CHAPTER

Aerodynamic Stability and Control Derivatives

13

13.1 Introduction

As is usual in aerodynamic analysis, for the purposes of obtaining simple expressions for the stability and control derivatives a wind axis reference system is assumed throughout. The choice of wind axes is convenient since it reduces the derivatives to their simplest possible description by retaining only the essential contributions and hence, maximises the *visibility* of the physical phenomena involved. It is therefore very important to remember that if the derivatives thus obtained are required for use in equations of motion referred to an alternative axis, then the appropriate axis transformation must be applied to the derivatives. Some useful transformations are given in Appendices 7 and 8. In all cases analytical expressions are obtained for the derivatives assuming subsonic flight conditions; it is then relatively straightforward to develop the expressions further to allow for the effects of Mach number as suggested in Section 12.4.

It has already been established that simple analytical expressions for the derivatives rarely give accurate estimates. Their usefulness is significantly more important as a means for explaining their physical origins, thereby providing the essential link between aircraft dynamics and airframe aerodynamics. The analytical procedure for obtaining simple derivative expressions has been well established for very many years, and the approach commonly encountered in the United Kingdom today is comprehensively described by Babister (1961) and, in less detail, by Babister (1980). The following paragraphs owe much to that work since it is unlikely that the treatment can be bettered. For the calculation of more reliable estimates of derivative values, reference to the ESDU data items is advised. The reader requiring a more detailed aerodynamic analysis of stability and control derivatives will find much useful material in Hancock (1995).

13.2 Longitudinal aerodynamic stability derivatives

For convenience, a summary of the derivative expressions derived in this paragraph is included in Table A8.1, Appendix 8.

13.2.1 Preliminary considerations

A number of expressions are required repeatedly in the derivative analysis, so it is convenient to assemble these expressions prior to embarking on the analysis. A longitudinal small perturbation is

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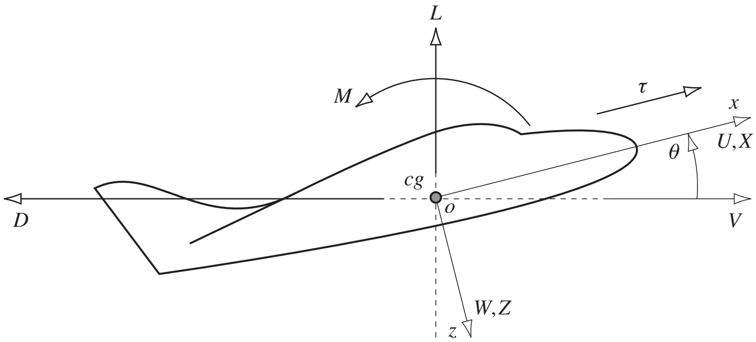

FIGURE 13.1 Perturbed wind axes.

shown in Fig. 13.1 in which the aircraft axes are wind axes and the initial condition assumes steady symmetric level flight at velocity V_0. Although not strictly an aerodynamic force, the thrust τ is shown since it may behave like an aerodynamic variable in a perturbation. As indicated, the thrust force is tied to the aircraft x axis and moves with it.

In the perturbation the total velocity becomes V with components U and W along the ox and oz axes respectively. Whence

$$V^2 = U^2 + W^2 \tag{13.1}$$

and

$$\begin{aligned} U &= U_e + u = V\cos\theta \\ W &= W_e + w = V\sin\theta \end{aligned} \tag{13.2}$$

Since wind axes are assumed, the pitch attitude perturbation θ and the incidence perturbation α are the same and are given by

$$\tan\theta \equiv \tan\alpha = \frac{W}{U} \tag{13.3}$$

Differentiate equation (13.1) with respect to U and W in turn to obtain the following partial derivatives:

$$\frac{\partial V}{\partial U} = \frac{U}{V} \quad \text{and} \quad \frac{\partial V}{\partial W} = \frac{W}{V} \tag{13.4}$$

Substitute for U and W from equation (13.2) to obtain

$$\frac{\partial V}{\partial U} = \cos\theta \cong 1 \quad \text{and} \quad \frac{\partial V}{\partial W} = \sin\theta \cong 0 \tag{13.5}$$

since, by definition, θ is a small angle.

In a similar way, differentiate equation (13.3) with respect to U and W in turn and substitute for U and W from equation (13.2) to obtain

$$\begin{aligned}\frac{\partial\theta}{\partial U} &\equiv \frac{\partial\alpha}{\partial U} = \frac{-\sin\theta}{V} \cong 0\\ \frac{\partial\theta}{\partial W} &\equiv \frac{\partial\alpha}{\partial W} = \frac{\cos\theta}{V} \cong \frac{1}{V}\end{aligned} \tag{13.6}$$

since again, by definition, θ is a small angle.

From equation (12.10),

$$\frac{\partial}{\partial V} = \frac{1}{a}\frac{\partial}{\partial \mathrm{M}} \tag{13.7}$$

which is useful for transforming from a velocity dependency to a Mach number dependency and where, here, a is the local speed of sound.

13.2.2 Aerodynamic force and moment components

With reference to Fig. 13.1 the lift and drag forces may be resolved into the disturbed aircraft axes to give the following components of aerodynamic force. The perturbed axial force is

$$X = L\sin\theta - D\cos\theta + \tau = \frac{1}{2}\rho V^2 S(C_L\sin\theta - C_D\cos\theta) + \tau \tag{13.8}$$

and the perturbed normal force is

$$Z = -L\cos\theta - D\sin\theta = -\frac{1}{2}\rho V^2 S(C_L\cos\theta + C_D\sin\theta) \tag{13.9}$$

In the initial steady trim condition, by definition the pitching moment M is zero. However, in the perturbation the transient pitching moment is nonzero and is given by

$$M = \frac{1}{2}\rho V^2 S\bar{\bar{c}}C_m \tag{13.10}$$

Note that considerable care is needed in order not to confuse pitching moment M and Mach number M.

13.2.3 Force derivatives due to velocity perturbations

$\overset{\circ}{X}_u = \dfrac{\partial X}{\partial U}$ ***Axial force due to axial velocity***

Differentiate equation (13.8) to obtain

$$\begin{aligned}\frac{\partial X}{\partial U} = &\frac{1}{2}\rho V^2 S\left(\frac{\partial C_L}{\partial U}\sin\theta + C_L\cos\theta\frac{\partial\theta}{\partial U} - \frac{\partial C_D}{\partial U}\cos\theta + C_D\sin\theta\frac{\partial\theta}{\partial U}\right)\\ &+ \rho VS\frac{\partial V}{\partial U}(C_L\sin\theta - C_D\cos\theta) + \frac{\partial\tau}{\partial U}\end{aligned} \tag{13.11}$$

Substitute for $\partial V/\partial U$ from equation (13.5) and for $\partial\theta/\partial U$ from equation (13.6). As θ is a small angle, in the limit, $\cos\theta \cong 1$ and $\sin\theta \cong 0$ and equation (13.11) simplifies to

$$\frac{\partial X}{\partial U} = -\frac{1}{2}\rho V^2 S\frac{\partial C_D}{\partial U} - \rho VSC_D + \frac{\partial \tau}{\partial U} \tag{13.12}$$

Now,

$$\frac{\partial C_D}{\partial U} = \frac{\partial C_D}{\partial V}\frac{\partial V}{\partial U} = \frac{\partial C_D}{\partial V} \tag{13.13}$$

and, similarly,

$$\frac{\partial \tau}{\partial U} = \frac{\partial \tau}{\partial V} \tag{13.14}$$

In the limit the total perturbation velocity tends to the equilibrium value and $V \cong V_0$. Whence equation (13.12) may be written as

$$\overset{\circ}{X}_u = \frac{\partial X}{\partial U} = -\rho V_0 SC_D - \frac{1}{2}\rho V_0^2 S\frac{\partial C_D}{\partial V} + \frac{\partial \tau}{\partial V} \tag{13.15}$$

With reference to Appendix 2, the dimensionless form of the derivative is given by

$$X_u = \frac{\overset{\circ}{X}_u}{\frac{1}{2}\rho V_0 S} = -2C_D - V_0\frac{\partial C_D}{\partial V} + \frac{1}{\frac{1}{2}\rho V_0 S}\frac{\partial \tau}{\partial V} \tag{13.16}$$

Alternatively, using equation (13.7), the dimensionless derivative may be expressed in terms of Mach number rather than velocity:

$$X_u = -2C_D - \frac{1}{\mathrm{M}_0}\frac{\partial C_D}{\partial \mathrm{M}} + \frac{1}{\frac{1}{2}\rho \mathrm{M}_0 Sa^2}\frac{\partial \tau}{\partial \mathrm{M}} \tag{13.17}$$

Expressions for the remaining force velocity derivatives are obtained in a similar way as follows.

$\overset{\circ}{Z}_u = \dfrac{\partial Z}{\partial U}$ *Normal force due to axial velocity*

By differentiating equation (13.9) with respect to U, it is easily shown that

$$\overset{\circ}{Z}_u = \frac{\partial Z}{\partial U} = -\rho VSC_L - \frac{1}{2}\rho V^2 S\frac{\partial C_L}{\partial U} \tag{13.18}$$

Now, in the manner of equation (13.13),

$$\frac{\partial C_L}{\partial U} = \frac{\partial C_L}{\partial V}\frac{\partial V}{\partial U} = \frac{\partial C_L}{\partial V} \tag{13.19}$$

Thus in the limit $V \cong V_0$ and equation (13.18) may be written as

$$\overset{\circ}{Z}_u = -\rho V_0 SC_L - \frac{1}{2}\rho V_0^2 S\frac{\partial C_L}{\partial V} \tag{13.20}$$

With reference to Appendix 2, the dimensionless form of the derivative is given by

$$Z_u = \frac{\overset{\circ}{Z}_u}{\frac{1}{2}\rho V_0 S} = -2C_L - V_0 \frac{\partial C_L}{\partial V} \tag{13.21}$$

or, alternatively, expressed in terms of Mach number rather than velocity,

$$Z_u = -2C_L - \mathrm{M}_0 \frac{\partial C_L}{\partial \mathrm{M}} \tag{13.22}$$

$\overset{\circ}{X}_w = \frac{\partial X}{\partial W}$ *Axial force due to normal velocity*

As before, it may be shown that by differentiating equation (13.8) with respect to W,

$$\overset{\circ}{X}_w = \frac{\partial X}{\partial W} = \frac{1}{2}\rho V^2 S\left(\frac{1}{V}C_L - \frac{\partial C_D}{\partial W}\right) + \frac{\partial \tau}{\partial W} \tag{13.23}$$

Now, with reference to equation (13.6) and noting that $\alpha = \theta$,

$$\frac{\partial C_D}{\partial W} = \frac{\partial C_D}{\partial \theta}\frac{\partial \theta}{\partial W} \equiv \frac{1}{V}\frac{\partial C_D}{\partial \alpha} \tag{13.24}$$

Similarly, it may be shown that

$$\frac{\partial \tau}{\partial W} \equiv \frac{1}{V}\frac{\partial \tau}{\partial \alpha} = 0 \tag{13.25}$$

since it is assumed that thrust variation resulting from small incidence perturbations is negligible. Thus, in the limit equation (13.23) may be written as

$$\overset{\circ}{X}_w = \frac{1}{2}\rho V_0 S\left(C_L - \frac{\partial C_D}{\partial \alpha}\right) \tag{13.26}$$

With reference to Appendix 2, the dimensionless form of the derivative is given by

$$X_w = \frac{\overset{\circ}{X}_w}{\frac{1}{2}\rho V_0 S} = \left(C_L - \frac{\partial C_D}{\partial \alpha}\right) \tag{13.27}$$

$\overset{\circ}{Z}_w = \frac{\partial Z}{\partial W}$ *Normal force due to normal velocity*

As before, by differentiating equation (13.9) with respect to W and with reference to equation (13.24) it may be shown that

$$\overset{\circ}{Z}_w = \frac{\partial Z}{\partial W} = -\frac{1}{2}\rho V^2 S\left(\frac{\partial C_L}{\partial W} + \frac{1}{V}C_D\right) = -\frac{1}{2}\rho V S\left(\frac{\partial C_L}{\partial \alpha} + C_D\right) \tag{13.28}$$

In the limit equation (13.28) may be rewritten as

$$\overset{\circ}{Z}_w = -\frac{1}{2}\rho V_0 S\left(\frac{\partial C_L}{\partial \alpha} + C_D\right) \tag{13.29}$$

With reference to Appendix 2, the dimensionless form of the derivative is given by

$$Z_w = \frac{\overset{\circ}{Z}_w}{\frac{1}{2}\rho V_0 S} = -\left(\frac{\partial C_L}{\partial \alpha} + C_D\right) \tag{13.30}$$

13.2.4 Moment derivatives due to velocity perturbations

$\overset{\circ}{M}_u = \dfrac{\partial M}{\partial U}$ ***Pitching moment due to axial velocity***

In a perturbation the pitching moment becomes nonzero and is given by equation (13.10). Differentiating equation (13.10) with respect to U,

$$\frac{\partial M}{\partial U} = \frac{1}{2}\rho V^2 S\bar{\bar{c}}\frac{\partial C_m}{\partial U} + \rho V S\bar{\bar{c}} C_m \tag{13.31}$$

and with reference to equation (13.5),

$$\frac{\partial C_m}{\partial U} = \frac{\partial C_m}{\partial V}\frac{\partial V}{\partial U} = \frac{\partial C_m}{\partial V} \tag{13.32}$$

Now, in the limit, as the perturbation tends to zero, the pitching moment coefficient C_m in the second term in equation (13.31) tends to the steady equilibrium value, which is, of course, zero. Therefore, in the limit equation (13.31) simplifies to

$$\overset{\circ}{M}_u = \frac{\partial M}{\partial U} = \frac{1}{2}\rho V_0^2 S\bar{\bar{c}}\frac{\partial C_m}{\partial V} \tag{13.33}$$

With reference to Appendix 2, the dimensionless form of the derivative is given by

$$M_u = \frac{\overset{\circ}{M}_u}{\frac{1}{2}\rho V_0 S\bar{\bar{c}}} = V_0\frac{\partial C_m}{\partial V} \tag{13.34}$$

Alternatively, using equation (13.7), the dimensionless derivative may be expressed in terms of Mach number rather than velocity:

$$M_u = \mathrm{M}_0\frac{\partial C_m}{\partial \mathrm{M}} \tag{13.35}$$

In subsonic flight the pitching moment coefficient C_m is very nearly independent of velocity, or Mach number; thus the derivative M_u is often assumed to be negligibly small for those flight conditions.

$\overset{\circ}{M}_w = \dfrac{\partial M}{\partial W}$ ***Pitching moment due to normal velocity***

As previously, differentiating equation (13.10) with respect to W and with reference to equation (13.24), it may be shown that

$$\overset{\circ}{M}_w = \frac{\partial M}{\partial W} = \frac{1}{2}\rho V^2 S\overline{\overline{c}}\frac{\partial C_m}{\partial W} = \frac{1}{2}\rho V S\overline{\overline{c}}\frac{\partial C_m}{\partial \alpha} \tag{13.36}$$

In the limit $V \cong V_0$ and equation (13.36) may be written as

$$\overset{\circ}{M}_w = \frac{1}{2}\rho V_0 S\overline{\overline{c}}\frac{\partial C_m}{\partial \alpha} \tag{13.37}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$M_w = \frac{\overset{\circ}{M}_w}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = \frac{\partial C_m}{\partial \alpha} \tag{13.38}$$

Further, assuming that linear aerodynamic conditions apply, such as are typical of subsonic flight, then with reference to equation (3.17),

$$M_w = \frac{dC_m}{d\alpha} = \frac{dC_L}{d\alpha}\frac{dC_m}{dC_L} = -aK_n \tag{13.39}$$

where, here, a denotes the lift curve slope and K_n is the controls-fixed static margin. As shown in Chapter 6 the derivative M_w is a measure of the *pitch stiffness* of the aeroplane and plays an important part in the determination of the longitudinal short-term dynamics.

13.2.5 Derivatives due to a pitch velocity perturbation

It is usually assumed that the longitudinal aerodynamic properties of an aeroplane are dominated by those of the wing and tailplane. However, when the disturbance is a small perturbation in pitch rate q it is assumed that the dominating aerodynamic properties are those of the tailplane. Thus, in the first instance the resulting aerodynamic changes contributing to the stability derivatives are assumed to arise entirely from tailplane effects. By so doing it is acknowledged that the wing contribution may not necessarily be small and that its omission will reduce the accuracy of the derivative estimates. However, experience has shown that the error incurred by adopting this assumption is usually acceptably small.

An aeroplane pitching through its equilibrium attitude with pitch rate perturbation q is shown in Fig. 13.2. Since the effect of the pitch rate is to cause the tailplane to experience a normal velocity component due to rotation about the cg, the resultant effect is a change in the local incidence α_T of the tailplane. The total perturbation velocity is V and the tailplane incidence perturbation is given by

$$\alpha_T \cong \tan\alpha_T = \frac{ql_T}{V} \tag{13.40}$$

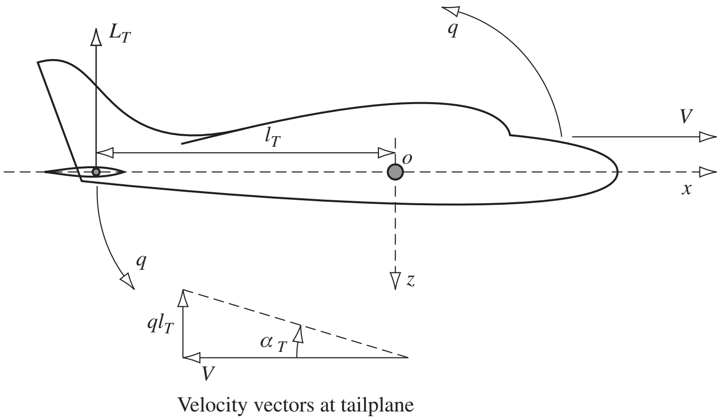

FIGURE 13.2 Tailplane incidence due to pitch rate.

since, by definition, α_T is a small angle. It is important to appreciate that α_T is the change, or increment, in tailplane incidence relative to its equilibrium value and, like the pitch rate perturbation, is transient in nature. From equation (13.40) it follows that

$$\frac{\partial \alpha_T}{\partial q} = \frac{l_T}{V} \tag{13.41}$$

$\overset{\circ}{X}_q = \dfrac{\partial X}{\partial q}$ *Axial force due to pitch rate*

In this instance, for the reasons given above, it is assumed that the axial force perturbation arises from the tailplane drag perturbation only; thus

$$X = -D_T = -\frac{1}{2}\rho V^2 S_T C_{D_T} \tag{13.42}$$

Assuming V to be independent of pitch rate q, differentiate equation (13.42) with respect to the perturbation variable q:

$$\frac{\partial X}{\partial q} = -\frac{1}{2}\rho V^2 S_T \frac{\partial C_{D_T}}{\partial q} \tag{13.43}$$

Now, with reference to equation (13.41) write

$$\frac{\partial C_{D_T}}{\partial q} = \frac{\partial C_{D_T}}{\partial \alpha_T}\frac{\partial \alpha_T}{\partial q} = \frac{l_T}{V}\frac{\partial C_{D_T}}{\partial \alpha_T} \tag{13.44}$$

Substitute equation (13.44) into equation (13.43); then, in the limit $V \cong V_0$ and equation (13.43) may be written as

$$\overset{\circ}{X}_q = \frac{\partial X}{\partial q} = -\frac{1}{2}\rho V_0 S_T l_T \frac{\partial C_{D_T}}{\partial \alpha_T} \tag{13.45}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$X_q = \frac{\overset{\circ}{X}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = -\overline{V}_T \frac{\partial C_{D_T}}{\partial \alpha_T} \tag{13.46}$$

where the *tail volume ratio* is given by

$$\overline{V}_T = \frac{S_T l_T}{S\overline{\overline{c}}} \tag{13.47}$$

Since the rate of change of tailplane drag with incidence is usually small, it is customary to assume that the derivative X_q is insignificantly small and it is frequently ignored in aircraft stability and control analysis.

$\overset{\circ}{Z}_q = \dfrac{\partial Z}{\partial q}$ *Normal force due to pitch rate*

Similarly, it is assumed that in a pitch rate perturbation the change in normal force arises from tailplane lift only; thus

$$Z = -L_T = -\frac{1}{2}\rho V^2 S_T C_{L_T} \tag{13.48}$$

Differentiate equation (13.48) with respect to q and with reference to equation (13.44); then

$$\frac{\partial Z}{\partial q} = -\frac{1}{2}\rho V S_T l_T \frac{\partial C_{L_T}}{\partial \alpha_T} = -\frac{1}{2}\rho V S_T l_T a_1 \tag{13.49}$$

where, again, it is assumed that V is independent of pitch rate q and that, additionally, the tailplane lift coefficient is a function of incidence only with lift curve slope denoted a_1. Whence in the limit $V \cong V_0$ and equation (13.49) may be written as

$$\overset{\circ}{Z}_q = \frac{\partial Z}{\partial q} = -\frac{1}{2}\rho V_0 S_T l_T a_1 \tag{13.50}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$Z_q = \frac{\overset{\circ}{Z}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = -\overline{V}_T a_1 \tag{13.51}$$

$\overset{\circ}{M}_q = \dfrac{\partial M}{\partial q}$ *Pitching moment due to pitch rate*

Again, in a pitch rate perturbation q, the pitching moment is assumed to arise entirely from the moment of the tailplane normal force perturbation, given by equation (13.48), about the *cg*. Thus, in the perturbation,

$$M = Z l_T = -\frac{1}{2}\rho V^2 S_T l_T C_{L_T} \tag{13.52}$$

Differentiate equation (13.52) with respect to q to obtain the relationship

$$\overset{\circ}{M}_q = \frac{\partial M}{\partial q} = l_T \frac{\partial Z}{\partial q} = l_T \overset{\circ}{Z}_q \tag{13.53}$$

It therefore follows that

$$\overset{\circ}{M}_q = -\frac{1}{2}\rho V_0 S_T l_T^2 a_1 \tag{13.54}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$M_q = \frac{\overset{\circ}{M}_q}{\frac{1}{2}\rho V_0 S \overline{\overline{c}}^2} = -\overline{V}_T \frac{l_T}{\overline{\overline{c}}} a_1 \equiv \frac{l_T}{\overline{\overline{c}}} Z_q \tag{13.55}$$

It is shown in Chapter 6 that M_q is the all important pitch damping derivative. Although this simple model illustrates the importance of the tailplane in determining the pitch damping characteristics of the aeroplane, wing and body contributions may also be significant. Equation (13.55) should therefore be regarded as the first estimate rather than the definitive estimate of the derivative. However, it is often good enough for preliminary analysis of stability and control.

