EQUATIONS OF MOTION 3

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Tug Aircraft: 141hp EuroFox 2K

Introduction

- We saw that the six dynamic equations of motion (EoM) – $\dot{U}, \dot{V}, \dot{W}, \dot{p}, \dot{q}$ and $\dot{r}$ – contain **inertial** terms and **aerodynamic** terms. In this lecture we discuss the inertial effects and then move on to the aerodynamic forces and moments.

- Any study of the dynamics of an aircraft must allow for inertial factors and these are not always easily understood. You need to become familiar with:

  - *moments of inertia*,

  - *products of inertia*,

  - *dynamic coupling of motions and of equations*

- i.e. need to know what to expect for the inertia terms in the equations of motion when they represent a typical aircraft configuration.

The Inertial Terms

$$I_{xx} = \sum m(y^2 + z^2) \longrightarrow I_x$$

$$I_{yy} = \sum m(x^2 + z^2) \longrightarrow I_y$$

$$I_{zz} = \sum m(x^2 + y^2) \longrightarrow I_z$$

$$I_{yz} = \sum myz \longrightarrow I_{zy}$$

$$I_{xz} = \sum mxz \longrightarrow I_{zx}$$

$$I_{xy} = \sum mxy \longrightarrow I_{yx}$$

Note: here x, y, z are positions of masses m relative to the coordinate origin.

Rolling example

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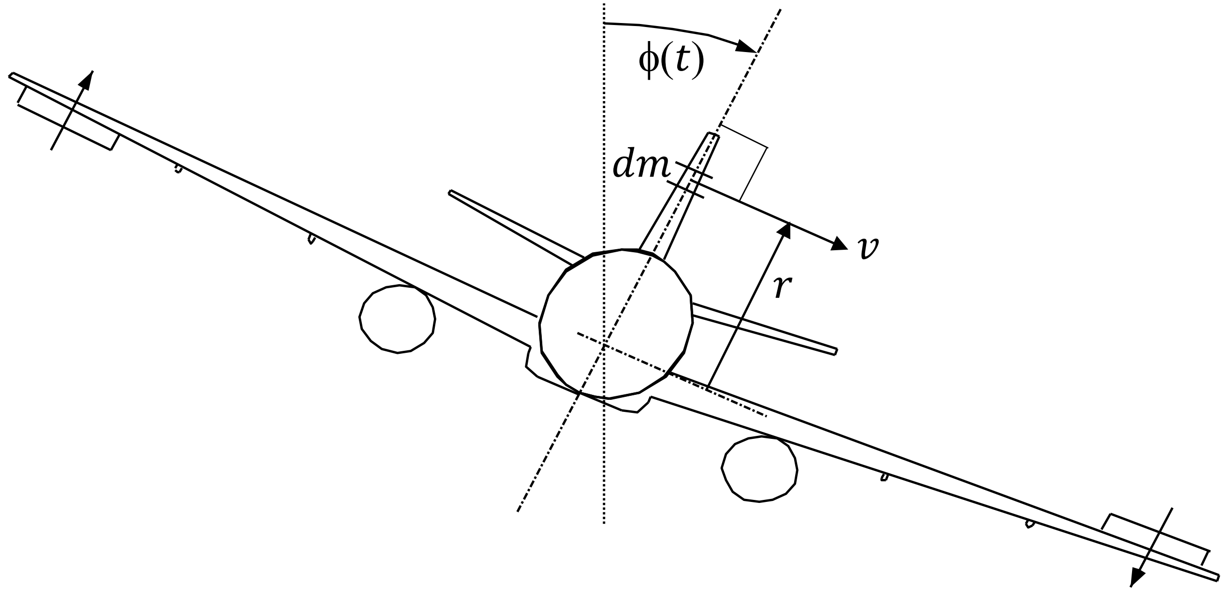
Rolling example



Aircraft view from behind

Rolling example

- Consider that a rolling moment has been applied via a deflection ξ [ksi] of the ailerons and is equal to $L(\xi)$. The differential element of momentum for a mass in the fin will be given by:

d(momentum) = dm x velocity (1)

- and the local velocity (about the roll axis) is simply $v = r\dot{\phi}$ so the differential linear momentum is:

d(momentum) = $r\dot{\phi}dm$ (2)

Rolling example

- the differential *moment* of momentum (or angular momentum) about the roll axis is simply the product r x d(momentum) or:

$$\text{d(angular momentum)} = r^2 \dot{\phi} dm \qquad (3)$$

- Allowing for the rolling moment $L(\xi)$ being applied to the whole of the aircraft mass we can show:

$$\boxed{L(\xi)} = \frac{d}{dt}\left(\int_{vehicle} r^2 \dot{\phi}\, dm \right) = \boxed{\ddot{\phi} \int_{veh.} r^2\, dm} \qquad (4)$$

