CHAPTER

Systems of Axes and Notation

2

Before commencing the main task of developing mathematical models of the aircraft, it is first necessary to put in place an appropriate and secure foundation on which to build the models. This foundation comprises a mathematical framework in which the equations of motion can be developed in an orderly and consistent way. Since aircraft have six degrees of freedom, the description of their motion can be relatively complex. Therefore, motion is usually described by a number of variables which are related to a suitably chosen system of axes. In the United Kingdom, the scheme of notation and nomenclature in common use is based on that developed by Hopkin (1970), and a simplified summary may be found in the appropriate ESDU (1987) data item. As far as is reasonably possible, the notation and nomenclature used throughout this book corresponds with Hopkin's.

By making the appropriate choice of axis systems, order and consistency may be introduced to the process of model building. The importance of order and consistency in the definition of the mathematical framework cannot be overemphasised because, without either, misunderstanding and chaos will surely follow. Only the most basic commonly used axis systems appropriate to aircraft are discussed in the following sections. In addition to the above named references, a more expansive treatment may be found in Etkin (1972) or in McRuer et al. (1973), for example.

2.1 Earth axes

Since only normal atmospheric flight is considered, it is usual to measure aircraft motion with reference to an earth-fixed framework. The accepted convention for defining earth axes determines that a reference point o_0 on the surface of the earth is the origin of a right-handed orthogonal system of axes $(o_0x_0y_0z_0)$, where o_0x_0 points to the north, o_0y_0 points to the east, and o_0z_0 points vertically "down" along the gravity vector. Conventional earth axes are illustrated in Fig. 2.1.

Clearly, the plane $(o_0x_0y_0)$ defines the local horizontal plane which is tangential to the surface of the earth. The flight path of an aircraft flying in the atmosphere in the vicinity of the reference point o_0 may thus be completely described by its coordinates in the axis system. This assumes a *flat earth* where the vertical is "tied" to the gravity vector. For localised flight this model is quite adequate, although it is best suited to navigation and performance applications where flight path trajectories are of primary interest.

For investigations involving trans-global navigation, the axis system described is inappropriate; a spherical coordinate system is preferred. Similarly, for trans-atmospheric flight involving the launch and reentry of space vehicles, a spherical coordinate system is more appropriate. However, since in such an application the angular velocity of the earth becomes important, it is necessary to define a fixed spatial axis system to which the spherical earth axis system may be referenced.

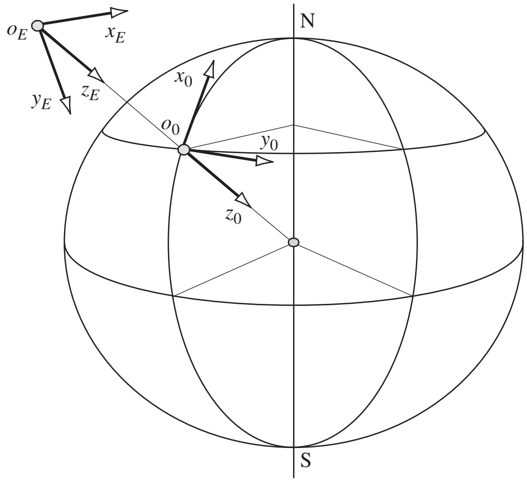

FIGURE 2.1 Conventional earth axes.

For flight dynamics applications a simpler definition of earth axes is preferred. Since only short-term motion is of interest, it is perfectly adequate to assume flight above a flat earth. The most common consideration is that of motion about *straight and level* flight. Straight and level flight assumes flight in a horizontal plane at a constant altitude; whatever the subsequent motion of the aircraft might be, the *attitude* is determined with respect to the horizontal. Referring again to Fig. 2.1, the horizontal plane is defined by ($o_E x_E y_E$) and is parallel to the plane ($o_0 x_0 y_0$) at the surface of the earth. The only difference is that the $o_E x_E$ axis points in the arbitrary direction of the flight of the aircraft rather than to the north. The $o_E z_E$ axis points vertically down as before. Therefore, it is only necessary to place the origin o_E in the atmosphere at the most convenient point, which is frequently coincident with the origin of the aircraft body−fixed axes. Earth axes ($o_E x_E y_E z_E$) defined in this way are called *datum-path earth axes*; they are "tied" to the earth by means of the gravity vector and provide the inertial reference frame for short-term aircraft motion.

2.2 Aircraft body−fixed axes

A number of aircraft body−fixed axis systems are in common use. However, for small perturbation analysis only the generalised body axes and wind, or stability, axes need be considered.

2.2.1 Generalised body axes

It is usual practice to define a right-handed orthogonal axis system fixed in the aircraft and constrained to move with it. Thus, when the aircraft is disturbed from its initial flight condition, the axes move with the airframe and the motion is quantified in terms of perturbation variables referred to the moving axes. The way in which the axes may be fixed in the airframe is arbitrary, although it is preferable to use an accepted standard orientation. The most general axis system is known as a

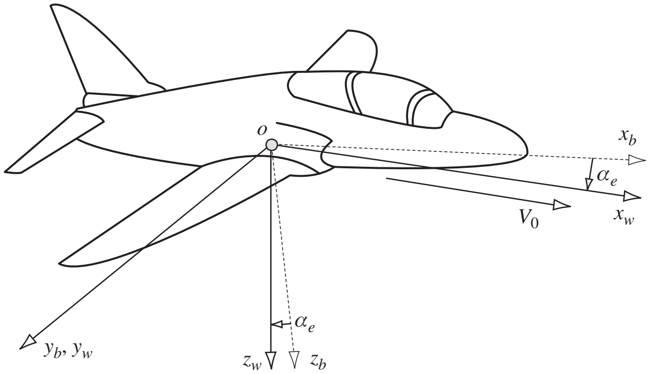

FIGURE 2.2 Moving-axis systems.

body axis system ($ox_by_bz_b$), which is fixed in the aircraft as shown in Fig. 2.2. The (ox_bz_b) plane defines the plane of symmetry of the aircraft, and it is convenient to arrange the ox_b axis such that it is parallel to the geometrical *horizontal fuselage datum*. Thus in normal flight attitudes the oy_b axis is directed to starboard and the oz_b axis is directed "downward." The origin o of the axes is fixed at a convenient reference point in the airframe, which is usually, but not necessarily, coincident with the centre of gravity (cg).

