8 Advanced Counting Techniques

§8.1 Applications of Recurrence Relations

the Fibonacci Numbers $f_n = f_{n-1} + f_{n-2}$

the Tower of Hanoi $H_n = 2H_{n-1} + 1$

example. find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s

End with a 1	Any bit string of length n-1 with no two consecutive 0s	1	a_{n-1}
End with a 0	Any bit string of length n-2 with no two consecutive 0s	1 0	a_{n-2}

§8.2 Solving Linear Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$\Downarrow$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \quad \text{Characteristic quation}$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

THEOREM 1 $r^2 - c_1 r - c_2 = 0$. has two distinct roots r_1, r_2

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

then $\begin{cases} a_0 = \alpha_1 + \alpha_2 \\ a_1 = \alpha_1 r_1 + \alpha_2 r_2 \end{cases}$ $\quad \alpha_1 = \frac{a_1 - a_0 r_2}{r_1 - r_2}$ $\quad \alpha_2 = \frac{a_0 r_1 - a_1}{r_1 - r_2}$

the formula for the Fibonacci number

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

THEOREM 2 $r^2 - c_1 r - c_2 = 0$ has only one root r_0.

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

THEOREM 3 $r^k - c_1 r^{k-1} - \cdots - c_k = 0$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

THEOREM 4 $r^k - c_1 r^{k-1} - \cdots - c_k = 0$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

linear nonhomogeneous recurrence relations with constant coefficients.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$

s is not a root of the characteristic equation

particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

s is a root of this characteristic equation and its multiplicity is m.

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

§8.4 Generating Function

the sequence $a_0, a_1, \cdots a_k, \cdots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

The generating function of 1, 1, 1, 1, 1, 1 is

$$1 + x + x^2 + x^3 + x^4 + x^5 = \frac{(x^6 - 1)}{x - 1}$$

Useful Facts About Power Series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

(1) $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

(2) $f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) x^k = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$

$= (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$

$= f(x) \cdot g(x)$

(3) $\alpha \cdot f(x) = \sum_{k=0}^{\infty} \alpha a_k x^k$

(4) $x \cdot f'(x) = \sum_{k=0}^{\infty} k a_k x^k$

(5) $f(\alpha x) = \sum_{k=0}^{\infty} \alpha^k a_k x^k$

$(1+x)^n \quad C_n^0, C_n^1, C_n^2 \cdots C_n^n$

$a_k = 1$

$G(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}$ converges for $|x| < 1$

$a_k = b$

$G(x) = \sum_{k=0}^{\infty} b x^k = b \sum_{k=0}^{\infty} x^k = b(1 + x + x^2 + \cdots) = \frac{b}{1-x}$

$a_k = b^k$

$G(x) = \sum_{k=0}^{\infty} b^k x^k = 1 + bx + (bx)^2 + \cdots = \frac{1}{1-bx} \quad \frac{1}{1-bx}$

$a_n = \frac{1}{n!}$

$G(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots \quad G(x) = e^x$

$a_n = \frac{1}{n}$ for odd n, $-\frac{1}{n}$ for even n

$G(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad G(x) = \ln(1+x)$

$C_n = n+1 \iff G(x) = \frac{1}{(1-x)^2}$