University of BRISTOL

ADVANCED STRUCTURES & MATERIALS

Finite Element Analysis Principles – Lecture 2

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Governing Equations of Linear Elasticity: Navier's Equations

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x},$$
$$\epsilon_{yy} = \frac{\partial u_y}{\partial y},$$
$$\epsilon_{zz} = \frac{\partial u_z}{\partial z},$$
$$\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x},$$
$$\epsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y},$$
$$\epsilon_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

$$\Rightarrow$$

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right)$$
$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right)$$
$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right)$$
$$\epsilon_{xy} = 2\frac{1+\nu}{E}\sigma_{xy}$$
$$\epsilon_{yz} = 2\frac{1+\nu}{E}\sigma_{yz}$$
$$\epsilon_{zx} = 2\frac{1+\nu}{E}\sigma_{zx}$$

$$\Rightarrow$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

+ BCs on constrained surface, S_c: $u = u_{bc}$ (*Geometric* or *Essential*)

+ BCs on free surface, S_f: applied pressures, $\boldsymbol{p}$ applied forces, $\boldsymbol{P}$ (*Force* or *Natural*)

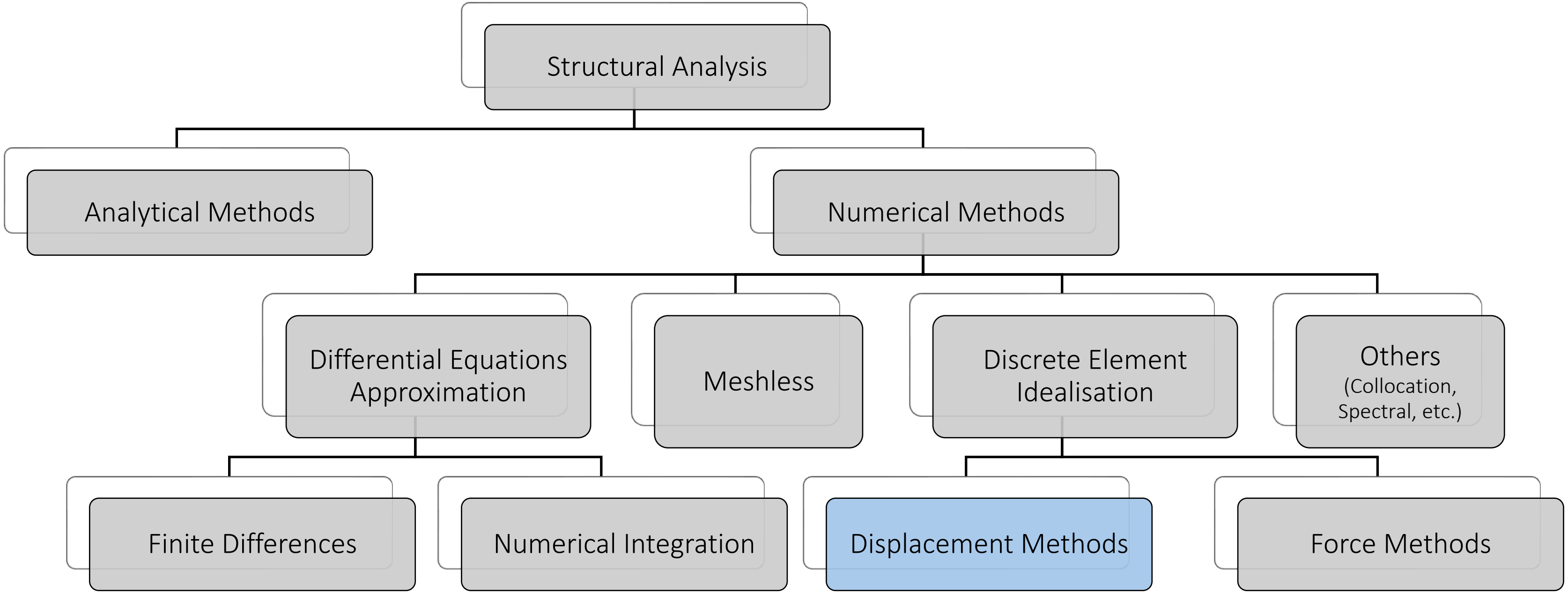
Navier’s Equations

- General, closed-form solutions do not exist.
- In other words, the system cannot be solved analytically.
- With appropriate simplifying assumptions, Navier’s equations can be specialised for 1D or 2D continua, resulting in beam and plate/shell theories.
  - Closed-form solutions exist for bodies with certain geometries and boundary conditions.
- Unless analytical solutions exist, one must resort to approximations of either the equations or the structure.

