Appendix 9: Transformation of Aerodynamic Stability Derivatives from a Body Axes Reference to a Wind Axes Reference

A9.1 Introduction

Aerodynamic stability derivatives are usually quoted with respect to either a system of body axes or a system of wind axes. When the derivatives are quoted with respect to one system, and it is desired to work with the equations of motion referred to a different system, the derivatives must be transformed to the system of interest. Fortunately, the transformation of aerodynamic derivatives from one axis system to another is a relatively straightforward procedure using the transformation relationships discussed in Chapter 2. The procedure for transforming derivatives from a body axes reference to a wind axes reference is illustrated here. However, it can be applied for transforming derivatives between any two systems of reference axes provided their angular relationship is known.

In steady level symmetric flight a system of body axes differs from that of wind axes only by the body incidence α_e, as shown in Fig. 2.2. In the following paragraphs small perturbation force and velocity components, X, Y, Z and u, v, w, respectively, are indicated in the usual way, where the subscript denotes the reference axes. Small perturbation moment and angular velocity components, L, M, N and p, q, r, respectively, are also most conveniently represented by vectors as described in Chapter 2. Again, the subscript denotes the reference axis system.

A9.2 Force and Moment Transformation

The transformation of the aerodynamic force components from a *body axes to a wind axes* reference may be obtained directly by the application of the inverse direction cosine matrix, as given by equation (2.13). Writing $\theta = \alpha_e$ and $\phi = \psi = 0$, since level symmetric flight is assumed, then

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} \cos \alpha_e & 0 & \sin \alpha_e \\ 0 & 1 & 0 \\ -\sin \alpha_e & 0 & \cos \alpha_e \end{bmatrix} \begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} \tag{A9.1}$$

Similarly, the aerodynamic moments transformation may be written as

$$\begin{bmatrix} L_w \\ M_w \\ N_w \end{bmatrix} = \begin{bmatrix} \cos \alpha_e & 0 & \sin \alpha_e \\ 0 & 1 & 0 \\ -\sin \alpha_e & 0 & \cos \alpha_e \end{bmatrix} \begin{bmatrix} L_b \\ M_b \\ N_b \end{bmatrix} \tag{A9.2}$$

A9.3 Aerodynamic Stability Derivative Transformations

A9.3.1 Force-Velocity Derivatives

Consider the situation when the aerodynamic force components comprise only those terms involving the force-velocity derivatives. Then, referred to wind axes,

$$
\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} \mathring{X}_{u_w} & 0 & \mathring{X}_{w_w} \\ 0 & \mathring{Y}_{v_w} & 0 \\ \mathring{Z}_{u_w} & 0 & \mathring{X}_{w_w} \end{bmatrix} \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix}
\tag{A9.3}
$$

and referred to body axes,

$$
\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} \mathring{X}_{u_b} & 0 & \mathring{X}_{w_b} \\ 0 & \mathring{Y}_{v_b} & 0 \\ \mathring{Z}_{u_b} & 0 & \mathring{X}_{w_b} \end{bmatrix} \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix}
\tag{A9.4}
$$

Substitute equations (A9.3) and (A9.4) into equation (A9.1):

$$
\begin{bmatrix} \mathring{X}_{u_w} & 0 & \mathring{X}_{w_w} \\ 0 & \mathring{Y}_{v_w} & 0 \\ \mathring{Z}_{u_w} & 0 & \mathring{X}_{w_w} \end{bmatrix} \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & \sin\alpha_e \\ 0 & 1 & 0 \\ -\sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} \mathring{X}_{u_b} & 0 & \mathring{X}_{w_b} \\ 0 & \mathring{Y}_{v_b} & 0 \\ \mathring{Z}_{u_b} & 0 & \mathring{X}_{w_b} \end{bmatrix} \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix}
\tag{A9.5}
$$

Now the transformation of linear velocity components from a *wind axes to a body axes* reference may be obtained directly from the application of the direction cosine matrix, equation (2.12), with the same constraints as previously:

$$
\begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & -\sin\alpha_e \\ 0 & 1 & 0 \\ \sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix}
\tag{A9.6}
$$

