Aerodynamics and Numerical Simulation Methods

Karman Momentum Integral Equation

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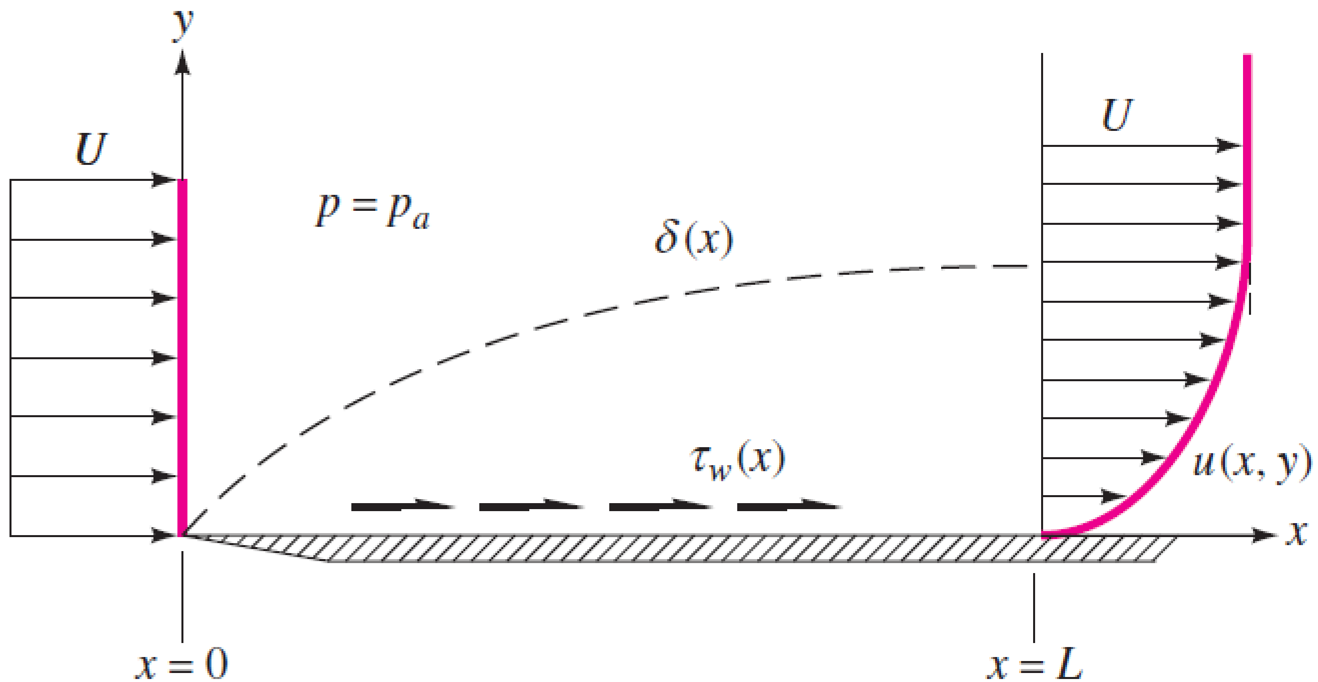
Topics for today

- Wall and edge boundary conditions
- Conservation of mass and momentum
- Momentum Integral Equation (MIE)
- Use of MIE

Flat-plate boundary layer

- Consider a steady 2D flow over a flat plate in the *x* direction



Source: Frank White pg 461

Wall Boundary Conditions:

- At the wall, *y=u=v=0*, and hence:

$$\cancel{u\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{dp}{dx}=\mu\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall}$$

Kinematic viscosity

Edge Boundary Conditions:

- At the boundary layer edge:
  - $u = u_e$, derivatives w.r.t $y = 0$ (by definition of the boundary layer edge). Hence:

$$u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{dp}{dx} + \cancel{\nu\frac{\partial^2 u}{\partial y^2}}$$

