CHAPTER

Shear of Beams

10

In Chapter 3 we saw that externally applied shear loads produce internal shear forces and bending moments in cross sections of a beam. The bending moments cause direct stress distributions in beam sections (Chapter 9); we shall now determine the corresponding distributions of shear stress. Initially, however, we shall examine the physical relationship between bending and shear; the mathematical relationships have already been defined in Eqs. (3.8), (9.57), and (9.58).

Suppose that a number of planks are laid one on top of the other and supported at each end as shown in Fig. 10.1(a). Applying a central concentrated load to the planks at mid span will cause them to bend as shown in Fig. 10.1(b). Due to bending the underside of each plank will stretch and the topside will shorten. It follows that there must be a relative sliding between the surfaces in contact. If now the planks are glued together they will bend as shown in Fig. 10.2. The glue has prevented the relative sliding of the adjacent surfaces and is therefore subjected to a shear force. This means that the application of a vertical shear load to a beam not only produces internal shear forces on cross sections of the beam but shear forces on horizontal planes as well. In fact, we have noted this earlier in Section 7.3 where we saw that shear stresses applied in one plane induce equal complementary shear stresses on perpendicular planes which is exactly the same situation as in the connected planks. This is important in the design of the connections between, say, a concrete slab and the flange of a steel I-section beam where the connections, usually steel studs, are subjected to this horizontal shear.

Shear stress distributions in beam cross sections depend upon the geometry of the beam section. We shall now determine this distribution for the general case of an unsymmetrical beam section before extending the theory to the simpler case of beam sections having at least one axis of symmetry. This is the reverse of our approach in Chapter 9 for bending but, here, the development of the theory is only marginally more complicated for the general case.

10.1 Shear stress distribution in a beam of unsymmetrical section

Consider an elemental length, δx, of a beam of arbitrary section subjected to internal shear forces S_z and S_y as shown in Fig. 10.3(a). The origin of the axes xyz coincides with the centroid G of the beam section. Let us suppose that the lines of action of S_z and S_y are such that no twisting of the beam occurs (see Section 10.4). The shear stresses induced are therefore due solely to shearing action and are not contributed to by torsion.

Imagine now that a 'slice' of width b_0 is taken through the length of the element. Let τ be the average shear stress along the edge, b_0, of the slice in a direction perpendicular to b_0 and in the plane of the cross section (Fig. 10.3(b)); note that τ is not necessarily the absolute value of shear stress at this position. We saw in Chapter 7 that shear stresses on given planes induce equal, complementary shear stresses on planes perpendicular to the given planes. Thus, τ on the cross-sectional face of the slice induces shear stresses τ on the flat longitudinal face of the slice. In addition, as we saw in Chapter 3, shear loads produce internal bending

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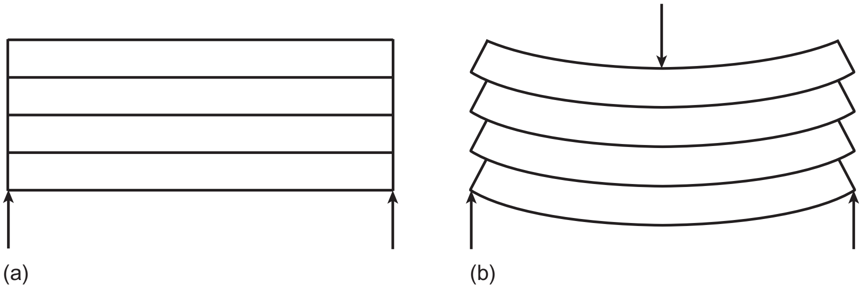

FIGURE 10.1

Bending of unconnected planks.

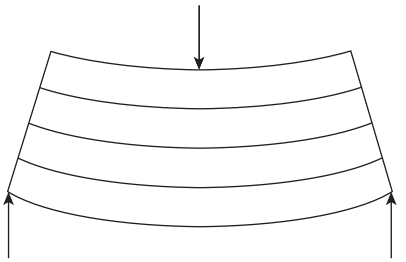

FIGURE 10.2

Bending of connected planks.

moments which, in turn, give rise to direct stresses in beam cross sections. Therefore on any filament, $\delta A'$, of the slice there is a direct stress σ_x at the section x and a direct stress $\sigma_x + (\partial\sigma_x/\partial x)\delta x$ at the section $x+\delta x$ (Fig. 10.3(b)). The slice is therefore in equilibrium in the x direction under the combined action of the direct stress due to bending and the complementary shear stress, τ. Hence

$$\tau b_0 \delta x - \int_{A'} \sigma_x \, \mathrm{d}A' + \int_{A'} \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \delta x \right) \mathrm{d}A' = 0$$

which, when simplified, becomes

$$\tau b_0 = -\int_{A'} \frac{\partial \sigma_x}{\partial x} \mathrm{d}A' \tag{10.1}$$

We shall assume (see Section 9.8) that the direct stresses produced by the bending action of shear loads are given by the theory developed for the pure bending of beams. Therefore, for a beam of unsymmetrical section and for coordinates referred to axes through the centroid of the section

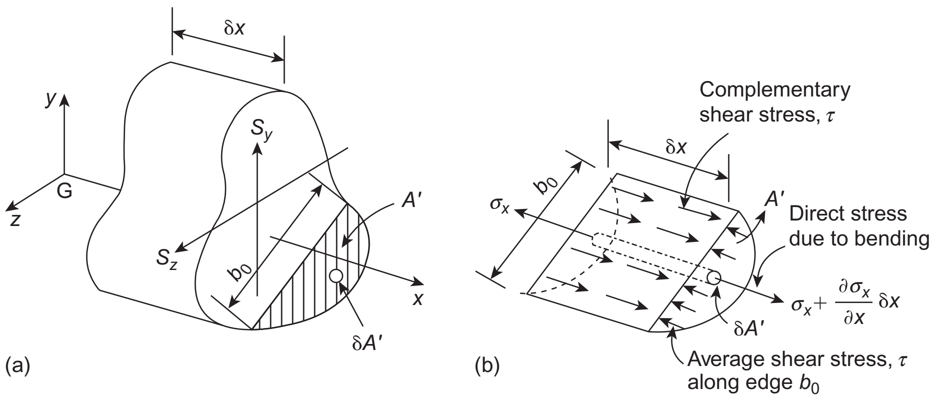

FIGURE 10.3

Determination of shear stress distribution in a beam of arbitrary cross section.

$$\sigma_x = -\left(\frac{M_y I_z - M_z I_{zy}}{I_z I_y - I_{zy}^2}\right) z - \left(\frac{M_z I_y - M_y I_{zy}}{I_z I_y - I_{zy}^2}\right) y \quad (i.\text{e. Eq. } (9.31))$$

Then

$$\frac{\partial \sigma_x}{\partial x} = -\left\{\frac{[(\partial M_y/\partial x)I_z - (\partial M_z/\partial x)I_{zy}]z + [(\partial M_z/\partial x)I_y - (\partial M_y/\partial x)I_{zy}]y}{I_z I_y - I_{zy}^2}\right\}$$

From Eqs (9.57) and (9.58)

$$\frac{\partial M_y}{\partial x} = -S_z \quad \frac{\partial M_z}{\partial x} = -S_y$$

so that

$$\frac{\partial \sigma_x}{\partial x} = -\left\{\frac{(-S_z I_z + S_y I_{zy})z + (-S_y I_y + S_z I_{zy})y}{I_z I_y - I_{zy}^2}\right\}$$

Substituting for $\partial\sigma_x/\partial x$ in Eq. (10.1) we obtain

$$\tau b_0 = \frac{S_y I_{zy} - S_z I_z}{I_z I_y - I_{zy}^2}\int_{A'} z\,\mathrm{d}A' + \frac{S_z I_{zy} - S_y I_y}{I_z I_y - I_{zy}^2}\int_{A'} y\,\mathrm{d}A'$$

or

$$\tau = \frac{S_y I_{zy} - S_z I_z}{b_0(I_z I_y - I_{zy}^2)}\int_{A'} z\,\mathrm{d}A' + \frac{S_z I_{zy} - S_y I_y}{b_0(I_z I_y - I_{zy}^2)}\int_{A'} y\,\mathrm{d}A' \tag{10.2}$$

The slice may be taken so that the average shear stress in any chosen direction can be determined.

10.2 Shear stress distribution in symmetrical sections

Generally in civil engineering we are not concerned with shear stresses in unsymmetrical sections except where they are of the thin-walled type (see Sections 10.4 and 10.5). 'Thick' beam sections usually possess at least one axis of symmetry and are subjected to shear loads in that direction.

Suppose that the beam section shown in Fig. 10.4 is subjected to a single shear load S_y. Since the y axis is an axis of symmetry, it follows that $I_{zy} = 0$ (Section 9.6). Therefore Eq. 10.2 reduces to

$$\tau = -\frac{S_y}{b_0 I_z}\int_{A'} y\,\mathrm{d}A' \tag{10.3}$$

The negative sign arises because the average shear stress τ along the base b_0 of the slice A' is directed towards b_0 from *within the slice* as shown in Fig. 10.3(b). Taking the slice above Gz, as in Fig. 10.4, means that τ is now directed downwards. Clearly a positive shear force S_y produces shear stresses in the positive y direction, hence the negative sign.

Clearly the important shear stresses in the beam section of Fig. 10.4 are in the direction of the load. To find the distribution of this shear stress throughout the depth of the beam we therefore take the slice, b_0, in a direction parallel to and at any distance y from the z axis. The integral term in Eq. (10.3) represents, mathematically, the first moment of the shaded area A' about the z axis. We may therefore rewrite Eq. (10.3) as

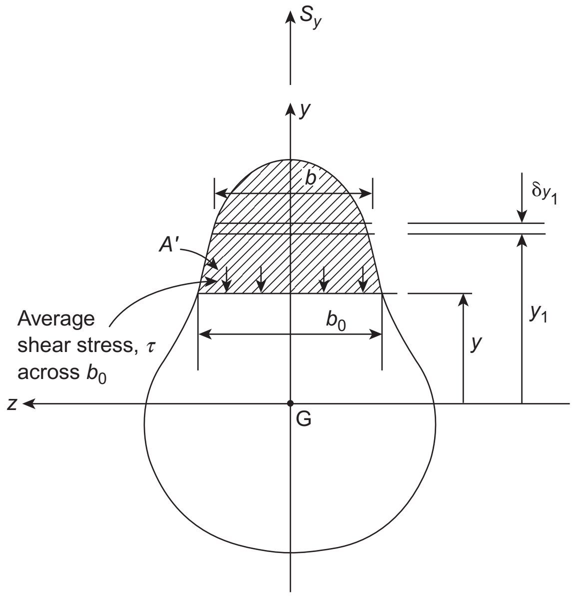

FIGURE 10.4

Shear stress distribution in a symmetrical section beam.

$$\tau = -\frac{S_y A' \bar{y}}{b_0 I_z} \tag{10.4}$$

where $\bar{y}$ is the distance of the centroid of the area A' from the z axis. Alternatively, if the value of $\bar{y}$ is not easily determined, say by inspection, then $\int_{A'} y \mathrm{d}A'$ may be found by calculating the first moment of area about the z axis of an elemental strip of length b, width δy_1 (Fig. 10.4), and integrating over the area A'. Equation (10.3) then becomes

$$\tau = -\frac{S_y}{b_0 I_z} \int_y^{y\,\max} b y_1 \,\mathrm{d}y_1 \tag{10.5}$$

Either of Eqs. (10.4) or (10.5) may be used to determine the distribution of vertical shear stress in a beam section possessing at least a horizontal or vertical axis of symmetry and subjected to a vertical shear load. The corresponding expressions for the horizontal shear stress due to a horizontal load are, by direct comparison with Eqs (10.4) and (10.5)

$$\tau = -\frac{S_z A' \bar{z}}{b_0 I_y} \quad \tau = -\frac{S_z}{b_0 I_y} \int_z^{z\,\max} b z_1 \,\mathrm{d}z_1 \tag{10.6}$$

in which b_0 is the length of the edge of a vertical slice.

