CHAPTER

The Solution of the Equations of Motion

5

5.1 Methods of solution

The primary reason for solving the equations of motion is to obtain a mathematical, and hence graphical, description of the time histories of all motion variables in response to a control input, or atmospheric disturbance, and to enable an assessment of stability. It is also important that the chosen method of solution provides good insight into the way in which the physical properties of the airframe influence the responses. Since the evolution of the development of the equations of motion and their solution followed in the wake of observation of aeroplane behaviour, it was no accident that practical constraints were applied which resulted in the decoupled small-perturbation equations. The longitudinal and lateral-directional decoupled equations of motion are each represented by a set of three simultaneous linear differential equations which have traditionally been solved using classical mathematical analysis methods. Although laborious to apply, the advantage of the traditional approach is that it is capable of providing excellent insight into the nature of aircraft stability and response. However, since traditional methods invariably involve the use of the dimensionless equations of motion, considerable care in the interpretation of the numerical results is required if confusion is to be avoided. A full discussion of these methods can be found in many of the earlier books on the subject, for example, Duncan (1952).

Operational methods have also enjoyed some popularity as a means for solving the equations of motion. In particular, the Laplace transform method has been, and continues to be, used extensively. By transforming the differential equations, they become algebraic equations expressed in terms of the Laplace operator s. Their manipulation to obtain a solution then becomes a relatively straightforward exercise in algebra. The problem is thus transformed into one of solving a set of simultaneous linear algebraic equations, a process that is readily accomplished by computational methods. Further, the input-output response relationship or transfer characteristic is described by a simple algebraic *transfer function* in terms of the Laplace operator. The time response then follows by finding the inverse Laplace transform of the transfer function for the input of interest.

Now, the transfer function as a means for describing the characteristics of a linear dynamic system is the principal tool of the control systems engineer, and a vast array of mathematical tools is available for transfer function analysis. With relative ease, analysis of the transfer function of a system enables a complete picture of its dynamic behaviour to be drawn. In particular, stability, time response, and frequency response information is readily obtained. Furthermore, obtaining the system transfer function is usually the prelude to design of a feedback control system, and an additional array of mathematical tools is available to support this task. Since most modern aeroplanes are dependent, to a greater or lesser extent, on feedback control for their continued proper operation, it would seem particularly advantageous to be able to describe them in terms of transfer functions. Fortunately, this is easily accomplished. The Laplace transform of the linearised small-perturbation equations of motion is readily obtained, and by the subsequent application of the appropriate mathematical tools the *response transfer functions* may be derived. An analysis of the dynamic properties of the aeroplane may then be made using control engineering tools as an alternative to the traditional methods of the aerodynamicist. Indeed, as described in Chapter 1, many computer software packages are available which facilitate the rapid and accurate analysis of linear dynamic systems and the design of automatic control systems. Today, access to computer software of this type is essential for the flight dynamicist.

The process of solution requires that the equations of motion be assembled in the appropriate format, numerical values for the derivatives and other parameters substituted, and then the whole model input to a suitable computer program. The output, which is usually obtained instantaneously, is most conveniently arranged in terms of response transfer functions. The objective can thus usually be achieved relatively easily, with great rapidity and good accuracy. A significant shortcoming of such computational methods is the lack of *visibility*; that is, the functional steps in the solution process are hidden from the investigator. Consequently, considerable care, and some skill, is required to analyse the solution correctly, which can be greatly facilitated if the investigator has a good understanding of the computational solution process. Indeed, it is considered essential to have an understanding of the steps involved in the solution of the equations of motion using the operational methods common to most computer software packages.

Much of the remainder of this chapter is concerned with the use of the Laplace transform for solving the small-perturbation equations of motion to obtain the response transfer functions. This is followed by a description of the computational process involving matrix methods which is normally undertaken with the aid of a suitable computer software package.

5.2 Cramer's rule

Cramer's rule describes a mathematical process for solving sets of simultaneous linear algebraic equations. It may be used to solve the equations of motion algebraically and is found in many degree-level mathematical texts, and in books devoted to the application of computational methods to linear algebra (e.g., Goult et al. (1974)). Since Cramer's rule involves the use of matrix algebra it is easily implemented in a digital computer.

To solve the system of n simultaneous linear algebraic equations described in matrix form as

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{5.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are column vectors and $\mathbf{A}$ is a matrix of constant coefficients, Cramer's rule states that

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \equiv \left(\frac{\text{Adjoint}\mathbf{A}}{\text{Det}\mathbf{A}}\right)\mathbf{y} \tag{5.2}$$

where the solution for x_i, the ith row of equation (5.2), is given by

$$x_i = \frac{1}{|\mathbf{A}|}(A_{1i}y_1 + A_{2i}y_2 + A_{3i}y_3 + \cdots\cdots + A_{ni}y_n) \tag{5.3}$$

The significant observation is that the numerator of equation (5.3) is equivalent to the determinant of $\mathbf{A}$, with the ith column replaced by the vector $\mathbf{y}$. Thus the solution of equations (5.1) to find all the x_i reduces to the relatively simple problem of evaluating $n + 1$ determinants.

EXAMPLE 5.1

To illustrate the use of Cramer's rule, consider the trivial example in which it is required to solve the simultaneous linear algebraic equations

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 \\ y_2 &= 2x_2 + 4x_2 + 5x_3 \\ y_3 &= 3x_1 + 5x_2 + 6x_3 \end{aligned}$$

or, in matrix notation,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Applying Cramer's rule to solve for x_i,

$$x_1 = \frac{\begin{vmatrix} y_1 & 2 & 3 \\ y_2 & 4 & 5 \\ y_3 & 5 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}} = \frac{y_1\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - y_2\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + y_3\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}}{-1} = y_1 - 3y_2 + 2y_3$$

$$x_2 = \frac{\begin{vmatrix} 1 & y_1 & 3 \\ 2 & y_2 & 5 \\ 3 & y_3 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}} = \frac{-y_1\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + y_2\begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} - y_3\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}}{-1} = -3y_1 + 3y_2 - y_3$$

and

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & y_1 \\ 2 & 4 & y_2 \\ 3 & 5 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}} = \frac{y_1\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} - y_2\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} + y_3\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}}{-1} = 2y_1 - y_2$$

Clearly, in this example, the numerator determinants are found by expanding about the column containing **y**. The denominator determinant may be found by expanding about the first row; thus

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = 1\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = -1 + 6 - 6 = -1$$

5.3 Aircraft response transfer functions

Aircraft response transfer functions describe the dynamic relationships between the input variables and the output variables. The relationships are indicated diagrammatically in Fig. 5.1 and it is clear that a number of possible input-output relationships exist. When the mathematical model of the aircraft comprises the decoupled small-perturbation equations of motion, transfer functions relating longitudinal input variables to lateral-directional output variables do not exist and vice versa. This may not necessarily be the case when the aircraft is described by a fully coupled set of small-perturbation equations of motion. For example, such a description is quite usual when modelling the helicopter.

All transfer functions are written as a ratio of two polynomials in the Laplace operator s. All *proper* transfer functions have a numerator polynomial which is at least one order less than the denominator polynomial, although occasionally *improper* transfer functions crop up in aircraft applications. For example, the transfer function describing acceleration response to an input variable is improper; the numerator and denominator polynomials are of the same order. Care is needed when working with improper transfer functions, as sometimes the computational tools are unable to deal with them correctly. Obviously, this is a situation where some understanding of the physical meaning of the transfer function can be of considerable advantage. A shorthand notation is used to represent aircraft response transfer functions in this book. For example, pitch attitude $\theta(s)$ response to elevator $\eta(s)$ is denoted

$$\frac{\theta(s)}{\eta(s)} \equiv \frac{N_\eta^\theta(s)}{\Delta(s)} \tag{5.4}$$

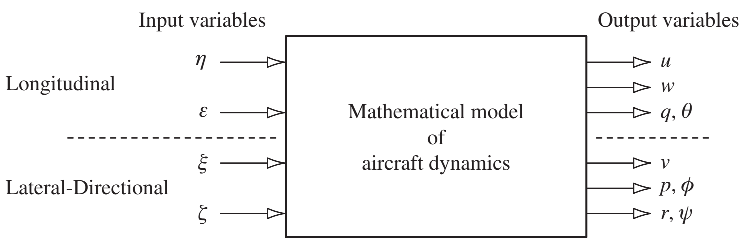

FIGURE 5.1 Aircraft input-output relationships.

where $N_\eta^\theta(s)$ is the unique numerator polynomial in s relating pitch attitude response to elevator input, and $\Delta(s)$ is the denominator polynomial in s which is common to all longitudinal response transfer functions. Similarly, for example, roll rate response to aileron is denoted

$$\frac{p(s)}{\xi(s)} \equiv \frac{N_\xi^p(s)}{\Delta(s)} \tag{5.5}$$

where in this instance $\Delta(s)$ is the denominator polynomial which is common to all of the lateral-directional response transfer functions. Since $\Delta(s)$ is context-dependent its correct identification does not usually present problems.

The denominator polynomial $\Delta(s)$ is called the *characteristic polynomial* and, when equated to zero, defines the *characteristic equation*. Thus $\Delta(s)$ completely describes the longitudinal or lateral-directional stability characteristics of the aeroplane as appropriate, and the roots, or *poles*, of $\Delta(s)$ describe its *stability modes*. Thus the stability characteristics of an aeroplane can be determined simply on inspection of the response transfer functions.

5.3.1 The longitudinal response transfer functions

The Laplace transforms of the differential quantities $\dot{x}(t)$ and $\ddot{x}(t)$, for example, are given by

$$\begin{aligned} \mathrm{L}\{\dot{x}(t)\} &= sx(s) - x(0) \\ \mathrm{L}\{\ddot{x}(t)\} &= s^2x(s) - sx(0) - \dot{x}(0) \end{aligned} \tag{5.6}$$

where $x(0)$ and $\dot{x}(0)$ are the initial values of $x(t)$ and $\dot{x}(t)$, respectively, at $t = 0$. Now, taking the Laplace transform of the longitudinal equations of motion (4.40), referred to body axes, assuming zero initial conditions, and since small-perturbation motion only is considered, write

$$\dot{\theta}(t) = q(t) \tag{5.7}$$

then

$$\begin{aligned} \left(ms - \mathring{X}_u\right)u(s) - \left(\mathring{X}_{\dot{w}}s + \mathring{X}_w\right)w(s) - \left(\left(\mathring{X}_q - mW_e\right)s - mg\cos\theta_e\right)\theta(s) &= \mathring{X}_\eta\,\eta(s) + \mathring{X}_\tau\,\tau(s) \\ -\mathring{Z}_u\,u(s) - \left(\left(\mathring{Z}_{\dot{w}} - m\right)s + \mathring{Z}_w\right)w(s) - \left(\left(\mathring{Z}_q + mU_e\right)s - mg\sin\theta_e\right)\theta(s) &= \mathring{Z}_\eta\,\eta(s) + \mathring{Z}_\tau\,\tau(s) \\ -\mathring{M}_u\,u(s) - \left(\mathring{M}_{\dot{w}}\,s + \mathring{M}_w\right)w(s) + \left(I_ys^2 - \mathring{M}_q\,s\right)\theta(s) &= \mathring{M}_\eta\,\eta(s) + \mathring{M}_\tau\,\tau(s) \end{aligned} \tag{5.8}$$

Writing equations (5.8) in matrix format,

$$\begin{bmatrix} \left(ms - \mathring{X}_u\right) & -\left(\mathring{X}_{\dot{w}}s + \mathring{X}_w\right) & -\left(\left(\mathring{X}_q - mW_e\right)s - mg\cos\theta_e\right) \\ -\mathring{Z}_u & -\left(\left(\mathring{Z}_{\dot{w}} - m\right)s + \mathring{Z}_w\right) & -\left(\left(\mathring{Z}_q + mU_e\right)s - mg\sin\theta_e\right) \\ -\mathring{M}_u & -\left(\mathring{M}_{\dot{w}}\,s + \mathring{M}_w\right) & \left(I_ys^2 - \mathring{M}_q\,s\right) \end{bmatrix} \begin{bmatrix} u(s) \\ w(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} \mathring{X}_\eta & \mathring{X}_\tau \\ \mathring{Z}_\eta & \mathring{Z}_\tau \\ \mathring{M}_\eta & \mathring{M}_\tau \end{bmatrix} \begin{bmatrix} \eta(s) \\ \tau(s) \end{bmatrix} \tag{5.9}$$

