Signals, Systems and Control

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1.2 Time domain metrics: means and moments

1.2.1 Considering signals in the time domain

Consider this signal in the time domain. (It's a signal recorded from an accelerometer, but that is not important).

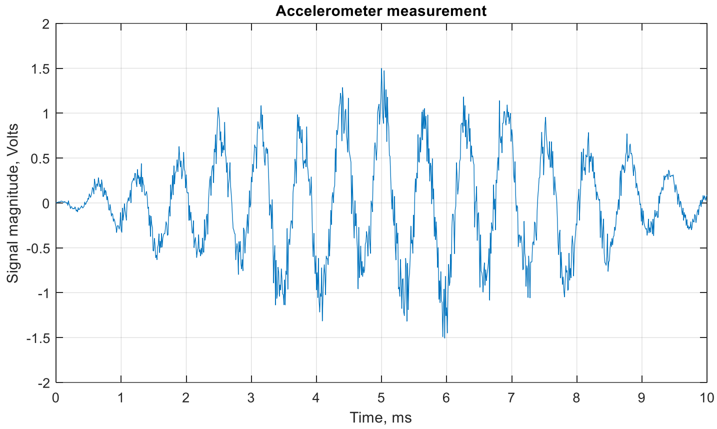


Fig.2.1. A typical 'real' signal

How could you characterise this signal?

1. Min/Max values
2. Peak-to-peak value
3. Mean value
4. Period of dominant components
5. Envelope rise times / fall times (envelope - imagine drawing a line joining all the peaks and another joining the troughs)
6. RMS
7. Moments

Take some time to think of each of these and what they might tell you about the signal.

All of these metrics can be useful but the ones we are going to focus on are the last two – a pair of related measures that tell you something about the ***average*** value of the signal over time. Averaging produces a single numerical value from a time varying quantity.

1.2.2 Statistical moments

The first form of average we are going to consider comprise a set of related measures known as 'Statistical Moments'. You have come across some of these before, but perhaps not considered them in the general way we will treat them here. As an aside, averaging has an interesting history if you want to do some reading outside the course.

Related concept: if you have studied second moment of area you will see this is related. Think why? What is the physical significance of the math?

The i^{th} statistical moment, m_i, (sometimes called the raw moment) is given by:

$$m_i = \frac{1}{T}\int_0^T [x(t)]^i dt$$

What is this equation doing? Integrating a continuous function over some range of time and then dividing by that time is a method to determine the ***mean***. In this case we are integrating the signal raised to the power i, so we are taking the mean of a signal that has had its amplitude values raised to a power.

When i = 1, this equation gives us the first moment, which is the ***mean*** of the signal x(t); when i = 2, we get the ***second moment*** or ***variance*** of the signal x(t).

You might also be anticipating a slight variation called the 'Central Statistical Moment':

$$m_i = \frac{1}{T}\int_0^T [x(t) - \bar{x}]^i dt$$

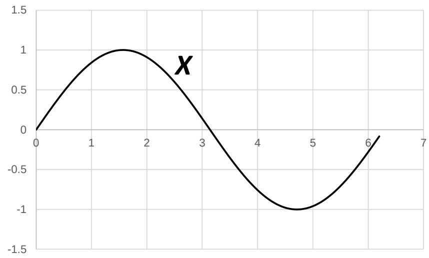
The central moment describes the moment about some expected value ($\bar{x}$). In the raw moment the expected value is zero; in the central moment it can be anything although often it is useful to set the expected value as the ***mean*** of x(t), if you do this the first central moment is zero.

So why are Moments useful? Well, what we are trying to come up with are individual (or a few) metrics that can be used to describe our signals, enabling us to characterise and compare them.

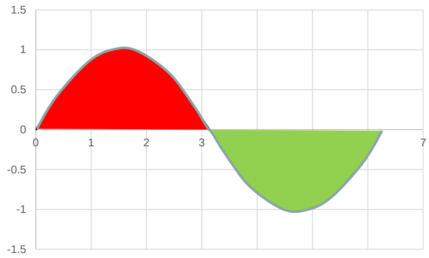
We can explore the effect of the moment order 'i' on some simple sine wave signals to illustrate what features of the signal we can capture.