EXAMPLE 10.1

Determine the distribution of vertical shear stress in the beam section shown in Fig. 10.5(a) due to a vertical shear load S_y.

In this example the value of $\bar{y}$ for the slice A' is found easily by inspection so that we may use Eq. (10.4). From Fig. 10.5(a) we see that

$$b_0 = b \quad I_z = \frac{bd^3}{12} \quad A' = b\left(\frac{d}{2} - y\right) \quad \bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence

$$\tau = -\frac{12 S_y}{b^2 d^3} b \left(\frac{d}{2} - y\right) \frac{1}{2}\left(\frac{d}{2} + y\right)$$

which simplifies to

$$\tau = -\frac{6 S_y}{bd^3}\left(\frac{d^2}{4} - y^2\right) \tag{10.7}$$

The distribution of vertical shear stress is therefore parabolic as shown in Fig. 10.5(b) and varies from $\tau = 0$ at $y = \pm d/2$ to $\tau = \tau_{max} = 3S_y/2bd$ at the neutral axis ($y = 0$) of the beam section. Note that $\tau_{\max} = 1.5\tau_{\mathrm{av}}$, where τ_{av}, the average vertical shear stress over the section, is given by $\tau_{\mathrm{av}} = S_y/bd$.

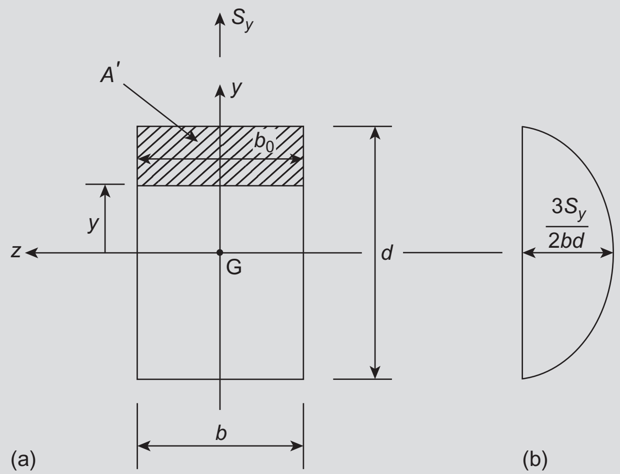

FIGURE 10.5

Shear stress distribution in a rectangular section beam.

EXAMPLE 10.2

Determine the distribution of vertical shear stress in the I-section beam of Fig. 10.6(a) produced by a vertical shear load, S_y.

It is clear from Fig. 10.6(a) that the geometry of each of the areas A'_f and A'_w formed by taking a slice of the beam in the flange (at $y = y_\mathrm{f}$) and in the web (at $y = y_\mathrm{w}$), respectively, are different and will therefore lead to different distributions of shear stress. First we shall consider the flange. The area A'_f is rectangular so that the distribution of vertical shear stress, τ_f, in the flange is, by direct comparison with Ex. 10.1

$$\tau_\mathrm{f} = -\frac{S_y}{BI_z}\frac{B}{2}\left(\frac{D}{2} - y_\mathrm{f}\right)\left(\frac{D}{2} + y_\mathrm{f}\right)$$

or

$$\tau_\mathrm{f} = -\frac{S_y}{2I_z}\left(\frac{D^2}{4} - y_\mathrm{f}^2\right) \tag{10.8}$$

where I_z is the second moment of area of the complete section about the centroidal axis Gz and is obtained by the methods of Section 9.6.

A difficulty arises in the interpretation of Eq. (10.8) which indicates a parabolic distribution of vertical shear stress in the flanges increasing from $\tau_\mathrm{f} = 0$ at $y_\mathrm{f} = \pm D/2$ to a value

$$\tau_\mathrm{f} = -\frac{S_y}{8I_z}(D^2 - d^2) \tag{10.9}$$

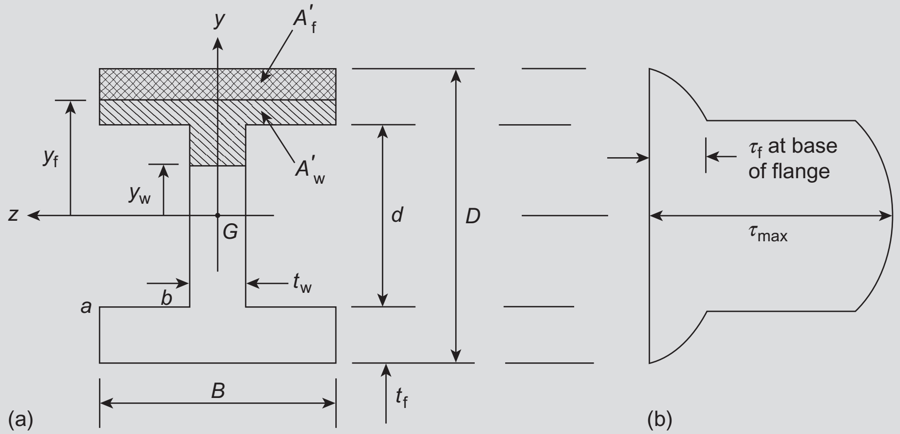

FIGURE 10.6

Shear stress distribution in an I-section beam.

at $y_f = \pm d/2$. However, the shear stress must also be zero at the inner surfaces ab, etc., of the flanges. Equation (10.8) therefore may only be taken to give an indication of the vertical shear stress distribution in the flanges *in the vicinity of the web*. Clearly if the flanges are thin so that d is close in value to D then τ_f *in the flanges* at the extremities of the web is small, as indicated in Fig. 10.6(b).

The area A'_w formed by taking a slice in the web at $y = y_w$ comprises two rectangles which may therefore be treated separately in determining $A'\bar{y}$ for the web.

Thus

$$\tau_w = -\frac{S_y}{t_w I_z}\left[B\left(\frac{D}{2}-\frac{d}{2}\right)\frac{1}{2}\left(\frac{D}{2}+\frac{d}{2}\right)+t_w\left(\frac{d}{2}-y_w\right)\frac{1}{2}\left(\frac{d}{2}+y_w\right)\right]$$

which simplifes to

$$\tau_w = -\frac{S_y}{t_w I_z}\left[\frac{B}{8}(D^2-d^2)+\frac{t_w}{2}\left(\frac{d^2}{4}-y_w^2\right)\right] \tag{10.10}$$

or

$$\tau_w = -\frac{S_y}{I_z}\left[\frac{B}{8t_w}(D^2-d^2)+\frac{1}{2}\left(\frac{d^2}{4}-y_w^2\right)\right] \tag{10.11}$$

Again the distribution is parabolic and increases from

$$\tau_w = -\frac{S_y}{I_z}\frac{B}{8t_w}(D^2-d^2) \tag{10.12}$$

at $y_w = \pm d/2$ to a maximum value, $\tau_{w,\max}$, given by

$$\tau_{w,\max} = -\frac{S_y}{I_z}\left[\frac{B}{8t_w}(D^2-d^2)+\frac{d^2}{8}\right] \tag{10.13}$$

at $y_w = 0$. Note that the value of τ_w at the extremities of the web (Eq. (10.12)) is greater than the corresponding values of τ_f by a factor B/t_w. The complete distribution is shown in Fig. 10.6(b). Note also that the negative sign indicates that τ is vertically upwards.

The value of $\tau_{w,max}$ (Eq. (10.13)) is not very much greater than that of τ_w at the extremities of the web. In design checks on shear stress values in I-section beams it is usual to assume that the maximum shear stress in the web is equal to the shear load divided by the web area. In most cases the result is only slightly different from the value given by Eq. (10.13). A typical value given in Codes of Practice for the maximum allowable value of shear stress in the web of an I-section, mild steel beam is 100 N/mm^2; this is applicable to sections having web thicknesses not exceeding 40 mm.

We have been concerned so far in this example with the distribution of vertical shear stress. We now consider the situation that arises if we take the slice across one of the flanges at $z = z_f$ as shown in Fig. 10.7(a). Equations (10.4) and (10.5) still apply, but in this case $b_0 = t_f$. Thus, using Eq. (10.4)

$$\tau_{f(h)} = -\frac{S_y}{t_f I_z} t_f \left(\frac{B}{2} - z_f\right) \frac{1}{2} \left(\frac{D}{2} + \frac{d}{2}\right)$$

where $\tau_{f(h)}$ is the distribution of horizontal shear stress in the flange. Simplifying the above equation we obtain

$$\tau_{f(h)} = -\frac{S_y(D+d)}{4I_z}\left(\frac{B}{2} - z_f\right) \tag{10.14}$$

Equation (10.14) shows that the horizontal shear stress varies linearly in the flanges from zero at $z_f = B/2$ to $-S_y(D+d)B/8I_z$ at $z_f = 0$.

We have defined a positive shear stress as being directed towards the edge b_0 of the slice away from the interior of the slice, Fig. 10.3(b). Since Eq. (10.14) is always negative for the upper flange, $\tau_{f(h)}$ in the upper flange is directed towards the edges of the flange. By a similar argument $\tau_{f(h)}$ in the lower flange is directed away

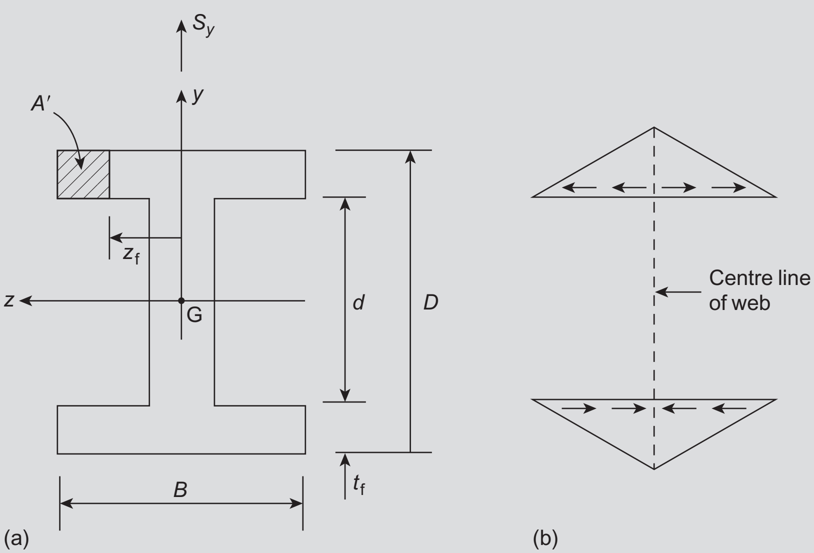

FIGURE 10.7

Distribution of horizontal shear stress in the flanges of an I-section beam.

from the edges of the flange because y for a slice in the lower flange is negative making Eq. (10.14) always positive. The distribution of horizontal shear stress in the flanges of the beam is shown in Fig. 10.7(b).

From Eq. (10.12) we see that the numerical value of shear stress at the extremities of the web multiplied by the web thickness is

$$\tau_w t_w = \frac{S_y}{I_z}\frac{B}{8}(D+d)(D-d) = \frac{S_y}{I_z}\frac{B}{8}(D+d)2t_f \tag{10.15}$$

The product of horizontal flange stress and flange thickness at the extremities of the web is, from Eq. (10.14)

$$\tau_{f(h)} t_f = \frac{S_y}{I_z}\frac{B}{8}(D+d)\, t_f \tag{10.16}$$

Comparing Eqs (10.15) and (10.16) we see that

$$\tau_w t_w = 2\tau_{f(h)}\, t_f \tag{10.17}$$

The product *stress* × *thickness* gives the *shear force per unit length* in the walls of the section and is known as the *shear flow*, a particularly useful parameter when considering thin-walled sections. In the above example we note that $\tau_{f(h)}t_f$ is the shear flow at the extremities of the web produced by considering one half of the complete flange. From symmetry there is an equal shear flow at the extremities of the web from the other half of the flange. Equation (10.17) therefore expresses the equilibrium of the shear flows at the web/flange junctions. We shall return to a more detailed consideration of shear flow when investigating the shear of thin-walled sections.