Kinematic viscosity

$$\frac{dp}{dx} = -\rho_e u_e \frac{du_e}{dx}$$

$$\boxed{\frac{dp}{dx} = -\rho_e u_e \frac{du_e}{dx} = \mu\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall}}$$

Flat Plate with no Pressure Gradient

- In uniform flow over a flat plate there is no pressure gradient, i.e.

$$\frac{dp}{dx} = 0$$

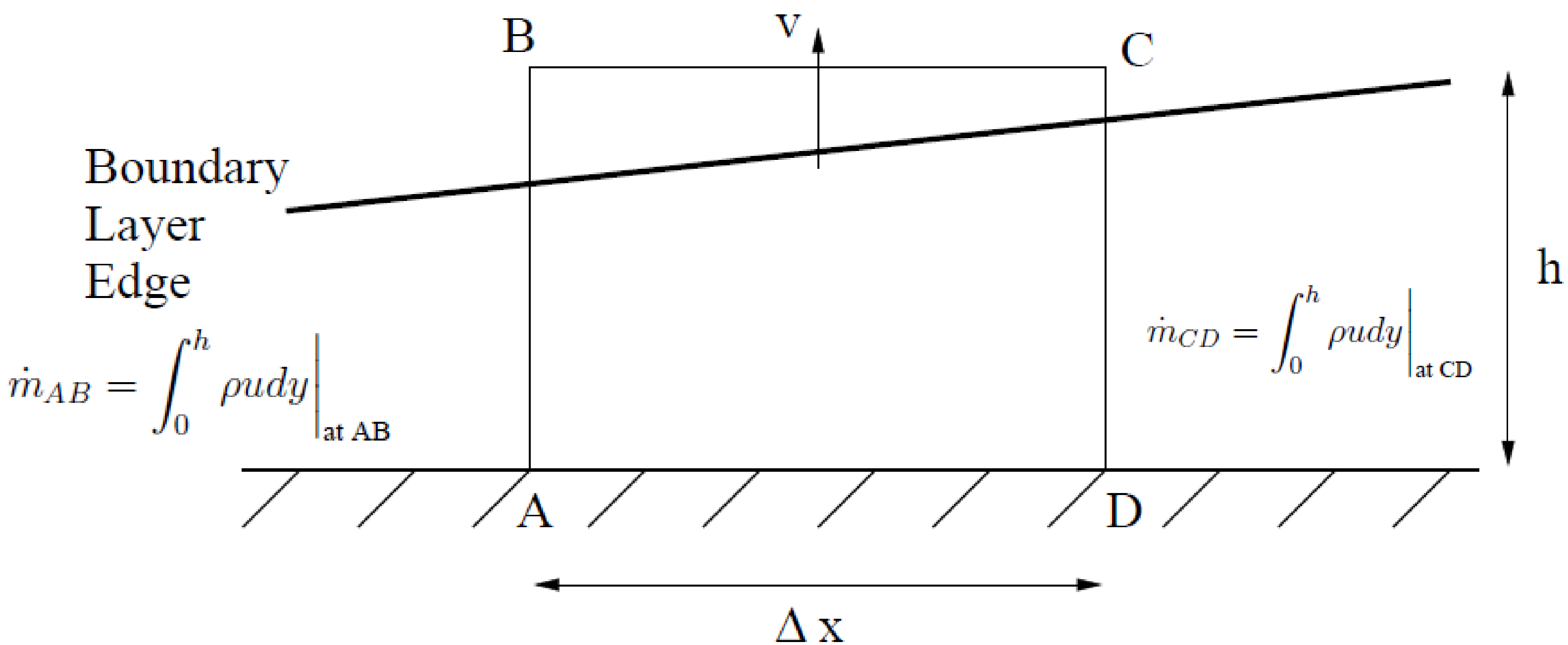
- and hence

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} = 0$$

- This makes it a special case, reducing the number of boundary conditions to be solved – allows an accurate solution to be derived

