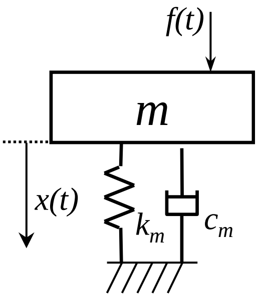
# Signals, **Systems** and Control

## Contents

## 2.3 Poles, Zeros and the Bode plot

### 2.3.1 Roots of the Transfer function – poles and zeros

In the last lecture we looked at how we could derive the transfer function of many common systems. For example, a mass/spring/damper system with an input of force applied to the mass and an output of displacement of the mass:




$$TF = \frac{output}{input} = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + c_m s + k_m}$$

We can generalise the transfer function to:

$$\mathrm{TF} = \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_2 s^2 + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_2 s^2 + b_1 s + b_0}$$

The highest order exponent (i.e. n or m) determines the ***order*** of the system, and is related to the number of independent energy storage elements in the system, e.g. a 2nd order system might have a mass and a spring; a circuit with 2 capacitors and an inductor would be 3rd order – although it is important to note that this only applies if the variables in the TF are root-power quantities, more of this later.

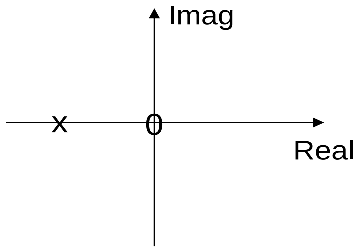
The system described by the transfer function is a ***proper system*** if m is greater or equal to n. Often it is stated that only a proper system in physically realisable. You might immediately spot that a differentiator is an ***improper system***, as is the system at the end of example b in section 2.2.2 where we swapped the signals that we defined as the input and output. The subtlety here is we can only approximate improper systems over some limited frequency range otherwise as the frequency tends to infinity, so would the transfer function. Practically, in a physical mechanical system this might manifest as the force required to produce the desired displacement might becoming infinite – and this obviously cannot be realised in practice.

The transfer function can be factorised:

$$\text{TF} = \frac{(s - z_1)(s - z_2) \dots \dots (s - z_n)}{(s - p_1)(s - p_2) \dots \dots (s - p_m)}$$

The roots of the numerator are called '***zeros***'; The roots of the denominator are called '***poles***'; the denominator is called the ***characteristic equation***

The poles and zeros give us insight into the system behaviour and they can be plotted on the complex plane – an important graphical tool in classical control theory. Poles are denoted by 'x', and zeros are denoted by '0'. In the example below we have a zero at the origin and a pole on the negative real axis. For a system second order and above, poles (or zeros) occur in conjugate pairs.




The characteristic equation is key in determining the time domain behaviour of the system. Poles on the real axis control decay or damping; the right-hand half of the plane being unstable. Oscillatory behaviour (resonance) occurs with pairs of poles. (Zeros don't cause unstable behaviour since they make the TF tend to zero, rather than infinity, hence we look at the CE for stability).

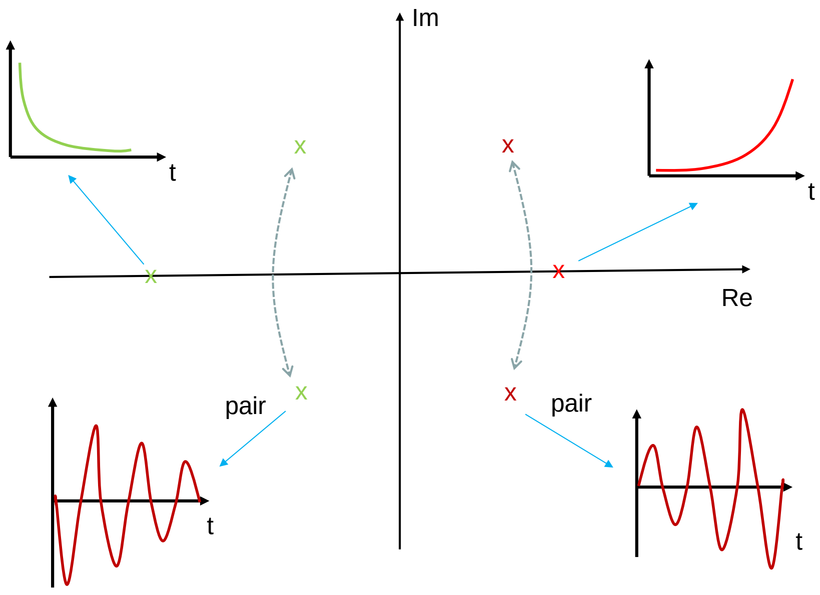



***Fig.1. Influence of pole location on the stability of a system. Where a pole has an imaginary component, it will always occur in a pair (and the system be minimum 2$^{nd}$ order).***

