CHAPTER

Torsion of Beams

11

Torsion in beams arises generally from the action of shear loads whose points of application do not coincide with the shear centre of the beam section. Examples of practical situations where this occurs are shown in Fig. 11.1 where, in Fig. 11.1(a), a concrete encased I-section steel beam supports an offset masonry wall and in Fig. 11.1(b) a floor slab, cast integrally with its supporting reinforced concrete beams, causes torsion of the beams as it deflects under load. Codes of Practice either imply or demand that torsional stresses and deflections be checked and provided for in design.

The solution of torsion problems is complex particularly in the case of beams of solid section and arbitrary shape for which exact solutions do not exist. Use is then made of empirical formulae which are conveniently expressed in terms of correction factors based on the geometry of a particular shape of cross section. The simplest case involving the torsion of solid section beams (as opposed to hollow cellular sections) is that of a circular section shaft or bar. Therefore, this case forms an instructive introduction to the more complex cases of the torsion of solid section, thin-walled open section and closed section beams.

11.1 Torsion of solid and hollow circular section bars

Initially, as in the cases of bending and shear, we shall examine the physical aspects of torsion.

Suppose that the circular section bar shown in Fig. 11.2(a) is cut at some point along its length and that the two parts of the bar are threaded onto a spindle along its axis. Now we draw a line ABC along the surface of the bar parallel to its axis and apply equal and opposite torques, T, at each end as shown in Fig. 11.2(b). The two parts of the bar will rotate relative to each other so that the line ABC becomes stepped. For this to occur there must be a relative slippage between the two internal surfaces in contact.

If, now, we glue the two parts of the bar together this relative slippage is prevented. The glue, therefore, produces an in-plane force which must, from a consideration of the equilibrium of either part of the bar, be equal to the applied torque T. This internal torque is distributed over each face of the cross section of the bar in the form of torsional shear stresses whose resultant must be a pure torque. It follows that the form of these internal shear stresses is that shown in Fig. 11.3 in which they act on a series of small elements positioned on an internal circle of radius r. Of course, there are an infinite number of elements on this circle and an infinite number of circles within the cross section.

Our discussion so far applies to all cross sections of the bar. The problem is to determine the distribution of shear stress and the actual twisting of the bar that the torque causes.

Figure 11.4(a) shows a circular section bar of length L subjected to equal and opposite torques, T, at each end. The torque at any section of the bar is therefore equal to T and is constant along its length. We shall assume that cross sections remain plane during twisting, that radii remain straight during twisting and that all normal cross sections equal distances apart suffer the same relative rotation.

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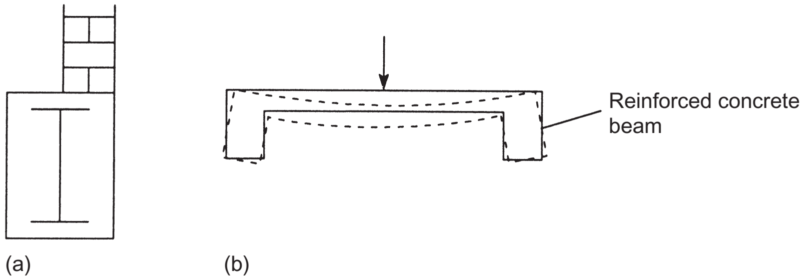

FIGURE 11.1

Causes of torsion in beams.

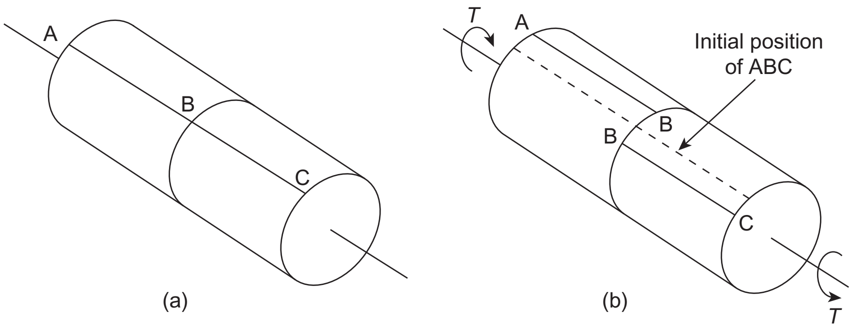

FIGURE 11.2

Torsion of a circular section bar.

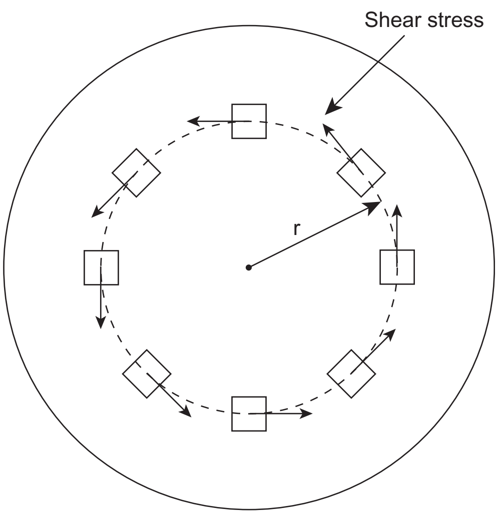

FIGURE 11.3

Shear stresses produced by a pure torque.

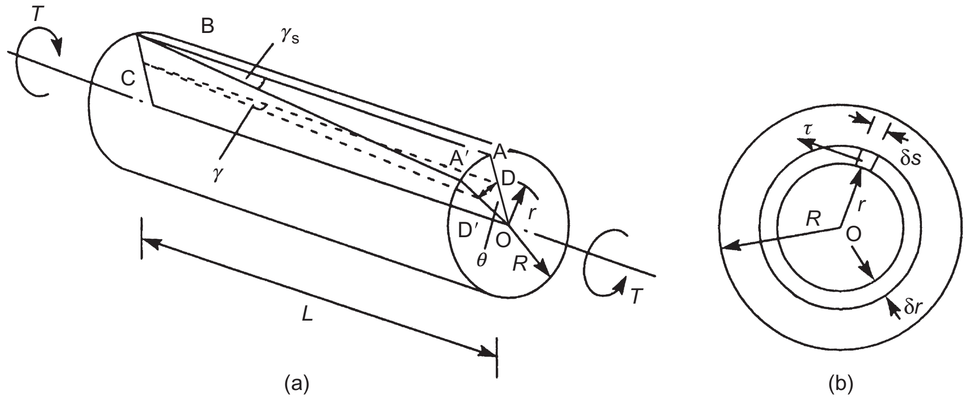

FIGURE 11.4

Torsion of a solid circular section bar.

Consider the generator AB on the surface of the bar and parallel to its longitudinal axis. Due to twisting, the end A is displaced to A′ so that the radius OA rotates through a small angle, θ, to OA′. The shear strain, γ_s, on the surface of the bar is then equal to the angle ABA′ in radians so that

$$\gamma_s = \frac{\mathrm{AA}'}{L} = \frac{R\theta}{L}$$

Similarly the shear strain, γ, at any radius r is given by the angle DCD′ so that

$$\gamma = \frac{\mathrm{DD}'}{L} = \frac{r\theta}{L}$$

The shear stress, τ, at the radius r is related to the shear strain γ by Eq. (7.9). Then

$$\gamma = \frac{\tau}{G} = \frac{r\theta}{L}$$

or, rearranging

$$\frac{\tau}{r} = G\frac{\theta}{L} \tag{11.1}$$

Consider now any cross section of the bar as shown in Fig. 11.4(b). The shear stress, τ, on an element δs of an annulus of radius r and width δr is tangential to the annulus, is in the plane of the cross section and is constant round the annulus since the cross section of the bar is perfectly symmetrical (see also Fig. 11.3). The shear force on the element δs of the annulus is then $\tau\,\delta s\,\delta r$ and its moment about the centre, O, of the section is $\tau\,\delta s\,\delta r r$. Summing the moments on all such elements of the annulus we obtain the torque, δT, on the annulus, i.e.

$$\delta T = \int_0^{2\pi r} \tau\,\delta r r\,\mathrm{d}s$$

which gives

$$\delta T = 2\pi r^2 \tau \delta r$$

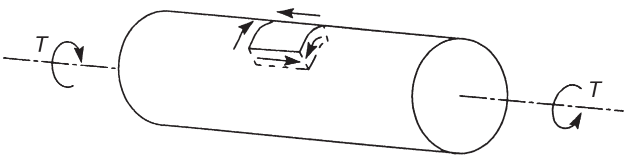

FIGURE 11.5

Shear and complementary shear stresses at the surface of a circular section bar subjected to torsion.