13.2.6 Derivatives due to acceleration perturbations

The derivatives due to the acceleration perturbations $\dot{u}$, $\dot{w}$, and $\dot{q}$ are not commonly encountered in the longitudinal equations of motion since their numerical values are usually insignificantly small. Their meaning is perhaps easier to appreciate when the longitudinal equations of motion are written in matrix form, equation (4.65), to include all of the acceleration derivatives. To recap, the state equation is given by

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{A}'\mathbf{x} + \mathbf{B}'\mathbf{u} \tag{13.56}$$

with state vector, $\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} u & w & q & \theta \end{bmatrix}$ and input vector $\mathbf{u}^{\mathrm{T}} = \begin{bmatrix} \eta & \tau \end{bmatrix}$. The state matrix $\boldsymbol{A}'$ and input matrix $\boldsymbol{B}'$ remain unchanged whereas the mass matrix $\mathbf{M}$ is modified to include all the additional acceleration derivatives:

$$\mathbf{M} = \begin{bmatrix} \left(m - \overset{\circ}{X}_{\dot{u}}\right) & -\overset{\circ}{X}_{\dot{w}} & -\overset{\circ}{X}_{\dot{q}} & 0 \\ -\overset{\circ}{Z}_{\dot{u}} & \left(m - \overset{\circ}{Z}_{\dot{w}}\right) & -\overset{\circ}{Z}_{\dot{q}} & 0 \\ -\overset{\circ}{M}_{\dot{u}} & -\overset{\circ}{M}_{\dot{w}} & \left(I_y - \overset{\circ}{M}_{\dot{q}}\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13.57}$$

Since all of the acceleration derivatives appear in the mass matrix alongside the normal mass and inertia terms, their effect is to change (increase) the apparent mass and inertia properties of the aircraft. For this reason they are sometimes referred to as *apparent* or *virtual* mass and inertia terms. Whenever the aeroplane moves, some of the surrounding displaced air mass is entrained and moves with the aircraft, and it is the mass and inertia of this air which modifies the apparent mass and inertia of the aeroplane. The acceleration derivatives quantify this effect. For most aircraft, since the mass of the displaced air is a small fraction of the mass of the aircraft, the acceleration derivatives are insignificantly small. An exception to this is the airship, for which the apparent mass and inertia can be as much as 50% larger than the actual physical value. Other vehicles in which these effects may be nonnegligible include balloons, parachutes, and underwater vehicles which operate in a much denser fluid medium.

For many modern high-performance aeroplanes the derivatives due to a rate of change of normal velocity perturbation $\dot{w}$ $(\dot{\alpha})$ may not be negligible. A rate of change of normal velocity perturbation causes a transient disturbance in the downwash field behind the wing which passes over the tailplane a short time later. The disturbance to the moving air mass in the vicinity of the wing is, in itself, insignificant for the reason given above. However, since the tailplane sees this as a transient in incidence, a short time later it responds accordingly and the effect on the airframe is not necessarily insignificant. This particular characteristic is known as the *downwash lag* effect.

An expression for the total incidence of the tailplane is given by equation (3.9), which, for the present application, may be written as

$$\alpha_T(t) = \alpha_e + \eta_T - \varepsilon(t) \tag{13.58}$$

where α_e is the steady equilibrium incidence of the wing, η_T is the tailplane setting angle, and $\varepsilon(t)$ is the downwash flow angle at the tailplane. Thus any change in downwash angle at the tailplane in otherwise steady conditions gives rise to a change in tailplane incidence of equal magnitude and opposite sign. It is important to appreciate that the perturbation at the tailplane is observed at time t and is due to an event on the wing which took place some time earlier. For this reason the flow conditions on the wing at time t are assumed to have recovered their steady equilibrium state.

With reference to Fig. 13.3, the point a in the flow field around the wing arrives at point b in the flow field around the tailplane at a time l_T/V_0 later, referred to as the *downwash lag* and where, for convenience, the mean distance travelled is assumed to be equal to the tail moment arm l_T. Thus a perturbation in $\dot{w}$ $(\dot{\alpha})$ causes a perturbation in the wing downwash field which arrives at the tailplane after the downwash lag time interval. Therefore, there is a short delay between cause and effect.

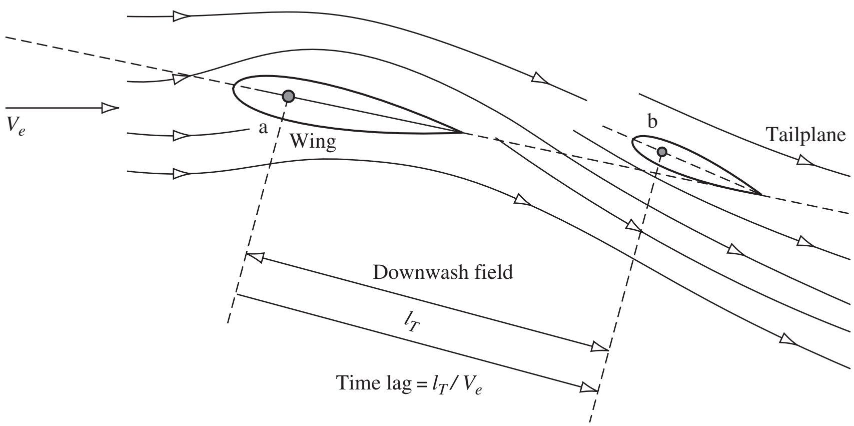

FIGURE 13.3 Typical downwash field.

The downwash angle $\varepsilon(t)$ at the tailplane at time t is therefore a function of the incidence of the wing at time $t' = t\text{-}l_T/V_0$ and may be expressed as

$$\begin{aligned}\varepsilon(t) &= \frac{d\varepsilon}{d\alpha}\alpha(t') = \frac{d\varepsilon}{d\alpha}\frac{d\alpha}{dt}\left(t - \frac{l_T}{V_0}\right)\\ &= \frac{d\varepsilon}{d\alpha}\alpha_e - \frac{d\varepsilon}{d\alpha}\frac{\dot{w}l_T}{V_0^2} = \varepsilon_e - \frac{d\varepsilon}{d\alpha}\frac{\dot{w}l_T}{V_0^2}\end{aligned} \tag{13.59}$$

since

$$\frac{d\alpha}{dt}t = \alpha(t) \equiv \alpha_e \tag{13.60}$$

and

$$\alpha \cong \tan\alpha = \frac{w}{V_0} \tag{13.61}$$

whence

$$\frac{d\alpha}{dt} = \frac{\dot{w}}{V_0} \tag{13.62}$$

Thus, with reference to equations (13.58) and (13.59) the total tailplane incidence during a downwash perturbation may be written as

$$\alpha_{T_e} + \alpha_T(t) = \alpha_e + \eta_T - \varepsilon_e + \frac{d\varepsilon}{d\alpha}\frac{\dot{w}l_T}{V_0^2} \tag{13.63}$$

The perturbation in tailplane incidence due to the downwash lag effect is therefore given by

$$\alpha_T(t) = \frac{d\varepsilon}{d\alpha}\frac{\dot{w}l_T}{V_0^2} \tag{13.64}$$

$\overset{\circ}{X}_{\dot{w}} = \dfrac{\partial X}{\partial \dot{w}}$ *Axial force due to rate of change of normal velocity*

In this instance, it is assumed that the axial force perturbation arises from the perturbation in tailplane drag due solely to the perturbation in incidence; whence

$$X = -D_T = -\frac{1}{2}\rho V^2 S_T C_{D_T} = -\frac{1}{2}\rho V^2 S_T \frac{\partial C_{D_T}}{\partial \alpha_T}\alpha_T \tag{13.65}$$

Now, by definition,

$$X = \overset{\circ}{X}_{\dot{w}}\,\dot{w} \tag{13.66}$$

and in the limit $V \cong V_0$. Thus substitute equation (13.64) into (13.65) and apply equation (13.66) to obtain

$$\overset{\circ}{X}_{\dot{w}} = -\frac{1}{2}\rho S_T l_T \frac{\partial C_{D_T}}{\partial \alpha_T}\frac{d\varepsilon}{d\alpha} \tag{13.67}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$X_{\dot{w}} = \frac{\overset{\circ}{X}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}} = -\overline{V}_T \frac{\partial C_{D_T}}{\partial \alpha_T}\frac{d\varepsilon}{d\alpha} \equiv X_q \frac{d\varepsilon}{d\alpha} \tag{13.68}$$

Since $X_q X_q$ is usually very small and $d\varepsilon/d\alpha < 1$, the derivative $X_{\dot{w}}$ is insignificantly small and is usually omitted from the equations of motion.

$\overset{\circ}{Z}_{\dot{w}} = \dfrac{\partial Z}{\partial \dot{w}}$ *Normal force due to rate of change of normal velocity*

Again, it is assumed that the normal force perturbation arises from the perturbation in tailplane lift due solely to the perturbation in incidence; whence

$$Z = -L_T = -\frac{1}{2}\rho V^2 S_T C_{L_T} = -\frac{1}{2}\rho V^2 S_T \frac{\partial C_{L_T}}{\partial \alpha_T}\alpha_T \tag{13.69}$$

Again, by definition,

$$Z = \overset{\circ}{Z}_{\dot{w}}\,\dot{w} \tag{13.70}$$

and in the limit $V \cong V_0$. Thus substitute equation (13.64) into (13.69) and apply equation (13.70) to obtain

$$\overset{\circ}{Z}_{\dot{w}} = -\frac{1}{2}\rho S_T l_T a_1 \frac{d\varepsilon}{d\alpha} \tag{13.71}$$

As in Section 13.2.4, it is assumed that the tailplane lift coefficient is a function of incidence only with lift curve slope denoted a_1. With reference to Appendix 2, the dimensionless form of the derivative is given by

$$Z_{\dot{w}} = \frac{\overset{\circ}{Z}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}} = -\overline{V}_T a_1 \frac{d\varepsilon}{d\alpha} \equiv Z_q \frac{d\varepsilon}{d\alpha} \tag{13.72}$$

Care should be exercised since $Z_{\dot{w}}$ is not always insignificant.

$\overset{\circ}{M}_{\dot{w}} = \dfrac{\partial M}{\partial \dot{w}}$ *Pitching moment due to rate of change of normal velocity*

In this instance the pitching moment is assumed to arise entirely from the moment of the tailplane normal force perturbation about the *cg* resulting from the perturbation in incidence, given by equation (13.64). Thus, in the perturbation,

$$M = Zl_T = -\frac{1}{2}\rho V^2 S_T l_T \frac{\partial C_{L_T}}{\partial \alpha_T}\alpha_T \tag{13.73}$$

Again, by definition,

$$M = \overset{\circ}{M}_{\dot{w}}\,\dot{w} \tag{13.74}$$

and in the limit $V \cong V_0$. Thus substitute equation (13.64) into (13.73) and apply equation (13.74) to obtain

$$\overset{\circ}{M}_{\dot{w}} = -\frac{1}{2}\rho S_T l_T^2 a_1 \frac{d\varepsilon}{d\alpha} \tag{13.75}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$M_{\dot{w}} = \frac{\overset{\circ}{M}_{\dot{w}}}{\frac{1}{2}\rho S \overline{\overline{c}}^2} = -\overline{V}_T \frac{l_T}{\overline{\overline{c}}} a_1 \frac{d\varepsilon}{d\alpha} \equiv M_q \frac{d\varepsilon}{d\alpha} \tag{13.76}$$

The derivative $M_{\dot{w}}$ is nearly always significant and makes an important contribution to the damping of the short-period pitching oscillation (see equation (6.21)).

13.3 Lateral-directional aerodynamic stability derivatives

For convenience, a summary of the derivative expressions derived in this paragraph is also included in Table A8.2, Appendix 8.

13.3.1 Preliminary considerations

Unlike the longitudinal aerodynamic stability derivatives, the lateral-directional derivatives are much more difficult to estimate with any degree of confidence. The problem arises from the mutual aerodynamic interference between the lifting surfaces, fuselage, power plant, undercarriage, and so forth in asymmetric flow conditions, which makes it difficult to identify the most significant contributions to a particular derivative with any degree of certainty. When a derivative cannot be estimated by the simplest analysis of the often complex aerodynamics, then, the use of *strip theory* is resorted to, which is a method of analysis which also tends to oversimplify the aerodynamic conditions in order that progress can be made. Either way, analytical estimates of the lateral-directional derivatives are often of poor accuracy and, for more reliable estimates, use of the ESDU data items is preferable. However, the simple theories used for the purpose do give a useful insight into the physical phenomena involved and, consequently, are a considerable asset to the proper understanding of aeroplane dynamics.

13.3.2 Derivatives due to sideslip

As seen by the pilot (and consistent with the notation), a positive sideslip is to the right (starboard) and is defined by the small-perturbation lateral velocity transient denoted v. The nature of a free positive sideslip disturbance is such that the right wing tends to drop and the nose tends to swing to the left of the incident *wind vector* as the aeroplane slips to the right. The reaction to the disturbance is stabilising if the aerodynamic forces and moments produced in response to the sideslip velocity tend to restore the aeroplane to a wings-level equilibrium state. The motions involved are discussed in greater detail in the context of lateral static stability in Section 3.4, in the context of directional static stability in Section 3.5, and in the context of dynamic stability in Section 7.2.

$\overset{\circ}{Y}_v = \dfrac{\partial Y}{\partial V}$ *Sideforce due to sideslip*

Sideforce due to sideslip arises mainly from the fuselage, the fin, the wing (especially a wing with dihedral), and engine nacelles in aircraft with external engines. The derivative is notoriously difficult to estimate with any degree of confidence and simple analysis assumes that the dominant contributions arise from the fuselage and fin only.

With reference to Fig. 13.4, the fuselage creates a sideforce Y_B in a sideslip, which may be regarded as *lateral drag* and is given by

$$Y_B = \frac{1}{2}\rho V_0^2 S_B \beta y_B \tag{13.77}$$

where S_B is the projected fuselage side area and y_B is a dimensionless coefficient. Note that the product βy_B is equivalent to a *lateral drag coefficient* for the fuselage. Further, since the disturbance is small the sideslip angle β is given by

$$\beta \cong \tan\beta = \frac{v}{V_0} \tag{13.78}$$

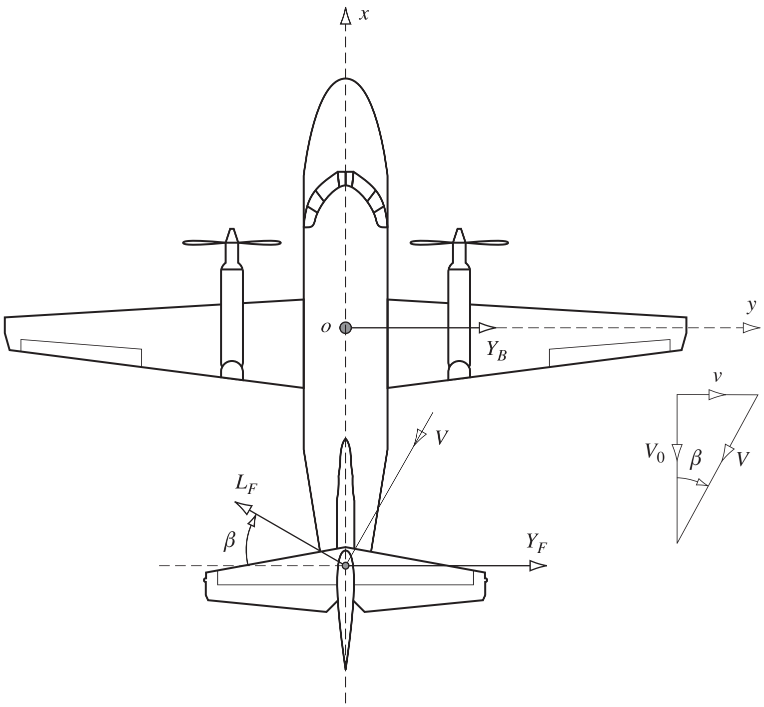

FIGURE 13.4 Sideforce generation in sideslip.

In a sideslip the fin is at incidence β and produces lift as indicated in Fig. 13.4. The fin lift resolves into a sideforce Y_F given by

$$Y_F = -\frac{1}{2}\rho V_0^2 S_F a_{1_F} \beta \cos\beta \cong -\frac{1}{2}\rho V_0^2 S_F a_{1_F} \beta \tag{13.79}$$

and since the sideslip angle β is small, $\cos\beta \cong 1$.

Let the total sideforce due to sideslip be denoted Y; then, by definition,

$$v \mathring{Y}_v = Y = Y_B + Y_F = \frac{1}{2}\rho V_0^2 (S_B y_B - S_F a_{1_F})\beta \tag{13.80}$$

Substitute the expression for β given by equation (13.78) into equation (13.80) to obtain an expression for the dimensional derivative,

$$\mathring{Y}_v = \frac{1}{2}\rho V_0 (S_B y_B - S_F a_{1_F}) \tag{13.81}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$Y_v = \frac{\mathring{Y}_v}{\frac{1}{2}\rho V_0 S} = \left(\frac{S_B}{S} y_B - \frac{S_F}{S} a_{1_F}\right) \tag{13.82}$$

$\mathring{L}_v = \dfrac{\partial L}{\partial V}$ *Rolling moment due to sideslip*

Rolling moment due to sideslip is one of the most important lateral stability derivatives since it quantifies the *lateral static stability* of the aeroplane, discussed in Section 3.4. It is one of the most difficult derivatives to estimate with any degree of confidence since it is numerically small and has many identifiable contributions. Preliminary estimates are based on the most significant contributions, which are usually assumed to arise from wing dihedral, wing sweep, wing-fuselage geometry, and the fin.

In many classical aeroplanes the wing dihedral makes the most significant contribution to the overall value of the derivative. Indeed, dihedral is one of the most important variables available to the aircraft designer with which to tailor the lateral static stability of the aeroplane. The derivative is therefore frequently referred to as *dihedral effect* irrespective of the magnitude of the other contributions. Since the tendency is for the right wing to drop in a positive sideslip disturbance, the associated disturbing rolling moment is also positive. A stabilising aerodynamic reaction is one in which the rolling moment due to sideslip is negative since this will tend to oppose the disturbing rolling moment. Dihedral effect is particularly beneficial in this respect.

In a positive sideslip disturbance to the right, the effect of dihedral is to increase the incidence of the right wing panel indicated in Fig. 13.5. The left wing panel "sees" a corresponding reduction in incidence. Thus the rolling moment is generated by the differential lift across the wing span.

Referring to Fig. 13.5, the component of sideslip velocity perpendicular to the plane of the wing panel is given by

$$v' = v\sin\Gamma \cong v\Gamma \tag{13.83}$$

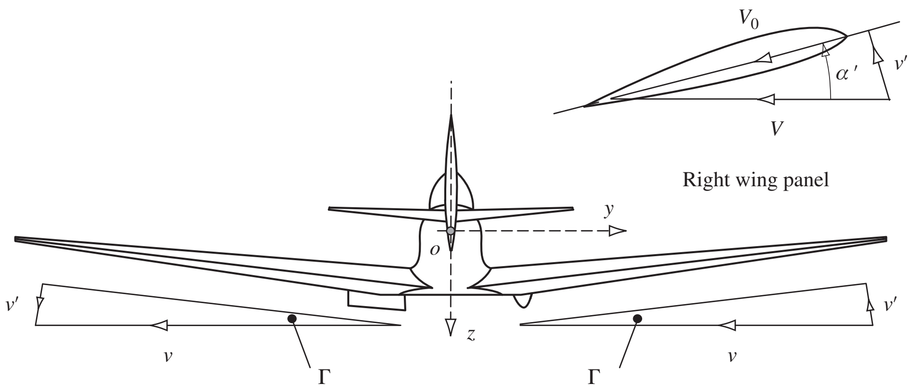

FIGURE 13.5 Incidence due to sideslip on a wing with dihedral.

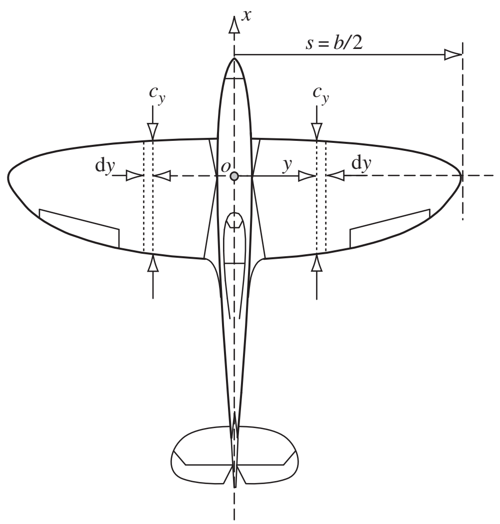

FIGURE 13.6 Chordwise strip element on the right wing panel.

since the dihedral angle Γ is usually small. The velocity component v' gives rise to a small increment in incidence α' as shown, where

$$\alpha' \cong \tan\alpha' = \frac{v'}{V_0} = \frac{v\Gamma}{V_0} \tag{13.84}$$

Consider the lift due to the increment in incidence on the chordwise strip element on the right wing panel as shown in Fig. 13.6. The strip is at spanwise coordinate y measured from the ox axis, and has elemental width dy and local chord c_y. The lift increment on the strip resolves into a normal force increment δZ given by

$$\delta Z_{right} = -\frac{1}{2}\rho V_0^2 c_y dy a_y \alpha' \cos\Gamma \cong -\frac{1}{2}\rho V_0 c_y a_y v \Gamma dy \tag{13.85}$$

where a_y is the local lift curve slope. The corresponding increment in rolling moment δL is given by

$$\delta L_{right} = \delta Z_{right} y = -\frac{1}{2}\rho V_0 c_y a_y v \Gamma y dy \tag{13.86}$$

The total rolling moment due to the right wing panel may be obtained by integrating equation (13.86) from the root to the tip, whence

$$L_{right} = -\frac{1}{2}\rho V_0 v \int_0^s c_y a_y \Gamma y dy \tag{13.87}$$

Similarly for the left hand wing panel,

$$\delta L_{left} = -\delta Z_{left} y = -y\left(\frac{1}{2}\rho V_0 c_y a_y v \Gamma dy\right) \tag{13.88}$$

Note that the sign of the normal force increment is reversed on the left wing panel since the incidence is in fact a decrement and that the sign of the moment arm is also reversed. Thus

$$L_{left} = -\frac{1}{2}\rho V_0 v \int_0^s c_y a_y \Gamma y dy \tag{13.89}$$

By definition the total rolling moment in the sideslip disturbance is given by

$$v \mathring{L}_v = L_{right} + L_{left} = L_{total} = -\rho V_0 v \int_0^s c_y a_y \Gamma y dy \tag{13.90}$$

Whence the contribution to the dimensional derivative due to dihedral is

$$\mathring{L}_{v(dihedral)} = -\rho V_0 \int_0^s c_y a_y \Gamma y dy \tag{13.91}$$

and with reference to Appendix 2, the dimensionless form of the contribution is given by

$$L_{v(dihedral)} = \frac{\mathring{L}_{v(dihedral)}}{\frac{1}{2}\rho V_0 S b} = -\frac{1}{Ss}\int_0^s c_y a_y \Gamma y dy \tag{13.92}$$

where $b = 2s$ is the wing span. It is clear that for a wing with dihedral the expression given by equation (13.92) will always be negative and hence stabilising. On the other hand, a wing with anhedral will be destabilising.