### 2.3.2 The Bode plot

An important characteristic of LTI systems – no matter what the order - is that they only change the magnitude and phase of an input signal, not the frequency content. Hence if the input to a system is:

$$x(t) = \sin(\omega t)$$

Then the output will be of the form:

$$y(t) = \mathrm{A}\sin(\omega t + \theta)$$

*Gain response*, *A(ω)*        *Phase response*, *Θ(ω)*

The gain and phase response are both functions of frequency and are captured in a diagram known as the **Bode Plot** (the plot is named after HW Bode). Bode had Dutch ancestry and the pronunciation of his surname would be 'boh-da', although it is recorded he used 'boh-dee' during his time at Bell labs. You will likely hear both pronunciations as well as the English, which rhymes with 'code' or 'mode')

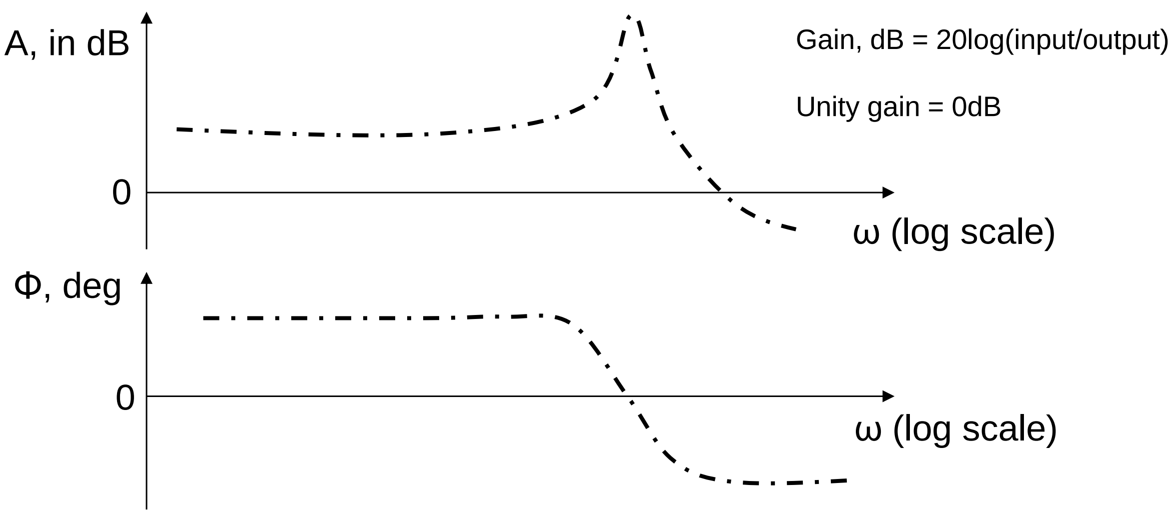



***Fig. 2. An illustrative Bode plot for a system. The gain response is at the top, measured in dB and with a logarithmic frequency axis – which can be in radians/s or Hertz. The phase plot is below with the frequency axis (x axis) aligned with the gain plot. Phase is in units of degrees***

### 2.3.3 LTI systems as filters

Irrespective of the order, LTI systems are simply filters, i.e. they amplify or attenuate a signal based in frequency. There are several classic gain responses that are worth quickly describing before we look at the Bode plots for several simple systems.

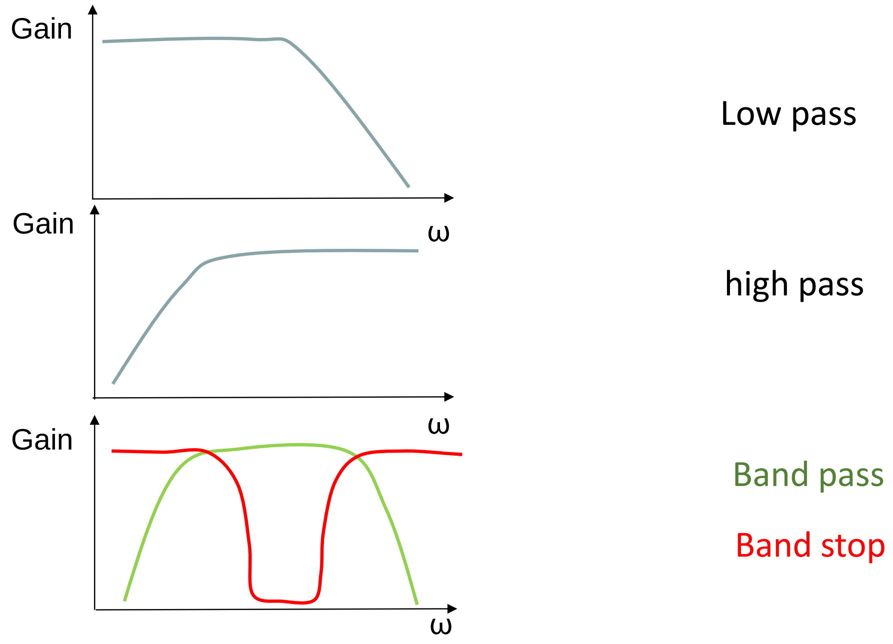



***Fig. 3. Sketches of some common gain responses***

In closed-loop control, the subject we will cover next in this unit, the phase response is also very important. A couple of important responses:

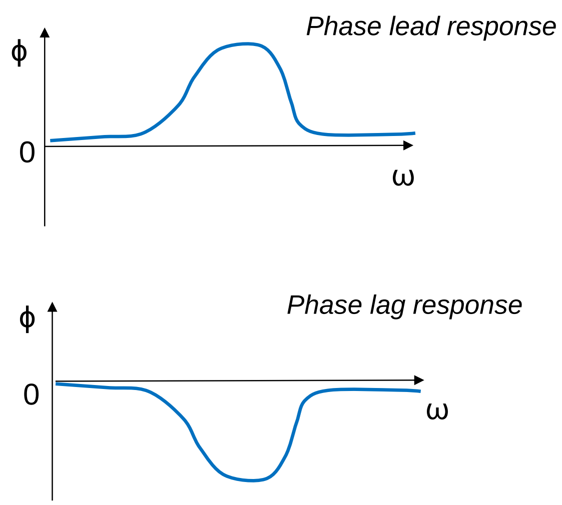
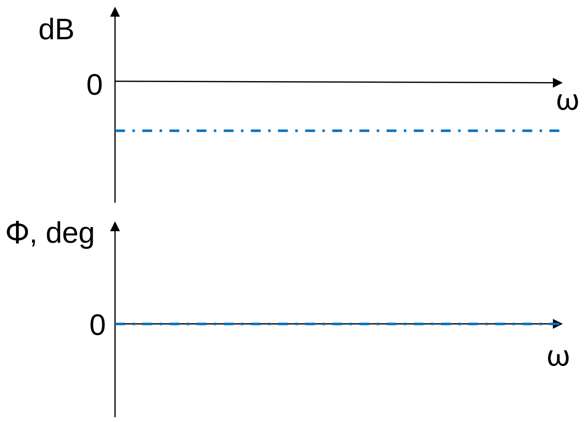



***Fig. 4. Sketches of some common phase responses***

Of course, it is important to note that any particular phase response will have an associated gain response and vice-versa (note the common gain responses shown are not paired with the common gain responses!)

### 2.3.4 Bode plots – Single element systems - Attenuation / Integrators / differentiators

*Attenuation and gain*



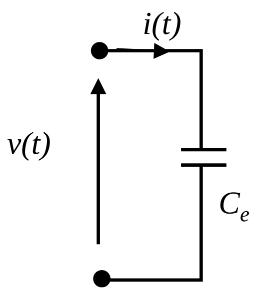

The most basic systems have only attenuation (or amplification). There are no 'dynamics', no poles or zeros.

Attenuation is produced by dashpots, resistors etc

Amplification generally requires active devices e.g. electronic amplifiers, (note passive systems with resonance can amplify at a fixed frequency)

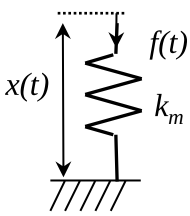
*Integrators*

The first interesting single element we will look at is the integrator, for example:




$$\mathrm{v}(t) = \frac{1}{C_e}\int i(t)dt$$

*Capacitor integrates current*



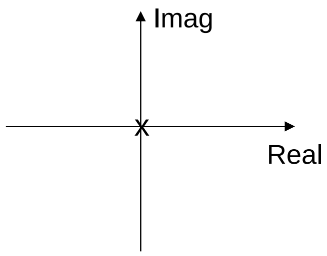

$$f(t) = k_e \int \dot{x}(t)dt$$

*Spring integrates velocity*

In the Laplace domain an integrator is given by:

$$\frac{Y(s)}{X(s)} = \frac{const.}{s}$$

Which is a single pole at the origin:




Now we can investigate how to extract the gain and phase response

The gain response is found by substituting s=jω (we are looking for the steady-state sinusoidal response) and evaluating the magnitude of the transfer function in dB. Remember we use ‘20log(a/b)’ because these are root power quantities.

$$H(s) = \frac{k}{s} \qquad\qquad H(j\omega) = \frac{k}{j\omega}$$

recall:

$$mag(H(j\omega)) = \sqrt{real(H)^2 + imag(H)^2}$$

$$mag\left(\frac{k}{j\omega}\right) = k/\omega$$

In decibels:

$$mag(\frac{k}{j\omega}) = 20\log(\frac{k}{\omega}) = 20\log(k) - 20\log(\omega)$$

*Constant term*

*Slope of -20 db / decade frequency*

*Sum is zero at ω=k*

The phase response is found from the argument of the transfer function:

$$\arg(H(j\omega)) = \tan^{-1}\left(\frac{imag(H)}{real(H)}\right) \qquad \arg\left(\frac{k}{j\omega}\right) = \tan^{-1}\left(\frac{-k/\omega}{0}\right) = -90°$$

*(remember 1/j = -j, so we are in effect taking the inverse tangent of negative infinity)*

Finally, we can plot the Bode diagram of an integrator and note important features:

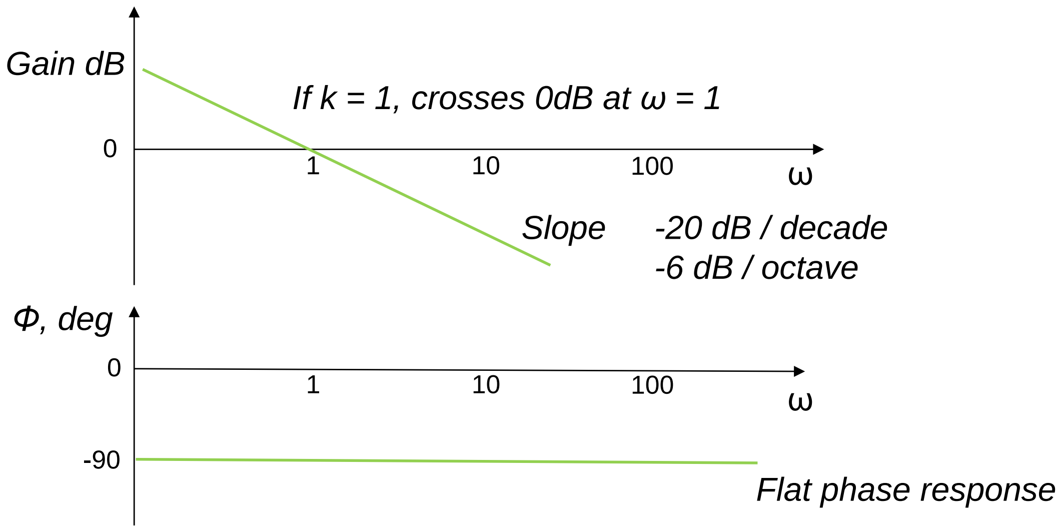
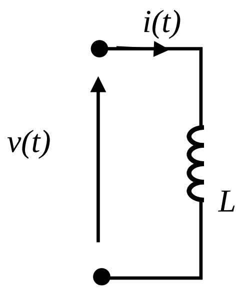


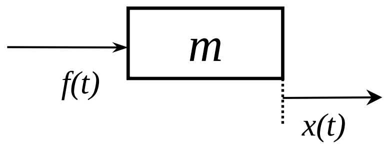

***Fig. 5. Bode plot of an integrator***

The zero crossing of the gain response is determined by the constant term ‘k’, i.e. if k = 10, the response would cross the x axis at ω = 10; An octave is a doubling in frequency; the TF has only imaginary terms hence the phase is a flat -90° shift i.e. the phase shift does not change with frequency.

### *Differentiators*




$$v(t) = L\frac{di}{dt}$$




$$f(t) = m\frac{d\dot{x}}{dt}$$

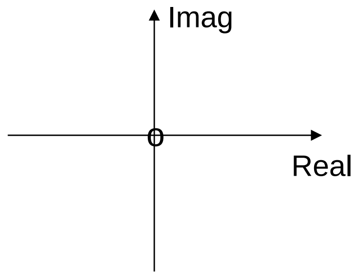
*Inductor differentiates current*

*Mass differentiates velocity*

*In the Laplace domain the general form of a differentiator is:*

$$\frac{Y(s)}{X(s)} = const.s$$

Which is a single zero at the origin:




As we did for the integrator previously, we derive the bode plot by making the substitution s=jω, and evaluating the magnitude and argument of the transfer function

$$H(s) = ks \qquad\qquad H(j\omega) = kj\omega$$

$$mag\big(H(j\omega)\big) = 20\log(k\omega)$$

*zero dB when at ω=1/k*

$$\arg\left(H(j\omega)\right) = \tan^{-1}\left(\frac{imag}{real}\right) \qquad\qquad \arg(j\omega k) = \tan^{-1}\left(\frac{\omega k}{0}\right) = 90°$$