The total torque on the bar is now obtained by summing the torques from each annulus in the cross section. Thus

$$T = \int_0^R 2\pi r^2 \tau \, \mathrm{d}r \tag{11.2}$$

Substituting for τ in Eq. (11.2) from Eq. (11.1) we have

$$T = \int_0^R 2\pi r^3 G \frac{\theta}{L} \mathrm{d}r$$

which gives

$$T = \frac{\pi R^4}{2} G \frac{\theta}{L}$$

or

$$T = JG\frac{\theta}{L} \tag{11.3}$$

where $J = \pi R^4/2$ $(= \pi D^4/32)$ is defined as the polar second moment of area of the cross section (see Eq. (9.42)). Combining Eqs (11.1) and (11.3) we have

$$\frac{T}{J} = \frac{\tau}{r} = G\frac{\theta}{L} \tag{11.4}$$

Note that for a given torque acting on a given bar the shear stress is a maximum at the outer surface of the bar. Note also that these shear stresses induce complementary shear stresses on planes parallel to the axis of the bar but not on the actual surface (Fig. 11.5).

Torsion of a circular section hollow bar

The preceding analysis may be applied directly to a hollow bar of circular section having outer and inner radii R_{o} and R_{i}, respectively. Equation (11.2) then becomes

$$T = \int_{R_{\mathrm{i}}}^{R_{\mathrm{o}}} 2\pi r^2 \tau \, \mathrm{d}r$$

Substituting for τ from Eq. (11.1) we have

$$T = \int_{R_{\mathrm{i}}}^{R_{\mathrm{o}}} 2\pi r^3 G \frac{\theta}{L} \mathrm{d}r$$

from which

$$T = \frac{\pi}{2}(R_{\mathrm{o}}^4 - R_{\mathrm{i}}^4) G \frac{\theta}{L}$$

The polar second moment of area, J, is then

$$J = \frac{\pi}{2}(R_o^4 - R_i^4) \tag{11.5}$$

EXAMPLE 11.1

A hollow shaft of outside diameter 220 mm and thickness 40 mm is required to transmit power at a speed of 80 rpm. If the maximum shear stress in the shaft is limited to 60 N/mm^2 determine the power transmitted and the angle of twist in a length of 10 m. Take $G = 80000$ N/mm^2.

(Note: Power P (watts)=Torque (Nm)×angular speed (rad/sec)).

From Eq. (11.5)

$$J = (\pi/2)(110^4 - 70^4) = 192.3 \times 10^6 \text{ mm}^4$$

From Eq. (11.4)

$$T = \frac{\tau_{\max}}{r_o} J = \frac{60 \times 192.3 \times 10^6}{110} = 104.9 \times 10^6 \text{ Nmm}$$

so that

$$T = 104.9 \times 10^3 \text{ Nm}$$

Then

$$P = 104.9 \times 10^3 \times 2\pi \times 80/60 = 878808.2 \text{ watts}$$

or

$$P = 878.8 \text{ kW}$$

Again, from Eq. (11.4)

$$\theta = \frac{TL}{GJ} = \frac{104.9 \times 10^6 \times 10 \times 10^3}{80000 \times 192.3 \times 10^6} = 0.068 \text{ rad}$$

or

$$\theta = 3.9°$$

EXAMPLE 11.2

If a solid shaft was used to transmit the same power as the hollow shaft in Ex.11.1 with the same length and limiting stress what would be the percentage increase in weight of the material used.

For a shaft diameter of D mm

$$J = \frac{\pi D^4}{32}$$

Then, substituting for the torque from Eq. (11.4) in the expression for power

$$P = \frac{2\tau_{\max} J}{D} \frac{\times 2\pi \times 80}{60}$$

that is

$$878.8 \times 10^3 = \frac{2 \times 60 \times \pi D^4}{10^3 \times 32 \times D} \times \frac{2\pi \times 80}{60}$$

which gives $D = 207.3$ mm

Since weight is proportional to cross sectional area the percentage weight increase is given by

$$\%\,\text{weight increase} = \frac{(\pi \times 207.3^2/4) - \pi(220^2 - 140^2)/4}{\pi(220^2 - 140^2)/4} \times 100$$

that is % weight increase = 49.0%.

EXAMPLE 11.3

Determine the angle of twist and the maximum shear stress in the tapered shaft shown in Fig. 11.6. The shear modulus of the material of the shaft is G.

Suppose that the diameter of the shaft at a distance x from the left-hand end is d. Then, from Eq. (11.4), the angle of twist, $\delta\theta$, over the length δx is given by

$$\delta\theta = \frac{T}{GJ}\delta x = \frac{T}{(G\pi d^4/32)}\delta x \tag{i}$$

If the change in diameter over the length δx is δd then

$$\frac{\delta d}{\delta x} = \frac{D_2 - D_1}{L} \quad \text{(Note that } \delta d \text{ is a decrease in diameter)}$$

Substituting for δx in Eq. (i)

$$\delta\theta = \frac{32TG}{\pi d^4}\frac{L}{D_2 - D_1}\delta d$$

The angle of twist over the complete length of the shaft is then given by

$$\theta = \frac{32TGL}{\pi(D_2 - D_1)}\int_{D_1}^{D_2} d^{-4}\,\mathrm{d}d = -\frac{32TGL}{3\pi(D_2 - D_1)}[d^{-3}]_{D1}^{D2}$$

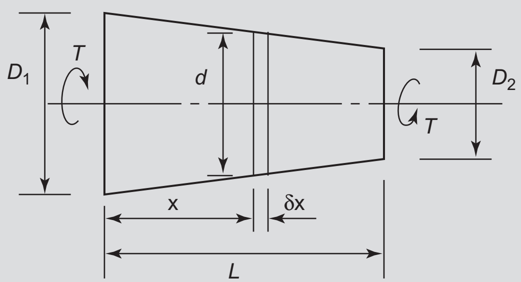

FIGURE 11.6

Tapered shaft of Ex 11.3.

which gives

$$\theta = \frac{-32TGL}{3\pi(D_2 - D_1)}\left[(1/D_2^3) - (1/D_1^3)\right]$$

which simplifies to

$$\theta = \frac{-32TGL}{3\pi D_1^3 D_2^3}(D_1^2 + D_1 D_2 + D_2^2) \tag{ii}$$

From Eq. (11.4)

$$\pi = \frac{Td/2}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

The maximum shear stress therefore occurs at the section where d is a minimum, that is where $d = D_2$. Then

$$\tau_{\max} = \frac{16T}{\pi D_2^3}$$

Statically indeterminate circular section bars under torsion

In many instances bars subjected to torsion are supported in such a way that the support reactions are statically indeterminate. These reactions must be determined, however, before values of maximum stress and angle of twist can be obtained.

Figure 11.7(a) shows a bar of uniform circular cross section firmly supported at each end and subjected to a concentrated torque at a point B along its length. From equilibrium we have

$$T = T_A + T_C \tag{11.6}$$

A second equation is obtained by considering the compatibility of displacement at B of the two lengths AB and BC. Thus the angle of twist at B in AB must equal the angle of twist at B in BC, i.e.