Cramer's rule can now be applied to obtain the longitudinal response transfer functions—for example, those describing response to elevator. Assume, therefore, that the thrust remains constant, which means that the throttle is fixed at its trim setting τ_e, and $\tau(s) = 0$. Therefore, after dividing through by $\eta(s)$ equation, (5.9) may be simplified to

$$\begin{bmatrix} \left(ms - \mathring{X}_u\right) & -\left(\mathring{X}_{\dot{w}}s + \mathring{X}_w\right) & -\left(\left(\mathring{X}_q - mW_e\right)s - mg\cos\theta_e\right) \\ -\mathring{Z}_u & -\left(\left(\mathring{Z}_{\dot{w}} - m\right)s + \mathring{Z}_w\right) & -\left(\left(\mathring{Z}_q + mU_e\right)s - mg\sin\theta_e\right) \\ -\mathring{M}_u & -\left(\mathring{M}_{\dot{w}}s + \mathring{M}_w\right) & \left(I_y s^2 - \mathring{M}_q s\right) \end{bmatrix} \begin{bmatrix} \dfrac{u(s)}{\eta(s)} \\ \dfrac{w(s)}{\eta(s)} \\ \dfrac{\theta(s)}{\eta(s)} \end{bmatrix} = \begin{bmatrix} \mathring{X}_\eta \\ \mathring{Z}_\eta \\ \mathring{M}_\eta \end{bmatrix} \tag{5.10}$$

Equation (5.10) is of the same form as equation (5.1). Cramer's rule may be applied directly, and the elevator response transfer functions are given by

$$\frac{u(s)}{\eta(s)} \equiv \frac{N_\eta^u(s)}{\Delta(s)} \qquad \frac{w(s)}{\eta(s)} \equiv \frac{N_\eta^w(s)}{\Delta(s)} \qquad \frac{\theta(s)}{\eta(s)} \equiv \frac{N_\eta^\theta(s)}{\Delta(s)} \tag{5.11}$$

Since the Laplace transform of equation (5.7) is $s\theta(s) = q(s)$, the pitch rate response transfer function follows directly:

$$\frac{q(s)}{\eta(s)} \equiv \frac{N_\eta^q(s)}{\Delta(s)} = \frac{sN_\eta^\theta(s)}{\Delta(s)} \tag{5.12}$$

The numerator polynomials are given by the following determinants:

$$N_\eta^u(s) = \begin{vmatrix} \mathring{X}_\eta & -\left(\mathring{X}_{\dot{w}}s + \mathring{X}_w\right) & -\left(\left(\mathring{X}_q - mW_e\right)s - mg\cos\theta_e\right) \\ \mathring{Z}_\eta & -\left(\left(\mathring{Z}_{\dot{w}} - m\right)s + \mathring{Z}_w\right) & -\left(\left(\mathring{Z}_q + mU_e\right)s - mg\sin\theta_e\right) \\ \mathring{M}_\eta & -\left(\mathring{M}_{\dot{w}}s + \mathring{M}_w\right) & \left(I_y s^2 - \mathring{M}_q s\right) \end{vmatrix} \tag{5.13}$$

$$N_\eta^w(s) = \begin{vmatrix} \left(ms - \mathring{X}_u\right) & \mathring{X}_\eta & -\left(\left(\mathring{X}_q - mW_e\right)s - mg\cos\theta_e\right) \\ -\mathring{Z}_u & \mathring{Z}_\eta & -\left(\left(\mathring{Z}_q + mU_e\right)s - mg\sin\theta_e\right) \\ -\mathring{M}_u & \mathring{M}_\eta & \left(I_y s^2 - \mathring{M}_q s\right) \end{vmatrix} \tag{5.14}$$

$$N_\eta^\theta(s) = \begin{vmatrix} \left(ms - \mathring{X}_u\right) & -\left(\mathring{X}_{\dot{w}}s + \mathring{X}_w\right) & \mathring{X}_\eta \\ -\mathring{Z}_u & -\left(\left(\mathring{Z}_{\dot{w}} - m\right)s + \mathring{Z}_w\right) & \mathring{Z}_\eta \\ -\mathring{M}_u & -\left(\mathring{M}_{\dot{w}}s + \mathring{M}_w\right) & \mathring{M}_\eta \end{vmatrix} \tag{5.15}$$

and the common denominator polynomial is given by the determinant

$$\Delta(s) = \begin{vmatrix} \left(ms - \mathring{X}_u\right) & -\left(\mathring{X}_{\dot{w}}s + \mathring{X}_w\right) & -\left(\left(\mathring{X}_q - mW_e\right)s - mg\cos\theta_e\right) \\ -\mathring{Z}_u & -\left(\left(\mathring{Z}_{\dot{w}} - m\right)s + \mathring{Z}_w\right) & -\left(\left(\mathring{Z}_q + mU_e\right)s - mg\sin\theta_e\right) \\ -\mathring{M}_u & -\left(\mathring{M}_{\dot{w}}s + \mathring{M}_w\right) & \left(I_y s^2 - \mathring{M}_q s\right) \end{vmatrix} \tag{5.16}$$

The thrust response transfer functions may be derived by assuming the elevator to be fixed at its trim value; thus $\eta(s) = 0$, and $\tau(s)$ is written in place of $\eta(s)$. Then the derivatives $\mathring{X}_\eta$, $\mathring{Z}_\eta$, and $\mathring{M}_\eta$ in equations (5.13), (5.14), and (5.15) are replaced by $\mathring{X}_\tau$, $\mathring{Z}_\tau$, and $\mathring{M}_\tau$, respectively. Since the polynomial expressions given by the determinants are substantial, they are set out in full in Appendix 3.

5.3.2 The lateral-directional response transfer functions

The lateral-directional response transfer functions may be obtained by exactly the same means as the longitudinal transfer functions. The Laplace transform, assuming zero initial conditions, of the lateral–directional equations of motion referred to body axes, equations (4.45), may be written in matrix form as follows:

$$\begin{bmatrix} \left(ms - \mathring{Y}_v\right) & -\left(\begin{matrix}\left(\mathring{Y}_p + mW_e\right)s \\ + mg\cos\theta_e\end{matrix}\right) & -\left(\begin{matrix}\left(\mathring{Y}_r - mU_e\right)s \\ + mg\sin\theta_e\end{matrix}\right) \\ -\mathring{L}_v & \left(I_x s^2 - \mathring{L}_p s\right) & -\left(I_{xz}s^2 + \mathring{L}_r s\right) \\ -\mathring{N}_v & -\left(I_{xz}s^2 + \mathring{N}_p s\right) & \left(I_z s^2 - \mathring{N}_r s\right) \end{bmatrix} \begin{bmatrix} v(s) \\ \phi(s) \\ \psi(s) \end{bmatrix} = \begin{bmatrix} \mathring{Y}_\xi & \mathring{Y}_\zeta \\ \mathring{L}_\xi & \mathring{L}_\zeta \\ \mathring{N}_\xi & \mathring{N}_\zeta \end{bmatrix} \begin{bmatrix} \xi(s) \\ \zeta(s) \end{bmatrix} \tag{5.17}$$

where $s\phi(s) = p(s)$ and $s\psi(s) = r(s)$. By holding the rudder at its trim setting, $\zeta(s) = 0$, the aileron response transfer functions may be obtained by applying Cramer's rule to equation (5.17). Similarly, by holding the ailerons at the trim setting, $\xi(s) = 0$, the rudder response transfer functions may be obtained. For example, roll rate response to aileron is given by

$$\frac{N_\xi^p(s)}{\Delta(s)} \equiv \frac{p(s)}{\xi(s)} = \frac{s\phi(s)}{\xi(s)} \equiv \frac{sN_\xi^\phi(s)}{\Delta(s)} \tag{5.18}$$

where the numerator polynomial is given by

$$N_\xi^p(s) = s\begin{vmatrix} \left(ms - \mathring{Y}_v\right) & \mathring{Y}_\xi & -\left(\begin{matrix}\left(\mathring{Y}_r - mU_e\right)s \\ + mg\sin\theta_e\end{matrix}\right) \\ -\mathring{L}_v & \mathring{L}_\xi & -\left(I_{xz}s^2 + \mathring{L}_r s\right) \\ -\mathring{N}_v & \mathring{N}_\xi & \left(I_z s^2 - \mathring{N}_r s\right) \end{vmatrix} \tag{5.19}$$

and the denominator polynomial is given by

$$\Delta(s) = \begin{vmatrix} \left(ms - \mathring{Y}_v\right) & -\left(\begin{matrix}\left(\mathring{Y}_p + mW_e\right)s \\ + mg\cos\theta_e\end{matrix}\right) & -\left(\begin{matrix}\left(\mathring{Y}_r - mU_e\right)s \\ + mg\sin\theta_e\end{matrix}\right) \\ -\mathring{L}_v & \left(I_x s^2 - \mathring{L}_p s\right) & -\left(I_{xz}s^2 + \mathring{L}_r s\right) \\ -\mathring{N}_v & -\left(I_{xz}s^2 + \mathring{N}_p s\right) & \left(I_z s^2 - \mathring{N}_r s\right) \end{vmatrix} \tag{5.20}$$

Again, since the polynomial expressions given by the determinants are substantial, they are also set out in full in Appendix 3.

EXAMPLE 5.2

We wish to obtain the transfer function describing pitch attitude response to elevator for the Lockheed F-104 Starfighter. The data were obtained from Teper (1969) and describe a sea-level flight condition. Inspection of the data revealed that $\theta_e = 0$; thus it was concluded that the equations of motion to which the data relate are referred to wind axes.

Air density ρ	$= 0.00238$ slug/ft^3
Axial velocity component U_e	$= 305$ ft/s
Aircraft mass m	$= 746$ slugs
Moment of inertia in pitch I_y	$= 65000$ slug ft^2
Gravitational constant g	$= 32.2$ ft/s^2

The dimensional aerodynamic stability and control derivatives follow. Derivatives which are not quoted are assumed to be insignificant and are given a zero value; whence

$$
\begin{array}{lll}
\overset{\circ}{X}_u = -26.26 \text{ slug/s} & \overset{\circ}{Z}_u = -159.64 \text{ slug/s} & \overset{\circ}{M}_u = 0 \\
\overset{\circ}{X}_w = 79.82 \text{ slug/s} & \overset{\circ}{Z}_w = -328.24 \text{ slug/s} & \overset{\circ}{M}_w = -1014.0 \text{ slug ft/s} \\
\overset{\circ}{X}_{\dot{w}} = 0 & \overset{\circ}{Z}_{\dot{w}} = 0 & \overset{\circ}{M}_{\dot{w}} = -36.4 \text{ slug ft} \\
\overset{\circ}{X}_q = 0 & \overset{\circ}{Z}_q = 0 & \overset{\circ}{M}_q = -18{,}135 \text{ slug ft}^2\text{/s} \\
\overset{\circ}{X}_\eta = 0 & \overset{\circ}{Z}_\eta = -16{,}502 \text{ slug ft/s}^2\text{/rad} & \overset{\circ}{M}_\eta = -303{,}575 \text{ slug ft/s}^2\text{/rad}
\end{array}
$$

The American imperial units are retained in this example since it is preferable to work with the equations of motion in the dimensional units appropriate to the source material. Conversion from one system of units to another often leads to confusion and error and is therefore not recommended. However, for information, factors for conversion from American imperial units to SI units are given in Appendix 4.

These numerical values are substituted into equation (5.10) to obtain

$$
\begin{bmatrix} 746s + 26.26 & -79.82 & 24021.2 \\ 159.64 & 746s + 328.24 & -227530s \\ 0 & 36.4s + 1014 & 65000s^2 + 18135s \end{bmatrix} \begin{bmatrix} u(s) \\ w(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -16502 \\ -303575 \end{bmatrix} \eta(s) \tag{5.21}
$$

Cramer's rule may be applied directly to equation (5.21) to obtain the transfer function of interest:

$$
\frac{N_\eta^\theta(s)}{\Delta(s)} = \frac{\begin{vmatrix} 746s + 26.26 & -79.82 & 0 \\ 159.64 & 746s + 328.24 & -16502 \\ 0 & 36.4s + 1014 & -303575 \end{vmatrix}}{\begin{vmatrix} 746s + 26.26 & -79.82 & 24021.2 \\ 159.64 & 746s + 328.24 & -227530s \\ 0 & 36.4s + 1014 & 65000s^2 + 18135s \end{vmatrix}} \text{ rad/rad} \tag{5.22}
$$

Whence

$$
\frac{N_\eta^\theta(s)}{\Delta(s)} = \frac{-16.850 \times 10^{10}(s^2 + 0.402s + 0.036)}{3.613 \times 10^{10}(s^4 + 0.925s^3 + 4.935s^2 + 0.182s + 0.108)} \text{ rad/rad} \tag{5.23}
$$

Or, in the preferable factorised form,

$$\frac{N_{\eta}^{\theta}(s)}{\Delta(s)}=\frac{-4.664(s+0.135)(s+0.267)}{(s^2+0.033s+0.022)(s^2+0.893s+4.884)}\ \text{rad/rad} \tag{5.24}$$

The denominator of equation (5.24) factorises into two pairs of complex roots (*poles*), each of which describes a longitudinal stability mode. The factors describing the modes may be written alternatively $(s^2+2\zeta\omega s+\omega^2)$, which is clearly the characteristic polynomial describing damped harmonic motion. The stability of each mode is determined by the damping ratio ζ; the undamped natural frequency, by ω. The lower-frequency mode is called the *phugoid* and the higher-frequency mode is called the *short-period pitching oscillation*. For the aeroplane to be completely longitudinally stable, the damping ratio of both modes must be positive.