Substitute the velocity vector referred to body axes, given by equation (A9.6), into equation (A9.5), and cancel the velocity vectors referred to wind axes to obtain

$$
\begin{bmatrix} \mathring{X}_{u_w} & 0 & \mathring{X}_{w_w} \\ 0 & \mathring{Y}_{v_w} & 0 \\ \mathring{Z}_{u_w} & 0 & \mathring{X}_{w_w} \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & \sin\alpha_e \\ 0 & 1 & 0 \\ -\sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} \mathring{X}_{u_b} & 0 & \mathring{X}_{w_b} \\ 0 & \mathring{Y}_{v_b} & 0 \\ \mathring{Z}_{u_b} & 0 & \mathring{X}_{w_b} \end{bmatrix} \begin{bmatrix} \cos\alpha_e & 0 & -\sin\alpha_e \\ 0 & 1 & 0 \\ \sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix}
$$

or, after multiplying the matrices on the right hand side, the following transformations are obtained:

$$
\mathring{X}_{u_w} = \mathring{X}_{u_b} \cos^2\alpha_e + \mathring{Z}_{w_b} \sin^2\alpha_e + \left(\mathring{X}_{w_b} + \mathring{Z}_{u_b} \right) \sin\alpha_e \cos\alpha_e
$$

$$
\mathring{X}_{w_w} = \mathring{X}_{w_b} \cos^2\alpha_e - \mathring{Z}_{u_b} \sin^2\alpha_e - \left(\mathring{X}_{u_b} - \mathring{Z}_{w_b} \right) \sin\alpha_e \cos\alpha_e
$$

$$
\mathring{Y}_{v_w} = \mathring{Y}_{v_b}
\tag{A9.7}
$$

$$
\mathring{Z}_{u_w} = \mathring{Z}_{u_b} \cos^2\alpha_e - \mathring{X}_{w_b} \sin^2\alpha_e - \left(\mathring{X}_{u_b} - \mathring{Z}_{w_b} \right) \sin\alpha_e \cos\alpha_e
$$

$$
\mathring{Z}_{w_w} = \mathring{Z}_{w_b} \cos^2\alpha_e + \mathring{X}_{u_b} \sin^2\alpha_e - \left(\mathring{X}_{w_b} + \mathring{Z}_{u_b} \right) \sin\alpha_e \cos\alpha_e
$$

A9.3.2 Moment-Velocity Derivatives

Consider now the situation when the aerodynamic moment components comprise only those terms involving the moment-velocity derivatives. Then, referred to wind axes,

$$
\begin{bmatrix} L_w \\ M_w \\ N_w \end{bmatrix} = \begin{bmatrix} 0 & \overset{\circ}{L}_{v_w} & 0 \\ \overset{\circ}{M}_{u_w} & 0 & \overset{\circ}{M}_{w_w} \\ 0 & \overset{\circ}{N}_{v_w} & 0 \end{bmatrix} \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix} \tag{A9.8}
$$

and referred to body axes,

$$
\begin{bmatrix} L_b \\ M_b \\ N_b \end{bmatrix} = \begin{bmatrix} 0 & \overset{\circ}{L}_{v_b} & 0 \\ \overset{\circ}{M}_{u_b} & 0 & \overset{\circ}{M}_{w_b} \\ 0 & \overset{\circ}{N}_{v_b} & 0 \end{bmatrix} \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix} \tag{A9.9}
$$

Substitute equations (A9.8) and (A9.9) into equation (A9.2):

$$
\begin{bmatrix} 0 & \overset{\circ}{L}_{v_w} & 0 \\ \overset{\circ}{M}_{u_w} & 0 & \overset{\circ}{M}_{w_w} \\ 0 & \overset{\circ}{N}_{v_w} & 0 \end{bmatrix} \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix} = \begin{bmatrix} \cos \alpha_e & 0 & \sin \alpha_e \\ 0 & 1 & 0 \\ -\sin \alpha_e & 0 & \cos \alpha_e \end{bmatrix} \begin{bmatrix} 0 & \overset{\circ}{L}_{v_b} & 0 \\ \overset{\circ}{M}_{u_b} & 0 & \overset{\circ}{M}_{w_b} \\ 0 & \overset{\circ}{N}_{v_b} & 0 \end{bmatrix} \begin{bmatrix} u_b \\ v_b \\ w_b \end{bmatrix} \tag{A9.10}
$$