2.2.2 Aerodynamic, wind, or stability axes

It is often convenient to define a set of aircraft fixed axes such that the ox axis is parallel to the total velocity vector V_0, as shown in Fig. 2.2. Such axes are called *aerodynamic*, *wind*, or *stability* axes. In steady symmetric flight, wind axes ($ox_wy_wz_w$) are just a particular version of body axes which are rotated about the oy_b axis through the steady body incidence angle α_e until the ox_w axis aligns with the velocity vector. Thus the plane (ox_wz_w) remains the plane of symmetry of the aircraft and the oy_w and the oy_b axes are coincident. There is a unique value of body incidence angle α_e for every flight condition; therefore, for every flight condition, the wind axes orientation in the airframe is different. However, for any given flight condition this orientation is defined and fixed in the aircraft at the outset and is constrained to move with it in subsequent disturbed flight. Typically, the body incidence might vary in the range $-10° \le \alpha_e \le 20°$ over a normal flight envelope.

2.2.3 Perturbation variables

The motion of the aircraft is described in terms of force, moment, linear and angular velocities, and attitude resolved into components with respect to the chosen aircraft fixed axis system. For convenience, it is preferable to assume a generalised *body axis* system in the first instance. Thus the aircraft is initially assumed to be in steady rectilinear, but not necessarily level, flight when the body incidence is α_e and the steady velocity V_0 resolves into components U_e, V_e, and W_e as indicated in Fig. 2.3. In steady nonaccelerating flight the aircraft is in equilibrium and the forces and moments acting on the airframe are in balance and sum to zero. This initial condition is usually referred to as *trimmed equilibrium*.

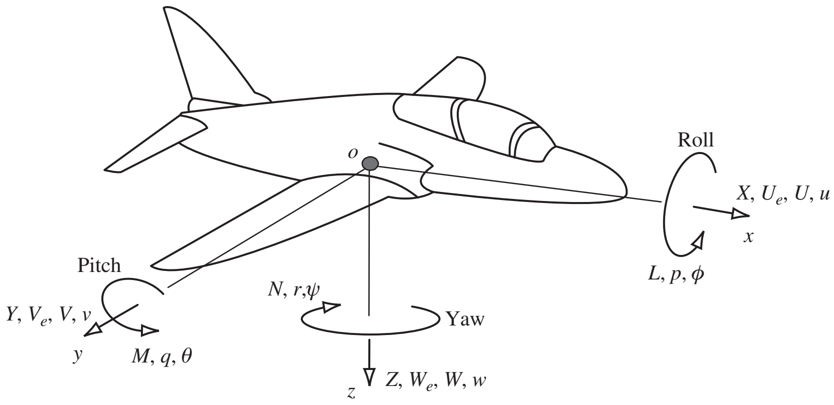

FIGURE 2.3 Motion variables notation.

Whenever the aircraft is disturbed from equilibrium, the force and moment balance is upset and the resulting transient motion is quantified in terms of the perturbation variables. The perturbation variables are shown in Fig. 2.3 and summarised in Table 2.1.

The positive sense of the variables is determined by the choice of a right-handed axis system. Components of linear quantities, force, velocity, and so forth are positive when their direction of action is the same as the direction of the axis to which they relate. The positive sense of the components of rotary quantities, moment, velocity, attitude, and so forth is a right-handed rotation and may be determined as follows. Positive *roll* about the *ox* axis is such that the *oy* axis moves toward the *oz* axis; positive *pitch* about the *oy* axis is such that the *oz* axis moves toward the *ox* axis; and positive *yaw* about the *oz* axis is such that the *ox* axis moves toward the *oy* axis. Therefore, positive roll is right wing down, positive pitch is nose up, and positive yaw is nose to the right as seen by the pilot.

A simple description of the perturbation variables is given in Table 2.2. The intention is to provide some insight into the physical meaning of the many variables used in the model. Note that the components of the total linear velocity perturbations (U,V,W) are given by the sum of the steady equilibrium components and the transient perturbation components (u,v,w); thus

$$\begin{aligned} U &= U_e + u \\ V &= V_e + v \\ W &= W_e + w \end{aligned} \tag{2.1}$$

2.2.4 Angular relationships in symmetric flight

Since it is assumed that the aircraft is in steady rectilinear, but not necessarily level, flight and that the axes fixed in the aircraft are body axes, it is useful to relate the steady and perturbed angles as shown in Fig. 2.4.

With reference to Fig. 2.4, the steady velocity vector V_0 defines the flight path and γ_e is the steady flight path angle. As before, α_e is the steady body incidence and θ_e is the steady pitch

Table 2.1 Summary of Motion Variables

	Trimmed Equilibrium			**Perturbed**		
Aircraft axis	*ox*	*oy*	*oz*	*ox*	*oy*	*oz*
Force	0	0	0	X	Y	Z
Moment	0	0	0	L	M	N
Linear velocity	U_e	V_e	W_e	U	V	W
Angular velocity	0	0	0	p	q	r
Attitude	0	θ_e	0	ϕ	θ	ψ

Table 2.2 Perturbation Variables

X	Axial "drag" force	Sum of the components of aerodynamic, thrust and weight forces
Y	Side force	
Z	Normal "lift" force	
L	Rolling moment	Sum of the components of aerodynamic, thrust and weight moments
M	Pitching moment	
N	Yawing moment	
p	Roll rate	Components of angular velocity
q	Pitch rate	
r	Yaw rate	
U	Axial velocity	Total linear velocity components of the *cg*
V	Lateral velocity	
W	Normal velocity	