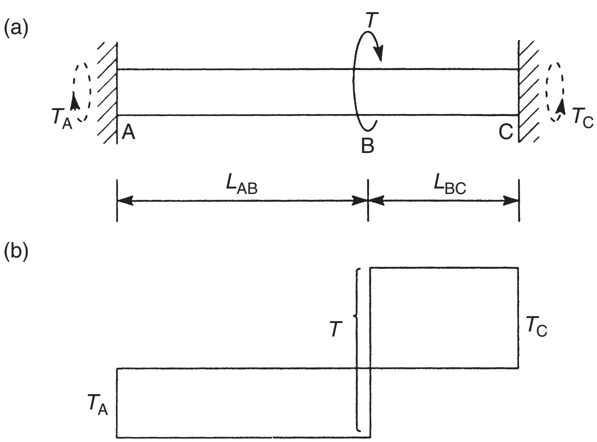

FIGURE 11.7

Torsion of a circular section bar with built-in ends.

$$\theta_{B(AB)} = \theta_{B(BC)}$$

or using Eq. (11.3)

$$\frac{T_A L_{AB}}{GJ} = \frac{T_C L_{BC}}{GJ}$$

whence

$$T_A = T_C \frac{L_{BC}}{L_{AB}}$$

Substituting in Eq. (11.6) for T_A we obtain

$$T = T_C \left(\frac{L_{BC}}{L_{AB}} + 1 \right)$$

which gives

$$T_C = \frac{L_{AB}}{L_{AB} + L_{BC}} T \tag{11.7}$$

Hence

$$T_A = \frac{L_{BC}}{L_{AB} + L_{BC}} T \tag{11.8}$$

The distribution of torque along the length of the bar is shown in Fig. 11.7(b). Note that if $L_{AB} > L_{BC}$, T_C is the maximum torque in the bar.

EXAMPLE 11.4

A bar of circular cross section is 2.5 m long (Fig. 11.8). For 2 m of its length its diameter is 200 mm while for the remaining 0.5 m its diameter is 100 mm. If the bar is firmly supported at its ends and subjected to a torque of 50 kNm applied at its change of section, calculate the maximum stress in the bar and the angle of twist at the point of application of the torque. Take $G = 80000\ \text{N/mm}^2$.

In this problem Eqs (11.7) and (11.8) cannot be used directly since the bar changes section at B. Thus from equilibrium

$$T = T_A + T_C \tag{i}$$

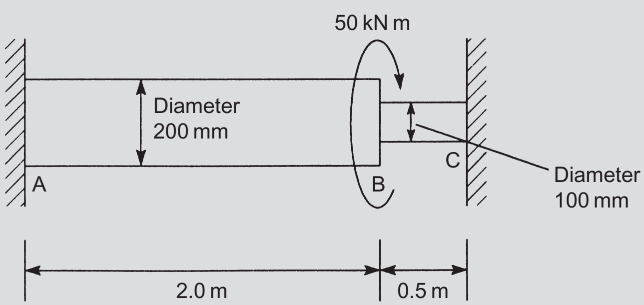

FIGURE 11.8

Bar of Ex. 11.4.

and from the compatibility of displacement at B in the lengths AB and BC

$$\theta_{B(AB)} = \theta_{B(BC)}$$

or using Eq. (11.3)

$$\frac{T_A L_{AB}}{G J_{AB}} = \frac{T_C L_{BC}}{G J_{BC}}$$

whence

$$T_A = \frac{L_{BC} J_{AB}}{L_{AB} J_{BC}} T_C \tag{ii}$$

Substituting in Eq. (i) we obtain

$$T = T_C \left(\frac{L_{BC} J_{AB}}{L_{AB} J_{BC}} + 1 \right)$$

or

$$50 = T_C \left[\frac{0.5}{2.0} \times \left(\frac{200 \times 10^{-3}}{100 \times 10^{-3}} \right)^4 + 1 \right]$$

from which

$$T_C = 10\,\text{kNm}$$

Hence, from Eq. (i)

$$T_A = 40\,\text{kNm}$$

Although the maximum torque occurs in the length AB, the length BC has the smaller diameter. It can be seen from Eq. (11.4) that shear stress is directly proportional to torque and inversely proportional to diameter (or radius) cubed. Therefore, we conclude that in this case the maximum shear stress occurs in the length BC of the bar and is given by

$$\tau_{max} = \frac{10 \times 10^6 \times 100 \times 32}{2 \times \pi \times 100^4} = 50.9\,\text{N/mm}^2$$

Also the rotation at B is given by either

$$\theta_B = \frac{T_A L_{AB}}{G J_{AB}} \quad \text{or} \quad \theta_B = \frac{T_C L_{BC}}{G J_{BC}}$$

Using the first of these expressions we have

$$\theta_B = \frac{40 \times 10^6 \times 2 \times 10^3 \times 32}{80\,000 \times \pi \times 200^4} = 0.0064\,\text{rad}$$

or

$$\theta_B = 0.37^\circ$$

11.2 Strain energy due to torsion

It can be seen from Eq. (11.3) that for a bar of a given material, a given length, L, and radius, R, the angle of twist is directly proportional to the applied torque. Therefore a torque–angle of twist graph is linear and for a gradually applied torque takes the form shown in Fig. 11.9. The work done by a gradually applied torque, T, is equal to the area under the torque–angle of twist curve and is given by

$$\text{Work done} = \frac{1}{2}T\theta$$

The corresponding strain energy stored, U, is therefore also given by

$$U = \frac{1}{2}T\theta$$

Substituting for T and θ from Eq. (11.4) in terms of the maximum shear stress, $\tau_{\max}$, on the surface of the bar we have

$$U = \frac{1}{2}\frac{\tau_{\max}J}{R} \times \frac{\tau_{\max}L}{GR}$$

or

$$U = \frac{1}{4}\frac{\tau_{\max}^2}{G}\pi R^2 L \quad \text{since } J = \frac{\pi R^4}{2}$$

Hence

$$U = \frac{\tau_{\max}^2}{4G} \times \text{volume of bar} \tag{11.9}$$

Alternatively, in terms of the applied torque T we have

$$U = \frac{1}{2}T\theta = \frac{T^2 L}{2GJ} \tag{11.10}$$

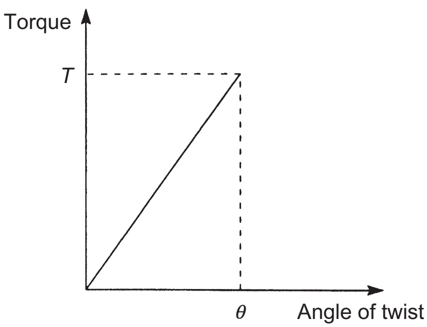

FIGURE 11.9

Torque–angle of twist relationship for a gradually applied torque.

EXAMPLE 11.5

Determine the angle of twist at the free end of the shaft shown in Fig. 11.10. Take $G = 80000\ \text{N/mm}^2$.

From Eq. (11.10) the total strain energy, U, in the shaft is

$$U = \frac{(4 \times 10^6)^2 \times 100 \times 32}{80000 \times \pi \times 50^4} + \frac{(4 \times 10^6)^2 \times 200 \times 32}{80000 \times \pi \times 100^4}$$

so that $U = 36669.3$ Nmm

The total strain energy in the shaft is equal to the work done by the applied torque. Therefore

$$4 \times 10^6\, \theta/2 = 36669.3$$

which gives

$$\theta = 0.018\ \text{rad} = 1.05°$$

Again, as in the case of trusses, strain energy can only be used directly when a shaft is subjected to a single applied torque. Further, it is only possible to obtain the angle of twist at the section where the torque is applied.

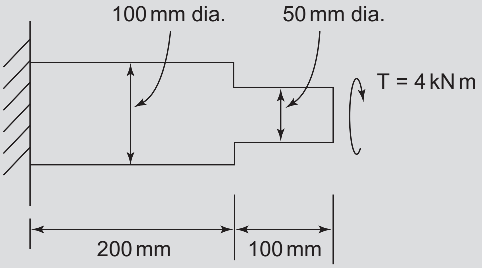

FIGURE 11.10

Shaft of Ex 11.5.