Navier's Equations *(let's revisit some concepts you know of)*

- There is a class of problems, **statically determinate** or **isostatic** problems, for which it is possible to uniquely derive all unknown reaction forces, including stresses, using only the equilibrium equations.
  - Strains and displacements can be found later using compatibility and constitutive equations.
- When the equilibrium equations alone are not sufficient to solve the problem of determining the stress state, the problem is said to be **statically indeterminate** or **hyperstatic**.
  - The equilibrium equations do not provide enough information to determine the actual stress state.
  - More than one, indeed an infinite number of equilibrated stress states can be identified.
  - Only by considering the other information (compatibility and constitutive equations) can the exact state be chosen among them.
- A structural analysis problem is said to be **statically impossible** when there is no possibility of equilibrium. This is the case with **mechanisms**, where the applied forces are not balanced.

Approximate Solution Methods



Alright, so we need to solve Navier’s equations…

…but actually, we won’t.

Instead, we solve its **weak form**.

Strong Form and Weak Form

Strong Form

- In the strong form, the solution must **satisfy the PDE** and boundary conditions **exactly** at **every point** within the domain.
- The **governing equations** are in their **original differential form**.
- The solution needs to be sufficiently smooth, typically requiring high levels of continuity and differentiability.

Weak Form

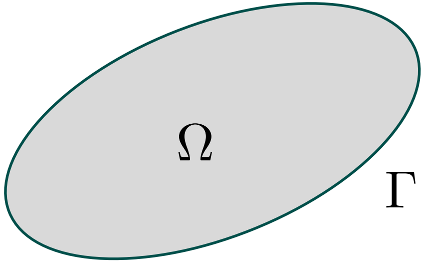
- The weak form is derived by multiplying the PDE by a test function and integrating, reducing the differentiability requirements.
- It **relaxes the pointwise conditions**, focusing on **satisfying the equation in an "average" sense** across the domain.

Weak Forms

Consider a domain $\Omega \subset \mathbb{R}^2$ with boundary Γ. Let us assume that the governing equation for a potential $\phi(x, y)$ in this region is given by Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y)$$

+ BCs on Γ



where $f(x, y)$ represents a source term.

Weak Forms

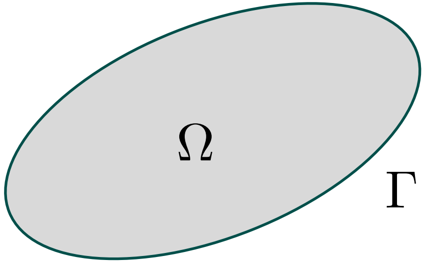
If $\bar{\phi}(x, y)$ is an exact solution

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} - f = 0$$

Let $\hat{\phi}$ be an approximate solution. Then

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} - f = R \neq 0$$

defines the residual R.

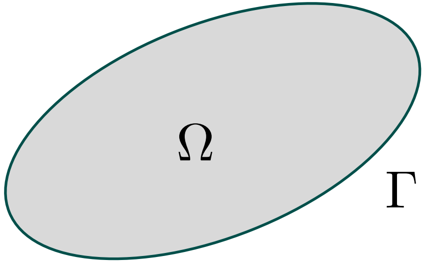


Weak Forms

Our objective becomes finding a $\hat{\phi}$ such that R is close to zero at each point in $\boldsymbol{\Omega}$.

Weighted Residual statement:

$$\int W(x,y)\left(\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}-f\right)d\Omega=0$$



W is a known weight function.

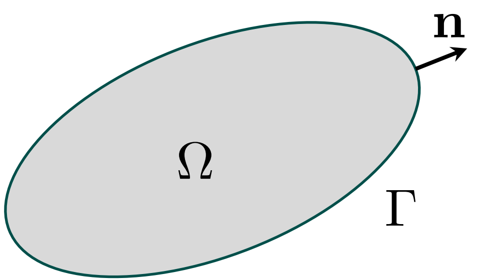
Weak Forms

Rewriting the Weighted Residual statement using the Laplacian operator and using Green's first identity:

$$\int W\left(\nabla^2\hat{\phi} - f\right) d\Omega = 0$$

$$\int W\,\nabla^2\hat{\phi}\,d\Omega - \int W f\,d\Omega = 0$$

$$\int W\,\nabla\hat{\phi}\cdot\mathbf{n}\,d\Gamma - \int \nabla W\cdot\nabla\hat{\phi}\,d\Omega - \int W f\,d\Omega = 0$$



Weak Forms

$$\boxed{\int W\,\nabla\hat{\phi}\cdot\mathbf{n}\,d\Gamma} - \int \nabla W\cdot\nabla\hat{\phi}\,d\Omega - \int W f\,d\Omega = 0$$

Term relating to
boundary conditions

This is where the magic starts, but no domain discretisation yet.