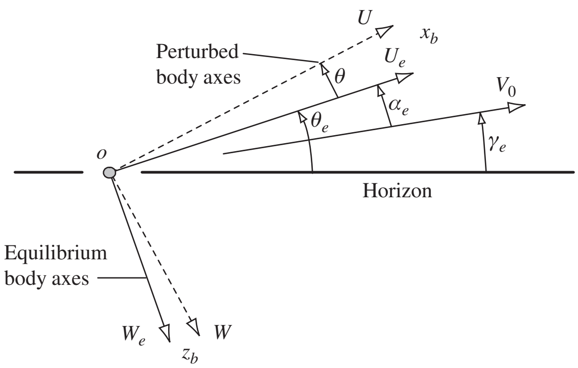

FIGURE 2.4 Generalised body axes in symmetric flight.

attitude of the aircraft. The relative angular change in a perturbation is also shown in Fig. 2.4, where it is implied that the axes have moved with the airframe and the motion is viewed at some instant during the disturbance. Thus the steady flight path angle is given by

$$\gamma_e = \theta_e - \alpha_e \tag{2.2}$$

In the case when the aircraft-fixed axes are wind axes rather than body axes, then

$$\alpha_e = 0 \tag{2.3}$$

and in the special case when the axes are wind axes and the initial condition is level flight,

$$\alpha_e = \theta_e = 0 \tag{2.4}$$

It is also useful to note that the perturbation in pitch attitude θ and the perturbation in body incidence α are the same, so it is convenient to write

$$\tan(\alpha_e + \theta) \equiv \tan(\alpha_e + \alpha) = \frac{W}{U} \equiv \frac{W_e + w}{U_e + u} \tag{2.5}$$

2.2.5 Choice of axes

Having reviewed the definition of aircraft–fixed axis systems, an obvious question must be: *When is it appropriate to use wind axes and when is it appropriate to use body axes*? The answer to this question depends on the use to which the equations of motion are to be put. The best choice of axes simply facilitates the analysis of the equations of motion. When starting from first principles, it is preferable to use generalised body axes since the resulting equations can serve for most applications. It is then reasonably straightforward to simplify the equations to a wind axis form if the application warrants it. On the other hand, to extend wind axis–based equations to serve the more general case is not as easy.

When dealing with numerical data for an existing aircraft, it is not always obvious which axis system has been used in the derivation of the model. However, by reference to equation (2.3) or (2.4) and the quoted values of α_e and θ_e, the system used should become clear.

When it is necessary to make experimental measurements in an actual aircraft, or in a model, which are to be used subsequently in the equations of motion, it is preferable to use a generalised body axis system. Since the measuring equipment is installed in the aircraft, its location is precisely known in terms of body axis coordinates, which therefore determines the best choice of axis system. In a similar way, most aerodynamic measurements and computations are referenced to the free stream velocity vector. For example, in wind tunnel work the obvious reference is the tunnel axis which is coincident with the velocity vector. Thus, for aerodynamic investigations involving the equations of motion, a wind axis reference is preferred. Traditionally, all aerodynamic data for use in the equations of motion are referenced to wind axes.

To summarise, it is not particularly important which axis system is chosen provided it models the flight condition to be investigated; the end result does not depend on this choice. However, when compiling data for use in the aircraft equations of motion, it is quite common for some data to be referred to wind axes and for some to be referred to body axes. It therefore becomes necessary to have available the mathematical tools for transforming data between different reference axes.

2.3 Euler angles and aircraft attitude

The angles defined by the right-handed rotation about the three axes of a right-handed system of axes are called *Euler angles*. The sense of the rotations and the order in which they are considered about the three axes are, in turn, very important because angles do not obey the commutative law. The attitude of an aircraft is defined as the angular orientation of the airframe-fixed axes with respect to earth axes. Attitude angles are therefore a particular application of Euler angles. With reference to Fig. 2.5, $(ox_0y_0z_0)$ are datum or reference axes and $(ox_3y_3z_3)$ are aircraft-fixed axes, either generalised body axes or wind axes. The attitude of the aircraft, with respect to the datum axes, may be established by considering the rotation about each axis in turn required to bring $(ox_3y_3z_3)$ into coincidence with $(ox_0y_0z_0)$. Thus, first rotate about ox_3 through the *roll* angle ϕ to $(ox_2y_2z_2)$. Second, rotate about oy_2 through the *pitch* angle θ to $(ox_1y_1z_1)$. Third, rotate about oz_1 through the *yaw* angle ψ to $(ox_0y_0z_0)$. Clearly, when the attitude of the aircraft is considered with respect to earth axes, $(ox_0y_0z_0)$ and $(ox_Ey_Ez_E)$ are coincident.

2.4 Axes transformations

It is frequently necessary to transform motion variables and other parameters from one system of axes to another. Clearly, the angular relationships used to describe attitude may be generalised to describe the angular orientation of one set of axes with respect to another. A typical example might be to transform components of linear velocity from wind axes to body axes. Thus, with reference to Fig. 2.5, $(ox_0y_0z_0)$ may be used to describe the velocity components in wind axes and $(ox_3y_3z_3)$ may be used to describe the components of velocity in body axes; the angles (ϕ,θ,ψ) then describe the generalised angular orientation of one set of axes with respect to the other. It is usual to retain the

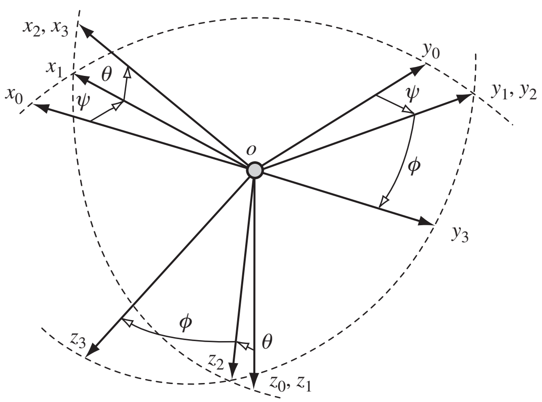

FIGURE 2.5 Euler angles.

angular description of *roll*, *pitch*, and *yaw*, although the angles do not necessarily describe attitude strictly in accordance with the definition given in Section 2.3.