As before, the transformation of linear velocity components from a *wind axes to a body axes* reference is given by equation (A9.6). Substitute the velocity vector referred to body axes, given by equation (A9.6), into equation (A9.10). Again, the velocity vectors referred to wind axes cancel and, after multiplying the matrices on the right hand side, the following transformations are obtained:

$$
\begin{aligned}
\overset{\circ}{L}_{v_w} &= \overset{\circ}{L}_{v_b} \cos \alpha_e + \overset{\circ}{N}_{v_b} \sin \alpha_e \\
\overset{\circ}{M}_{u_w} &= \overset{\circ}{M}_{u_b} \cos \alpha_e + \overset{\circ}{M}_{w_b} \sin \alpha_e \\
\overset{\circ}{M}_{w_w} &= \overset{\circ}{M}_{w_b} \cos \alpha_e - \overset{\circ}{M}_{u_b} \sin \alpha_e \\
\overset{\circ}{N}_{v_w} &= \overset{\circ}{N}_{v_b} \cos \alpha_e - \overset{\circ}{L}_{v_b} \sin \alpha_e
\end{aligned} \tag{A9.11}
$$

A9.3.3 Force-Rotary Derivatives

Consider now the situation in which the aerodynamic force components comprise only those terms involving the force angular velocity derivatives, more commonly referred to as force-rotary derivatives. Then, referred to wind axes,

$$
\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} 0 & \overset{\circ}{X}_{q_w} & 0 \\ \overset{\circ}{Y}_{p_w} & 0 & \overset{\circ}{Y}_{r_w} \\ 0 & \overset{\circ}{Z}_{q_w} & 0 \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} \tag{A9.12}
$$

and referred to body axes,

$$
\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} 0 & \overset{\circ}{X}_{q_b} & 0 \\ \overset{\circ}{Y}_{p_b} & 0 & \overset{\circ}{Y}_{r_b} \\ 0 & \overset{\circ}{Z}_{q_b} & 0 \end{bmatrix} \begin{bmatrix} p_b \\ q_b \\ r_b \end{bmatrix}
\tag{A9.13}
$$

Substitute equations (A9.12) and (A9.13) into equation (A9.1):

$$
\begin{bmatrix} 0 & \overset{\circ}{X}_{q_w} & 0 \\ \overset{\circ}{Y}_{p_w} & 0 & \overset{\circ}{Y}_{r_w} \\ 0 & \overset{\circ}{Z}_{q_w} & 0 \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & \sin\alpha_e \\ 0 & 1 & 0 \\ -\sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} 0 & \overset{\circ}{X}_{q_b} & 0 \\ \overset{\circ}{Y}_{p_b} & 0 & \overset{\circ}{Y}_{r_b} \\ 0 & \overset{\circ}{Z}_{q_b} & 0 \end{bmatrix} \begin{bmatrix} p_b \\ q_b \\ r_b \end{bmatrix}
\tag{A9.14}
$$

Now, with reference to Chapter 2, the treatment of angular velocity components as vectors enables their transformation from a *wind axes to a body axes* reference to be obtained as before by the direct application of the direction cosine matrix, equation (2.12). Thus, with the same constraints as previously,

$$
\begin{bmatrix} p_b \\ q_b \\ r_b \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & -\sin\alpha_e \\ 0 & 1 & 0 \\ \sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix}
\tag{A9.15}
$$

Substitute the angular velocity vector referred to body axes, given by equation (A9.15), into equation (A9.14), and cancel the velocity vectors referred to wind axes to obtain

$$
\begin{bmatrix} 0 & \overset{\circ}{X}_{q_w} & 0 \\ \overset{\circ}{Y}_{p_w} & 0 & \overset{\circ}{Y}_{r_w} \\ 0 & \overset{\circ}{Z}_{q_w} & 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha_e & 0 & \sin\alpha_e \\ 0 & 1 & 0 \\ -\sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix} \begin{bmatrix} 0 & \overset{\circ}{X}_{q_b} & 0 \\ \overset{\circ}{Y}_{p_b} & 0 & \overset{\circ}{Y}_{r_b} \\ 0 & \overset{\circ}{Z}_{q_b} & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha_e & 0 & -\sin\alpha_e \\ 0 & 1 & 0 \\ \sin\alpha_e & 0 & \cos\alpha_e \end{bmatrix}
$$

or, after multiplying the matrices on the right hand side, the following transformations are obtained:

$$
\begin{aligned}
\overset{\circ}{X}_{q_w} &= \overset{\circ}{X}_{q_b} \cos\alpha_e + \overset{\circ}{Z}_{q_b} \sin\alpha_e \\
\overset{\circ}{Y}_{p_w} &= \overset{\circ}{Y}_{p_b} \cos\alpha_e + \overset{\circ}{Y}_{r_b} \sin\alpha_e \\
\overset{\circ}{Y}_{r_w} &= \overset{\circ}{Y}_{r_b} \cos\alpha_e - \overset{\circ}{Y}_{p_b} \sin\alpha_e \\
\overset{\circ}{Z}_{q_w} &= \overset{\circ}{Z}_{q_b} \cos\alpha_e - \overset{\circ}{X}_{q_b} \sin\alpha_e
\end{aligned}
\tag{A9.16}
$$

A9.3.4 Moment-Rotary Derivatives

Consider now the situation in which the aerodynamic moment components comprise only those terms involving the moment-angular velocity, or moment-rotary, derivatives. Then, referred to wind axes,

$$
\begin{bmatrix} L_w \\ M_w \\ N_w \end{bmatrix} = \begin{bmatrix} \mathring{L}_{p_w} & 0 & \mathring{L}_{r_w} \\ 0 & \mathring{M}_{q_w} & 0 \\ \mathring{N}_{p_w} & 0 & \mathring{N}_{r_w} \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix}
\tag{A9.17}
$$

and referred to body axes,

$$
\begin{bmatrix} L_b \\ M_b \\ N_b \end{bmatrix} = \begin{bmatrix} \mathring{L}_{p_b} & 0 & \mathring{L}_{r_b} \\ 0 & \mathring{M}_{q_b} & 0 \\ \mathring{N}_{p_b} & 0 & \mathring{N}_{r_b} \end{bmatrix} \begin{bmatrix} p_b \\ q_b \\ r_b \end{bmatrix}
\tag{A9.18}
$$

Substitute equations (A9.17) and (A9.18) into equation (A9.2):

$$
\begin{bmatrix} \mathring{L}_{p_w} & 0 & \mathring{L}_{r_w} \\ 0 & \mathring{M}_{q_w} & 0 \\ \mathring{N}_{p_w} & 0 & \mathring{N}_{r_w} \end{bmatrix} \begin{bmatrix} p_w \\ q_w \\ r_w \end{bmatrix} = \begin{bmatrix} \cos \alpha_e & 0 & \sin \alpha_e \\ 0 & 1 & 0 \\ -\sin \alpha_e & 0 & \cos \alpha_e \end{bmatrix} \begin{bmatrix} \mathring{L}_{p_b} & 0 & \mathring{L}_{r_b} \\ 0 & \mathring{M}_{q_b} & 0 \\ \mathring{N}_{p_b} & 0 & \mathring{N}_{r_b} \end{bmatrix} \begin{bmatrix} p_b \\ q_b \\ r_b \end{bmatrix}
\tag{A9.19}
$$

As before, the transformation of angular velocity components from a *wind axes to a body axes* reference is given by equation (A9.15). Substitute the angular velocity vector referred to body axes, given by equation (A9.15), into equation (A9.19). Again, the angular velocity vectors referred to wind axes cancel and, after multiplying the matrices on the right hand side, the following transformations are obtained:

$$
\begin{aligned}
\mathring{L}_{p_w} &= \mathring{L}_{p_b} \cos^2 \alpha_e + \mathring{N}_{r_b} \sin^2 \alpha_e + \left(\mathring{L}_{r_b} + \mathring{N}_{p_b} \right) \sin \alpha_e \cos \alpha_e \\
\mathring{L}_{r_w} &= \mathring{L}_{r_b} \cos^2 \alpha_e - \mathring{N}_{p_b} \sin^2 \alpha_e - \left(\mathring{L}_{p_b} - \mathring{N}_{r_b} \right) \sin \alpha_e \cos \alpha_e \\
\mathring{M}_{q_w} &= \mathring{M}_{q_b} \\
\mathring{N}_{p_w} &= \mathring{N}_{p_b} \cos^2 \alpha_e - \mathring{L}_{r_b} \sin^2 \alpha_e - \left(\mathring{L}_{p_b} - \mathring{N}_{r_b} \right) \sin \alpha_e \cos \alpha_e \\
\mathring{N}_{r_w} &= \mathring{N}_{r_b} \cos^2 \alpha_e + \mathring{L}_{p_b} \sin^2 \alpha_e - \left(\mathring{L}_{r_b} + \mathring{N}_{p_b} \right) \sin \alpha_e \cos \alpha_e
\end{aligned}
\tag{A9.20}
$$