11.3 Plastic torsion of circular section bars

Equations (11.4) apply only if the shear stress–shear strain curve for the material of the bar in torsion is linear. Stresses greater than the yield shear stress, τ_Y, induce plasticity in the outer region of the bar and this extends radially inwards as the torque is increased. It is assumed, in the plastic analysis of a circular section bar subjected to torsion, that cross sections of the bar remain plane and that radii remain straight.

For a material, such as mild steel, which has a definite yield point the shear stress–shear strain curve may be idealized in a similar manner to that for direct stress (see Fig. 18.1) as shown in Fig. 11.11. Thus, after yield, the shear strain increases at a more or less constant value of shear stress. It follows that the shear stress in the plastic region of a mild steel bar is constant and equal to τ_Y. Figure 11.12 illustrates the various stages in the development of full plasticity in a mild steel bar of circular section. In Fig. 11.12(a) the maximum stress at the outer surface of the bar has reached the yield stress, τ_Y. Equations (11.4) still apply, therefore, so that at the outer surface of the bar

$$\frac{T_Y}{J} = \frac{\tau_Y}{R}$$

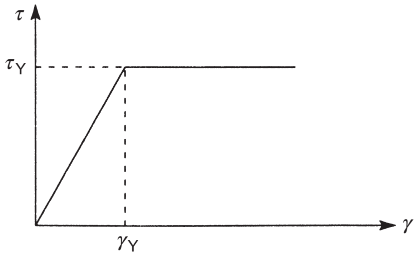

FIGURE 11.11

Idealized shear stress–shear strain curve for a mild steel bar.

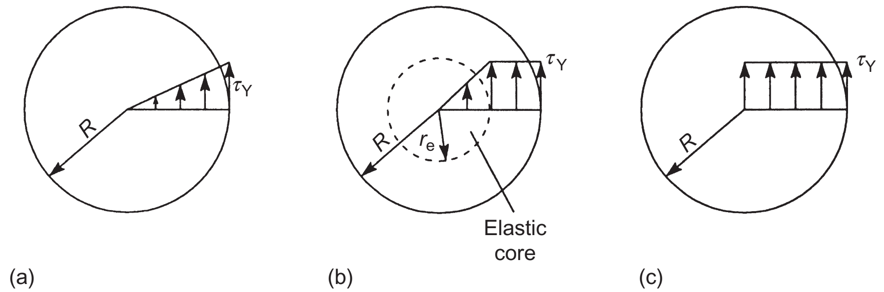

FIGURE 11.12

Plastic torsion of a circular section bar.

or

$$T_Y = \frac{\pi R^3}{2}\tau_Y \tag{11.11}$$

where T_Y is the torque producing yield. In Fig. 11.12(b) the torque has increased above the value T_Y so that the plastic region extends inwards to a radius r_e. Within r_e the material remains elastic and forms an *elastic core*. At this stage the total torque is the sum of the contributions from the elastic core and the plastic zone, i.e.

$$T = \frac{\tau_Y J_e}{r_e} + \int_{r_e}^{R} 2\pi r^2 \tau_Y \mathrm{d}r$$

where J_e is the polar second moment of area of the elastic core and the contribution from the plastic zone is derived in an identical manner to Eq. (11.2) but in which $\tau = \tau_Y =$ constant. Hence

$$T = \frac{\tau_Y \pi r_e^3}{2} + \frac{2}{3}\pi\tau_Y(R^3 - r_e^3)$$

which simplifies to

$$T = \frac{2\pi R^3}{3}\tau_Y\left(1 - \frac{r_e^3}{4R^3}\right) \tag{11.12}$$

Note that for a given value of torque, Eq. (11.12) fixes the radius of the elastic core of the section. In stage three (Fig. 11.12(c)) the cross section of the bar is completely plastic so that r_e in Eq. (11.12) is zero and the ultimate torque or fully plastic torque, T_P, is given by

$$T_P = \frac{2\pi R^3}{3}\tau_Y \tag{11.13}$$

Comparing Eqs (11.11) and (11.13) we see that

$$\frac{T_P}{T_Y} = \frac{4}{3} \tag{11.14}$$

so that only a one-third increase in torque is required after yielding to bring the bar to its ultimate load-carrying capacity.

Since we have assumed that radii remain straight during plastic torsion, the angle of twist of the bar must be equal to the angle of twist of the elastic core which may be obtained directly from Eq. (11.3) in which the torque is T_e the portion of the total torque carried by the elastic core. Thus for a bar of length L and shear modulus G,

$$\theta = \frac{T_e L}{GJ_e} = \frac{2T_e L}{\pi G r_e^4} \tag{11.15}$$

or, in terms of the shear stress, τ_Y, at the outer surface of the elastic core

$$\theta = \frac{\tau_Y L}{G r_e} \tag{11.16}$$

Either of Eq. (11.15) or (11.16) shows that θ is inversely proportional to the radius, r_e, of the elastic core. Clearly, when the bar becomes fully plastic, $r_e \rightarrow 0$ and θ becomes, theoretically, infinite. In practical terms this means that twisting continues with no increase in torque in the fully plastic state.

EXAMPLE 11.6

A solid circular section bar has a diameter of 100 mm and the material of the bar has a yield stress in shear of 150 N/mm^2. Determine the maximum torque the bar can transmit without yielding occurring and also the radius of the elastic core of the section if this torque is increased by 20%. What is the value of torque which would result in the section becoming fully plastic?

From Eq. (11.11)

$$T_Y = (\pi R^3/2)\tau_Y = (\pi \times 50^3/2) \times 150$$

which gives

$$T_Y = 29.5 \times 10^6 \text{Nmm}$$

Then, from Eq. (11.12)

$$1.2 \times 29.5 \times 10^6 = (2\pi \times 50^3/3) \times 150[1 - r_e^3/(4 \times 50^3)]$$

from which $r_e = 36.7$ mm

From Eq. (11.14) (or Eq. (11.13))

$$T_p = 4 \times 29.5 \times 10^6/3 = 39.3 \times 10^6 \text{ Nmm}$$

EXAMPLE 11.7

If the bar in Ex.11.6 is subjected to a torque of 32.5 kNm calculate the angle of twist in the bar over a length of 3 m. Take $G = 80000$ N/mm^2.

From Eq. (11.12)

$$32.5 \times 10^6 = (2\pi \times 50^3/3) \times 150[1 - r_e^3/(4 \times 50^3)]$$

from which $r_e = 44.1$ mm

The torque resisted by the elastic core (and therefore the torque producing the twist) is then given by

$$T_e = \frac{\tau_Y J_e}{r_e} = \frac{\tau_Y \pi r_e^3}{2} = \frac{150 \times \pi \times 44.1^3}{2}$$

i.e. $T_e = 20.2 \times 10^6$ Nmm

Then, from Eq. (11.15) (or Eq. (11.16))

$$\theta = \frac{2 \times 20.2 \times 10^6 \times 3 \times 10^3}{\pi \times 80000 \times 44.1^4} = 0.127 \text{ rad}$$

or $\theta = 7.3°$

EXAMPLE 11.8

A hollow circular section bar has external and internal radii, R_o and R_i, respectively and carries a torque, T. If the yield stress in shear of the material of the bar is τ_Y and the value of T is sufficient to cause the plastic region to penetrate to a radius, r_e, ($R_o > r_e > R_i$) derive an expression for r_e.

The total torque is, as for a solid bar, equal to the sum of the contributions from the elastic and plastic cores. Then

$$T = (\tau_Y J_e)/r_e + \int_{re}^{Ro} 2\pi r^2 \tau_Y \, dr$$

so that

$$T = \tau_Y \pi ({r_e}^4 - {R_i}^4)/2r_e + 2\pi\tau_Y (R_o^3 - r_e^3)/3$$

Simplifying

$$T = \frac{\pi\tau_Y}{2r_e}\left(\frac{4}{3}R_o^3 r_e - \frac{1}{3}{r_e}^4 - {R_i}^4\right) \tag{i}$$

from which, for given values of T, τ_Y, R_o and R_i , r_e can be determined.