What is the physical significance of i = 1?

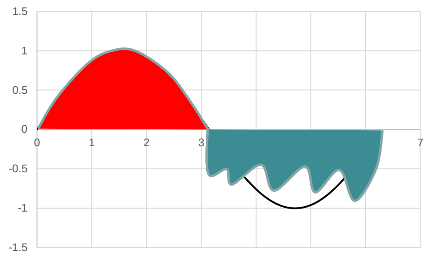
This is an easy one, so we won't spend too much time on it! When i=1 we capture the mean of the signal x(t). If our signal is a sinewave then the mean is zero



All we are doing is summing the areas on both sides of the x -axis, i.e summing the red (+ve) and green (-ve) areas. If they are equal, then the first moment is zero.

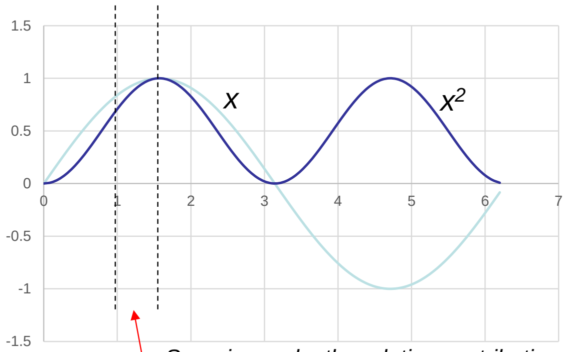


There are two useful observations we can tease-out of this case, first that the magnitude of the excursions from the expected value does not matter, and second that the shape of the signal either side of the expected value does not matter – all that matters is the relative area of the positive and negative excursions.



What is the physical significance of i = 2?

We know that the second moment is the variance, but let's look to see what is actually happening. From the graph of x^2 we can see that we are now averaging a signal that is uni-polar (i.e. all positive) and so will have a non-zero mean even if the original signal (x(t)) had a mean of zero.



Squaring make the relative contribution to the average from t=1 less than t =1.5

Also, perhaps more subtle, is that fact that now higher values of the original signal will make a greater relative contribution to the average in comparison to a sine wave– we have accentuated the influence of higher values.

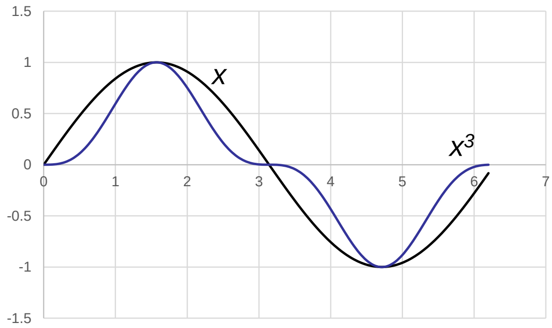
This has the effect of capturing some information about how far the original signals deviates from the expected value, (but we do not know if this deviation is balanced or one sided).

Another interesting feature evident here is that $x(t)^2$ has twice the frequency of x(t). This is because the operation of squaring is not linear. We will see later than linear operations (and the systems that can be modelled by them) cannot change the frequency content of a signal.

What is the physical significance of i = 3?

The 3rd moment further accentuates the relative weighting on higher signal values in the average, but it can produce a zero value if the weighted positive and negative excursions are similar.

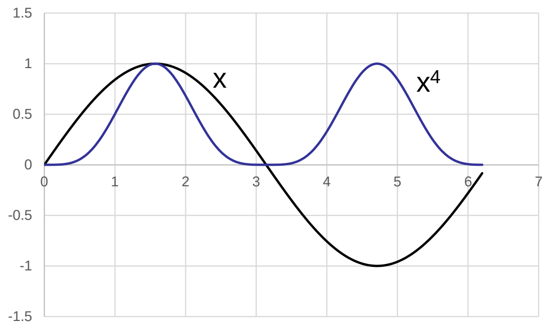
We are going to start think about this accentuation of the higher values as providing us information about the shape of the original signal, and the ability to produce a zero value as some measure of the symmetry of the signal.