Wing sweep also makes a significant contribution to L_v. The lift on a yawed wing is determined by the component of velocity normal to the quarter-chord line in subsonic flight and normal to the leading edge in supersonic flight. A swept wing is therefore treated as a yawed wing. With reference to Fig. 13.7, consider an elemental chordwise strip on the right wing panel which is

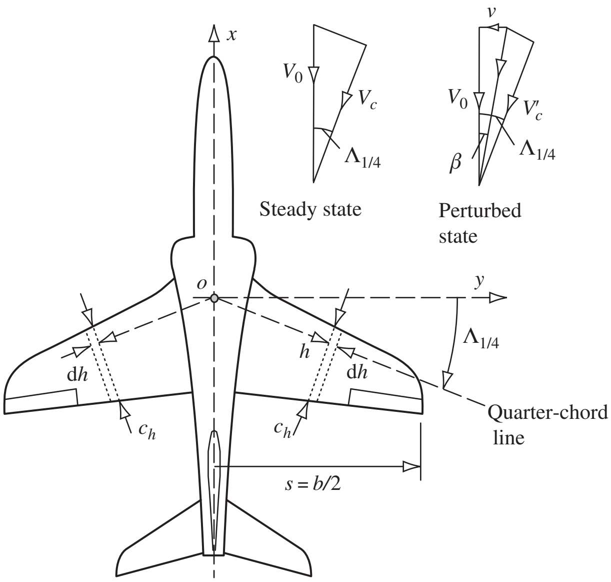

FIGURE 13.7 Swept wing in sideslip.

perpendicular to the quarter-chord line. Subsonic flow conditions are therefore assumed and the flow direction is parallel to the chord line. The strip element is at spanwise distance h from the ox axis, measured along the quarter-chord line, the local chord is c_h, and the width of the strip is dh. In the steady equilibrium flight condition the chordwise component of velocity is given by

$$V_c = V_0 \cos \Lambda_{1/4} \tag{13.93}$$

and in the presence of a positive sideslip disturbance this becomes

$$V'_c = \frac{V_0}{\cos\beta}\cos(\Lambda_{1/4} - \beta) \cong V_0 \cos(\Lambda_{1/4} - \beta) \tag{13.94}$$

where β is the sideslip angle, which is small by definition. The increment in normal force δZ on the chordwise strip due to the sideslip disturbance arises from the difference in lift between the steady flight condition and the perturbed condition and is given by

$$\delta Z_{right} = -\left(\frac{1}{2}\rho V'_c 2 c_h dh a_h \alpha - \frac{1}{2}\rho V_c^2 c_h dh a_h \alpha\right) = -\frac{1}{2}\rho(V_c'^2 - V_c^2) c_h dh a_h \alpha \tag{13.95}$$

Substitute the velocity expressions, equations (13.93) and (13.94), into equation (13.95), rearrange, and make small angle approximations where appropriate to obtain

$$\delta Z_{right} = -\frac{1}{2}\rho V_0^2 (\beta^2 \sin^2 \Lambda_{1/4} + 2\beta \sin\Lambda_{1/4} \cos\Lambda_{1/4}) a_h \alpha c_h dh \tag{13.96}$$

Thus the resulting increment in rolling moment is

$$\begin{aligned}\delta L_{right} &= h\cos\Lambda_{1/4}\delta Z_{right}\\ &= -\frac{1}{2}\rho V_0^2\cos\Lambda_{1/4}(\beta^2\sin^2\Lambda_{1/4} + 2\beta\sin\Lambda_{1/4}\cos\Lambda_{1/4})a_h\alpha c_h hdh\end{aligned} \tag{13.97}$$

On the corresponding strip element on the left-hand wing panel the chordwise velocity in the sideslip disturbance is given by

$$V_c' = \frac{V_0}{\cos\beta}\cos(\Lambda_{1/4} + \beta) \cong V_0\cos(\Lambda_{1/4} + \beta) \tag{13.98}$$

It therefore follows that the resulting increment in rolling moment arising from the left wing panel is

$$\begin{aligned}\delta L_{left} &= -h\cos\Lambda_{1/4}\delta Z_{left}\\ &= \frac{1}{2}\rho V_0^2\cos\Lambda_{1/4}(\beta^2\sin^2\Lambda_{1/4} - 2\beta\sin\Lambda_{1/4}\cos\Lambda_{1/4})a_h\alpha c_h hdh\end{aligned} \tag{13.99}$$

The total increment in rolling moment is given by the sum of the right and left wing panel contributions, (13.97) and (13.99); substituting for β from equation (13.78), then

$$L_{total} = \delta L_{right} + \delta L_{left} = -2\rho V_0 v\sin\Lambda_{1/4}\cos^2\Lambda_{1/4}a_h\alpha c_h hdh \tag{13.100}$$

Thus the total rolling moment due to the sideslip disturbance is given by integrating equation (13.100) along the quarter-chord line from the root to the wing tip. By definition the total rolling moment due to sweep is given by

$$v\mathring{L}_{v(sweep)} = \int \delta L_{total} = -2\rho V_0 v\sin\Lambda_{1/4}\cos^2\Lambda_{1/4}\int_0^{s\sec\Lambda_{1/4}} a_h\alpha c_h hdh \tag{13.101}$$

or

$$\mathring{L}_{v(sweep)} = -2\rho V_0\sin\Lambda_{1/4}\cos^2\Lambda_{1/4}\int_0^{s\sec\Lambda_{1/4}} a_h\alpha c_h hdh \tag{13.102}$$

Now it is more convenient to express the geometric variables in equation (13.102) in terms of spanwise and chordwise parameters measured parallel to the oy and ox axes, respectively. The geometry of the wing determines that $c_y = c_h\cos\Lambda_{1/4}$, $dy = dh\cos\Lambda_{1/4}$, $y = h\cos\Lambda_{1/4}$, and the integral limit $s\sec\Lambda_{1/4}$ becomes s. Equation (13.102) may then be written as

$$\mathring{L}_{v(sweep)} = -2\rho V_0\tan\Lambda_{1/4}\int_0^s C_{L_y}c_y ydy \tag{13.103}$$

where $C_{L_y} = a_h\alpha$ is the local lift coefficient. However, in the interests of practicality the constant mean lift coefficient for the wing is often assumed and equation (13.103) then simplifies to

$$\mathring{L}_{v(sweep)} = -2\rho V_0 C_L\tan\Lambda_{1/4}\int_0^s c_y ydy \tag{13.104}$$

and with reference to Appendix 2, the dimensionless form of the contribution is given by

$$L_{v(sweep)} = \frac{\mathring{L}_{v(sweep)}}{\frac{1}{2}\rho V_0 Sb} = -\frac{2C_L \tan\Lambda_{1/4}}{Ss}\int_0^s c_y y dy \tag{13.105}$$

where $b = 2s$ is the wing span. Again, it is clear that for a wing with aft sweep the expression given by equation (13.105) will always be negative and hence stabilising. Thus wing sweep is equivalent to dihedral as a mechanism for improving lateral stability. On the other hand, a wing with forward sweep will be laterally destabilising.

The geometry of the wing and fuselage in combination may also make a significant contribution to dihedral effect since in a sideslip condition the lateral cross-flow in the vicinity of the wing root gives rise to differential lift, which, in turn, gives rise to rolling moment.

As shown in Fig. 13.8 in a positive sideslip perturbation the aeroplane "sees" the lateral sideslip velocity component approaching from the right, it being implied that the right wing starts to drop at the onset of the disturbance. The lateral flow around the fuselage is approximately as indicated, thereby giving rise to small perturbations in upwash and downwash in the vicinity of the wing root. As a consequence of the flow condition the high wing configuration experiences a transient increase in incidence at the right wing root and a corresponding decrease in incidence at the left wing root.

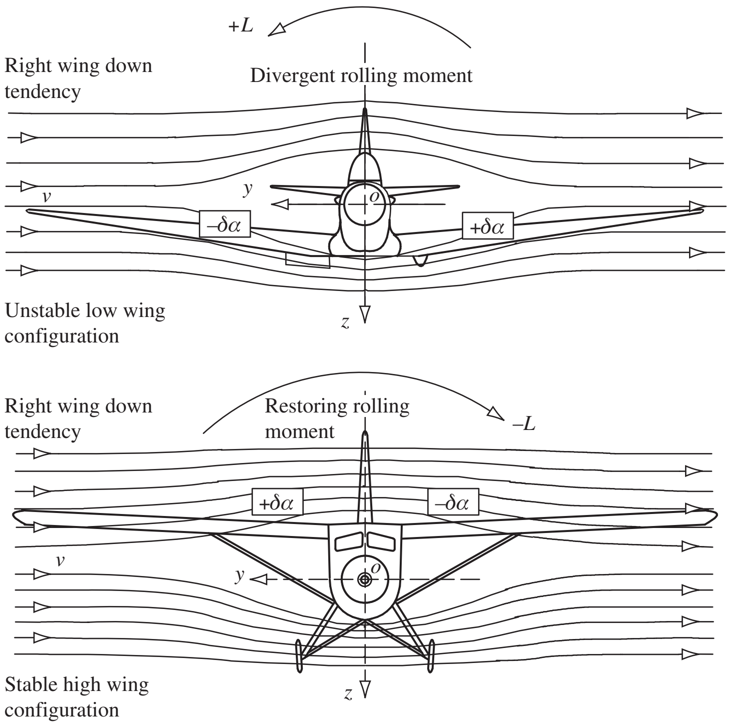

FIGURE 13.8 Lateral cross-flow in sideslip.

The differential lift thus created causes a negative rolling moment, and since this will tend to "pick up" the right wing the effect is stabilising. Clearly, as indicated, a low wing configuration behaves in the opposite manner and the rolling moment due to sideslip is definitely destabilising. Thus a high wing configuration enjoys an additional stabilising contribution to dihedral effect whereas a low wing configuration makes a destabilising contribution.

It is not generally possible to develop simple aerodynamic expressions to quantify the wing-fuselage geometry contribution to rolling moment due to sideslip. The aerodynamic phenomena involved are rather too complex to be modelled simply. It is known, for example, that the magnitude of the contribution is increased with an increase in fuselage width or depth and with an increase in aspect ratio. Reliable values for the contribution are best obtained by measurement or by reference to source documents such as ESDU data items.

The fin contribution to rolling moment due to sideslip arises from the way in which the lift developed on the fin in a sideslip perturbation acts on the airframe. The lift acts at the aerodynamic centre of the fin, which may be above or below the roll axis, thereby giving rise to a rolling moment. A typical situation is shown in Fig. 13.9.

The sideforce Y_F resulting from the lift developed by the fin in a sideslip perturbation is given by equation (13.79), and if the moment arm of the aerodynamic centre about the roll axis (ox axis) is denoted h_F, then in the perturbation by definition

$$v\mathring{L}_{v(fin)} = L = Y_F h_F = -\frac{1}{2}\rho V_0^2 S_F a_{1_F} \beta h_F \tag{13.106}$$

Substitute for β from equation (13.78) to obtain the following expression for the dimensional contribution to the derivative:

$$\mathring{L}_{v(fin)} = -\frac{1}{2}\rho V_0 S_F a_{1_F} h_F \tag{13.107}$$

With reference to Appendix 2, the dimensionless form of the contribution is given by

$$L_{v(fin)} = \frac{\mathring{L}_{v(fin)}}{\frac{1}{2}\rho V_0 S b} = -\frac{S_F h_F}{Sb} a_{1_F} = -\overline{V}_F \frac{h_F}{l_F} a_{1_F} \tag{13.108}$$

where the *fin volume ratio* is given by

$$\overline{V}_F = \frac{S_F l_F}{Sb} \tag{13.109}$$

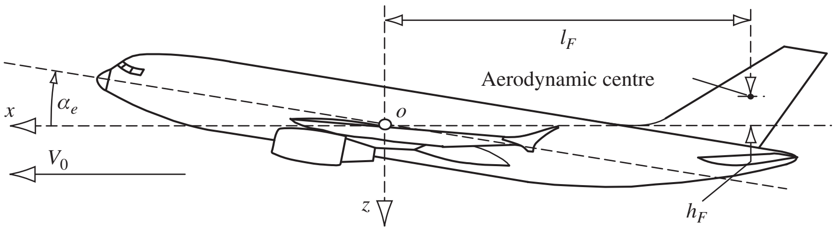

FIGURE 13.9 Rolling moment due to fin lift in sideslip.

When the aerodynamic centre of the fin is above the roll axis, h_F is positive and the expression given by equation (13.108) will be negative and hence stabilising. However, it is evident that, depending on aircraft geometry, h_F may be small and may even change sign at extreme aircraft attitude. Thus at certain flight conditions the contribution to rolling moment due to sideslip arising from the fin may become positive and hence laterally destabilising.

An estimate of the total value of the derivative L_v is obtained by summing the estimates of all the contributions for which a value can be obtained. Since the value of the derivative is usually small and negative, and hence stabilising, even small inaccuracies in the estimated values of the contributions can lead to a very misleading conclusion. Since the derivative is so important in the determination of the lateral stability and control characteristics of an aeroplane, the ESDU data items include a comprehensive procedure for estimating meaningful values of the significant contributions. Although, collectively, all the contributions probably embrace the most complex aerodynamics of all the derivatives, it is, fortunately, relatively easy to measure in both a wind tunnel test and in a flight test.

$\overset{\circ}{N}_v = \dfrac{\partial N}{\partial V}$ *Yawing moment due to sideslip*

The *weathercock*, or directional static stability, of an aircraft is determined by the yawing moment due to sideslip derivative. It quantifies the tendency of the aeroplane to turn *into wind* in the presence of a sideslip disturbance. Directional static stability is also discussed in greater detail in Section 3.5. In a sideslip disturbance the resulting lift increments arising from wing dihedral, wing sweep, wing-fuselage geometry, and so forth, as described previously, also give rise to associated increments in induced drag. The differential drag effects across the wing span give rise in turn to contributions to yawing moment due to sideslip. However, these contributions are often regarded as insignificant compared with that due to the fin, at least for preliminary estimates. Note that in practice the additional contributions may well be significant and that by ignoring them a degree of inaccuracy is implied in the derivative estimate.

With reference to Fig. 13.4 and Fig. 13.9, consider only the fin contribution which arises from the turning moment in yaw caused by the fin sideforce resulting from the sideslip. By definition this may be quantified as follows:

$$v\overset{\circ}{N}_{v(fin)} = -l_F Y_F = \frac{1}{2}\rho V_0^2 S_F a_{1_F} \beta l_F \tag{13.110}$$

where the fin sideforce due to sideslip is given by equation (13.79). Substitute for β from equation (13.78) to obtain the expression for the dimensional derivative:

$$\overset{\circ}{N}_{v(fin)} = \frac{1}{2}\rho V_0 S_F a_{1_F} l_F \tag{13.111}$$

With reference to Appendix 2, the dimensionless form of the derivative is given by

$$N_{v(fin)} = \frac{\overset{\circ}{N}_{v(fin)}}{\frac{1}{2}\rho V_0 S b} = \overline{V}_F a_{1_F} \tag{13.112}$$

Note that the sign of N_v is positive, which indicates that it is stabilising in effect. In a positive sideslip the incident wind vector is offset to the right of the nose (see Fig. 13.4), and the stabilising yawing moment due to sideslip results in a positive yaw response to turn the aircraft to the right until the aircraft aligns directionally with the wind vector. The yawing effect of the sideslip is thus nullified. The contribution from the wing due to differential drag effects is also usually stabilising and may well become the most significant contribution at high angles of attack since a large part of the fin may become immersed in the forebody wake, with the consequent reduction in its aerodynamic effectiveness. The contribution from the *lateral drag* effects on the gross side area ahead of and behind the *cg* may also be significant. However, it is commonly found that the yawing moment due to sideslip arising from the side area is often negative and hence destabilising. For certain classes of aircraft, such as large transport aeroplanes, this destabilising contribution can be very significant and requires a very large fin to ensure a reasonable degree of aerodynamic directional stability.

13.3.3 Derivatives due to rate of roll

As seen by the pilot, positive roll is to the right, consistent with a down-going right wing, and the small-perturbation roll rate transient is denoted p. The nature of a free positive roll rate disturbance is such that as the right wing tends to drop it is accompanied by a tendency for the nose to turn to the right and for the aeroplane to sideslip to the right. The reaction to the roll rate disturbance is stabilising if the aerodynamic forces and moments produced in response tend to restore the aeroplane to a wings-level zero sideslip equilibrium state.

$\overset{\circ}{Y}_p = \dfrac{\partial Y}{\partial p}$ *Sideforce due to roll rate*

The sideforce due to roll rate is usually considered to be negligible except for aircraft with a large high-aspect-ratio fin. Even then, the effect may well be small. Thus the fin contribution is assumed to be the only significant contribution to the derivative and may be estimated as follows.

With reference to Fig. 13.10, consider the chordwise strip element on the fin of width dh and at coordinate h measured upwards from the ox axis. When the aeroplane experiences a positive roll rate disturbance p, the strip element on the fin experiences a lateral velocity component ph. The resultant total velocity transient V is at incidence α' to the fin and, since the incidence transient is small by definition,

$$\alpha' \cong \tan\alpha' = \frac{ph}{V_0} \tag{13.113}$$

The incidence transient causes a fin lift transient, which resolves into a lateral force increment δY on the chordwise strip element and is given by

$$\delta Y = -\frac{1}{2}\rho V_0^2 c_h dh a_h \alpha' = -\frac{1}{2}\rho V_0 p a_h c_h h dh \tag{13.114}$$

where a_h is the local lift curve slope and c_h is the local chord. The total sideforce transient acting on the fin in the roll rate disturbance is given by integrating equation (13.114) from the root to the tip of the fin and, by definition,

$$p\overset{\circ}{Y}_{p(fin)} = Y_F = -\frac{1}{2}\rho V_0 p \int_0^{H_F} a_h c_h h dh \tag{13.115}$$

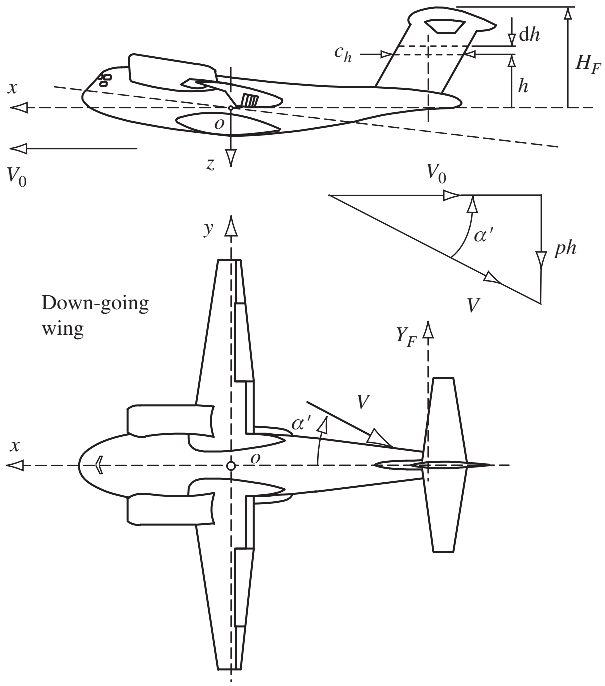

FIGURE 13.10 Fin sideforce generation in rolling flight.

where H_F is the fin span measured from the *ox* axis. The expression for the fin contribution to the dimensional derivative is therefore given by

$$\overset{\circ}{Y}_{p(fin)} = -\frac{1}{2}\rho V_0 \int_0^{H_F} a_h c_h h dh \tag{13.116}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$Y_{p(fin)} = \frac{\overset{\circ}{Y}_{p(fin)}}{\frac{1}{2}\rho V_0 S b} = -\frac{1}{Sb}\int_0^{H_F} a_h c_h h dh \tag{13.117}$$

$\overset{\circ}{L}_p = \dfrac{\partial L}{\partial p}$ Rolling moment due to roll rate

Rolling moment due to roll rate arises largely from the wing with smaller contributions from the fuselage, tailplane, and fin. This derivative is most important since it quantifies the damping in roll and is therefore significant in determining the dynamic characteristics of the roll subsidence mode, discussed in some detail in Section 7.2. The following analysis considers the wing contribution only.

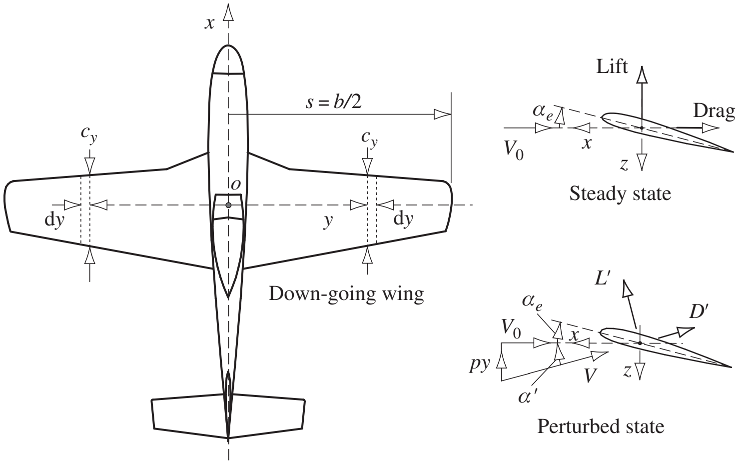

FIGURE 13.11 Wing incidence in rolling flight.

