D_{22} = a$, $D_{12} = D_{21} = b$, $D_{13} = D_{23} = D_{31} = D_{32} = 0$ and $D_{33} = c$, derive the stiffness matrix for the element.

Ans.

$$[K^e] = \frac{1}{4}\begin{bmatrix} a+c & & & & & \\ b+c & a+c & & & & \\ -a+c & -b+c & a+c & & & \\ b-c & a-c & -b-c & a+c & & \\ -2c & -2c & -2c & 2c & 4c & \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

P.17.21 The following interpolation formula is suggested as a displacement function for deriving the stiffness matrix of a plane stress rectangular element of uniform thickness t shown in Fig. P.17.21

$$u = \frac{1}{4ab}[(a-x)(b-y)u_1 + (a+x)(b-y)u_2 + (a+x)(b+y)u_3 + (a-x)(b+y)u_4]$$

Form the strain matrix and obtain the stiffness coefficients K_{11} and K_{12} in terms of the material constants c, d and e where, in the elasticity matrix $[D]$, $D_{11} = D_{22} = c$, $D_{12} = d$, $D_{33} = e$ and $D_{13} = D_{23} = 0$.

Ans. $K_{11} = t(4c + e)/6$, $K_{12} = t(d + e)/4$

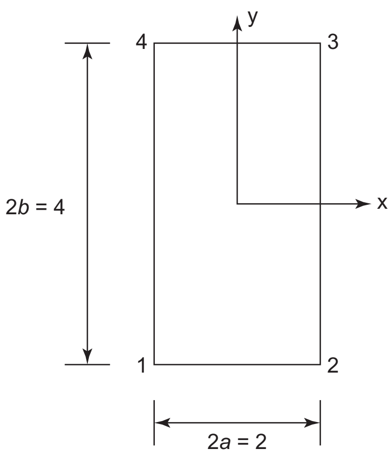

FIGURE P.17.21