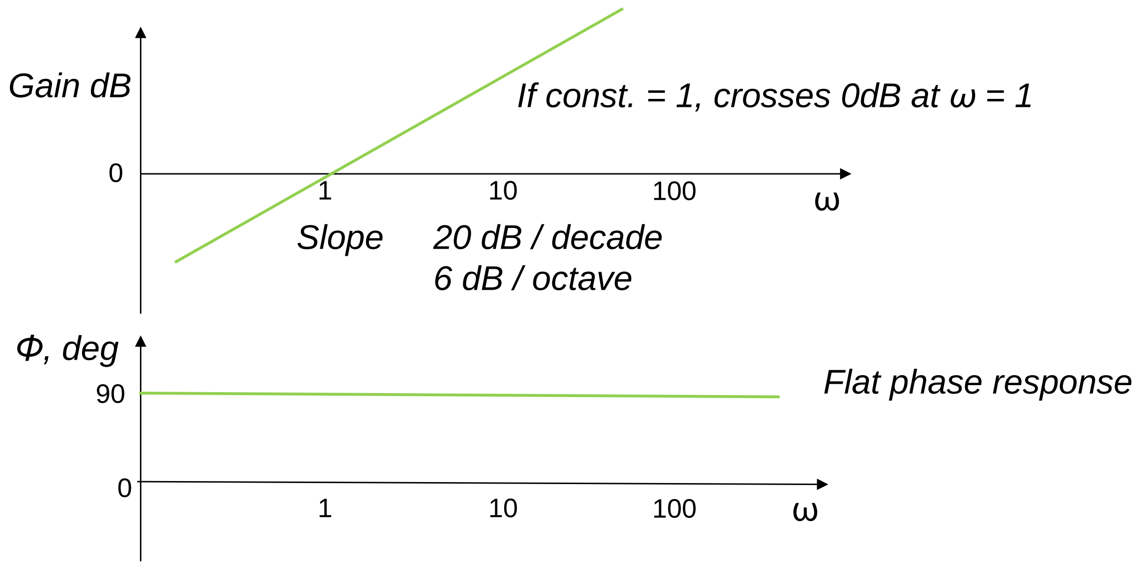



***Fig. 6. Bode plot of a differentiator***

*Practical integration and differentiation*

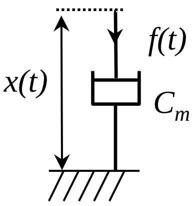
Integrators and differentiators are common building blocks in signal processing or control systems; however, implementations of these ideal systems often require some thought and a bit of care.

Integrators can be physically realised without too many problems but the infinitely high gain at DC (low frequency) can lead to 'wind-up' – a saturation of the output over time.

Differentiators are harder to implement because the gain becomes infinite as the frequency increases. This can exaggerate noise in a signal and result in unstable or simply unknown behaviour at high frequencies as it is impossible to construct a system with an ideal response up to an infinitely high frequency. Another way of looking at this is that the differentiation is not a proper system as the exponent of the numerator is higher than the denominator.

You may have spotted that integrators behave a little like low pass filters, whilst differentiators are a little like high pass filters. In practical applications filters are often used to approximate integrators and all practical differentiators have to be based on filter response.

Finally, something to cogitate on:




$$f(t) = C_m \frac{dx}{dt}$$

*Dashpot differentiates position*

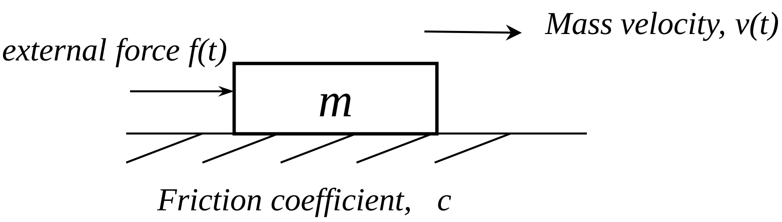
How does this work? We have previously suggested that 'order' of the TF comes from the number of independent energy storage elements and this is a first order system with a dashpot (a damper), and we know dampers do not store energy.

This illustrates how the math is general but when we apply it to physical systems there may be some constraints imposed – for example, recall how the application of the Fourier transform to real signals made us have to think about the units of the time/frequency domains. Here, the approximation that the order is determined by the number of energy storage element holds only if the variables are root-power quantities (position is not a root-power quantity).

### 2.3.5 First order systems

The pure integrator and pure differentiator that we saw in the last section are idealised systems. Even leaving to one side the difficulties with their idealised behaviours (wind-up, infinitely high gain at high frequencies), they could not be implemented with physical components because of the parasitic losses inherent in all physical components. Hence in many cases the most simple system model that can be applied is two element first order.

#### *1st order low-pass*




Consider a mass resting on a surface with friction coefficient ‘c’. We can write out the differential equation as before:

$$f(t) = m\frac{dv(t)}{dt} + cv(t)$$

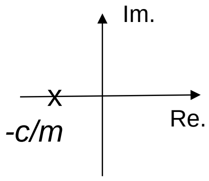
and convert into the Laplace domain:

$$F(s) = msV(s) + cV(s)$$

then rearrange to express as a transfer function:

$$\frac{V(s)}{F(s)} = \frac{1}{ms + c} = \frac{\frac{1}{c}}{\frac{m}{c}s+1} = \frac{\frac{1}{m}}{s + \frac{c}{m}}$$

The pole zero diagram for this is a single pole at -c/m




We can also say that when s=jω=0, (i.e. zero frequency, DC) the TF reduces to *1/c* – this is the ***sensitivity*** of the system. The system ***cut-off frequency*** is c/m and the reciprocal, m/c, is the system ***time constant.***

The Bode plot for this system can be calculated in exactly the same way as we did for integrators and differentiators – by evaluating the magnitude and argument of the TF over a range of frequencies. The bode plot now has a few more features than for the integrator/differentiator. The system starts to attenuate above a certain frequency and note how the phase of the system changes over the range $0.1\frac{c}{m} < \omega < 10\frac{c}{m}$.

