CHAPTER

Composite Beams

12

Frequently in civil engineering construction beams are fabricated from comparatively inexpensive materials of low strength which are reinforced by small amounts of high-strength material, such as steel. In this way a timber beam of rectangular section may have steel plates bolted to its sides or to its top and bottom surfaces. Again, concrete beams are reinforced in their weak tension zones and also, if necessary, in their compression zones, by steel-reinforcing bars. Other instances arise where steel beams support concrete floor slabs in which the strength of the concrete may be allowed for in the design of the beams. The design of reinforced concrete beams, and concrete and steel beams is covered by Codes of Practice and relies, as in the case of steel beams, on ultimate load analysis. The design of steel-reinforced timber beams is not covered by a code, and we shall therefore limit the analysis of this type of beam to an elastic approach.

12.1 Steel-reinforced timber beams

The timber joist of breadth b and depth d shown in Fig. 12.1 is reinforced by two steel plates bolted to its sides, each plate being of thickness t and depth d. Let us suppose that the beam is bent to a radius R at this section by a positive bending moment, M. Clearly, since the steel plates are firmly attached to the sides of the timber joist, both are bent to the same radius, R. Then, from Eq. (9.7), the bending moment, M_t, carried by the timber joist is

$$M_t = \frac{E_t I_t}{R} \tag{12.1}$$

where E_t is Young's modulus for the timber and I_t is the second moment of area of the timber section about the centroidal axis, Gz. Similarly for the steel plates

$$M_s = \frac{E_s I_s}{R} \tag{12.2}$$

in which I_s is the combined second moment of area about Gz of the two plates. The total bending moment is then

$$M = M_t + M_s = \frac{1}{R}(E_t I_t + E_s I_s)$$

from which

$$\frac{1}{R} = \frac{M}{E_t I_t + E_s I_s} \tag{12.3}$$

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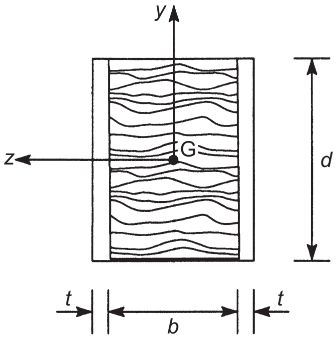

FIGURE 12.1

Steel-reinforced timber beam.

From a comparison of Eqs (12.3) and (9.7) we see that the composite beam behaves as a homogeneous beam of bending stiffness EI where

$$EI = E_t I_t + E_s I_s$$

or

$$EI = E_t\left(I_t + \frac{E_s}{E_t} I_s\right) \tag{12.4}$$

The composite beam may therefore be treated wholly as a timber beam having a total second moment of area

$$I_t + \frac{E_s}{E_t} I_s$$

This is equivalent to replacing the steel-reinforcing plates by timber 'plates' each having a thickness $(E_s/E_t)t$ as shown in Fig. 12.2(a). Alternatively, the beam may be transformed into a wholly steel beam by writing Eq. (12.4) as

$$EI = E_s\left(\frac{E_t}{E_s} I_t + I_s\right)$$

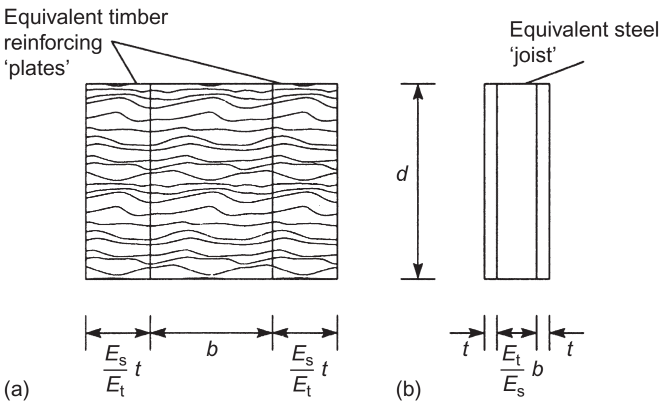

FIGURE 12.2

Equivalent beam sections.

so that the second moment of area of the equivalent steel beam is

$$\frac{E_t}{E_s}I_t + I_s$$

which is equivalent to replacing the timber joist by a steel 'joist' of breadth $(E_t/E_s)b$ (Fig. 12.2(b)). Note that the transformed sections of Fig. 12.2 apply only to the case of bending about the horizontal axis, Gz. Note also that the depth, d, of the beam is unchanged by either transformation.

The direct stress due to bending in the timber joist is obtained using Eq. (9.9), i.e.

$$\sigma_t = -\frac{M_t y}{I_t} \tag{12.5}$$

From Eqs (12.1) and (12.3)

$$M_t = \frac{E_t I_t}{E_t I_t + E_s I_s} M$$

or

$$M_t = \frac{M}{1 + \dfrac{E_s I_s}{E_t I_t}} \tag{12.6}$$

Substituting in Eq. (12.5) from Eq. (12.6) we have

$$\sigma_t = -\frac{My}{I_t + \dfrac{E_s}{E_t} I_s} \tag{12.7}$$

Equation (12.7) could in fact have been deduced directly from Eq. (9.9) since $I_t + (E_s/E_t)I_s$ is the second moment of area of the equivalent timber beam of Fig. 12.2(a). Similarly, by considering the equivalent steel beam of Fig. 12.2(b), we obtain the direct stress distribution in the steel, i.e.

$$\sigma_s = -\frac{My}{I_s + \dfrac{E_t}{E_s} I_t} \tag{12.8}$$

EXAMPLE 12.1

A beam is formed by connecting two timber joists each 100 mm×400 mm with a steel plate 12 mm×300 mm placed symmetrically between them (Fig. 12.3). If the beam is subjected to a bending moment of 50 kNm, determine the maximum stresses in the steel and in the timber. The ratio of Young's modulus for steel to that of timber is 12 : 1.

The second moments of area of the timber and steel about the centroidal axis, Gz, are

$$I_t = 2 \times 100 \times \frac{400^3}{12} = 1067 \times 10^6 \text{ mm}^4$$

and

$$I_s = 12 \times \frac{300^3}{12} = 27 \times 10^6 \text{ mm}^4$$

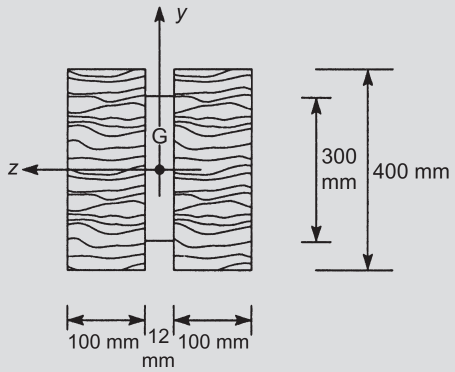

FIGURE 12.3

Steel-reinforced timber beam of Ex. 12.1.

respectively. Therefore, from Eq. (12.7) we have

$$\sigma_t = \pm \frac{50 \times 10^6 \times 200}{1067 \times 10^6 + 12 \times 27 \times 10^6} = \pm 7.2\,\mathrm{N/mm^2}$$

and from Eq. (12.8)

$$\sigma_s = \pm \frac{50 \times 10^6 \times 150}{27 \times 10^6 + 1067 \times 10^6/12} = \pm 64.7\,\mathrm{N/mm^2}$$