Conservation of mass

0=(mass in)-(mass out)



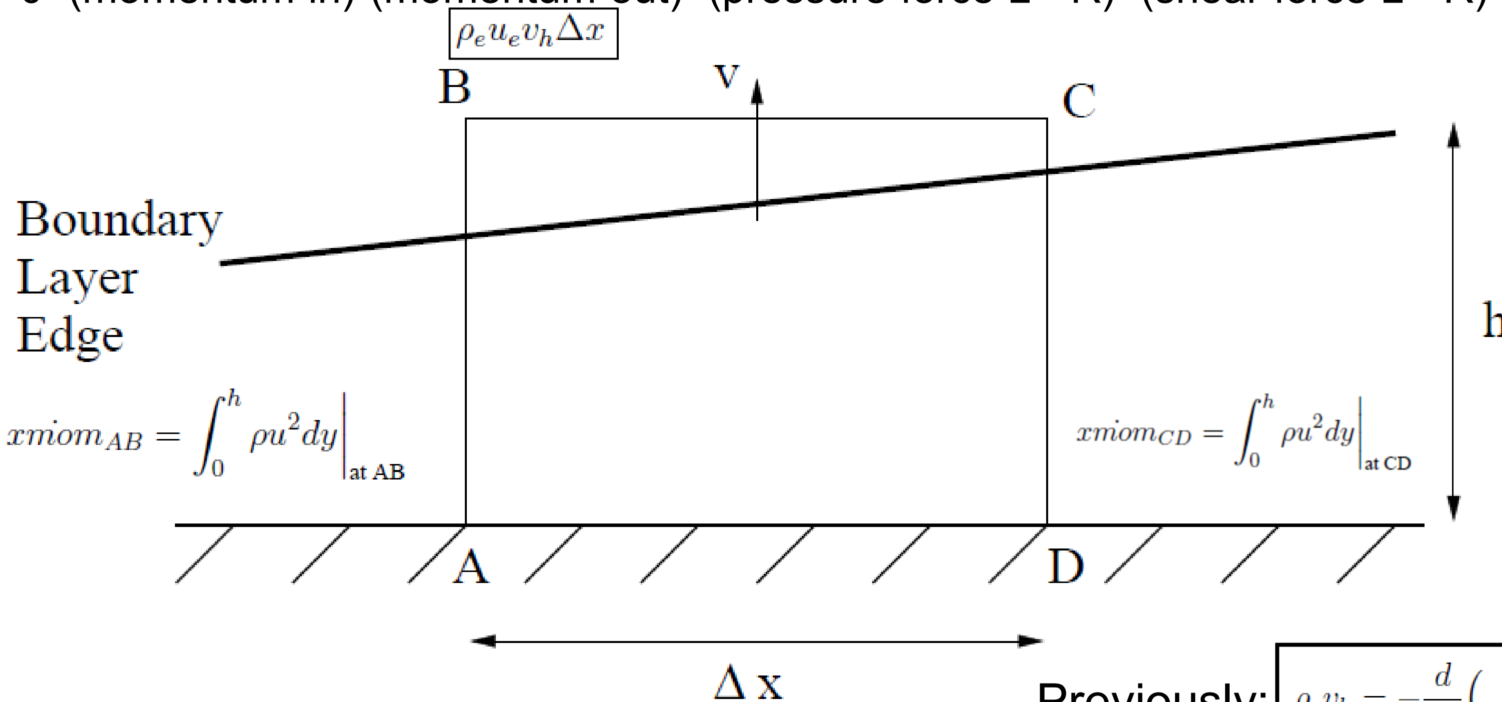
$$\int_0^h \rho u dy \Big|_{\text{at AB}} - \int_0^h \rho u dy \Big|_{\text{at CD}} = \rho_e v_h \Delta x$$

$$\boxed{\rho_e v_h = -\frac{d}{dx}\left(\int_0^h \rho u dy\right)}$$

Conservation of momentum

$$\int_0^h \rho u^2 dy\Big|_{\text{at AB}} \qquad \int_0^h \rho u^2 dy\Big|_{\text{at CD}} \qquad -(p_{CD}-p_{AB})h \qquad -\tau_w \Delta x$$