Rolling example

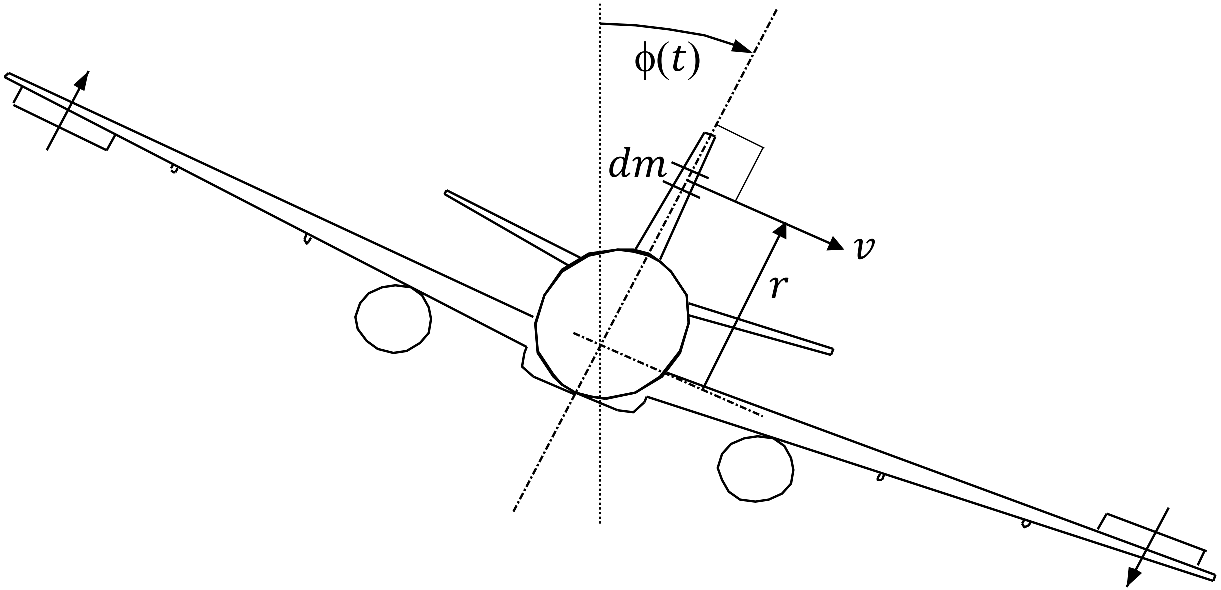
- *Note: the previous equation assumes that neither r (for any element dm), nor dm (for any part of the vehicle) changes with time.*

- Recognising that $\ddot{\phi}$ is an angular acceleration, it is clear that

$$\int_{vehicle} r^2 \, dm$$

- is the inertial factor that we need for the *angular* or *rotational inertia I* in the equation moment of inertia $\times$ ang. accel. = applied moment. Clearly, it has the dimensions of (mass) x (length)^2.

Rolling example



Aircraft view from behind

Rolling example

- This is the general rule for determining the mass moment of inertia (i.e. the angular inertia) of mass rotating about a known axis, a value r being needed for every identifiable mass, e.g.

$$I_{roll} = I_{XX} = \sum_{vehicle} m_i \, r_i^2 \qquad (5)$$

- Obviously, the same set of masses with two different sets of r_i would be used for calculating I about the yaw (zz) or pitch axes (yy).

Parallel Axis Theorem

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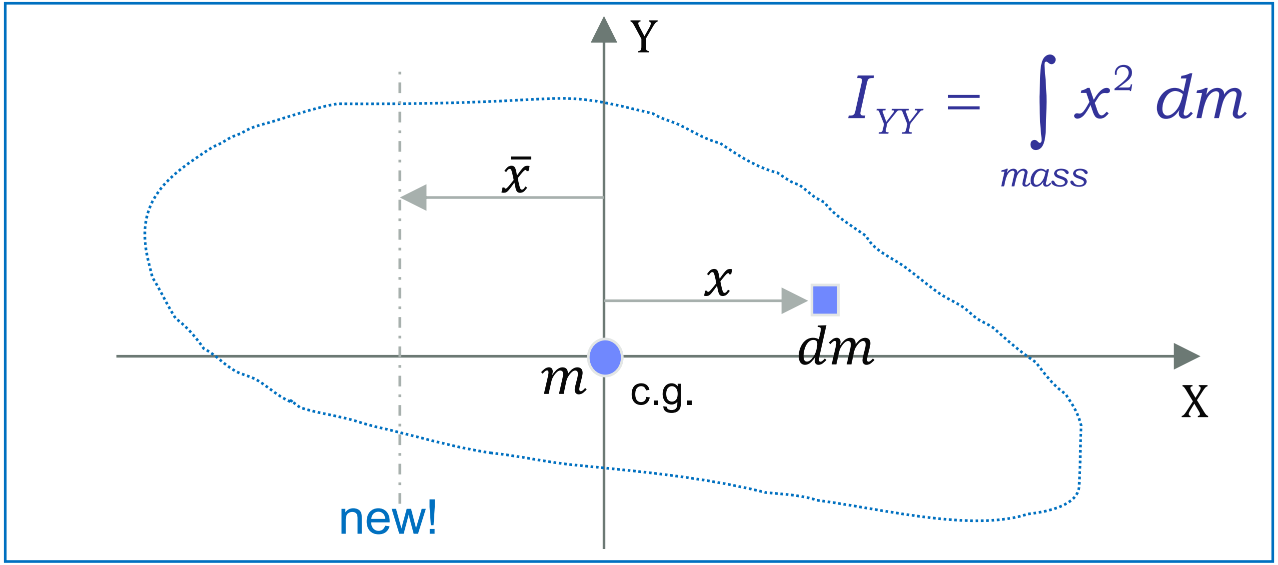
Parallel Axis Theorem



The theorem says essentially that for a new axis, parallel with that through the c.g., the new value of I can be found to be

$$I_{new} = I_{CG} + m\bar{x}^2$$

Parallel Axis Theorem

- Need to be careful applying this, easy as it may be to employ, because it must not be interpreted as

$$\cancel{I_{new} = I_{current} + m\bar{x}^2} \qquad !$$

where $\bar{x}$ is the distance through which the axis is to be moved.