In 'thick' I-section beams the horizontal flange shear stress is not of great importance since, as can be seen from Eq. (10.17), it is of the order of half the magnitude of the vertical shear stress at the extremities of the web if $t_w \simeq t_f$. In thin-walled I-sections (and other sections too) this horizontal shear stress can produce shear distortions of sufficient magnitude to redistribute the direct stresses due to bending, thereby seriously affecting the accuracy of the basic bending theory described in Chapter 9. This phenomenon is known as *shear lag*.

EXAMPLE 10.3

A steel beam has the cross section shown in Fig. 10.8(a) and carries a load of W kN in its vertical plane of symmetry. Calculate and sketch the distribution of shear stress in the cross section of the beam and hence determine the maximum allowable value of W if the shear stress in the beam is limited to 100 N/mm^2.

The second moment of area of the beam section about the z axis is given by (see Section 9.6)

$$I_z = \frac{120 \times 240^3}{12} - \frac{105 \times 200^3}{12} = 6.8 \times 10^7 \text{ mm}^4$$

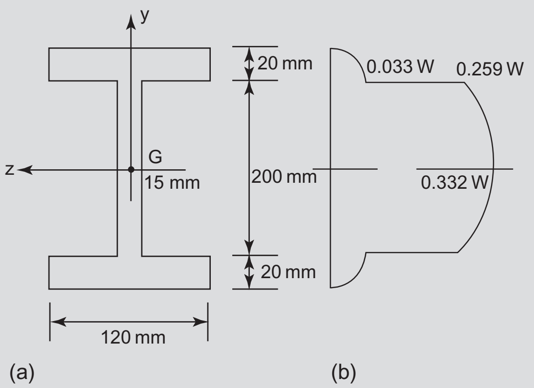

FIGURE 10.8

Shear stress distribution in beam section of Ex. 10.3.

Then, from Eq. (10.8), the shear stress distribution in the flange is

$$\tau_f = -\frac{W \times 10^3}{2 \times 6.8 \times 10^7}\left(\frac{240^2}{4} - y_f^2\right)$$

so that

$$\tau_f = -7.4 \times 10^{-6}\, W(120^2 - y_f^2)\,\mathrm{N/mm^2} \quad \text{(i)}$$

Then, when $y_f = 120$ mm, $\tau_f = 0$ and when $y_f = 100$ mm, $\tau_f = -0.033W$ N/mm^2.

The shear stress distribution in the web is obtained directly from Eq. (10.11) and is

$$\tau_w = -\frac{W \times 10^3}{6.8 \times 10^7}\left[\frac{120}{8 \times 15}(240^2 - 200^2) + \frac{1}{2}\left(\frac{200^2}{4} - y_w^2\right)\right]$$

which gives

$$\tau_w = -9.8 \times 10^{-7}\, W(339000 - 7.5 y_w^2) \quad \text{(ii)}$$

From Eq. (ii), when $y_w = 100$ mm, $\tau_w = -0.259\, W$ N/mm^2 and when $y_w = 0$, $\tau_w = -0.332W$ N/mm$^2 = \tau_{max}$. The complete distribution is shown in Fig. 10.8(b).

For a limiting value of shear stress of 100 N/mm^2,

$$100 = 0.332W$$

so that

$$W = 301\,\mathrm{kN}.$$

In this example we have applied Eqs (10.8) and (10.11) directly. However, rather than attempt to commit these lengthy formulae to memory it is generally better to work from first principles.

EXAMPLE 10.4

The beam whose cross section is shown in Fig. 10.9(a) is subjected to a shear load of 15 kN applied in its vertical plane of symmetry. Calculate and sketch the distribution of shear stress in the cross section of the beam inserting all principal values.

Initially we need to find the position of the centroid of area, G, which will lie on the vertical axis of symmetry of the beam section. Taking moments of area about the base of the flange

$$(50 \times 20 + 60 \times 10)\,\bar{y} = 50 \times 20 \times 10 + 60 \times 10 \times 50$$

which gives

$$\bar{y} = 25\,\mathrm{mm}$$

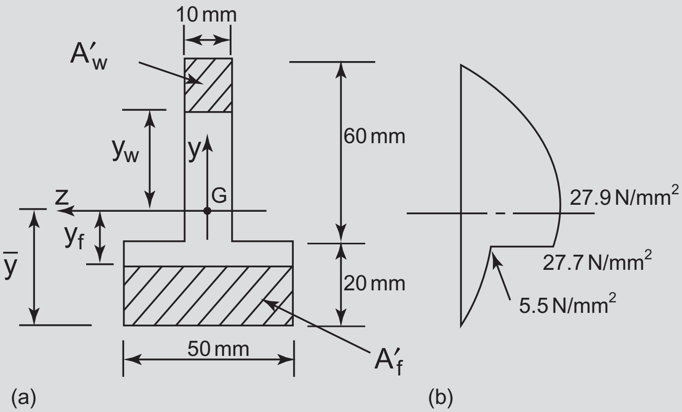

FIGURE 10.9

Shear stress distribution in beam of Ex. 10.4.

The second moment of area of the beam section about Gz is then

$$I_z = \frac{50 \times 20^3}{12} + 50 \times 20 \times 15^2 + \frac{10 \times 60^3}{12} + 10 \times 60 \times 25^2$$

so that

$$I_z = 8.13 \times 10^5 \text{ mm}^4$$

We now take a slice $A_w{}'$ in the web as shown in Fig. 10.9(a). Then, from Eq. (10.4)

$$\tau_w = -\frac{15 \times 10^3 \times 10}{10 \times 8.13 \times 10^5}(55 - y_w)\frac{1}{2}(55 + y_w)$$

from which

$$\tau_w = -9.23 \times 10^{-3}(55^2 - y_w{}^2) \tag{i}$$

From Eq. (i), when $y_w = 55$ mm, $\tau_w = 0$ and when $y_w = -5$ mm, $\tau_w = -27.7$ N/mm^2. The maximum value of shear stress occurs when $y_w = 0$ and is -27.9 N/mm^2.

To find the shear stress distribution in the flange of the beam section we could consider the inverted T-shape above the slice. However, a simpler solution is obtained by considering the part of the flange below the slice as shown in Fig. 10.9(a). Then, from Eq. (10.4)

$$\tau_f = -\frac{15 \times 10^3 \times 50}{50 \times 8.13 \times 10^5}(25 - y_f)\frac{1}{2}(25 + y_f)$$

which simplifies to

$$\tau_f = -9.23 \times 10^{-3}(25^2 - y_f^2) \tag{ii}$$

From Eq. (ii), when $y_f = -25$ mm, $\tau_f = 0$ and when $y_f = -5$ mm, $\tau_f = -5.54$ N/mm^2. The complete distribution is shown in Fig. 10.9(b).

If the inverted T-shape above the slice in the flange had been considered as $A_f{}'$ then

$$A_f{}'\,\bar{y}_f = 60 \times 10 \times 25 - 50(y_f - 5)(1/2)(y_f + 5)$$

so that

$$A_f' \, \bar{y}_f = 15000 - 25(y_f^{\,2} - 5)$$

Note that the contribution to $A_f' \, \bar{y}_f$ from the leg of the T- and from the flange have opposite signs due to the fact that their centroids lie on opposite sides of the Gz axis.

EXAMPLE 10.5

Determine the distribution of vertical shear stress in a beam of circular cross section when it is subjected to a shear force S_y (Fig. 10.10).

The area A' of the slice in this problem is a segment of a circle and therefore does not lend itself to the simple treatment of the previous two examples. We shall therefore use Eq. (10.5) to determine the distribution of vertical shear stress. Thus

$$\tau = -\frac{S_y}{b_0 I_z} \int_y^{D/2} b y_1 \, \mathrm{d}y_1 \tag{10.18}$$

where

$$I_z = \frac{\pi D^4}{64} \, (\text{Eq. } (9.40))$$

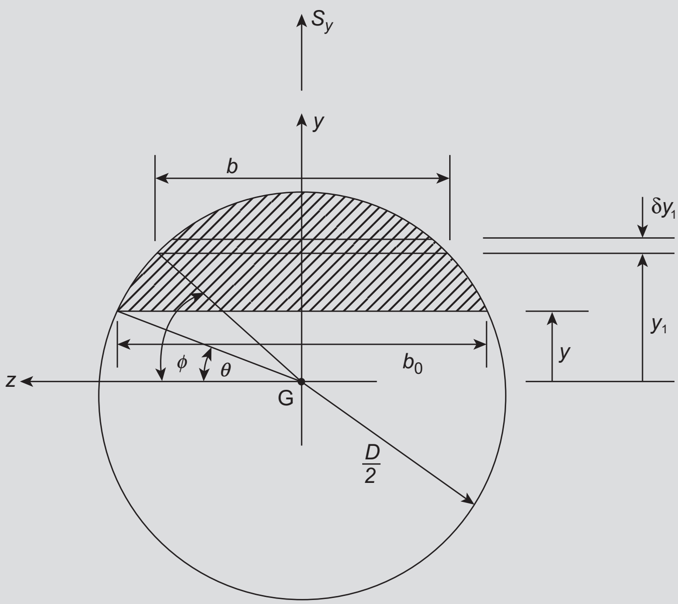

FIGURE 10.10

Distribution of shear stress in a beam of circular cross section.

Integration of Eq. (10.18) is simplified if angular variables are used; thus, from Fig. 10.10

$$b_0 = 2 \times \frac{D}{2}\cos\theta \quad b = 2 \times \frac{D}{2}\cos\phi \quad y_1 = \frac{D}{2}\sin\phi \quad \mathrm{d}y_1 = \frac{D}{2}\cos\phi\,\mathrm{d}\phi$$

Equation (10.18) then becomes

$$\tau = -\frac{16S_y}{\pi D^2\cos\theta}\int_{\theta}^{\pi/2}\cos^2\phi\sin\phi\,\mathrm{d}\phi$$

Integrating we obtain

$$\tau = -\frac{16S_y}{\pi D^2\cos\theta}\left[-\frac{\cos^3\phi}{3}\right]_{\theta}^{\pi/2}$$

which gives

$$\tau = -\frac{16S_y}{3\pi D^2}\cos^2\theta$$

But

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{y}{D/2}\right)^2$$

Therefore

$$\tau = -\frac{16S_y}{3\pi D^2}\left(1 - \frac{4y^2}{D^2}\right) \tag{10.19}$$

The distribution of shear stress is parabolic with values of $\tau = 0$ at $y = \pm D/2$ and $\tau = \tau_{\max} = -16S_y/3\pi D^2$ at $y = 0$, the neutral axis of the section.

10.3 Strain energy due to shear

Consider a small rectangular element of material of side δx, δy and thickness t subjected to a shear stress and complementary shear stress system, τ (Fig. 10.11(a)); τ produces a shear strain γ in the element so that distortion occurs as shown in Fig. 10.11(b), where displacements are relative to the side CD. The horizontal displacement of the side AB is $\gamma\,\delta y$ so that the shear force on the face AB moves through this distance and therefore does work. If the shear loads producing the shear stress are gradually applied, then the work done by the shear force on the element and hence the strain energy stored, δU, is given by

$$\delta U = \frac{1}{2}\tau t\,\delta x \gamma\,\delta y$$

or

$$\delta U = \frac{1}{2}\tau\gamma t\,\delta x\,\delta y$$

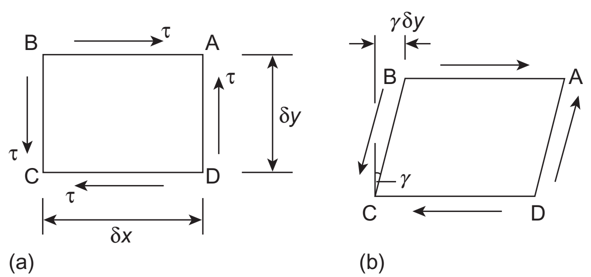

FIGURE 10.11

Determination of strain energy due to shear.