0=(momentum in)-(momentum out)+(pressure force L->R)+(shear force L->R)



Previously: $\boxed{\rho_e v_h = -\frac{d}{dx}\Big(\int_0^h \rho u dy\Big)}$

$$\underbrace{-\int_0^h \rho u^2 dy\Big|_{\text{at AB}} + \int_0^h \rho u^2 dy\Big|_{\text{at CD}}} + \underbrace{\rho_e u_e v_h \Delta x} = -(p_{CD}-p_{AB})h - \tau_w \Delta x$$

$$\boxed{\frac{d}{dx}\Big(\int_0^h \rho u^2 dy\Big) - u_e \frac{d}{dx}\Big(\int_0^h \rho u dy\Big) = -h\frac{dp}{dx} - \tau_w}$$

$$\boxed{\frac{d}{dx}\left(\int_0^h \rho u^2 dy\right) - u_e \frac{d}{dx}\left(\int_0^h \rho u dy\right) = -h\frac{dp}{dx} - \tau_w}$$

$$\frac{d}{dx}\left(u_e \int_0^h \rho u dy\right) = u_e \frac{d}{dx}\left(\int_0^h \rho u dy\right) + \frac{du_e}{dx}\int_0^h \rho u dy$$

$$u_e \frac{d}{dx}\left(\int_0^h \rho u dy\right) = \boxed{\frac{d}{dx}\left(u_e \int_0^h \rho u dy\right) - \frac{du_e}{dx}\int_0^h \rho u dy}$$

Replace using product rule

$$-\frac{dp}{dx} = \rho_e u_e \frac{du_e}{d_x}$$

(from edge BC)

$$\frac{d}{dx}\left[\int_0^h \rho u(u-u_e)dy\right] + \frac{du_e}{dx}\left[\int_0^h \rho u dy\right] = h\rho_e u_e \frac{du_e}{dx} - \tau_w$$

$$\delta^* = \int_0^h \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$

$$\theta = \int_0^h \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$$

Introduce δ^* and θ

$$h\rho_e u_e \frac{du_e}{dx} = \frac{du_e}{dx}\int_0^h \rho_e u_e dy$$

$$\boxed{\frac{d}{dx}(\rho_e u_e^2 \theta) + \frac{du_e}{dx}\rho_e u_e \delta^* = \tau_w}$$

$$\boxed{\frac{d}{dx}(\rho_e u_e^2\theta) + \frac{du_e}{dx}\rho_e u_e \delta^* = \tau_w}$$

$$\rho_e u_e^2 \frac{d\theta}{dx} + \theta\left(u_e^2 \frac{d\rho_e}{dx} + \rho_e 2u_e \frac{du_e}{dx}\right) + \frac{du_e}{dx}\rho_e u_e \delta^* = \tau_w$$

Expanding derivative

$$\frac{d\theta}{dx} + \theta\left(\frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{2}{u_e}\frac{du_e}{dx}\right) + \frac{1}{u_e}\frac{du_e}{dx}\delta^* = \frac{\tau_w}{\rho_e u_e^2}$$

Divide by $\rho_e u_e^2$

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) + \frac{\theta}{\rho_e}\frac{d\rho_e}{dx} = \frac{\tau_w}{\rho_e u_e^2}$$

Define shape factor: $H = \dfrac{\delta^*}{\theta}$

$$\boxed{\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{\tau_w}{\rho_e u_e^2} = \frac{\frac{1}{2}\tau_w}{\frac{1}{2}\rho_e u_e^2} = \frac{c_f}{2}}$$

Incompressible (no density derivatives)

Momentum integral equation

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$$

MIE

- The Momentum Integral Equation is a key part of integral methods
- Read the derivation and make sure you understand the steps
- There are other integral equations, e.g. *kinetic energy integral equation,* and *total energy integral equation*
- These are used in compressible analysis, heat transfer problems, etc

Use of The MIE, worked example

Worked Example: Calculate the drag coefficient of a flat plate at zero incidence with steady flow on one side only, length 1 metre, in a sea level flow at 40 m/s. Assume incompressible, laminar flow, and a linear velocity profile within the boundary layer.