In this bode plot we have also shown the convention of plotting straight-line approximations (dashed) to the true responses - this is commonly used in Bode diagrams.

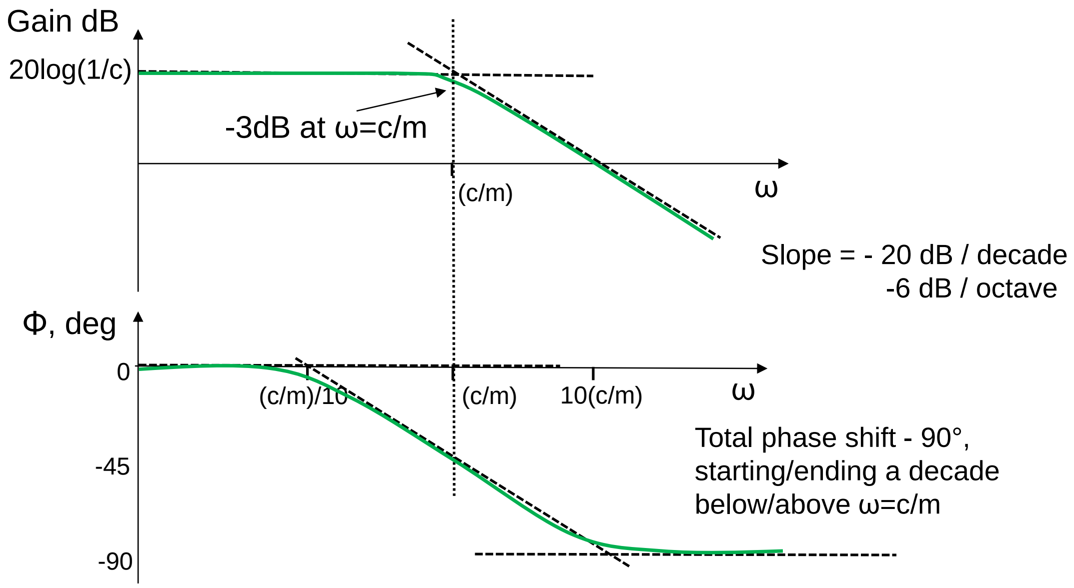
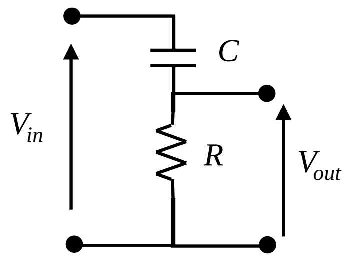



***Fig. 7. Bode plot of a first order low pass response***

The system we have considered here is one form of a mechanical low pass filter, however there are, of course, other mechanical arrangements and electrical equivalents with this response.

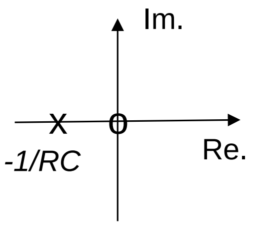
### *1st order high pass filter*




$$\frac{V_{out}}{V_{in}} = \frac{R}{R + X_c} = \frac{R}{R + \frac{1}{SC}} = \frac{s}{s + \frac{1}{RC}}$$

In this electrical system we see something slightly different – the input and output variable are of the same type (both voltages). Remember the math does not know what real-world significance we assign to variables; we are still working with root-power quantities so our one energy storage element per system order still holds. The transfer function in this case can be derived using the potential divider rule (where $X_c$ is the reactance of the capacitor)

In the system there is a zero at s = 0, and a pole at s = -1/RC. Giving a pole/zero diagram:




In this system when s=jω=0, the TF reduces to 0 and as s→∞, TF →1.

The system cut-off frequency is 1/RC and RC is the system time constant. Again, we can calculate the TF by evaluating the magnitude and argument of the TF over a range of frequencies.

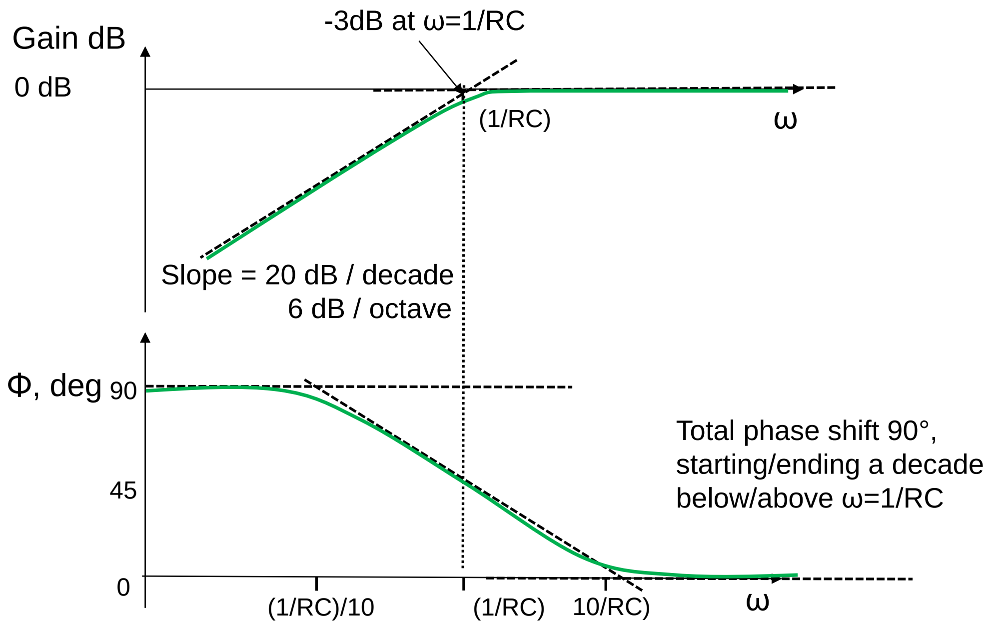
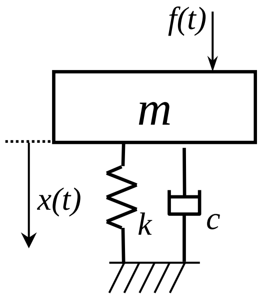