Consider now the steel-reinforced timber beam of Fig. 12.4(a) in which the steel plates are attached to the top and bottom surfaces of the timber. The section may be transformed into an equivalent timber beam (Fig. 12.4(b)) or steel beam (Fig. 12.4(c)) by the methods used for the beam of Fig. 12.1. The direct stress distributions are then obtained from Eqs (12.7) and (12.8). There is, however, one important difference between the beam of Fig. 12.1 and that of Fig. 12.4(a). In the latter case, when the beam is subjected to shear loads, the connection between the timber and steel must resist horizontal complementary shear stresses as shown in Fig. 12.5. Generally, it is sufficiently accurate to assume that the timber joist resists all the vertical shear and then calculate an average value of shear stress, τ_{av}, i.e.

$$\tau_{av} = \frac{S_y}{bd}$$

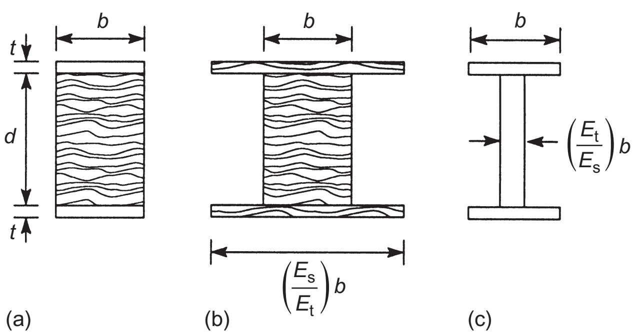

FIGURE 12.4

Reinforced timber beam with steel plates attached to its top and bottom surfaces.

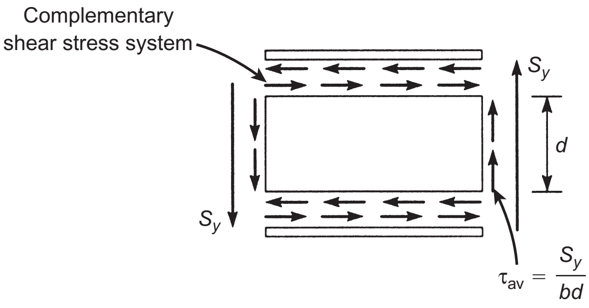

FIGURE 12.5

Shear stresses between steel plates and timber beam (side view of a length of beam).

so that, based on this approximation, the horizontal complementary shear stress is S_y/bd and the shear force per unit length resisted by the timber/steel connection is S_y/d.

EXAMPLE 12.2

A timber joist 100 mm×200 mm is reinforced on its top and bottom surfaces by steel plates 15 mm thick×100 mm wide. The composite beam is simply supported over a span of 4 m and carries a uniformly distributed load of 10 kN/m. Determine the maximum direct stress in the timber and in the steel and also the shear force per unit length transmitted by the timber/steel connection. Take $E_s/E_t = 15$.

The second moments of area of the timber and steel about a horizontal axis through the centroid of the beam are

$$I_t = \frac{100 \times 200^3}{12} = 66.7 \times 10^6 \text{ mm}^4$$

and

$$I_s = 2 \times 15 \times 100 \times 107.5^2 = 34.7 \times 10^6 \text{mm}^4$$

respectively. Note that the second moment of area of a steel plate about an axis through its own centroid is negligibly small. The maximum bending moment in the beam occurs at mid-span and is

$$M_{\max} = \frac{10 \times 4^2}{8} = 20 \text{ kNm}$$

From Eq. (12.7)

$$\sigma_{t,\max} = \pm \frac{20 \times 10^6 \times 100}{66.7 \times 10^6 + 15 \times 34.7 \times 10^6} = \pm 3.4 \text{ N/mm}^2$$

and from Eq. (12.8)

$$\sigma_{s,\max} = \pm \frac{20 \times 10^6 \times 115}{34.7 \times 10^6 + 66.7 \times 10^6/15} = \pm 58.8 \text{ N/mm}^2$$

The maximum shear force in the beam occurs at the supports and is equal to 10×4/2 = 20 kN. The average shear stress in the timber joist is then

$$\tau_{av} = \frac{20 \times 10^3}{100 \times 200} = 1\,\text{N/mm}^2$$

It follows that the shear force per unit length in the timber/steel connection is $1 \times 100 = 100$ N/mm or 100 kN/m. Note that this value is an approximation for design purposes since, as we saw in Chapter 10, the distribution of shear stress through the depth of a beam of rectangular section is not uniform.

12.2 Reinforced concrete beams

As we have noted in Chapter 8, concrete is a brittle material which is weak in tension. It follows that a beam comprised solely of concrete would have very little bending strength since the concrete in the tension zone of the beam would crack at very low values of load. Concrete beams are therefore reinforced in their tension zones (and sometimes in their compression zones) by steel bars embedded in the concrete. Generally, whether the beam is precast or forms part of a slab/beam structure, the bars are positioned in a mould (usually fabricated from timber and called formwork) into which the concrete is poured. On setting, the concrete shrinks and grips the steel bars; the adhesion or *bond* between the bars and the concrete transmits bending and shear loads from the concrete to the steel.

In the design of reinforced concrete beams the elastic method has been superseded by the ultimate load method. We shall, however, for completeness, consider both methods.

Elastic theory

Consider the concrete beam section shown in Fig. 12.6(a). The beam is subjected to a bending moment, M, and is reinforced in its tension zone by a number of steel bars of total cross-sectional area A_s. The centroid of the reinforcement is at a depth d_1 from the upper surface of the beam; d_1 is known as the *effective depth* of the beam. The bending moment, M, produces compression in the concrete above the neutral axis whose position is at some, as yet unknown, depth, n, below the upper surface of the beam. Below the neutral axis the concrete is in tension and is assumed to crack so that its contribution to the bending strength of the beam is negligible. All tensile forces are therefore resisted by the reinforcing steel.

The reinforced concrete beam section may be conveniently analysed by the method employed in Section 12.1 for steel-reinforced beams. The steel reinforcement is, therefore, transformed into an equivalent area, mA_s, of concrete in which m, the *modular ratio*, is given by

$$m = \frac{E_s}{E_c}$$

where E_s and E_c are Young's moduli for steel and concrete, respectively. The transformed section is shown in Fig. 12.6(b). Taking moments of areas about the neutral axis we have

$$bn\frac{n}{2} = mA_s(d_1 - n)$$

which, when rearranged, gives a quadratic equation in n, i.e.

$$\frac{bn^2}{2} + mA_s n - mA_s d_1 = 0 \qquad (12.9)$$

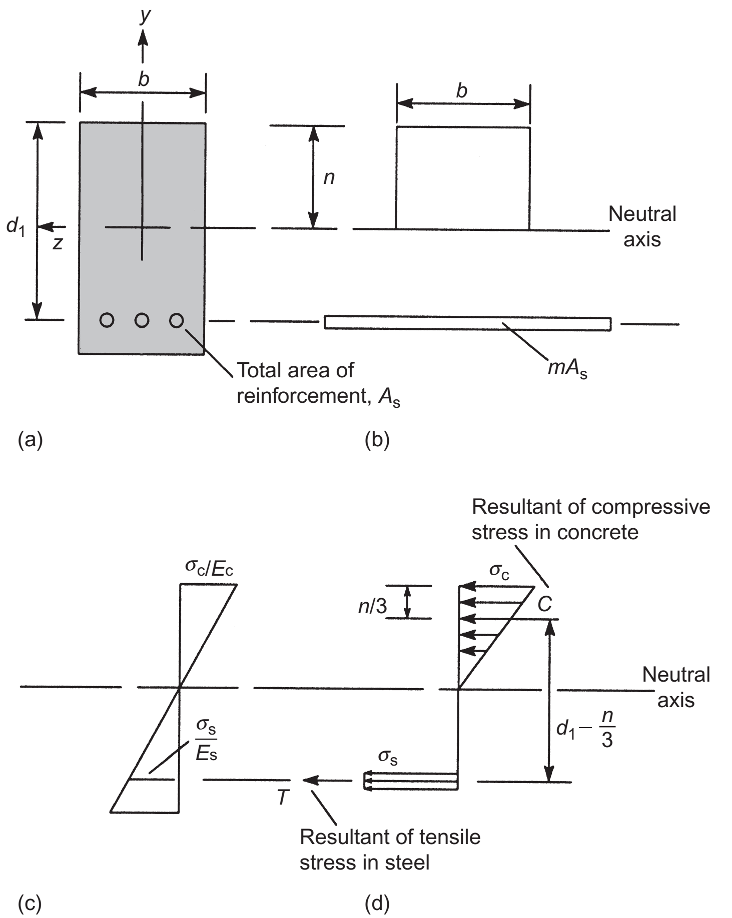

FIGURE 12.6

Reinforced concrete beam.

