# Week 1: Optimization For Data Science

## Convex Optimization

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# Contents

# 1 Module 0: Preliminaries (The Toolkit)

Before diving into affine and convex sets, we must rigorously define the space we are working in and how we measure size and direction.

## 1.1 Vector Spaces and Subspaces

**Definition 1.1** (Linear Subspace). *A* ***nonempty*** *subset* $V \subseteq \mathbb{R}^n$ *is a* ***linear subspace*** *if it is closed under linear combinations:*

$$\forall u, v \in V,\ \forall \alpha, \beta \in \mathbb{R} \ \Rightarrow\ \alpha u + \beta v \in V.$$

**Crucial Property:** Every linear subspace contains the origin $\mathbf{0}$.
*Proof:* Since $V \neq \emptyset$, pick any $v \in V$. Then $0 \cdot v + 0 \cdot v = \mathbf{0} \in V$.

## 1.2 Norms and Balls

To define "circles" or "balls" in higher dimensions, we need a norm.

**Definition 1.2** (Norm). *A function* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is a norm if, for all* $x, y \in \mathbb{R}^n$ *and all* $t \in \mathbb{R}$,

1. *(Positive definiteness)* $\|x\| \geq 0$ *and* $\|x\| = 0 \iff x = \mathbf{0}$.
2. *(Homogeneity)* $\|tx\| = |t|\,\|x\|$.
3. *(Triangle inequality)* $\|x + y\| \leq \|x\| + \|y\|$.

### 1.2.1 Common Norms

For $x = (x_1, \dots, x_n)$:

- **$\ell_1$-Norm (Manhattan):** $\|x\|_1 = \sum |x_i|$. The unit ball is a diamond (square tilted by 45° in 2D).
- **$\ell_2$-Norm (Euclidean):** $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$. The unit ball is the (solid) Euclidean ball $\{x \mid \|x\|_2 \leq 1\}$. Its boundary $\{x \mid \|x\|_2 = 1\}$ is the sphere.
- **$\ell_\infty$-Norm (Max):** $\|x\|_\infty = \max |x_i|$. The unit ball is an axis-aligned square/cube.

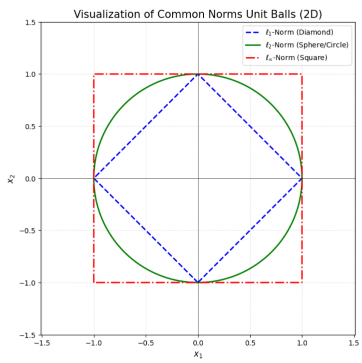


**Figure 1:** Unit balls under common norms.

## 1.3 Span vs. Affine Hull vs. Convex Hull

This distinction is the core of the entire lecture. Let $S = \{x_1, \dots, x_k\}$.

| Concept | Coefficients Constraint | Geometric Meaning |
|---|---|---|
| **Linear Span** span($S$) | No constraint $\sum \theta_i x_i,\ \theta_i \in \mathbb{R}$ | The infinite subspace generated by $S$ (passes through $\mathbf{0}$). |
| **Affine Hull** $\mathbf{aff}(S)$ | $\sum \theta_i = 1$ $\theta_i \in \mathbb{R}$ | The infinite flat plane/line passing through $S$. |
| **Convex Hull** $\mathbf{conv}(S)$ | $\sum \theta_i = 1$ $\theta_i \geq 0$ | The smallest solid polygon/polytope enclosing $S$. |

# 2 Module 1: Affine Sets

## 2.1 Definition and Geometry

An affine set is a "linear subspace that has been shifted." It does not need to pass through the origin.

**Definition 2.1** (Affine Set). *A set $C \subseteq \mathbb{R}^n$ is affine if the line through any two distinct points in $C$ lies in $C$.*

$$\forall x_1, x_2 \in C,\ \forall \theta \in \mathbb{R} \implies \theta x_1 + (1-\theta)x_2 \in C.$$

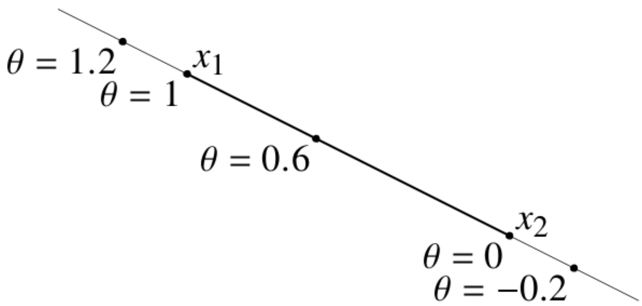


**Figure 2:** Illustration of an affine set as a shifted subspace.

**Example 2.1** (Line through two points in $\mathbb{R}^2$). *Let $x_1 = (1,2)$ and $x_2 = (4,1)$. The line through them is*

$$L = \{\theta x_1 + (1-\theta)x_2 \mid \theta \in \mathbb{R}\}.$$

- $\theta \in [0,1]$*: points on the* ***segment*** *between $x_1$ and $x_2$.*
- $\theta < 0$ *or* $\theta > 1$*: points* ***beyond*** *the segment (still on the same line).*