The true separation between a new axis and the original ***c.g. axis*** *must always be found.*

- Note the implication that whatever the orientation of the axis, e.g. pitch, roll, yaw, the minimum value of I for rotation about that axis will exist when the axis goes through the c.g.

Product of Inertia

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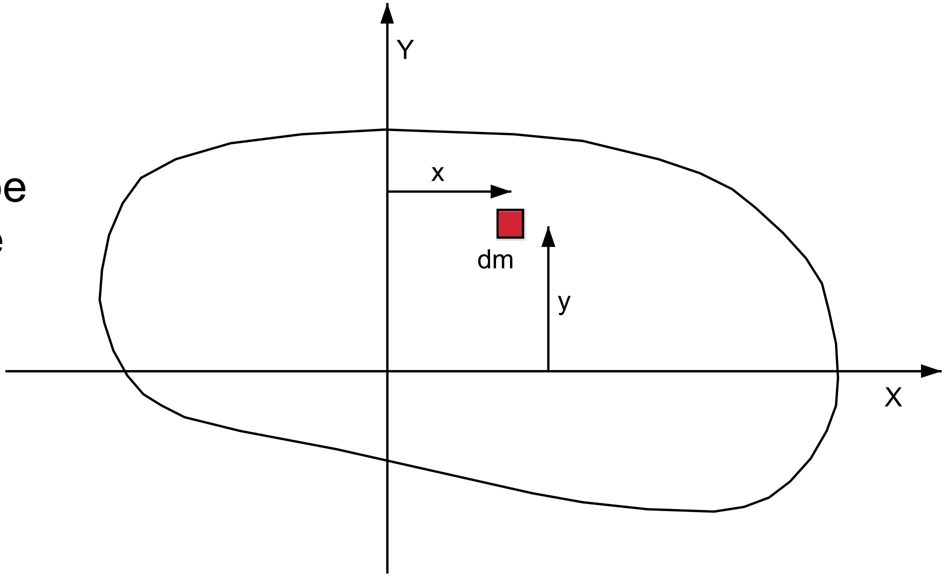
Product of Inertia

- The concept of a cross-inertia can be more difficult to understand than the moment of inertia *I*.
- Start from known concepts.
- First and second moments.



Product of Inertia: first moment

- The *first moment*, for the body in the figure on the previous slide, is given by

$$\int_{body} y\, dm \quad or \quad \int_{body} x\, dm$$

and is associated with the task of finding a centre-of-mass or c.g.

- With regard to the problem of finding a centre-of-mass, when the axis passes through the c.g.,

$$\int_{body} x\, dm = 0 \quad or \quad \sum_i m_i x_i = 0$$

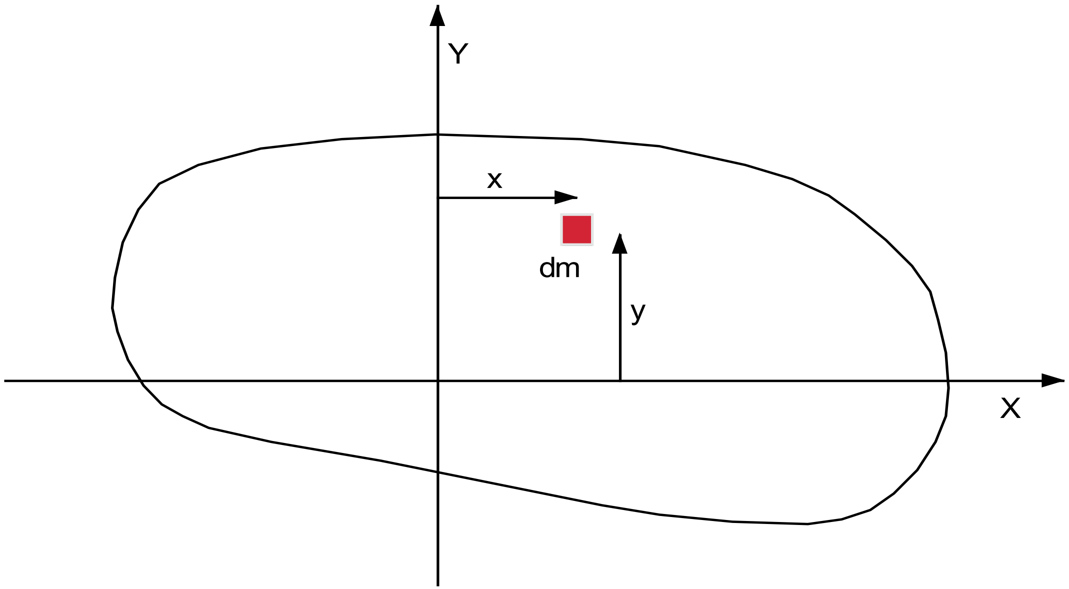
- Clearly, if both the X and Y axes give a zero moment of mass, the chosen origin would be at the c.g.

Note that the first moment is not a true moment until multiplied by the gravitational factor.