2.4.1 Linear quantities transformation

As an example, let (ox_3, oy_3, oz_3) represent components of a linear quantity in the axis system $(ox_3y_3z_3)$, and let (ox_0, oy_0, oz_0) represent components of the same linear quantity transformed into the axis system $(ox_0y_0z_0)$. The linear quantities of interest might be acceleration, velocity, displacement, and so forth. Resolving through each rotation in turn and in the correct order, then, with reference to Fig. 2.5 it may be shown that, after *rolling* about ox_3 through the angle ϕ,

$$\begin{aligned} ox_3 &= ox_2 \\ oy_3 &= oy_2 \cos\phi + oz_2 \sin\phi \\ oz_3 &= -oy_2 \sin\phi + oz_2 \cos\phi \end{aligned} \tag{2.6}$$

Alternatively, writing equations (2.6) in the more convenient matrix form,

$$\begin{bmatrix} ox_3 \\ oy_3 \\ oz_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} ox_2 \\ oy_2 \\ oz_2 \end{bmatrix} \tag{2.7}$$

Similarly, after *pitching* about oy_2 through the angle θ,

$$\begin{bmatrix} ox_2 \\ oy_2 \\ oz_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} ox_1 \\ oy_1 \\ oz_1 \end{bmatrix} \tag{2.8}$$

and after *yawing* about oz_1 through the angle ψ,

$$\begin{bmatrix} ox_1 \\ oy_1 \\ oz_1 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ox_0 \\ oy_0 \\ oz_0 \end{bmatrix} \tag{2.9}$$

By repeated substitution, equations (2.7), (2.8), and (2.9) may be combined to give the required transformation relationship:

$$\begin{bmatrix} ox_3 \\ oy_3 \\ oz_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ox_0 \\ oy_0 \\ oz_0 \end{bmatrix} \tag{2.10}$$

or

$$\begin{bmatrix} ox_3 \\ oy_3 \\ oz_3 \end{bmatrix} = \mathbf{D} \begin{bmatrix} ox_0 \\ oy_0 \\ oz_0 \end{bmatrix} \tag{2.11}$$

where the *direction cosine matrix* **D** is given by

$$\mathbf{D} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi & \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ -\cos\phi\sin\psi & +\cos\phi\cos\psi & \\ \cos\phi\sin\theta\cos\psi & \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \\ +\sin\phi\sin\psi & -\sin\phi\cos\psi & \end{bmatrix} \tag{2.12}$$

As shown, equation (2.11) transforms linear quantities from $(ox_0y_0z_0)$ to $(ox_3y_3z_3)$. By inverting the direction cosine matrix **D**, the transformation from $(ox_3y_3z_3)$ to $(ox_0y_0z_0)$ is obtained:

$$\begin{bmatrix} ox_0 \\ oy_0 \\ oz_0 \end{bmatrix} = \mathbf{D}^{-1}\begin{bmatrix} ox_3 \\ oy_3 \\ oz_3 \end{bmatrix} \tag{2.13}$$

EXAMPLE 2.1

To illustrate the use of equation (2.11), consider the very simple example in which it is required to resolve the velocity of the aircraft through both the incidence angle and the sideslip angle into aircraft axes. The situation prevailing is assumed to be steady and is shown in Fig. 2.6.

The axes $(oxyz)$ are generalised aircraft body axes with velocity components U_e, V_e, and W_e, respectively. The free stream velocity vector is V_0, and the angles of incidence and sideslip are α_e and β_e, respectively. With reference to equation (2.11), axes $(oxyz)$ correspond with axes $(ox_3y_3z_3)$ and V_0 corresponds with ox_0 of axes $(ox_0y_0z_0)$. Therefore, the following vector substitutions may be made:

$$(ox_0,\ oy_0,\ oz_0) = (V_0, 0, 0) \text{ and } (ox_3,\ oy_3,\ oz_3) = (U_e, V_e, W_e)$$

The angular correspondence means that the following substitution may be made:

$$(\phi,\ \theta,\ \psi) = (0, \alpha_e, -\beta_e)$$

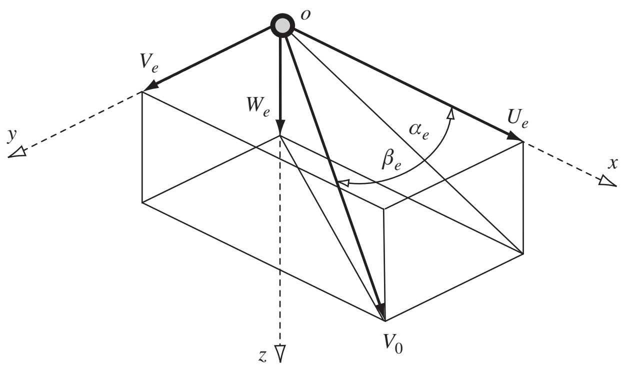

FIGURE 2.6 Resolution of velocity through incidence and sideslip angles.