Now $\gamma = \tau/G$, where G is the shear modulus and $t\ \delta x\ \delta y$ is the volume of the element. Hence

$$\delta U = \frac{1}{2}\frac{\tau^2}{G} \times \text{volume of element}$$

The total strain energy, U, due to shear in a structural member in which the shear stress, τ, is uniform is then given by

$$U = \frac{\tau^2}{2G} \times \text{volume of member} \tag{10.20}$$

We shall use this expression for the strain energy due to shear in the determination of beam deflections due to shear in Chapter 13.

10.4 Shear stress distribution in thin-walled open section beams

In considering the shear stress distribution in thin-walled open section beams we shall make identical assumptions regarding the calculation of section properties as were made in Section 9.6. In addition we shall assume that shear stresses in the plane of the cross section and parallel to the tangent at any point on the beam wall are constant across the thickness (Fig. 10.12(a)), whereas shear stresses normal to the tangent are negligible (Fig. 10.12(b)). The validity of the latter assumption is evident when it is realized that these normal shear stresses must be zero on the inner and outer surfaces of the section and that the walls are thin. We shall further assume that the wall thickness can vary round the section but is constant along the length of the member.

Figure 10.13 shows a length of a thin-walled beam of arbitrary section subjected to shear loads S_z and S_y which are applied such that no twisting of the beam occurs. In addition to shear stresses, direct stresses due to the bending action of the shear loads are present so that an element $\delta s \times \delta x$ of the beam wall is in equilibrium under the stress system shown in Fig. 10.14(a). The shear stress τ is assumed to be positive in the positive

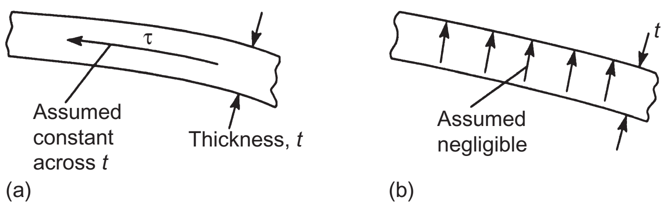

FIGURE 10.12

Assumptions in thin-walled open section beams.

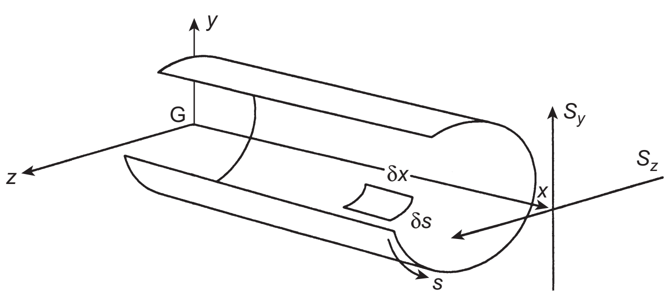

FIGURE 10.13

Shear of a thin-walled open section beam.

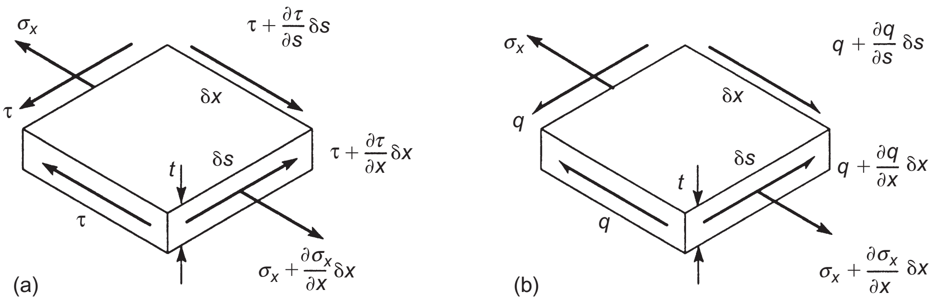

FIGURE 10.14

Equilibrium of beam element.

direction of s, the distance round the profile of the section measured from an open edge. Although we have specified that the thickness t may vary with s, this variation is small for most thin-walled sections so that we may reasonably make the approximation that t is constant over the length δs. As stated in Ex. 10.2 it is convenient, when considering thin-walled sections, to work in terms of shear flow to which we assign the symbol q ($=\tau t$). Figure 10.14(b) shows the shear stress system of Fig. 10.14(a) represented in terms of q. Thus for equilibrium of the element in the x direction

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\delta x\right) t\,\delta s - \sigma_x t\,\delta s + \left(q + \frac{\partial q}{\partial s}\delta s\right)\delta x - q\,\delta x = 0$$

which gives

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_x}{\partial x} = 0 \tag{10.21}$$

Again we assume that the direct stresses are given by Eq. (9.31). Then, substituting in Eq. (10.21) for $\partial\sigma_x/\partial x$ from the derivation of Eq. (10.2)

$$\frac{\partial q}{\partial s} = \frac{(S_y I_{zy} - S_z I_z)}{I_z I_y - I_{zy}^2} tz + \frac{(S_z I_{zy} - S_y I_y)}{I_z I_y - I_{zy}^2} ty$$

Integrating this expression from $s=0$ (where $q=0$ on the open edge of the section) to any point s we have

$$q_s = \frac{(S_y I_{zy} - S_z I_z)}{I_z I_y - I_{zy}^2}\int_0^s tz\,\mathrm{d}s + \frac{(S_z I_{zy} - S_y I_y)}{I_z I_y - I_{zy}^2}\int_0^s ty\,\mathrm{d}s \tag{10.22}$$

The shear stress at any point in the beam section wall is then obtained by dividing q_s by the wall thickness at that point, i.e.

$$\tau_s = \frac{q_s}{t_s} \tag{10.23}$$

EXAMPLE 10.6

Determine the shear flow distribution in the thin-walled Z-section beam shown in Fig. 10.15 produced by a shear load S_y applied in the plane of the web.

The origin for our system of reference axes coincides with the centroid of the section at the mid-point of the web. The centroid is also the centre of antisymmetry of the section so that the shear load, applied through this point, causes no twisting of the section and the shear flow distribution is given by Eq. (10.22) in which $S_z = 0$, i.e

$$q_s = \frac{S_y I_{zy}}{I_z I_y - I_{zy}^2} \int_0^s tz \, ds - \frac{S_y I_y}{I_z I_y - I_{zy}^2} \int_0^s ty \, ds \tag{i}$$

The second moments of area of the section about the z and y axes have previously been calculated in Ex. 9.17 and are

$$I_z = \frac{h^3 t}{3} \quad I_y = \frac{h^3 t}{12} \quad I_{zy} = \frac{h^3 t}{8}$$

Substituting these values in Eq. (i) we obtain

$$q_s = \frac{S_y}{h^3} \int_0^s (10.29z - 6.86y) ds$$

On the upper flange AB, $y = +h/2$ and $z = h/2 - s_A$ where $0 \le s_A \le h/2$. Therefore

$$q_{AB} = \frac{S_y}{h^3} \int_0^{s_A} (1.72\,h - 10.29 s_A) ds_A$$

which gives

$$q_{AB} = \frac{S_y}{h^3} (1.72 h s_A - 5.15 s_A^2) \tag{ii}$$

Thus at A($s_A = 0$), $q_A = 0$ and at B ($s_A = h/2$), $q_B = -0.43\,S_y/h$. Note that the order of the suffixes of q in Eq. (ii) denotes the positive direction of q (and s_A). An examination of Eq. (ii) shows that the shear flow distribution on the upper flange is parabolic with a change of sign (i.e. direction) at $s_A = 0.33\,h$. For values of $s_A < 0.33\,h$, q_{AB} is positive and is therefore in the same direction as s_A. Furthermore, q_{AB} has a turning value between $s_A = 0$ and $s_A = 0.33\,h$ at a value of s_A given by

$$\frac{dq_{AB}}{ds_A} = 1.72\,h - 10.29 s_A = 0$$

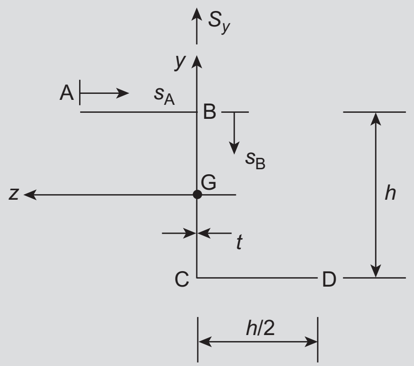

FIGURE 10.15

Beam section of Ex. 10.6.

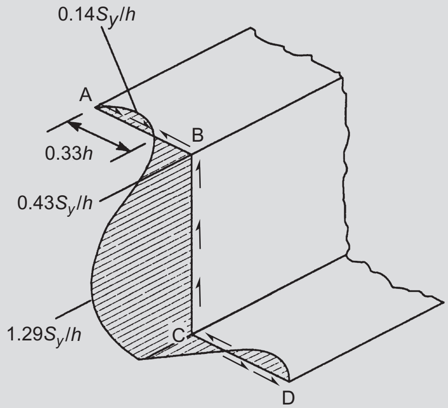

FIGURE 10.16

Shear flow distribution in beam section of Ex. 10.6.

i.e. at $s_A = 0.17\ h$. The corresponding value of q_{AB} is then, from Eq. (ii), $q_{AB} = 0.14S_y/h$.

In the web BC, $y = +h/2 - s_B$ where $0 \le s_B \le h$ and $z = 0$. Thus

$$q_{BC} = \frac{S_y}{h^3}\int_0^{s_B}(6.86s_B - 3.43\ h)\mathrm{d}s_B + q_B \tag{iii}$$

Note that in Eq. (iii), q_{BC} is not zero when $s_B = 0$ but equal to the value obtained by inserting $s_A = h/2$ in Eq. (ii), i.e. $q_B = -0.43\ S_y/h$. Integrating the first two terms on the right-hand side of Eq. (iii) we obtain

$$q_{BC} = \frac{S_y}{h^3}(3.43s_B^2 - 3.43\ hs_B - 0.43\ h^2) \tag{iv}$$

Equation (iv) gives a parabolic shear flow distribution in the web, symmetrical about Gz and with a maximum value at $s_B = h/2$ equal to $-1.29S_y/h$; q_{AB} is negative at all points in the web.

The shear flow distribution in the lower flange may be deduced from antisymmetry; the complete distribution is shown in Fig. 10.16.

Shear centre

We have specified in the previous analysis that the lines of action of the shear loads S_z and S_y must not cause twisting of the section. For this to be the case, S_z and S_y must pass through the *shear centre* of the section. Clearly in many practical situations this is not so and torsion as well as shear is induced. These problems may be simplified by replacing the shear loads by shear loads acting through the shear centre, plus a pure torque, as illustrated in Fig. 10.17 for the simple case of a channel section subjected to a vertical shear load S_y applied in the line of the web. The shear stresses corresponding to the separate loading cases are then added by superposition.

Where a section possesses an axis of symmetry, the shear centre must lie on this axis. For cruciform, T and angle sections of the type shown in Fig. 10.18 the shear centre is located at the intersection of the walls since the resultant internal shear loads all pass through this point. In fact in any beam section in which the walls are straight and intersect at just one point, that point is the shear centre of the section.

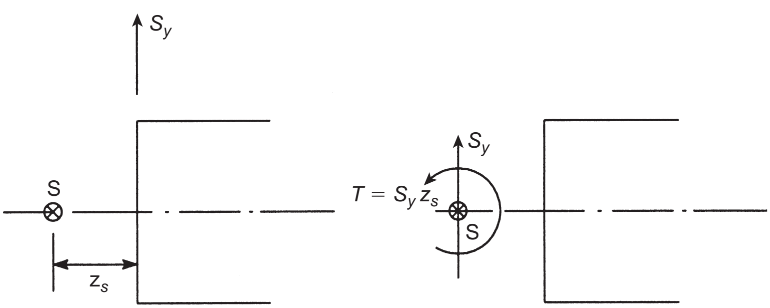

FIGURE 10.17

Replacement of a shear load by a shear load acting through the shear centre plus a torque.