The units of the transfer function given in equation (5.24) are rad/rad or, equivalently deg/deg. Angular measure is usually, and correctly, quantified in radians, and care must be applied when interpreting transfer functions since the radian is a very large angular quantity in the context of aircraft small-perturbation motion. This becomes especially important when dealing with transfer functions in which the input and output variables have different units. For example, the transfer function describing speed response to elevator for the F-104 has units ft/s/rad, and one radian of elevator input is impossibly large! It is therefore very important to remember that one radian is equivalent to 57.3 degrees. It is also important to remember that all transfer functions have units, which should always be indicated if confusion is to be avoided.

The transfer function given by equation (5.24) provides a complete description of the longitudinal stability characteristics and the dynamic pitch response to elevator of the F-104 at the flight condition in question. It is interesting that the transfer function has a negative sign. This means that a positive elevator deflection results in a negative pitch response, which is completely in accordance with the notation defined in Chapter 2. Clearly, the remaining longitudinal response transfer functions can be obtained by applying Cramer's rule to equation (5.21) for each of the remaining motion variables. A comprehensive review of aeroplane dynamics based on transfer function analysis can be found in Chapters 6 and 7.

The complexity of this example is such that, although tedious, the entire computation is easily undertaken manually to produce a result of acceptable accuracy. Alternatively, transfer function (5.23) can be calculated merely by substituting the values of the derivative and other data into the appropriate polynomial expressions given in Appendix 3.

5.4 Response to controls

Time histories for the aircraft response to controls are readily obtained by finding the inverse Laplace transform of the appropriate transfer function expression. For example, the roll rate response to aileron is given by equation (5.5) as

$$p(s)=\frac{N_{\xi}^{p}(s)}{\Delta(s)}\xi(s) \tag{5.25}$$

assuming that the aeroplane is initially in trimmed flight. The numerator polynomial $N_{\xi}^{p}(s)$ and denominator polynomial $\Delta(s)$ are given in Appendix 3. The aileron input $\xi(s)$ is simply the Laplace transform of the required input function. For example, two commonly used inputs are the *impulse* and *step* functions, where

Impulse of magnitude k is given by $\xi(s) = k$
Step of magnitude k is given by $\xi(s) = k/s$

Other useful input functions include the *ramp*, *pulse* (or step) of finite length, *doublet*, and *sinusoid*. However, the Laplace transform of these functions is not quite so straightforward. Fortunately, most computer programs for handling transfer function problems have the most commonly used functions "built in."

To continue with the example, the roll rate response to an aileron step input of magnitude k is given by

$$p(t) = \mathrm{L}^{-1}\left\{\frac{k}{s}\frac{N_{\xi}^{p}(s)}{\Delta(s)}\right\} \tag{5.26}$$

Solution of equation (5.26) to obtain the time response involves finding the inverse Laplace transform of the expression on the right-hand side, which may be accomplished manually with a table of standard transforms. However, this calculation is painlessly achieved with an appropriate computer software package such as MATLAB or Program CC, for example. Even so, it is instructive to review the mathematical procedure since this provides valuable insight into the correct interpretation of a computer solution, and this is most easily achieved by example, as follows.

EXAMPLE 5.3

To obtain the pitch response of the F-104 aircraft to a unit step elevator input at the flight condition evaluated in Example 5.2. Assuming the unit step input to be in degree units, then from equation (5.24),

$$\theta(t) = \mathrm{L}^{-1}\left\{\frac{-4.664(s+0.135)(s+0.267)}{s(s^2+0.033s+0.022)(s^2+0.893s+4.884)}\right\}\ \mathrm{deg} \tag{5.27}$$

Before the inverse Laplace transform of the expression in parentheses can be found, it is first necessary to reduce it to partial fractions. Thus

$$\frac{-4.664(s^2+0.402s+0.036)}{s(s^2+0.033s+0.022)(s^2+0.893s+4.884)} = -4.664\left(\frac{A}{s}+\frac{Bs+C}{(s^2+0.033s+0.022)} + \frac{Ds+E}{(s^2+0.893s+4.884)}\right) \tag{5.28}$$

To determine the values for A, B, C, D, and E, multiply out the fractions on the right-hand side and equate the numerator coefficients from both sides of the equation for like powers of s to obtain

$$\begin{aligned} 0 &= (A+B+D)s^4 \\ 0 &= (0.925A + 0.893B + C + 0.033D + E)s^3 \\ s^2 &= (4.935A + 4.884B + 0.893C + 0.022D + 0.033E)s^2 \\ 0.402s &= (0.182A + 4.884C + 0.022E)s \\ 0.036 &= 0.108A \end{aligned}$$

These simultaneous linear algebraic equations are easily solved using Cramer's rule if they are first written in matrix form:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0.925 & 0.893 & 1 & 0.033 & 1 \\ 4.935 & 4.884 & 0.893 & 0.022 & 0.033 \\ 0.182 & 0 & 4.884 & 0 & 0.022 \\ 0.108 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0.402 \\ 0.036 \end{bmatrix} \tag{5.29}$$

Thus $A = 0.333$, $B = -0.143$, $C = 0.071$, $D = -0.191$, $E = -0.246$ and equation (5.27) may be written as

$$\theta(t) = \mathrm{L}^{-1}\left\{-4.664\left(\frac{0.333}{s} - \frac{(0.143s - 0.071)}{(s^2 + 0.033s + 0.022)} - \frac{(0.191s + 0.246)}{(s^2 + 0.893s + 4.884)}\right)\right\} \text{ deg} \tag{5.30}$$

A very short table of Laplace transforms relevant to this problem is given in Appendix 5. Inspection of the table of transforms determines that equation (5.30) needs some rearrangement before its inverse transform can be found. When solving problems of this type, it is useful to appreciate that the solution contains terms describing damped harmonic motion; the required form of the terms in equation (5.30) is thus more easily established. With reference to Appendix 5, transform pairs 1, 5, and 6 appear to be most applicable. Therefore, rearranging equation (5.30) to suit,

$$\theta(t) = \mathrm{L}^{-1}\left\{-4.664\left(\begin{aligned} &\frac{0.333}{s} - \left(\frac{0.143(s+0.017)}{(s+0.017)^2 + 0.148^2} - \frac{0.496(0.148)}{(s+0.017)^2 + 0.148^2}\right) \\ &- \left(\frac{0.191(s+0.447)}{(s+0.447)^2 + 2.164^2} + \frac{0.074(2.164)}{(s+0.447)^2 + 2.164^2}\right) \end{aligned}\right)\right\} \text{ deg} \tag{5.31}$$

Using transform pairs 1, 5, and 6, equation (5.31) may be evaluated to give the time response:

$$\begin{aligned} \theta(t) = &-1.553 + 0.667e^{-0.017t}(\cos 0.148t - 3.469 \sin 0.148t) \\ &+ 0.891e^{-0.447t}(\cos 2.164t + 0.389 \sin 2.164t) \text{ deg} \end{aligned} \tag{5.32}$$

The solution given by equation (5.32) comprises three terms which may be interpreted as follows:

The first term, −1.553 deg, is the constant steady-state pitch attitude (gain) of the aeroplane.
The second term describes the contribution made by the phugoid dynamics, the undamped natural frequency $\omega_p = 0.148$ rad/s, and since $\zeta_p\omega_p = 0.017$ rad/s, the damping ratio is $\zeta_p = 0.115$.
The third term describes the contribution made by the short-period pitching oscillation dynamics, the undamped natural frequency $\omega_s = 2.164$ rad/s, and since $\zeta_s\omega_s = 0.447$ rad/s, the damping ratio is $\zeta_s = 0.207$.

The time response described by equation (5.32) is shown in Fig. 5.2, in which the two dynamic modes are clearly visible. It is also clear that the pitch attitude eventually settles to the steady-state value predicted previously.

Example 5.3 illustrates that it is not necessary to have a complete time response solution merely to obtain the characteristics of the dynamic modes. The principal mode characteristics, damping ratio and natural frequency, are directly obtainable on inspection of the characteristic polynomial $\Delta(s)$ in any aircraft transfer function. The steady-state gain is also readily established by application of the *final value theorem*, which states that

$$f(t)_{t\to\infty} = \operatorname*{Lim}_{s\to 0}(sf(s)) \tag{5.33}$$

The corresponding *initial value theorem*, also a valuable tool, states that

$$f(t)_{t\to 0} = \operatorname*{Lim}_{s\to\infty}(sf(s)) \tag{5.34}$$

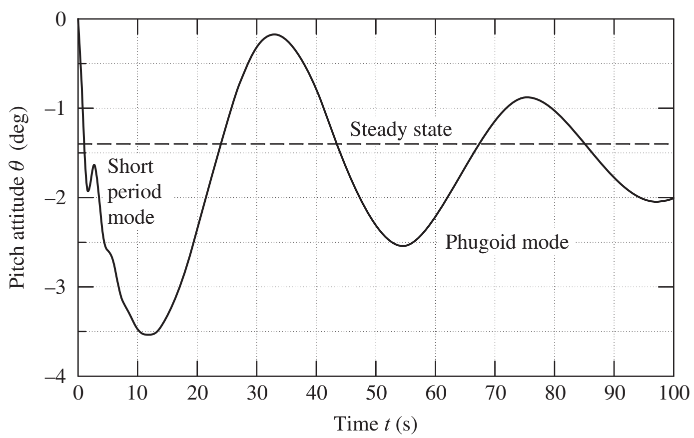

FIGURE 5.2 F-104 pitch attitude response to a 1° step of elevator.

A complete discussion of these theorems may be found in most books on control theory—for example, Shinners (1980).

EXAMPLE 5.4

Applying the initial value and final value theorems to find the initial and steady values of the pitch attitude response of the F-104 of the previous examples. From equation (5.27), the Laplace transform of the unit step response is given by

$$\theta(s) = \frac{-4.664(s+0.135)(s+0.267)}{s(s^2+0.033s+0.022)(s^2+0.893s+4.884)} \text{ deg} \tag{5.35}$$

Applying the final value theorem, we obtain

$$\theta(t)_{t\to\infty} = \lim_{s\to 0}\left(\frac{-4.664(s+0.135)(s+0.267)}{(s^2+0.033s+0.022)(s^2+0.893s+4.884)}\right) \text{ deg} = -1.565 \text{ deg} \tag{5.36}$$

and applying the initial value theorem, we obtain

$$\theta(t)_{t\to 0} = \lim_{s\to\infty}\left(\frac{-4.664(s+0.135)(s+0.267)}{(s^2+0.033s+0.022)(s^2+0.893s+4.884)}\right) \text{ deg} = 0 \text{ deg} \tag{5.37}$$

The values given by equations (5.36) and (5.37) clearly correlate reasonably well with the known pitch attitude response calculated in Example 5.3. Bear in mind that, in all calculations, numbers have been rounded to three decimal places for convenience.

5.5 Acceleration response transfer functions

Acceleration response transfer functions are frequently required but are not given directly by the solution of the equations of motion described previously. Expressions for the components of inertial acceleration are given in equations (4.9) and clearly comprise a number of motion variable contributions. Assuming small-perturbation motion such that the usual simplifications can be made, equations (4.9) may be restated as

$$\begin{aligned} a_x &= \dot{u} - rV_e + qW_e - y\dot{r} + z\dot{q} \\ a_y &= \dot{v} - pW_e + rU_e + x\dot{r} - z\dot{p} \\ a_z &= \dot{w} - qU_e + pV_e - x\dot{q} + y\dot{p} \end{aligned} \tag{5.38}$$

If, for example, the normal acceleration response to elevator referred to the *cg* is required ($x = y = z = 0$) and if fully decoupled motion is assumed ($pV_e = 0$), then the equation for normal acceleration simplifies to

$$a_z = \dot{w} - qU_e \tag{5.39}$$

The Laplace transform of equation (5.39), assuming zero initial conditions, may be written as

$$a_z(s) = sw(s) - s\theta(s)U_e \tag{5.40}$$

Or, expressing equation (5.40) in terms of elevator response transfer functions,

$$a_z(s) = s\frac{N_\eta^w(s)}{\Delta(s)}\eta(s) - sU_e\frac{N_\eta^\theta(s)}{\Delta(s)}\eta(s) = \frac{s(N_\eta^w(s) - U_eN_\eta^\theta(s))\eta(s)}{\Delta(s)} \tag{5.41}$$

Thus the required normal acceleration response transfer function may be written as

$$\frac{N_\eta^{a_z}(s)}{\Delta(s)} \equiv \frac{a_z(s)}{\eta(s)} = \frac{s(N_\eta^w(s) - U_eN_\eta^\theta(s))}{\Delta(s)} \tag{5.42}$$

Transfer functions for the remaining acceleration response components may be derived in a similar manner.