How does this equation lead to the Finite Element formulation?

How do we obtain a system of algebraic equations from here?

Weak Forms

$$\boxed{\int W \, \nabla\hat{\phi} \cdot \mathbf{n} \, d\Gamma} - \int \nabla W \cdot \nabla\hat{\phi} \, d\Omega - \int W f \, d\Omega = 0$$

Term relating to
boundary conditions

Let's answer the second question first.

How do we obtain a system of algebraic equations from here?

Ans: by choosing W and decomposing $\hat{\phi}$ with a finite-dimensional series.

Weak Forms

$$\boxed{\int W\,\nabla\hat{\phi}\cdot\mathbf{n}\,d\Gamma} - \int \nabla W\cdot\nabla\hat{\phi}\,d\Omega - \int W f\,d\Omega = 0$$

Boundary term relating
to boundary conditions

Onto the first question now.

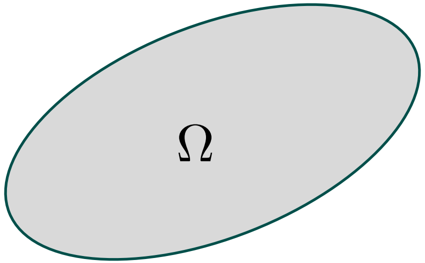
How does this equation lead to the Finite Element formulation?

Ans: by discretisation of Ω into smaller domains with simple shapes over which the weak form with the decomposed $\hat{\phi}$ can be integrated.

The Finite Element Process

The Finite Element Process

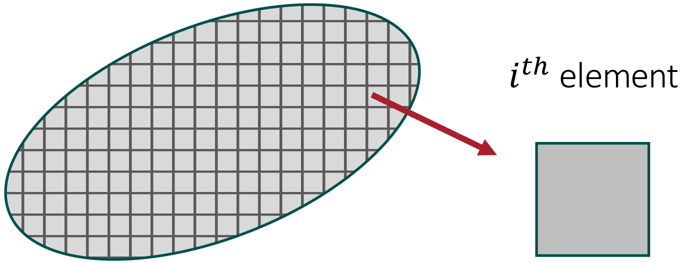
1. The process starts with a weak form over the domain.



$$\int W\,\nabla\hat{\phi}\cdot\mathbf{n}\,d\Gamma - \int \nabla W\cdot\nabla\hat{\phi}\,d\Omega - \int W f\,d\Omega = 0$$

The Finite Element Process

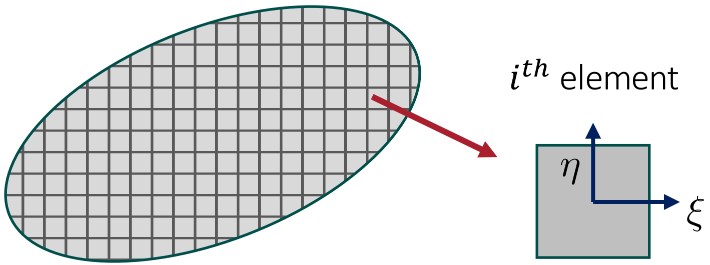
2. Domain discretisation. Weak form over individual elements.



$$\left(\int W \, \nabla \hat{\phi} \cdot \mathbf{n} \, d\Gamma - \int \nabla W \cdot \nabla \hat{\phi} \, d\Omega - \int W f \, d\Omega \right)_i = 0$$

The Finite Element Process

3. Shape functions and elementwise integration → Algebraic equations.



i^{th} element

Nodal d.o.f.
The new unknowns

Weak form integration

$$\mathbf{A}_i \boldsymbol{\phi}_i = \mathbf{f}_i \longleftarrow \hat{\phi}^{(e)}(\xi, \eta) = \sum_j N_j(\xi, \eta) \cdot \phi_j^{\text{nodal}}$$

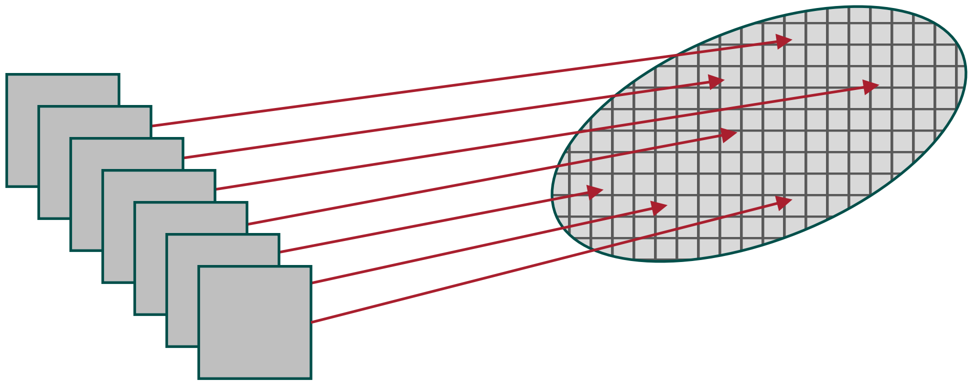
This is the decomposition of $\hat{\phi}$ mentioned earlier