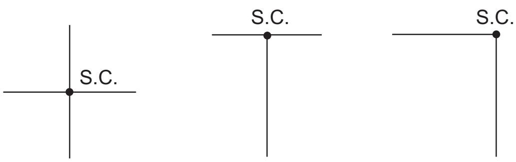

FIGURE 10.18

Special cases of shear centre (S.C.) position.

EXAMPLE 10.7

Determine the position of the shear centre of the thin-walled channel section shown in Fig. 10.19.

The shear centre S lies on the horizontal axis of symmetry at some distance z_S, say, from the web. If an arbitrary shear load, S_y, is applied through the shear centre, then the shear flow distribution is given by Eq. (10.22) and the moment about any point in the cross section produced by these shear flows is *equivalent* to the moment of the applied shear load about the same point; S_y appears on both sides of the resulting equation and may therefore be eliminated to leave z_S as the unknown.

For the channel section, G*z* is an axis of symmetry so that $I_{zy}=0$. Equation (10.22) therefore simplifies to

$$q_s = -\frac{S_y}{I_z}\int_0^s ty\,\mathrm{d}s$$

where

$$I_z = \frac{th^3}{12} + 2bt\left(\frac{h}{2}\right)^2 = \frac{th^3}{12}\left(1+6\frac{b}{h}\right)$$

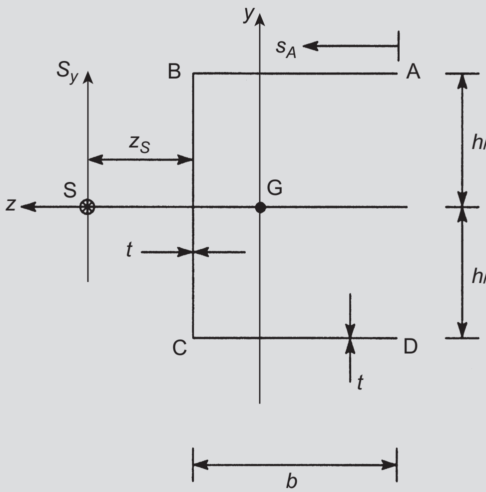

FIGURE 10.19

Channel section beam of Ex. 10.7.

Substituting for I_z and noting that t is constant round the section, we have

$$q_s = -\frac{12S_y}{h^3(1+6b/h)}\int_0^s y\,\mathrm{d}s \qquad \text{(i)}$$

The solution of this type of problem may be reduced in length by giving some thought to what is required. We are asked, in this case, to obtain the position of the shear centre and not a complete shear flow distribution. From symmetry it can be seen that the moments of the resultant shear forces on the upper and lower flanges about the mid-point of the web are numerically equal and act in the same sense. Furthermore, the moment of the web shear about the same point is zero. Therefore it is only necessary to obtain the shear flow distribution on either the upper or lower flange for a solution. Alternatively, the choice of either flange/web junction as the moment centre leads to the same conclusion.

On the upper flange, $y = +h/2$ so that from Eq. (i) we obtain

$$q_{\mathrm{AB}} = -\frac{6S_y}{h^2(1+6b/h)}s_{\mathrm{A}} \qquad \text{(ii)}$$

Equating the anticlockwise moments of the internal shear forces about the mid-point of the web to the clockwise moment of the applied shear load about the same point gives

$$S_y z_{\mathrm{S}} = -\int_0^b q_{\mathrm{AB}}\frac{h}{2}\mathrm{d}s_{\mathrm{A}}$$

Substituting for q_{AB} from Eq. (ii) we have

$$S_y z_{\mathrm{S}} = 2\int_0^b \frac{6S_y}{h^2(1+6b/h)}\frac{h}{2}s_{\mathrm{A}}\,\mathrm{d}s_{\mathrm{A}}$$

from which

$$z_{\mathrm{S}} = \frac{3b^2}{h(1+6b/h)}$$

In the case of an unsymmetrical section, the coordinates $(z_{\mathrm{S}}, y_{\mathrm{S}})$ of the shear centre referred to some convenient point in the cross section are obtained by first determining z_{S} in a similar manner to that described above and then calculating y_{S} by applying a shear load S_z through the shear centre.

10.5 Shear stress distribution in thin-walled closed section beams

The shear flow and shear stress distributions in a closed section, thin-walled beam are determined in a manner similar to that described in Section 10.4 for an open section beam but with two important differences. Firstly, the shear loads may be applied at points in the cross section other than the shear centre so that shear and torsion occur simultaneously. We shall see that a solution may be obtained for this case without separating the shear and torsional effects, although such an approach is an acceptable alternative, particularly if the position of the shear centre is required. Secondly, it is not generally possible to choose an origin for s

that coincides with a known value of shear flow. A closed section beam under shear is therefore singly redundant as far as the internal force system is concerned and requires an equation additional to the equilibrium equation (Eq. (10.21)). Identical assumptions are made regarding section properties, wall thickness and shear stress distribution as were made for the open section beam.

The thin-walled beam of arbitrary closed section shown in Fig. 10.20 is subjected to shear loads S_z and S_y applied through any point in the cross section. These shear loads produce direct and shear stresses on any element in the beam wall identical to those shown in Fig. 10.14. The equilibrium equation (Eq. (10.21)) is therefore applicable and is

$$\frac{\partial q}{\partial s}+t\frac{\partial \sigma_x}{\partial x}=0$$

Substituting for $\partial\sigma_x/\partial x$ from the derivation of Eq. (10.2) and integrating we obtain, in an identical manner to that for an open section beam

$$q_s=\frac{S_yI_{zy}-S_zI_z}{I_zI_y-I_{zy}^2}\int_0^s tz\,\mathrm{d}s+\frac{S_zI_{zy}-S_yI_y}{I_zI_y-I_{zy}^2}\int_0^s ty\,\mathrm{d}s+q_{s,0} \tag{10.24}$$

where $q_{s,0}$ is the value of shear flow at the origin of s.

It is clear from a comparison of Eqs (10.22) and (10.24) that the first two terms of the right-hand side of Eq. (10.24) represent the shear flow distribution in an open section beam with the shear loads applied through its shear centre. We shall denote this 'open section' or 'basic' shear flow distribution by q_{b} and rewrite Eq. (10.24) as

$$q_s=q_{\mathrm{b}}+q_{s,0}$$

We obtain q_{b} by supposing that the closed section beam is 'cut' at some convenient point, thereby producing an 'open section' beam as shown in Fig. 10.21(b); we take the 'cut' as the origin for s. The shear flow distribution round this 'open section' beam is given by Eq. (10.22), i.e.

$$q_{\mathrm{b}}=\frac{S_yI_{zy}-S_zI_z}{I_zI_y-I_{zy}^2}\int_0^s tz\,\mathrm{d}s+\frac{S_zI_{zy}-S_yI_y}{I_zI_y-I_{zy}^2}\int_0^s ty\,\mathrm{d}s$$

Equation (10.22) is valid only if the shear loads produce no twist; in other words, S_z and S_y must be applied through the shear centre of the 'open section' beam. Thus by 'cutting' the closed section beam to determine q_{b} we are, in effect, transferring the lines of action of S_z and S_y to the shear centre, $S_{s,0}$, of the resulting 'open section' beam. The implication is, therefore, that when we 'cut' the section we must

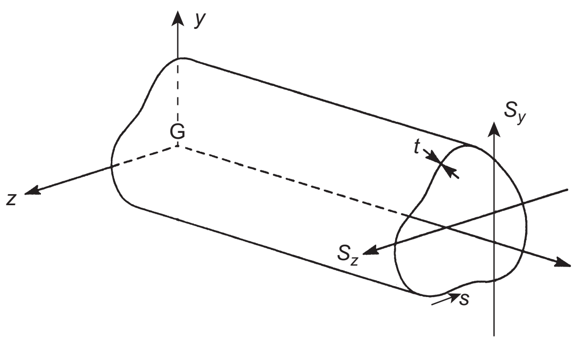

FIGURE 10.20

Shear of a thin-walled closed section beam.

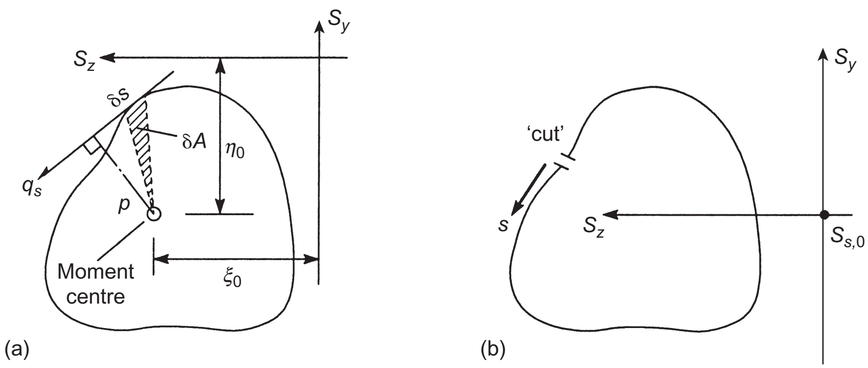

FIGURE 10.21

Determination of shear flow value at the origin for s in a closed section beam.

simultaneously introduce a pure torque to compensate for the transference of S_z and S_y. We shall show in Chapter 11 that the application of a pure torque to a closed section beam results in a constant shear flow round the walls of the beam. In this case $q_{s,0}$, which is effectively a constant shear flow round the section, corresponds to the pure torque produced by the shear load transference. Clearly different positions of the 'cut' will result in different values for $q_{s,0}$ since the corresponding 'open section' beams have different shear centre positions.

Equating internal and external moments in Fig. 10.21(a), we have

$$S_z\eta_0 + S_y\xi_0 = \oint pq_s\,\mathrm{d}s = \oint pq_\mathrm{b}\,\mathrm{d}s + q_{s,0}\oint p\,\mathrm{d}s$$

where $\oint$ denotes integration taken completely round the section. In Fig. 10.21(a) the elemental area δA is given by

$$\delta A = \frac{1}{2}p\delta s$$

Thus

$$\oint p\,\mathrm{d}s = 2\oint \mathrm{d}A$$

or

$$\oint p\,\mathrm{d}s = 2A$$

where A is the area enclosed by the mid-line of the section wall. Hence

$$S_x\eta_0 + S_y\xi_0 = \oint pq_\mathrm{b}\,\mathrm{d}s + 2Aq_{s,0} \tag{10.25}$$

If the moment centre coincides with the lines of action of S_z and S_y then Eq. (10.25) reduces to

$$0 = \oint pq_\mathrm{b}\,\mathrm{d}s + 2Aq_{s,0} \tag{10.26}$$

The unknown shear flow $q_{s,0}$ follows from either of Eqs. (10.25) or (10.26). Note that the signs of the moment contributions of S_z and S_y on the left-hand side of Eq. (10.25) depend upon the position of their lines of action relative to the moment centre. The values given in Eq. (10.25) apply only to Fig. 10.21(a) and could change for different moment centres and/or differently positioned shear loads.

EXAMPLE 10.8

Determine the shear flow distribution in the walls of the thin-walled closed section beam shown in Fig. 10.22, the wall thickness, t, is constant throughout.