What is the physical significance of i = 4?

Like the 2nd moment the 4th moment has a non-zero value even if the mean is zero (as long as the signal itself is not zero for all time!). However, the high values are even more accentuated – so we can think of the 4th moment as an exaggeration of the variance.

We don't have any information on the symmetry of the waveform.



Hopefully you will have spotted the underlying effect here:

- **If 'i' is odd – we get some measure of symmetry of a waveform.**

  If 'i' is even, symmetry is irrelevant

- **As 'i' increases we get increasing sensitivity to the higher values of a waveform or the 'peakyness' of the signal.**

1.2.3 What are the limitations of statistical moments?

The main limitation of moments was hidden in the previous examples by choosing a maximum magnitude for x(t) of unity (1). If we consider the powers of a signal with peak magnitude '2' we see a potential issue:

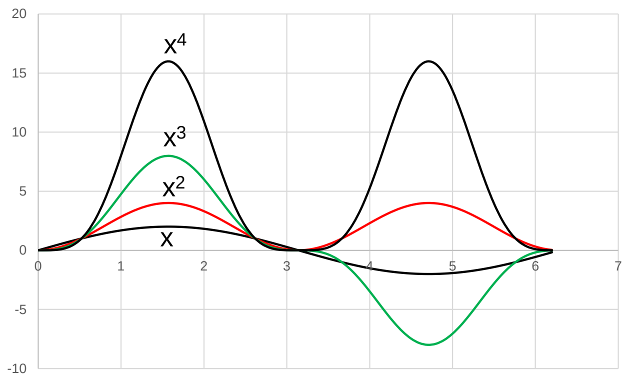


Fig.2.2. Raising the signal to a power has the effect of changing its magnitude

Raising the signal to some power has now had the effect of amplifying the signal. This can also be seen by considering the units of statistical moments – they are:

units_of_original_signal^i

So, if your original signal was measured in volts, the 3rd moment would have units of volts^3

Remember Second Moment of Area? The units of that are area^2 e.g. m^4

1.2.4 Power Means

In some applications the amplification of moments is unhelpful as we would like to be able to compare values in the same units as the original signal. The solution is to employ the 'power mean':

$$i^{th}powermean = \left(\frac{1}{T}\int_0^T x^i(t)dt\right)^{\frac{1}{i}}$$

The power mean takes the i^{th} root of the i^{th} moment. Often the power mean is shown using the raw moment, but the central moment could be used if it is appropriate for what you are trying to determine.

You will hopefully see that the power mean will have the same units as the original signal.

The power means commonly encountered are RMS (Root-Mean Square) and standard deviation – both versions of the second power mean.

Many of you may have a good understanding of RMS already, but briefly it is often used to calculate average power in respect of a time varying field quantity (or root power quantity – hence the name). For example - consider the car damper from the previous section: the ***average*** power dissipated by the damper is proportional to the second power mean (RMS) of the instantaneous velocity across the damper (or indeed the RMS of the instantaneous force through the damper); Consider a resistor – the power dissipated by the resistor is proportional to the second power mean (RMS) of the instantaneous current through the resistor, or the second power mean (RMS) of the instantaneous voltage across the resistor.

You will notice I have started referring to 'instantaneous' values to distinguish the time varying signals from the average values. The instantaneous values are functions of time and very – they will be represented by lower case letters; the average values will be a singular value and represented by capital letters.

We have explained the utility of the second power mean, but you might well ask what use are other orders? There are some quantities that are proportional to the cube root of power for example air velocity in an acoustic system – however these are less common.

1.2.5 Standardised Moments

With power means we saw that we could divide our moment by another well-chosen metric to return the units to those of the original signal – so what if we took this one step further and tried to remove units altogether?