Another useful transfer function which is often required in handling qualities studies gives the normal acceleration response to elevator measured at the pilot's seat. In this special case, x in equations (5.38) represents the distance measured from the cg to the pilot's seat; the normal acceleration is therefore given by

$$a_z = \dot{w} - qU_e - x\dot{q} \tag{5.43}$$

As before, the transfer function is easily derived:

$$\left.\frac{N_\eta^{a_z}(s)}{\Delta(s)}\right|_{\text{pilot}} = \frac{s(N_\eta^w(s) - (U_e + xs)N_\eta^\theta(s))}{\Delta(s)} \tag{5.44}$$

EXAMPLE 5.5

To calculate the normal acceleration response to elevator at the cg for the F-104 Starfighter at the flight condition defined in Example 5.2. At this flight condition the steady axial velocity component $U_e = 305$ ft/s and the pitch attitude and normal velocity transfer functions describing response to elevator are given by

$$\frac{N_\eta^\theta(s)}{\Delta(s)} = \frac{-4.664(s+0.135)(s+0.267)}{(s^2+0.033s+0.022)(s^2+0.893s+4.884)} \text{ rad/rad} \tag{5.45}$$

and

$$\frac{N_\eta^w(s)}{\Delta(s)} = \frac{-22.147(s^2+0.035s+0.022)(s+64.675)}{(s^2+0.033s+0.022)(s^2+0.893s+4.884)} \text{ ft/s/rad} \tag{5.46}$$

Substitute equations (5.45) and (5.46) together with U_e into equation (5.42), paying particular attention to the units, and multiply out the numerator and factorise the result to obtain the required transfer function:

$$\frac{N_\eta^{a_z}(s)}{\Delta(s)} = \frac{-22.147s(s+0.037)(s-4.673)(s+5.081)}{(s^2+0.033s+0.022)(s^2+0.893s+4.884)} \text{ ft/s}^2\text{/rad} \tag{5.47}$$

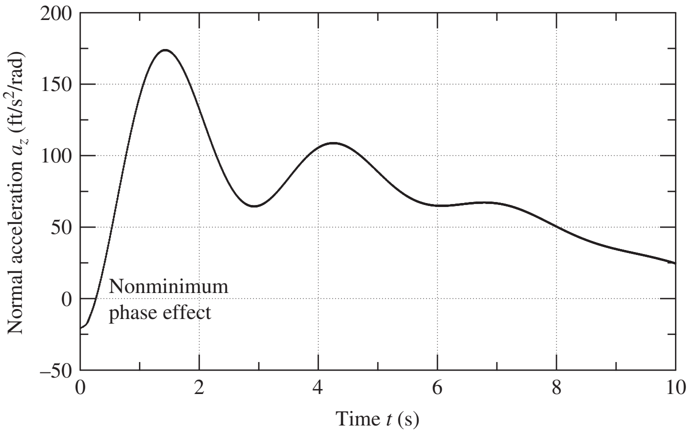

FIGURE 5.3 Normal acceleration response at the *cg* to an elevator unit step input.

Note that, since the numerator and denominator are of the same order, the acceleration transfer function from equation (5.47) is *improper*. The positive numerator root, or *zero*, implies that the transfer function is *nonminimum phase*, which is typical of aircraft acceleration transfer functions. The nonminimum phase effect is illustrated in the unit (1 rad) step response time history shown in Fig. 5.3 and causes the initial response to be in the wrong sense. Only the first few seconds of the response are shown and, as may be determined by application of the final value theorem, the steady-state acceleration is zero.

5.6 The state-space method

The use of the state-space method greatly facilitates solution of the small-perturbation equations of motion. Since the computational mechanism is based on the use of matrix algebra, it is most conveniently handled by a digital computer and, as already indicated, many suitable software packages are available. Most commercial software is intended for problems in modern control, so some care is needed to ensure that the aircraft equations of motion are correctly assembled before a solution is computed using these tools. However, the available tools are generally very powerful, and their use for the solution of the equations of motion of aircraft is particularly simple.

5.6.1 The transfer function matrix

The general state equations, (4.60) and (4.61), describing a linear dynamic system may be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{5.48}$$

The assembly of the equations of motion in this form, for the particular application to aircraft, was explained in Section 4.4.2. Since $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are matrices of constant coefficients, the Laplace transform of equations (5.48), assuming zero initial conditions, is

$$\begin{aligned} s\mathbf{x}(s) &= \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) \\ \mathbf{y}(s) &= \mathbf{C}\mathbf{x}(s) + \mathbf{D}\mathbf{u}(s) \end{aligned} \tag{5.49}$$

The state equation may be rearranged and written as

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s) \tag{5.50}$$

where $\mathbf{I}$ is the identity matrix and is the same order as $\mathbf{A}$. Thus, eliminating $\mathbf{x}(s)$, the state vector, by combining the output equation and equation (5.50), the output vector $\mathbf{y}(s)$ is given by

$$\mathbf{y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{u}(s) = \mathbf{G}(s)\mathbf{u}(s) \tag{5.51}$$

where $\mathbf{G}(s)$ is called the *transfer function matrix*. In general, the transfer function matrix has the form

$$\mathbf{G}(s) = \frac{1}{\Delta(s)}\mathbf{N}(s) \tag{5.52}$$

and $\mathbf{N}(s)$ is a polynomial matrix whose elements are all of the response transfer function numerators. The denominator $\Delta(s)$ is the characteristic polynomial and is common to all transfer functions. Thus the application of the state-space method to the solution of the equations of motion enables all response transfer functions to be obtained in a single computation.

Now, as explained in Section 4.4.2, when dealing with the solution of the equations of motion it is usually required that $\mathbf{y}(s) = \mathbf{x}(s)$, that is, the output vector and state vector are chosen to be the same. In this case equation (5.51) may be simplified since $\mathbf{C} = \mathbf{I}$ and $\mathbf{D} = 0$; therefore,

$$\mathbf{G}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{\mathrm{Adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}}{|s\mathbf{I} - \mathbf{A}|} \tag{5.53}$$

and equation (5.53) is equivalent to the multivariable application of Cramer's rule as discussed in Section 5.3.

Comparing equation (5.53) with equation (5.52), it is evident that the polynomial numerator matrix is given by

$$\mathbf{N}(s) = \mathrm{Adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}$$

and the characteristic polynomial is given by

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}|$$

5.6.2 The longitudinal transfer function matrix

The concise longitudinal state equations were given by equations (4.67) and (4.68). Substituting for **A**, **B**, and **I** in equation (5.53), the longitudinal transfer function matrix is thus given by

$$\mathbf{G}(s)=\begin{bmatrix} s-x_u & -x_w & -x_q & -x_\theta \\ -z_u & s-z_w & -z_q & -z_\theta \\ -m_u & -m_w & s-m_q & -m_\theta \\ 0 & 0 & -1 & s \end{bmatrix}^{-1}\begin{bmatrix} x_\eta & x_\tau \\ z_\eta & z_\tau \\ m_\eta & m_\tau \\ 0 & 0 \end{bmatrix} \tag{5.54}$$

Algebraic manipulation of equation (5.54) leads to

$$\mathbf{G}(s)=\frac{1}{\Delta(s)}\begin{bmatrix} N_\eta^u(s) & N_\tau^u(s) \\ N_\eta^w(s) & N_\tau^w(s) \\ N_\eta^q(s) & N_\tau^q(s) \\ N_\eta^\theta(s) & N_\tau^\theta(s) \end{bmatrix} \tag{5.55}$$

In this case the numerator and denominator polynomials are expressed in terms of the concise derivatives. A complete listing of the longitudinal algebraic transfer functions in this form is given in Appendix 3.

5.6.3 The lateral-directional transfer function matrix

The lateral-directional state equation is given in terms of normalised derivatives by equation (4.69). Thus substituting for **A**, **B**, and **I** in equation (5.53), the lateral transfer function matrix is given by

$$\mathbf{G}(s)=\begin{bmatrix} s-y_v & -y_p & -y_r & -y_\phi & -y_\psi \\ -l_v & s-l_p & -l_r & -l_\phi & -l_\psi \\ -n_v & -n_p & s-n_r & -n_\phi & -n_\psi \\ 0 & -1 & 0 & s & 0 \\ 0 & 0 & -1 & 0 & s \end{bmatrix}^{-1}\begin{bmatrix} y_\xi & y_\zeta \\ l_\xi & l_\zeta \\ n_\xi & n_\zeta \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.56}$$

As for the longitudinal solution, the lateral-directional transfer function matrix may be written as

$$\mathbf{G}(s)=\frac{1}{\Delta(s)}\begin{bmatrix} N_\xi^v(s) & N_\zeta^v(s) \\ N_\xi^p(s) & N_\zeta^p(s) \\ N_\xi^r(s) & N_\zeta^r(s) \\ N_\xi^\phi(s) & N_\zeta^\phi(s) \\ N_\xi^\psi(s) & N_\zeta^\psi(s) \end{bmatrix} \tag{5.57}$$

Again, the numerator and denominator polynomials are expressed in terms of the concise derivatives. A complete listing of the lateral algebraic transfer functions in this form is given in Appendix 3.

EXAMPLE 5.6

This example illustrates the use of the state space method for obtaining the lateral-directional transfer function matrix. Data for the Lockheed C-5A were obtained from Heffley and Jewell (1972). The data relate to a flight condition at an altitude of 20,000 ft and a Mach number of 0.6, and are referred to aircraft body axes. Although the data are given in American imperial units, they are converted to SI units simply for illustration. The normalised derivatives were derived from the data, great care being exercised to ensure the correct units. The derivatives are listed here, and, as in previous examples, missing derivatives are assumed to be insignificant and made equal to zero.

$$
\begin{array}{lll}
y_v = -0.1060\ 1/\mathrm{s} & l_v = -0.0070\ 1/\mathrm{m/s} & n_v = 0.0023\ 1/\mathrm{m/s} \\
y_p = 0 & l_p = -0.9880\ 1/\mathrm{s} & n_p = -0.0921\ 1/\mathrm{s} \\
y_r = -189.586\ \mathrm{m/s} & l_r = 0.2820\ 1/\mathrm{s} & n_r = -0.2030\ 1/\mathrm{s} \\
y_\phi = 9.8073\ \mathrm{m/s^2} & l_\phi = 0 & n_\phi = 0 \\
y_\psi = 0.3768\ \mathrm{m/s^2} & l_\psi = 0 & n_\psi = 0 \\
y_\xi = -0.0178\ \mathrm{m/s^2} & l_\xi = 0.4340\ 1/\mathrm{s^2} & n_\xi = 0.0343\ 1/\mathrm{s^2} \\
y_\zeta = 3.3936\ \mathrm{m/s^2} & l_\zeta = 0.1870\ 1/\mathrm{s^2} & n_\zeta = -0.5220\ 1/\mathrm{s^2}
\end{array}
$$

The lateral-directional state equation is obtained by substituting the derivative values into equation (4.69):

$$
\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -0.106 & 0 & -189.586 & 9.8073 & 0.3768 \\ -0.007 & -0.988 & 0.282 & 0 & 0 \\ 0.0023 & -0.0921 & -0.203 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} -0.0178 & 3.3936 \\ 0.434 & 0.187 \\ 0.0343 & -0.522 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{5.58}
$$

The output equation, written out in full, is

$$
\begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{5.59}
$$

The transfer function matrix was calculated using Program CC. The matrices **A**, **B**, **C**, and **D** are input to the program, and the command for finding the transfer function matrix is invoked. A printout of the result produces the following:

$$
\mathbf{G}(s) = \frac{1}{\Delta(s)}\mathbf{N}(s) \tag{5.60}
$$

where equation (5.60) is the shorthand version of equation (5.57) and

$$\mathbf{N}(s)=\begin{bmatrix} -0.018s(s+0.15)(s-0.98)(s+367.35) & 3.394s(s-0.012)(s+1.05)(s+2.31) \\ 0.434s(s-0.002)(s^2+0.33s+0.57) & 0.187s(s-0.002)(s+1.55)(s-2.16) \\ 0.343s(s+0.69)(s^2-0.77s+0.51) & -0.522s(s+1.08)(s^2+0.031s+0.056) \\ 0.434(s-0.002)(s^2+0.33s+0.57) & 0.187(s-0.002)(s+1.55)(s-2.16) \\ 0.343(s+0.69)(s^2-0.77s+0.51) & -0.522(s+1.08)(s^2+0.031s+0.056) \end{bmatrix} \tag{5.61}$$

The common denominator, the lateral characteristic polynomial, is given by

$$\Delta(s) = s(s+0.01)(s+1.11)(s^2+0.18s+0.58) \tag{5.62}$$

The lateral-directional characteristic polynomial factorises into three real roots and a complex pair of roots. Its roots, or poles, provide a complete description of the lateral-directional stability characteristics of the aeroplane. The zero root indicates *neutral stability* in yaw, the first non-zero real root describes the *spiral mode*, the second real root describes the *roll subsidence mode*, and the complex pair of roots describe the *oscillatory dutch roll mode*.

