Aerodynamics and Numerical Simulation Methods

Analytical solutions for fluid flows

University of BRISTOL

Topics for today

- Couette flow
  - Assumptions about the flow
  - Analytical solution

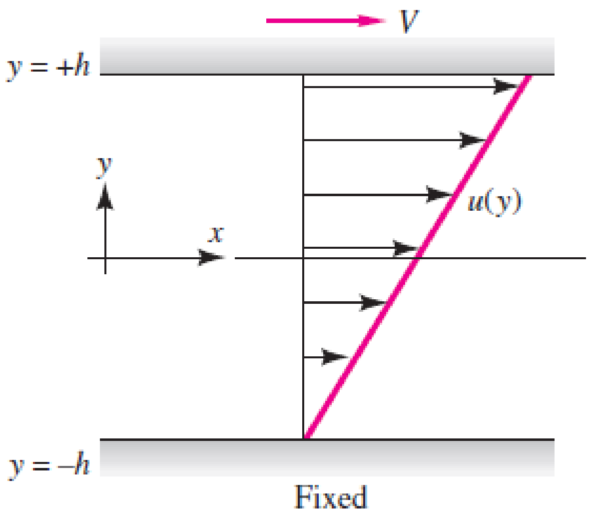
- Flow between Two Fixed Plates
  - Assumptions about the flow
  - Analytical solution

Couette Flow between a Fixed and Moving Plate

- 2D, incompressible, plane, viscous, axial flow
- Zero pressure gradient and gravity effects
- Fully developed flow (sufficiently far from entrance)

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

- u=u(y) from continuity. But how do we solve for u(y)?

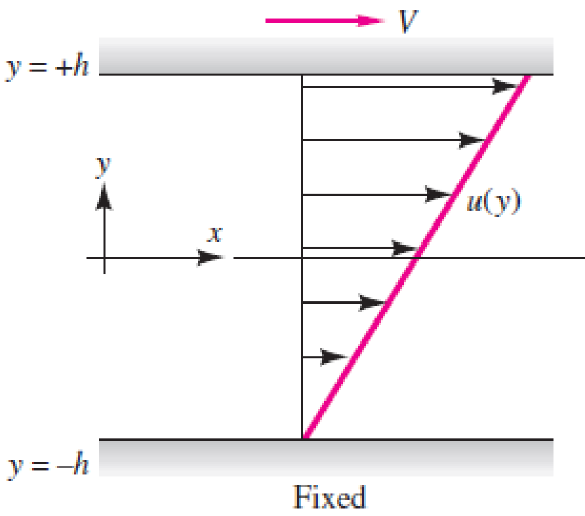


Couette Flow between a Fixed and Moving Plate

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho \frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho \frac{dw}{dt}$$



- Exercise: What are the terms we can drop based on the assumptions?

Source: Frank White pg 268

Couette Flow between a Fixed and Moving Plate

- We can drop almost all the terms! The momentum equation reduces to:

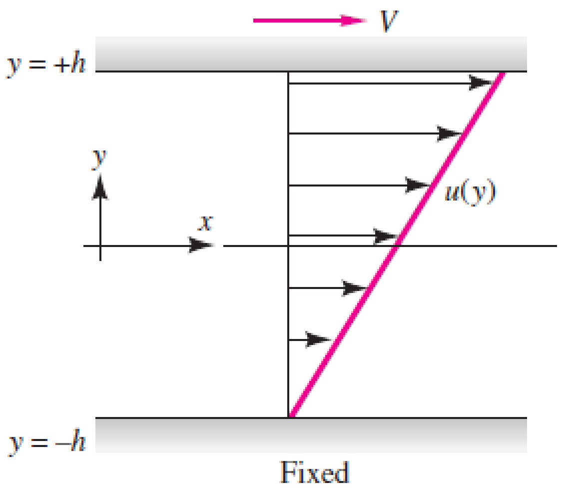
$$\frac{\partial^2 u}{\partial y^2} = 0 \qquad\qquad u = C_1 y + C_2$$

- How do we solve for C_1 and C_2?
- Boundary conditions!
  1. At y=+h, we know U=V
  2. At y=-h, we know U=0

Source: Frank White pg 268

Couette Flow between a Fixed and Moving Plate

- Solving $u = C_1 y + C_2$ using the two boundary conditions, we arrive at the solution for the Couette flow due to a moving wall:

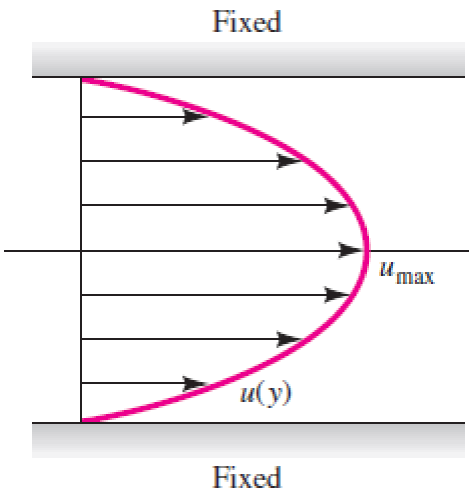


$$u = \frac{V}{2h} y + \frac{V}{2} \qquad valid\ for\ -h \leq y \leq +h$$

Source: Frank White pg 268

Flow due to Pressure Gradient between Two Fixed Plates

- 2D, incompressible, plane, viscous, axial flow
- Zero gravity effects, flow driven by pressure gradient
- Fully developed flow (sufficiently far from entrance)



$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

- u=u(y) from continuity. But how do we solve for u(y)?