With reference to Fig. 13.11, when the right wing panel experiences a positive perturbation in roll rate p, assuming the aircraft rolls about the ox axis, then the small increase in incidence α' at the chordwise strip element is given by

$$\alpha' \cong \tan\alpha' = \frac{py}{V_0} \tag{13.118}$$

There is, of course, a reduction in incidence on the corresponding chordwise strip element on the left wing panel. Denoting the total lift and drag increments in the disturbance on the chordwise strip element on the right wing panel L' and D', respectively, then

$$L' = \frac{1}{2}\rho V_0^2 c_y dy a_y(\alpha_e + \alpha') \tag{13.119}$$

and

$$D' = \frac{1}{2}\rho V_0^2 c_y dy C_{D_y} \tag{13.120}$$

The normal force increment $\delta Z_{(right)}$ acting at the chordwise strip element in the roll rate perturbation is given by

$$\delta Z_{(right)} = -L'\cos\alpha' - D'\sin\alpha' \cong -L' - D'\alpha' \tag{13.121}$$

since α' is a small angle. Substitute for L', D', and α' from equations (13.119), (13.120), and (13.118), respectively, to obtain

$$\delta Z_{(right)} = -\frac{1}{2}\rho V_0^2\left(a_y\alpha_e + (a_y + C_{D_y})\frac{py}{V_0}\right)c_y dy \tag{13.122}$$

The resulting increment in rolling moment is then given by

$$\delta L_{(right)} = y\delta Z_{(right)} = -\frac{1}{2}\rho V_0^2\left(a_y\alpha_e + (a_y + C_{D_y})\frac{py}{V_0}\right)c_y y dy \tag{13.123}$$

and the corresponding increment in rolling moment arising from the left wing panel, where the incidence is reduced by α' since the panel is rising with respect to the incident air flow, is given by

$$\delta L_{(left)} = -y\delta Z_{(left)} = \frac{1}{2}\rho V_0^2\left(a_y\alpha_e - (a_y + C_{D_y})\frac{py}{V_0}\right)c_y y dy \tag{13.124}$$

The total rolling moment due to roll rate is obtained by summing the increments from the right and left chordwise strips, given by equations (13.123) and (13.124), respectively, and integrating from the root to the tip of the wing,

$$L_{total} = \int_{span}\left(\delta L_{(left)} + \delta L_{(right)}\right) = -\rho V_0 p\int_0^s (a_y + C_{D_y})c_y y^2 dy \tag{13.125}$$

and, by definition,

$$p\overset{\circ}{L}_p = L_{total} = -\rho V_0 p\int_0^s (a_y + C_{D_y})c_y y^2 dy \tag{13.126}$$

Whence the dimensional derivative expression is given by

$$\overset{\circ}{L}_p = -\rho V_0\int_0^s (a_y + C_{D_y})c_y y^2 dy \tag{13.127}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$L_p = \frac{\overset{\circ}{L}_p}{\frac{1}{2}\rho V_0 S b^2} = -\frac{1}{2Ss^2}\int_0^s (a_y + C_{D_y})c_y y^2 dy \tag{13.128}$$

where $b = 2s$ is the wing span.

$\overset{\circ}{N}_p = \dfrac{\partial N}{\partial p}$ *Yawing moment due to roll rate*

Yawing moment due to roll rate is almost entirely determined by the wing contribution, although in some aircraft a large fin may give rise to a significant additional contribution. Only the wing contribution is considered here.

It is shown in Fig. 13.11 that in a roll rate perturbation the chordwise strip element on the right (down-going) wing experiences an incremental increase in lift and induced drag, given by equations (13.119) and (13.120), whilst there is an equal decrease in lift and induced drag on the corresponding strip on the left (up-going) wing. The differential drag thereby produced gives rise to the yawing moment perturbation.

With reference to Fig. 13.11, the longitudinal axial force increment acting on the chordwise strip element on the right wing panel is given by

$$\delta X_{(right)} = L'\sin\alpha' - D'\cos\alpha' \cong L'\alpha' - D' \tag{13.129}$$

Substitute for L' and D' from equations (13.119) and (13.120), respectively, and write

$$C_{D_y} = \frac{dC_D}{d\alpha_y}(\alpha_e + \alpha') \tag{13.130}$$

to obtain

$$\delta X_{(right)} = \frac{1}{2}\rho V_0^2\left(a_y\alpha' - \frac{dC_D}{d\alpha_y}\right)(\alpha_e + \alpha')c_y dy \tag{13.131}$$

The incremental axial force gives rise to a negative increment in yawing moment given by

$$\delta N_{(right)} = -y\delta X_{(right)} = -\frac{1}{2}\rho V_0^2\left(a_y\alpha' - \frac{dC_D}{d\alpha_y}\right)(\alpha_e + \alpha')c_y y dy \tag{13.132}$$

The reduction in incidence due to roll rate on the corresponding chordwise strip element on the left wing panel gives rise to a positive increment in yawing moment and, in a similar way, it may be shown that

$$\delta N_{(left)} = y\delta X_{(left)} = -\frac{1}{2}\rho V_0^2\left(a_y\alpha' + \frac{dC_D}{d\alpha_y}\right)(\alpha_e - \alpha')c_y y dy \tag{13.133}$$

The total yawing moment increment due to roll rate is given by summing equations (13.132) and (13.135) and substituting for α' from equation (13.118):

$$\delta N_{total} = \delta N_{(left)} + \delta N_{(right)} = -\rho V_0 p\left(a_y\alpha_e - \frac{dC_D}{d\alpha_y}\right)c_y y^2 dy \tag{13.134}$$

By definition, the total yawing moment due to roll rate is given by

$$p\,\overset{\circ}{N}_p = N_{total} = \int_{\substack{semi\\span}} \delta N_{total} = -\rho V_0 p\int_0^s \left(a_y\alpha_e - \frac{dC_D}{d\alpha_y}\right)c_y y^2 dy \tag{13.135}$$

Whence the expression for the dimensional derivative is given by

$$\overset{\circ}{N}_p = -\rho V_0\int_0^s \left(C_{L_y} - \frac{dC_D}{d\alpha_y}\right)c_y y^2 dy \tag{13.136}$$

where $C_{L_y} = a_y\alpha_e$ is the equilibrium local lift coefficient. With reference to Appendix 2, the dimensionless form of the derivative is given by

$$N_p = \frac{\overset{\circ}{N}_p}{\frac{1}{2}\rho V_0 S b^2} = -\frac{1}{2Ss^2}\int_0^s \left(C_{L_y} - \frac{dC_D}{d\alpha_y}\right)c_y y^2 dy \tag{13.137}$$

13.3.4 Derivatives due to rate of yaw

As seen by the pilot, a positive yaw rate is such that the nose of the aeroplane swings to the right and the small-perturbation yaw rate transient is denoted r. The nature of a free positive yaw rate disturbance is such that as the nose swings to the right the right wing tends to drop and the aeroplane sideslips to the right. The reaction to the yaw rate disturbance is stabilising if the aerodynamic forces and moments produced in response tend to restore the aeroplane to a symmetric wings-level equilibrium flight condition.

$\overset{\circ}{Y}_r = \dfrac{\partial Y}{\partial r}$ *Sideforce due to yaw rate*

For most conventional aeroplanes the sideforce due to yaw rate is insignificant unless the fin is relatively large. In such cases the fin lift generated by the yawing motion gives rise to a sideforce of significant magnitude.

Referring to Fig. 13.12, in a yaw rate perturbation the transient incidence of the fin may be written as

$$\alpha' \cong \tan\alpha' = \frac{rl_F}{V_0} \tag{13.138}$$

where l_F is the moment arm of the fin aerodynamic centre about the centre of rotation in yaw, the cg, and, by definition, the incidence transient is a small angle. The resultant transient fin lift L'_F gives rise to a sideforce Y_F,

$$Y_F = L'_F \cos\alpha' \cong \frac{1}{2}\rho V_0^2 S_F a_{1_F}\alpha' = \frac{1}{2}\rho V_0 S_F l_F a_{1_F} r \tag{13.139}$$

By definition, the sideforce arising in a yaw rate disturbance is given by

$$r\overset{\circ}{Y}_r = Y_F = \frac{1}{2}\rho V_0 S_F l_F a_{1_F} r \tag{13.140}$$

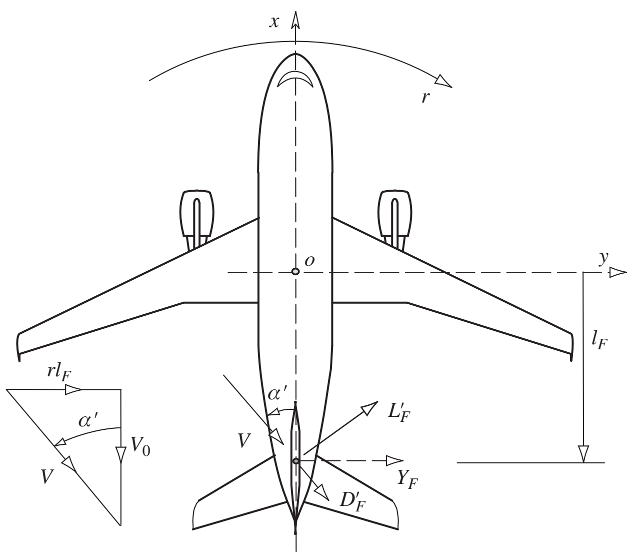

FIGURE 13.12 Fin incidence due to yaw rate.

Whence the expression for the dimensional sideforce due to yaw rate derivative is given by

$$\overset{\circ}{Y}_r = \frac{1}{2}\rho V_0 S_F l_F a_{1_F} \tag{13.141}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$Y_r = \frac{\overset{\circ}{Y}_r}{\frac{1}{2}\rho V_0 S b} = \overline{V}_F a_{1_F} \tag{13.142}$$

where the *fin volume ratio* $\overline{V}_F$ is given by equation (13.109). Clearly, the resolved component of the induced drag transient on the fin D'_F will also make a contribution to the total sideforce transient. However, this is usually considered to be insignificantly small compared with the lift contribution.

$\overset{\circ}{L}_r = \dfrac{\partial L}{\partial r}$ *Rolling moment due to yaw rate*

In positive yawing motion the relative velocity of the air flowing over the right wing panel is decreased whilst the velocity over the left wing panel is increased. This gives rise to an increase in lift and induced drag on the port wing with a corresponding decrease in lift and drag on the starboard wing. The force increments thus produced result in a rolling moment and a yawing moment about the *cg*. A contribution to rolling moment also arises due to the sideforce generated by the fin in yawing motion, although it is generally smaller than the wing contribution.

With reference to Fig. 13.13, the velocity at the chordwise strip element on the right wing during a yaw rate perturbation is given by

$$V = V_0 - ry \tag{13.143}$$

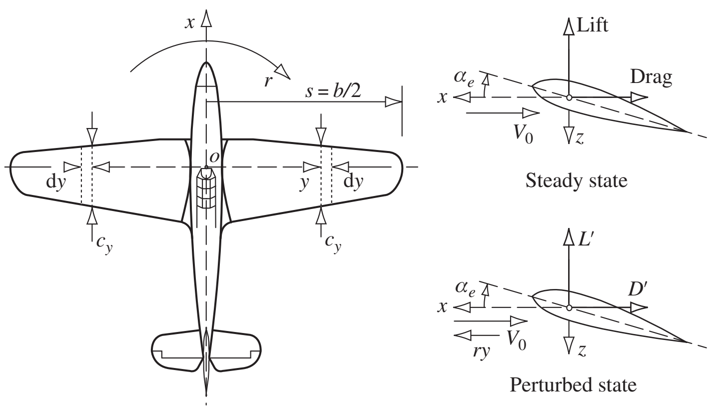

FIGURE 13.13 Wing forces due to yaw rate.

and the total lift on the chordwise strip element during the perturbation is given by

$$\begin{aligned}\delta L'_{(right)} &= \frac{1}{2}\rho V^2 c_y dy C_{L_y} = \frac{1}{2}\rho(V_0 - ry)^2 c_y dy C_{L_y} \\ &= \frac{1}{2}\rho(V_0^2 - 2ryV_0)c_y dy C_{L_y}\end{aligned} \tag{13.144}$$

when products of small quantities are neglected. The rolling moment due to the lift on the chordwise strip element on the right wing is therefore given by

$$\delta L_{(right)} = -\delta L'_{(right)} y = -\frac{1}{2}\rho(V_0^2 - 2ryV_0)c_y y dy C_{L_y} \tag{13.145}$$

Similarly, the rolling moment due to the lift on the chordwise strip element on the left wing is given by

$$\delta L_{(left)} = \delta L'_{(left)} y = \frac{1}{2}\rho(V_0^2 + 2ryV_0)c_y y dy C_{L_y} \tag{13.146}$$

Thus the total rolling moment due to yaw rate arising from the wing is given by integrating the sum of the components due to the chordwise strip elements, equations (13.145) and (13.146), over the semi-span:

$$L_{wing} = \int_{\substack{semi \\ span}} \left(\delta L_{(left)} + \delta L_{(right)}\right) = 2\rho V_0 r \int_0^s C_{L_y} c_y y^2 dy \tag{13.147}$$

By definition, the rolling moment due to wing lift in a yaw rate disturbance is given by

$$r\overset{\circ}{L}_{r(wing)} = L_{wing} = 2\rho V_0 r \int_0^s C_{L_y} c_y y^2 dy \tag{13.148}$$

Whence the expression for the wing contribution to the dimensional rolling moment due to yaw rate derivative is

$$\overset{\circ}{L}_{r(wing)} = 2\rho V_0 \int_0^s C_{L_y} c_y y^2 dy \tag{13.149}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$L_{r(wing)} = \frac{\overset{\circ}{L}_{r(wing)}}{\frac{1}{2}\rho V_0 S b^2} = \frac{1}{Ss^2}\int_0^s C_{L_y} c_y y^2 dy \tag{13.150}$$

where $b = 2s$ is the wing span.

Note that for a large-aspect-ratio rectangular wing it may be assumed that $C_{L_y} = C_L$, the lift coefficient for the whole wing, and that $c_y = c$, the constant geometric chord of the wing. For this special case it is easily shown, from equation (13.150), that

$$L_r = \frac{1}{6}C_L \tag{13.151}$$

However, it should be appreciated that the assumption relating to constant lift coefficient across the span is rather crude, and consequently the result given by equation (13.151) is very approximate, although it can be useful as a guide for checking estimated values of the derivative.

The fin contribution to the rolling moment due to yaw rate derivative arises from the moment about the roll axis of the sideforce generated by the fin in yaw. The sideforce is generated by the mechanism illustrated in Fig. 13.12 and acts at the aerodynamic centre of the fin, which is usually above the roll axis, and hence gives rise to a positive rolling moment. The situation prevailing is illustrated in Fig. 13.14.

With reference to Fig. 13.14, a rolling moment is developed by the fin sideforce due to yaw rate Y_F, which is given by equation (13.139), acting at the aerodynamic centre, which is located h_F above the roll axis. Whence the rolling moment is given by

$$L_{fin} = Y_F h_F = \frac{1}{2}\rho V_0 S_F l_F a_{1_F} r h_F \tag{13.152}$$

By definition, the rolling moment due to fin sideforce in a yaw rate disturbance is given by

$$r\overset{\circ}{L}_{r(fin)} = L_{fin} = \frac{1}{2}\rho V_0 S_F l_F a_{1_F} r h_F \tag{13.153}$$

Whence the fin contribution to the dimensional derivative is given by

$$\overset{\circ}{L}_{r(fin)} = \frac{1}{2}\rho V_0 S_F l_F a_{1_F} h_F \tag{13.154}$$

As before, and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$L_{r(fin)} = \frac{\overset{\circ}{L}_{r(fin)}}{\frac{1}{2}\rho V_0 S b^2} = a_{1_F}\overline{V}_F \frac{h_F}{b} \equiv -L_{v(fin)}\frac{l_F}{b} \tag{13.155}$$

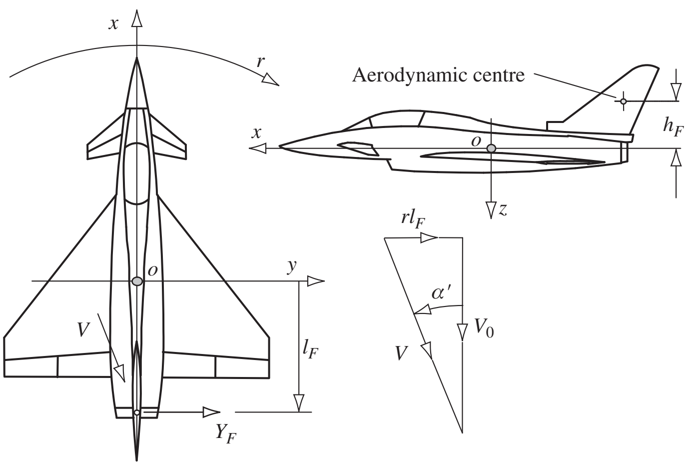

FIGURE 13.14 Rolling moment due to yaw rate arising from fin.

The total value of the rolling moment due to yaw rate derivative is then given by the sum of all the significant contributions.

$\mathring{N}_r = \dfrac{\partial N}{\partial r}$ *Yawing moment due to yaw rate*

The yawing moment due to yaw rate derivative is an important parameter in the determination of aircraft directional stability. In particular it is a measure of the damping in yaw and is therefore dominant in determining the stability of the oscillatory dutch roll mode. This significance of this derivative to lateral-directional dynamics is discussed in detail in Section 7.2. The most easily identified contributions to yaw damping arise from the fin and from the wing. However, it is generally accepted that the most significant contribution arises from the fin, although in some aircraft the fin contribution may become significantly reduced at high angles of attack, in which case the wing contribution becomes more important.

Consider the wing contribution first. This arises as a result of the differential drag effect in yawing motion as illustrated in Fig. 13.13. Referring to Fig. 13.13, the total drag on the chordwise strip element on the right wing subject to a steady yaw rate r is reduced for the same reason as the lift, given by equation (13.144), and may be written as

$$\delta D'_{(right)} = \frac{1}{2}\rho(V_0^2 - 2ryV_0)c_y dy C_{D_y} \tag{13.156}$$

The yawing moment about the cg generated by the drag on the chordwise strip element is

$$\delta N_{(right)} = \delta D'_{(right)} y = \frac{1}{2}\rho(V_0^2 - 2ryV_0)c_y y dy C_{D_y} \tag{13.157}$$

and similarly for the yawing moment arising at the corresponding chordwise strip on the left wing:

$$\delta N_{(left)} = -\delta D'_{(left)} y = -\frac{1}{2}\rho(V_0^2 + 2ryV_0)c_y y dy C_{D_y} \tag{13.158}$$

Thus the total yawing moment due to yaw rate arising from the wing is given by integrating the sum of the components due to the chordwise strip elements, equations (13.157) and (13.158), over the semi-span:

$$N_{wing} = \int_{\substack{semi\\span}} \left(\delta N_{(right)} - \delta N_{(left)}\right) = -2\rho V_0 r \int_0^s C_{D_y} c_y y^2 dy \tag{13.159}$$

By definition, the yawing moment due to differential wing drag in a yaw rate perturbation is given by

$$r\mathring{N}_{r(wing)} = N_{wing} = -2\rho V_0 r \int_0^s C_{D_y} c_y y^2 dy \tag{13.160}$$

Whence the expression for the wing contribution to the dimensional yawing moment due to yaw rate derivative is

$$\mathring{N}_{r(wing)} = -2\rho V_0 \int_0^s C_{D_y} c_y y^2 dy \tag{13.161}$$

and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$N_{r(wing)} = \frac{\overset{\circ}{N}_{r(wing)}}{\frac{1}{2}\rho V_0 S b^2} = -\frac{1}{Ss^2}\int_0^s C_{D_y} c_y y^2 dy \tag{13.162}$$

where $b = 2s$ is the wing span.

As for the derivative L_r, for a large-aspect-ratio rectangular wing it may be assumed that $C_{D_y} = C_D$, the drag coefficient for the whole wing, and that $c_y = c$, the constant geometric chord of the wing. For this special case it is easily shown, from equation (13.162), that

$$N_{r(wing)} = \frac{1}{6}C_D \tag{13.163}$$

Although the result given by equation (13.163) is rather approximate and subject to the assumptions made, it is useful as a guide for checking the value of an estimated contribution to the derivative.

The fin contribution to yawing moment due to yaw rate is generated by the yawing moment of the fin sideforce due to yaw rate. The mechanism for the generation of fin sideforce is illustrated in Fig. 13.12 and, with reference to that figure and to equation (13.139), the yawing moment thereby generated is given by

$$N_{fin} = -Y_F l_F = -\frac{1}{2}\rho V_0 S_F l_F^2 a_{1_F} r \tag{13.164}$$

By definition, the yawing moment due to the fin in a yaw rate perturbation is given by

$$r\overset{\circ}{N}_{r(fin)} = N_{fin} = -\frac{1}{2}\rho V_0 S_F l_F^2 a_{1_F} r \tag{13.165}$$

Whence the expression for the fin contribution to the dimensional yawing moment due to yaw rate derivative is

$$\overset{\circ}{N}_{r(fin)} = -\frac{1}{2}\rho V_0 S_F l_F^2 a_{1_F} \tag{13.166}$$

As before, and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$N_{r(fin)} = \frac{\overset{\circ}{N}_{r(fin)}}{\frac{1}{2}\rho V_0 S b^2} = -a_{1_F}\overline{V}_F\frac{l_F}{b} = -\frac{l_F}{b}N_{v(fin)} \tag{13.167}$$

The fin volume ratio $\overline{V}_F$ is given by equation (13.109). The total value of the yawing moment due to yaw rate derivative is therefore given by the sum of all the significant contributions.

13.4 Aerodynamic control derivatives

Estimates may be made for the aerodynamic control derivatives provided that the controller in question is a simple flap-like device and provided that its aerodynamic properties can be modelled with a reasonable degree of confidence. However, estimates of the aileron and rudder control derivatives obtained from simple models are unlikely to be accurate since it is very difficult to describe the aerodynamic conditions applying in sufficient detail. Estimates for the lateral-directional aerodynamic control derivatives are best obtained from the appropriate ESDU data items or, preferably, by experimental measurement. However, simple models for the aileron and rudder control derivatives are given here for completeness and in order to illustrate the principles of lateral-directional control.

For convenience, a summary of the derivative expressions derived in the following paragraphs is included in Table A8.3 and Table A8.4, Appendix 8.