A9.3.5 Force-Acceleration Derivatives

The force-acceleration derivatives are calculated in exactly the same way as the force-velocity derivatives. However, in this case the aerodynamic force components referred to wind axes are given by

$$
\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathring{X}_{\dot{w}_w} \\ 0 & 0 & 0 \\ 0 & 0 & \mathring{Z}_{\dot{w}_w} \end{bmatrix} \begin{bmatrix} \dot{u}_w \\ \dot{v}_w \\ \dot{w}_w \end{bmatrix}
\tag{A9.21}
$$

and referred to body axes are given by

$$\begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathring{X}_{\dot{w}_b} \\ 0 & 0 & 0 \\ 0 & 0 & \mathring{Z}_{\dot{w}_b} \end{bmatrix} \begin{bmatrix} \dot{u}_b \\ \dot{v}_b \\ \dot{w}_b \end{bmatrix} \tag{A9.22}$$

Equations (A9.21) and (A9.22) are substituted into equation (A9.1), velocity vectors become acceleration vectors, and, after some algebraic manipulation, the following transformations are obtained:

$$\begin{aligned} \mathring{X}_{\dot{w}_w} &= \mathring{X}_{\dot{w}_b} \cos^2 \alpha_e + \mathring{Z}_{\dot{w}_b} \sin\alpha_e \cos \alpha_e \\ \mathring{Z}_{\dot{w}_w} &= \mathring{Z}_{\dot{w}_b} \cos^2 \alpha_e - \mathring{X}_{\dot{w}_b} \sin\alpha_e \cos \alpha_e \end{aligned} \tag{A9.23}$$

A9.3.6 Moment-Acceleration Derivatives

The moment-acceleration derivatives are calculated in exactly the same way as the moment-velocity derivatives. However, in this case the aerodynamic moment components referred to wind axes are given by

$$\begin{bmatrix} L_w \\ M_w \\ N_w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathring{M}_{\dot{w}_w} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_w \\ \dot{v}_w \\ \dot{w}_w \end{bmatrix} \tag{A9.24}$$

and referred to body axes are given by

$$\begin{bmatrix} L_b \\ M_b \\ N_b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathring{M}_{\dot{w}_b} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_b \\ \dot{v}_b \\ \dot{w}_b \end{bmatrix} \tag{A9.25}$$

Equations (A9.24) and (A9.25) are substituted into equation (A9.2), the velocity vectors become acceleration vectors, and, after some algebraic manipulation, the following transformation is obtained:

$$\mathring{M}_{\dot{w}_w} = \mathring{M}_{\dot{w}_b} \cos \alpha_e \tag{A9.26}$$

A9.3.7 Aerodynamic Control Derivatives

The aerodynamic control derivatives are most easily dealt with by denoting a general control input δ. The transformation of the control force derivatives from a *body axis* reference *to a wind axes* reference then follows directly from equation (A9.1):

$$\begin{bmatrix} \mathring{X}_{\delta_w} \\ \mathring{Y}_{\delta_w} \\ \mathring{Z}_{\delta_w} \end{bmatrix} \delta = \begin{bmatrix} \cos \alpha_e & 0 & \sin \alpha_e \\ 0 & 1 & 0 \\ -\sin \alpha_e & 0 & \cos \alpha_e \end{bmatrix} \begin{bmatrix} \mathring{X}_{\delta_b} \\ \mathring{Y}_{\delta_b} \\ \mathring{Z}_{\delta_b} \end{bmatrix} \delta \tag{A9.27}$$

The corresponding transformation of the control moment derivatives follows directly from equation (A9.2):

$$\begin{bmatrix} \mathring{L}_{\delta_w} \\ \mathring{M}_{\delta_w} \\ \mathring{N}_{\delta_w} \end{bmatrix} \delta = \begin{bmatrix} \cos \alpha_e & 0 & \sin \alpha_e \\ 0 & 1 & 0 \\ -\sin \alpha_e & 0 & \cos \alpha_e \end{bmatrix} \begin{bmatrix} \mathring{L}_{\delta_b} \\ \mathring{M}_{\delta_b} \\ \mathring{N}_{\delta_b} \end{bmatrix} \delta \tag{A9.28}$$