It is very important to remember the units of the transfer functions comprising the transfer function matrix, which are

$$\text{units of } \mathbf{G}(s) = \frac{1}{\Delta(s)}\begin{bmatrix} N_\xi^v(s) & N_\zeta^v(s) \\ N_\xi^p(s) & N_\zeta^p(s) \\ N_\xi^r(s) & N_\zeta^r(s) \\ N_\xi^\phi(s) & N_\zeta^\phi(s) \\ N_\xi^\psi(s) & N_\zeta^\psi(s) \end{bmatrix} = \begin{bmatrix} \text{m/s/rad} & \text{m/s/rad} \\ \text{rad/s/rad} & \text{rad/s/rad} \\ \text{rad/s/rad} & \text{rad/s/rad} \\ \text{rad/rad} & \text{rad/rad} \\ \text{rad/rad} & \text{rad/rad} \end{bmatrix} \tag{5.63}$$

Thus the transfer functions of interest can be obtained from inspection of equation (5.61) together with equation (5.62). For example, the transfer function describing sideslip velocity response to rudder is given by

$$\frac{v(s)}{\zeta(s)} = \frac{N_\zeta^v(s)}{\Delta(s)} = \frac{3.394(s-0.012)(s+1.05)(s+29.31)}{(s+0.01)(s+1.11)(s^2+0.18s+0.58)}\ \text{m/s/rad} \tag{5.64}$$

Comparison of these results with those of the original source material in Heffley and Jewell (1972) reveals a number of small numerical discrepancies. These are due in part to the numerical rounding employed to keep this illustration to a reasonable size and in part to the differences in the computational algorithms used to obtain the solutions. However, in both cases the accuracy is adequate for most practical purposes.

It is worth noting that many matrix inversion algorithms introduce numerical errors which accumulate rapidly with increasing matrix order, and it is possible to obtain seriously inaccurate results with some poorly conditioned matrices. The typical aircraft state matrix has a tendency to fall into this category, so it is advisable to check the result of a transfer function matrix computation for reasonableness when accuracy is in doubt. This may be done, for example, by making a test calculation using the expressions given in Appendix 3. For this reason, Program CC includes two algorithms for calculating the transfer function matrix. In Example 5.6 it was found that the *Generalised Eigenvalue Problem* algorithm gave obviously incorrect values for some transfer function numerators, whereas the *Fadeeva* algorithm gave the entirely correct solution. Thus when using computer tools for handling aircraft stability and control problems, it is advisable to input the aircraft derivative and other data at the accuracy given.

5.6.4 Response in terms of state description

The main reasons for the use of state-space modelling tools are the extreme power and convenience of machine solution of the equations of motion and the fact that the solution is obtained in a form which readily lends itself to further analysis in the context of flight control. The solution process is usually completely hidden from the investigator. However, it is important to be aware of the mathematical procedures implemented in the software algorithms for the reasons mentioned previously. A description of the methods of solution of the state equations describing a general system may be found in many books on modern control or system theory. For example, descriptions may be found in Barnett (1975), Shinners (1980), and Owens (1981). The following description is a summary of the solution of the aircraft state equations and includes only those aspects of the process which are most relevant to the aircraft application. For a more comprehensive review, the reader should consult the references.

The Laplace transform of the state equations (5.49) may be restated for the general case in which non-zero initial conditions are assumed:

$$\begin{aligned} s\mathbf{x}(s) - \mathbf{x}(0) &= \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) \\ \mathbf{y}(s) &= \mathbf{C}\mathbf{x}(s) + \mathbf{D}\mathbf{u}(s) \end{aligned} \tag{5.65}$$

Thus the state equation may be written as

$$\mathbf{x}(s) = [s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}\mathbf{u}(s) \tag{5.66}$$

or

$$\mathbf{x}(s) = \mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{\Phi}(s)\mathbf{B}\mathbf{u}(s) \tag{5.67}$$

where $\mathbf{\Phi}(s)$ is called the *resolvent* of $\mathbf{A}$. The most general expression for the state vector $\mathbf{x}(t)$ is determined by finding the inverse Laplace transform of equation (5.67), which is written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{5.68}$$

The *state transition matrix* $\mathbf{\Phi}(t - t_0)$ is defined as

$$\mathbf{\Phi}(t - t_0) = \mathrm{L}^{-1}\{[s\mathbf{I}-\mathbf{A}]^{-1}\} = \mathrm{e}^{\mathbf{A}(t-t_0)} \tag{5.69}$$

It is equivalent to the *matrix exponential* and describes the transition in the state response $x(t)$ from time t_0 to time t. The state transition matrix has the following special properties:

$$\mathbf{\Phi}(0) = \mathrm{e}^{\mathbf{A}t}_{t=0} = \mathbf{I}$$

$$
\begin{aligned}
\boldsymbol{\Phi}(\infty) &= \mathrm{e}^{\mathbf{A}t}\Big|_{t=\infty} = \mathbf{0} \\
\boldsymbol{\Phi}(t+\tau) &= \boldsymbol{\Phi}(t)\boldsymbol{\Phi}(\tau) = \mathrm{e}^{\mathbf{A}t}\mathrm{e}^{\mathbf{A}\tau} \\
\boldsymbol{\Phi}(t_2-t_0) &= \boldsymbol{\Phi}(t_2-t_1)\boldsymbol{\Phi}(t_1-t_0) = \mathrm{e}^{\mathbf{A}(t_2-t_1)}\mathrm{e}^{\mathbf{A}(t_1-t_0)} \\
\boldsymbol{\Phi}^{-1}(t) &= \boldsymbol{\Phi}(-t) = \mathrm{e}^{-\mathbf{A}t}
\end{aligned} \tag{5.70}
$$

The integral term in equation (5.68) is a *convolution integral* whose properties are well known and discussed in most texts on linear systems theory. A very accessible explanation of the role of the convolution integral in determining system response may be found in Auslander et al. (1974).

For aircraft applications it is usual to measure time from $t_0 = 0$, so equation (5.68) may be written as

$$
\begin{aligned}
\mathbf{x}(t) &= \boldsymbol{\Phi}(t)\mathbf{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \\
&= \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau
\end{aligned} \tag{5.71}
$$

The output response vector $\mathbf{y}(t)$ is determined by substituting the state vector $\mathbf{x}(t)$, obtained from equation (5.71), into the output equation:

$$
\begin{aligned}
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \\
&= \mathbf{C}\mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t)
\end{aligned} \tag{5.72}
$$

Analytical solution of the state equation (5.71) is only possible when the form of the input vector $\mathbf{u}(t)$ is known; therefore, further limited progress can be made only for specified applications. Three solutions are of particular interest in aircraft applications: the *unforced* or *homogeneous* response, the *impulse* response, and the *step* response.

Eigenvalues and eigenvectors

The characteristic equation is given by equating the characteristic polynomial to zero:

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = 0 \tag{5.73}$$

The roots or *zeros* of this equation, denoted λ_i, are the *eigenvalues* of the state matrix $\mathbf{A}$. An eigenvalue λ_i and its corresponding non-zero *eigenvector* $\mathbf{v}_i$ are such that

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i \tag{5.74}$$

whence

$$[\lambda_i\mathbf{I} - \mathbf{A}]\mathbf{v}_i = 0 \tag{5.75}$$

Since $\mathbf{v}_i \neq 0$, $[\lambda_i\mathbf{I} - \mathbf{A}]$ is singular. The eigenvectors $\mathbf{v}_i$ are always linearly independent provided the eigenvalues λ_i are *distinct*—that is, the characteristic equation (5.73) has no repeated roots. When an eigenvalue is complex its corresponding eigenvector is also complex, and the complex conjugate λ_i^* corresponds with the complex conjugate $\mathbf{v}_i^*$.

The *eigenvector* or *modal matrix* comprises all of the eigenvectors and is defined as

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \end{bmatrix} \tag{5.76}$$

It follows directly from equation (5.74) that

$$\mathbf{AV} = \mathbf{V}\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & \lambda_m \end{bmatrix} \equiv \mathbf{V\Lambda} \tag{5.77}$$

where $\mathbf{\Lambda}$ is the diagonal *eigenvalue matrix*. Thus

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{\Lambda} \tag{5.78}$$

and $\mathbf{A}$ is said to be *similar* to the diagonal eigenvalue matrix $\mathbf{\Lambda}$. The mathematical operation on the state matrix $\mathbf{A}$ described by equation (5.78) is referred to as a *similarity transform*. Similar matrices possess the special property that their eigenvalues are the same. When the state equations are transformed into a similar form such that the state matrix $\mathbf{A}$ is replaced by the diagonal eigenvalue matrix $\mathbf{\Lambda}$, their solution is greatly facilitated. Presented in this way, the state equations are said to be in *modal form*.

Eigenvectors may be determined as follows. By definition,

$$[\lambda_i\mathbf{I}-\mathbf{A}]^{-1} = \frac{\mathrm{Adj}[\lambda_i\mathbf{I}-\mathbf{A}]}{|\lambda_i\mathbf{I}-\mathbf{A}|} \tag{5.79}$$

and since, for any eigenvalue λ_i, $|\lambda_i\mathbf{I}-\mathbf{A}| = \mathbf{0}$, equation (5.79) may be rearranged and written as

$$[\lambda_i\mathbf{I}-\mathbf{A}]\mathrm{Adj}[\lambda_i\mathbf{I}-\mathbf{A}] = |\lambda_i\mathbf{I}-\mathbf{A}|\mathbf{I} = \mathbf{0} \tag{5.80}$$

Comparing equation (5.80) with equation (5.75), the eigenvector $\mathbf{v}_i$ corresponding to the eigenvalue λ_i is defined as

$$\mathbf{v}_i = \mathrm{Adj}[\lambda_i\mathbf{I}-\mathbf{A}] \tag{5.81}$$

Any nonzero column of the adjoint matrix is an eigenvector, and if there is more than one column they differ only by a constant factor. Eigenvectors are therefore unique in direction only, not in magnitude. However, the dynamic characteristics of a system determine the unique relationship between each of its eigenvectors.