Product of Inertia: second moment

In the previous section we used the definition

$$\int r^2 \, dm$$



Product of Inertia: definition

- The classical definition quoted for this inertial factor is usually a form of

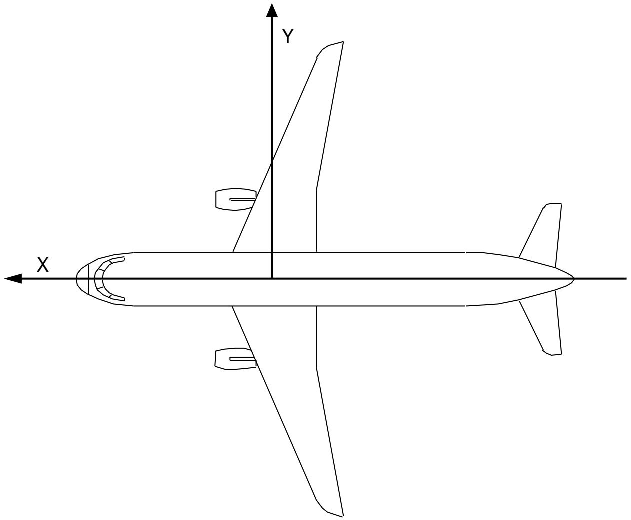
$$I_{XY} = \int_{body} xy\, dm$$

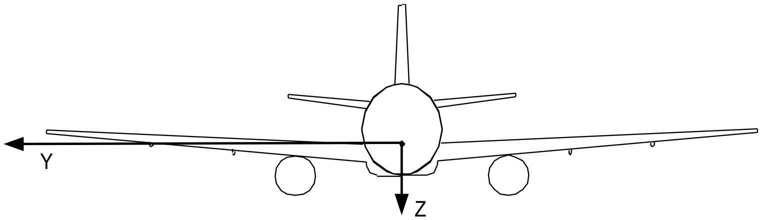
- This does not reveal the physical significance of I_{XY} and that is the challenge that remains!
- Under what conditions is this cross-inertia zero?

Product of Inertia: physical significance (symmetric bodies)

Symmetric Bodies

- The easy part of this exercise is to consider how the equation for I_{XY} relates to a body which shows symmetry.
- consider a normal aircraft configuration:

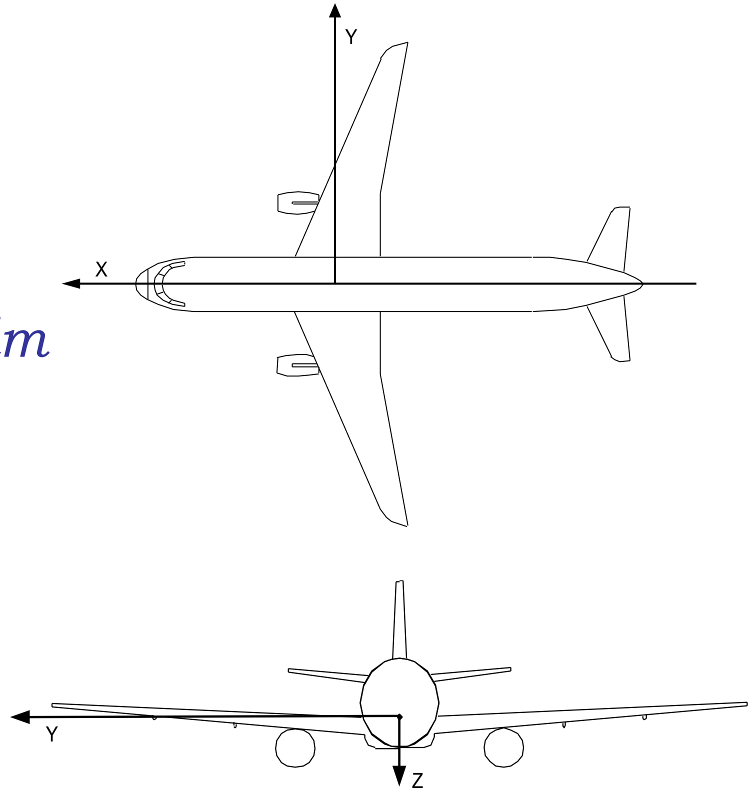




Product of Inertia: physical significance (symmetric bodies)

Symmetric Bodies:

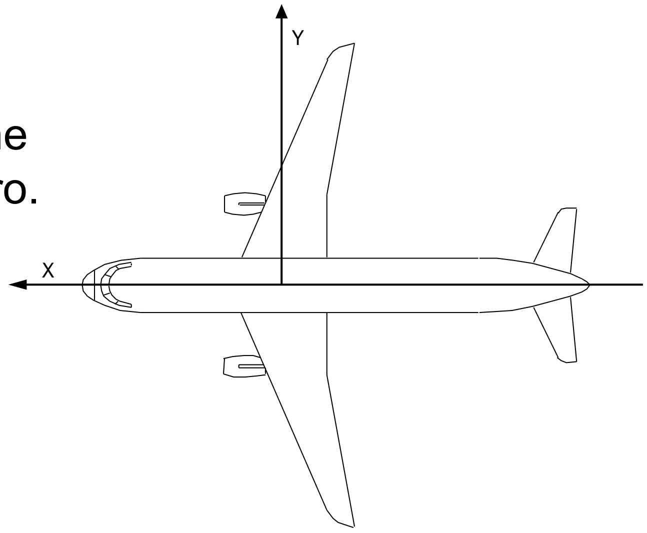
- If we think of the integral $I_{XY} = \int_{vehicle} xy \, dm$
- and recognise that both x and y can be of either sign then the following must be true:



Product of Inertia: physical significance (symmetric bodies)