solving gives

$$n = \frac{mA_s}{b}\left(\sqrt{1 + \frac{2bd_1}{mA_s}} - 1\right) \tag{12.10}$$

Note that the negative solution of Eq. (12.9) has no practical significance and is therefore ignored. The second moment of area, I_c, of the transformed section is

$$I_c = \frac{bn^3}{3} + mA_s(d_1 - n)^2 \tag{12.11}$$

and the maximum stress, σ_c, induced in the concrete is

$$\sigma_c = -\frac{Mn}{I_c} \tag{12.12}$$

The stress, σ_s, in the steel may be deduced from the strain diagram (Fig. 12.6(c)) which is linear throughout the depth of the beam since the beam section is assumed to remain plane during bending. Then

$$\frac{\sigma_s/E_s}{d_1 - n} = -\frac{\sigma_c/E_c}{n} \quad \text{(note : strains are of opposite sign)}$$

from which

$$\sigma_s = -\sigma_c \frac{E_s}{E_c}\left(\frac{d_1 - n}{n}\right) = -\sigma_c m \left(\frac{d_1 - n}{n}\right) \tag{12.13}$$

Substituting for σ_c from Eq. (12.12) we obtain

$$\sigma_s = \frac{mM}{I_c}(d_1 - n) \tag{12.14}$$

Frequently, instead of determining stresses in a given beam section subjected to a given applied bending moment, we wish to calculate the moment of resistance of a beam when either the stress in the concrete or the steel reaches a maximum allowable value. Equations (12.12) and (12.14) may be used to solve this type of problem but an alternative and more direct method considers moments due to the resultant loads in the concrete and steel. From the stress diagram of Fig. 12.6(d)

$$M = C\left(d_1 - \frac{n}{3}\right)$$

so that

$$M = \frac{\sigma_c}{2} bn \left(d_1 - \frac{n}{3}\right) \tag{12.15}$$

Alternatively, taking moments about the centroid of the concrete stress diagram

$$M = T\left(d_1 - \frac{n}{3}\right)$$

or

$$M = \sigma_s A_s \left(d_1 - \frac{n}{3}\right) \tag{12.16}$$

Equation (12.16) may also be used in conjunction with Eq. (12.13) to 'design' the area of reinforcing steel in a beam section subjected to a given bending moment so that the stresses in the concrete and steel attain their maximum allowable values simultaneously. Such a section is known as a *critical* or *economic* section. The position of the neutral axis is obtained directly from Eq. (12.13) in which σ_s, σ_c, m and d_1 are known. The required area of steel is then determined from Eq. (12.16).

EXAMPLE 12.3

A rectangular section reinforced concrete beam has a breadth of 200 mm and is 350 mm deep to the centroid of the steel reinforcement which consists of two steel bars each having a diameter of 20 mm. If the beam is subjected to a bending moment of 30 kNm, calculate the stress in the concrete and in the steel. The modular ratio m is 15.

The area A_s of the steel reinforcement is given by

$$A_s = 2 \times \frac{\pi}{4} \times 20^2 = 628.3\ \text{mm}^2$$

The position of the neutral axis is obtained from Eq. (12.10) and is

$$n = \frac{15 \times 628.3}{200}\left(\sqrt{1 + \frac{2 \times 200 \times 350}{15 \times 628.3}} - 1\right) = 140.5\,\text{mm}$$

Now using Eq. (12.11)

$$I_c = \frac{200 \times 140.5^3}{3} + 15 \times 628.3(350 - 140.5)^2 = 598.5 \times 10^6\,\text{mm}^4$$

The maximum stress in the concrete follows from Eq. (12.12), i.e.

$$\sigma_c = -\frac{30 \times 10^6 \times 140.5}{598.5 \times 10^6} = -7.0\,\text{N/mm}^2(\text{compression})$$

and from Eq. (12.14)

$$\sigma_s = \frac{15 \times 30 \times 10^6}{598.5 \times 10^6}(350 - 140.5) = 157.5\,\text{N/mm}^2(\text{tension})$$

EXAMPLE 12.4

A reinforced concrete beam has a rectangular section of breadth 250 mm and a depth of 400 mm to the steel reinforcement, which consists of three 20 mm diameter bars. If the maximum allowable stresses in the concrete and steel are 7.0 N/mm^2 and 140 N/mm^2, respectively, determine the moment of resistance of the beam. The modular ratio $m = 15$.

The area, A_s, of steel reinforcement is

$$A_s = 3 \times \frac{\pi}{4} \times 20^2 = 942.5\,\text{mm}^2$$

From Eq. (12.10)

$$n = \frac{15 \times 942.5}{250}\left(\sqrt{1 + \frac{2 \times 250 \times 400}{15 \times 942.5}} - 1\right) = 163.5\,\text{mm}$$

The maximum bending moment that can be applied such that the permissible stress in the concrete is not exceeded is given by Eq. (12.15). Thus

$$M = \frac{7}{2} \times 250 \times 163.5\left(400 - \frac{163.5}{3}\right) \times 10^{-6} = 49.4\,\text{kNm}$$

Similarly, from Eq. (12.16) the stress in the steel limits the applied moment to

$$M = 140 \times 942.5\left(400 - \frac{163.5}{3}\right) \times 10^{-6} = 45.6\,\text{kNm}$$

The steel is therefore the limiting material and the moment of resistance of the beam is 45.6 kNm.

EXAMPLE 12.5

A rectangular section reinforced concrete beam is required to support a bending moment of 40 kNm and is to have dimensions of breadth 250 mm and effective depth 400 mm. The maximum allowable stresses in the steel and concrete are 120 N/mm^2 and 6.5 N/mm^2, respectively; the modular ratio is 15. Determine the required area of reinforcement such that the limiting stresses in the steel and concrete are attained simultaneously.

Using Eq. (12.13) we have

$$120 = 6.5 \times 15\left(\frac{400}{n} - 1\right)$$

from which $n = 179.3$ mm.

The required area of steel is now obtained from Eq. (12.16); hence

$$A_s = \frac{M}{\sigma_s(d_1 - n/3)}$$

i.e.

$$A_s = \frac{40 \times 10^6}{120(400 - 179.3/3)} = 979.7\ \text{mm}^2$$

It may be seen from Ex. 12.4 that for a beam of given cross-sectional dimensions, increases in the area of steel reinforcement do not result in increases in the moment of resistance after a certain value has been attained. When this stage is reached the concrete becomes the limiting material, so that additional steel reinforcement only serves to reduce the stress in the steel. However, the moment of resistance of a beam of a given cross section may be increased above the value corresponding to the limiting concrete stress by the addition of steel in the compression zone of the beam.