Since the z axis is an axis of symmetry, $I_{zy}=0$, and since $S_z=0$, Eq. (10.24) reduces to

$$q_s = -(S_y/I_z)\int_0^s ty\,\mathrm{d}s + q_{s,0}$$

where

$$I_z = (\pi t r^3/2) + 2\times 2rt\times r^2 + [t(2r)^3/12] = 6.24tr^3$$

We now “cut” the beam section at 1. Any point may be chosen for the “cut” but the amount of computation will be reduced if a point is chosen which coincides with the axis of symmetry. Then

$$q_{\mathrm{b},12} = -(S_y/I_z)\int_0^\theta tr\sin\theta\, r\mathrm{d}\theta$$

which gives

$$q_{\mathrm{b},12} = 0.16(S_y/r)[\cos\theta]_0^\theta$$

so that

$$q_{\mathrm{b},12} = 0.16(S_y/r)(\cos\theta - 1) \tag{i}$$

When $\theta=\pi/2$, $q_{\mathrm{b},2}=-0.16(S_y/r)$. The shear flow in the wall 23 is then

$$q_{\mathrm{b},23} = -(S_y/I_z)\int_0^{S_1} tr\,\mathrm{d}s_1 - 0.16(S_y/r)$$

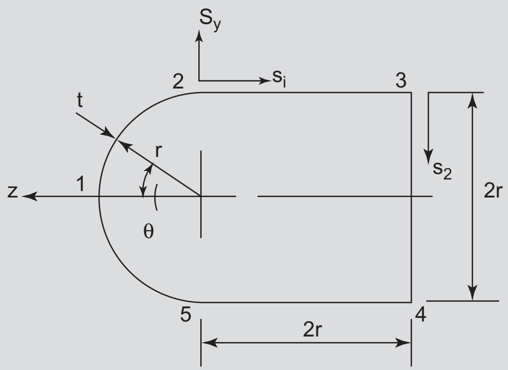

FIGURE 10.22

Beam section of Ex. 10.8.

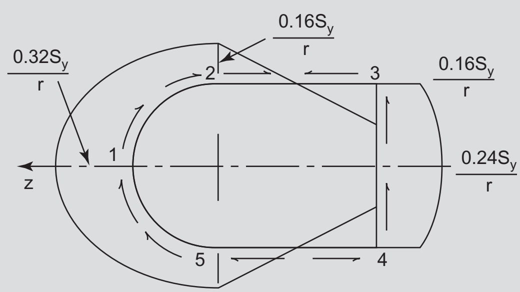

FIGURE 10.23

Shear flow distribution in beam of Ex. 10.8.

so that

$$q_{b,23} = -0.16(S_y/r^2)(s_1 + r) \tag{ii}$$

and when $s_1 = 2r$, $q_{b,3} = -0.48(S_y/r)$. Then, in the wall 34

$$q_{b,34} = -0.16(S_y/tr^3)\int_0^{S_2} t(r - s_2)\,ds_2 - 0.48(S_y/r)$$

so that

$$q_{b,34} = -0.16(S_y/r)(rs_2 - 0.5\,s_2^2 + 3r^2) \tag{iii}$$

The remaining distribution follows from symmetry. Now taking moments about O and using Eq. (10.26)

$$0 = 2\left[\int_0^{\pi/2} q_{b,12}\,r^2\,d\theta + \int_0^{2r} q_{b,23}\,r\,ds_1 + \int_0^{r} q_{b,34}\,2r\,ds_2\right] + 2[4r^2 + (\pi r^2/2)]q_{s,0} \tag{iv}$$

Substituting in Eq. (iv) for $q_{b,12}$ etc from Eqs (i), (ii) and (iii) gives

$$q_{s,0} = 0.32 S_y/r$$

Adding $q_{s,0}$ to the q_b distributions of Eqs (i), (ii) and (iii) gives

$$q_{12} = 0.16(S_y/r^3)(r^2\cos\theta + r^2)$$
$$q_{23} = 0.16(S_y/r^3)(r^2 - rs_1)$$
$$q_{34} = 0.16(S_y/r^3)(0.5s_2^2 - rs_2 - r^2)$$

Note that q_{23} changes sign at $s_1 = r$. The shear flow distribution in the lower half of the section follows from symmetry and the complete distribution is shown in Fig. 10.23.

Shear centre

A complication arises in the determination of the position of the shear centre of a closed section beam since the line of action of the arbitrary shear load (applied through the shear centre as in Ex. 10.7) must be known before $q_{s,0}$ can be determined from either of Eqs. (10.25) or (10.26). However, before the position of the shear centre can be found, $q_{s,0}$ must be obtained. Thus an alternative method of determining $q_{s,0}$ is required. We therefore consider the rate of twist of the beam which, when the shear loads act through the shear centre, is zero.

Consider an element, $\delta s \times \delta x$, of the wall of the beam subjected to a system of shear and complementary shear stresses as shown in Fig. 10.24(a). These shear stresses induce a shear strain, γ, in the element which is given by

$$\gamma = \phi_1 + \phi_2$$

irrespective of whether direct stresses (due to bending action) are present or not. If the linear displacements of the sides of the element in the s and x directions are δv_t (i.e. a tangential displacement) and δw, respectively, then as both δs and δx become infinitely small

$$\gamma = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial x} \tag{10.27}$$

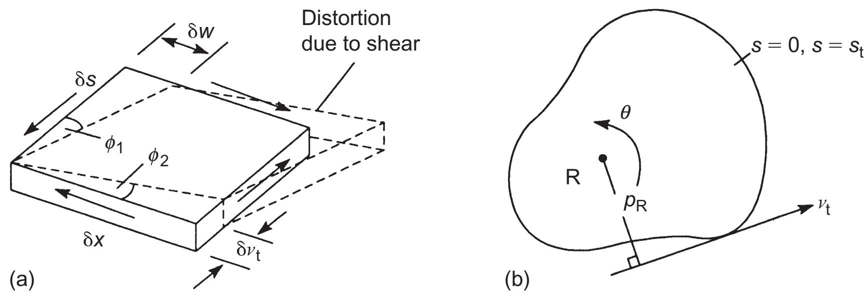

FIGURE 10.24

Rate of twist in a thin-walled closed section beam.

Suppose now that the beam section is given a small angle of twist, θ, about its centre of twist, R. If we assume that the shape of the cross section of the beam is unchanged by this rotation (i.e. it moves as a rigid body), then from Fig. 10.24(b) it can be seen that the tangential displacement, v_t, of a point in the wall of the beam section is given by

$$v_t = p_R \theta$$

Hence

$$\frac{\partial v_t}{\partial x} = p_R \frac{\partial \theta}{\partial x}$$

Since we are assuming that the section rotates as a rigid body, it follows that θ is a function of x only so that the above equation may be written

$$\frac{\partial v_t}{\partial x} = p_R \frac{d\theta}{dx}$$

Substituting for $\partial v_t/\partial x$ in Eq. (10.27) we have

$$\gamma = \frac{\partial w}{\partial s} + p_R \frac{d\theta}{dx}$$

Now

$$\gamma = \frac{\tau}{G} = \frac{q_s}{Gt}$$

Thus

$$\frac{q_s}{Gt} = \frac{\partial w}{\partial s} + p_R \frac{d\theta}{dx}$$

Integrating both sides of this equation completely round the cross section of the beam, i.e. from $s = 0$ to $s = s_t$ (see Fig. 10.24(b))

$$\oint \frac{q_s}{Gt} ds = \oint \frac{\partial w}{\partial s} ds + \frac{d\theta}{dx} \oint p_R \, ds$$

which gives

$$\oint \frac{q_s}{Gt}\mathrm{d}s = [w]_{s=0}^{s=s_t} + \frac{\mathrm{d}\theta}{\mathrm{d}x}2A$$

The axial displacement, w, must have the same value at $s=0$ and $s=s_t$. Therefore the above expression reduces to

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{2A}\oint \frac{q_s}{Gt}\mathrm{d}s \tag{10.28}$$

For shear loads applied through the shear centre, $\mathrm{d}\theta/\mathrm{d}x = 0$ so that

$$0 = \oint \frac{q_s}{Gt}\mathrm{d}s$$

which may be written

$$0 = \oint \frac{1}{Gt}(q_\mathrm{b} + q_{s,0})\mathrm{d}s$$

Hence

$$q_{s,0} = -\frac{\oint (q_\mathrm{b}/Gt)\mathrm{d}s}{\oint \mathrm{d}s/Gt} \tag{10.29}$$

If G is constant then Eq. (10.29) simplifies to

$$q_{s,0} = -\frac{\oint (q_\mathrm{b}/t)\mathrm{d}s}{\oint \mathrm{d}s/t} \tag{10.30}$$

EXAMPLE 10.9

A thin-walled, closed section beam has the singly symmetrical, trapezoidal cross section shown in Fig. 10.25. Calculate the distance of the shear centre from the wall AD. The shear modulus G is constant throughout the section.

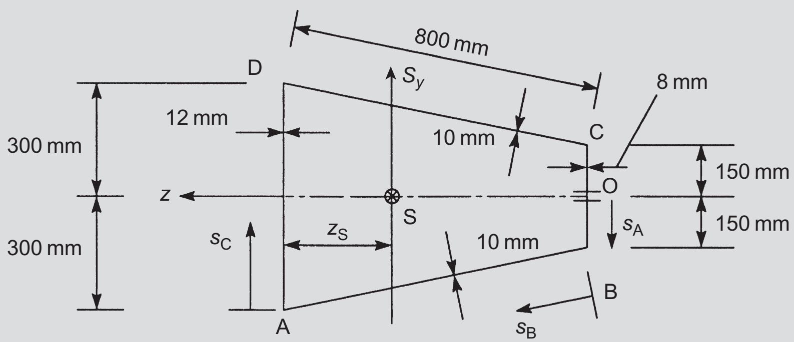

FIGURE 10.25

Closed section beam of Ex. 10.9.

The shear centre lies on the horizontal axis of symmetry so that it is only necessary to apply a shear load S_y through S to determine z_S. Furthermore the axis of symmetry coincides with the centroidal reference axis Gz so that $I_{zy}=0$. Equation (10.24) therefore simplifies to

$$q_s=-\frac{S_y}{I_z}\int_0^s ty\,\mathrm{d}s+q_{s,0} \tag{i}$$

Note that in Eq. (i) only the second moment of area about the z axis and coordinates of points referred to the z axis are required so that it is unnecessary to calculate the position of the centroid on the z axis. It will not, in general, and in this case in particular, coincide with S.

The second moment of area of the section about the z axis is given by

$$I_z=\frac{12\times 600^3}{12}+\frac{8\times 300^3}{12}+2\left[\int_0^{800}10\left(150+\frac{150}{800}s\right)^2\mathrm{d}s\right]$$

from which $I_z=1074\times 10^6\ \mathrm{mm}^4$. Alternatively, the second moment of area of each inclined wall about an axis through its own centroid may be found using the method described in Section 9.6 and then transferred to the z axis by the parallel axes theorem.

We now obtain the q_{b} shear flow distribution by 'cutting' the beam section at the mid-point O of the wall CB. Thus, since $y=-s_{\mathrm{A}}$ we have

$$q_{\mathrm{b,OB}}=\frac{S_y}{I_z}\int_0^{s_A}8s_{\mathrm{A}}\mathrm{d}s_{\mathrm{A}}$$

which gives

$$q_{\mathrm{b,OB}}=\frac{S_y}{I_z}4s_{\mathrm{A}}^2 \tag{ii}$$

Thus

$$q_{\mathrm{b,B}}=\frac{S_y}{I_z}\times 9\times 10^4$$

For the wall BA where $y=-150-150s_{\mathrm{B}}/800$

$$q_{\mathrm{b,BA}}=\frac{S_y}{I_z}\left[\int_0^{s_{\mathrm{B}}}10\left(150+\frac{150}{800}s_{\mathrm{B}}\right)\mathrm{d}s_{\mathrm{B}}+9\times 10^4\right]$$

from which

$$q_{\mathrm{b,BA}}=\frac{S_y}{I_z}\left(1500s_{\mathrm{B}}+\frac{15}{16}s_{\mathrm{B}}^2+9\times 10^4\right) \tag{iii}$$

Then

$$q_{\mathrm{b,A}}=\frac{S_y}{I_z}\times 189\times 10^4$$

In the wall AD, $y = -300 + s_C$ so that

$$q_{b,AD} = \frac{S_y}{I_z}\left[\int_0^{s_C} 12(300 - s_C)\,ds_C + 189 \times 10^4\right]$$

which gives

$$q_{b,AD} = \frac{S_y}{I_z}(3600 s_C - 6 s_C^2 + 189 \times 10^4) \tag{iv}$$

The remainder of the q_b distribution follows from symmetry.

