Problem Sheet: Maximum Likelihood and Bayesian Inference

1. **Gaussian / Exponential model**

   Let a single observation x satisfy
   $$x \sim \mathcal{N}(\theta, \sigma^2),$$
   where σ^2 is known. We choose the prior
   $$\theta \sim \mathcal{N}(m, \rho\sigma^2).$$

   (a) What is the posterior distribution of θ given x?

   (b) Answer the same question in the exponential model: suppose now that $X \sim \mathcal{E}(\lambda)$ and that the prior on λ is
   $$\lambda \sim \mathcal{G}(a, b).$$
   What is the posterior distribution of λ given x?

2. **Maximum Likelihood Estimator (MLE)**

   Let (x_1, x_2) be two random observations of a pair (X_1, X_2). We have two candidates for the joint law of these observations:

   (i) X_1, X_2 are i.i.d. with $X_i \sim \mathcal{N}(\theta, 1)$;

   (ii) the pair has joint density
   $$g(x_1, x_2 \mid \theta) = \pi^{-3/2} \frac{\exp\big\{-(x_1 + x_2 - 2\theta)^2/4\big\}}{1 + (x_1 - x_2)^2}.$$

   In each of the two cases, find the maximum likelihood estimator (MLE) of θ. What do you observe?

3. **Bernoulli model.**

   Consider the Bernoulli model
   $$\mathcal{P} = \big\{\mathcal{B}(\theta)^{\otimes n} : \theta \in [0, 1]\big\},$$
   and an i.i.d. sample $X = (X_1, \ldots, X_n)$ with $X_i \sim \mathcal{B}(\theta)$. Let
   $$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$
   be the empirical mean.

   (a) Show that the maximum likelihood estimator $\hat{\theta}(X)$ is unique.

(b) Show that

$$\hat{\theta}(X) = \overline{X}_n.$$

4. **Genetics (from Lindley, 1965)**

   Suppose that in each individual of a large population there is a pair of genes, each of which can be either x or X, that controls eye color: individuals with genotype xx have blue eyes, while heterozygotes (those with Xx or xX) and those with XX have brown eyes. The proportion of blue–eyed individuals is p^2 and the proportion of heterozygotes is $2p(1-p)$, where $0 < p < 1$.

   Each parent transmits one of its own genes to the child; if a parent is a heterozygote, the probability that it transmits the gene of type X is $1/2$.

   (a) Assuming random mating, show that among brown–eyed children of brown–eyed parents, the expected proportion of heterozygotes is

   $$\frac{2p}{1+p^2}.$$

   (b) Suppose Judy, a brown–eyed child of brown–eyed parents, marries a heterozygote, and they have n children, all brown–eyed. Find the posterior probability that Judy is a heterozygote.

   (c) Using your result in (b), find the probability that Judy's first grandchild has blue eyes.

5. **Twins and Elvis Presley**

   Approximately $1/125$ of all births are fraternal twins and $1/300$ of births are identical twins. Elvis Presley had a twin brother (who died at birth).

   (a) What is the probability that Elvis was an identical twin?

   (b) You may approximate the probability of a boy or girl birth as $1/2$.

6. **Monty Hall / Let's Make a Deal.**

   The following problem is loosely based on the television game show *Let's Make a Deal.* At the end of the show, a contestant is asked to choose one of three large boxes, where one box contains a fabulous prize and the other two boxes contain lesser prizes.

   After the contestant chooses a box, Monty Hall, the host of the show, opens one of the two boxes containing the smaller prizes. (In order to keep the conclusion suspenseful, Monty does not open the box selected by the contestant.) Monty then offers the contestant the opportunity to switch from the chosen box to the remaining unopened box.

   (a) Should the contestant switch or stay with the original choice?

   (b) Calculate the probability that the contestant wins under each strategy.

   (c) (*Evil Monty variant*) Suppose now that Monty behaves as follows: if the contestant's initial choice is the winning box, Monty offers the opportunity to switch only with probability p (and otherwise simply reveals the prize and ends the game); if the contestant's initial choice is a losing box, Monty always offers the opportunity to switch. Conditioning on the event that Monty actually offers the contestant the chance to switch, does this change your answer to parts (a)–(b)? Justify your reasoning.

   This exercise is about being clear which information should be conditioned on when constructing a probability judgment.