Figure 12.7(a) shows a concrete beam reinforced in both its tension and compression zones. The centroid of the compression steel of area A_{sc} is at a depth d_2 below the upper surface of the beam, while the tension steel of area A_{st} is at a depth d_1. The section may again be transformed into an equivalent concrete section as shown in Fig. 12.7(b).

However, when determining the second moment of area of the transformed section it must be remembered that the area of concrete in the compression zone is reduced due to the presence of the steel. Thus taking moments of areas about the neutral axis we have

$$\frac{bn^2}{2} - A_{sc}(n - d_2) + mA_{sc}(n - d_2) = mA_{st}(d_1 - n)$$

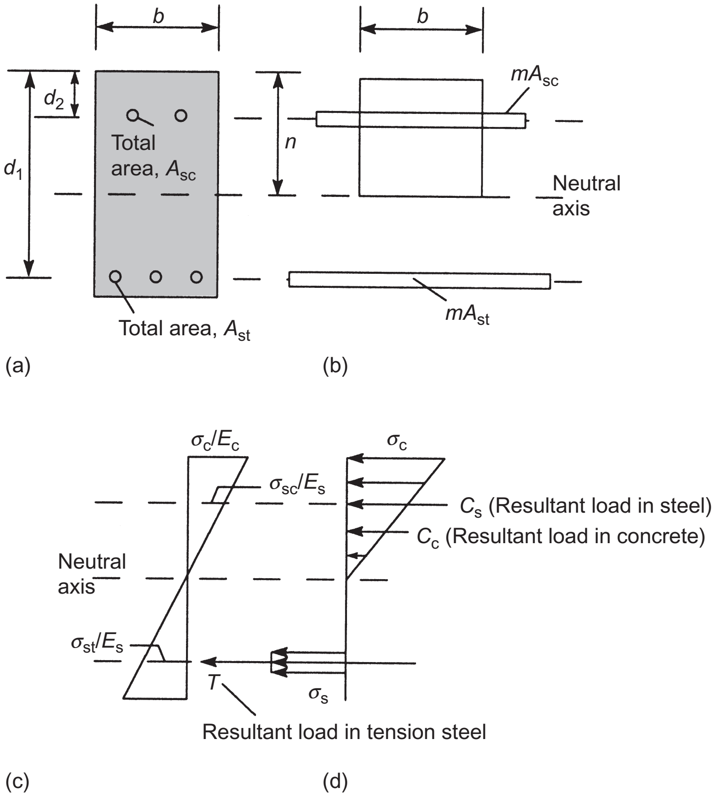

FIGURE 12.7

Reinforced concrete beam with steel in tension and compression zones.

or, rearranging

$$\frac{bn^2}{2} + (m-1)A_{sc}(n-d_2) = mA_{st}(d_1 - n) \tag{12.17}$$

It can be seen from Eq. (12.17) that multiplying A_{sc} by $(m-1)$ in the transformation process rather than m automatically allows for the reduction in the area of concrete caused by the presence of the compression steel. Thus the second moment of area of the transformed section is

$$I_c = \frac{bn^3}{3} + (m-1)A_{sc}(n-d_2)^2 + mA_{st}(d_1 - n)^2 \tag{12.18}$$

The maximum stress in the concrete is then

$$\sigma_c = -\frac{Mn}{I_c} \quad \text{(see Eq. (12.12))}$$

The stress in the tension steel and in the compression steel are obtained from the strain diagram of Fig. 12.7(c). Hence

$$\frac{\sigma_{sc}/E_s}{n-d_2} = \frac{\sigma_c/E_c}{n} \quad \text{(both strains have the same sign)} \tag{12.19}$$

so that

$$\sigma_{sc} = \frac{m(n - d_2)}{n}\sigma_c = -\frac{mM(n - d_2)}{I_c} \tag{12.20}$$

and

$$\sigma_{st} = \frac{mM}{I_c}(d_1 - n) \text{ as before} \tag{12.21}$$

An alternative expression for the moment of resistance of the beam is derived by taking moments of the resultant steel and concrete loads about the compressive reinforcement. Therefore from the stress diagram of Fig. 12.7(d)

$$M = T(d_1 - d_2) - C_c\left(\frac{n}{3} - d_2\right)$$

whence

$$M = \sigma_{st}A_{st}(d_1 - d_2) - \frac{\sigma_c}{2}bn\left(\frac{n}{3} - d_2\right) \tag{12.22}$$

EXAMPLE 12.6

A rectangular section concrete beam is 180 mm wide and has a depth of 360 mm to its tensile reinforcement. It is subjected to a bending moment of 45 kNm and carries additional steel reinforcement in its compression zone at a depth of 40 mm from the upper surface of the beam. Determine the necessary areas of reinforcement if the stress in the concrete is limited to 8.5 N/mm^2 and that in the steel to 140 N/mm^2. The modular ratio $E_s/E_c = 15$.

Assuming that the stress in the tensile reinforcement and that in the concrete attain their limiting values we can determine the position of the neutral axis using Eq. (12.13). Thus

$$140 = 8.5 \times 15\left(\frac{360}{n} - 1\right)$$

from which

$$n = 171.6\,\text{mm}$$

Substituting this value of n in Eq. (12.22) we have

$$45 \times 10^6 = 140A_{st}(360 - 40) + \frac{8.5}{2} \times 180 \times 171.6\left(\frac{171.6}{3} - 40\right)$$

which gives

$$A_{st} = 954\,\text{mm}^2$$

We can now use Eq. (12.17) to determine A_{sc} or, alternatively, we could equate the load in the tensile steel to the combined compressive load in the concrete and compression steel. Substituting for n and A_{st} in Eq. (12.17) we have

$$\frac{180 \times 171.6^2}{2} + (15 - 1)A_{sc}(171.6 - 40) = 15 \times 954(360 - 171.6)$$

from which

$$A_{sc} = 24.9\,\text{mm}^2$$

The stress in the compression steel may be obtained from Eq. (12.20), i.e.

$$\sigma_{sc} = -15\frac{(171.6-40)}{171.6}\times 8.5 = -97.8\,\text{N/mm}^2 \quad \text{(compression)}$$

In many practical situations reinforced concrete beams are cast integrally with floor slabs, as shown in Fig. 12.8. Clearly, the floor slab contributes to the overall strength of the structure so that the part of the slab adjacent to a beam may be regarded as forming part of the beam. The result is a T-beam whose flange, or the major portion of it, is in compression. The assumed width, B, of the flange cannot be greater than L, the distance between the beam centres; in most instances B is specified in Codes of Practice.

It is usual to assume in the analysis of T-beams that the neutral axis lies within the flange or coincides with its under surface. In either case the beam behaves as a rectangular section concrete beam of width B and effective depth d_1 (Fig. 12.9). Therefore, the previous analysis of rectangular section beams still applies.

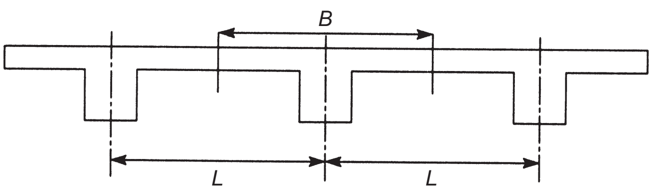

FIGURE 12.8

Slab-reinforced concrete beam arrangement.