Note that a positive sideslip angle is equivalent to a negative yaw angle. Thus, making the substitutions in equation (2.9),

$$\begin{bmatrix} U_e \\ V_e \\ W_e \end{bmatrix} = \begin{bmatrix} \cos\alpha_e \cos\beta_e & -\cos\alpha_e \sin\beta_e & -\sin\alpha_e \\ \sin\beta_e & \cos\beta_e & 0 \\ \sin\alpha_e \cos\beta_e & -\sin\alpha_e \sin\beta_e & \cos\alpha_e \end{bmatrix} \begin{bmatrix} V_0 \\ 0 \\ 0 \end{bmatrix} \tag{2.14}$$

Whence, equivalently,

$$\begin{aligned} U_e &= V_0 \cos\alpha_e \cos\beta_e \\ V_e &= V_0 \sin\beta_e \\ W_e &= V_0 \sin\alpha_e \cos\beta_e \end{aligned} \tag{2.15}$$

EXAMPLE 2.2

Another very useful application of the direction cosine matrix is to calculate height perturbations in terms of aircraft motion. Equation (2.13) may be used to relate the velocity components in aircraft axes to the corresponding components in earth axes as follows:

$$\begin{bmatrix} U_E \\ V_E \\ W_E \end{bmatrix} = \mathbf{D}^{-1} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & \begin{matrix}\cos\psi\sin\theta\sin\phi \\ -\sin\psi\cos\phi\end{matrix} & \begin{matrix}\cos\psi\sin\theta\cos\phi \\ +\sin\psi\sin\phi\end{matrix} \\ \sin\psi\cos\theta & \begin{matrix}\sin\psi\sin\theta\sin\phi \\ +\cos\psi\cos\phi\end{matrix} & \begin{matrix}\sin\psi\sin\theta\cos\phi \\ -\cos\psi\sin\phi\end{matrix} \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} \tag{2.16}$$

where, U_E, V_E, and W_E are the perturbed total velocity components referred to earth axes. Now, since height is measured positive in the "upward" direction, the rate of change of height due to the perturbation in aircraft motion is given by

$$\dot{h} = -W_E$$

Whence, from equation (2.16),

$$\dot{h} = U\sin\theta - V\cos\theta\sin\phi - W\cos\theta\cos\phi \tag{2.17}$$

2.4.2 Angular velocities transformation

Probably the most useful angular quantities transformation relates the angular velocities p, q, r of the aircraft-fixed axes to the resolved components of angular velocity, the attitude rates $\dot{\phi}, \dot{\theta}, \dot{\psi}$ with respect to datum axes. The easiest way to deal with the algebra of this transformation whilst retaining a good grasp of the physical implications is to superimpose the angular rate vectors onto the axes shown in Fig. 2.5; the result of this is shown in Fig. 2.7.

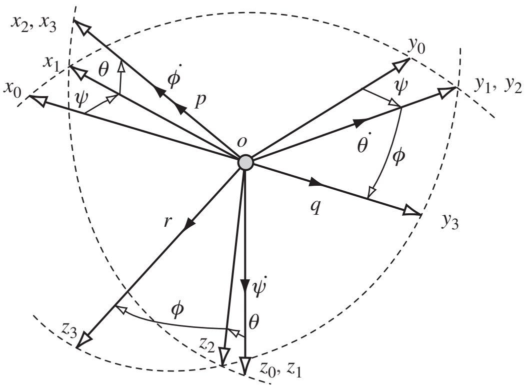

FIGURE 2.7 Angular rates transformation.

The angular *body rates* p,q,r are shown in the aircraft axes ($ox_3y_3z_3$), then, considering each rotation in turn necessary to bring the aircraft axes into coincidence with the datum axes ($ox_0y_0z_0$). First, *roll* about ox_3 through the angle ϕ with angular velocity $\dot{\phi}$. Second, *pitch* about oy_2 through the angle θ with angular velocity $\dot{\theta}$. Third, *yaw* about oz_1 through the angle ψ with angular velocity $\dot{\psi}$. Again, it is most useful to refer the attitude rates to earth axes, in which case datum axes ($ox_0y_0z_0$) are coincident with earth axes ($o_Ex_Ey_Ez_E$). The *attitude rate* vectors are clearly shown in Fig. 2.7. The relationship between the aircraft body rates and the attitude rates, referred to datum axes, is readily established as follows:

Roll rate p is equal to the sum of the components of $\dot{\phi}, \dot{\theta}, \dot{\psi}$ resolved along ox_3:

$$p = \dot{\phi} - \dot{\psi} \sin \theta \tag{2.18}$$

Pitch rate q is equal to the sum of the components of $\dot{\phi}, \dot{\theta}, \dot{\psi}$ resolved along oy_3:

$$q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta \tag{2.19}$$

Yaw rate r is equal to the sum of the components of $\dot{\phi}, \dot{\theta}, \dot{\psi}$ resolved along oz_3:

$$r = \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi \tag{2.20}$$

Equations (2.18), (2.19), and (2.20) may be combined into the convenient matrix notation

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{2.21}$$

and the inverse of equation (2.21) is

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{2.22}$$

When the aircraft perturbations are small, such that (ϕ, θ, ψ) may be treated as small angles, equations (2.21) and (2.22) may be approximated by

$$\begin{aligned} p &= \dot{\phi} \\ q &= \dot{\theta} \\ r &= \dot{\psi} \end{aligned} \tag{2.23}$$

EXAMPLE 2.3

To illustrate the use of the angular velocities transformation, consider the situation when an aircraft is flying in a steady level coordinated turn at a speed of 250 m/s at a bank angle of 60°. It is required to calculate the turn rate $\dot{\psi}$, the yaw rate r, and the pitch rate q. The forces acting on the aircraft are shown in Fig. 2.8.

By resolving the forces acting on the aircraft vertically and horizontally, and eliminating the lift L between the two resulting equations, it is easily shown that the radius of turn is given by

$$R = \frac{V_0^2}{g \tan \phi} \tag{2.24}$$

The time to complete one turn is given by

$$t = \frac{2\pi R}{V_0} = \frac{2\pi V_0}{g \tan \phi} \tag{2.25}$$

Therefore, the rate of turn is given by

$$\dot{\psi} = \frac{2\pi}{t} = \frac{g \tan \phi}{V_0} \tag{2.26}$$

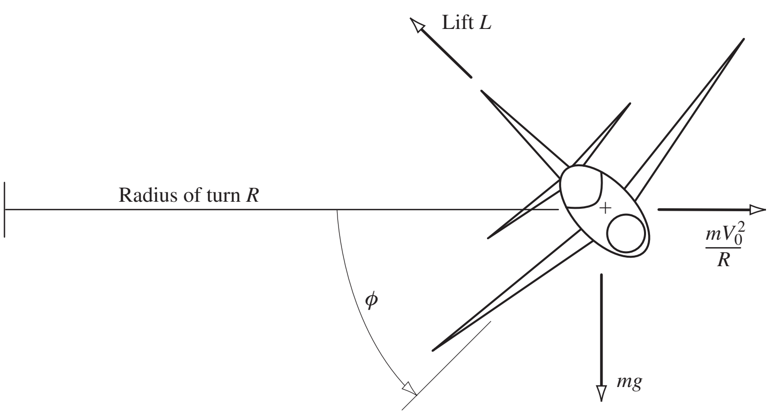

FIGURE 2.8 Aircraft in a steady banked turn.