***Fig.85. Bode plot of a first order high pass response***

### 2.3.6 Second order systems

Adding an additional element to our system will make it a second order system – you should be at least slightly familiar with these from previous courses.

Let us consider a simple mass/spring/damper system:




As before we start with the differential equation derived from a force balance approach:

$$f(t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

Then, convert to Laplace domain, factorise and rearrange

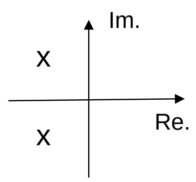
$$\frac{F(s)}{X(s)} = ms^2 + cs + k$$

Finally express in terms of the input/output we want:

$$TF = \frac{1}{ms^2 + cs + k} = \frac{\frac{1}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

When s=jω=0, the TF reduces to 1/k

The system has two poles, which may be complex



You should have already encountered the characteristic equation for a second order system:

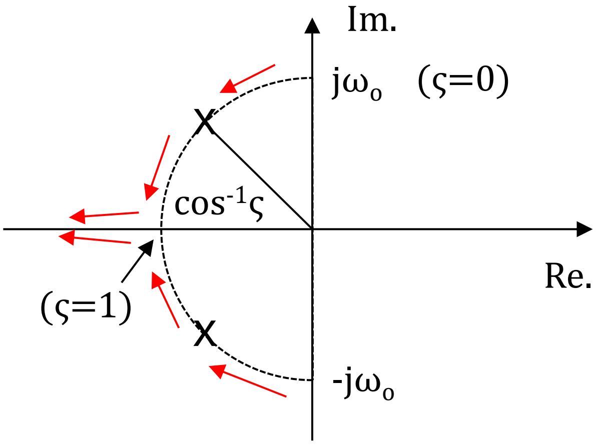
$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \frac{k}{m}x$$

We can make comparison with the characteristic equation we have derived in the Laplace domain to link physical parameters mass, compliance and damping with resonant frequency ω and damping ratio ζ.

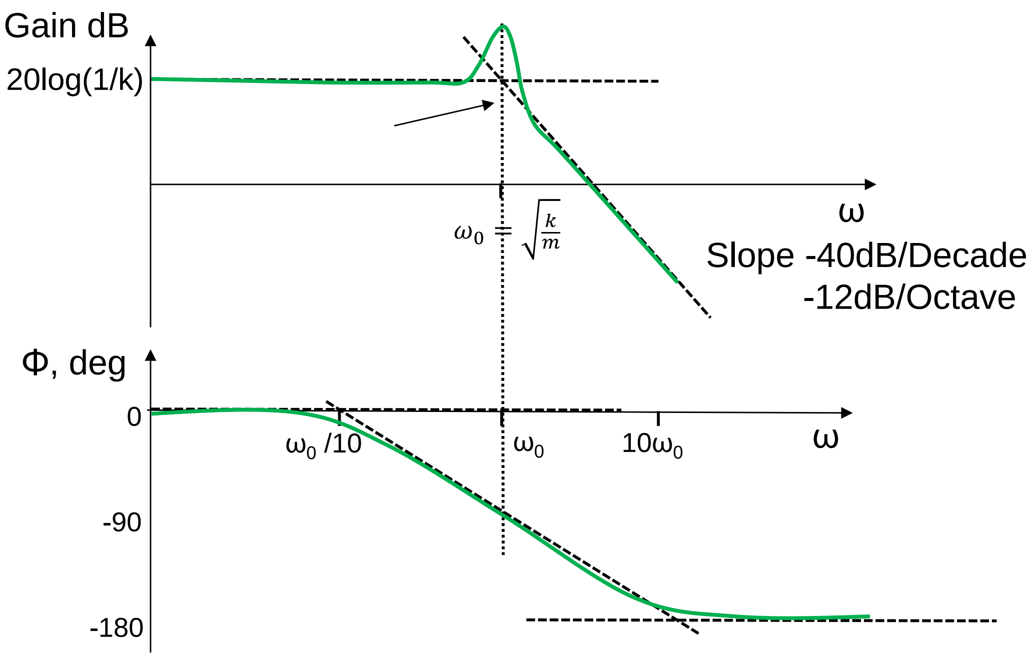
We find that:

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad\qquad \zeta = \frac{c}{2\sqrt{mk}}$$

If we vary a parameter (i.e. mass etc) the poles of the system move on the complex plane and the path they follow is called the ***root locus.*** The root locus below shows what happens as we increase damping; eventually the poles join the real axis and the system natural response is no longer oscillatory (i.e. it is critically damped and will not 'ring' in response to a step input).