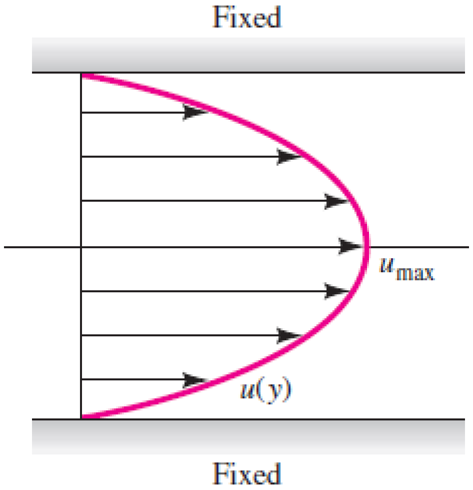
Source: Frank White pg 269

Flow due to Pressure Gradient between Two Fixed Plates

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho \frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho \frac{dw}{dt}$$



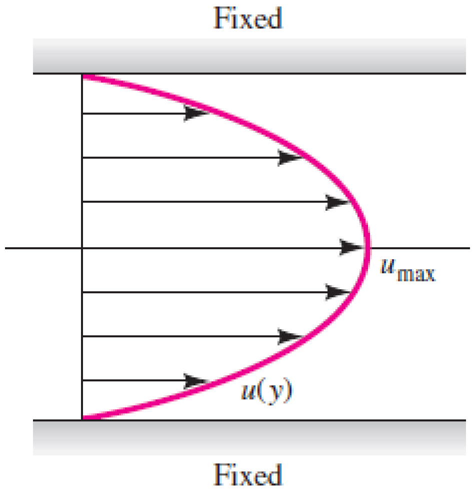
- Exercise: What are the terms we can drop based on the assumptions?

Source: Frank White pg 268

Flow due to Pressure Gradient between Two Fixed Plates

- The momentum equation should reduce to:

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \qquad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$$



- Hence,

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} = constant < 0$$

- Integrating twice with respect to y, we get:

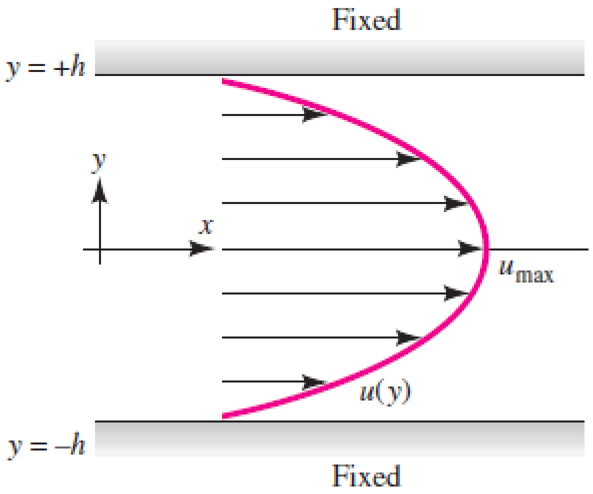
$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

Source: Frank White pg 268

Flow due to Pressure Gradient between Two Fixed Plates

- To solve the equation:

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$



- We apply the boundary condition:
  1. At y=+h, u=0
  2. At y=-h, u=0
- We get:

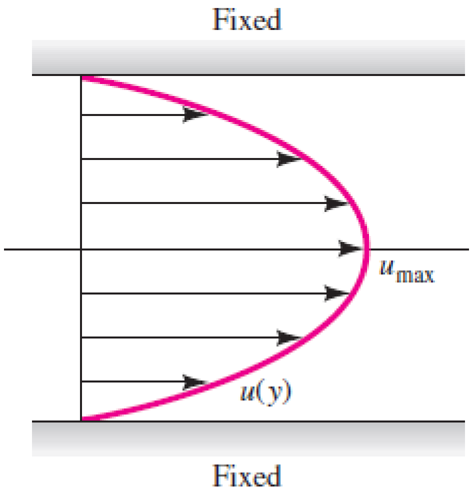
$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2} + C_1 h + C_2 \qquad u = \frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2} - C_1 h + C_2$$

Source: Frank White pg 268

Flow due to Pressure Gradient between Two Fixed Plates

- This just means:

$$C_1 = 0 \qquad\qquad C_2 = -\frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2}$$



- Therefore, the solution is:

$$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right)$$

Source: Frank White pg 268