The shear load S_y is applied through the shear centre of the section so that we must use Eq. (10.30) to determine $q_{s,0}$. Now

$$\oint \frac{ds}{t} = \frac{600}{12} + \frac{2 \times 800}{10} + \frac{300}{8} = 247.5$$

Therefore

$$q_{s,0} = -\frac{2}{247.5}\left(\int_0^{150} \frac{q_{b,OB}}{8}\,ds_A + \int_0^{800} \frac{q_{b,BA}}{10}\,ds_B + \int_0^{300} \frac{q_{b,AD}}{12}\,ds_C\right) \tag{v}$$

Substituting for $q_{b,OB}$, $q_{b,BA}$ and $q_{b,AD}$ in Eq. (v) from Eqs (ii), (iii) and (iv), respectively, we obtain

$$q_{s,0} = -\frac{2S_y}{247.5 I_z}\left[\int_0^{150} \frac{s_A^2}{2}\,ds_A + \int_0^{800}\left(150 s_B + \frac{15}{160} s_B^2 + 9 \times 10^3\right) ds_B + \int_0^{300}\left(300 s_C - \frac{1}{2} s_C^2 + \frac{189 \times 10^4}{12}\right) ds_C\right]$$

from which

$$q_{s,0} = -\frac{S_y}{I_z} \times 1.04 \times 10^6$$

Taking moments about the mid-point of the wall AD we have

$$-S_y z_S = 2\left(\int_0^{150} 786 q_{OB}\,ds_A + \int_0^{800} 294 q_{BA}\,ds_B\right) \tag{vi}$$

Noting that $q_{OB} = q_{b,OB} + q_{s,0}$ and $q_{BA} = q_{b,BA} + q_{s,0}$ we rewrite Eq. (vi) as

$$S_y z_S = \frac{2S_y}{I_z}\left[\int_0^{150} 786(+4 ds_A^2 - 1.4 \times 10^6)\,ds_A + \int_0^{800} 294(+1500 s_B + \frac{15}{16} s_B^2 - 0.95 \times 10^6)\,ds_B\right] \tag{vii}$$

Integrating Eq. (vii) and eliminating S_y gives

$$z_S = 282\ \text{mm}.$$

PROBLEMS

P.10.1 A cantilever has the inverted T-section shown in Fig. P.10.1. It carries a vertical shear load of 4 kN in a downward direction. Determine the distribution of vertical shear stress in its cross-section.

Ans. In web: $\tau = 0.004(44^2 - y^2)\text{N/mm}^2$, in flange: $\tau = 0.004(26^2 - y^2)\text{N/mm}^2$

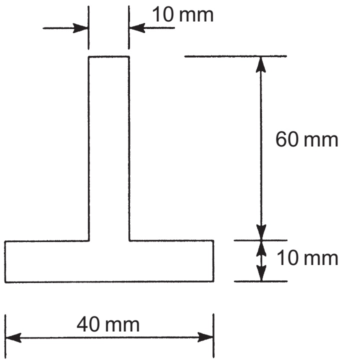

FIGURE P.10.1

P.10.2 An I-section beam having the cross-sectional dimensions shown in Fig. P.10.2 carries a vertical shear load of 80 kN. Calculate and sketch the distribution of vertical shear stress across the beam section and determine the percentage of the total shear load carried by the web.

Ans. τ (base of flanges) $= 1.1$ N/mm^2, τ (ends of web) $= 11.1$ N/mm^2,
τ (neutral axis) $= 15.77$ N/mm^2, 95.9%.

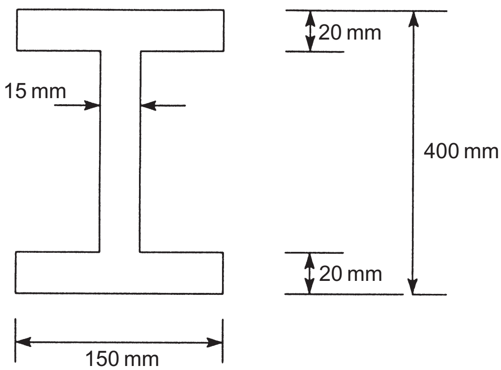

FIGURE P.10.2

P.10.3 A doubly symmetrical I-section beam is reinforced by a flat plate attached to the upper flange as shown in Fig. P.10.3. If the resulting compound beam is subjected to a vertical shear load of 200 kN, determine the distribution of shear stress in the portion of the cross section that extends from the top of the plate to the neutral axis. Calculate also the shear force per unit length of beam resisted by the shear connection between the plate and the flange of the I-section beam.

Ans. τ (top of plate) = 0
τ (bottom of plate) = 0.68 N/mm^2
τ (top of flange) = 1.36 N/mm^2
τ (bottom of flange) = 1.78 N/mm^2
τ (top of web) = 14.22 N/mm^2
τ (neutral axis) = 15.15 N/mm^2
Shear force per unit length = 272 kN/m.

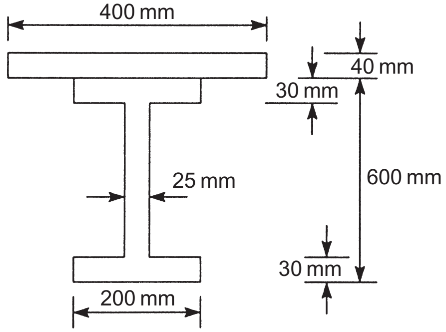

FIGURE P.10.3

P.10.4 A timber beam has a rectangular cross section, 150 mm wide by 300 mm deep, and is simply supported over a span of 4 m. The beam is subjected to a two-point loading at the quarter span points. If the beam fails in shear when the total of the two concentrated loads is 180 kN, determine the maximum shear stress at failure.

Ans. 3 N/mm^2.

P.10.5 A steel box girder is simply supported over a span of 5 m and is to be constructed by bolting flat plates to the flanges of two channel sections as shown in Fig. P.10.5. If the girder is required to

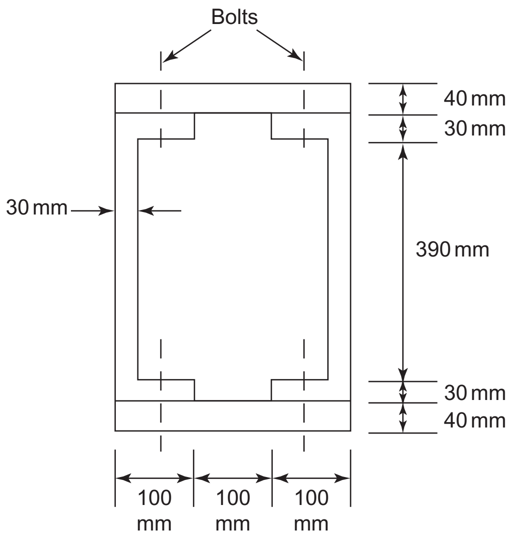

FIGURE P.10.5

support a load of 500 kN at its mid-span calculate the number of bolts required/metre length of the beam. Take the allowable load/bolt as 150 kN and make full allowance for the self-weight of the plates and channel sections; the density of the steel is 77 kN/m^3.

Ans. 1 bolt/metre in each channel section flange.

P.10.6 If the maximum allowable shear stress in the box girder of P.10.5 is 100 N/mm^2 and the maximum allowable direct stress due to bending is 200 N/mm^2 determine the maximum concentrated mid-span load the box girder can carry and state which is the limiting case.

Ans. 1359 kN. Bending is the limiting case.

P.10.7 A beam has the singly symmetrical thin-walled cross section shown in Fig. P.10.7. Each wall of the section is flat and has the same length, a, and thickness, t. Determine the shear flow distribution round the section due to a vertical shear load, S_y, applied through the shear centre and find the distance of the shear centre from the point C.

Ans. $q_{AB} = -3S_y(2as_A - s_A^2/2)/16a^3 \sin\alpha$
$q_{BC} = -3S_y(3/2 + s_B/a - s_B^2/2a^2)/16a \sin\alpha$
S.C. is $5a\cos\alpha/8$ from C.

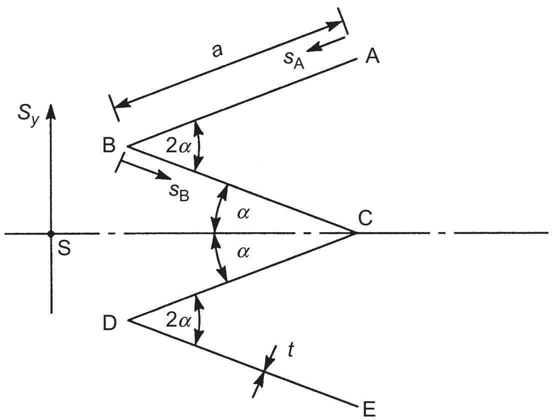

FIGURE P.10.7

P.10.8 Calculate the shear flow distribution in the thin-walled open section shown in Fig. P.10.8 produced by a vertical shear load, S_y, acting through its shear centre.

Ans. $q_\theta = (S_y/\pi r)(\cos\theta - 1)$.

P.10.9 A beam has the singly symmetrical, thin-walled cross section shown in Fig. P.10.9. The thickness t of the walls is constant throughout. Show that the distance of the shear centre from the web is given by

$$z_S = d\frac{\rho^2 \sin\alpha\cos\alpha}{1 + 6\rho + 2\rho^3\sin^2\alpha} \quad \text{where } \rho = d/h$$

P.10.10 Determine the position of the shear centre of the thin-walled open section shown in Fig. P.10.10. The thickness t is constant.

Ans. $\pi r/3$ from the junction of the two semi-circular portions.

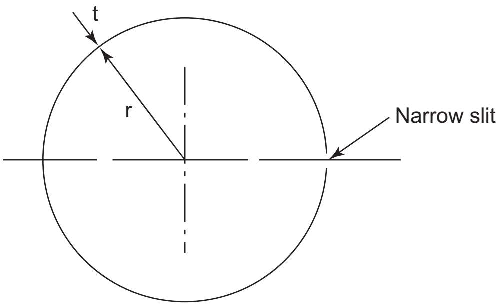

FIGURE P.10.8

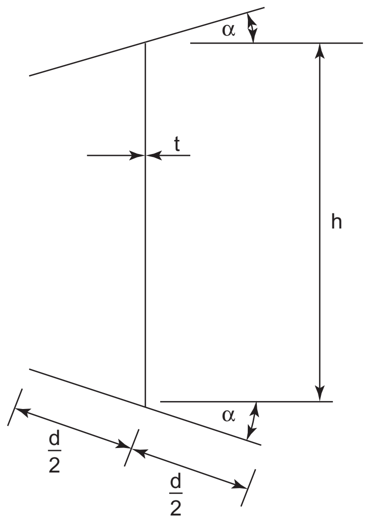

FIGURE P.10.9

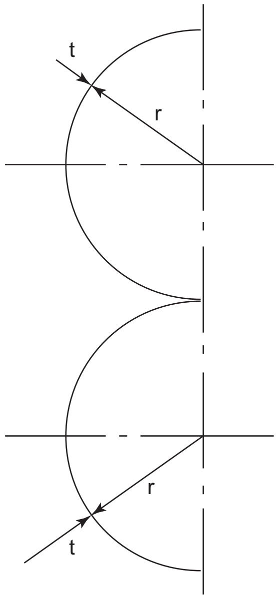

FIGURE P.10.10

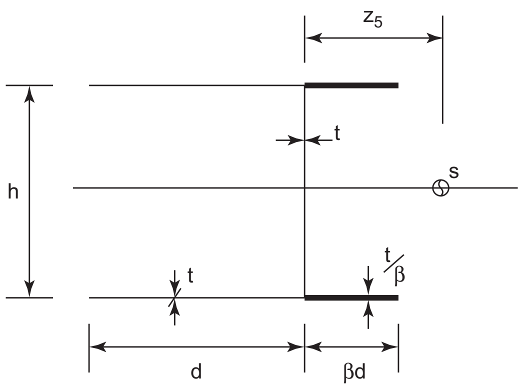

FIGURE P.10.11

P.10.11 Figure P.10.11 shows the cross section of a thin-walled, singly symmetrical I-section beam. Show that the distance z_S of the shear centre from the vertical is given by

$$\frac{z_S}{d} = \frac{3\rho(1-\beta)}{(1+12\rho)} \quad \text{where } \rho = d/h$$

P.10.12 Find the position of the shear centre of the thin-walled beam section shown in Fig. P.10.12.