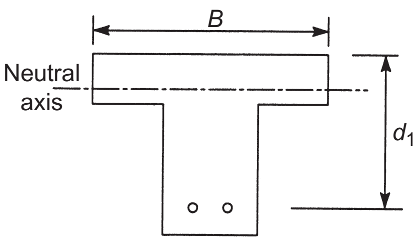

FIGURE 12.9

Analysis of a reinforced concrete T-beam.

Ultimate load theory

We have previously noted in this chapter and also in Chapter 8 that the modern design of reinforced concrete structures relies on ultimate load theory. The calculated moment of resistance of a beam section is therefore based on the failure strength of concrete in compression and the yield strength of the steel reinforcement in tension modified by suitable factors of safety. Typical values are 1.5 for concrete (based on its 28-day cube strength) and 1.15 for steel. However, failure of the concrete in compression could occur suddenly in a reinforced concrete beam, whereas failure of the steel by yielding would be gradual. It is therefore preferable that failure occurs in the reinforcement rather than in the concrete. Thus, in design, the capacity of the concrete is underestimated to ensure that the preferred form of failure occurs. A further factor affecting the design stress for concrete stems from tests in which it has been found that concrete subjected to compressive stress due to bending always fails before attaining a compressive stress equal to the 28-day cube strength.

The characteristic strength of concrete in compression is therefore taken as two-thirds of the 28-day cube strength. A typical design strength for concrete in compression is then

$$\frac{\sigma_{\text{cu}}}{1.5} \times 0.67 = 0.45\sigma_{\text{cu}}$$

where σ_{cu} is the 28-day cube strength. The corresponding figure for steel is

$$\frac{\sigma_{\text{Y}}}{1.15} = 0.87\sigma_{\text{Y}}$$

In the ultimate load analysis of reinforced concrete beams it is assumed that plane sections remain plane during bending and that there is no contribution to the bending strength of the beam from the concrete in tension. From the first of these assumptions we deduce that the strain varies linearly through the depth of the beam as shown in Fig. 12.10(b). However, the stress diagram in the concrete is not linear but has the rectangular–parabolic shape shown in Fig. 12.10(c). Design charts in Codes of Practice are based on this stress distribution, but for direct calculation purposes a reasonably accurate approximation can be made in which the rectangular–parabolic stress distribution of Fig. 12.10(c) is replaced by an equivalent rectangular distribution as shown in Fig. 12.11(b) in which the compressive stress in the concrete is assumed to extend down to the mid-effective depth of the section at the maximum condition, i.e. at the ultimate moment of resistance, M_{u}, of the section.

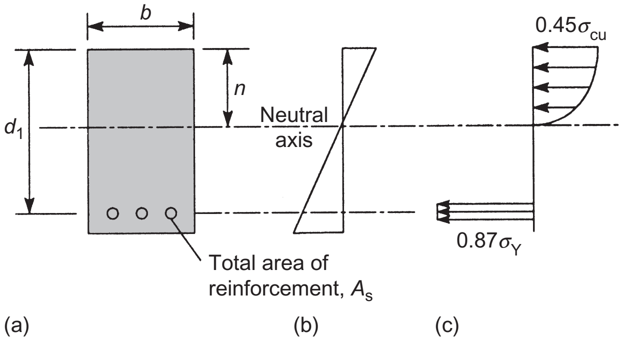

FIGURE 12.10

Stress and strain distributions in a reinforced concrete beam.

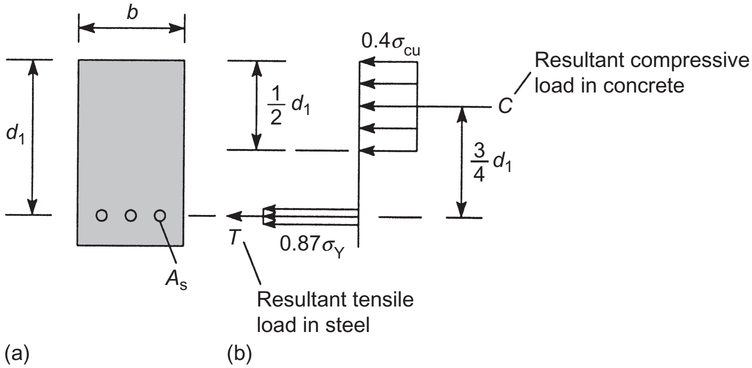

FIGURE 12.11

Approximation of stress distribution in concrete.

M_u is then given by

$$M_u = C\tfrac{3}{4}d_1 = 0.40\sigma_{cu}b\tfrac{1}{2}d_1\tfrac{3}{4}d_1$$

which gives

$$M_u = 0.15\sigma_{cu}b(d_1)^2 \tag{12.23}$$

or

$$M_u = T\tfrac{3}{4}d_1 = 0.87\sigma_Y A_s\tfrac{3}{4}d_1$$

from which

$$M_u = 0.65\sigma_Y A_s d_1 \tag{12.24}$$

whichever is the lesser. For applied bending moments less than M_u a rectangular stress block may be assumed for the concrete in which the stress is $0.4\sigma_{cu}$ but in which the depth of the neutral axis must be calculated. For beam sections in which the applied bending moment is greater than M_u, compressive reinforcement is required.

EXAMPLE 12.7

A reinforced concrete beam having an effective depth of 600 mm and a breadth of 250 mm is subjected to a bending moment of 350 kNm. If the 28-day cube strength of the concrete is 30 N/mm^2 and the yield stress in tension of steel is 400 N/mm^2, determine the required area of reinforcement.

First it is necessary to check whether or not the applied moment exceeds the ultimate moment of resistance provided by the concrete. Hence, using Eq. (12.23)

$$M_u = 0.15 \times 30 \times 250 \times 600^2 \times 10^{-6} = 405\,\text{kNm}$$

Since this is greater than the applied moment, the beam section does not require compression reinforcement.

We now assume the stress distribution shown in Fig. 12.12 in which the neutral axis of the section is at a depth n below the upper surface of the section. Thus, taking moments about the tensile reinforcement we have

$$350 \times 10^6 = 0.4 \times 30 \times 250n\left(600 - \frac{n}{2}\right)$$

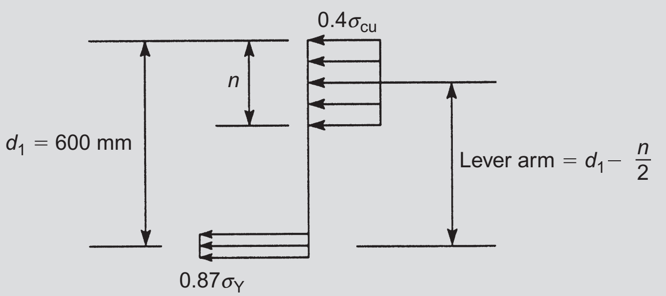

FIGURE 12.12

Stress distribution in beam of Ex. 12.7.

from which

$$n = 243.3\,\text{mm}$$

The lever arm is therefore equal to 600 – 243.3/2 = 478.4 mm. Now taking moments about the centroid of the concrete we have

$$0.87 \times 400 \times A_s \times 478.4 = 350 \times 10^6$$

which gives

$$A_s = 2102.3\,\text{mm}^2$$

EXAMPLE 12.8

A reinforced concrete beam of breadth 250 mm is required to have an effective depth as small as possible. Design the beam and reinforcement to support a bending moment of 350 kNm assuming that $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 400$ N/mm^2.