Note that Eq. (i) reduces to Eq. (11.12) for the case of a solid bar for which $R_i = 0$.

11.4 Torsion of a thin-walled closed section beam

Although the analysis of torsion problems is generally complex and in some instances relies on empirical methods for a solution, the torsion of a thin-walled beam of arbitrary closed section is relatively straightforward.

Figure 11.13(a) shows a thin-walled closed section beam subjected to a torque, T. The thickness, t, is constant along the length of the beam but may vary round the cross section. The torque T induces a stress system in the walls of the beam which consists solely of shear stresses if the applied loading comprises only a

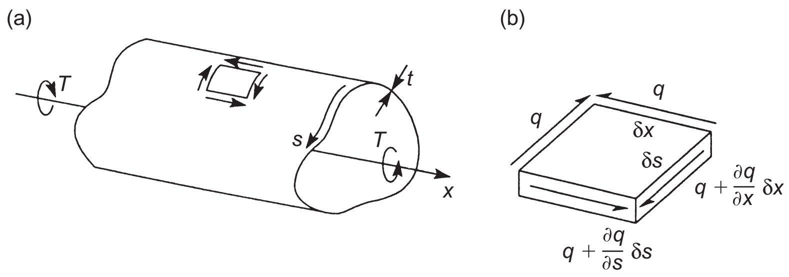

FIGURE 11.13

Torsion of a thin-walled closed section beam.

pure torque. In some cases structural or loading discontinuities or the method of support produce a system of direct stresses in the walls of the beam even though the loading consists of torsion only. These effects, known as axial constraint effects, are considered in more advanced texts.

The shear stress system on an element of the beam wall may be represented in terms of the shear flow, q, (see Section 10.4) as shown in Fig. 11.13(b). Again we are assuming that the variation of t over the side δs of the element may be neglected. For equilibrium of the element in the x direction we have

$$\left(q + \frac{\partial q}{\partial s}\delta s\right)\delta x - q\,\delta x = 0$$

which gives

$$\frac{\partial q}{\partial s} = 0 \tag{11.17}$$

Considering equilibrium in the s direction

$$\left(q + \frac{\partial q}{\partial x}\delta x\right)\delta s - q\,\delta s = 0$$

from which

$$\frac{\partial q}{\partial x} = 0 \tag{11.18}$$

Equations (11.17) and (11.18) may only be satisfied simultaneously by a constant value of q. We deduce, therefore, that the application of a pure torque to a thin-walled closed section beam results in the development of a constant shear flow in the beam wall. However, the shear stress, τ, may vary round the cross section since we allow the wall thickness, t, to be a function of s.

The relationship between the applied torque and this constant shear flow may be derived by considering the torsional equilibrium of the section shown in Fig. 11.14. The torque produced by the shear flow acting on the element, δs, of the beam wall is $q\,\delta s\,p$. Hence

$$T = \oint pq\,\mathrm{d}s$$

or, since $q =$ constant

$$T = q\oint p\,\mathrm{d}s \tag{11.19}$$

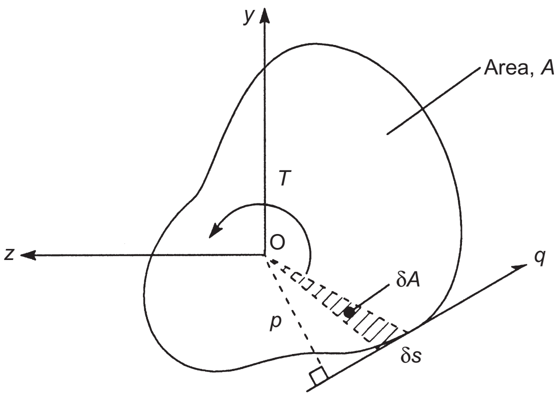

FIGURE 11.14

Torque–shear flow relationship in a thin-walled closed section beam.

We have seen in Section 10.5 that $\oint p\,\mathrm{d}s = 2A$ where A is the area enclosed by the midline of the beam wall. Hence

$$T = 2Aq \tag{11.20}$$

The theory of the torsion of thin-walled closed section beams is known as the *Bredt-Batho theory* and Eq. (11.20) is often referred to as the *Bredt-Batho formula.*

It follows from Eq. (11.20) that

$$\tau = \frac{q}{t} = \frac{T}{2At} \tag{11.21}$$

and that the maximum shear stress in a beam subjected to torsion will occur at the section where the torque is a maximum and at the point in that section where the thickness is a minimum. Thus

$$\tau_{\max} = \frac{T_{\max}}{2At_{\min}} \tag{11.22}$$

In Section 10.5 we derived an expression (Eq. (10.28)) for the rate of twist, $\mathrm{d}\theta/\mathrm{d}x$, in a shear-loaded thin-walled closed section beam. Equation (10.28) also applies to the case of a closed section beam under torsion in which the shear flow is constant if it is assumed that, as in the case of the shear-loaded beam, cross sections remain undistorted after loading. Thus, rewriting Eq. (10.28) for the case $q_s = q =$ constant, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{q}{2A}\oint \frac{\mathrm{d}s}{Gt} \tag{11.23}$$

Substituting for q from Eq. (11.20) we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{T}{4A^2}\oint \frac{\mathrm{d}s}{Gt} \tag{11.24}$$

or, if G, the shear modulus, is constant round the section

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{T}{4A^2G}\oint \frac{\mathrm{d}s}{t} \tag{11.25}$$

EXAMPLE 11.9

A thin-walled circular section beam has a diameter of 200 mm and is 2 m long; it is firmly restrained against rotation at each end. A concentrated torque of 30 kNm is applied to the beam at its mid-span point. If the maximum shear stress in the beam is limited to 200 N/mm^2 and the maximum angle of twist to 2°, calculate the minimum thickness of the beam walls. Take $G = 25000$ N/mm^2.

The minimum thickness of the beam corresponding to the maximum allowable shear stress of 200 N/mm^2 is obtained directly using Eq. (11.22) in which $T_{\max} = 15$ kNm. Thus

$$t_{\min} = \frac{15 \times 10^6 \times 4}{2 \times \pi \times 200^2 \times 200} = 1.2\,\text{mm}$$

The rate of twist along the beam is given by Eq. (11.25) in which

$$\oint \frac{\mathrm{d}s}{t} = \frac{\pi \times 200}{t_{\min}}$$

Hence

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{T}{4A^2G} \times \frac{\pi \times 200}{t_{\min}} \tag{i}$$

Taking the origin for x at one of the fixed ends and integrating Eq. (i) for half the length of the beam we obtain

$$\theta = \frac{T}{4A^2G} \times \frac{200\pi}{t_{\min}} x + C_1$$

where C_1 is a constant of integration. At the fixed end where $x = 0$, $\theta = 0$ so that $C_1 = 0$. Hence

$$\theta = \frac{T}{4A^2G} \times \frac{200\pi}{t_{\min}} x$$

The maximum angle of twist occurs at the mid-span of the beam where $x = 1$ m. Hence

$$t_{\min} = \frac{15 \times 10^6 \times 200 \times \pi \times 1 \times 10^3 \times 180}{4 \times (\pi \times 200^2/4)^2 \times 25\,000 \times 2 \times \pi} = 2.7\text{mm}$$

The minimum allowable thickness that satisfies both conditions is therefore 2.7 mm.

11.5 Torsion of solid section beams

Generally, by solid section beams, we mean beam sections in which the walls do not form a closed loop system. Examples of such sections are shown in Fig. 11.15. An obvious exception is the hollow circular section bar which is, however, a special case of the solid circular section bar. The prediction of stress distributions and angles of twist produced by the torsion of such sections is complex and relies on the St. Venant warping function or Prandtl stress function methods of solution. Both of these methods are based on the theory of elasticity which may be found in advanced texts devoted solely to this topic. Even so, exact solutions exist for only a few practical cases, one of which is the circular section bar.