Before we look at this, take a few moments to think about why we would want to do this. There are two related concepts to mention here: ***Normalising*** – where we make all measurements relative to some value; The second concept is ***Non-dimensionalisation***, which involves normalising in such a way so that the result has no units – it is dimensionless. For example, expressing things as percentages removes dimensions. We do this when we want to be able to capture some property that can be compared across measurements of differing magnitudes or classifications.

With moments we call the dimensionless form Standardised Moments, and they are given by:

$$K_i = \frac{\frac{1}{T}\int_0^T [x(t) - \bar{x}]^i \, dt}{\sigma^i}$$

This is i^{th} moment divided by the standard deviation (or RMS) raised to the power I, and the goal here is to capture something fundamental about the ***shape*** of the signal.

It is the 3^{rd} and 4^{th} standardised moments that are most useful, known as 'skewness' and 'kurtosis' (sometimes 'peekiness') respectively. Kurtosis is routinely used in fault detection.

$$K_4 = \frac{\frac{1}{T}\int_0^T [x(t) - \bar{x}]^4 \, dt}{\sigma^4} = \frac{\frac{1}{T}\int_0^T [x(t) - \bar{x}]^4 \, dt}{x_{rms}^4}$$

1.2.6 Moments in Probability Distributions

You may well be familiar with moments being used to describe probability distributions – they are used, as we are here, to extract some essential information about the distribution shape. It's not something you need for this course, but let's have a quick recap now which might help you reconcile the same mathematical concept applied to two different applications.

A probability density function describes the relative likelihood that a random sample will have a particular value.

Gaussian (or Normal) distribution

In probability theory, the probability density function often exhibits a Gaussian distribution.

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\sigma)^2}$$

where μ is the mean (first moment) and σ is the standard deviation (second power mean).

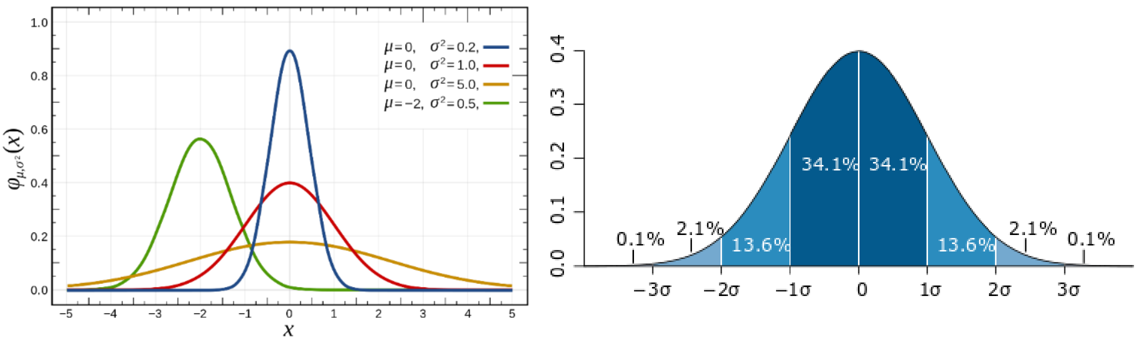


Fig 2.3. Illustrations of Normal probability distributions, (reproduced from Wikipedia)

The uncertainty is expressed in relation to the confidence interval. For example, the uncertainty for a confidence interval of 68% is plus/minus the standard deviation $\pm\sigma$, while for a confidence interval of 95%, the uncertainty is $\pm 2\sigma$.

Skewness and Kurtosis of probability distributions

Skewness (3rd standardised moment) appears as a shift to the left (or right) of the normal distribution – (and you might need to do a bit of mental gymnastics here as with our signals we are looking at symmetry either side of a mean value – you need to rotate through 90°).