Element coordinates

Shape Functions (known)

The Finite Element Process

4. Assembly of global matrix equation and solution.



$$\mathbf{A}\phi = \mathbf{f}$$

$N \times N$ matrix

$N \times 1$ vectors

$$\phi = \mathbf{A}^{-1}\mathbf{f}$$

N = total number of d.o.f.

The assembly technique exploits the continuity of ϕ and the balance of f. More on this later.

Convergence and Accuracy

- The **accuracy** of the finite element model improves as the size of the elements used in the discretization decreases.
- As (well-formulated) finite elements become smaller, the solution **converges asymptotically** and **monotonically** to that of the real, continuous structure. → ***h-refinement***
- As the order of the shape functions, i.e. the number of degrees of freedom characterising the behaviour of each element, increases, the solution becomes more accurate. → ***p-refinement***
- The trade-off between computational cost and solution accuracy can be controlled by adjusting the discretization.

The Principle of Virtual Work

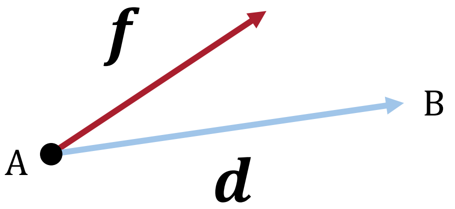
- In Solid Mechanics special weak forms exists:
  - The **Principle of Virtual Displacements**
    - A special case of the Principle of Virtual Work
  - The **Minimum Total Potential Energy Principle**

- Before we consider the principle of virtual work, let’s refresh our memory on what is meant by work.

The Principle of Virtual Work

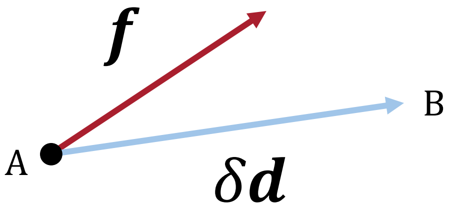
- Work is done when a force moves its point of application.

- For a particle, $W = \boldsymbol{f} \cdot \boldsymbol{d} = \boldsymbol{f}^{\top}\boldsymbol{d}$.



The Principle of Virtual Displacements (PVD)

*A **particle** is in equilibrium under the action of a system of forces if the total work done by the forces is zero for any virtual displacement of the particle.*



$$\delta W = \boldsymbol{f}^{\top} \delta \boldsymbol{d} = 0$$

Variational Statement
of equilibrium

Virtual: compatible but not necessarily equilibrated

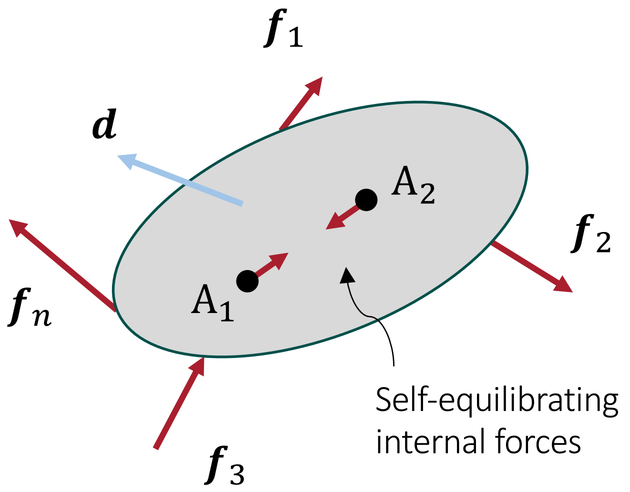
The Principle of Virtual Displacements

*A **rigid body** is in equilibrium under the action of a system of forces if the total work done by the forces is zero for any virtual displacement of the body.*