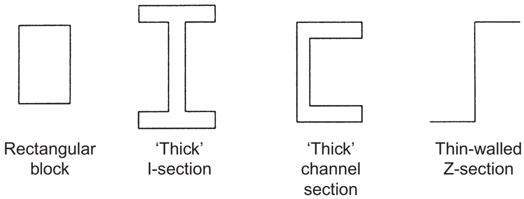

FIGURE 11.15

Examples of solid beam sections.

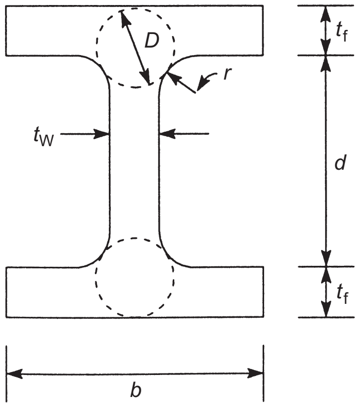

FIGURE 11.16

Torsion constant for a 'thick' I-section beam.

In all torsion problems, however, it is found that the torque, T, and the rate of twist, $d\theta/dx$, are related by the equation

$$T = GJ\frac{d\theta}{dx} \tag{11.26}$$

where G is the shear modulus and J is the *torsion constant.* For a circular section bar J is the polar second moment of area of the section (see Eq. (11.3)) while for a thin-walled closed section beam J, from Eq. (11.25), is seen to be equal to $4A^2 \oint(ds/t)$. It is J, in fact, that distinguishes one torsion problem from another.

For 'thick' sections of the type shown in Fig. 11.15 J is obtained empirically in terms of the dimensions of the particular section. For example, the torsion constant of the 'thick' I-section shown in Fig. 11.16 is given by

$$J = 2J_1 + J_2 + 2\alpha D^4$$

where

$$J_1 = \frac{bt_f^3}{3}\left[1 - 0.63\frac{t_f}{b}\left(1 - \frac{t_f^4}{12b^4}\right)\right]$$

$$J_2 = \frac{1}{3}dt_w^3$$

$$\alpha = \frac{t_1}{t_2}\left(0.15 + 0.1\frac{r}{t_f}\right)$$

in which $t_1 = t_f$ and $t_2 = t_w$ if $t_f < t_w$, or $t_1 = t_w$ and $t_2 = t_f$ if $t_f > t_w$.

It can be seen from the above that J_1 and J_2, which are the torsion constants of the flanges and web, respectively, are each equal to one-third of the product of their length and their thickness cubed multiplied, in the case of the flanges, by an empirical constant. The torsion constant for the complete section is then the

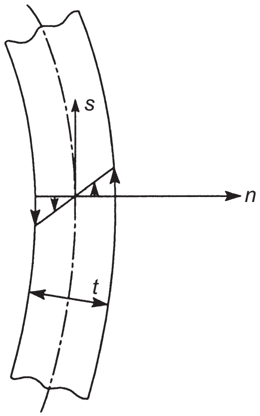

FIGURE 11.17

Shear stress distribution due to torsion in a thin-walled open section beam.

sum of the torsion constants of the components plus a contribution from the material at the web/flange junction. If the section were thin-walled, $t_f \ll b$ and D^4 would be negligibly small, in which case

$$J \simeq 2\frac{bt_f^3}{3} + \frac{dt_w^3}{3}$$

Generally, for thin-walled sections the torsion constant J may be written as

$$J = \frac{1}{3}\sum st^3 \tag{11.27}$$

in which s is the length and t the thickness of each component in the cross section or if t varies with s

$$J = \frac{1}{3}\int_{\text{section}} t^3 \,\mathrm{d}s \tag{11.28}$$

The shear stress distribution in a thin-walled open section beam (Fig. 11.17) may be shown to be related to the rate of twist by the expression

$$\tau = 2Gn\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{11.29}$$

where n is the distance to any point in the section wall measured normally from its midline. The distribution is therefore linear across the thickness as shown in Fig. 11.17 and is zero at the midline of the wall. An alternative expression for shear stress distribution is obtained, in terms of the applied torque, by substituting for $\mathrm{d}\theta/\mathrm{d}x$ in Eq. (11.29) from Eq. (11.26). Thus

$$\tau = 2n\frac{T}{J} \tag{11.30}$$

It is clear from either of Eqs. (11.29) or (11.30) that the maximum value of shear stress occurs at the outer surfaces of the wall when $n = \pm t/2$. Hence

$$\tau_{\max} = \pm Gt\frac{\mathrm{d}\theta}{\mathrm{d}x} = \pm\frac{Tt}{J} \tag{11.31}$$

The positive and negative signs in Eq. (11.31) indicate the direction of the shear stress in relation to the assumed direction for s.

The behaviour of closed and open section beams under torsional loads is similar in that they twist and develop internal shear stress systems. However, the manner in which each resists torsion is different. It is clear from the preceding discussion that a pure torque applied to a beam section produces a closed, continuous shear stress system since the resultant of any other shear stress system would generally be a shear force unless, of course, the system were self-equilibrating. In a closed section beam this closed loop system of shear stresses is allowed to develop in a continuous path round the cross section, whereas in an open section beam it can only develop within the thickness of the walls; examples of both systems are shown in Fig. 11.18. Here, then,

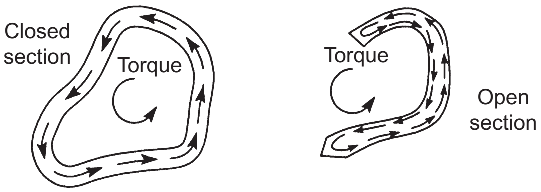

FIGURE 11.18

Shear stress development in closed and open section beams subjected to torsion.

lies the basic difference in the manner in which torsion is resisted by closed and open section beams and the reason for the comparatively low torsional stiffness of thin-walled open sections. Clearly the development of a closed loop system of shear stresses in an open section is restricted by the thinness of the walls.

EXAMPLE 11.10

The thin-walled section shown in Fig. 11.19 is symmetrical about a horizontal axis through O. The thickness t_0 of the centre web CD is constant, while the thickness of the other walls varies linearly from t_0 at points C and D to zero at the open ends A, F, G and H. Determine the torsion constant J for the section and also the maximum shear stress produced by a torque T.

Since the thickness of the section varies round its profile except for the central web, we use both Eqs (11.27) and (11.28) to determine the torsion constant. Thus,

$$J = \frac{2at_0^3}{3} + 2\times\frac{1}{3}\int_0^a \left(\frac{s_A t_0}{a}\right)^3 \mathrm{d}s_A + 2\times\frac{1}{3}\int_0^{3a}\left(\frac{s_B t_0}{3a}\right)^3 \mathrm{d}s_B$$

which gives

$$J = \frac{4at_0^3}{3}$$

The maximum shear stress is now obtained using Eq. (11.31), i.e.

$$\tau_{\max} = \pm\frac{Tt_0}{J} = \pm\frac{3Tt_0}{4at_0^3} = \pm\frac{3T}{4at_0^2}$$

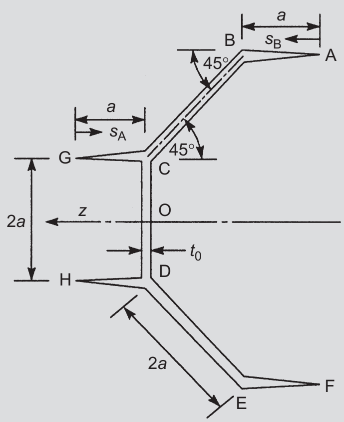

FIGURE 11.19

Beam section of Ex. 11.10.

