

Extended binomial coefficient

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)\cdots(u-k+1)}{k!} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

$|x| < 1$, u is a real number, then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Sequence

(1) $C(n, k)$

(2) $C(n, k) a^k$

(3) $\frac{1}{1, 1, \dots}$

(4) $1, 1, \dots$

(5) a^k

(6) $b+k$

(7) $C(n+k-1, k)$

(8) $(-1)^k C(n+k-1, k)$

(9) $C(n+k-1, k) a^k$

(10) $\frac{1}{k!} b^{k+1}$

(11) $\frac{(-1)^k}{k}$

Generating function

$$\sum_{k=0}^{\infty} C(n, k) x^k = (1+x)^n$$

$$1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$$

$$\frac{1}{1-x}$$

$$\frac{1}{1-ax}$$

$$\frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^n}$$

$$\frac{1}{(1+x)^n}$$

$$\frac{1}{(1-ax)^n}$$

$$e^x$$

$$n(1+x)$$

Solving Recurrence Relations

$$a_n = 3a_{n-1} \quad \text{with initial conditions } a_0 = 2$$

$$a_n = 3a_{n-1}$$

$$a_n x^n = 3a_{n-1} x^n, \quad n=1, 2, \dots$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1} x^n$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$G(x) = 3xG(x)$$

$$G(x) = \frac{2}{1-3x} \geq 2 \sum_{k=0}^{\infty} (3x)^k$$

$$= \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

$$a_n = 2 \cdot 3^n$$



Combination

Find the number of ways to select r objects of n different if we must select at least one of each kind

$$G(x) = (x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)\cdots$$

$$= (x+x^2+\cdots)^n$$

$$= x^n \left(\frac{1}{1-x} \right)^n$$

$$= x^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$$= x^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} C(n+k-1, k) x^{k+n}$$

$$a_r = C(n+r-1, r-n) = C(n, r-n)$$

permutation

$$\{a_n\} \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

How many different strings with four characters can be formed from the letter in CHANCE AND CHOICE

$$4C_5, 2H_5, 2A_5, 2N_5, 2, E_5, 1D, 1O, 1I$$

$$G(x) = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right) (1+x+\frac{x^2}{2!})^4 (1+x)$$

We need the coefficient $\frac{x^9}{9!}$



§8.3 介治算法

将规模 n 的问题分成 a 个子问题, 其中每个子问题的规模是 $\frac{n}{b}$, 又需

$g(n)$ 的额外运算组合子问题

$$\therefore f(n) = af\left(\frac{n}{b}\right) + g(n)$$

$$\text{如果 } f(n) = af\left(\frac{n}{b}\right) + g(n) \text{ 是 } \begin{cases} O(n^{\log_b a}) & \text{如果 } a > 1 \\ O(\log n) & \text{如果 } a = 1 \end{cases} \quad f(n) = a^k f\left(\frac{n}{a^k}\right) + \frac{c}{a-1} \left| \frac{n}{a^k} - \frac{c}{a-1} \right|$$

$$\star \text{ 定理: } f(n) = af\left(\frac{n}{b}\right) + cnd$$

$$f(n) \text{ 是 } \begin{cases} O(n^d) & a < b^d \\ O(n^d \log n) & a = b^d \\ O(n^{\log_b a}) & a > b^d \end{cases}$$

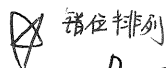
§8.5 容斥原理

A_1, A_2, \dots, A_n 是有限集.

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

$$N(P_1' P_2' \dots P_n') = N - \sum_{1 \leq i \leq n} N(P_i) + \sum_{1 \leq i < j \leq n} N(P_i P_j) - \sum_{1 \leq i < j < k \leq n} N(P_i P_j P_k) + \dots + (-1)^n N(P_1 P_2 \dots P_n)$$

应用 埃拉托斯特尼



错位排列

$$D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right]$$

$$= n! - C_n^1 (n-1)! + C_n^2 (n-2)! - \dots + (-1)^n C_n^n (n-n)!$$