The modal equations

Define the transform

$$\mathbf{x}(t) = \mathbf{Vz}(t) \equiv \mathbf{v}_1 z_1(t) + \mathbf{v}_2 z_2(t) + \cdots\cdots + \mathbf{v}_m z_m(t) = \sum_{i=1}^{i=m} \mathbf{v}_i z_i(t) \tag{5.82}$$

Then the state equations (5.48) may be rewritten in modal form as

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \boldsymbol{\Lambda}\mathbf{z}(t) + \mathbf{V}^{-1}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{V}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{5.83}$$

Unforced response

With reference to equation (5.71), the solution to the state equation in modal form, equation (5.83), is given by

$$\mathbf{z}(t) = \mathrm{e}^{\boldsymbol{\Lambda}t}\mathbf{z}(0) + \int_0^t \mathrm{e}^{\boldsymbol{\Lambda}(t-\tau)}\mathbf{V}^{-1}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{5.84}$$

The *matrix exponential* $\mathrm{e}^{\boldsymbol{\Lambda}t}$ in diagonal form is defined as

$$\mathrm{e}^{\boldsymbol{\Lambda}t} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & & & \\ & \mathrm{e}^{\lambda_2 t} & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \ddots & \\ & & & & \mathrm{e}^{\lambda_m t} \end{bmatrix} \tag{5.85}$$

and since it is diagonal, the solution of the transformed state variables $z_i(t)$ given by equation (5.84) are uncoupled, which is the principal advantage of the transform. Thus

$$z_i(t) = \mathrm{e}^{\lambda_i t}z_i(0) + \int_0^t \mathrm{e}^{\lambda_i(t-\tau)}\mathbf{V}^{-1}\mathbf{B}u_i(\tau)\mathrm{d}\tau \tag{5.86}$$

The unforced response is given by equation (5.84) when $\mathbf{u}(t) = 0$; whence

$$\mathbf{z}(t) = \mathrm{e}^{\boldsymbol{\Lambda}t}\mathbf{z}(0) \tag{5.87}$$

Or, substituting equation (5.87) into equation (5.82), the unforced state trajectory $\mathbf{x}(t)$ may be derived:

$$\mathbf{x}(t) = \mathbf{V}\mathrm{e}^{\boldsymbol{\Lambda}t}\mathbf{z}(0) = \sum_{i=1}^{i=m} \mathbf{v}_i\mathrm{e}^{\lambda_i t}z_i(0) = \sum_{i=1}^{i=m} \mathbf{v}_i\mathrm{e}^{\lambda_i t}\mathbf{V}^{-1}x_i(0) \tag{5.88}$$

or

$$\mathbf{x}(t) = \mathbf{V}\mathrm{e}^{\boldsymbol{\Lambda}t}\mathbf{V}^{-1}\mathbf{x}(0) \equiv \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) \tag{5.89}$$

From equation (5.72) the output response follows:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = \mathbf{C}\mathbf{V}\mathrm{e}^{\boldsymbol{\Lambda}t}\mathbf{V}^{-1}\mathbf{x}(0) \equiv \mathbf{C}\mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) \tag{5.90}$$

Clearly the system behaviour is governed by the *system modes* $\mathrm{e}^{\lambda_i t}$, the *eigenfunctions* $\mathbf{v}_i\mathrm{e}^{\lambda_i t}$, and the *initial state* $\mathbf{z}(0) = \mathbf{V}^{-1}\mathbf{x}(0)$.

Impulse response

The *unit impulse function* or *Dirac delta function*, denoted $\delta(t)$, is usually taken to mean a rectangular pulse of unit area, and in the limit the width of the pulse tends to zero whilst its magnitude tends to infinity. Thus the special property of the unit impulse function is

$$\int_{-\infty}^{+\infty} \delta(t - t_0)\mathrm{d}t = 1 \tag{5.91}$$

where t_0 is the time at which the impulse commences.

The solution of the modal state equation in response to a unit impulse follows from equation (5.84):

$$\mathbf{z}(t) = \mathrm{e}^{\mathbf{\Lambda} t}\mathbf{z}(0) + \int_0^t \mathrm{e}^{\mathbf{\Lambda}(t-\tau)}\mathbf{V}^{-1}\mathbf{B}\mathbf{u}_\delta(\tau)\mathrm{d}\tau \tag{5.92}$$

where $\mathbf{u}_\delta(\tau)$ is a unit impulse vector. The property of the unit impulse function enables the convolution integral to be solved, and

$$\mathbf{z}(t) = \mathrm{e}^{\mathbf{\Lambda} t}\mathbf{z}(0) + \mathrm{e}^{\mathbf{\Lambda} t}\mathbf{V}^{-1}\mathbf{B} = \mathrm{e}^{\mathbf{\Lambda} t}[\mathbf{z}(0) + \mathbf{V}^{-1}\mathbf{B}] \tag{5.93}$$

Thus the transform, equation (5.82), enables the state vector to be determined:

$$\mathbf{x}(t) = \mathbf{V}\mathrm{e}^{\mathbf{\Lambda} t}\mathbf{V}^{-1}[\mathbf{x}(0) + \mathbf{B}] \equiv \mathrm{e}^{\mathbf{A} t}[\mathbf{x}(0) + \mathbf{B}] \tag{5.94}$$

The corresponding output response vector is given by

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\mathbf{V}\mathrm{e}^{\mathbf{\Lambda} t}\mathbf{V}^{-1}[\mathbf{x}(0) + \mathbf{B}] + \mathbf{D}u_\delta(t) \\ &\equiv \mathbf{C}\mathrm{e}^{\mathbf{A} t}[\mathbf{x}(0) + \mathbf{B}] + \mathbf{D}u_\delta(t) \end{aligned} \tag{5.95}$$

For application to aeroplanes, it was established in Chapter 4 (Section 4.4.2) that the direct matrix $\mathbf{D}$ is zero. Comparing equations (5.95) and (5.90), it is seen that the impulse response is the same as the unforced response with initial condition $[\mathbf{x}(0) + \mathbf{B}]$.

Step response

When the vector input to the system is a step of constant magnitude, denoted $\mathbf{u}_k$, applied at time $t_0 = 0$, the state equation (5.84) may be written as

$$\mathbf{z}(t) = \mathrm{e}^{\mathbf{\Lambda} t}\mathbf{z}(0) + \int_0^t \mathrm{e}^{\mathbf{\Lambda}(t-\tau)}\mathbf{V}^{-1}\mathbf{B}\mathbf{u}_k\mathrm{d}\tau \tag{5.96}$$

Since the input is constant, the convolution integral is easily evaluated and

$$\mathbf{z}(t) = \mathrm{e}^{\mathbf{\Lambda} t}\mathbf{z}(0) + \mathbf{\Lambda}^{-1}[\mathrm{e}^{\mathbf{\Lambda} t} - \mathbf{I}]\mathbf{V}^{-1}\mathbf{B}\mathbf{u}_k \tag{5.97}$$

Thus the transform, equation (5.82), enables the state vector to be determined:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{V}\mathrm{e}^{\mathbf{\Lambda} t}[\mathbf{V}^{-1}\mathbf{x}(0) + \mathbf{\Lambda}^{-1}\mathbf{V}^{-1}\mathbf{B}\mathbf{u}_k] - \mathbf{A}^{-1}\mathbf{B}\mathbf{u}_k \\ &\equiv \mathrm{e}^{\mathbf{A} t}[\mathbf{x}(0) + \mathbf{A}^{-1}\mathbf{B}\mathbf{u}_k] - \mathbf{A}^{-1}\mathbf{B}\mathbf{u}_k \end{aligned} \tag{5.98}$$

The derivation of equation (5.98) makes use of the following property of the matrix exponential:

$$\mathbf{\Lambda}^{-1}\mathrm{e}^{\mathbf{\Lambda} t} \equiv \mathrm{e}^{\mathbf{\Lambda} t}\mathbf{\Lambda}^{-1} \tag{5.99}$$

It also draws on the similarity transform:

$$\mathbf{A}^{-1} = \mathbf{V}\boldsymbol{\Lambda}^{-1}\mathbf{V}^{-1} \tag{5.100}$$

Again, the output response is obtained by substituting the state vector $\mathbf{x}(t)$, equation (5.98), into the output equation to give

$$\begin{aligned}\mathbf{y}(t) &= \mathbf{C}\mathbf{V}e^{\boldsymbol{\Lambda}t}[\mathbf{V}^{-1}\mathbf{x}(0) + \boldsymbol{\Lambda}^{-1}\mathbf{V}^{-1}\mathbf{B}\mathbf{u}_k] - [\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D}]\mathbf{u}_k \\ &\equiv \mathbf{C}e^{\mathbf{A}t}[\mathbf{x}(0) + \mathbf{A}^{-1}\mathbf{B}\mathbf{u}_k] - [\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{D}]\mathbf{u}_k\end{aligned} \tag{5.101}$$

Since the direct matrix $\mathbf{D}$ is zero for aeroplanes, comparing equations (5.101) and (5.95) shows that the step response is the same as the impulse response with initial condition $[\mathbf{x}(0) + \mathbf{A}^{-1}\mathbf{B}\mathbf{u}_k]$ superimposed on the constant output $-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\mathbf{u}_k$.

Response shapes

With reference to equations (5.90), (5.95), and (5.101), it is clear that, irrespective of the input, the transient output response shapes are governed by the system eigenfunctions $\mathbf{V}e^{\boldsymbol{\Lambda}t}$ or, alternatively, by the eigenvectors and eigenvalues. Most computer solutions of the state equations produce output response in the form of time history data together with the eigenvalues and eigenvectors. Thus, in aircraft response analysis, the system modes and eigenfunctions may be calculated if required. The value of this facility is that it provides a very effective means for gaining insight into the key physical properties governing the response. In particular, it enables the mode content in any response variable to be assessed merely by inspection of the corresponding eigenvectors.

The output response to other input functions may also be calculated algebraically provided the input function can be expressed in a suitable analytic form. Typical examples include the ramp function and various sinusoidal functions. Computer software packages intended for analysing system response always include a number of common input functions and usually provide a means for creating others. However, in aircraft response analysis input functions other than those discussed in detail previously are generally of less interest.

EXAMPLE 5.7

The longitudinal equations of motion for the Lockheed F-104 Starfighter given in Example 5.2 may be written in state form as described in Chapter 4 (Section 4.4.2). Thus

$$\begin{bmatrix} 746 & 0 & 0 & 0 \\ 0 & 746 & 0 & 0 \\ 0 & 36.4 & 65000 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -26.26 & 79.82 & 0 & -24021.2 \\ -159.64 & -328.24 & 227530 & 0 \\ 0 & -1014 & -18135 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -16502 \\ -303575 \\ 0 \end{bmatrix}\eta \tag{5.102}$$

Premultiplying this equation by the inverse of the mass matrix results in the usual form of the state equation in terms of the concise derivatives:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.0352 & 0.1070 & 0 & -32.2 \\ -0.2140 & -0.4400 & 305 & 0 \\ 1.198E-04 & -0.0154 & -0.4498 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -22.1206 \\ -4.6580 \\ 0 \end{bmatrix} \eta \tag{5.103}$$

or, in algebraic form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{5.104}$$

which defines the matrices **A** and **B** and the vectors $\mathbf{x}(t)$ and $\mathbf{u}(t)$. Using the computer software package MATLAB interactively, the diagonal eigenvalue matrix is calculated:

$$\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} -0.4459+2.1644j & 0 & 0 & 0 \\ 0 & -0.4459-2.1644j & 0 & 0 \\ 0 & 0 & -0.0166+0.1474j & 0 \\ 0 & 0 & 0 & -0.0166-0.1474j \end{bmatrix} \\ &\equiv \begin{bmatrix} \lambda_s & 0 & 0 & 0 \\ 0 & \lambda_s^* & 0 & 0 \\ 0 & 0 & \lambda_p & 0 \\ 0 & 0 & 0 & \lambda_p^* \end{bmatrix} \end{aligned} \tag{5.105}$$

and the corresponding eigenvector matrix is calculated:

$$\mathbf{V} = \begin{bmatrix} 0.0071-0.0067j & 0.0071+0.0067j & -0.9242-0.3816j & -0.9242+0.3816j \\ 0.9556-0.2944j & 0.9556+0.2944j & 0.0085+0.0102j & 0.0085-0.0102j \\ 0.0021+0.0068j & 0.0021-0.0068j & -0.0006-0.0002j & -0.0006+0.0002j \\ 0.0028-0.0015j & 0.0028+0.0015j & -0.0012+0.0045j & -0.0012-0.0045j \end{bmatrix} \tag{5.106}$$

λ_s, λ_p, and their complex conjugates, λ_s^*, λ_p^*, are the eigenvalues corresponding to the short-period pitching oscillation and the phugoid, respectively. The corresponding matrix exponential is given by

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{(-0.4459+2.1644j)t} & 0 & 0 & 0 \\ 0 & e^{(-0.4459-2.1644j)t} & 0 & 0 \\ 0 & 0 & e^{(-0.0166+0.1474j)t} & 0 \\ 0 & 0 & 0 & e^{(-0.0166-0.1474j)t} \end{bmatrix} \tag{5.107}$$

The eigenfunction matrix $\mathbf{V}e^{\mathbf{\Lambda}t}$ therefore has complex nonzero elements, and each row describes the dynamic content of the state variable to which it relates. For example, the *first row* describes the dynamic content of the velocity perturbation u and comprises the following four elements:

$$\begin{array}{r}(0.0071 - 0.0067j)\mathrm{e}^{(-0.4459+2.1644j)t} \\ (0.0071 + 0.0067j)\mathrm{e}^{(-0.4459-2.1644j)t} \\ (-0.9242 - 0.3816j)\mathrm{e}^{(-0.0166+0.1474j)t} \\ (-0.9242 + 0.3816j)\mathrm{e}^{(-0.0166-0.1474j)t}\end{array} \tag{5.108}$$

The first two elements in (5.108) describe the short-period pitching oscillation content in a velocity perturbation, and the second two elements describe the phugoid content. The relative *magnitudes* of the eigenvectors, the terms in parentheses, associated with the phugoid dynamics are the largest and clearly indicate that the phugoid dynamics are dominant in a velocity perturbation. The short-period pitching oscillation, on the other hand, is barely visible. Obviously, this kind of observation can be made for all of the state variables simply by inspection of the eigenvector and eigenvalue matrices only. This is a very useful facility for investigating the response properties of an aeroplane, especially when the behaviour is not conventional, when stability modes are obscured, or when a significant degree of mode coupling is present.