The specific control derivative transformations are then obtained by substituting elevator angle η, aileron angle ξ, rudder angle ζ, thrust τ, and so on, in place of δ in equations (A9.27) and (A9.28). Bearing in mind that the longitudinal and lateral-directional equations of motion are decoupled, it follows that

$$\begin{aligned} \mathring{X}_{\eta_w} &= \mathring{X}_{\eta_b} \cos \alpha_e + \mathring{Z}_{\eta_b} \sin \alpha_e \\ \mathring{Z}_{\eta_w} &= \mathring{Z}_{\eta_b} \cos \alpha_e - \mathring{X}_{\eta_b} \sin \alpha_e \\ \mathring{M}_{\eta_w} &= \mathring{M}_{\eta_b} \end{aligned} \tag{A9.29}$$

and

$$\begin{aligned} \mathring{Y}_{\xi_w} &= \mathring{Y}_{\xi_b} \\ \mathring{L}_{\xi_w} &= \mathring{L}_{\xi_b} \cos \alpha_e + \mathring{N}_{\xi_b} \sin \alpha_e \\ \mathring{N}_{\xi_w} &= \mathring{N}_{\xi_b} \cos \alpha_e - \mathring{L}_{\xi_b} \sin \alpha_e \end{aligned} \tag{A9.30}$$

By substituting τ for η in equation (A9.29), the thrust control derivative transformations are obtained. Similarly, by substituting ζ for ξ in equation (A9.30), the rudder control derivative transformations are obtained.

A9.4 Summary

The derivative transformations of *body axes to wind axes* described here are summarised in Table A9.1. The transformations from *wind axes to body axes* are easily obtained by the inverse procedure, and these are summarised for convenience in Table A9.2. The corresponding control derivative transformations are summarised in Tables A9.3 and A9.4.

Table A9.1 Body to wind axes derivative transformations

Wind Axes	Body Axes
$\mathring{X}_{u_w}$	$\mathring{X}_{u_b} \cos^2 \alpha_e + \mathring{Z}_{w_b} \sin^2 \alpha_e + \left(\mathring{X}_{w_b} + \mathring{Z}_{u_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{X}_{w_w}$	$\mathring{X}_{w_b} \cos^2 \alpha_e - \mathring{Z}_{u_b} \sin^2 \alpha_e - \left(\mathring{X}_{u_b} - \mathring{Z}_{w_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{Y}_{v_w}$	$\mathring{Y}_{v_b}$
$\mathring{Z}_{u_w}$	$\mathring{Z}_{u_b} \cos^2 \alpha_e - \mathring{X}_{w_b} \sin^2 \alpha_e - \left(\mathring{X}_{u_b} - \mathring{Z}_{w_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{Z}_{w_w}$	$\mathring{Z}_{w_b} \cos^2 \alpha_e + \mathring{X}_{u_b} \sin^2 \alpha_e - \left(\mathring{X}_{w_b} + \mathring{Z}_{u_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{L}_{v_w}$	$\mathring{L}_{v_b} \cos \alpha_e + \mathring{N}_{v_b} \sin \alpha_e$
$\mathring{M}_{u_w}$	$\mathring{M}_{u_b} \cos \alpha_e + \mathring{M}_{w_b} \sin \alpha_e$
$\mathring{M}_{w_w}$	$\mathring{M}_{w_b} \cos \alpha_e - \mathring{M}_{u_b} \sin \alpha_e$
$\mathring{N}_{v_w}$	$\mathring{N}_{v_b} \cos \alpha_e - \mathring{L}_{v_b} \sin \alpha_e$
$\mathring{X}_{q_w}$	$\mathring{X}_{q_b} \cos \alpha_e + \mathring{Z}_{q_b} \sin \alpha_e$
$\mathring{Y}_{p_w}$	$\mathring{Y}_{p_b} \cos \alpha_e + \mathring{Y}_{r_b} \sin \alpha_e$
$\mathring{Y}_{r_w}$	$\mathring{Y}_{r_b} \cos \alpha_e - \mathring{Y}_{p_b} \sin \alpha_e$
$\mathring{Z}_{q_w}$	$\mathring{Z}_{q_b} \cos \alpha_e - \mathring{X}_{q_b} \sin \alpha_e$
$\mathring{L}_{p_w}$	$\mathring{L}_{p_b} \cos^2 \alpha_e + \mathring{N}_{r_b} \sin^2 \alpha_e + \left(\mathring{L}_{r_b} + \mathring{N}_{p_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{L}_{r_w}$	$\mathring{L}_{r_b} \cos^2 \alpha_e - \mathring{N}_{p_b} \sin^2 \alpha_e - \left(\mathring{L}_{p_b} - \mathring{N}_{r_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{M}_{q_w}$	$\mathring{M}_{q_b}$
$\mathring{N}_{p_w}$	$\mathring{N}_{p_b} \cos^2 \alpha_e - \mathring{L}_{r_b} \sin^2 \alpha_e - \left(\mathring{L}_{p_b} - \mathring{N}_{r_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{N}_{r_w}$	$\mathring{N}_{r_b} \cos^2 \alpha_e + \mathring{L}_{p_b} \sin^2 \alpha_e - \left(\mathring{L}_{r_b} + \mathring{N}_{p_b} \right) \sin \alpha_e \cos \alpha_e$
$\mathring{X}_{\dot{w}_w}$	$\mathring{X}_{\dot{w}_b} \cos^2 \alpha_e + \mathring{Z}_{\dot{w}_b} \sin \alpha_e \cos \alpha_e$
$\mathring{Z}_{\dot{w}_w}$	$\mathring{Z}_{\dot{w}_b} \cos^2 \alpha_e - \mathring{X}_{\dot{w}_b} \sin \alpha_e \cos \alpha_e$
$\mathring{M}_{\dot{w}_w}$	$\mathring{M}_{\dot{w}_b} \cos \alpha_e$