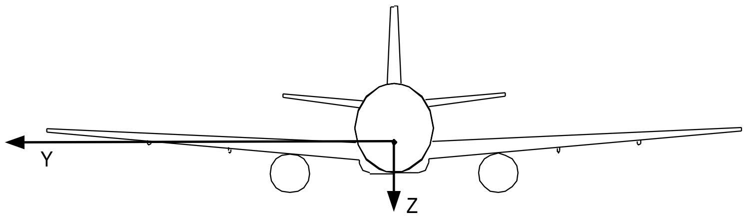
- if, for any x, there are equal masses at $\pm y$ then the summation or integral in the equation must be zero.

$$I_{XY} = \int_{vehicle} xy\,dm = 0$$



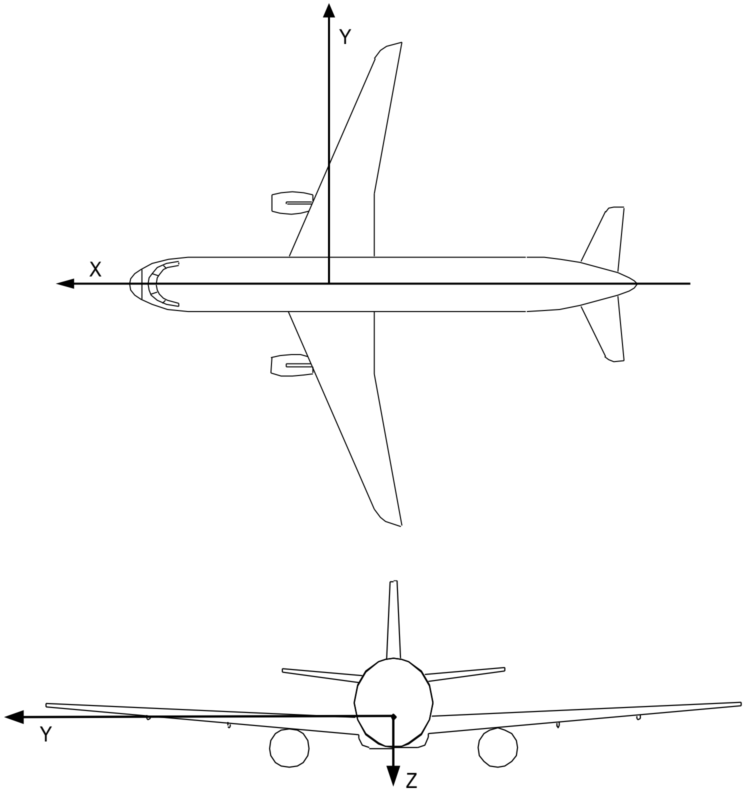
- Similarly, if there are equal masses at $\pm y$ for any chosen z then

$$I_{YZ} = \int_{vehicle} yz\,dm = 0$$



Product of Inertia: physical significance (symmetric bodies)

- Thus we have a plane of symmetry for a normal aircraft
- What is the physical significance?

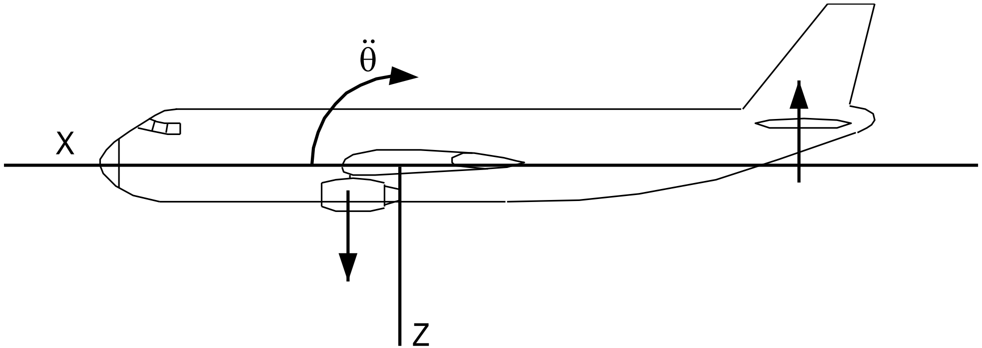


Product of Inertia: physical significance (symmetric bodies)

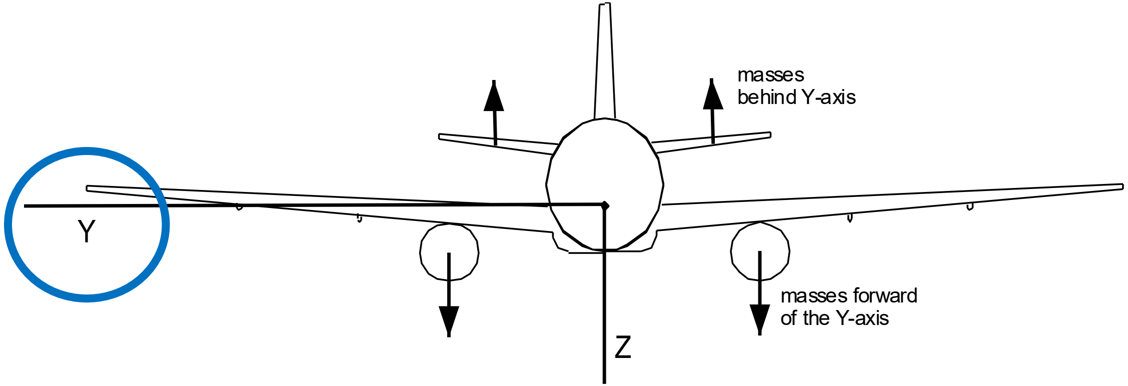
- if we were to impose a pitching acceleration $\ddot{\theta}$ about the Y-axis and then look at

$$\ddot{\theta} I_{XY} = \int_{veh.} \ddot{\theta} xy \, dm \quad or \quad \int_{veh.} y \, \ddot{\theta} x \, dm$$

- the product $\ddot{\theta} x$ being the local vertical acceleration, then $\ddot{\theta} x \, dm$ would be a local inertial force.