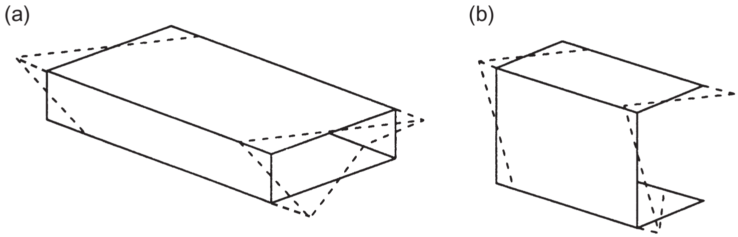

FIGURE 11.20

Warping of beam sections due to torsion.

11.6 Warping of cross sections under torsion

Although we have assumed that the shapes of closed and open beam sections remain undistorted during torsion, they do not remain plane. Thus, for example, the cross section of a rectangular section box beam, although remaining rectangular when twisted, warps out of its plane as shown in Fig. 11.20(a), as does the channel section of Fig. 11.20(b). The calculation of warping displacements is covered in more advanced texts and is clearly of importance if a beam is, say, built into a rigid foundation at one end. In such a situation the warping is suppressed and direct tensile and compressive stresses are induced which must be investigated in design particularly if a beam is of concrete where even low tensile stresses can cause severe cracking.

Some beam sections do not warp under torsion; these include solid (and hollow) circular section bars and square box sections of constant thickness.

PROBLEMS

P.11.1 The solid bar of circular cross section shown in Fig. P.11.1 is subjected to a torque of 1 kNm at its free end and a torque of 3 kNm at its change of section. Calculate the maximum shear stress in the bar and the angle of twist at its free end. $G = 70000$ N/mm^2.

Ans. 40.7 N/mm^2, 0.6°.

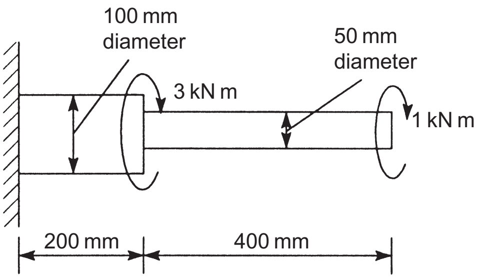

FIGURE P.11.1

P.11.2 A hollow circular section shaft 2 m long is firmly supported at each end and has an outside diameter of 80 mm. The shaft is subjected to a torque of 12 kNm applied at a point 1.5 m from one end. If the shear stress in the shaft is limited to 150 N/mm^2 and the angle of twist to 1.5°, calculate the maximum allowable internal diameter. The shear modulus $G = 80000$ N/mm^2.

Ans. 63.7 mm.

P.11.3 A bar ABCD of circular cross section having a diameter of 50 mm is firmly supported at each end and carries two concentrated torques at B and C as shown in Fig. P.11.3. Calculate the maximum shear stress in the bar and the maximum angle of twist. Take $G = 70000\ \text{N/mm}^2$.

Ans. 66.2 N/mm^2 in CD, 2.3° at B.

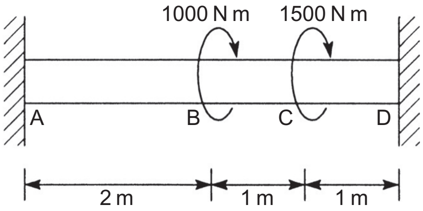

FIGURE P.11.3

P.11.4 A bar ABCD has a circular cross section of 75 mm diameter over half its length and 50 mm diameter over the remaining half of its length. A torque of 1 kNm is applied at C midway between B and D as shown in Fig. P.11.4. Sketch the distribution of torque along the length of the bar and calculate the maximum shear stress and the maximum angle of twist in the bar. $G = 70\ 000\ \text{N/mm}^2$.

Ans. $\tau_{\max} = 23.2\ \text{N/mm}^2$ in CD, 0.38° at C.

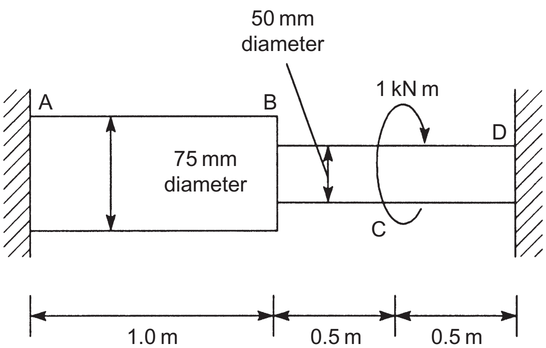

FIGURE P.11.4

P.11.5 A solid shaft has a circular cross section of diameter 150 mm and is required to transmit power at 90 rpm. If the maximum shear stress in the shaft is limited to 85 N/mm^2 calculate the power transmitted and the angle twist in a length of 5 m. Take $G = 80000\ \text{N/mm}^2$.

Ans. 531 kW, 4.1°.

P.11.6 If the solid shaft of P.11.5 is replaced by a hollow shaft of the same external diameter and having walls 30 mm thick calculate the percentage reduction in power transmitted for the same limiting value of shear stress. Calculate also the percentage reduction in weight.

Ans. 12.8%, 36%.

P.11.7 The bar shown in Fig. P.11.7 carries a single torque of 10 kNm applied mid-way along its length. Use strain energy to calculate the angle of twist under the applied torque and hence the angle of twist at its free end. Take $G = 75000\ \text{N/mm}^2$.

Ans. 2.9° at both points.

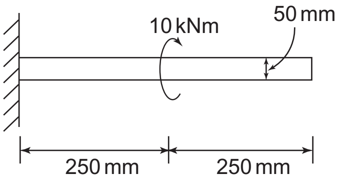

FIGURE P.11.7

P.11.8 Determine the minimum diameter of a solid bar which is required to transmit a torque of 40 kNm if the yield stress of the material of the bar is 120 N/mm^2.

Ans. 119.2 mm.

P.11.9 If the torque on the bar of P.11.8 is increased to 45 kNm calculate the diameter of the elastic core and the angle of twist of the bar over a length of 5 m.

Take $G = 80000$ N/mm^2.

Ans. 101.4 mm, 8.5°.

P.11.10 A hollow section bar has an outside diameter of 120 mm and an inside diameter of 60 mm. If the shear stress at yield in the material of the bar is 100 N/mm^2 calculate the maximum torque the bar can transmit without yielding occurring. If this torque is increased by 20% determine the outer radius of the elastic core of the section and the angle of twist over a length of 5 m. Take $G = 80000$ N/mm^2.

Ans. 42.8 mm (by trial and error), 8.3°.

P.11.11 A thin-walled rectangular section box girder carries a uniformly distributed torque loading of 1 kNm/mm over the outer half of its length as shown in Fig. P.11.11. Calculate the maximum shear stress in the walls of the box girder and also the distribution of angle of twist along its length; illustrate your answer with a sketch. Take $G = 70\ 000$ N/mm^2.

Ans. 133.3 N/mm^2. In AB, $\theta = 3.81 \times 10^{-6} x$ rad.

In BC, $\theta = 1.905 \times 10^{-9}(4000x - x^2/2) - 0.00381$ rad.

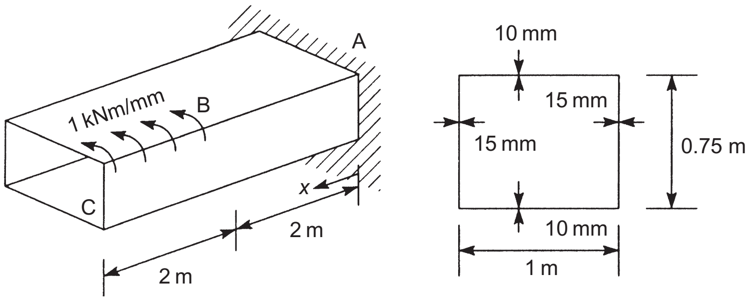

FIGURE P.11.11

P.11.12 The closed section thin-walled beam shown in Fig. P.11.12 is subjected to a torque of 4.5 kNm. For the curved wall 12 the thickness is 2 mm and the shear modulus is 22000 N/mm^2. For the walls 23, 34 and 41 the corresponding figures are 1.6 mm and 27500 N/mm^2. (Note: Gt = constant). Calculate the rate of twist of the beam section.