$$\delta W_t = \delta W_e + \cancel{\delta W_i} = \delta W_e = 0$$

total work = work of internal and external forces

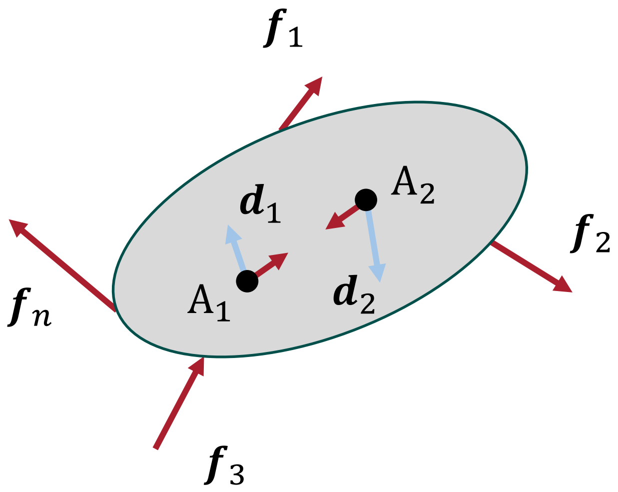
The sum of the virtual work done on A_1 and A_2 is zero.



The Principle of Virtual Displacements

For elastic bodies, the strain field over the deformable continuum makes $\delta W_i \neq 0$. Hence

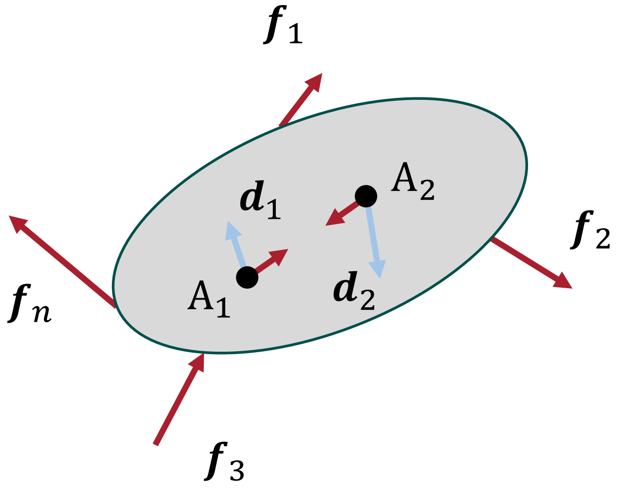
$$\delta W_t = \delta W_e + \delta W_i = 0$$



The Principle of Virtual Displacements

Consider a body in equilibrium under the action of external forces (e.g., body forces, $\boldsymbol{b}$, surface tractions, $\boldsymbol{p}$) and internal stresses.

If an infinitesimal, virtual change in configuration is applied to the body, which respects all compatibility conditions but not necessarily equilibrium, and this virtual change is represented by a set of virtual displacements $\delta\boldsymbol{u}$, then the system is in equilibrium if and only if the total virtual work done by the external and internal forces during any virtual displacement is equal to zero.

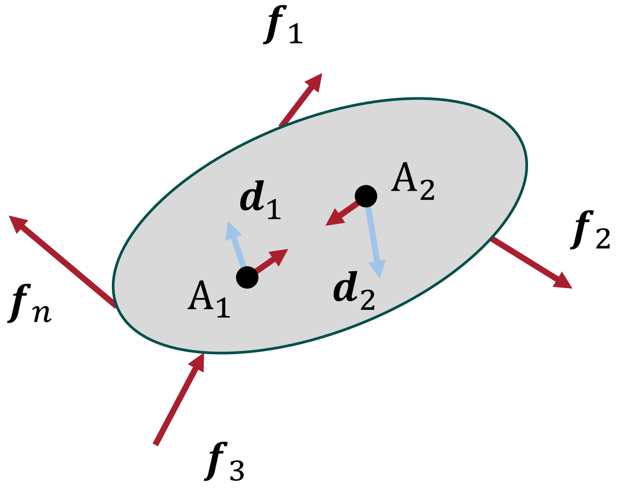


The Principle of Virtual Displacements

Note that in the above arguments only the conditions of equilibrium and the concept of work are employed.

Thus, the PVD does not require the deformable body to be linearly elastic (i.e. it need not obey Hooke's law) so that the principle of virtual work may be applied to any body that is rigid, elastic or plastic.

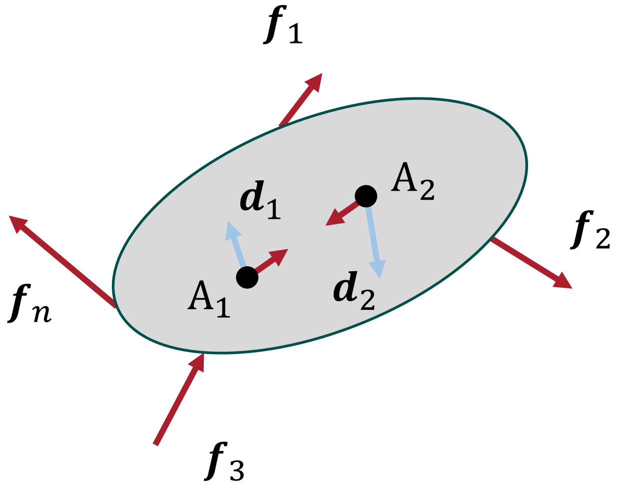
The PVD is also valid for large displacements.



