Extended binomial coefficient

$$C_u^k = \binom{u}{k} = \begin{cases} \frac{u(u-1)\cdots(u-k+1)}{k!} & \text{if } k>0 \\ 1 & \text{if } k=0 \end{cases}$$

$|x|<1$, u is a real number, Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

	Sequence	Generating function
(1)	$C(n,k)$	$\sum_{k=0}^{\infty} C(n,k)x^k = (1+x)^n$
(2)	$C(n,k)a^k$	$(1+ax)^n$
(3)	$\overbrace{1,1,\cdots 1}^{n}$	$(1+x)(1+x^2+\cdots+x^n) = \frac{1-x^{n+1}}{1-x}$
(4)	$1,1,\cdots$	$\frac{1}{1-x}$
(5)	a^k	$\frac{1}{1-ax}$
(6)	$k+1$	$\frac{1}{(1-x)^2}$
(7)	C_{n+k-1}^k	$\frac{1}{(1-x)^n}$
(8)	$(-1)^k C_{n+k-1}^k$	$\frac{1}{(1+x)^n}$
(9)	$C_{n+k-1}^k a^k$	$\frac{1}{(1-ax)^n}$
(10)	$\frac{1}{k!}$	e^x
(11)	$\frac{(-1)^{k+1}}{k}$	$\ln(1+x)$

Solving Recurrence Relations

$a_n = 3a_{n-1}$ with initial conditions $a_0 = 2$

$$a_n = 3a_{n-1}$$

$$a_n x^n = 3a_{n-1}x^n, \quad n=1,2\cdots$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1}x^n$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$G(x) - a_0 = 3x\,G(x)$$

$$G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty}(3x)^k$$

$$= \sum_{k=0}^{\infty} 2\cdot 3^k x^k$$

$$a_n = 2\cdot 3^n$$

Combination

Find the number of ways to select r objects of n different if we must select at least one of each kind

$$G(x) = (x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)\cdots$$

$$= (x+x^2+\cdots)^n$$

$$= x^n\left(\frac{1}{1-x}\right)^n$$

$$= x^n\sum_{k=0}^{\infty}\binom{-n}{k}x^k$$

$$= x^n\sum_{k=0}^{\infty}C_{n+k-1}^k x^k = \sum_{k=0}^{\infty} C(k-1,k-n)x^k$$

$$a_r = C(n+r-n-1, r-n)$$

$$= C(r-1, r-n)$$

$$= x^r\sum_{k=0}^{\infty} C_{-r}^k x^k$$

$$= x^r\sum_{k=0}^{\infty}(-1)^k C_{n+k-1}^k(-1)^k x^{k+n}$$

$$= x^r\sum_{k=0}^{\infty} C_{n+k-1}^k x^{k+n}$$

$$= x^n\sum_{t=n}^{\infty} C_{t-1}^{t-n} x^t$$

$$= x^n\sum_{t=n}^{\infty} C_{t-1}^{t-n} x^t$$

permutation

$$\{a_n\} \quad \sum_{n=0}^{\infty}\frac{a_n}{n!}x^n$$

How many different strings with four characters can be formed from the letter in CHANCE AND CHOICE

4 Cs, 2 Hs, 2As, 2Ns, 2 Es, 1D, 1O, 1I.

$$G(x) = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\left(1+x+\frac{x^2}{2!}\right)^4(1+x)^3$$

We need the coefficient $\frac{x^4}{4!}$

§8.3 分治算法

将规模n的问题分成a个子问题，其中每个子问题的规模是$\frac{n}{b}$，又需要$g(n)$的额外运算组合子问题

$$\therefore f(n) = af\left(\frac{n}{b}\right) + g(n)$$

如果 $f(n) = af\left(\frac{n}{b}\right)+c$，$f(n)$是 $\begin{cases} O(n^{\log_b a}) & \text{如果 } a>1 \\ O(\log n) & \text{如果 } a=1 \end{cases}$ $\quad f(n) = a^k\left(f(1)+\frac{c}{a-1}\right)-\frac{c}{a-1}$

主定理：$f(n) = af\left(\frac{n}{b}\right) + cn^d$

$$f(n)\text{是} \begin{cases} O(n^d) & a<b^d \\ O(n^d\log n) & a=b^d \\ O(n^{\log_b a}) & \cancel{a>b} \ a>b^d \end{cases}$$

§8.5 容斥原理

$A_1, A_2, \cdots A_n$是有穷集.

$$|A_1\cup A_2\cup\cdots\cup A_n| = \sum_{1\le i\le n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k|$$

$$-\cdots+(-1)^{n+1}|A_1\cap A_2\cap\cdots\cap A_n|$$

$$N(P_1'P_2'\cdots P_n') = N - \sum_{1\le i\le n}N(P_i) + \sum_{1\le i<j\le n}N(P_iP_j) - \sum_{1\le i<j<k\le n}N(P_iP_jP_k)$$

应用 埃拉托色尼筛 $\qquad + \cdots + (-1)^n N(P_1P_2\cdots P_n)$

错位排列

$$D_n = n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right]$$

$$= n! - C_n^1(n-1)! + C_n^2(n-2)! - \cdots + (-1)^n C_n^n(n-n)!$$