When it is recalled that

$$\mathrm{e}^{j\pi t} = \cos \pi t + j \sin \pi t \tag{5.109}$$

where π represents an arbitrary scalar variable, the velocity eigenfunctions, (5.108), may be written alternatively as

$$\begin{array}{r}(0.0071 - 0.0067j)\mathrm{e}^{-0.4459t}(\cos 2.1644t + j \sin 2.1644t) \\ (0.0071 + 0.0067j)\mathrm{e}^{-0.4459t}(\cos 2.1644t - j \sin 2.1644t) \\ (-0.9242 - 0.3816j)\mathrm{e}^{-0.0166t}(\cos 0.1474t + j \sin 0.1474t) \\ (-0.9242 + 0.3816j)\mathrm{e}^{-0.0166t}(\cos 0.1474t - j \sin 0.1474t)\end{array} \tag{5.110}$$

Since the elements in (5.110) include sine and cosine functions of time, the origins of the oscillatory response characteristics in the overall solution of the equations of motion are identified.

As described in Examples 5.2 and 5.3, the damping ratio and undamped natural frequency characterise the stability modes. This information comprises the eigenvalues, included in the matrix equation (5.105), and is interpreted as follows:

For the short-period pitching oscillation, the higher frequency mode

$$\begin{aligned}\text{Undamped natural frequency } \omega_s &= 2.1644 \text{ rad/s} \\ \zeta_s \omega_s &= 0.4459 \text{ rad/s} \\ \text{Damping ratio } \zeta_s &= 0.206\end{aligned}$$

For the phugoid oscillation, the lower frequency mode

$$\text{Undamped natural frequency } \omega_p = 0.1474 \text{ rad/s}$$
$$\zeta_p \omega_p = 0.0166 \text{ rad/s}$$
$$\text{Damping ratio } \zeta_p = 0.1126$$

It is instructive to calculate the pitch attitude response to a unit elevator step input using the state-space method for comparison with the method described in Example 5.3. The step response is given by equation (5.101), which, for zero initial conditions, a zero direct matrix **D**, and output matrix **C** replaced with the identity matrix **I**, reduces to

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{IV}e^{\Lambda t}\boldsymbol{\Lambda}^{-1}\mathbf{V}^{-1}\mathbf{Bu}_k - \mathbf{IA}^{-1}\mathbf{Bu}_k \\ &= \mathbf{V}e^{\Lambda t}\Lambda^{-1}\mathbf{V}^{-1}\mathbf{b} - \mathbf{A}^{-1}\mathbf{b} \end{aligned} \tag{5.111}$$

Since the single elevator input is a unit step, $\mathbf{u}_k = 1$ and the input matrix **B** becomes the column matrix **b**. The expression on the right-hand side of equation (5.111) is a (4×1) column matrix, the elements of which describe u, w, q, and θ responses to the input. With the aid of Matlab the following were calculated:

$$\boldsymbol{\Lambda}^{-1}\mathbf{V}^{-1}\mathbf{b} = \begin{bmatrix} 147.36 + 19.07j \\ 147.36 - 19.07j \\ 223.33 - 133.29j \\ 223.33 + 133.29j \end{bmatrix} \quad \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -512.2005 \\ 299.3836 \\ 0 \\ 1.5548 \end{bmatrix} \tag{5.112}$$

The remainder of the calculation of the first term on the right-hand side of equation (5.111) was completed by hand, an exercise which is definitely not recommended! Pitch attitude response is given by the fourth row of the resulting column matrix $\mathbf{y}(t)$ and is

$$\begin{aligned} \theta(t) = {} & 0.664e^{-0.017t}(\cos 0.147t - 3.510 \sin 0.147t) \\ & + 0.882e^{-0.446t}(\cos 2.164t + 0.380 \sin 2.164t) - 1.5548 \end{aligned} \tag{5.113}$$

This equation compares very favourably with equation (5.32) and may be interpreted in exactly the same way.

This example is intended to illustrate the role of the various elements contributing to the solution and, as such, are not normally undertaken on a routine basis. Machine computation simply produces the result in the most accessible form, which is usually graphical, although the investigator can obtain additional information in much the same way as shown in this example.

5.7 State-space model augmentation

It is frequently necessary to obtain response characteristics for variables which are not included in the aeroplane equations of motion. Provided that the variables of interest can be expressed as functions of the basic aeroplane motion variables, then response transfer functions can be derived in the same way as the acceleration response transfer functions described in Section 5.5. However, when the additional transfer functions of interest are strictly proper, they can also be obtained by extending, or *augmenting*, the state description of the aeroplane and solving in the usual way as described. This latter course of action is extremely convenient as it extends the usefulness of the aeroplane state-space model and requires little additional effort on behalf of the investigator.

For some additional variables, such as height, it is necessary to create a new state variable and to augment the state equation accordingly. For others, such as flight path angle, which may be expressed as the simple sum of basic aeroplane state variables, it is only necessary to create an additional output variable and to augment the output equation accordingly. It is also a straightforward matter to augment the state description to include the additional dynamics of components such as engines and control surface actuators. In this case all of the response transfer functions obtained in the solution of the equations of motion implicitly include the effects of the additional dynamics.

5.7.1 Height response transfer function

An expression for height rate was given by equation (2.17), which for small perturbations may be written as

$$\dot{h} = U\theta - V\phi - W \tag{5.114}$$

Substitute for (U, V, W) from equation (2.1) and note that, for symmetric flight, $V_e = 0$. Since the products of small quantities are insignificantly small, they may be ignored and equation (5.114) may be written as

$$\dot{h} = U_e\theta - W_e - w \tag{5.115}$$

With reference to Fig. 2.4, assuming α_e to be small, $U_e \cong V_0$, $W_e \cong 0$, and equation (5.114) may be written, to a good approximation, as

$$\dot{h} = V_0\theta - w \tag{5.116}$$

The decoupled longitudinal state equation in concise form, equation (4.67), may be augmented to include the height variable by the inclusion of equation (5.116):

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} x_u & x_w & x_q & x_\theta & 0 \\ z_u & z_w & z_q & z_\theta & 0 \\ m_u & m_w & m_q & m_\theta & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & V_0 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} x_\eta & x_\tau \\ z_\eta & z_\tau \\ m_\eta & m_\tau \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \tau \end{bmatrix} \tag{5.117}$$

Alternatively, this may be written in a more compact form:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{h}(t) \end{bmatrix} = \begin{bmatrix} & & \mathbf{A} & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & V_0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \cdots \\ h(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \cdots \\ 0 \quad 0 \end{bmatrix} \mathbf{u}(t) \tag{5.118}$$

where $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are the state and input vectors, respectively, and $\mathbf{A}$ and $\mathbf{B}$ are the state and input matrices, respectively, of the basic aircraft state equation (4.67). Solution of equation (5.118) to obtain the longitudinal response transfer functions now results in two additional transfer functions describing the height response to an elevator perturbation and the height response to a thrust perturbation.

5.7.2 Incidence and sideslip response transfer functions

Dealing first with the inclusion of incidence angle in the longitudinal decoupled equations of motion. It follows from equation (2.5) that for small-perturbation motion incidence, α is given by

$$\alpha \cong \tan\alpha = \frac{w}{V_0} \tag{5.119}$$

since $U_e \to V_0$ as the perturbation tends to zero. Thus incidence α is equivalent to normal velocity w divided by the steady free stream velocity. Incidence can be included in the longitudinal state equations in two ways. Either it can be added to the output vector $\mathbf{y}(t)$ without changing the state vector or it can replace normal velocity w in the state vector. When the output equation is augmented, the longitudinal state equations (4.67) and (4.68) are written as

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\
\mathbf{y}(t) &= \begin{bmatrix} u \\ w \\ q \\ \theta \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/V_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} & \mathbf{I} & & \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1/V_0 & 0 & 0 \end{bmatrix} \mathbf{x}(t)
\end{aligned} \tag{5.120}$$

When incidence replaces normal velocity, it is first necessary to note that equation (5.119) may be differentiated to give $\dot{\alpha} = \dot{w}/V_0$. Thus the longitudinal state equation (4.67) may be rewritten as

$$\begin{bmatrix} \dot{u} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_u & x_w V_0 & x_q & x_\theta \\ z_u/V_0 & z_w & z_q/V_0 & z_\theta/V_0 \\ m_u & m_w V_0 & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} x_\eta & x_\tau \\ z_\eta/V_0 & z_\tau/V_0 \\ m_\eta & m_\tau \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \tau \end{bmatrix} \tag{5.121}$$

The output equation (4.68) remains unchanged except that the output vector $\mathbf{y}(t)$ now includes α instead of w; thus

$$\mathbf{y}^{\mathrm{T}}(t) = \begin{bmatrix} u & \alpha & q & \theta \end{bmatrix} \tag{5.122}$$

In a similar way it is easily shown that in a lateral perturbation the sideslip angle β is given by

$$\beta \cong \tan\beta = \frac{v}{V_0} \tag{5.123}$$

and the lateral small-perturbation equations can be modified in the same way as the longitudinal equations in order to incorporate sideslip angle β in the output equation; alternatively, it may replace lateral velocity v in the state equation. When the output equation is augmented, the lateral-directional state equations may be written as

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\
\mathbf{y}(t) &= \begin{bmatrix} v \\ p \\ r \\ \phi \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/V_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} = \begin{bmatrix} & \mathbf{I} & & \\ \cdots & \cdots & \cdots & \cdots \\ 1/V_0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t)
\end{aligned} \tag{5.124}$$

where the lateral-directional state equation is given by equation (4.70). When sideslip angle β replaces lateral velocity v in the lateral-directional state equation (4.70), it is written as

$$\begin{bmatrix}\dot{\beta}\\ \dot{p}\\ \dot{r}\\ \dot{\phi}\end{bmatrix}=\begin{bmatrix}y_v & y_p/V_0 & y_r/V_0 & y_\phi/V_0\\ l_vV_0 & l_p & l_r & l_\phi\\ n_vV_0 & n_p & n_r & n_\phi\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\beta\\ p\\ r\\ \phi\end{bmatrix}+\begin{bmatrix}y_\xi/V_0 & y_\zeta/V_0\\ l_\xi & l_\zeta\\ n_\xi & n_\zeta\\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi\\ \zeta\end{bmatrix} \quad (5.125)$$

Again, for this alternative the lateral-directional output vector $\mathbf{y}(t)$ remains unchanged except that sideslip angle β replaces lateral velocity v; thus

$$\mathbf{y}^{\mathrm{T}}(t)=\begin{bmatrix}\beta & p & r & \phi\end{bmatrix} \quad (5.126)$$

Solution of the longitudinal or lateral-directional state equations will produce the transfer function matrix in the usual way. In every case, transfer functions are calculated to correspond with the particular set of variables that make up the output vector.

5.7.3 Flight path angle response transfer function

Sometimes flight path angle γ response to controls is required, especially when handling qualities in the approach flight condition are under consideration. Perturbations in flight path angle γ may be expressed in terms of perturbations in pitch attitude θ and incidence α, as indicated for the steady-state case by equation (2.2). Whence

$$\gamma=\theta-\alpha\cong\theta-\frac{w}{V_0} \quad (5.127)$$

Thus the longitudinal output equation (4.68) may be augmented to include flight path angle as an additional output variable. The form of the longitudinal state equations is then similar to that of equations (5.120), and

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \begin{bmatrix}u\\ w\\ q\\ \theta\\ \gamma\end{bmatrix}=\begin{bmatrix} & & \mathbf{I} & \\ \cdots & \cdots & \cdots & \cdots\\ 0 & -1/V_0 & 0 & 1\end{bmatrix}\mathbf{x}(t)\end{aligned} \quad (5.128)$$

where the state vector $\mathbf{x}(t)$ remains unchanged:

$$\mathbf{x}^{\mathrm{T}}(t)=\begin{bmatrix}u & w & q & \theta\end{bmatrix} \quad (5.129)$$

5.7.4 Addition of engine dynamics

Provided that the thrust-producing devices can be modelled by a linear transfer function then, in general, it can be integrated into the aircraft state description. This then enables the combined engine and airframe dynamics to be modelled by the overall system response transfer functions. A very simple engine thrust model was described by equation (2.34), with transfer function

$$\frac{\tau(s)}{\varepsilon(s)} = \frac{k_\tau}{(1 + sT_\tau)} \tag{5.130}$$

where $\tau(t)$ is the thrust perturbation in response to a perturbation in throttle lever angle $\varepsilon(t)$. The transfer function equation (5.130) may be rearranged thus:

$$s\tau(s) = \frac{k_\tau}{T_\tau}\varepsilon(s) - \frac{1}{T_\tau}\tau(s) \tag{5.131}$$

This is the Laplace transform, assuming zero initial conditions, of the following time domain equation:

$$\dot{\tau}(t) = \frac{k_\tau}{T_\tau}\varepsilon(t) - \frac{1}{T_\tau}\tau(t) \tag{5.132}$$

The longitudinal state equation (4.67) may be augmented to include the engine dynamics described by equation (5.132), which, after some rearrangement, may be written as

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} x_u & x_w & x_q & x_\theta & x_\tau \\ z_u & z_w & z_q & z_\theta & z_\tau \\ m_u & m_w & m_q & m_\theta & m_\tau \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/T_\tau \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ \tau \end{bmatrix} + \begin{bmatrix} x_\eta & 0 \\ z_\eta & 0 \\ m_\eta & 0 \\ 0 & 0 \\ 0 & k_\tau/T_\tau \end{bmatrix} \begin{bmatrix} \eta \\ \varepsilon \end{bmatrix} \tag{5.133}$$

Thus the longitudinal state equation has been augmented to include thrust as an additional state, and the second input variable is now throttle lever angle ε. The output equation (4.68) remains unchanged except that the **C** matrix is increased in order to the (5×5) identity matrix **I** to provide the additional output variable corresponding to the extra state variable τ.