The Principle of Virtual Displacements

Assuming linear **elastic material** properties and **small displacements**, i.e. linearity, as well as that the system is subject only to **conservative forces**, the PVD can be expressed as a special case known as the **Principle of Minimum Total Potential Energy**.

These assumptions allow us to express the principles using the same field variables employed in Navier's equations.



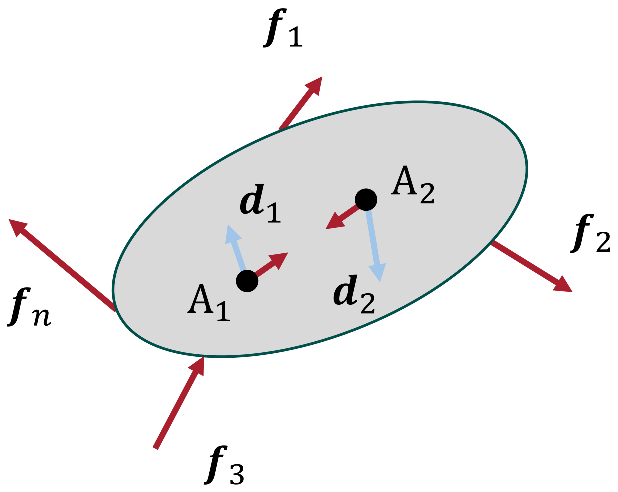
The Principle of Minimum Total Potential Energy

The **total potential energy**, Π, is the sum of the **elastic strain energy**, U, stored in the deformed body and the **potential energy**, V, associated to the applied forces.

$$\Pi = V + U$$

The total potential energy of an elastic system has a stationary value for all virtual displacements when the system is in equilibrium; further, the equilibrium is stable if the stationary value is a minimum.

$$\delta\Pi = \delta V + \delta U = 0$$



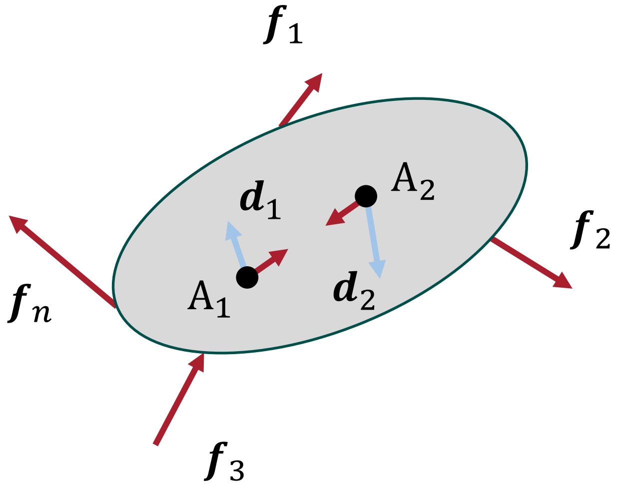
The Principle of Minimum Total Potential Energy

$$\delta\Pi = \delta V + \delta U = 0$$

and

$$\delta W_t = \delta W_e + \delta W_i = 0$$

are equivalent.



The Principle of Minimum Total Potential Energy

$$\delta\Pi = \delta V + \delta U = 0 \quad \Longrightarrow \quad \delta U = -\delta V$$

which can be written as

$$\int_S \boldsymbol{p}^{\mathsf{T}} \delta \boldsymbol{u} \, dS + \int_V \boldsymbol{b}^{\mathsf{T}} \delta \boldsymbol{u} \, dV = \int_V \boldsymbol{\sigma}^{\mathsf{T}} \delta \boldsymbol{\varepsilon} \, dV$$

- $\boldsymbol{\sigma}$ is the stress tensor,
- $\delta\boldsymbol{\varepsilon}$ represents the virtual strain corresponding to the virtual displacement $\delta\boldsymbol{u}$,
- $\boldsymbol{p}$ are surface tractions,
- $\boldsymbol{b}$ represents body forces.

The Principle of Minimum Total Potential Energy

$$\int_S \boldsymbol{p}^\mathsf{T} \delta \boldsymbol{u} \, dS + \int_V \boldsymbol{b}^\mathsf{T} \delta \boldsymbol{u} \, dV = \int_V \boldsymbol{\sigma}^\mathsf{T} \delta \boldsymbol{\varepsilon} \, dV$$

This is the weak form we were looking for!

It can be show to be an alternative formulation of the equilibrium equations.