In the Bode plot for this particular second order system we see that it is a low-pass response. The second order system produces a resonant rise in the gain response at the cut-off frequency, $\omega_0$ (for damping ratio less than 1), but notably the roll-off beyond the cut-off is now -40 dB/decade, and the phase shift is 180° total.




## 2.3.7 Constructing Bode plots from the sum of individual poles and zeros

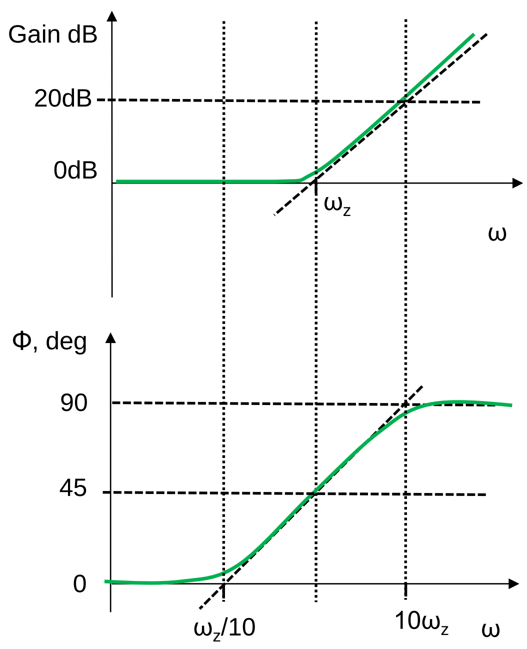
One of the powerful aspects of the Bode plot as a tool for control system design is the ability to construct the response of a system by adding the effect of all the individual poles and zeros – the gains (in dB) and phases simply add. This is normally done with the asymptotic approximations to the Bode responses.

As well as determining the Bode of a system by adding together the contributions of individual poles and zeros, we can also design a system based on a required Bode plot – you will see this when we move onto control.

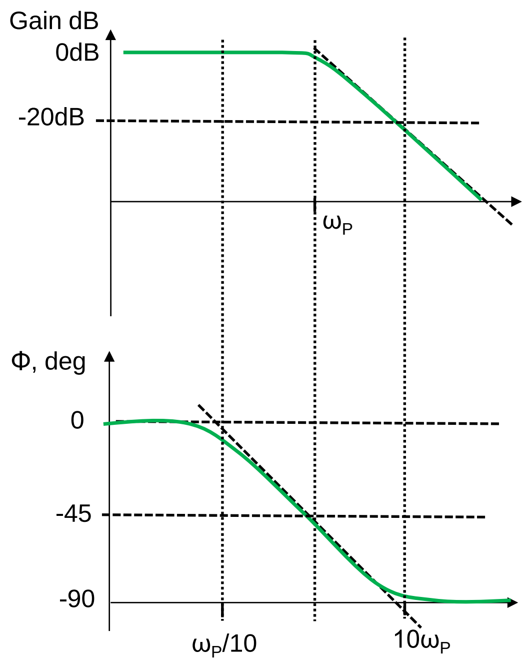
So far, the realisable systems we have seen are:

- 1st order low pass - 1 pole
- 1st order high pass – 1 pole & 1 zero
- 2nd order low pass – pair of poles.

So why have we not seen a system with 1 zero? This is the same issue that we had with the differentiator previously – it is not a proper system (n>m) and so can't be realised on its own, but fortunately we can identify the contribution made by a single zero when it occurs in a more complex system.




*Bode of an individual zero at $\omega_z$*




*Bode of an individual pole at $\omega_p$*

### 2.3.8 test yourself

1) What determines the ‘order’ of a system, mathematically?
2) How can you estimate the order of the system from the physical components which form it?
3) Why can’t systems with numerator of higher order than the denominator be realised physically?
4) What happens if a system has poles on the right hand (positive real) side of the pole/zero diagram?
5) What are the two graphs that make up the Bode plot of a system?
6) Which common mathematical operations are captured with: a single zero?; a single pole?
7) How is the gain response of a transfer function found?
8) How is the phase response of a transfer function found?
9) What is the ‘time constant’ of a system?
10) As a rule of thumb, over what range (relative to the frequency of a pole/zero) does the phase response change?
11) Why does expressing gain in dB help when calculating the bode plot of multiple poles/zeros.