13.4.1 Derivatives due to elevator

Typically, the lift coefficient for a tailplane with elevator control is given by

$$C_{L_T} = a_0 + a_1\alpha_T + a_2\eta \tag{13.168}$$

where a_1 is the lift curve slope of the tailplane and a_2 is the lift curve slope with respect to elevator angle η. The corresponding drag coefficient may be expressed as

$$C_{D_T} = C_{D_{0_T}} + k_T C_{L_T}^2 \tag{13.169}$$

where all of the parameters in equation (13.169) are tailplane-dependent.

$\overset{\circ}{X}_\eta = \dfrac{\partial X}{\partial \eta}$ *Axial force due to elevator*

It is assumed that for a small elevator deflection, consistent with a small perturbation, the resulting axial force perturbation arises from the drag change associated with the tailplane only. Whence

$$X \equiv X_T = -D_T = -\frac{1}{2}\rho V^2 S_T C_{D_T} \tag{13.170}$$

Thus

$$\overset{\circ}{X}_\eta = \frac{\partial X_T}{\partial \eta} = -\frac{1}{2}\rho V^2 S_T \frac{\partial C_{D_T}}{\partial \eta} \tag{13.171}$$

Substitute for C_{D_T}, from equation (13.169), in equation (13.171) to obtain

$$\overset{\circ}{X}_\eta = \frac{\partial X_T}{\partial \eta} = -\rho V^2 S_T k_T C_{L_T} \frac{\partial C_{L_T}}{\partial \eta} \tag{13.172}$$

For a small perturbation, in the limit $V \cong V_0$, from equation (13.168) $\partial C_{L_T}/\partial \eta \cong a_2$ and equation (13.172) may be written as

$$\overset{\circ}{X}_\eta = -\rho V_0^2 S_T k_T C_{L_T} a_2 \tag{13.173}$$

With reference to Appendix 2 the dimensionless form of the derivative is given by

$$X_\eta = \frac{\overset{\circ}{X}_\eta}{\frac{1}{2}\rho V_0^2 S} = -2\frac{S_T}{S} k_T C_{L_T} a_2 \tag{13.174}$$

$\overset{\circ}{Z}_\eta = \dfrac{\partial Z}{\partial \eta}$ *Normal force due to elevator*

As before, it is assumed that for a small elevator deflection the resulting normal force perturbation arises from the lift change associated with the tailplane only. Whence

$$Z \equiv Z_T = -L_T = -\frac{1}{2}\rho V^2 S_T C_{L_T} \tag{13.175}$$

Thus

$$\overset{\circ}{Z}_\eta = \frac{\partial Z_T}{\partial \eta} = -\frac{1}{2}\rho V^2 S_T \frac{\partial C_{L_T}}{\partial \eta} \tag{13.176}$$

Substitute for C_{L_T}, from equation (13.168), to obtain

$$\overset{\circ}{Z}_\eta = \frac{\partial Z_T}{\partial \eta} = -\frac{1}{2}\rho V^2 S_T a_2 \tag{13.177}$$

For a small perturbation, in the limit $V \cong V_0$ and with reference to Appendix 2, the dimensionless form of the derivative is given by

$$Z_\eta = \frac{\overset{\circ}{Z}_\eta}{\frac{1}{2}\rho V_0^2 S} = -\frac{S_T}{S} a_2 \tag{13.178}$$

$\overset{\circ}{M}_\eta = \dfrac{\partial M}{\partial \eta}$ *Pitching moment due to elevator*

It is assumed that the pitching moment resulting from elevator deflection is due entirely to the moment of the tailplane lift about the *cg*. Whence

$$M \equiv M_T = -L_T l_T = -\frac{1}{2}\rho V^2 S_T l_T C_{L_T} \tag{13.179}$$

Thus it follows that

$$\overset{\circ}{M}_\eta = \frac{\partial M_T}{\partial \eta} = -\frac{1}{2}\rho V^2 S_T l_T \frac{\partial C_{L_T}}{\partial \eta} = \overset{\circ}{Z}_\eta l_T \tag{13.180}$$

With reference to Appendix 2 the dimensionless form of the derivative is given by

$$M_\eta = \frac{\overset{\circ}{M}_\eta}{\frac{1}{2}\rho V_0^2 S \overline{\overline{c}}} = -\frac{S_T l_T}{S \overline{\overline{c}}} a_2 = -\overline{V}_T a_2 \tag{13.181}$$

where $\overline{V}_T$ is the tail volume ratio.

13.4.2 Derivatives due to aileron

Typical aileron geometry is shown in Fig. 13.15 and comprises a part span flap in the outboard sections of both port and starboard wings. Differential deflection of the flaps creates the desired

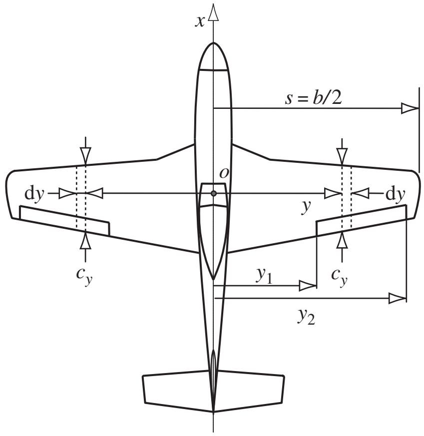

FIGURE 13.15 Aileron control geometry.

control in roll. As described in Section 2.6, a positive aileron deflection results in the starboard (right) surface deflecting trailing edge down and the port (left) surface trailing edge up, and aileron angle ξ is taken to be the mean of the two surface angles. Thus, referring to equation (13.168), the local lift coefficient at spanwise coordinate y is given by

$$
\begin{aligned}
C_{L_y}\big|_{right} &= a_0 + a_y\alpha + a_{2_A}\xi \\
C_{L_y}\big|_{left} &= a_0 + a_y\alpha - a_{2_A}\xi
\end{aligned}
\tag{13.182}
$$

Since it is not practical to define a simple model for the increment in local drag coefficient due to aileron deflection, let it be defined more generally as

$$
\begin{aligned}
C_{D_y}\big|_{right} &= \frac{\partial C_{D_y}}{\partial \xi}\xi \\
C_{D_y}\big|_{left} &= -\frac{\partial C_{D_y}}{\partial \xi}\xi
\end{aligned}
\tag{13.183}
$$

where it is assumed that, for small aileron angles, the change in drag $\partial C_{D_y}/\partial\xi$ is dominated by induced drag effects and may vary over the aileron span.

$\overset{\circ}{Y}_\xi = \dfrac{\partial Y}{\partial \xi}$ *Sideforce due to aileron*

For aeroplanes of conventional layout the sideforce due to aileron is zero or insignificantly small. However, for aeroplanes of unconventional layout, with highly swept wings or that utilise differential canard surfaces for roll control, this may not be the case. In such cases, simple analytical models would not be the most appropriate means for obtaining an estimate of the derivative value.

$\overset{\circ}{L}_\xi = \dfrac{\partial L}{\partial \xi}$ *Rolling moment due to aileron*

This derivative describes the roll control property of the aeroplane, and an accurate estimate of its value is important to flight dynamics analysis. With reference to equations (13.182) and Fig. 3.15, the application of simple strip theory enables the rolling moment due to starboard aileron deflection to be written as

$$L_{right} = -\frac{1}{2}\rho V^2 \int_{y_1}^{y_2} C_{L_y}\Big|_{right} c_y y dy = -\frac{1}{2}\rho V^2 \int_{y_1}^{y_2} (a_0 + a_y\alpha + a_{2_A}\xi) c_y y dy \tag{13.184}$$

where a_{2_A} is the aileron lift curve slope, which is assumed to be constant over the span of the aileron. Similarly, the rolling moment due to port aileron deflection may be written as

$$L_{left} = \frac{1}{2}\rho V^2 \int_{y_1}^{y_2} C_{L_y}\Big|_{left} c_y y dy = \frac{1}{2}\rho V^2 \int_{y_1}^{y_2} (a_0 + a_y\alpha - a_{2_A}\xi) c_y y dy \tag{13.185}$$

It follows that the total rolling moment may be written as

$$\overset{\circ}{L}_\xi \xi = L_{right} + L_{left} = -\rho V^2 \left(a_{2_A}\xi\right) \int_{y_1}^{y_2} c_y y dy \tag{13.186}$$

Whence the simple expression for the dimensional derivative is given by

$$\overset{\circ}{L}_\xi = -\rho V^2 a_{2_A} \int_{y_1}^{y_2} c_y y dy \tag{13.187}$$

Alternatively, with reference to Appendix 2, the dimensionless derivative may be written as

$$L_\xi = \frac{\overset{\circ}{L}_\xi}{\frac{1}{2}\rho V_0^2 S b} = -\frac{1}{Ss} a_{2_A} \int_{y_1}^{y_2} c_y y dy \tag{13.188}$$

where $b = 2s$ is the wing span.

$\overset{\circ}{N}_\xi = \dfrac{\partial N}{\partial \xi}$ *Yawing moment due to aileron*

This derivative describes the adverse yaw property of the aeroplane in response to aileron roll commands. With reference to equations (13.183) and Fig. 3.15, the application of simple strip theory enables the yawing moment due to starboard aileron deflection to be written as

$$N_{right} = \frac{1}{2}\rho V^2 \int_{y_1}^{y_2} C_{D_y}\Big|_{right} c_y y dy = \frac{1}{2}\rho V^2 \int_{y_1}^{y_2} \left(\frac{\partial C_{D_y}}{\partial \xi}\xi\right) c_y y dy \tag{13.189}$$

and, similarly, the yawing moment due to port aileron deflection may be written as

$$N_{left} = -\frac{1}{2}\rho V^2 \int_{y_1}^{y_2} C_{D_y}\Big|_{left} c_y y dy = -\frac{1}{2}\rho V^2 \int_{y_1}^{y_2} \left(-\frac{\partial C_{D_y}}{\partial \xi}\xi\right) c_y y dy \tag{13.190}$$

It follows that the total yawing moment may be written as

$$\overset{\circ}{N}_{\xi}\,\xi = N_{right} + N_{left} = \rho V^2 \int_{y_1}^{y_2} \left(\frac{\partial C_{D_y}}{\partial \xi}\xi\right) c_y y dy \tag{13.191}$$

Whence a simple expression for the dimensional derivative is given by

$$\overset{\circ}{N}_{\xi} = \rho V^2 \int_{y_1}^{y_2} \left(\frac{\partial C_{D_y}}{\partial \xi}\right) c_y y dy \tag{13.192}$$

Alternatively, with reference to Appendix 2, the dimensionless derivative may be written as

$$N_{\xi} = \frac{\overset{\circ}{N}_{\xi}}{\frac{1}{2}\rho V_0^2 Sb} = \frac{1}{Ss}\int_{y_1}^{y_2} \left(\frac{\partial C_{D_y}}{\partial \xi}\right) c_y y dy \tag{13.193}$$

13.4.3 Derivatives due to rudder

In normal trimmed flight the fin and rudder generate zero sideforce. Deflection of the rudder ζ (in the notation a positive rudder angle is trailing edge to the left) generates a positive sideforce, which gives rise to both rolling and yawing moments. With reference to Fig. 13.14, for example, it is assumed that the sideforce acts at the aerodynamic centre of the fin, which is located a distance l_F aft of the cg and a distance h_F above the cg. Since the aerodynamics of the fin and rudder will inevitably be significantly influenced by the presence of the aft fuselage and the horizontal tailplane, the accuracy of the following models is likely to be poor.

$\overset{\circ}{Y}_{\zeta} = \dfrac{\partial Y}{\partial \zeta}$ ***Sideforce due to rudder***

The sideforce generated by the fin when the rudder angle is ζ is given approximately by

$$Y = \frac{1}{2}\rho V^2 S_F a_{2_R}\zeta \tag{13.194}$$

and, by definition,

$$\overset{\circ}{Y}_{\zeta}\,\zeta = Y = \frac{1}{2}\rho V^2 S_F a_{2_R}\zeta \tag{13.195}$$

where a_{2_R} is the rudder lift curve slope, which is assumed to be constant over the span of the fin and rudder. Whence the very simple expression for the dimensional derivative is

$$\overset{\circ}{Y}_{\zeta} = \frac{1}{2}\rho V^2 S_F a_{2_R} \tag{13.196}$$

Alternatively, with reference to Appendix 2, the dimensionless derivative may be written as

$$Y_{\zeta} = \frac{\overset{\circ}{Y}_{\zeta}}{\frac{1}{2}\rho V_0^2 S} = \frac{S_F}{S}a_{2_R} \tag{13.197}$$

$\overset{\circ}{L}_\zeta = \dfrac{\partial L}{\partial \zeta}$ ***Rolling moment due to rudder***

This derivative describes the adverse roll property of the aeroplane in response to rudder yaw commands. Since the sideforce due to rudder acts above the roll axis, the rolling moment due to rudder follows directly:

$$\overset{\circ}{L}_\zeta \zeta = Yh_F = \frac{1}{2}\rho V^2 S_F h_F a_{2_R} \zeta \tag{13.198}$$

Thus a simple expression for the dimensional derivative is

$$\overset{\circ}{L}_\zeta = \frac{1}{2}\rho V^2 S_F h_F a_{2_R} \tag{13.199}$$

and, with reference to Appendix 2, an expression for the dimensionless derivative may be written as

$$L_\zeta = \frac{\overset{\circ}{L}_\zeta}{\frac{1}{2}\rho V_0^2 Sb} = \frac{S_F h_F}{Sb} a_{2_R} \equiv \overline{V}_F \frac{h_F}{l_F} a_{2_R} \tag{13.200}$$

where $\overline{V}_F$ is the fin volume ratio.

$\overset{\circ}{N}_\zeta = \dfrac{\partial N}{\partial \zeta}$ ***Yawing moment due to rudder***

This derivative describes the yaw control property of the aeroplane and, again, an accurate estimate of its value is important to flight dynamics analysis. Since the sideforce due to rudder acts well behind the cg, it generates a yawing moment described as follows:

$$\overset{\circ}{N}_\zeta \zeta = -Yl_F = -\frac{1}{2}\rho V^2 S_F l_F a_{2_R} \zeta \tag{13.201}$$

Thus the simple expression for the dimensional derivative is

$$\overset{\circ}{N}_\zeta = \frac{1}{2}\rho V^2 S_F l_F a_{2_R} \tag{13.202}$$

and, with reference to Appendix 2, an expression for the dimensionless derivative may be written as

$$N_\zeta = \frac{\overset{\circ}{N}_\zeta}{\frac{1}{2}\rho V_0^2 Sb} = -\frac{S_F l_F}{Sb} a_{2_R} \equiv -\overline{V}_F a_{2_R} \tag{13.203}$$

13.5 North American derivative coefficient notation

An alternative notation for the dimensionless aerodynamic stability and control derivatives, based on the derivatives of aerodynamic force and moment coefficients, is the standard notation used in

North America and commonly used in Europe and elsewhere. Interpretation of the derivatives as quasi-static representations of continuously varying aerodynamic properties of the aircraft remains the same as described in Section 12.2.

To illustrate the mathematical derivation of the coefficient notation, it is useful to remember that in a nonsteady flight condition, with perturbed velocity V, the lift and drag force are described in terms of the dimensionless lift and drag coefficients respectively, namely

$$L = \frac{1}{2}\rho V^2 S C_L$$

$$D = \frac{1}{2}\rho V^2 S C_D$$

In a similar way, the aerodynamic forces and moments in the American normalised dimensional equations of motion (4.72) and (4.73) may be written in terms of dimensionless coefficients. With reference to expressions (4.75), the normalised longitudinal equations of motion (4.72) may thus be written as

$$\begin{aligned}
\dot{u} + qW_e &= \frac{X}{m} = \frac{1}{m}\left(\frac{1}{2}\rho V^2 S C_x\right) - g\theta\cos\theta_e \\
\dot{w} - qU_e &= \frac{Z}{m} = \frac{1}{m}\left(\frac{1}{2}\rho V^2 S C_z\right) - g\theta\sin\theta_e \\
\dot{q} &= \frac{M}{I_y} = \frac{1}{I_y}\left(\frac{1}{2}\rho V^2 S\bar{\bar{c}} C_m\right)
\end{aligned} \tag{13.204}$$

and, with reference to expressions (4.78), the normalised lateral-directional equations of motion (4.73) may be written as

$$\begin{aligned}
\dot{v} - pW_e + rU_e &= \frac{Y}{m} = \frac{1}{m}\left(\frac{1}{2}\rho V^2 S C_y\right) + g\phi\cos\theta_e + g\psi\sin\theta_e \\
\dot{p} - \frac{I_{xz}}{I_x}\dot{r} &= \frac{L}{I_x} = \frac{1}{I_x}\left(\frac{1}{2}\rho V^2 S b C_l\right) \\
\dot{r} - \frac{I_{xz}}{I_z}\dot{p} &= \frac{N}{I_z} = \frac{1}{I_z}\left(\frac{1}{2}\rho V^2 S b C_n\right)
\end{aligned} \tag{13.205}$$

Note that, as written, equations (13.204) and (13.205) are referenced to a general aircraft body axis system. However, as in the British notation, it is usual (and preferable) to refer the dimensionless aerodynamic derivative coefficients to aircraft wind axes.

13.5.1 The longitudinal aerodynamic derivative coefficients

Consider the longitudinal equations of motion, and by comparing equations (4.77) with (13.204), in a perturbation the aerodynamic, thrust, and control forces and moments may be written as

$$\frac{1}{m}\left(\frac{1}{2}\rho V^2 S C_x\right) = \mathrm{X}_u u + \mathrm{X}_{\dot{w}}\dot{w} + \mathrm{X}_w w + \mathrm{X}_q q + \mathrm{X}_{\delta_e}\delta_e + \mathrm{X}_{\delta_{th}}\delta_{th}$$

$$\frac{1}{m}\left(\frac{1}{2}\rho V^2 S C_z\right) = \mathrm{Z}_u u + \mathrm{Z}_{\dot{w}}\dot{w} + \mathrm{Z}_w w + \mathrm{Z}_q q + \mathrm{Z}_{\delta_e}\delta_e + \mathrm{Z}_{\delta_{th}}\delta_{th} \tag{13.206}$$

$$\frac{1}{I_y}\left(\frac{1}{2}\rho V^2 S\bar{\bar{c}} C_m\right) = \mathrm{M}_u u + \mathrm{M}_{\dot{w}}\dot{w} + \mathrm{M}_w w + \mathrm{M}_q q + \mathrm{M}_{\delta_e}\delta_e + \mathrm{M}_{\delta_{th}}\delta_{th}$$

In order to identify the dimensionless derivative coefficients the left-hand sides of equations (13.206) may be expanded in terms of partial derivative functions of the perturbation variables. This mathematical procedure and its application to aerodynamic modelling of aircraft is described in Section 4.2.2. Thus, for example, the axial force equation in (13.204) may be written as

$$\frac{1}{m}\left(\frac{1}{2}\rho V^2 S C_x\right) = \frac{\rho S}{2m}\left(\begin{array}{l}\dfrac{\partial(V^2C_x)}{\partial U}u + \dfrac{\partial(V^2C_x)}{\partial \dot{W}}\dot{w} + \dfrac{\partial(V^2C_x)}{\partial W}w \\[2ex] + \dfrac{\partial(V^2C_x)}{\partial q}q + \dfrac{\partial(V^2C_x)}{\partial \delta_e}\delta_e + \dfrac{\partial(V^2C_x)}{\partial \delta_{th}}\delta_{th}\end{array}\right) \tag{13.207}$$

$$= \mathrm{X}_u u + \mathrm{X}_{\dot{w}}\dot{w} + \mathrm{X}_w w + \mathrm{X}_q q + X_{\delta_e}\delta_e + \mathrm{X}_{\delta_{th}}\delta_{th}$$

Equating equivalent terms in equation (13.207), expressions for the normalised derivatives, referred to aircraft wind axes, may thus be derived. Recall the following derivative relationships for small perturbations from equations (13.5) and (13.6):

$$\frac{\partial V}{\partial U} = 1 \quad \frac{\partial V}{\partial W} = 0 \quad \frac{\partial \theta}{\partial U} = 0 \quad \frac{\partial \theta}{\partial W} = \frac{1}{V} \tag{13.208}$$

Expressions for the force coefficients follow directly from equations (13.8) and (13.9), retaining the aerodynamic terms only:

$$C_x = C_L \sin\theta - C_D \cos\theta \tag{13.209}$$

$$C_z = -C_L \cos\theta - C_D \sin\theta \tag{13.210}$$

Referring to expressions (13.208) and equation (13.209), it follows that

$$
\begin{aligned}
\mathrm{X}_u &= \frac{\rho S}{2m}\frac{\partial\left(V^2 C_x\right)}{\partial U} = \frac{\rho VS}{2m}\left(V\frac{\partial C_x}{\partial U} + 2C_x\right) \\
&= \frac{\rho VS}{2m}\left(V\frac{\partial C_L}{\partial U}\sin\theta - V\frac{\partial C_D}{\partial U}\cos\theta + 2C_L\sin\theta - 2C_D\cos\theta\right)
\end{aligned}
\tag{13.211}
$$

In the limit, let the perturbation become vanishingly small such that $\theta \rightarrow 0$, $V \rightarrow V_0$, and Mach number $\mathrm{M}_0 = V_0/a$; then

$$
\mathrm{X}_u = -\frac{\rho VS}{2m}\left(V\frac{\partial C_D}{\partial V} + 2C_D\right) = -\frac{\rho VS}{2m}\left(\mathrm{M}\frac{\partial C_D}{\partial \mathrm{M}} + 2C_D\right) = -\frac{\rho V_0 S}{2m}\left(\mathrm{M}_0 C_{D_\mathrm{M}} + 2C_D\right) \tag{13.212}
$$

It should be noted that $\partial C_D/\partial V$ is not dimensionless and in order to render the derivative dimensionless, velocity dependency is replaced by Mach number dependency. This also has the advantage that the model is then not limited to subsonic flight applications only. Note that equation (13.212) is the direct equivalent of that given in the British notation as equation (12.9) and by equation (13.17).