In this example the effective depth of the beam will be as small as possible when the applied moment is equal to the ultimate moment of resistance of the beam. Then, using Eq. (12.23)

$$350 \times 10^6 = 0.15 \times 30 \times 250 \times d_1^2$$

which gives

$$d_1 = 557.8\,\text{mm}$$

This is not a practical dimension since it would be extremely difficult to position the reinforcement to such accuracy. We therefore assume $d_1 = 558$ mm. Since the section is stressed to the limit, we see from Fig. 12.11(b) that the lever arm is

$$\frac{3}{4}d_1 = \frac{3}{4} \times 558 = 418.5\,\text{mm}$$

Hence, from Eq. (12.24)

$$350 \times 10^6 = 0.87 \times 400A_s \times 418.5$$

from which

$$A_s = 2403.2\,\text{mm}^2$$

A comparison of Exs 12.7 and 12.8 shows that the reduction in effective depth is only made possible by an increase in the area of steel reinforcement.

We have noted that the ultimate moment of resistance of a beam section of given dimensions can only be increased by the addition of compression reinforcement. However, although the design stress for tension reinforcement is $0.87\sigma_Y$, compression reinforcement is designed to a stress of $0.72\sigma_Y$ to avoid the possibility of the reinforcement buckling between the binders or stirrups. The method of designing a beam section to include compression reinforcement is simply treated as an extension of the singly reinforced case and is best illustrated by an example.

EXAMPLE 12.9

A reinforced concrete beam has a breadth of 300 mm and an effective depth to the tension reinforcement of 618 mm. Compression reinforcement, if required, will be placed at a depth of 60 mm. If $\sigma_{cu} = 30\ \text{N/mm}^2$ and $\sigma_Y = 410\ \text{N/mm}^2$, design the steel reinforcement if the beam is to support a bending moment of 650 kNm.

The ultimate moment of resistance provided by the concrete is obtained using Eq. (12.23) and is

$$M_u = 0.15 \times 30 \times 300 \times 618^2 \times 10^{-6} = 515.6\,\text{kNm}$$

This is less than the applied moment so that compression reinforcement is required to resist the excess moment of 650–515·6 = 134·4 kNm. If A_{sc} is the area of compression reinforcement

$$134.4 \times 10^6 = \text{lever arm} \times 0.72 \times 410 A_{sc}$$

i.e.

$$134.4 \times 10^6 = (618 - 60) \times 0.72 \times 410 A_{sc}$$

which gives

$$A_{sc} = 815.9\,\text{mm}^2$$

The tension reinforcement, A_{st}, is required to resist the moment of 515.6 kNm (as though the beam were singly reinforced) plus the excess moment of 134.4 kNm. Hence

$$A_{st} = \frac{515.6 \times 10^6}{0.75 \times 618 \times 0.87 \times 410} + \frac{134.4 \times 10^6}{(618 - 60) \times 0.87 \times 410}$$

from which

$$A_{st} = 3793.8\,\text{mm}^2$$

The ultimate load analysis of reinforced concrete T-beams is simplified in a similar manner to the elastic analysis by assuming that the neutral axis does not lie below the lower surface of the flange. The ultimate moment of a T-beam therefore corresponds to a neutral axis position coincident with the lower surface of the flange as shown in Fig. 12.13(a). M_u is then the lesser of the two values given by

$$M_u = 0.4\sigma_{cu} B h_f \left(d_1 - \frac{h_f}{2} \right) \tag{12.25}$$

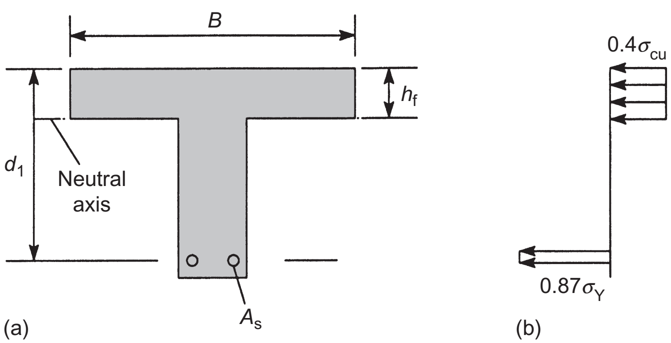

FIGURE 12.13

Ultimate load analysis of a reinforced concrete T-beam.

or

$$M_u = 0.87\sigma_Y A_s \left(d_1 - \frac{h_f}{2} \right) \tag{12.26}$$

For T-beams subjected to bending moments less than M_u, the neutral axis lies within the flange and must be found before, say, the amount of tension reinforcement can be determined. Compression reinforcement is rarely required in T-beams due to the comparatively large areas of concrete in compression.

EXAMPLE 12.10

A reinforced concrete T-beam has a flange width of 1200 mm and an effective depth of 618 mm; the thickness of the flange is 150 mm. Determine the required area of reinforcement if the beam is required to resist a bending moment of 500 kNm. Take $\sigma_{cu} = 30\ \text{N/mm}^2$ and $\sigma_Y = 410\ \text{N/mm}^2$.

M_u for this beam section may be determined using Eq. (12.25), i.e.

$$M_u = 0.4 \times 30 \times 1200 \times 150 \left(618 - \frac{150}{2} \right) \times 10^{-6} = 1173\ \text{kNm}$$

Since this is greater than the applied moment, we deduce that the neutral axis lies within the flange. Then from Fig. 12.14

$$500 \times 10^6 = 0.4 \times 30 \times 1200n \left(618 - \frac{n}{2} \right)$$

the solution of which gives

$$n = 59\ \text{mm}$$

Now taking moments about the centroid of the compression concrete we have

$$500 \times 10^6 = 0.87 \times 410 \times A_s \left(618 - \frac{59}{2} \right)$$

which gives

$$A_s = 2381.9\ \text{mm}^2$$

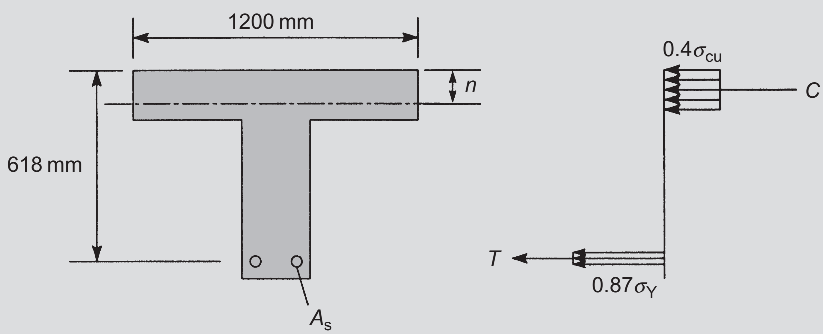

FIGURE 12.14

Reinforced concrete T-beam of Ex. 12.10.

EXAMPLE 12.11

A concrete floor slab whose partial cross section is shown in Fig. 12.15(a) is required to carry a uniformly distributed load of 100 kN/m^2. The beams supporting the slab are themselves simply supported over a span of 5 m. If $\sigma_{cu} = 25$ N/mm^2 and $\sigma_Y = 400$ N/mm^2 determine the required depth of the slab and the area of steel reinforcement.