Thus $\dot{\psi} = 0.068$ rad/s. For the conditions applying to the turn, $\dot{\phi} = \dot{\theta} = \theta = 0$ and thus equation (2.21) may now be used to find the values of r and q:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & \sin 60^\circ \\ 0 & -\sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

Therefore, $p = 0$, $q = 0.059$ rad/s, and $r = 0.034$ rad/s. Note that p, q, and r are the angular velocities that would be measured by rate gyros fixed in the aircraft with their sensitive axes aligned with the ox, oy, and oz aircraft axes, respectively.

2.5 Aircraft reference geometry

A description of the geometric layout of an aircraft is an essential part of the mathematical modelling process. For the purposes of flight dynamics analysis, it is convenient that the geometry of the aircraft can be adequately described by a small number of dimensional reference parameters, which are defined next and illustrated in Fig. 2.9.

2.5.1 Wing area

The reference area is usually the gross plan area of the wing, including that part within the fuselage, and is denoted S, where

$$S = b\bar{c} \tag{2.27}$$

b is the wing span, and $\bar{c}$ is the standard mean chord of the wing.

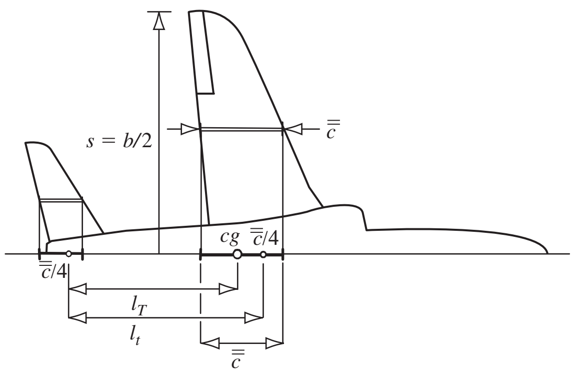

FIGURE 2.9 Longitudinal reference geometry.

2.5.2 Mean aerodynamic chord

The *mean aerodynamic chord* of the wing (*mac*) is denoted $\bar{\bar{c}}$ and is defined as

$$\bar{\bar{c}} = \frac{\int_{-s}^{s} c_y^2 \, dy}{\int_{-s}^{s} c_y \, dy} \tag{2.28}$$

The reference *mac* is located on the centre line of the aircraft by projecting $\bar{\bar{c}}$ from its spanwise location as shown in Fig. 2.9.

For a swept wing, then, the leading edge of the *mac* lies aft of the leading edge of the root chord of the wing. The *mac* represents the location of the root chord of a rectangular wing that has the same aerodynamic influence on the aircraft as the actual wing. Traditionally, the *mac* is used in stability and control studies since a number of important aerodynamic reference centres are located on it.

2.5.3 Standard mean chord

The *standard mean chord* of the wing (*smc*) is effectively the same as the geometric mean chord and is denoted $\bar{c}$. For a wing of symmetric planform it is defined as

$$\bar{c} = \frac{\int_{-s}^{s} c_y \, dy}{\int_{-s}^{s} dy} \tag{2.29}$$

where $s = b/2$ is the semi-span and c_y is the local chord at spanwise coordinate y. For a straight tapered wing, equation (2.29) simplifies to

$$\bar{c} = \frac{S}{b} \tag{2.30}$$

The reference *smc* is located on the centre line of the aircraft by projecting $\bar{c}$ from its spanwise location in the same way that the *mac* is located. Thus for a swept wing the leading edge of the *smc* also lies aft of the leading edge of the root chord of the wing. The *smc* is the mean chord preferred by aircraft designers since it relates very simply to the geometry of the aircraft. For most aircraft the *smc* and *mac* are sufficiently similar in length and location to be practically interchangeable. It is quite common to find references that quote a mean chord without specifying *smc* or *mac*. This is not good practice, although the error incurred by assuming the wrong chord is rarely serious. However, the reference chord used in any application should always be clearly defined at the outset.

2.5.4 Aspect ratio

The aspect ratio of the aircraft wing is a measure of its spanwise slenderness and is denoted A. It is defined as follows:

$$A = \frac{b^2}{S} = \frac{b}{\bar{c}} \tag{2.31}$$

2.5.5 Location of centre of gravity

The centre of gravity of an aircraft, *cg*, is usually located on the reference chord as indicated in Fig. 2.9. Its position is quoted as a fraction of $\bar{\bar{c}}$ (or $\bar{c}$), denoted h, and is measured from the leading edge of the reference chord as shown. The *cg* position varies as a function of aircraft loading; the typical variation is in the range of 10% to 40% of $\bar{\bar{c}}$ or, equivalently, $0.1 \le h \le 0.4$.

2.5.6 Tail moment arm and tail volume ratio

The *mac* of the horizontal tailplane, or foreplane, is defined and located in the airframe in the same way as the *mac* of the wing, as indicated in Fig. 2.9. The wing and tailplane aerodynamic forces and moments are assumed to act at their respective aerodynamic centres, which, to a good approximation, lie at the quarter-chord points of the *mac* of the wing and tailplane, respectively. The tail moment arm l_T is defined as the longitudinal distance between the centre of gravity and the aerodynamic centre of the tailplane, as shown in Fig. 2.9. The *tail volume ratio* $\overline{V}_T$ is an important geometric parameter and is defined as

$$\overline{V}_T = \frac{S_T l_T}{S\bar{\bar{c}}} \tag{2.32}$$

where S_T is the gross area of the tailplane and *mac* $\bar{\bar{c}}$ is the longitudinal reference length. Typically, the tail volume ratio has a value in the range $0.5 \le \overline{V}_T \le 1.3$ and is a measure of the aerodynamic effectiveness of the tailplane as a stabilising device.