Similarly, referring to the expressions (13.208) and equation (13.209), it follows that

$$
\begin{aligned}
\mathrm{X}_w &= \frac{\rho S}{2m}\frac{\partial\left(V^2 C_x\right)}{\partial W} = \frac{\rho VS}{2m}\left(V\frac{\partial C_x}{\partial W}\right) \\
&= \frac{\rho VS}{2m}\left(V\frac{\partial C_L}{\partial W}\sin\theta - V\frac{\partial C_D}{\partial W}\cos\theta + C_L\cos\theta - C_D\sin\theta\right) \\
&= \frac{\rho VS}{2m}\left(\frac{\partial C_L}{\partial \alpha}\sin\theta - \frac{\partial C_D}{\partial \alpha}\cos\theta + C_L\cos\theta - C_D\sin\theta\right)
\end{aligned}
\tag{13.213}
$$

Since a wind axes reference is assumed, $W = W_e + w = w$ and $w/V = \tan\alpha \cong \alpha$ for small perturbations. As before, let the perturbation become vanishingly small such that $\theta \rightarrow 0$ and $V \rightarrow V_0$; then

$$
\mathrm{X}_w = \frac{\rho V_0 S}{2m}\left(C_L - \frac{\partial C_D}{\partial \alpha}\right) = \frac{\rho V_0 S}{2m}\left(C_L - C_{D_\alpha}\right) \tag{13.214}
$$

Although the derivative $\mathrm{X}_{\dot{w}}$ is usually negligibly small, its derivation in terms of the dimensionless derivative coefficient is illustrated for completeness:

$$
\begin{aligned}
\mathrm{X}_{\dot{w}} &= \frac{\rho S}{2m}\frac{\partial\left(V^2 C_x\right)}{\partial \dot{W}} = \frac{\rho VS}{2m}\left(V\frac{\partial C_x}{\partial \dot{w}}\right) \equiv \frac{\rho VS}{2m}\left(\frac{\partial C_x}{\partial \dot{\alpha}}\right) = \frac{\rho VS}{2m}\left(\frac{\bar{\bar{c}}P}{2V}\right)\left(\frac{\partial C_x}{\partial\left(\dot{\alpha}\bar{\bar{c}}/2V\right)}\right) \\
\mathrm{X}_{\dot{w}} &= \frac{\rho S\bar{\bar{c}}}{4m}C_{x_{\dot{\alpha}}}
\end{aligned}
\tag{13.215}
$$

Since a wind axes reference is assumed, $W = W_e + w = w$ and $\dot{W} = \dot{w}$. Also, $w/V = \tan\alpha \cong \alpha$ for small perturbations, and it follows that $\dot{w}/V \cong \dot{\alpha}$. Now the derivative $\partial C_x/\partial\dot{\alpha}$ is not dimensionless as $\dot{\alpha}$ has units rad/s, and in order to render $\partial C_x/\partial\dot{\alpha}$ dimensionless it is necessary to multiply $\dot{\alpha}$ by a *longitudinal reference time* value. The value used for this purpose is $\bar{\bar{c}}/2V$, the time taken for the aircraft to traverse one half-chord length. It is not possible to take this reduction further without significant additional analysis to define the dependency of C_x on $\dot{\alpha}$. In the context of the derivation of the equivalent British notation, the analysis is set out in Section 13.2.6, where it is seen to describe tailplane aerodynamic response to downwash lag following a perturbation in normal acceleration $\dot{w}$.

The derivative and X_q are derived in a similar manner, and since C_{x_q} is negligibly small for small perturbations it is usual to omit X_q from the linear longitudinal aircraft model:

$$\begin{aligned}
\mathrm{X}_q &= \frac{\rho S}{2m}\frac{\partial\left(V^2 C_x\right)}{\partial q} = \frac{\rho V^2 S}{2m}\frac{\partial C_x}{\partial q} = \frac{\rho V^2 S}{2m}\left(\frac{\bar{\bar{c}}}{2V}\right)\left(\frac{\partial C_x}{\partial\left(q\bar{\bar{c}}/2V\right)}\right) \\
\mathrm{X}_q &= \frac{\rho V_0 S\bar{\bar{c}}}{4m}C_{x_q}
\end{aligned} \qquad (13.216)$$

Again, it is not practical to take this reduction further without significant additional analysis to define the dependency of C_x on pitch rate q. However, the analysis relating to the equivalent British notation is set out in Section 13.2.5 where it is seen to describe tailplane aerodynamic response to a perturbation in pitch rate q.

The control derivatives may be dealt with generically as the independent variable is control angle δ, with a subscript denoting the surface to which it relates, and since it is measured in radians it is treated as dimensionless. When δ is used to denote thrust control, it is also treated as dimensionless and may be considered as a fraction of maximum thrust or, equivalently, as a perturbation in throttle lever angle. Otherwise, the derivation of X_{δ_e} and $\mathrm{X}_{\delta_{th}}$ follows the same procedure:

$$\begin{aligned}
\mathrm{X}_\delta &= \frac{\rho S}{2m}\frac{\partial\left(V^2 C_x\right)}{\partial \delta} = \frac{\rho V^2 S}{2m}\frac{\partial C_x}{\partial \delta} \\
\mathrm{X}_\delta &= \frac{\rho V_0^2 S}{2m}C_{x_\delta}
\end{aligned} \qquad (13.217)$$

Again, it is not practical to take the derivation any further since it depends explicitly on the aerodynamic and thrust control layout of a given aeroplane.

The normalised axial force derivative expressions given by equations (13.212) through (13.217) are summarised in Appendix 7.

In a similar way, expressions for the normalised normal force derivatives in equations (13.206) may be derived in terms of dimensionless derivative coefficients, and noting that the normal force coefficient C_z is given by equation (13.210). The results of the derivations follow and are also summarised in Appendix 7:

$$
\begin{aligned}
\mathrm{Z}_u &= -\frac{\rho V_0 S}{2m}\left(\mathrm{M}_0 C_{L_\mathrm{M}} + 2C_L\right) \\
\mathrm{Z}_{\dot{w}} &= \frac{\rho S\bar{\bar{c}}}{4m} C_{z_{\dot{\alpha}}} \\
\mathrm{Z}_w &= -\frac{\rho V_0 S}{2m}\left(C_D + C_{L_\alpha}\right) \\
\mathrm{Z}_q &= \frac{\rho V_0 S\bar{\bar{c}}}{m} C_{z_q} \\
\mathrm{Z}_\delta &= \frac{\rho V_0^2 S}{2m} C_{z_\delta}
\end{aligned}
\tag{13.218}
$$

Since there are some small differences in the derivation of the moment derivatives, it is useful to review the normalised pitching moment derivative definitions. As before, the left-hand side of the pitching moment equation from equations (13.206) may be expanded in terms of partial derivative functions of the perturbation variables to give

$$
\frac{1}{I_y}\left(\frac{1}{2}\rho V^2 S\bar{\bar{c}} C_m\right) = \frac{\rho S\bar{\bar{c}}}{2I_y}\left(\begin{array}{l} \dfrac{\partial\left(V^2 C_m\right)}{\partial U}u + \dfrac{\partial\left(V^2 C_m\right)}{\partial \dot{W}}\dot{w} + \dfrac{\partial\left(V^2 C_m\right)}{\partial W}w \\ + \dfrac{\partial\left(V^2 C_m\right)}{\partial q}q + \dfrac{\partial\left(V^2 C_m\right)}{\partial \delta_e}\delta_e + \dfrac{\partial\left(V^2 C_m\right)}{\partial \delta_{th}}\delta_{th} \end{array}\right)
\tag{13.219}
$$

$$
= \mathrm{M}_u u + \mathrm{M}_{\dot{w}}\dot{w} + \mathrm{M}_w w + \mathrm{M}_q q + \mathrm{M}_{\delta_e}\delta_e + \mathrm{M}_{\delta_{th}}\delta_{th}
$$

Equating equivalent terms in equation (13.219), expressions for the normalised pitching moment derivatives, referred to aircraft wind axes, may be derived as follows:

$$
\begin{aligned}
\mathrm{M}_u &= \frac{\rho S\bar{\bar{c}}}{2I_y}\frac{\partial\left(V^2 C_m\right)}{\partial U} = \frac{\rho V S\bar{\bar{c}}}{2I_y}\left(V\frac{\partial C_m}{\partial V} + 2C_m\right) \\
&\equiv \frac{\rho V S\bar{\bar{c}}}{2I_y}\left(\mathrm{M}\frac{\partial C_m}{\partial \mathrm{M}} + 2C_m\right) = \frac{\rho V S\bar{\bar{c}}}{2I_y}\left(\mathrm{M}C_{m_\mathrm{M}} + 2C_m\right)
\end{aligned}
\tag{13.220}
$$

Again, remember that for a wind axes reference $U = U_e + u \equiv V$ and Mach number $\mathrm{M} = V/a$. To define the derivative M_u at the flight condition of interest, let the perturbation become vanishingly small such that $u \to 0$, hence $V \to V_0$ and $\mathrm{M} \to \mathrm{M}_0$. Further, since the perturbation is a disturbance about trim, and when the aircraft is in trim the pitching moment is zero, as the perturbation becomes vanishingly small $C_m \to 0$. Whence

$$\mathrm{M}_u = \frac{\rho V_0 S \bar{\bar{c}}}{2I_y}\left(\mathrm{M}_0 C_{m_{\mathrm{M}}}\right) \tag{13.221}$$

The remaining normalised pitching moment derivative expressions may be derived in a similar manner; for example,

$$\begin{aligned}
\mathrm{M}_{\dot{w}} &= \frac{\rho S \bar{\bar{c}}}{2I_y}\frac{\partial\left(V^2 C_m\right)}{\partial \dot{W}} = \frac{\rho V^2 S \bar{\bar{c}}}{2I_y}\frac{\partial C_m}{\partial \dot{w}} = \frac{\rho V S \bar{\bar{c}}}{2I_y}\frac{\partial C_m}{\partial \dot{\alpha}} = \frac{\rho V S \bar{\bar{c}}}{2I_y}\left(\frac{\bar{\bar{c}}}{2V}\right)\frac{\partial C_m}{\partial(\dot{\alpha}\bar{\bar{c}}/2V)} \\
\mathrm{M}_{\dot{w}} &= \frac{\rho S \bar{\bar{c}}^2}{4I_y} C_{m_{\dot{\alpha}}}
\end{aligned} \tag{13.222}$$

and so on. Whence

$$\begin{aligned}
\mathrm{M}_w &= \frac{\rho V_0 S \bar{\bar{c}}}{2I_y} C_{m_\alpha} \\
\mathrm{M}_q &= \frac{\rho V_0 S \bar{\bar{c}}^2}{4I_y} C_{m_q} \\
\mathrm{M}_\delta &= \frac{\rho V_0^2 S \bar{\bar{c}}}{2I_y} C_{m_\delta}
\end{aligned} \tag{13.223}$$

13.5.2 The lateral-directional aerodynamic derivative coefficients

Similarly, now consider the lateral-directional equations of motion; by comparing equations (4.78), (4.79), and (4.80) with (13.183), in a perturbation the aerodynamic and control forces and moments may be written as

$$\begin{aligned}
\frac{1}{m}\left(\frac{1}{2}\rho V^2 S C_y\right) &= \mathrm{Y}_v v + \mathrm{Y}_p p + \mathrm{Y}_r r + \mathrm{Y}_{\delta_\xi}\delta_\xi + \mathrm{Y}_{\delta_\zeta}\delta_\zeta \\
\frac{1}{I_x}\left(\frac{1}{2}\rho V^2 S b C_l\right) &= \mathrm{L}_v v + \mathrm{L}_p p + \mathrm{L}_r r + \mathrm{L}_{\delta_\xi}\delta_\xi + \mathrm{L}_{\delta_\zeta}\delta_\zeta \\
\frac{1}{I_z}\left(\frac{1}{2}\rho V^2 S b C_n\right) &= \mathrm{N}_v v + \mathrm{N}_p p + \mathrm{N}_r r + \mathrm{N}_{\delta_\xi}\delta_\xi + \mathrm{N}_{\delta_\zeta}\delta_\zeta
\end{aligned} \tag{13.224}$$

As before, the left-hand sides of equations (13.224) may be expanded in terms of partial derivative functions of the perturbation variables. Thus, for example, the lateral sideforce equation in (13.224) may be written as

$$\frac{1}{m}\left(\frac{1}{2}\rho V^2 SC_y\right) = \frac{\rho V^2 S}{2m}\left(\frac{\partial C_y}{\partial v}v + \frac{\partial C_y}{\partial p}p + \frac{\partial C_y}{\partial r}r + \frac{\partial C_y}{\partial \delta_\xi}\delta_\xi + \frac{\partial C_y}{\partial \delta_\zeta}\delta_\zeta\right)$$
$$= \mathrm{Y}_v v + \mathrm{Y}_p p + \mathrm{Y}_r r + \mathrm{Y}_{\delta_\xi}\delta_\xi + \mathrm{Y}_{\delta_\zeta}\delta_\zeta \tag{13.225}$$

and it is assumed that in a small lateral-directional perturbation, the motion is decoupled from longitudinal motion such that velocity V is independent of the perturbation variables. As before, equating terms in equation (13.225), expressions for the normalised derivatives referred to aircraft wind axes may be derived. For a wind axes reference $v/V = \tan\beta \cong \beta$, the small perturbation in sideslip angle. Whence

$$\mathrm{Y}_v = \frac{\rho V^2 S}{2m}\frac{\partial C_y}{\partial v} \equiv \frac{\rho VS}{2m}\frac{\partial C_y}{\partial \beta}$$
$$\mathrm{Y}_v = \frac{\rho V_0 S}{2m}C_{y_\beta} \tag{13.226}$$

or, equivalently,

$$\mathrm{Y}_\beta = \frac{\rho V_0^2 S}{2m}C_{y_\beta} \tag{13.227}$$

Although the derivatives Y_p and Y_r are usually insignificantly small, it is instructive to review their derivation in terms of dimensionless coefficients:

$$\mathrm{Y}_p = \frac{\rho V^2 S}{2m}\frac{\partial C_y}{\partial p} = \frac{\rho V^2 S}{2m}\left(\frac{b}{2V}\right)\frac{\partial C_y}{\partial (pb/2V)} = \frac{\rho VSb}{4m}C_{y_p}$$
$$\mathrm{Y}_r = \frac{\rho V^2 S}{2m}\frac{\partial C_y}{\partial r} = \frac{\rho V^2 S}{2m}\left(\frac{b}{2V}\right)\frac{\partial C_y}{\partial (rb/2V)} = \frac{\rho VSb}{4m}C_{y_r}$$

Since the derivatives $\partial C_p/\partial p$ and $\partial C_r/\partial r$ are not dimensionless, it is necessary to introduce the *lateral reference time* value $b/2V$ as shown. Again, let the perturbation become vanishingly small such that $V \to V_0$; then

$$\mathrm{Y}_p = \frac{\rho V_0 Sb}{4m}C_{y_p} \tag{13.228}$$

$$\mathrm{Y}_r = \frac{\rho V_0 Sb}{4m}C_{y_r} \tag{13.229}$$

Similarly, the control derivatives may be derived:

$$\mathrm{Y}_\delta = \frac{\rho V_0^2 S}{2m}C_{y_\delta} \tag{13.230}$$

By repeating this process, the roll and yawing moment normalised derivative expressions may also be derived. For the rolling moment equation in equations (13.224),

$$\mathrm{L}_v = \frac{\rho V_0 S b}{2 I_x} C_{l_\beta}$$

or equivalently

$$\begin{aligned}
\mathrm{L}_\beta &= \frac{\rho V_0^2 S b}{2 I_x} C_{l_\beta} \\
\mathrm{L}_p &= \frac{\rho V_0 S b^2}{4 I_x} C_{l_p} \\
\mathrm{L}_r &= \frac{\rho V_0 S b^2}{4 I_x} C_{l_r} \\
\mathrm{L}_\delta &= \frac{\rho V_0^2 S b}{2 I_x} C_{l_\delta}
\end{aligned} \tag{13.231}$$

and for the yawing moment equation in equations (13.224),

$$\mathrm{N}_v = \frac{\rho V_0 S b}{2 I_z} C_{n_\beta}$$

or equivalently

$$\begin{aligned}
\mathrm{N}_\beta &= \frac{\rho V_0^2 S b}{2 I_z} C_{n_\beta} \\
\mathrm{N}_p &= \frac{\rho V_0 S b^2}{4 I_z} C_{n_p} \\
\mathrm{N}_r &= \frac{\rho V_0 S b^2}{4 I_z} C_{n_r} \\
\mathrm{N}_\delta &= \frac{\rho V_0^2 S b}{2 I_z} C_{n_\delta}
\end{aligned} \tag{13.232}$$

The lateral-directional derivative expressions given by equations (13.226) through (13.232) are also summarised in Appendix 7.

13.5.3 Comments

Today it is common for the equations of motion to be presented either in British notation or in American notation, and, further, the units in either notational style can be Imperial or SI. In general, the notational style will depend on the source of the aerodynamic model of the aircraft. It is therefore important that the modern flight dynamicist become sufficiently dextrous to deal with the equations of motion and aerodynamic model in whatever form they are presented. It is evident from the foregoing that the differences between, for example, British dimensionless aerodynamic derivatives and American aerodynamic derivative coefficients can be subtle, and care must be exercised in their interpretation. It can be tempting to convert from one notational style to another for various reasons; however, it cannot be emphasised enough that this process can be fraught with pitfalls unless extreme care is exercised. Experience shows that it is very easy to confuse the differences between the various dimensionless derivative forms and even to make errors in the translation from one style of units to another. It is therefore most advisable to conduct any flight dynamics analysis of an aeroplane using the equations of motion and aerodynamic model in the notational style presented by the source of that information. After all, irrespective of notational style, all equations of motion appear in a similar format once reduced to the state-space form, and their solution differs only by the applicable units.

EXAMPLE 13.1

The estimation of the stability and control derivatives for an aircraft by analytical means is generally a complex and substantial task, the procedure is laborious, and the results are often of poor accuracy and hence of limited usefulness. In general, the best results are obtained for simple aircraft with moderate flight envelopes and whose properties can be reasonably well defined by classical aerodynamics. The estimation of the stability and control derivatives for advanced-technology aircraft having large flight envelopes, multiple engines, and extensive configuration variables for aerodynamic optimisation is best dealt with by wind tunnel testing, modern computational methods, and/or flight test—as discussed in Chapter 12. However, when it is possible and appropriate to do so, estimation by analytical means can give considerable insight into the way in which the aerodynamic properties of the aircraft determine the stability and control characteristics seen by the pilot. Such background information is also especially valuable for the flight control system designer when it is intended that the aircraft is to incorporate advanced flight control systems.

The Slingsby T51 Dart sailplane is chosen to illustrate the analytical estimation of the aerodynamic stability and control derivatives of an aircraft. A configuration drawing, after Coates (1978), is shown in Fig. 13.16. The reasons for this choice are that the aircraft is aerodynamically simple, it has a limited subsonic flight envelope, and, most important, geometric, mass, aerodynamic, and performance data are readily available in the public domain. It is also clear that the stability and control characteristics of the aircraft are entirely dependent on the aerodynamic configuration. The Dart is a mid-1960s design, manufactured by Slingsby in the United

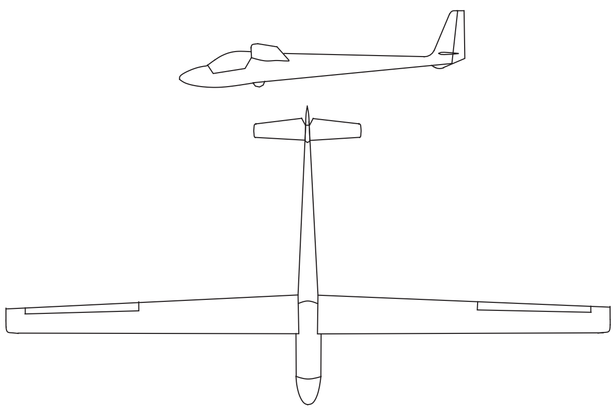

FIGURE 13.16 Slingsby Dart T51 configuration.

Kingdom, and is one of the last competition sailplanes to be built using traditional wood construction. The configuration was designed to meet the requirements of the standard 15 m span, single-seat class of high-performance sailplanes.

Basic aircraft data

Basic configuration, performance, and aerodynamic data were obtained from Taylor (1965−66), Coates (1978), and Ellison (1971). Of particular relevance to the aerodynamic modelling, the wing section is given—NACA $64_3618/615$—and its properties are available in Abbott and von Doenhoff (1959). Moment of inertia data are not given in any of the above references. However, the thesis by Millman (1971) reports on the design of a single-seat sailplane to meet the requirements of the 15 m-span standard class. Since the sailplane design has the same all-up weight (AUW) as the Dart and a configuration sufficiently similar, the same mass distribution was assumed. The inertia values computed by Millman (1971) were therefore considered to be sufficiently representative of the Dart and were adopted for this example.

The essential basic data were assembled and set out in Tables 13.1 through 13.3. Additional geometric and dimensional information not given in the references was estimated from the scaled three-view drawing of the aircraft given in Coates (1978). It is important to note that the inertia data obtained from Millman (1971) assumes an aircraft body axes reference. The aspect ratios of the wing and tailplane are sufficiently large that the calculation of their aerodynamic properties based on the simple theory is adequate. However, this is not the case for the fin, since it is subject to the end plate effect of the tailplane above which it is mounted. Calculation of the effective fin aspect ratio was made using information given in the relevant ESDU (2006) data item as noted with the data.

Table 13.1 T51 Dart 15 Performance at AUW of 318 kg

Maximum wing loading	24.8 kg/m^2
Best glide ratio	33.5:1 at 45.8 kt
Minimum sink speed	0.67 m/s at 41.5 kt
Maximum speed in smooth air	116 kt
Maximum speed in rough air	80 kt
Stall speed at maximum AUW (332 kg)	35 kt

Table 13.2 Mass and Inertias

Mass	m	318 kg
Roll inertia	I_{x_b}	1368 kgm^2
Pitch inertia	I_{y_b}	432 kgm^2
Yaw inertia	I_{z_b}	1778 kgm^2
Product of inertia	I_{xz_b}	−4.1 kgm^2
cg position (assumed)	h	0.3

Basic aerodynamic data for the wing, tailplane, and fin are given in Table 13.4 and have been calculated from two-dimensional section data given in Abbott and von Doenhoff (1959). The aerodynamic characteristics of the wing tail and fin were calculated using the methods given in Chapter 1 of Abbott and von Doenhoff (1959). The aerofoil sections used on the tailplane and fin are not given in any of the references, and representative symmetrical NACA sections have been assumed as indicated. The tailplane was assumed to have 6% thickness-to-chord ratio and the fin to have 15% thickness-to-chord ratio. It is also important to note that the Dart has an all-moving tailplane for control instead of a conventional elevator.