Ans. 29.3 rad/mm.

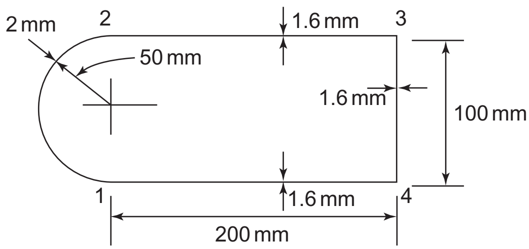

FIGURE P.11.12

P.11.13 The thin-walled box section beam ABCD shown in Fig. P.11.13 is attached at each end to supports which allow rotation of the ends of the beam in the longitudinal vertical plane of symmetry but prevent rotation of the ends in vertical planes perpendicular to the longitudinal axis of the beam. The beam is subjected to a uniform torque loading of 20 Nm/mm over the portion BC of its span. Calculate the maximum shear stress in the cross section of the beam and the distribution of angle of twist along its length; $G = 70\ 000$ N/mm^2.

Ans. 71.4 N/mm^2, $\theta_B = \theta_C = 0.36°$, θ at mid-span $= 0.72°$.

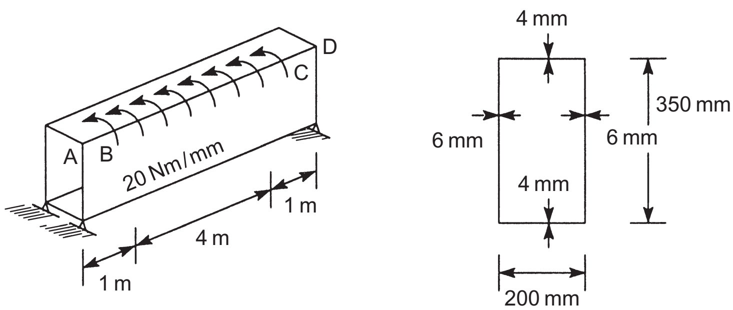

FIGURE P.11.13

P.11.14 Figure P.11.14 shows a thin-walled cantilever box-beam having a constant width of 50 mm and a depth which decreases linearly from 200 mm at the built-in end to 150 mm at the free end. If the beam is subjected to a torque of 1 kNm at its free end, plot the angle of twist of the beam at 500 mm intervals along its length and determine the maximum shear stress in the beam section. Take $G = 25\ 000$ N/mm^2.

Ans. $\tau_{max} = 33.3$ N/mm^2.

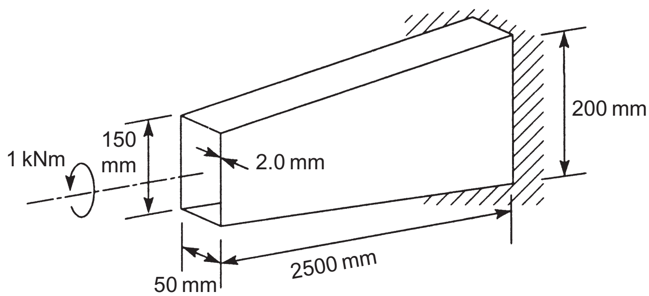

FIGURE P.11.14

P.11.15 The cold-formed section shown in Fig. P.11.15 is subjected to a torque of 50 Nm. Calculate the maximum shear stress in the section and its rate of twist. $G = 25\ 000\ \text{N/mm}^2$.

Ans. $\tau_{max} = 220.6\ \text{N/mm}^2$, $d\theta/dx = 0.0044$ rad/mm.

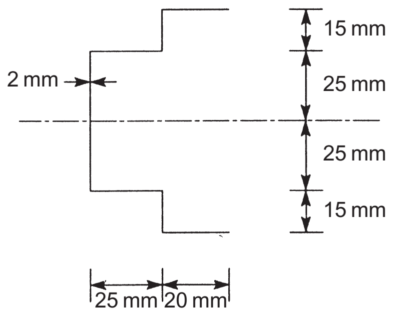

FIGURE P.11.15

P.11.16 The thin-walled angle section shown in Fig. P.11.16 supports shear loads that produce both shear and torsional effects. Determine the maximum shear stress in the cross section of the angle, stating clearly the point at which it acts.

Ans. $18.0\ \text{N/mm}^2$ on the inside of flange BC at 16.5 mm from point B.

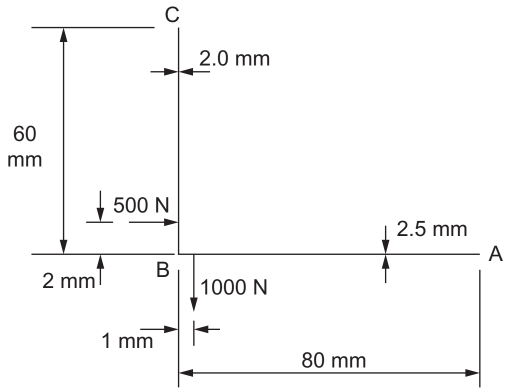

FIGURE P.11.16

P.11.17 Figure P.11.17 shows the cross section of a thin-walled inwardly lipped channel. The lips are of constant thickness while the flanges increase linearly in thickness from 1.27 mm, where they

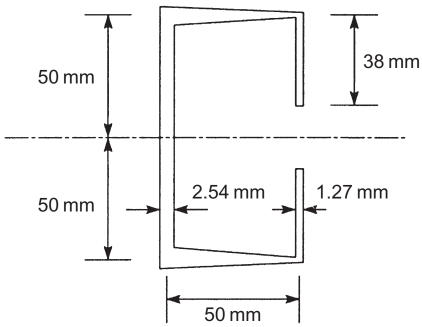

FIGURE P.11.17

meet the lips, to 2.54 mm at their junctions with the web. The web has a constant thickness of 2.54 mm and the shear modulus G is 26 700 N/mm^2. Calculate the maximum shear stress in the section and also its rate of twist if it is subjected to a torque of 100 Nm.

Ans. $\tau_{max} = \pm 297.4$ N/mm^2, $d\theta/dx = 0.0044$ rad/mm.

P.11.18 The thin-walled section shown in Fig. P.11.18 is subjected to a unit torque. Calculate the maximum shear stress in the section.

Ans. $\pm\ 0.42/rt^2$.

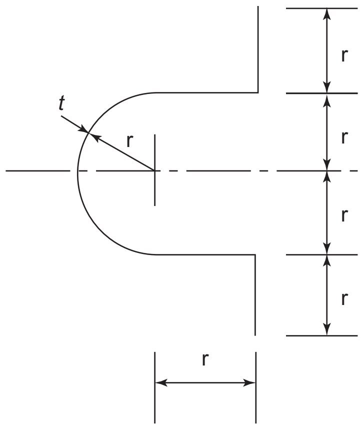

FIGURE P.11.18

P.11.19 Determine the maximum shear stress in the beam section shown in Fig. P.11.19 stating clearly the point at which it occurs. Calculate also the rate of twist of the section if the shear modulus $G = 25000$ N/mm^2.

Ans. 70.2 N/mm^2 on the underside of 24 at 2 or on the upper surface of 32 at 2.
9.0×10^{-4} rad/mm.

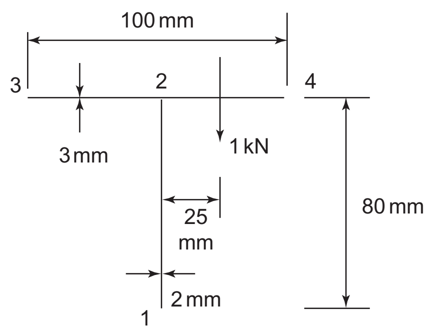

FIGURE P.11.19