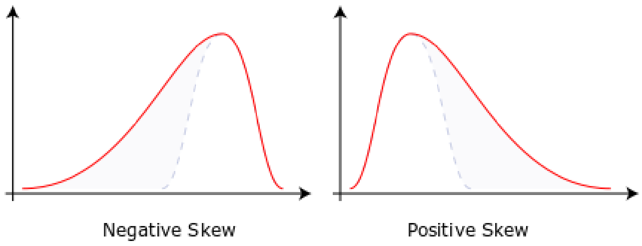


Fig 2.4. Illustration of skewness in a probability distribution function, (reproduced from Wikipedia)

The kurtosis of all normal distributions is ‘3’. This means that all the four pdfs in fig 2.3 (red, blue, green and yellow curves) have the same kurtosis. 3 is often used as a refence to normalise other probability distributions – and the term excess kurtosis is used. i.e. if the excess kurtosis is greater than zero, the shape is ‘peekier’ that the normal distribution, if it is less than zero it is flatter.

For further reading on this try:

https://en.wikipedia.org/wiki/Probability_distribution#Continuous_probability_distribution

Now, back to signals for us!

1.2.7 Moments of example signals

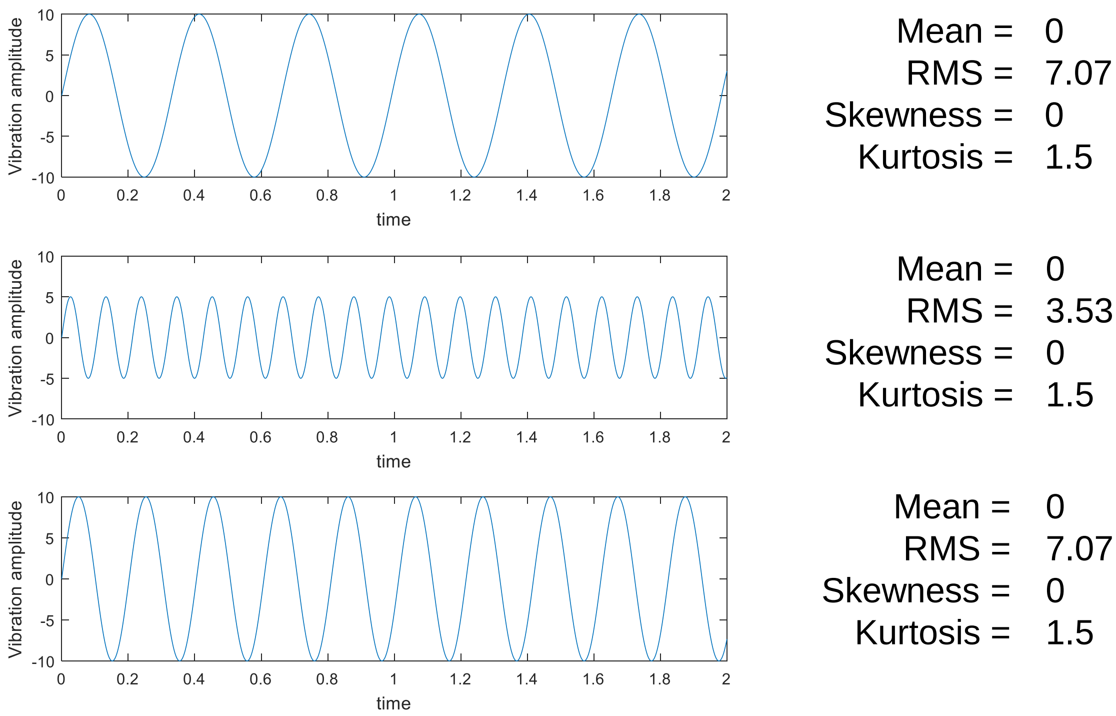


Fig. 2.5. These sine waves show how the standardised moments do not depend on frequency or magnitude