Sometimes, especially in stability and control studies, it is convenient to measure the longitudinal tail moment about the aerodynamic centre of the wing *mac*. In this case the tail moment arm is denoted l_t, as shown in Fig. 2.9, and a slightly modified tail volume ratio is defined.

2.5.7 Fin moment arm and fin volume ratio

The *mac* of the fin is defined and located in the airframe in the same way as the *mac* of the wing, as indicated in Fig. 2.10. As for the tailplane, the fin moment arm l_F is defined as the longitudinal distance between the centre of gravity and the aerodynamic centre of the fin, as shown in Fig. 2.10. The *fin volume ratio* $\overline{V}_F$ is an important geometric parameter and is defined as

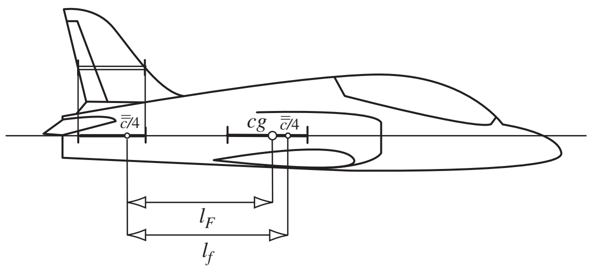

FIGURE 2.10 Fin moment arm.

$$\overline{V}_F = \frac{S_F l_F}{Sb} \tag{2.33}$$

where S_F is the gross area of the fin and the wing span b is the lateral-directional reference length. Again, the fin volume ratio is a measure of the aerodynamic effectiveness of the fin as a directional stabilising device.

As stated previously, it is sometimes convenient to measure the longitudinal moment of the aerodynamic forces acting at the fin about the aerodynamic centre of the wing *mac*. In this case the fin moment arm is denoted l_f, as shown in Fig. 2.10.

2.6 Controls notation

The definition of the sense of controls and pilot control actions is often vague at best, and so it is appropriate to review the commonly accepted notation for aerodynamic and propulsion controls.

2.6.1 Aerodynamic controls

Sometimes there appears to be some confusion with respect to the correct notation for aerodynamic controls, especially when unconventional control surfaces are used. Hopkin (1970) defines a notation which is intended to be generally applicable, but, since a very large number of combinations of control motivators are possible, the notation relating to control inceptors may become ill defined and hence application-dependent. However, for the conventional aircraft there is a universally accepted notation, which accords with Hopkin, and it is simple to apply. Generally, a positive *control action* by the pilot gives rise to a positive aircraft response, whereas a positive *control surface displacement* gives rise to a negative aircraft response. Thus

In roll: positive right push force on the stick $\Rightarrow$ positive stick displacement $\Rightarrow$ right aileron up and left aileron down (negative mean) $\Rightarrow$ right wing down roll response (positive).
In pitch: positive pull force on the stick $\Rightarrow$ positive aft stick displacement $\Rightarrow$ elevator trailing edge up (negative) $\Rightarrow$ nose-up pitch response (positive).
In yaw: positive push force on the right rudder pedal $\Rightarrow$ positive rudder bar displacement $\Rightarrow$ rudder trailing edge displaced to the right (negative) $\Rightarrow$ nose to the right yaw response (positive).

Roll and pitch control stick displacements are denoted δ_ξ and δ_η, respectively, and rudder pedal displacement is denoted δ_ζ. Aileron, elevator, and rudder surface displacements are denoted ξ, η, and ζ, respectively, as indicated in Fig. 2.11. It should be noted that since ailerons act differentially, the displacement ξ is usually taken as the mean value of the separate displacements of each aileron.

2.6.2 Engine control

Engine thrust τ is controlled by throttle lever displacement ε. Positive throttle lever displacement is usually in the forward push sense and results in a positive increase in thrust. For a turbojet engine the relationship between thrust and throttle lever angle is approximated by a simple first-order lag transfer function,

$$\frac{\tau(s)}{\varepsilon(s)} = \frac{k_\tau}{(1 + sT_\tau)} \tag{2.34}$$

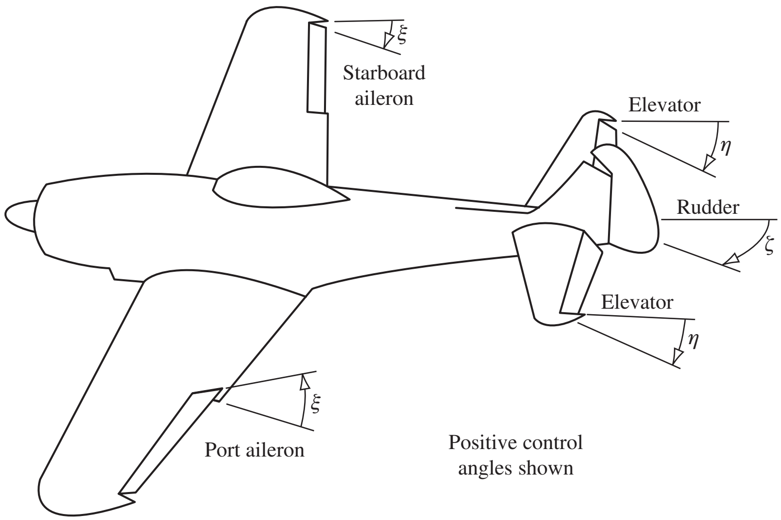

FIGURE 2.11 Aerodynamic controls notation.

where k_τ is a suitable gain constant and T_τ is the lag time constant, which is typically of the order of two to three seconds.