Table A9.2 Wind to body axes derivative transformations

Body Axes	Wind Axes
$\overset{\circ}{X}_{u_b}$	$\overset{\circ}{X}_{u_w}\,\cos^2\alpha_e + \overset{\circ}{Z}_{w_w}\,\sin^2\alpha_e - \left(\overset{\circ}{X}_{w_w} + \overset{\circ}{Z}_{u_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{X}_{w_b}$	$\overset{\circ}{X}_{w_w}\,\cos^2\alpha_e - \overset{\circ}{Z}_{u_w}\,\sin^2\alpha_e + \left(\overset{\circ}{X}_{u_w} - \overset{\circ}{Z}_{w_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{Y}_{v_b}$	$\overset{\circ}{Y}_{v_w}$
$\overset{\circ}{Z}_{u_b}$	$\overset{\circ}{Z}_{u_w}\,\cos^2\alpha_e - \overset{\circ}{X}_{w_w}\,\sin^2\alpha_e + \left(\overset{\circ}{X}_{u_w} - \overset{\circ}{Z}_{w_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{Z}_{w_b}$	$\overset{\circ}{Z}_{w_w}\,\cos^2\alpha_e + \overset{\circ}{X}_{u_w}\,\sin^2\alpha_e + \left(\overset{\circ}{X}_{w_w} + \overset{\circ}{Z}_{u_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{L}_{v_b}$	$\overset{\circ}{L}_{v_w}\,\cos\alpha_e - \overset{\circ}{N}_{v_w}\,\sin\alpha_e$
$\overset{\circ}{M}_{u_b}$	$\overset{\circ}{M}_{u_w}\,\cos\alpha_e - \overset{\circ}{M}_{w_w}\,\sin\alpha_e$
$\overset{\circ}{M}_{w_b}$	$\overset{\circ}{M}_{w_w}\,\cos\alpha_e + \overset{\circ}{M}_{u_w}\,\sin\alpha_e$
$\overset{\circ}{N}_{v_b}$	$\overset{\circ}{N}_{v_w}\,\cos\alpha_e + \overset{\circ}{L}_{v_w}\,\sin\alpha_e$
$\overset{\circ}{X}_{q_b}$	$\overset{\circ}{X}_{q_w}\,\cos\alpha_e - \overset{\circ}{Z}_{q_w}\,\sin\alpha_e$
$\overset{\circ}{Y}_{p_b}$	$\overset{\circ}{Y}_{p_w}\,\cos\alpha_e - \overset{\circ}{Y}_{r_w}\,\sin\alpha_e$
$\overset{\circ}{Y}_{r_b}$	$\overset{\circ}{Y}_{r_w}\,\cos\alpha_e + \overset{\circ}{Y}_{p_w}\,\sin\alpha_e$
$\overset{\circ}{Z}_{q_b}$	$\overset{\circ}{Z}_{q_w}\,\cos\alpha_e + \overset{\circ}{X}_{q_w}\,\sin\alpha_e$
$\overset{\circ}{L}_{p_b}$	$\overset{\circ}{L}_{p_w}\,\cos^2\alpha_e + \overset{\circ}{N}_{r_w}\,\sin^2\alpha_e - \left(\overset{\circ}{L}_{r_w} + \overset{\circ}{N}_{p_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{L}_{r_b}$	$\overset{\circ}{L}_{r_w}\,\cos^2\alpha_e - \overset{\circ}{N}_{p_w}\,\sin^2\alpha_e + \left(\overset{\circ}{L}_{p_w} - \overset{\circ}{N}_{r_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{M}_{q_b}$	$\overset{\circ}{M}_{q_w}$
$\overset{\circ}{N}_{p_b}$	$\overset{\circ}{N}_{p_w}\,\cos^2\alpha_e - \overset{\circ}{L}_{r_w}\,\sin^2\alpha_e + \left(\overset{\circ}{L}_{p_w} - \overset{\circ}{N}_{r_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{N}_{r_b}$	$\overset{\circ}{N}_{r_w}\,\cos^2\alpha_e + \overset{\circ}{L}_{p_w}\,\sin^2\alpha_e + \left(\overset{\circ}{L}_{r_w} + \overset{\circ}{N}_{p_w}\right)\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{X}_{\dot{w}_b}$	$\overset{\circ}{X}_{\dot{w}_w}\,\cos^2\alpha_e - \overset{\circ}{Z}_{\dot{w}_w}\,\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{Z}_{\dot{w}_b}$	$\overset{\circ}{Z}_{\dot{w}_w}\,\cos^2\alpha_e + \overset{\circ}{X}_{\dot{w}_w}\,\sin\alpha_e\cos\alpha_e$
$\overset{\circ}{M}_{\dot{w}_b}$	$\overset{\circ}{M}_{\dot{w}_w}\,\cos\alpha_e$