Ans. 1.2r on the axis of symmetry to the left of the section.

P.10.13 Determine the horizontal distance from O of the shear centre of the thin-walled beam section shown in Fig. P.10.13. Note that the position of the centroid of area, G, is given.

Ans. 2.0r to the left of O.

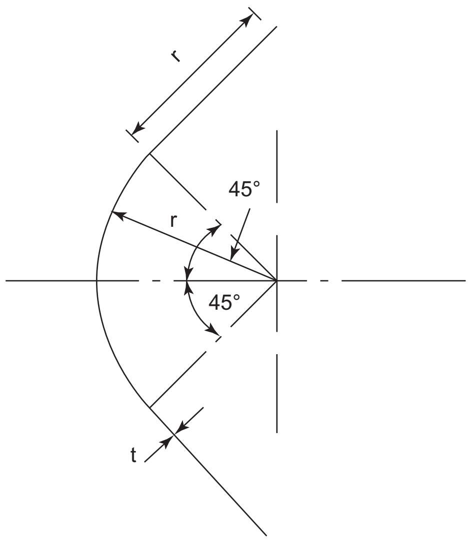

FIGURE P.10.12

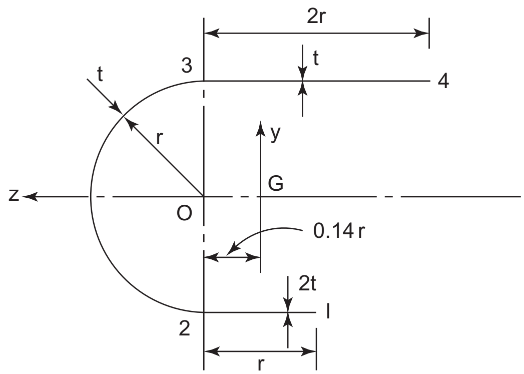

FIGURE P.10.13

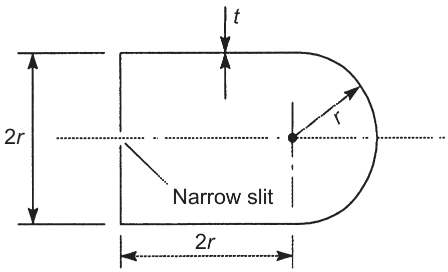

FIGURE P.10.14

P.10.14 Define the term 'shear centre' of a thin-walled open section and determine the position of the shear centre of the thin-walled open section shown in Fig. P.10.14.

Ans. $2.66r$ from centre of semicircular wall.

P.10.15 Determine the position of the shear centre of the cold-formed, thin-walled section shown in Fig. P.10.15. The thickness of the section is constant throughout.

Ans. 87.5 mm above centre of semicircular wall.

P.10.16 The thin-walled channel section shown in Fig. P.10.16 has flanges that decrease linearly in thickness from $2t_0$ at the tip to t_0 at their junction with the web. The web has a constant thickness t_0. Determine the distribution of shear flow round the section due to a shear load S_y applied through the shear centre S. Determine also the position of the shear centre.

Ans.

$$q_{AB} = -S_y t_0 h(s_A - s_A^2/4d)/I_z, \quad q_{BC} = -S_y t_0 (hs_B - s_B^2 + 3hd/2)/2I_z,$$

where $I_x = t_0 h^2(h + 9d)/12$; $5d^2/(h + 9d)$ from mid-point of web.

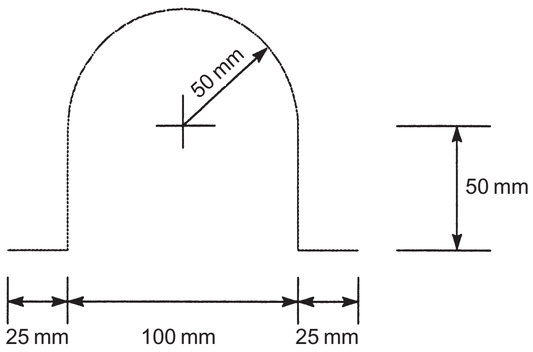

FIGURE P.10.15

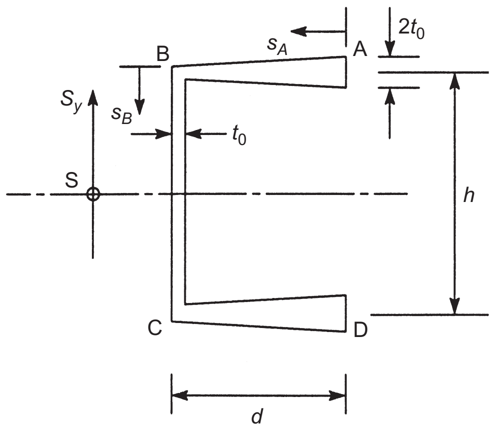

FIGURE P.10.16

P.10.17 Calculate the position of the shear centre of the thin-walled unsymmetrical channel section shown in Fig. P.10.17.

Ans. 23.1 mm from web BC, 76.3 mm from flange CD.

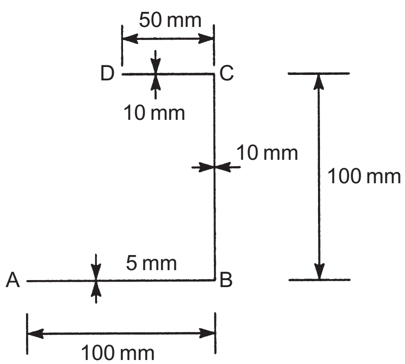

FIGURE P.10.17

P.10.18 A plate girder is one of two supporting a railway bridge deck, which has a simply supported span of 12 m. In a worst-case scenario, a stationary railway engine produces two concentrated loads of 1500 kN, 4 m apart on the bridge deck with each load being 4 m from the end of the bridge. If the cross section of a girder is that shown in Fig. P.10.18 where the flange plates are welded to the girder web, calculate the shear force/mm on a weld and recommend a suitable weld size (see Section 22.2) assuming an allowable weld stress of 100 N/mm^2. For the purposes of calculation, regard the cross section of the girder as thin walled and take the density of steel to be 7750 kg/mm^3.

Ans. 232 N/mm. Weld size −4 mm.

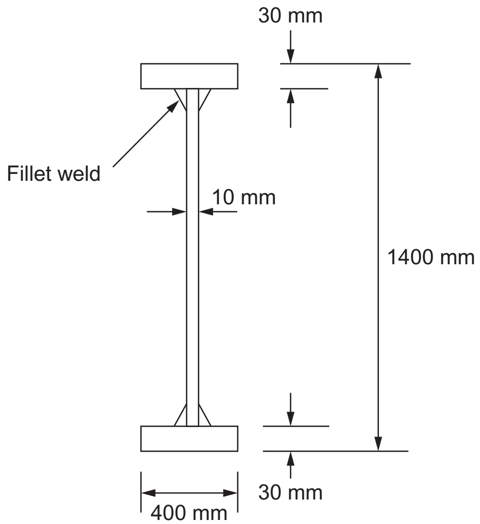

FIGURE P.10.18

P.10.19 The closed, thin-walled, hexagonal section shown in Fig. P.10.19 supports a shear load of 30 kN applied along one side. Determine the shear flow distribution round the section if the walls are of constant thickness throughout.

Ans. $q_{AB} = 1.2s_A – 0.003\,s_A^2 + 50$
$q_{BC} = 0.6s_B – 0.006\,s_B^2 + 140$
$q_{CD} = -0.6s_C – 0.003\,s_C^2 + 140.$
Remainder of distribution follows by symmetry. All shear flows in N/mm.

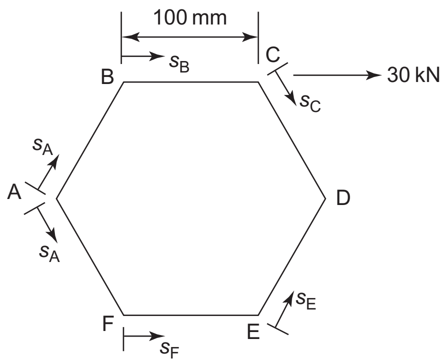

FIGURE P.10.19

P.10.20 A closed section, thin-walled beam has the shape of a quadrant of a circle and is subjected to a shear load S applied tangentially to its curved side as shown in Fig. P.10.20. If the walls are of constant thickness throughout determine the shear flow distribution round the section.

Ans. $q_{OA} = S(1.61\cos\theta - 0.81)/r$, $\quad q_{AB} = S(0.57s^2 - 1.14rs - 0.33)/r$.

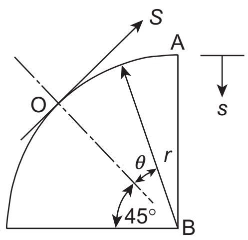

FIGURE P.10.20

P.10.21 Calculate the position of the shear centre of the beam section shown in Fig. P.10.20.

Ans. $0.61r$ from B on OB.

P.10.22 An overhead crane runs on tracks supported by a thin-walled beam whose closed cross section has the shape of an isosceles triangle (Fig. P.10.22). If the walls of the section are of constant thickness throughout determine the position of its shear centre.

Ans. 0.7 m from horizontal wall.

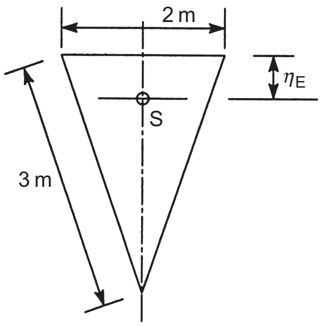

FIGURE P.10.22

P.10.23 A box girder has the singly symmetrical trapezoidal cross section shown in Fig. P.10.23. It supports a vertical shear load of 500 kN applied through its shear centre and in a direction perpendicular to its parallel sides. Calculate the shear flow distribution and the maximum shear stress in the section.

Ans. $q_{OA}=0.25s_A$
$q_{AB}=0.21s_B–2.14\times10^{-4}s_B^2+250$
$q_{BC}=-0.17s_C+246$
$\tau_{max}=30.2\ \text{N/mm}^2$.

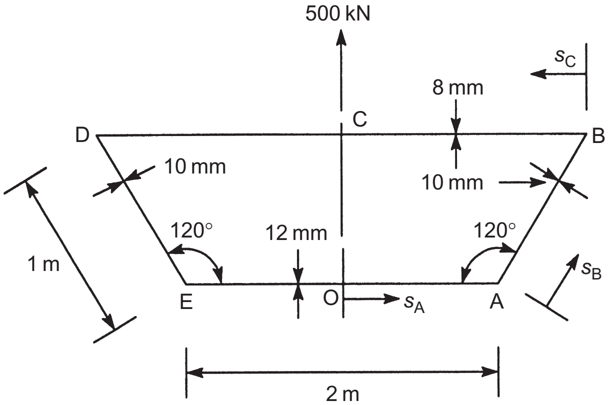

FIGURE P.10.23