2.7 Aerodynamic reference centres

With reference to Fig. 2.12, the *centre of pressure* of an aerofoil, wing, or complete aircraft, *cp,* is the point at which the resultant aerodynamic force F acts. It is usual to resolve the force into the *lift* component perpendicular to the velocity vector and the *drag* component parallel to the velocity vector, denoted L and D, respectively, in the usual way. The *cp* is located on the *mac* and thereby determines an important aerodynamic reference centre.

Simple theory establishes that the resultant aerodynamic force F generated by an aerofoil comprises two components—that due to camber F_c and that due to angle of attack F_α—both of which resolve into lift and drag forces as indicated. The aerodynamic force due to camber is constant and acts at the midpoint of the aerofoil chord; for a symmetric aerofoil section this force is zero. The aerodynamic force due to angle of attack acts at the quarter-chord point and varies directly with angle of attack at angles below the stall. This also explains why the zero-lift angle of attack of a cambered aerofoil is usually a small negative value because, in this condition, the lift due to camber is equal and opposite to the lift due to angle of attack. Thus at low speeds, when the angle of attack is generally large, most of the aerodynamic force is due to the angle of attack−dependent contribution and *cp* is nearer to the quarter-chord point.

On the other hand, at high speeds, when the angle of attack is generally small, a larger contribution to the aerodynamic force is due to the camber-dependent component and *cp* is nearer to the

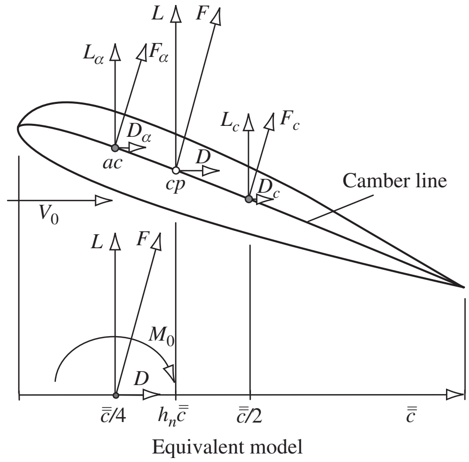

FIGURE 2.12 Aerodynamic reference centres.

midpoint of the chord. Thus, in the limit, the aerofoil *cp* generally lies between the quarter-chord and mid-chord points. More generally, the interpretation for an aircraft recognises that *cp* moves as a function of angle of attack, Mach number, and configuration. For example, at low angles of attack and high Mach numbers, *cp* tends to move aft and vice versa. Consequently, *cp* is of limited use as an aerodynamic reference point in stability and control studies. It should be noted that the *cp* of the complete aircraft in trimmed-equilibrium flight corresponds to the *controls-fixed neutral point* $h_n\bar{\bar{c}}$, which is discussed in Chapter 3.

If, instead of *cp*, another *fixed* point on the *mac* is chosen as an aerodynamic reference point, at this point the total aerodynamic force remains the same but is accompanied by a pitching moment about the point. Clearly, the most convenient reference point on the *mac* is the quarter-chord point since the pitching moment is the moment of the aerodynamic force due to camber and remains constant with variation in angle of attack. This point is called the *aerodynamic centre*, denoted *ac*, which at low Mach numbers lies at, or very close to, the quarter-chord point $\bar{\bar{c}}/4$. It is for this reason that *ac*, or equivalently the quarter-chord point of the reference chord, is preferred as a reference point. The corresponding equivalent aerofoil model is shown in Fig. 2.12. Since *ac* remains essentially fixed in position during small perturbations about a given flight condition, and since the pitching moment is nominally constant about the *ac*, it is used as a reference point in stability and control studies. It is important to appreciate the fact that, as the flight condition Mach number is increased, the *ac* moves aft; in supersonic-flow conditions it is located at, or very near, $\bar{\bar{c}}/2$.

The definition of aerodynamic centre given above applies most strictly to the location of the *ac* on the chord of an aerofoil. However, it also applies reasonably well to the location of the *ac* on the *mac* of a wing and is used extensively for locating the *ac* on the *mac* of a wing-body combination without too much loss of validity. It should be appreciated that the complex aerodynamics of a wing-body combination might result in an *ac* location which is not at the quarter-chord point, although typically it would not be too far removed from that point.

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PROBLEMS

2.1 A tailless aircraft of 9072 kg mass has a delta wing with aspect ratio 1 and area 37 m^2. Show that the aerodynamic mean chord of a delta wing,

$$\bar{\bar{c}} = \frac{\int_0^{\frac{b}{2}} c^2 \, \mathrm{d}y}{\int_0^{\frac{b}{2}} c \, \mathrm{d}y}$$

is two-thirds of its root chord and that for this wing it is 5.73 m.

(CU 1983)

2.2 With the aid of a diagram, describe the axis systems used in aircraft stability and control analysis. State the conditions under which each axis system might be preferred.

(CU 1982)

2.3 Show that, in a longitudinal symmetric small perturbation, the components of aircraft weight resolved into the *ox* and *oz* axes are given by

$$\begin{aligned} X_g &= -mg\theta \cos\theta_e - mg \sin\theta_e \\ Z_g &= mg\cos\theta_e - mg\theta \sin\theta_e \end{aligned}$$

where θ is the perturbation in pitch attitude and θ_e is the equilibrium pitch attitude.

(CU 1982)

2.4 With the aid of a diagram showing a generalised set of aircraft body axes, define the parameter notation used in the mathematical modelling of aircraft motion.

(CU 1982)

2.5 In the context of aircraft motion, what are the Euler angles? If the standard right-handed aircraft axis set is rotated through pitch θ and yaw ψ angles only, show that the initial vector quantity (x_0, y_0, z_0) is related to the transformed vector quantity (x, y, z) as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

(CU 1982)

2.6 Define the span, gross area, aspect ratio, and mean aerodynamic chord of an aircraft wing.

(CU 2001)

2.7 Distinguish between the centre of pressure and the aerodynamic centre of an aerofoil. Explain why the pitching moment about the quarter-chord point of an aerofoil is nominally constant in subsonic flight.

(CU 2001)