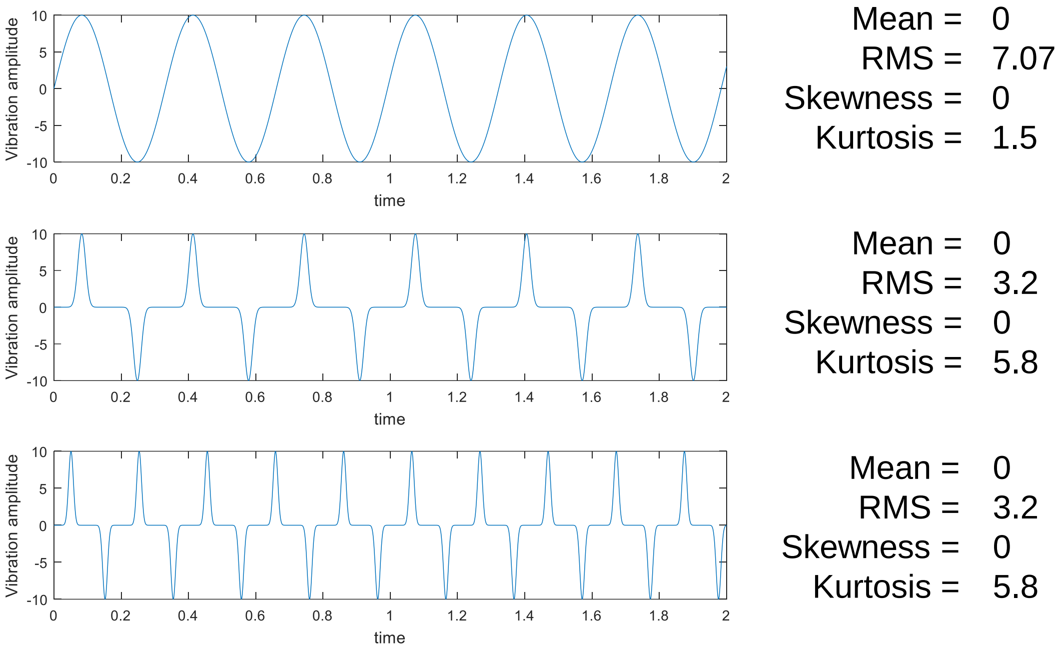


Fig. 2.6. These waveforms demonstrate kurtosis is not dependent on frequency – again it is a measure of shape.