The beam/slab arrangement may be designed as a T-beam having the cross section shown in Fig. 12.15(b). The maximum bending moment occurs at the mid-span of the beam and is given by

$$M_{max} = \frac{100 \times 2 \times 5^2}{8} = 625\ \text{kNm} \quad (\text{see Ex.3.8})$$

Then, assuming that the neutral axis coincides with the base of the slab, from Eq.(12.25)

$$625 \times 10^6 = 0.4 \times 25 \times 2 \times 10^3 h_f (500 - h_f/2)$$

which simplifies to the quadratic equation

$$h_f^2 - 1000 h_f + 62500 = 0$$

Solving

$$h_f = 67\ \text{mm}$$

Then, from Eq.(12.26)

$$625 \times 10^6 = 0.87 \times 400 A_s (500 - 67/2)$$

from which

$$A_s = 3849.9\ \text{mm}^2$$

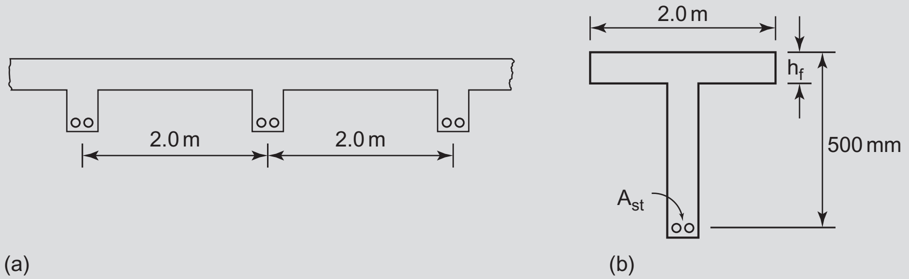

FIGURE 12.15

Beam/slab arrangement of Ex. 12.11.

12.3 Steel and concrete beams

In many instances concrete slabs are supported on steel beams, the two being joined together by shear connectors to form a composite structure. We therefore have a similar situation to that of the reinforced concrete T-beam in which the flange of the beam is concrete but the leg is a standard steel section.

Ultimate load theory is used to analyse steel and concrete beams with stress limits identical to those applying in the ultimate load analysis of reinforced concrete beams; plane sections are also assumed to remain plane.

Consider the steel and concrete beam shown in Fig. 12.16(a) and let us suppose that the neutral axis lies within the concrete flange. We ignore the contribution of the concrete in the tension zone of the beam to its bending strength, so that the assumed stress distribution takes the form shown in Fig. 12.16(b). A convenient method of designing the cross section to resist a bending moment, M, is to assume the lever arm to be $(h_c + h_s)/2$ and then to determine the area of steel from the moment equation

$$M = 0.87\sigma_Y A_s \frac{(h_c + h_s)}{2} \tag{12.27}$$

The available compressive force in the concrete slab, $0.4\,\sigma_{cu} b h_c$, is then checked to ensure that it exceeds the tensile force, $0.87\sigma_Y A_s$, in the steel. If it does not, the neutral axis of the section lies within the steel and A_s given by Eq. (12.27) will be too small. If the neutral axis lies within the concrete slab the moment of resistance of the beam is determined by first calculating the position of the neutral axis. Thus, since the compressive force in the concrete is equal to the tensile force in the steel

$$0.4\sigma_{cu} b n_1 = 0.87\sigma_Y A_s \tag{12.28}$$

Then, from Fig. 12.16

$$M_u = 0.87\sigma_Y A_s\left(d - \frac{n_1}{2}\right) \tag{12.29}$$

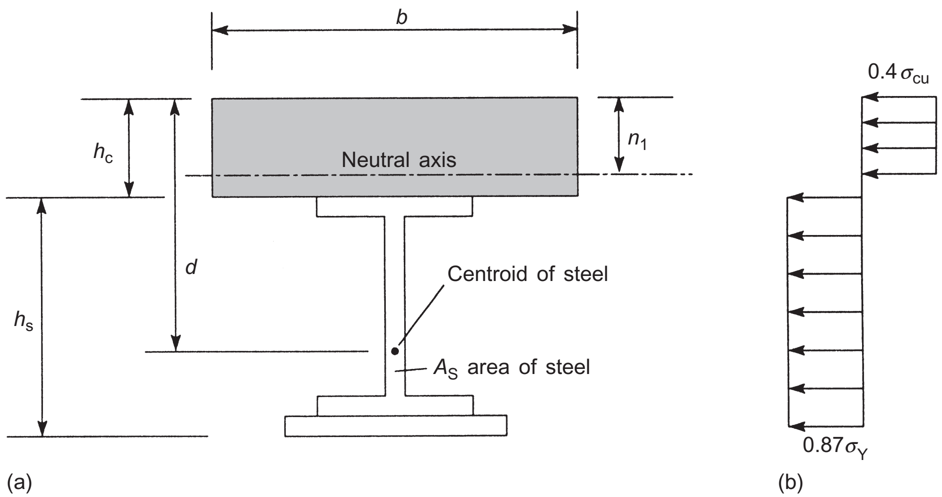

FIGURE 12.16

Ultimate load analysis of a steel and concrete beam, neutral axis within the concrete.

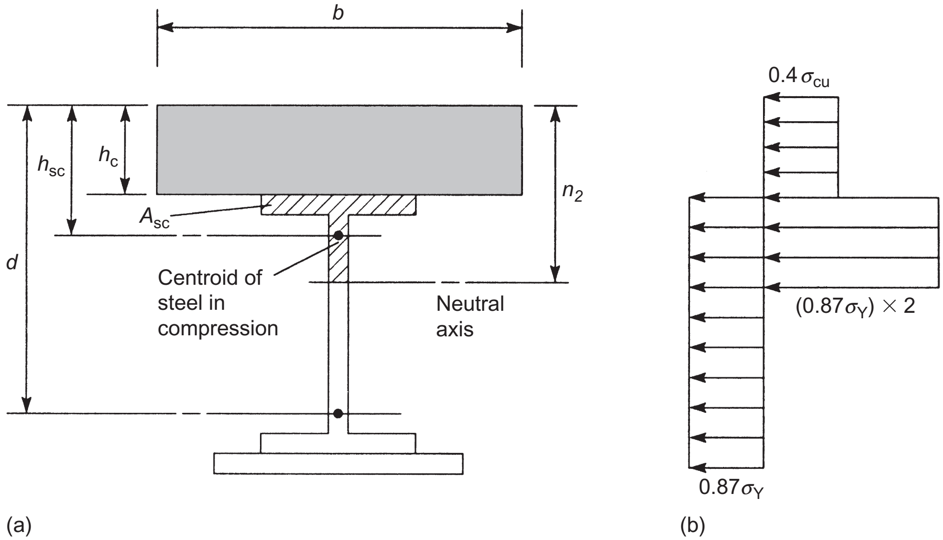

FIGURE 12.17

Ultimate load analysis of a steel and concrete beam, neutral axis within the steel.

If the neutral axis lies within the steel, the stress distribution shown in Fig. 12.17(b) is assumed in which the compressive stress in the steel above the neutral axis is the resultant of the tensile stress and twice the compressive stress. Thus, if the area of steel in compression is A_{sc}, we have, equating compressive and tensile forces

$$0.4\sigma_{cu}bh_c + 2 \times (0.87\sigma_Y)A_{sc} = 0.87\sigma_Y A_s \tag{12.30}$$

which gives A_{sc} and hence h_{sc}. Now taking moments

$$M_u = 0.87\sigma_Y A_s\left(d - \frac{h_c}{2}\right) - 2 \times (0.87\sigma_Y)A_{sc}\left(h_{sc} - \frac{h_c}{2}\right) \tag{12.31}$$

EXAMPLE 12.12

A concrete slab 150 mm thick is 1.8 m wide and is to be supported by a steel beam. The total depth of the steel/concrete composite beam is limited to 562 mm. Find a suitable beam section if the composite beam is required to resist a bending moment of 709 kNm. Take $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 350$ N/mm^2.