Governing Equations in Matrix Form

Governing Equations in Matrix Form

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x},$$
$$\epsilon_{yy} = \frac{\partial u_y}{\partial y},$$
$$\epsilon_{zz} = \frac{\partial u_z}{\partial z},$$
$$\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x},$$
$$\epsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y},$$
$$\epsilon_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

$$\Rightarrow \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \Rightarrow \boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{u}$$

Governing Equations in Matrix Form

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ \epsilon_{xy}\\ \epsilon_{yz}\\ \epsilon_{zx}\end{bmatrix} \Rightarrow \boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}$$

Governing Equations in Matrix Form

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = - \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\boldsymbol{D}^{\mathsf{T}} \boldsymbol{\sigma} = -\boldsymbol{b}$$

Governing Equations in Matrix Form

- Navier's equations:

$$\underbrace{\boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{u} \quad \Rightarrow \quad \boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon} \quad \Rightarrow \quad \boldsymbol{D}^{\mathsf{T}}\boldsymbol{\sigma} = -\boldsymbol{b}}$$

$$\boldsymbol{D}^{\mathsf{T}}\boldsymbol{E}\boldsymbol{D}\boldsymbol{u} = -\boldsymbol{b}$$

Governing Equations in Matrix Form

- TPE Principle:

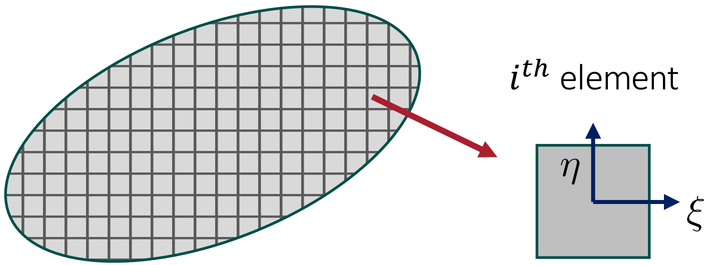
$$\int_S \boldsymbol{p}^\mathsf{T} \delta \boldsymbol{u} \, dS + \int_V \boldsymbol{b}^\mathsf{T} \delta \boldsymbol{u} \, dV = \int_V \boldsymbol{\sigma}^\mathsf{T} \delta \boldsymbol{\varepsilon} \, dV$$

$$\int_S \boldsymbol{p}^\mathsf{T} \delta \boldsymbol{u} \, dS + \int_V \boldsymbol{b}^\mathsf{T} \delta \boldsymbol{u} \, dV = \int_V \boldsymbol{u}^\mathsf{T} \boldsymbol{D}^\mathsf{T} \boldsymbol{E} \boldsymbol{D} \, \delta \boldsymbol{u} \, dV$$

Finite Element Formulation

Recall step 3 of the Finite Element Process

3. Shape functions and elementwise integration → Algebraic equations.



Nodal d.o.f.
The new unknowns

Weak form integration

$$\mathbf{A}_i\boldsymbol{\phi}_i = \mathbf{f}_i \quad \longleftarrow \quad \hat{\phi}^{(e)}(\xi,\eta) = \sum_j N_j(\xi,\eta)\cdot\phi_j^{\text{nodal}}$$

This is the decomposition of $\hat{\phi}$ mentioned earlier

Element coordinates

Shape Functions (known)

Finite Element Formulation

The first step to calculating the properties of a finite element is to suppose that the internal displacements $\boldsymbol{u}$ are expressed in terms of discrete nodal variables through the matrix relation

$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{d}$$

where $\boldsymbol{N}$ is a matrix of shape functions that account for the deformed configuration of the element as a function of nodal parameters, $\boldsymbol{d}$.

Finite Element Formulation

Substituting $$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{d}$$

into $$\int_S \boldsymbol{p}^\mathsf{T} \delta\boldsymbol{u}\, dS + \int_V \boldsymbol{b}^\mathsf{T} \delta\boldsymbol{u}\, dV = \int_V \boldsymbol{u}^\mathsf{T} \boldsymbol{D}^\mathsf{T} \boldsymbol{E}\boldsymbol{D}\, \delta\boldsymbol{u}\, dV$$

$$\int_S \boldsymbol{p}^\mathsf{T} \boldsymbol{N}\delta\boldsymbol{d}\, dS + \int_V \boldsymbol{b}^\mathsf{T} \boldsymbol{N}\delta\boldsymbol{d}\, dV = \int_V \boldsymbol{d}^\mathsf{T} \boldsymbol{N}^\mathsf{T} \boldsymbol{D}^\mathsf{T} \boldsymbol{E}\boldsymbol{D}\, \boldsymbol{N}\delta\boldsymbol{d}\, dV$$