Table 13.3 Geometric Data

Wing Geometry					
Span	b	15 m	Root chord	c_{rw}	1.04 m
Area	S	12.7 m^2	Tip chord	c_{tw}	0.63 m
Mean chord	$\overline{\overline{c}}$	0.835 m	Taper ratio	λ_w	0.606
Dihedral	Γ	2°	Aspect ratio	A	17.8
Rigging angle	α_{wr}	9°	Sweep $c_{1/4}$	Λ	−0.8°
Tailplane Geometry					
Span	b_T	2.61 m	Root chord	c_{rT}	0.574 m
Area	S_T	1.14 m^2	Tip chord	c_{tT}	0.304 m
Tail arm	l_t	4.63 m	Taper ratio	λ_T	0.53
Rigging angle	η_T	0°	Aspect ratio	A_T	5.98
Fin Geometry					
Height	h_F	1.25 m	Root chord	c_{rF}	0.878 m
Net area	S_f	0.54 m^2	Tip chord	c_{tF}	0.405 m
Total area	S_F	0.96 m^2	Taper ratio	λ_F	0.47
Rudder area	S_{FR}	0.4 2 m^2	Sweep $c_{1/4}$	Λ_F	16°
Effective aspect ratio (ESDU 82010)				A_F	3.26

Table 13.4 Basic Aerodynamic Data

		Wing-Body		Tailplane		Fin
Aerofoil section		NACA 64_3618		NACA 65 006		NACA 63_2015
Lift curve slope	a	5.55 rad^{-1}	a_1	4.303 rad^{-1}	a_{1_F}	3.68 rad^{-1}
Maximum lift coefficient	C_{Lmax}	1.29		0.8		1.5
Aerodynamic centre	h_0	0.25		0.25		0.25
Zero-lift pitch moment	C_{m0}	−0.11		0		0
Zero-lift angle of attack	α_0	−4°		0		0

Aerodynamic operating condition

Since the aerodynamic stability and control derivatives are calculated at each trimmed flight condition, the procedure is illustrated here for a range of trimmed operating conditions extending over the flight envelope of the Dart. The Mathcad program AeroTrim, described in Section 3.6.4, with some minor adjustments to suit this application, was used for the purpose.

Key to the representational accuracy of the aerodynamic modelling is the measured drag polar and associated performance data reported by Torode (1973). The drag coefficient was then calculated using the following expression:

$$C_D = 0.013 + 1.13\frac{C_L^2}{\pi A} \tag{13.233}$$

Similarly, simple performance calculations were made using the standard expressions fitted to measured flight data by Torode (1973). Sink rate, and hence flight path angle, was calculated using

$$v_s = 0.0123\left(\frac{\frac{1}{2}\rho V^3 S}{mg}\right) + 1.21\left(\frac{\frac{1}{2}mg\pi}{\rho VSA}\right) \tag{13.234}$$

and minimum drag speed was calculated using

$$V_{md} = \sqrt{\frac{mg}{\frac{1}{2}\rho S}}\left(\frac{1.13}{0.013\pi A}\right)^{0.25} \tag{13.235}$$

The operating conditions assumed for the example cover a large part of the flight envelope from 35 knots to 85 knots at 5-knot intervals at a constant altitude of 1000 ft. AeroTrim was used to solve the trim equations as set out in Sections 3.6.1, 3.6.2, and 3.6.3 using the drag, sink rate, and minimum drag speed models given in equations (13.233), (13.234), and (13.235), setting the thrust-dependent terms to zero and simplifying the elevator longitudinal control terms to represent an all-moving tailplane. The calculated trim data required for derivative estimation are summarised in Table 13.5.

Note that the derivative $dC_D/d\alpha$ was estimated from the gradient of the C_D-α_e plot and the computation was incorporated into AeroTrim for this example. This shows the flexibility offered by the use of AeroTrim, which is easily modified or adapted to deal with a wide variety of routine trim computations.

Note also that in this instance α_e represents the trimmed value of the wing incidence—the lift-producing incidence. However, since the wing rigging angle $\alpha_{wr} = 9°$, the aircraft attitude appears significantly nose down, and this is confirmed by reference to Fig. 13.16. Since a glider always descends with flight path angle γ_e in normal trimmed flight and pitch attitude is given by equation (5.127),

$$\gamma = \theta - \alpha$$

where, in this instance, α is the aircraft *body axes incidence*, and when *wind axes* are specified, as for this example, $\alpha = 0$, and it follows that $\theta_e = \gamma_e$.

Table 13.5 Calculated Longitudinal Trim Data

V_0 kt	V_0 m/s	α_e deg	η_e deg	C_L	C_D	γ_e deg	$dC_D/d\alpha$ rad^{-1}
35	18.025	9.209	−8.403	1.271	0.0457	−2.057	0.254
40	20.6	6.161	−6.432	0.973	0.0321	−1.892	0.226
45	23.175	4.072	−5.08	0.769	0.0249	−1.858	0.177
50	25.75	2.577	−4.113	0.623	0.0208	−1.916	0.143
55	28.325	1.471	−3.398	0.515	0.0184	−2.042	0.118
60	30.9	0.63	−2.854	0.433	0.0168	−2.222	0.099
65	33.475	−0.025	−2.431	0.369	0.0157	−2.446	0.084
70	36.05	−0.544	−2.095	0.318	0.0150	−2.71	0.073
75	38.625	−0.963	−1.824	0.277	0.0146	−3.009	0.063
80	41.2	−1.306	−1.602	0.243	0.0142	−3.34	0.055
85	43.775	−1.59	−1.418	0.216	0.0139	−3.701	0.052

Other useful aerodynamic data output by AeroTrim includes static margin K_n, minimum drag speed V_{md}, and rate of change of downwash with incidence at the tail $d\varepsilon/d\alpha$ with the following values:

$K_n = 0.247$
$V_{md} = 44.06$ knots
$d\varepsilon/d\alpha = 0.223$

Note that the minimum drag speed correlates well with the published performance data for the aircraft, and this confers a degree of confidence on the aerodynamic modelling thus far.

Since the moments of inertia assumed for the model are referred to body axes aligned to the *horizontal fuselage datum* axes (HFD), the remaining preparatory work requires that they be converted to a wind, or a stability, axes reference. The conversion formulae are given in Appendix 10, Table A10.1, and the correct value for α_e here is the angle between the aircraft body and wind axes, that is, the value of α_e given in Table 13.5 minus the wing rigging angle $\alpha_{wr} = 9°$ for each trim condition. The results are given in Table 13.6.

It will be seen that the variations in the moments of inertia over the speed envelope are negligibly small and that for all practical purposes a single value for each would be adequate. For example,

$$I_{x_w} = 1370\,\text{kg m}^2$$

$$I_{y_w} = 432\,\text{kg m}^2$$

$$I_{z_w} = 1770\,\text{kg m}^2$$

The value of the product of inertia is relatively small, but it does vary significantly over the speed envelope and it is preferable to use the values shown in Table 13.6.

Table 13.6 Inertias Referred to Wind Axes

V_0 kt	I_{x_w} kgm^2	I_{y_w} kgm^2	I_{z_w} kgm^2	I_{xz_w} kgm^2
35	1.368×10^3	432	1.778×10^3	-5.599
40	1.369×10^3	432	1.777×10^3	16.199
45	1.370×10^3	432	1.776×10^3	31.052
50	1.372×10^3	432	1.774×10^3	41.58
55	1.374×10^3	432	1.772×10^3	49.298
60	1.375×10^3	432	1.771×10^3	55.12
65	1.377×10^3	432	1.769×10^3	59.618
70	1.378×10^3	432	1.768×10^3	63.165
75	1.379×10^3	432	1.767×10^3	66.011
80	1.380×10^3	432	1.766×10^3	68.33
85	1.380×10^3	432	1.766×10^3	70.245

Estimation of the longitudinal stability and control derivatives

The dimensionless longitudinal stability and control derivatives are estimated using the expressions developed in Section 13.2 subject to a number of simplifying assumptions. In particular, since the speed envelope is very small, 35 knots to 85 knots, and since compressibility effects are negligible, lift, drag, and pitching moment coefficient dependency on speed is assumed insignificant, and the relevant terms are omitted from the derivative expressions. Accordingly, the derivative expressions used are listed below and their evaluation at 50 knots is also illustrated.

$$X_u = -2C_D = -0.042$$

$$X_w = C_L - \frac{\mathrm{d}C_D}{\mathrm{d}\alpha} = 0.623 - 0.143 = 0.48$$

$$X_q = 0$$

$$X_{\dot{w}} = 0$$

$$X_\eta = 0$$

$$Z_u = -2C_L = -1.246$$

$$Z_w = -(a + C_D) = -(5.55 + 0.02084) = -5.571$$

$$Z_q = -\overline{V}_T a_1 = -0.493 \times 4.303 = -2.122$$

$$Z_{\dot{w}} = Z_q \frac{d\varepsilon}{d\alpha} = -2.121 \times 0.223 = -0.472$$

$$Z_\eta = -\frac{S_T}{S} a_2 = -\frac{1.14}{12.7} \times 4.303 = -0.386$$

$$M_u = 0$$
$$M_w = -aK_n = -5.55 \times 0.247 = -1.373$$
$$M_q = Z_q \frac{l_T}{\overline{\overline{c}}} = -2.121 \frac{4.59}{0.835} = -11.663$$
$$M_{\dot{w}} = M_q \frac{d\varepsilon}{d\alpha} = -11.66 \times 0.223 = -2.595$$
$$M_\eta = -\overline{V}_T a_2 = -0.493 \times 4.303 = -2.122$$

In addition to the data assembled, in order to complete the derivative calculations shown, it is necessary to calculate a value for the tail volume ratio $\overline{V}_T$. The expression for this is given by equation (2.32) and the definition requires that the tail moment arm be calculated from the *cg* to the quarter-chord point on the tailplane *mac* (see Fig. 2.9). Accordingly,

$$l_T = l_t - (h - h_0)\overline{\overline{c}} = 4.63 - (0.3 - 0.25)0.835 = 4.588 \text{ m} \tag{13.236}$$

Thus the tail volume ratio follows:

$$\overline{V}_T = \frac{S_T l_T}{S\overline{\overline{c}}} = \frac{1.14 \times 4.59}{12.7 \times 0.835} = 0.493 \tag{13.237}$$

For the assumptions made, all derivatives for this model are constant with flight condition with the exception of X_u, X_w, Z_u, and Z_w, which vary with trim speed. The values for these derivatives are given in Table 13.7.

Note again that the value of the derivative Z_w varies little over the speed envelope and that for all practical purposes its value may be taken as $Z_w = -5.6$. This observation should not be surprising as the value of lift curve slope *a* dominates and for the Dart it remains essentially constant over the flight envelope.

Table 13.7 Longitudinal Velocity-Dependent Derivatives

V_0 kt	X_u	X_w	Z_u	Z_w
35	−0.091	1.017	−2.542	−5.596
40	−0.064	0.748	−1.946	−5.582
45	−0.050	0.592	−1.538	−5.575
50	−0.042	0.480	−1.246	−5.571
55	−0.037	0.397	−1.029	−5.568
60	−0.034	0.334	−0.865	−5.567
65	−0.031	0.284	−0.737	−5.566
70	−0.030	0.245	−0.636	−5.565
75	−0.029	0.214	−0.554	−5.565
80	−0.028	0.188	−0.487	−5.564
85	−0.028	0.164	−0.431	−5.564

This concludes the estimation of the dimensionless longitudinal stability and control derivatives for the Slingsby Dart T51. However, at this stage it is easy to write the derivatives in alternative formats as required. For example, the dimensionless derivatives, evaluated at 50 knots, may be written in American coefficient format as defined in Appendix 7, Table A7.2, to give

$$
\begin{array}{lll}
C_{x_u} \equiv X_u = -0.042 & C_{z_u} \equiv Z_u = -1.246 & C_{m_u} \equiv M_u = 0 \\
C_{x_\alpha} \equiv X_w = 0.48 & C_{z_\alpha} \equiv Z_w = -5.571 & C_{m_\alpha} \equiv M_w = -1.373 \\
C_{x_q} \equiv 2X_q = 0 & C_{z_q} \equiv 2Z_q = -4.244 & C_{m_q} \equiv 2M_q = -23.326 \\
C_{x_{\dot\alpha}} \equiv 2X_{\dot w} = 0 & C_{z_{\dot\alpha}} \equiv 2Z_{\dot w} = -0.944 & C_{m_{\dot\alpha}} \equiv 2M_{\dot w} = -5.19 \\
C_{x_{\delta e}} \equiv X_\eta = 0 & C_{z_{\delta e}} \equiv Z_\eta = -0.386 & C_{m_{\delta e}} \equiv M_\eta = -2.122
\end{array}
$$

Estimation of the lateral-directional stability and control derivatives

The dimensionless lateral-directional stability and control derivatives are estimated using the expressions developed in Section 13.3 subject to the same simplifying assumptions referred to previously. However, estimation of the lateral-directional stability and control derivatives by any means usually poses a challenge and the analytical models described in Section 13.3 produce estimates of uncertain accuracy. For this reason recourse is also made to ESDU data items since these can produce significantly better estimates, especially for a conventional aircraft configuration like the Slingsby Dart. Since the Dart is a very-high-aspect-ratio aircraft and its flight envelope is very subsonic, the simple lateral-directional derivative expressions can be applied with a higher degree of confidence than is usually the case. The procedure used for estimating the derivatives requires that the various contributions to each derivative are separately evaluated, as far as that is possible with the information available, and the sum of the contributions for each derivative is then taken to be the best estimate. To illustrate the procedure the derivatives are, once again, evaluated at 50 knots.

Wing contributions

Since the aspect ratio of the wing exceeds the range covered by the ESDU data items, the following derivatives can only be estimated analytically by application of the simple expressions given in Section 13.3. In the following computations it is assumed that lift curve slope a, lift coefficient C_L, drag coefficient C_D, and the derivative of drag coefficient with respect to incidence $dC_D/d\alpha$ are constant over the span of the wing.

$L_{v(wing)}$—*Rolling moment due to sideslip* This comprises two components, one due to dihedral and a very much smaller component due to wing sweep. The dihedral component is given by equation (13.92), which may be interpreted in terms of the aerodynamic and geometric properties of the Dart:

$$L_{v(dihedral)} = -\frac{a\Gamma}{Ss}\int_0^s \left(\frac{(c_{tw}-c_{rw})}{s}y + c_{rw}\right) y\,dy = -\frac{a\Gamma s}{S}\left(\frac{c_{tw}}{3}+\frac{c_{rw}}{6}\right) \qquad (13.238)$$

Substituting geometric and aerodynamic data for 50 knots flight speed into equation (13.238), taking care that dihedral angle has radian units, gives

$$L_{v(dihedral)} = -0.0439$$

The wing sweep component is given by equation (13.105), which may similarly be interpreted as

$$L_{v(sweep)} = -\frac{2C_L \tan\Lambda}{Ss}\int_0^s \left(\frac{(c_{tw}-c_{rw})}{s}y + c_{rw}\right) y\,dy = -\frac{2C_L s\tan\Lambda}{S}\left(\frac{c_{tw}}{3}+\frac{c_{rw}}{6}\right) \tag{13.239}$$

and, evaluated at 50 knots,

$$L_{v(sweep)} = 0.00394$$

This contribution is very small since the wing sweep angle is small and it is positive (destabilising) since the wing has forward, rather than aft sweep.

Thus the total wing contribution is given by

$$L_{v(wing)} = L_{v(dihedral)} + L_{v(sweep)} = -0.03996$$

$L_{p(wing)}$—*Rolling moment due to roll rate* The wing contribution is given by equation (13.128), which may be interpreted as follows:

$$L_{p(wing)} = -\frac{(a+C_D)}{2Ss^2}\int_0^s \left(\frac{(c_{tw}-c_{rw})}{s}y + c_{rw}\right) y^2 dy = -\frac{(a+C_D)s}{2S}\left(\frac{c_{tw}}{4}+\frac{c_{rw}}{12}\right) \tag{13.240}$$

and, evaluated at 50 knots,

$$L_{p(wing)} = -0.4017$$

$N_{p(wing)}$—*Yawing moment due to roll rate* The wing contribution is given by equation (13.137), which may be interpreted as follows:

$$N_{p(wing)} = -\frac{\left(C_L - \frac{dC_D}{d\alpha}\right)}{2Ss^2}\int_0^s \left(\frac{(c_{tw}-c_{rw})}{s}y + c_{rw}\right) y^2 dy = -\frac{\left(C_L - \frac{dC_D}{d\alpha}\right)s}{2S}\left(\frac{c_{tw}}{4}+\frac{c_{rw}}{12}\right) \tag{13.241}$$

and, evaluated at 50 knots,

$$N_{p(wing)} = -0.0346$$

$L_{r(wing)}$—*Rolling moment due to yaw rate* The wing contribution is given by equation (13.150), which may be interpreted as follows:

$$L_{r(wing)} = \frac{C_L}{Ss^2}\int_0^s \left(\frac{(c_{tw}-c_{rw})}{s}y + c_{rw}\right) y^2 dy = \frac{C_L s}{S}\left(\frac{c_{tw}}{4}+\frac{c_{rw}}{12}\right) \tag{13.242}$$

and, evaluated at 50 knots,

$$L_{r(wing)} = 0.08985$$

Note that this contribution is positive, and hence destabilising in roll, in accordance with the flight physical understanding. (Most stability derivatives are negative for a stable aircraft.)

$N_{r(wing)}$—*Yawing moment due to yaw rate* The wing contribution is given by equation (13.162), which may be interpreted as follows:

$$N_{r(wing)} = -\frac{C_D}{Ss^2}\int_0^s \left(\frac{(c_{tw} - c_{rw})}{s}y + c_{rw}\right)y^2 \mathrm{d}y = -\frac{C_D s}{S}\left(\frac{c_{tw}}{4} + \frac{c_{rw}}{12}\right) \tag{13.243}$$

and, evaluated at 50 knots,

$$N_{r(wing)} = -0.00301$$

Fin contributions

In the interests of achieving the best possible accuracy, all fin derivatives are evaluated using the relevant ESDU data items. Since the notation used in the calculation process is not always compatible with that adopted throughout this book, where necessary the ESDU notation takes precedence in this example. Application of the calculation methods described in the ESDU data items requires detailed dimensional information relating to the relative geometric location of the centre of gravity, the wing, the tailplane, and the fin. Some of the dimensional data is a part of the general aircraft description as listed at the start of the example. Additional geometric information is required, and for this example these dimensions were estimated from an enlarged copy of the scale three-view drawing of the aircraft given in Coates (1978). It must be emphasized that the accuracy of these dimensions is not likely to be particularly good. In a more formal aircraft design situation, this kind of dimensional information would be available at high accuracy.

In the notation of ESDU 82010, additional fin geometry parameters, assuming a fin-mounted tailplane, were estimated as follows:

Fin moment arm measured from the cg to the $c_{1/4F}m_F = 4.595$ m

$$h_{BF} = 0.13\ \mathrm{m} \quad h_{BW} = 0.777\ \mathrm{m} \quad z_{crF} = 0.118\ \mathrm{m} \quad z_W = -0.237\ \mathrm{m} \quad z_T = 0.135\ \mathrm{m}$$

In accordance with the data item, the following expressions are evaluated:

$$\frac{h_{BF}}{h_{BF} + h_F} = 0.094 \quad \frac{b_T}{h_F} = 2.088 \quad \frac{z_W}{h_{BW}} = -0.305$$

$$\frac{z_T}{h_F} = 0.108 \quad \left(\frac{z_T}{h_F}\right)^2 = 0.012$$

It is important to note that the geometric datum axis is the *horizontal fuselage datum* (HFD), or longitudinal body axis, and that the incidence angle α used in the following calculations is the incidence of the HFD to the airflow—that is, as before, the angle between the aircraft body and wind axes.

With this numerical information the nomograms given in the data item are used to determine the following parameters:

$$J_B = 0.8 \quad J_T = 1.0 \quad J_W = 0.81 \quad \bar{z}_F = 0.4 \, h_F = 0.5\ \mathrm{m}$$

An expression for a modified fin volume ratio is given which allows empirically for the aerodynamic adjustment due to fin sweep.

$$\overline{V}_F = \frac{S_F\left(m_F + 0.7\overline{z}_F \tan\Lambda_{1/4F}\right)}{Sb} = 0.024 \tag{13.244}$$

Using the dimensional parameters defined for the aircraft, they are substituted into various expressions in order to arrive at estimated values for the fin derivatives due to sideslip.

$Y_{v(fin)}$—*Sideforce due to sideslip* The fin contribution is given by

$$Y_{v(fin)} = -J_B J_T J_W a_{1F} \frac{S_F}{S} \tag{13.245}$$

and this has a constant value at all speeds:

$$Y_{v(fin)} = -0.18$$

$L_{v(fin)}$—*Rolling moment due to sideslip* The fin contribution is given in the data item:

$$L_{v(fin)} = Y_{v(fin)} \frac{(z_{crF} + 0.85\overline{z}_F)\cos\alpha - \left(m_F + 0.7\overline{z}_F \tan\Lambda_{1/4F}\right)\sin\alpha}{b} \tag{13.246}$$

The value of this derivative varies with incidence α over the flight envelope and, evaluated at 50 knots, is

$$L_{v(fin)} = -0.0128$$

$N_{v(fin)}$—*Yawing moment due to sideslip* The fin contribution is given in the data item:

$$N_{v(fin)} = -Y_{v(fin)} \frac{(z_{crF} + 0.85\overline{z}_F)\sin\alpha + \left(m_F + 0.7\overline{z}_F \tan\Lambda_{1/4F}\right)\cos\alpha}{b} \tag{13.247}$$

The value of this derivative also varies with incidence α over the flight envelope and, evaluated at 50 knots, is

$$N_{v(fin)} = 0.0553$$

Fin contributions arising from yaw rate are set out in ESDU 82017, and the expressions given make use of the same fin geometry parameters as set out above.