Using Eq. (12.27)

$$A_s = \frac{2 \times 709 \times 10^6}{0.87 \times 350 \times 562} = 8286\,\text{mm}^2$$

The tensile force in the steel is then

$$0.87 \times 350 \times 8286 \times 10^{-3} = 2523\,\text{kN}$$

and the compressive force in the concrete is

$$0.4 \times 1.8 \times 10^3 \times 150 \times 30 \times 10^{-3} = 3240\,\text{kN}$$

The neutral axis therefore lies within the concrete slab so that the area of steel in tension is, in fact, equal to A_s. From Steel Tables we see that a Universal Beam of nominal size 406 mm×152 mm×67 kg/m has an actual overall depth of 412 mm and a cross-sectional area of 8530 mm^2. The position of the neutral axis of the composite beam incorporating this beam section is obtained from Eq. (12.28); hence

$$0.4 \times 30 \times 1800 n_1 = 0.87 \times 350 \times 8530$$

which gives

$$n_1 = 120\,\text{mm}$$

Substituting for n_1 in Eq. (12.29) we obtain the moment of resistance of the composite beam

$$M_u = 0.87 \times 350 \times 8530(356 - 60) \times 10^{-6} = 769\,\text{kNm}$$

Since this is greater than the applied moment we deduce that the beam section is satisfactory.

EXAMPLE 12.13

If the concrete in the steel/concrete composite beam of Ex.12.12 has a reduced strength of 20 N/mm^2 determine whether or not the composite beam section is still satisfactory.

The cross sectional area of the steel beam chosen from Steel Tables is 8530 mm^2. The tensile force in the steel is then

$$0.87 \times 350 \times 8530 \times 10^{-3} = 2597.4\,\text{kN}$$

The compressive force in the concrete is

$$0.4 \times 1.8 \times 10^3 \times 150 \times 20 \times 10^{-3} = 2160\,\text{kN}$$

Since this is less than the tensile force in the steel the neutral axis of the beam section lies within the steel. Then, from Eq.(12.30)

$$0.4 \times 20 \times 1.8 \times 10^3 \times 150 + 2 \times 0.87 \times 350 A_{sc} = 0.87 \times 350 \times 8530$$

from which

$$A_{sc} = 718.2\,\text{mm}^2$$

From Steel Tables, the Universal Beam has a flange width of 153 mm and a flange thickness of 16 mm. Therefore, by inspection, the neutral axis lies within the flange of the steel beam. Then

$$158 h_f = 718.2$$

so that

$$h_f = 4.7\,\text{mm}$$

where h_f is the depth of the flange in compression. Then

$$h_{sc} = 4.7 + 150 = 154.7 \text{ mm}$$

From Eq.(12.31)

$$M_u = 0.87 \times 350 \times 8530\,(356 - 150/2) - 2 \times 0.87 \times 350 \times 718.2\,(154.7 - 150/2)$$

which gives

$$M_u = 695 \text{ kNm}$$

This is less than the applied bending moment so that the beam section is no longer satisfactory.

PROBLEMS

P.12.1 A timber beam 200 mm wide by 300 mm deep is reinforced on its top and bottom surfaces by steel plates each 12 mm thick by 200 mm wide. If the allowable stress in the timber is 8 N/mm^2 and that in the steel is 110 N/mm^2, find the allowable bending moment. The ratio of the modulus of elasticity of steel to that of timber is 20.

Ans. 94.7 kNm.

P.12.2 A simply supported beam of span 3.5 m carries a uniformly distributed load of 46.5 kN/m. The beam has the box section shown in Fig. P.12.2. Determine the required thickness of the steel plates if the allowable stresses are 124 N/mm^2 for the steel and 8 N/mm^2 for the timber. The modular ratio of steel to timber is 20.

Ans. 17 mm.

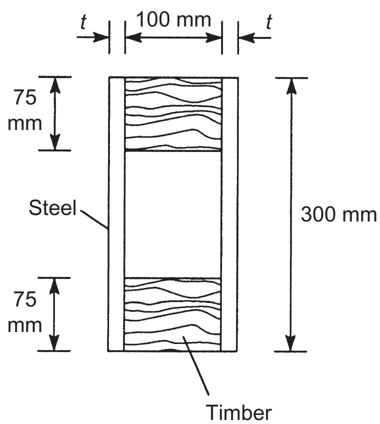

FIGURE P.12.2

P.12.3 A timber beam 150 mm wide by 300 mm deep is reinforced by a steel plate 150 mm wide and 12 mm thick which is securely attached to its lower surface. Determine the percentage increase in the moment of resistance of the beam produced by the steel-reinforcing plate. The allowable stress in the timber is 12 N/mm^2 and in the steel, 155 N/mm^2. The modular ratio is 20.

Ans. 176%.

P.12.4 A singly reinforced rectangular concrete beam of effective span 4.5 m is required to carry a uniformly distributed load of 16.8 kN/m. The overall depth, D, is to be twice the breadth and the centre of the steel is to be at $0.1D$ from the underside of the beam. Using elastic theory find the dimensions of the beam and the area of steel reinforcement required if the stresses are limited to 8 N/mm^2 in the concrete and 140 N/mm^2 in the steel. Take $m = 15$.

Ans. $D = 406.7$ mm, $A_s = 980.6$ mm^2.

P.12.5 A reinforced concrete beam is of rectangular section 300 mm wide by 775 mm deep. It has five 25 mm diameter bars as tensile reinforcement in one layer with 25 mm cover and three 25 mm diameter bars as compression reinforcement, also in one layer with 25 mm cover. Find the moment of resistance of the section using elastic theory if the allowable stresses are 7.5 N/mm^2 and 125 N/mm^2 in the concrete and steel, respectively. The modular ratio is 16.

Ans. 214.5 kNm.

P.12.6 A reinforced concrete T-beam is required to carry a uniformly distributed load of 42 kN/m on a simply supported span of 6 m. The slab is 125 mm thick, the rib is 250 mm wide and the effective depth to the tensile reinforcement is 550 mm. The working stresses are 8.5 N/mm^2 in the concrete and 140 N/mm^2 in the steel; the modular ratio is 15. Making a reasonable assumption as to the position of the neutral axis find the area of steel reinforcement required and the breadth of the compression flange.

Ans. 2655.7 mm^2, 700 mm (neutral axis coincides with base of slab).

P.12.7 Repeat P.12.4 using ultimate load theory assuming $\sigma_{cu} = 24$ N/mm^2 and $\sigma_Y = 280$ N/mm^2.

Ans. $D = 307.8$ mm, $A_s = 843$ mm^2.

P.12.8 Repeat P.12.5 using ultimate load theory and take $\sigma_{cu} = 22.5$ N/mm^2, $\sigma_Y = 250$ N/mm^2.

Ans. 222.5 kNm.

P.12.9 Repeat P.12.6 using ultimate load theory. Assume $\sigma_{cu} = 25.5$ N/mm^2 and $\sigma_Y = 280$ N/mm^2.

Ans. 1592 mm^2, 304 mm (neutral axis coincides with base of slab).

P.12.10 A concrete slab 175 mm thick and 2 m wide is supported by, and firmly connected to, a 457 mm$\times$152 mm$\times$74 kg/m Universal Beam whose actual depth is 461.3 mm and whose cross-sectional area is 9490 mm^2. If $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 350$ N/mm^2, find the moment of resistance of the resultant steel and concrete beam.

Ans. 919.5 kNm.

P.12.11 If the concrete in the composite beam in P.12.10 has a reduced strength of 15 N/mm^2 determine its resulting moment of resistance.

Ans. 843 kNm.