Product of Inertia: physical significance (symmetric bodies)

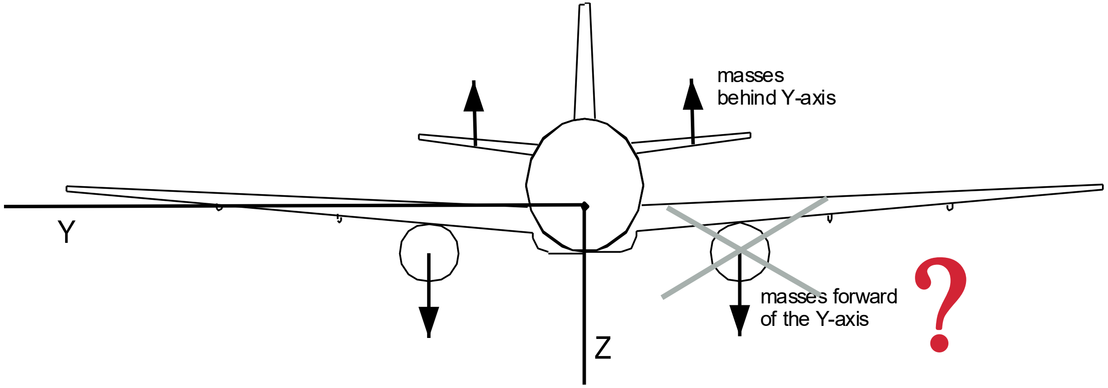


Reversed-effective forces are shown, a consequence of $\ddot{\theta}$.

Product of Inertia: physical significance (symmetric bodies)

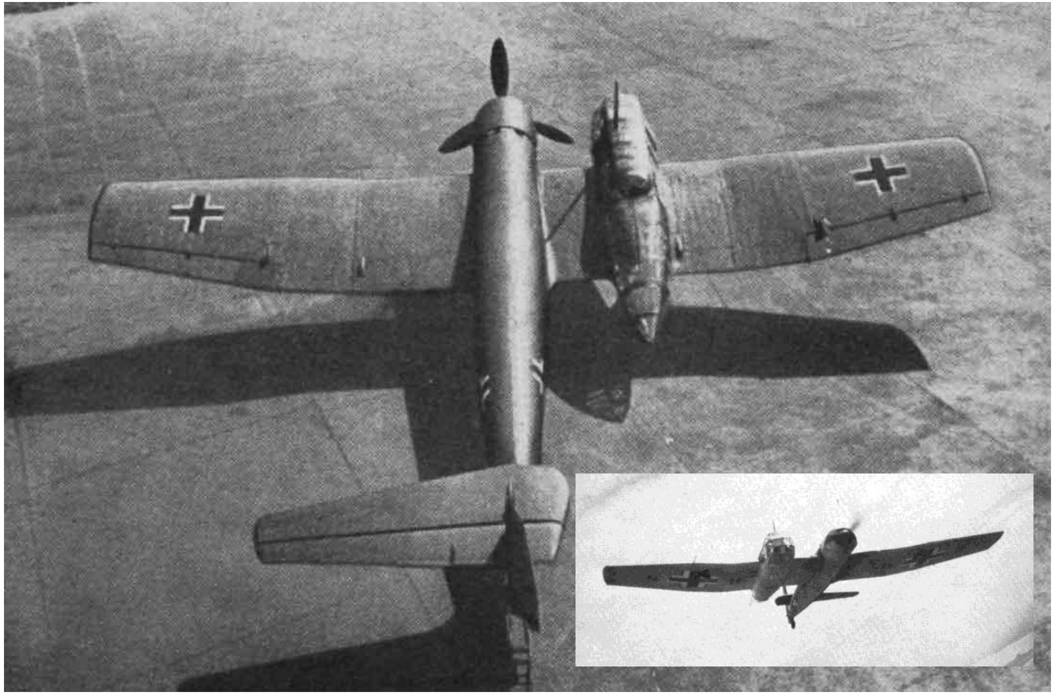
- Because of the symmetric distribution of masses, there will also be a (laterally) symmetric distribution of inertial forces and

  no overall rolling moment will be induced by the imposition of a pitching acceleration.



Product of Inertia: physical significance

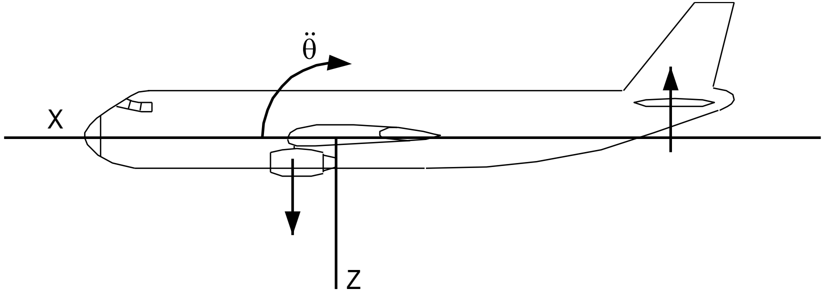
A rare exception: Blohm & Voss BV 141 (World War II German tactical reconnaissance aircraft designed for superior all-round visibility)



Product of Inertia: physical significance (symmetric bodies)

- Using a similar argument:

  the imposition of a pitching acceleration about the Y-axis does not induce a yawing moment about the Z-axis.