$Y_{r(fin)}$—*Sideforce due to yaw rate*

$$Y_{r(fin)} = -\frac{Y_{v(fin)}}{J_W} \frac{(z_{crF} + 0.85\overline{z}_F)\sin\alpha + \left(m_F + 0.7\overline{z}_F \tan\Lambda_{1/4F}\right)\cos\alpha}{b} \tag{13.248}$$

and, evaluated at 50 knots,

$$Y_{rF} = 0.068$$

$L_{r(fin)}$—*Rolling moment due to yaw rate*

$$L_{r(fin)} = Y_{r(fin)} \frac{(z_{crF} + 0.85\overline{z}_F)\cos\alpha - \left(m_F + 0.7\overline{z}_F \tan\Lambda_{1/4F}\right)\sin\alpha}{b} \tag{13.249}$$

and, evaluated at 50 knots,

$$L_{rF} = 0.00485$$

$N_{r(fin)}$—*Yawing moment due to yaw rate*

$$N_{r(fin)} = -Y_{r(fin)} \frac{(z_{crF} + 0.85\bar{z}_F)\sin\alpha + (m_F + 0.7\bar{z}_F \tan\Lambda_F)\cos\alpha}{b} \tag{13.250}$$

and, evaluated at 50 knots,

$$N_{rF} = -0.021$$

Body (fuselage) contributions

Body contributions to the lateral-directional stability derivatives are notoriously difficult to estimate with any degree of confidence. Contributions due to roll rate and yaw rate are generally considered to be sufficiently small that they can be ignored. However, sideforce and yawing moment due to sideslip are significant. Undoubtedly, the most important contribution is sideforce due to sideslip, and this was estimated for the Dart using ESDU 79006; even this is considered to be of limited accuracy in some applications.

Following ESDU 79006, and in the notation of the data item, the following geometric parameters were estimated:

$$h = 0.88\text{ m} \quad S_B = 3.88\text{ m}^2 \quad z = 0.24\text{ m} \quad d = 0.676\text{ m}$$

Again, a scaled side view of the aircraft was used from which to estimate the dimensions, and the side view was drawn on squared paper to facilitate the estimation of body side area S_B.

With this numerical information the nomograms given in the data item were used to determine the following parameters:

$$F = 0\; F_W = 0.6 \quad \frac{b}{d} = 22.189$$

Substituting these dimensional parameters into the following expression, an estimate of the sideforce due to sideslip for the body was made:

$$Y_{v(body)} = -\left(0.00714 + 0.674\frac{h^2}{S_B} + \frac{hbFF_W}{S_B}\left(4.95\frac{|z|}{h} - 0.12\right)\right)\frac{S_B}{S} - 0.006|\Gamma| \tag{13.251}$$

The value is constant for all flight conditions:

$$Y_{v(body)} = -0.055$$

Aileron control derivatives

Estimation of the aileron control derivatives depends on the availability of a representative estimate for the aileron lift curve slope a_{2A}. For a part span aileron, as is the case here, the problem concerns the aileron end losses, which are difficult, if not impossible, to quantify analytically. Thus recourse is made to the empirical estimation of a_{2A} using the appropriate ESDU data item.

Y_ξ—*Sideforce due to aileron* For conventional aircraft this derivative is usually assumed to be negligibly small; accordingly its value is taken as zero.

L_ξ—*Rolling moment due to aileron* For conventional aircraft of reasonable aspect ratio, the simple analytical model given by equation (13.188) provides an acceptable estimate provided the aileron lift curve slope a_{2A} can be estimated with sufficient accuracy. Since the Slingsby Dart has a very-high-aspect-ratio wing with part span ailerons, this approach is considered more than adequate. With reference to Fig. 13.15, equation (13.188) may be interpreted as follows:

$$\begin{aligned} L_\xi &= -\frac{1}{Ss} a_{2_A} \int_{y_1}^{y_2} c_y y \mathrm{d}y = -\frac{1}{Ss} a_{2_A} \int_{y_1}^{y_2} \left(\frac{(c_{tw} - c_{rw})}{s} y + c_{rw} \right) y \mathrm{d}y \\ &= -\frac{1}{Ss} a_{2_A} \left(\frac{(c_{tw} - c_{rw})}{s} \left(\frac{y_2^3 - y_1^3}{3} \right) - c_{rw} \left(\frac{y_2^2 - y_1^2}{2} \right) \right) \end{aligned} \tag{13.252}$$

Referring to ESDU controls 01.01.03, in the notation of the data item the following geometric parameters were calculated from available data:

Wing section thickness to chord ratio $t/c = 0.18$

Mean aileron chord to mean wing chord ratio $c_f/c_w = 0.275$

Referring to Abbott and von Doenhoff (1959), the ratio of the two-dimensional wing section lift curve slope to the three-dimensional lift curve slope for the wing was determined as follows, again in the notation of the ESDU data item:

$$\frac{a_0}{a_{0T}} = \frac{6.19}{6.45} = 0.96$$

With this numerical information the nomograms given in the data item were used to determine the following parameters:

$$a_{2_{0T}} = 4.4\,\mathrm{rad}^{-1} \quad a_{2_A} = 0.94 a_{2_{0T}} = 4.136\,\mathrm{rad}^{-1}$$

The aileron span variables y_1 and y_2 are self-evident in Fig. 13.15. Their values were estimated from the scaled drawing of the Dart given in Coates (1978) as fractions of the span, giving

$$y_1 = 0.56s \; y_2 = 0.94s$$

Having estimated values for the required numerical data, substitution of the appropriate values into equation (13.252) gives the constant value for the derivative:

$$L_\xi = -0.505$$

N_ξ—*Yawing moment due to aileron* This derivative was estimated using the data item ESDU controls 88029. The assumptions made were that the wing has zero spanwise twist, and aileron deflection is symmetric, that is, nondifferential. It was also found necessary to extrapolate the data since the aspect ratio of the Dart exceeds the range given in the data item. Using the previously calculated geometric and aerodynamic parameters, the nomograms given in the data item gave the following parameters:

$$G_1 = 0.066 \quad G_2 = 0.048$$

The yawing moment due to aileron derivative is given by the expression

$$N_\xi = (G_1 - G_2)C_L L_\xi \tag{13.253}$$

and, evaluated at 50 knots,

$$N_\xi = 0.00057$$

Rudder control derivatives

As for the aileron control derivatives, it is first necessary to estimate a representative value for the rudder lift curve slope a_{2R}. The fin section, NACA 63_2015, lift curve slope data was also obtained from Abbott and von Doenhoff (1959), the ratio of the two-dimensional fin section lift curve slope to the three-dimensional lift curve slope was determined as follows, again in the notation of the ESDU data item:

$$\frac{a_0}{a_{0T}} = \frac{5.75}{6.27} = 0.917$$

Again, referring to ESDU controls 01.01.03, in the notation of the data item the following geometric parameters were calculated from available data:

Fin section thickness to chord ratio $t/c = 0.15$
Mean rudder chord to mean fin chord ratio $c_f/c_w = 0.5$

With this numerical information, the nomograms given in the data item were used to determine the rudder lift curve slope as follows:

$$a_{2_{0T}} = 5.94\ \text{rad}^{-1} \quad a_{2_R} = 0.91 a_{2_{0T}} = 5.405\ \text{rad}^{-1}$$

Since the fin has a small aspect ratio, its effect on rudder control power is significant and must be corrected. The method described in Abbot and von Doenhoff (1959) was used for this purpose:

$$a_{2_R}(corrected) = f\frac{a_{2_R}}{\left(1 + a_{2_R}/\pi A_F\right)} \tag{13.254}$$

where $f = 1.0$ is an empirical correction factor determined from the reference. Thus the best estimate for rudder lift curves slope is

$$a_{2_R} = 3.538\ \text{rad}^{-1}$$

The simple expressions given in Section 13.4.3 were then evaluated for the rudder control derivatives as follows.

Y_ζ—*Sideforce due to rudder* From equations (13.197) and (13.82), it follows that

$$Y_\zeta = -\frac{a_{2R}}{a_{1F}} Y_{v(fin)} \tag{13.255}$$

Substitution of the appropriate values into equation (13.255) gives the constant value for the derivative:

$$Y_z = 0.173$$

L_ζ—*Rolling moment due to rudder* From equations (13.200) and (13.108) it follows that

$$L_\zeta = -\frac{a_{2R}}{a_{1F}} L_{v(fin)} \tag{13.256}$$

and evaluating at 50 knots gives

$$L_\zeta = 0.0125$$

N_ζ—*Yawing moment due to rudder* From equations (13.203) and (13.112) it follows that

$$N_\zeta = -\frac{a_{2R}}{a_{1F}} N_{v(fin)} \tag{13.257}$$

and evaluating at 50 knots gives

$$N_\zeta = -0.0529$$

Thus the total value of each of the stability and control derivatives comprises the sum of all the identifiable components, and each is very much an estimate as it is not possible to evaluate some of the more intangible contributions. For the assumptions made, most of the lateral-directional derivatives vary with trim speed as set out in Table 13.8. However, the derivatives Y_v, Y_p, Y_ζ, and L_ξ remain constant over the flight envelope and take the values calculated above.

Inspection of Table 13.8 indicates that most of the derivatives are near constant over the flight envelope; this is consistent with the view that for subsonic aircraft the lateral-directional stability characteristics remain fixed, approximately, by the symmetric design properties of the aircraft.

Table 13.8 Lateral-Directional Velocity-Dependent Derivatives

V_0 kt	L_v	L_p	L_r	L_ζ	N_v	N_p	N_r	N_ξ	N_ζ	Y_r
35	−0.042	−0.403	0.187	0.006	0.056	−0.074	−0.028	0.0120	−0.054	0.070
40	−0.047	−0.402	0.145	0.009	0.056	−0.055	−0.026	0.0089	−0.054	0.069
45	−0.05	−0.402	0.117	0.011	0.056	−0.044	−0.025	0.0070	−0.054	0.069
50	−0.053	−0.402	0.097	0.012	0.055	−0.036	−0.024	0.0057	−0.053	0.068
55	−0.054	−0.401	0.082	0.013	0.055	−0.030	−0.023	0.0047	−0.053	0.068
60	−0.056	−0.401	0.070	0.014	0.055	−0.025	−0.023	0.0039	−0.053	0.068
65	−0.057	−0.401	0.061	0.015	0.055	−0.022	−0.023	0.0035	−0.053	0.068
70	−0.058	−0.401	0.054	0.015	0.055	−0.019	−0.023	0.0029	−0.052	0.067
75	−0.058	−0.401	0.048	0.016	0.054	−0.017	−0.022	0.0025	−0.052	0.067
80	−0.059	−0.401	0.044	0.016	0.054	−0.015	−0.022	0.0022	−0.052	0.067
85	−0.059	−0.401	0.040	0.016	0.054	−0.013	−0.022	0.0020	−0.052	0.067

This concludes the estimation of the dimensionless lateral-directional stability and control derivatives for the Slingsby Dart T51. As before, writing the derivatives in American coefficient format as defined in Appendix 7, Table A7.4 and evaluating at 50 knots gives

$$C_{y_v} \equiv Y_v = -0.236 \qquad C_{l_v} \equiv L_v = -0.053 \qquad C_{n_v} \equiv N_v = 0.055$$

$$C_{y_p} \equiv 2Y_p = 0 \qquad C_{l_p} \equiv 2L_p = -0.804 \qquad C_{n_p} \equiv 2N_p = -0.072$$

$$C_{y_r} \equiv 2Y_r = 0.136 \qquad C_{l_r} \equiv 2L_r = 0.194 \qquad C_{n_r} \equiv 2N_r = -0.048$$

$$C_{y_{\delta A}} \equiv Y_\xi = 0 \qquad C_{l_{\delta A}} \equiv L_\xi = -0.505 \qquad C_{n_{\delta A}} \equiv N_\xi = 0.0057$$

$$C_{y_{\delta R}} \equiv Y_\zeta = 0.173 \qquad C_{l_{\delta R}} \equiv L_\zeta = 0.012 \qquad C_{n_{\delta R}} \equiv N_\zeta = -0.053$$

The concise derivatives

Having achieved a complete set of the estimated dimensionless stability and control derivatives it is usual to go on and study the flight dynamics of the aircraft. The starting point for this process is the assembly of the concise equations of motion, equations (4.67) and (4.69), which requires the derivatives to be in concise format. The concise derivatives may be derived using Appendix 2, Tables A2.5 to A2.8, which, evaluated for the Dart at 50 knots, gives the data shown in Tables 13.9 and 13.10.

Alternatively, the longitudinal and lateral-directional state equations may be assembled using matrix manipulation, and that procedure is illustrated in Examples 4.3 and 4.4.

Table 13.9 Longitudinal Concise Derivatives

x_u	−0.0257	z_u	−0.7550	m_u	0.0239
x_w	0.2936	z_w	−3.3764	m_w	−0.4093
x_q	0	z_q	24.442	m_q	−4.4344
x_θ	−9.8045	z_θ	0.325	m_θ	−0.0103
x_η	0	z_η	−6.0239	m_η	−20.351

Table 13.10 Lateral-Directional Concise Derivatives

y_v	−0.0144	l_v	−0.1101	n_v	0.0879
y_p	0	l_p	−12.864	n_p	−1.1899
y_r	−25.126	l_r	3.079	n_r	−0.52
y_ϕ	9.8045				
y_ψ	−0.3280				
y_ξ	0	l_ξ	−27.676	n_ξ	−0.4089
y_ζ	2.7246	l_ζ	0.5897	n_ζ	−2.2313

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PROBLEMS

13.1 The centre of gravity of an aircraft is moved aft through a distance δh. Derive expressions relating the dimensionless longitudinal aerodynamic stability derivatives before and after the *cg* shift.

(a) For an aircraft in steady level flight, using the relevant reduced-order solution of the equations of motion, calculate the aft *cg* limit at which the phugoid mode becomes unstable.

(b) The aircraft aerodynamic data, referred to wind axes, are as follows:

Air density $\rho = 1.225$ kg/m^3	Wing area $S = 24.15$ m^2
Velocity $V_0 = 560$ kts	Mean chord $\bar{\bar{c}} = 3.35$ m
Aircraft mass $m = 7930$ kg	Tail moment arm $l_T = 4.57$ m
cg location $h = 0.25$	Pitch inertia $I_y = 35251$ kg.m^2

(c) The dimensionless stability derivatives are

$X_u = -0.03$	$Z_q = 0$
$X_w = -0.02$	$M_u = 0.01$
$X_q = 0$	$M_w = -0.15$
$Z_u = -0.09$	$M_{\dot{w}} = 0$
$Z_w = -2.07$	$M_q = -0.58$

(d) Assume that the moment of inertia in pitch remains constant. (CU 1979)

13.2 Show that, to a good approximation, the following expressions hold for the dimensionless derivatives of a rigid aircraft in a glide with the engine off, $X_u \cong -2C_D$ and $Z_u \cong -2C_L$.

(a) With power on in level flight, the lift coefficient at the minimum power speed is given by

$$C_{L_{mp}} = \sqrt{3\pi A e C_{D_0}}$$

and the minimum power speed is given by

$$V_{mp} = \sqrt{\frac{mg}{\rho S C_{L_{mp}}}}$$

(b) Show that at the minimum power speed the thrust τ_{mp} and velocity V_{mp} satisfy the relation

$$\tau_{mp} V_{mp}^{n} = k$$

and find corresponding values for the constants k and n. Hence show that

$$X_u = -(n+2)C_D$$

(CU 1982)

13.3 Do the following:

(a) Explain the physical significance of the aerodynamic stability derivative.
(b) Discuss the dependence of L_v and N_v on the general layout of an aircraft.
(c) Show that for an unswept wing with dihedral angle Γ, the effect of sideslip angle β is to increase the incidence of the starboard wing by $\beta\Gamma$, where both β and Γ are small angles.
(d) Show that fin contribution to N_v is given by

$$N_v = a_{1_F}\overline{V}_F$$

(CU 1982)

13.4 The Navion light aircraft is a cantilevered low-wing monoplane. The unswept wing has a span of 10.59 m, a planform area of 17.12 m^2, and dihedral angle 7.5°. The fin has a planform area of 0.64 m^2 and an aspect ratio of 3.0; its aerodynamic centre is 5.5 m aft of the *cg*.

(a) Derive an expression for the dimensionless derivative N_v. Find its value for the Navion aircraft given that the lift curve slope a_1 of a lifting panel may be approximated by $a_0 A/(A+2)$, where a_0 is its two dimensional lift curve slope and A its aspect ratio.

(CU 1983)

13.5 Do the following:

(a) Explain roll damping and what its main sources are on an aeroplane.
(b) Assuming roll damping to be produced entirely by the wing of an aeroplane, show that the dimensionless roll damping stability derivative L_p is given by

$$L_p = -\frac{1}{2Ss^2}\int_0^s \left(a_y + C_{D_y}\right) c_y y^2 dy$$

(c) The Navion light aeroplane has a straight tapered wing of span 33.4 ft, area 184 ft^2, and root chord length 8 ft. At the given flight condition the wing drag coefficient is 0.02 and the lift curve slope is 4.44 per radian. Estimate a value for the dimensionless roll damping stability derivative L_p. Show all work and state any assumptions made.

(CU 1984)

13.6 Explain the physical significance of the aerodynamic stability derivatives M_u and M_w.

(a) The pitching moment coefficient for a wing is given by

$$C_m = \frac{k}{\gamma}\sin\alpha$$

and

$$\gamma = \sqrt{\left(1 - \frac{V^2}{a^2}\right)}$$

where k and a are constants. Obtain expressions for the wing contributions to M_u and M_w.
(CU 1985)

13.7 What is roll damping? Explain why the aerodynamic stability derivative L_p must always be negative.

(a) An aircraft of mass 5100 kg, aspect ratio 10, and wing span 16 m is in level flight at an equivalent airspeed of 150 kts when an aileron deflection of 5° results in a steady roll rate of 15°/s. The aileron aerodynamic characteristics are such that a 10° deflection produces a lift increment of 0.8, the aileron centres of pressure being at 6.5 m from the aircraft longitudinal axis. What is the value of L_p for this aircraft? Take $\rho = 1.225$ kg/m^3 and 1 kt = 0.515 m/s.
(CU 1985)

13.8 What is dihedral effect? Explain the effect of the dihedral angle of a wing on the dimensionless rolling moment due to sideslip derivative L_v.

(a) Show that, for a straight tapered wing of semi-span s, dihedral angle Γ, root chord c_r, and tip chord c_t,

$$L_v = -\frac{\Gamma}{6}\frac{dC_L}{d\alpha}\left(\frac{c_r + 2c_t}{c_r + c_t}\right)$$

assuming the lift curve slope to be constant with span.

(b) Calculate the aileron angle required to maintain a steady forced wings-level sideslip angle of 5° given that the dimensionless rolling moment due to aileron control derivative has the value -0.197 per radian. It may be assumed that the rolling moment due to sideslip is entirely due to dihedral effect. The straight tapered wing of the aircraft has a mean chord of 2.4 m, a taper ratio of 1.8, and a dihedral angle of 4°. The lift curve slope of the wing may be assumed to have the constant value of 5.0 per radian. (CU 1987)

13.9 What is the significance of roll damping to the flying and handling qualities of an aircraft?

(a) Derive from first principles an expression for the stability derivative L_p, and hence show that for an aircraft with a high-aspect-ratio rectangular wing the derivative is given approximately by

$$L_p = -\frac{1}{12}\left(C_D + \frac{dC_L}{d\alpha}\right)$$

State clearly the assumptions made in arriving at this result.

(b) The Lockheed NT-33A aircraft has a straight tapered wing of span 11.5 m and is fitted with wing tip fuel tanks each of which has a capacity of 1045 litres. With one tank empty and the other full it is found that the minimum speed at which wings-level flight can be maintained is 168 kts with the maximum aileron deflection of 15°. When both wing tip tanks contain equal quantities of fuel an aileron deflection of 5° results in a steady rate of roll of 17°/s at a velocity of 150 kts. What is the value of the derivative L_p?
Assume an air density of 1.225 kg/m^3, a fuel density of 0.8 kg/litre, a wing area of 22.23 m^2, and that 1 knot is equivalent to 0.52 m/s.

(CU 1990)

13.10 Show that for a wing with sweepback Λ and dihedral Γ,

$$L_v = -\frac{2}{Sb}(\alpha\sin\Lambda + \Gamma\cos\Lambda)\int_0^{s\sec\Lambda}\left(\frac{dC_L}{d\alpha}\right)_h c_h h\,dh$$

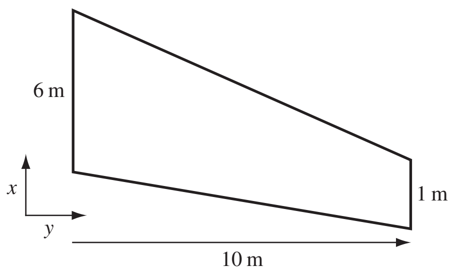
where h is the spanwise coordinate along the quarter-chord line. Assume the following for a chordwise element of the wing:

Chordwise velocity $V_c = V(\cos\Lambda + \beta\sin\Lambda)$
Velocity normal to wing $V_a = V(\alpha + \beta\Gamma)$
An aircraft has a wing with the following planform:

Sweep angle at quarter-chord = 55°
Dihedral angle = 3°
Lift curve slope 5.8/rad at $y = 0$
Lift curve slope 0/rad at $y = 10$ m
Calculate the value of L_v when the aircraft is flying at an incidence of 2°.

(LU 2001)

13.11 A three-surface aeroplane has the following characteristics:



Foreplane	*Wing*
Span = 4.0 m	Span = 12.5 m
Area = 2 m^2	Area = 25 m^2
Lift curve slope = 3.2/rad	Lift curve slope = 5.4/rad
Moment arm about cg = 5 m	Aerodynamic centre = $0.2\bar{\bar{c}}$
Tailplane	*Complete aircraft*
Span = 5.0 m	$cg = 0.4\bar{\bar{c}}$

Area = 6.0 m
Lift curve slope = 3.5/rad
Moment arm about cg = 6.0 m
Tailplane efficiency $(1 - d\varepsilon/d\alpha) = 0.95$

When flying at a true airspeed of 100 m/s and an altitude where $\sigma = 0.7$, determine for the complete aircraft (i.e. all three surfaces) values for $\mathring{M}_w$ and $\mathring{M}_q$. (LU 2002)

13.12 A tailless aircraft has the following characteristics:
Aerodynamic mean chord $\bar{\bar{c}} = 3.0$ m
cg position $h = 0.15\,\bar{\bar{c}}$
Aerodynamic centre $h_0 = 0.30\,\bar{\bar{c}}$
Wing area $S = 24$ m^2
Lift curve slope $a = 5.4$/rad

(a) When the aircraft is flying at 200 m/s at sea level, calculate $\mathring{M}_w$ and $\mathring{M}_q$. Where appropriate, assume $\partial V/\partial U = 0$, $\partial\alpha/\partial U = 0$, $\partial V/\partial W = 0$, and $\partial\alpha/\partial W = 1/V$.

(b) When the aircraft is in cruising flight with $C_L = 0.3$ and $C_{m_0} = 0.045$, determine a value for M_u. Assume that C_m is invariant with forward speed.

(LU 2004)