Finite Element Formulation

The expression needs further work.

$$\int_S \boldsymbol{p}^\top \boldsymbol{N} \delta \boldsymbol{d} \, dS + \int_V \boldsymbol{b}^\top \boldsymbol{N} \delta \boldsymbol{d} \, dV = \int_V \boldsymbol{d}^\top \boldsymbol{N}^\top \boldsymbol{D}^\top \boldsymbol{E} \boldsymbol{D} \, \boldsymbol{N} \delta \boldsymbol{d} \, dV$$

$$\left(\int_S \boldsymbol{p}^\top \boldsymbol{N} \, dS \right) \cancel{\delta \boldsymbol{d}} + \left(\int_V \boldsymbol{b}^\top \boldsymbol{N} \, dV \right) \cancel{\delta \boldsymbol{d}} = \boldsymbol{d}^\top \left(\int_V \boldsymbol{N}^\top \boldsymbol{D}^\top \boldsymbol{E} \boldsymbol{D} \, \boldsymbol{N} \, dV \right) \cancel{\delta \boldsymbol{d}}$$

Finite Element Formulation

Recall $(AB)^\top = B^\top A^\top$.

$$\left(\int_S \boldsymbol{p}^\top \boldsymbol{N}\, dS\right) \cancel{\delta \boldsymbol{d}} + \left(\int_V \boldsymbol{b}^\top \boldsymbol{N}\, dV\right) \cancel{\delta \boldsymbol{d}} = \boldsymbol{d}^\top \left(\int_V \boldsymbol{N}^\top \boldsymbol{D}^\top \boldsymbol{E} \boldsymbol{D}\, \boldsymbol{N}\, dV\right) \cancel{\delta \boldsymbol{d}}$$

$$\left(\int_S \boldsymbol{p}^\top \boldsymbol{N}\, dS\right) + \left(\int_V \boldsymbol{b}^\top \boldsymbol{N}\, dV\right) = \boldsymbol{d}^\top \left(\int_V (\boldsymbol{D}\boldsymbol{N})^\top \boldsymbol{E}\, (\boldsymbol{D}\boldsymbol{N})\, dV\right)$$

Finite Element Formulation

Commonly,

$$\boldsymbol{D}\boldsymbol{N} = \boldsymbol{B}$$

Hence,

$$\left(\int_S \boldsymbol{p}^\mathsf{T}\boldsymbol{N}\,dS\right) + \left(\int_V \boldsymbol{b}^\mathsf{T}\boldsymbol{N}\,dV\right) = \boldsymbol{d}^\mathsf{T}\left(\int_V \left(\boldsymbol{D}\boldsymbol{N}\right)^\mathsf{T}\boldsymbol{E}\left(\boldsymbol{D}\boldsymbol{N}\right)\,dV\right)$$

$$\left(\int_S \boldsymbol{p}^\mathsf{T}\boldsymbol{N}\,dS\right) + \left(\int_V \boldsymbol{b}^\mathsf{T}\boldsymbol{N}\,dV\right) = \boldsymbol{d}^\mathsf{T}\left(\int_V \boldsymbol{B}^\mathsf{T}\boldsymbol{E}\boldsymbol{B}\,dV\right)$$

Finite Element Formulation

Finally,

$$\left(\int_S \boldsymbol{p}^\mathsf{T}\boldsymbol{N}\,dS\right) + \left(\int_V \boldsymbol{b}^\mathsf{T}\boldsymbol{N}\,dV\right) = \boldsymbol{d}^\mathsf{T}\left(\int_V \boldsymbol{B}^\mathsf{T}\boldsymbol{E}\boldsymbol{B}\,dV\right)$$

$$\boldsymbol{f}^\mathsf{T} = \boldsymbol{d}^\mathsf{T}\boldsymbol{K}^\mathsf{T} \qquad \text{or} \qquad \boldsymbol{f} = \boldsymbol{K}\boldsymbol{d}$$

where $\boldsymbol{K} = \int_V \boldsymbol{B}^\mathsf{T}\boldsymbol{E}\boldsymbol{B}\,dV$ $\qquad$ $\boldsymbol{f} = \int_S \boldsymbol{N}^\mathsf{T}\boldsymbol{p}\,dS + \int_V \boldsymbol{N}^\mathsf{T}\boldsymbol{b}\,dV$

Finite Element Formulation

In conclusion, we have derived the algebraic equations governing the behaviour of the generic finite element:

$$\boldsymbol{f} = \boldsymbol{K}\boldsymbol{d}$$

$\boldsymbol{f}$ is a vector of equivalent nodal forces

$\boldsymbol{d}$ is the nodal displacements vector

$\boldsymbol{K}$ is the element stiffness matrix