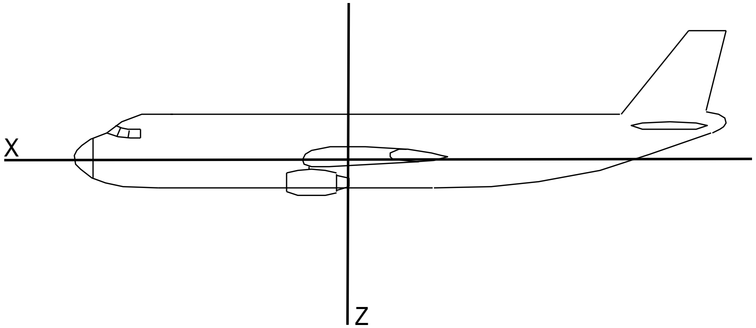


- These are special cases for a body which displays symmetry when we consider the directions YZ and XY.

- The same cannot be said for the axis pair XZ.

Product of Inertia: physical significance (asymmetric body)

An aircraft viewed from the side has several asymmetries:



- the fin has no *'ventral'* counterpart (around the X-axis),
- the wing and engines below X are not repeated above,
- both the above have no counterpart for the opposite sign of X (around the Z-axis),

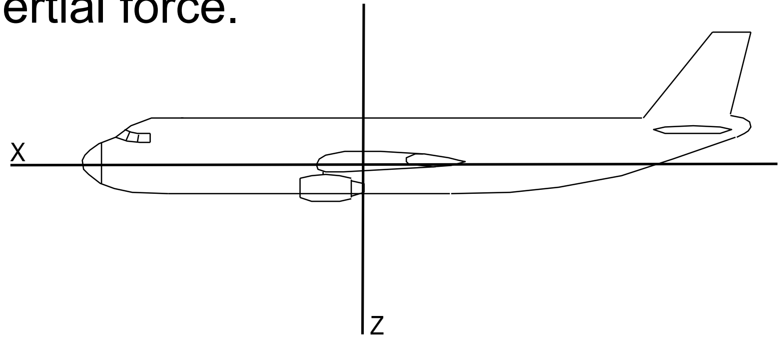
i.e. asymmetry is present!

Product of Inertia: physical significance (asymmetric body)

- if we were to impose a yawing acceleration $\dot{r}$ about the Z-axis and then look at

$$\dot{r}\ I_{XZ} = \int_{veh.} \dot{r}\, xz\, dm \qquad or \qquad \int_{veh.} z\ \ \dot{r}x\, dm$$

- the product $\dot{r}x$ being the local horizontal acceleration, then $\dot{r}x\ dm$ would be a local inertial force.



- This will, in general, cause rotation about the x axis (and vice versa).

Principal Axes

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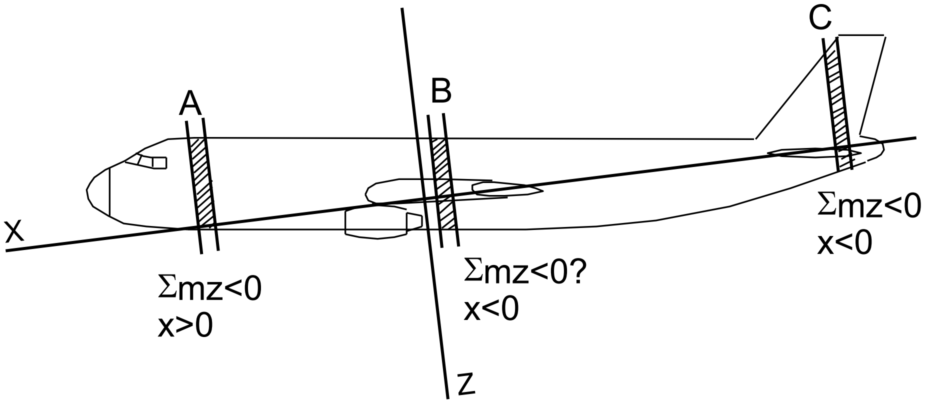
Principal Axes: an asymmetric body

- For convenience, we shall assume that our axis origin is at the c.g.
- We shall concentrate on the inertial properties of only a few slices of the aircraft, each at a different value of x, while trying to establish the significance of

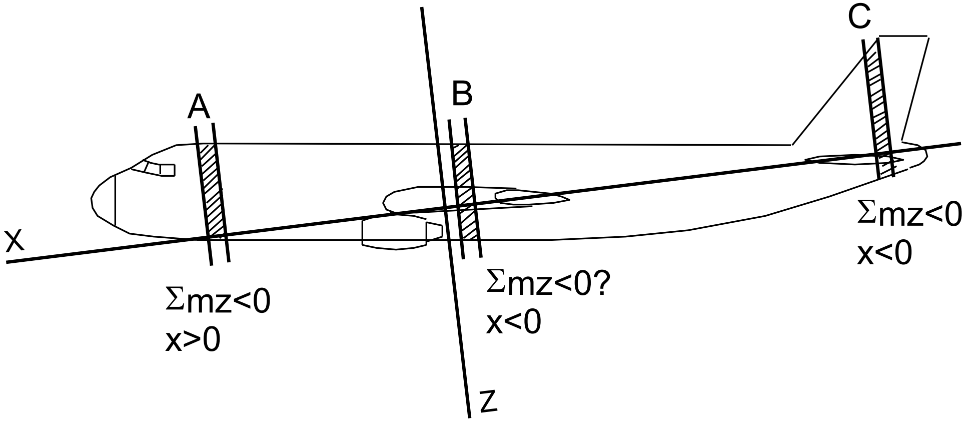
$$I_{XZ} = \int_{vehicle} xz \, dm$$



Principal Axes: an asymmetric body

A more practical version of this is to evaluate I_{xz} using a list of discrete component masses and their x, z positions:

$$I_{XZ} = \sum_{vehicle} xz\, \Delta m$$



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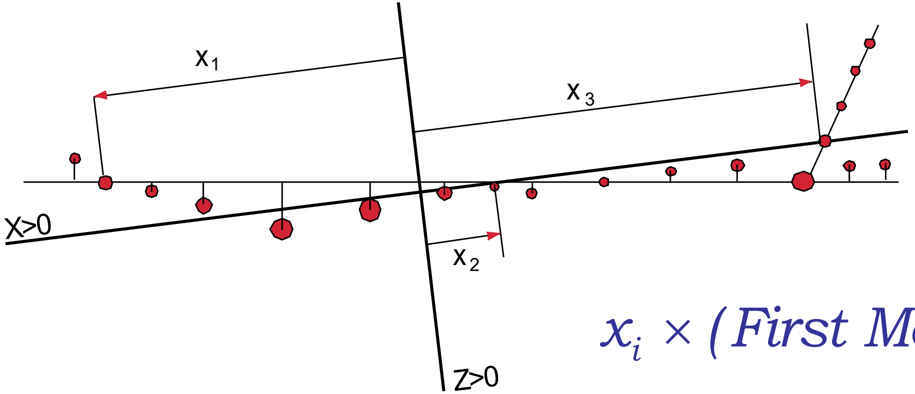
Principal Axes: an asymmetric body

- We can then reduce the aircraft to a skeletal "equivalent". Locally, every first moment is found (*some positive, some negative with respect to the chosen X-Z-axes*) and then "weighted" using:



$$x_i \times (First\ Moment)_i$$

- where the slices farthest away from the origin (largest x) have the greatest influence on the sum and on the ultimate sign of I_{XZ}.

Principal Axes: an asymmetric body

- It should be clear that some particular inclination of the X-Z axes will produce

$$I_{XZ} = 0$$

- even though the *likely* choice of orientation for these axes for flight mechanics studies is not going to produce a zero value for this cross-inertia.
- In most cases a small inclination away from the X body-axis (the likely convenient choice of *X-axis*) will give that zero value.

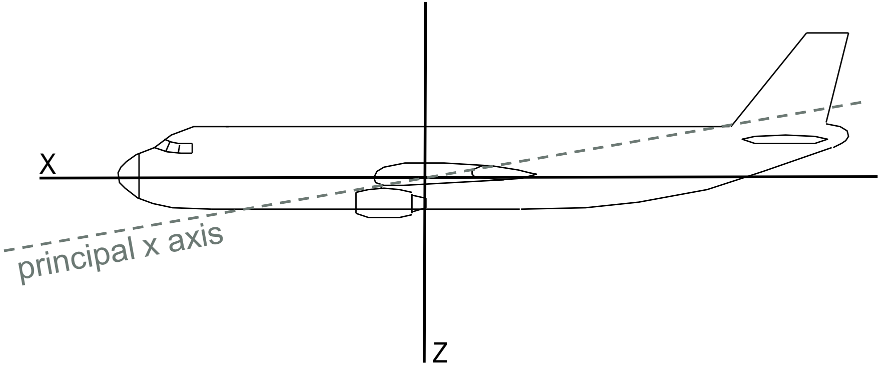
Principal Axes

- all off-diagonal terms of the square matrix are coupling terms: the products of inertia.

$$\begin{bmatrix} I_{XX} & -I_{XY} & -I_{XZ} \\ -I_{YX} & I_{YY} & -I_{YZ} \\ -I_{ZX} & -I_{ZY} & I_{ZZ} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

- The previous slides showed that I_{xy} and I_{yz} were zero, whereas a very careful choice of orientation for the X-axis could also produce $I_{xz}=0$.

- Normally, however, the more convenient choice for the X-axis (along the fuselage) would produce $I_{xz} \neq 0$.



Principal Axes

- When the choice of orientations for the axes produces zero values for all three products of inertia, the axes are said to be **principal axes** and clearly the square matrix below would become diagonal.

$$\begin{bmatrix} I_{XX} & \cancel{-I_{XY}} & \cancel{-I_{XZ}} \\ \cancel{-I_{YX}} & I_{YY} & \cancel{-I_{YZ}} \\ \cancel{-I_{ZX}} & \cancel{-I_{ZY}} & I_{ZZ} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

- The three rotational dynamic freedoms would then be de-coupled. *Every acceleration around one of these three 'special' axes would lead to an inertial reaction around only that one axis,* i.e. there would be direct inertial restraint against the acceleration. No other rotational response would be induced about another axis.

Conclusions

- It should be evident now that for the standard set of rigid body aircraft equations of motion, we *should* expect to find

$$I_{XZ} \neq 0$$

- whereas symmetry will cause the other two cross-inertias to be zero.
- The positions of the two factors I_{xz} and I_{zx} (equal though having reversed subscripts) in the equations imply that
  - ***when a rolling acceleration exists, there will be a yawing moment induced;***
  - ***when a yawing acceleration exists, there will be a rolling moment induced.***

Next Session
Equations of Motion 4

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