MIE -> $$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{c_f}{2}$$

no pressure gradients, so no velocity gradients in external flow

$$\frac{d\theta}{dx} = \frac{c_f}{2} = \frac{\tau_{wall}}{\rho u_e^2}$$

assume linear profile given by:

$$\frac{u}{u_e} = \frac{y}{\delta}$$

Integral properties:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy = \left[y - \frac{y^2}{2\delta}\right]_0^\delta = \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

$$\theta = \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

and

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}$$

and

$$u = \frac{y}{\delta} u_e \Rightarrow \frac{\partial u}{\partial y} = \frac{u_e}{\delta}$$

substitute into the MIE: $\dfrac{d\theta}{dx} = \dfrac{c_f}{2} = \dfrac{\tau_{wall}}{\rho u_e^2}$

Kinematic viscosity

$$\frac{d\theta}{dx} = \frac{\mu u_e}{\delta \rho_e u_e^2} \Rightarrow \frac{d\theta}{d\delta}\frac{d\delta}{dx} = \frac{\nu}{\delta u_e} \Rightarrow \frac{1}{6}\frac{d\delta}{dx} = \frac{\nu}{\delta u_e} \Rightarrow \delta d\delta = \frac{6\nu}{u_e} dx$$

integrate, and ignore constants of integration as delta = 0 when x = 0:

Kinematic viscosity

$$\delta^2 = \frac{12\nu x}{u_e} \Rightarrow \boxed{\frac{\delta}{x} = \sqrt{\frac{12\nu}{x u_e}} = \sqrt{12} Re^{-\frac{1}{2}}}$$

then

$$\frac{\delta^*}{x} = \frac{1}{2}\frac{\delta}{x} = 1.732 Re^{-\frac{1}{2}},$$

$$\frac{\theta}{x} = \frac{1}{6}\frac{\delta}{x} = 0.557 Re^{-\frac{1}{2}},$$

Shape factor: $H = 3.0$

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}$$

$$\frac{\delta}{x} = \sqrt{\frac{12\nu}{x u_e}} = \sqrt{12} Re^{-\frac{1}{2}}$$

and based on linear velocity profile:

$$u = \frac{y}{\delta} u_e \Rightarrow \frac{\partial u}{\partial y} = \frac{u_e}{\delta}$$

Kinematic viscosity

$$c_f = \frac{2\tau_{wall}}{\rho_e u_e^2} = \frac{2\nu}{u_e \delta} = \frac{2\nu}{\sqrt{12} Re^{-\frac{1}{2}} x u_e} = \frac{2}{\sqrt{12}} \frac{Re^{\frac{1}{2}}}{Re} = \frac{\theta}{x} = 0.577 Re^{-\frac{1}{2}}$$

this is for one side of the flat plate, locally

$$C_D = C_F = \frac{1}{c} \int_0^c c_f dx = \frac{0.577}{c} \int_0^c \sqrt{\frac{\nu}{u_e x}} dx = \frac{0.577}{c} \left[2\sqrt{\frac{x\nu}{u_e}} \right]_0^c$$

Kinematic viscosity

$$= \frac{1.155}{c} \sqrt{\frac{\nu c}{u_e}} = 1.155 Re_c^{-\frac{1}{2}}$$

$$Re_c = \frac{40 * 1}{1.461 * 10^{-5}} = 2.738 * 10^6$$

$$C_D = 0.00070$$

$\times$ 2 to include both sides (C_D=0.0014)