The procedure described above, in which a transfer function model of engine dynamics is converted to a form suitable for augmenting the state equation, is known as *system realisation*. More generally, relatively complex higher-order transfer functions can be *realised* as state equations, although the procedure for so doing is rather more involved than that illustrated here for a particularly simple example. The mathematical methods required are described in most books on modern control theory. The advantage and power of this relatively straightforward procedure is considerable since it literally enables the state equation describing a very complex system, such as an aircraft with advanced flight controls, to be built by repeated augmentation. The state descriptions of the various system components are simply added to the matrix state equation until the overall system dynamics are fully represented. Typically, this might mean, for example, that the basic longitudinal or lateral-directional (4×4) airframe state matrix is augmented to a much higher order of perhaps (12×12) or more, depending on the complexity of the engine model, control system, surface actuators, and so on. Whatever the result, the equations are easily solved using the tools described.

EXAMPLE 5.8

To illustrate the procedure for augmenting an aeroplane state model, let the longitudinal model for the Lockheed F-104 Starfighter of Example 5.2 be augmented to include height h and flight path angle γ and to replace normal velocity w with incidence α. The longitudinal state equation expressed in terms of concise derivatives is given by equation (5.103), which is modified in accordance with equation (5.121) to replace normal velocity w with incidence α:

$$\begin{bmatrix} \dot{u} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.0352 & 32.6342 & 0 & -32.2 \\ -7.016E-04 & -0.4400 & 1 & 0 \\ 1.198E-04 & -4.6829 & -0.4498 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -0.0725 \\ -4.6580 \\ 0 \end{bmatrix} \eta \tag{5.134}$$

Equation (5.134) is now augmented by the addition of equation (5.116), the height equation expressed in terms of incidence α and pitch attitude θ:

$$\dot{h} = V_0(\theta - \alpha) = 305\theta - 305\alpha \tag{5.135}$$

Thus the augmented state equation is written as

$$\begin{bmatrix} \dot{u} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -0.0352 & 32.6342 & 0 & -32.2 & 0 \\ -7.016E-04 & -0.4400 & 1 & 0 & 0 \\ 1.198E-04 & -4.6829 & -0.4498 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -305 & 0 & 305 & 0 \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} 0 \\ -0.0725 \\ -4.6580 \\ 0 \\ 0 \end{bmatrix} \eta \tag{5.136}$$

The corresponding output equation is augmented to included flight path angle γ as given by equation (5.127) and is then written as

$$\begin{bmatrix} u \\ \alpha \\ q \\ \theta \\ h \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \\ \theta \\ h \end{bmatrix} \tag{5.137}$$

This, of course, assumes the direct matrix **D** to be zero, as discussed earlier. Equations (5.136) and (5.137) together provide the complete state description of the Lockheed F-104 as required. Solving these equations with the aid of Program CC results in the six transfer functions describing the response to elevator.

The common denominator polynomial (the characteristic polynomial) is given by

$$\Delta(s) = s(s^2 + 0.033s + 0.022)(s^2 + 0.892s + 4.883) \tag{5.138}$$

The numerator polynomials are given by

$$\begin{aligned}
N_\eta^u(s) &= -2.367s(s-4.215)(s+5.519)\ \text{ft/s/rad}\\
N_\eta^\alpha(s) &= -0.073s(s+64.675)(s^2+0.035s+0.023)\ \text{rad/rad}\\
N_\eta^q(s) &= -4.658s^2(s+0.134)(s+0.269)\ \text{rad/s/rad}\\
N_\eta^\theta(s) &= -4.658s(s+0.134)(s+0.269)\ \text{rad/rad}\\
N_\eta^h(s) &= 22.121(s+0.036)(s-4.636)(s+5.085)\ \text{ft/rad}\\
N_\eta^\gamma(s) &= 0.073s(s+0.036)(s-4.636)(s+5.085)\ \text{rad/rad}
\end{aligned} \tag{5.139}$$

Note that the additional zero pole in the denominator is due to the increase in order of the state equation from four to five; it represents the *height integration*. This is easily interpreted since an elevator step input will cause the aeroplane to climb or descend steadily after the transient has died away, at which point the response becomes similar to that of a simple integrator. Note also that the denominator zero cancels with a zero in all numerator polynomials except that describing the height response. Thus the response transfer functions describing the basic aircraft motion variables u, α, q, and θ are identical to those obtained from the basic fourth-order state equations. The reason for the similarity between the height and flight path angle response numerators becomes obvious if the expression for the height, equation (5.135), is compared with the expression for flight path angle, equation (5.127).

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PROBLEMS

5.1 The free response $x(t)$ of a linear second-order system after release from an initial displacement A is given by

$$x(t) = \frac{1}{2}Ae^{-\omega\zeta t}\left(\left(1 + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\right)\mathrm{e}^{-\omega t\sqrt{\zeta^2-1}} + \left(1 - \frac{\zeta}{\sqrt{\zeta^2 - 1}}\right)\mathrm{e}^{\omega t\sqrt{\zeta^2-1}}\right)$$

where ω is the undamped natural frequency and ζ is the damping ratio.

(a) With the aid of sketches, show the possible forms of the motion as ζ varies from zero to a value greater than 1.

(b) How is the motion dependent on the sign of ζ?

(c) How do the time response shapes relate to the solution of the equations of motion of an aircraft?

(d) Define the damped natural frequency and explain how it depends on damping ratio ζ.

(CU 1982)

5.2 For an aircraft in steady rectilinear flight, describe flight path angle, incidence, and attitude, and show how they are related.

(CU 1986)

5.3 Write the Laplace transform of the longitudinal small-perturbation equations of motion of an aircraft for the special case when the phugoid motion is suppressed. It may be assumed that the equations are referred to wind axes and that the influence of the derivatives $\mathring{Z}_q$, $\mathring{Z}_{\dot{w}}$, and $\mathring{M}_{\dot{w}}$ is negligible. State all other assumptions made.

(a) By the application of Cramer's rule, obtain algebraic expressions for the pitch rate response and incidence angle response to elevator transfer functions.

(b) Derivative data for the Republic Thunderchief F-105B aircraft flying at an altitude of 35,000 ft and a speed of 518 kt are

$$\frac{\mathring{Z}_w}{m} = -0.4\ \mathrm{s}^{-1} \quad \frac{\mathring{M}_w}{I_y} = -0.0082\ \mathrm{ft}^{-1}\,\mathrm{s}^{-1} \quad \frac{\mathring{M}_q}{I_y} = -0.485\ \mathrm{s}^{-1}$$

$$\frac{\mathring{M}_\eta}{I_y} = -12.03\ \mathrm{s}^{-2} \quad \frac{\mathring{Z}_\eta}{m} = -65.19\ \mathrm{ft.s}^{-2}$$

Evaluate the transfer functions for this aircraft and calculate values for the longitudinal short-period frequency and damping.

(c) Sketch the pitch rate response to a 1 deg step of elevator angle and indicate the significant features of the response.

(CU 1990)

5.4 The roll response to aileron control of the Douglas DC-8 airliner in an approach flight condition is given by the following transfer function:

$$\frac{\phi(s)}{\xi(s)} = \frac{-0.726(s^2 + 0.421s + 0.889)}{(s - 0.013)(s + 1.121)(s^2 + 0.22s + 0.99)}$$

Realise the transfer function in terms of its partial fractions and, by calculating the inverse Laplace transform, obtain an expression for the roll time history in response to a unit aileron impulse. State all assumptions.

5.5 Describe the methods by which the normal acceleration response to elevator transfer function may be calculated. Using the Republic Thunderchief F-105B model given in Problem 5.3, calculate the transfer function $a_z(s)/\eta(s)$.

(a) With the aid of MATLAB, Program CC, or a similar software tool, obtain a normal acceleration time history response plot for a unit elevator step input. Choose a time scale of about 10 s.

(b) Calculate the inverse Laplace transform of $a_z(s)/\eta(s)$ for a unit step elevator input. Plot the time history given by the response function and compare with that obtained in Problem 5.5(a).

5.6 The lateral-directional equations of motion for the Boeing B-747 cruising at Mach 0.8 at 40,000 ft are given by Heffley and Jewell (1972) as follows:

$$\begin{bmatrix} (s+0.0558) & -\dfrac{(62.074s+32.1)}{774} & \dfrac{(771.51s-2.576)}{774} \\ 3.05 & s(s+0.465) & -0.388 \\ -0.598 & 0.0318s & (s+0.115) \end{bmatrix} \begin{bmatrix} \beta \\ \dfrac{p}{s} \\ r \end{bmatrix} = \begin{bmatrix} 0 & 0.00729 \\ 0.143 & 0.153 \\ 0.00775 & -0.475 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$$

where s is the Laplace operator and all angles are in radians. Using Cramer's rule, calculate all of the response transfer functions and factorise the numerators and common denominator. What are the stability mode characteristics at this flight condition?

5.7 The longitudinal equations of motion as given by Heffley and Jewell (1972) are

$$\begin{bmatrix} (1-X_{\dot{u}})s - X_u^* & -X_{\dot{w}}s - X_w^* & (-X_q + W_e)s + g\cos\theta_e \\ -Z_{\dot{u}}s - Z_u^* & (1-Z_{\dot{w}})s - Z_w & (-Z_q - U_e)s + g\sin\theta_e \\ -M_{\dot{u}}s - M_u^* & -(M_{\dot{w}}s + M_w) & s^2 - M_q s \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix} = \begin{bmatrix} X_\eta \\ Z_\eta \\ M_\eta \end{bmatrix} \eta$$

$$q = s\theta$$

$$\dot{h} = -w\cos\theta_e + u\sin\theta_e + (U_e\cos\theta_e + W_e\sin\theta_e)$$

$$a_z = sw - U_e q + (g\sin\theta_e)\theta$$

Note that the derivatives are in American notation and represent the mass- or inertia-divided dimensional derivatives as appropriate. The * symbol on the speed-dependent derivatives indicates that they include thrust effects as well as the usual aerodynamic characteristics.

All other symbols have their usual meanings. Rearrange these equations into the state space format:

$$\mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{A}'\mathbf{x}(t) + \mathbf{B}'\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

with state vector $\mathbf{x} = [u\ w\ q\ \theta\ h]$, input vector $\mathbf{u} = \eta$, and output vector $\mathbf{y} = [u\ w\ q\ \theta\ h\ a_z]$. State all assumptions made.

5.8 Longitudinal data for the Douglas A-4D Skyhawk flying at Mach 1.0 at 15,000 ft are given in Teper (1969) as follows:

Trim Pitch Attitude			$0.4°$
Speed of Sound at 15,000 ft			1058 ft/s
X_w	-0.0251 1/s	$M_{\dot{w}}$	-0.000683 1/ft
X_u	-0.1343 1/s	M_q	-2.455 1/s
Z_w	-1.892 1/s	X_η	-15.289 ft/rad s^2
Z_u	-0.0487 1/s	Z_η	-94.606 ft/rad s^2
M_w	-0.1072 1/ft s	M_η	-31.773 1/s^2
M_u	0.00263 1/ft s		

Using the state-space model derived in Problem 5.7, obtain the state equations for the Skyhawk in the following format:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Using MATLAB or Program CC, solve the state equations to obtain the response transfer functions for all output variables. What are the longitudinal stability characteristics of the Skyhawk at this flight condition?