In Tables A9.3 and A9.4, it is simply necessary to write η, τ, ξ, or ζ in place of δ as appropriate.

Table A9.3 Body axes to wind axes control derivative transformations

Wind Axes	Body Axes
$\overset{\circ}{X}_{\delta_w}$	$\overset{\circ}{X}_{\delta_b} \cos \alpha_e + \overset{\circ}{Z}_{\delta_b} \sin \alpha_e$
$\overset{\circ}{Y}_{\delta_w}$	$\overset{\circ}{Y}_{\delta_b}$
$\overset{\circ}{Z}_{\delta_w}$	$\overset{\circ}{Z}_{\delta_b} \cos \alpha_e - \overset{\circ}{X}_{\delta_b} \sin \alpha_e$
$\overset{\circ}{L}_{\delta_w}$	$\overset{\circ}{L}_{\delta_b} \cos \alpha_e + \overset{\circ}{N}_{\delta_b} \sin \alpha_e$
$\overset{\circ}{M}_{\delta_w}$	$\overset{\circ}{M}_{\delta_b}$
$\overset{\circ}{N}_{\delta_w}$	$\overset{\circ}{N}_{\delta_b} \cos \alpha_e - \overset{\circ}{L}_{\delta_b} \sin \alpha_e$

Table A9.4 Wind axes to body axes control derivative transformations

Body Axes	Wind Axes
$\overset{\circ}{X}_{\delta_b}$	$\overset{\circ}{X}_{\delta_w} \cos \alpha_e - \overset{\circ}{Z}_{\delta_w} \sin \alpha_e$
$\overset{\circ}{Y}_{\delta_b}$	$\overset{\circ}{Y}_{\delta_w}$
$\overset{\circ}{Z}_{\delta_b}$	$\overset{\circ}{Z}_{\delta_w} \cos \alpha_e + \overset{\circ}{X}_{\delta_w} \sin \alpha_e$
$\overset{\circ}{L}_{\delta_b}$	$\overset{\circ}{L}_{\delta_w} \cos \alpha_e - \overset{\circ}{N}_{\delta_w} \sin \alpha_e$
$\overset{\circ}{M}_{\delta_b}$	$\overset{\circ}{M}_{\delta_w}$
$\overset{\circ}{N}_{\delta_b}$	$\overset{\circ}{N}_{\delta_w} \cos \alpha_e + \overset{\circ}{L}_{\delta_w} \sin \alpha_e$