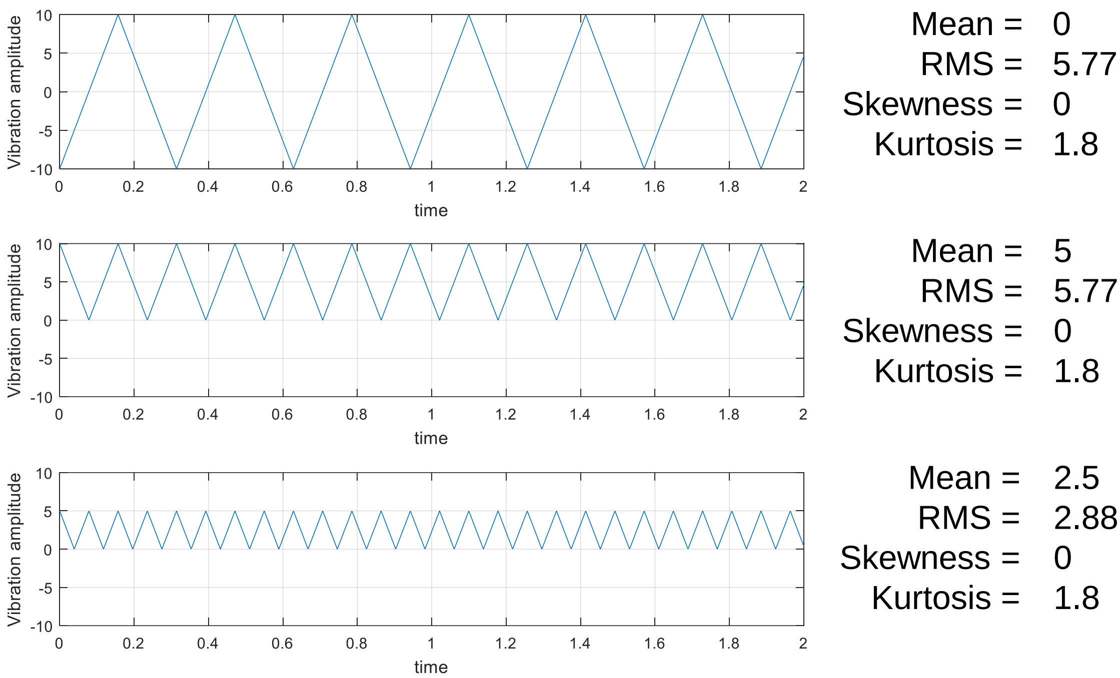


Fig. 2.7. Because the standardised moments are taken around the expected value of the mean, these waveforms all have zero skewness and the same kurtosis (note we could set the expected value to something other than the mean and we would see skewness).

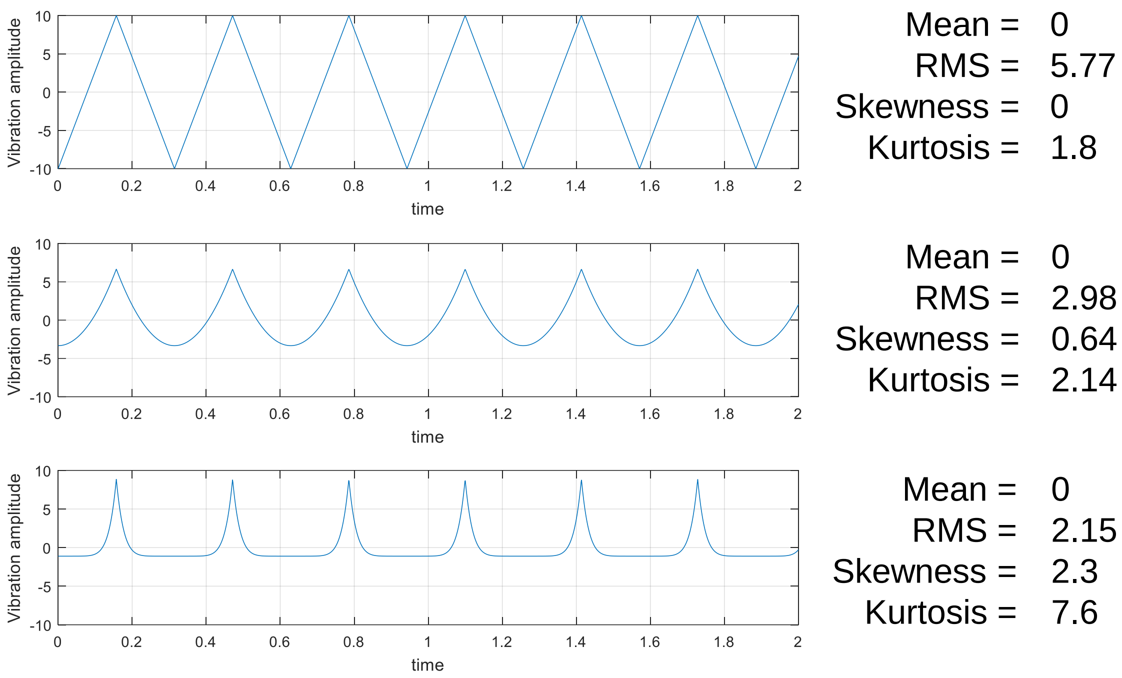


Fig. 2.8. These waveforms have non-symmetry around the mean, hence produce skewness

1.2.8 Test yourself section 1.2

Have a go at answering these question – if you can answer them all then you can move on to the next section. If you can't find an answer in the notes that 'clicks' for you, look it up – it is often good to have several explanations!

1) What is the envelope of a signal?
2) What mathematical technique are moments a form of?
3) Which moments tell us about the symmetry of a signal?
4) What are the units of power means?
5) What is the second power mean commonly known as?
6) Why is it useful to express moments in a non-dimensional form?
7) What is the Kurtosis of a normal distribution?
8) Which waveform has a higher kurtosis – a triangle or a sine wave?