**Example 2.2** (Hyperplane (affine but not a linear subspace)).

$$H = \{x \in \mathbb{R}^3 \mid 2x_1 - x_2 + x_3 = 5\}$$

*is an affine set. It is* ***not*** *a linear subspace because $0 \notin H$ (since $0 \neq 5$).*

**Example 2.3** (General solution form of $\{x \mid Ax = b\}$). *If the system $Ax = b$ is consistent, then the solution set is*

$$\{x \mid Ax = b\} = x_p + \mathcal{N}(A),$$

*where $x_p$ is any particular solution and $\mathcal{N}(A)$ is the nullspace.*

*Concrete example.*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

*Solve $Ax = b$: pick $x_2 = t$, then $x_1 = 1 - t$ and $x_3 = 2 - t$. So the affine set is*

$$\{(1 - t,\ t,\ 2 - t) \mid t \in \mathbb{R}\}.$$

**Example 2.4** (Affine hull of finitely many points). *For points $x_1, \dots, x_k$, the **affine hull** is*

$$\operatorname{aff}\{x_1, \dots, x_k\} = \left\{ \sum_{i=1}^{k} \theta_i x_i \;\middle|\; \sum_{i=1}^{k} \theta_i = 1 \right\}.$$

*Example in $\mathbb{R}^3$. Let $p_1 = (0,0,0)$, $p_2 = (1,0,0)$, $p_3 = (0,1,1)$. Then $\operatorname{aff}\{p_1, p_2, p_3\}$ is a **plane** (unless the points are collinear).*

## 2.2 Algebraic Representation (Ax = b)

This connects the geometry to linear algebra (Solving systems of equations).

**Theorem 2.1** (Solution Sets). *If the system $Ax = b$ is consistent (i.e., $C = \{x \mid Ax = b\} \neq \emptyset$), then $C$ is an affine set.*

*Proof.* Let $x_1, x_2 \in C$, meaning $Ax_1 = b$ and $Ax_2 = b$. Consider $y = \theta x_1 + (1 - \theta)x_2$.

$$A(\theta x_1 + (1 - \theta)x_2) = \theta A x_1 + (1 - \theta)A x_2 = \theta b + (1 - \theta)b = b.$$

Thus, $y \in C$. □

**Geometric Interpretation:** If $x_0$ is a particular solution ($Ax_0 = b$), then the entire set can be written as:

$$C = x_0 + \mathcal{N}(A)$$

where $\mathcal{N}(A) = \{z \mid Az = 0\}$ is the Nullspace (a Linear Subspace).

- If $C \neq \emptyset$ and $\operatorname{rank}(A) = r$, then $\dim(C) = \dim(\mathcal{N}(A)) = n - r$.

## 2.3 Projection onto Affine Sets

The projection of $y$ onto $C = \{x \mid Ax = b\}$. This is a constrained optimization problem:

$$\min_x \frac{1}{2} \|x - y\|_2^2 \quad \text{s.t.} \quad Ax = b$$

**Derivation Step-by-Step:** 1. Form the Lagrangian: $L(x, \nu) = \frac{1}{2} \|x - y\|_2^2 + \nu^T(Ax - b)$. 2. Take gradient w.r.t $x$: $\nabla_x L = (x - y) + A^T \nu = 0 \implies x = y - A^T \nu$. 3. Substitute $x$ into constraint $Ax = b$:

$$A(y - A^T \nu) = b \implies Ay - AA^T \nu = b \implies AA^T \nu = Ay - b.$$

4. Assuming $A$ is full row rank, $AA^T$ is invertible:

$$\nu = (AA^T)^{-1}(Ay - b).$$

5. Substitute $\nu$ back into the expression for $x$:

$$\boxed{x^* = y - A^T(AA^T)^{-1}(Ay - b)}$$

# 3 Module 2: Convex Sets

## 3.1 Definition and Geometry

**Definition 3.1** (Convex Set)**.** *A set $C$ is convex if the line* ***segment*** *between any two points in $C$ lies in $C$.*

$$\forall x_1, x_2 \in C,\ \forall \theta \in [0,1] \implies \theta x_1 + (1-\theta)x_2 \in C.$$

**Definition 3.2** (Convex Combination)**.** *Given points $x_1, \dots, x_k \in \mathbb{R}^n$, a point $x$ is a* ***convex combination*** *of $\{x_1, \dots, x_k\}$ if*

$$x = \sum_{i=1}^{k} \theta_i x_i, \qquad \theta_i \geq 0, \qquad \sum_{i=1}^{k} \theta_i = 1.$$

**Intuition:** Convex sets have no ”dents” or ”holes.” If you wrap a rubber band around the set, it touches the boundary everywhere.

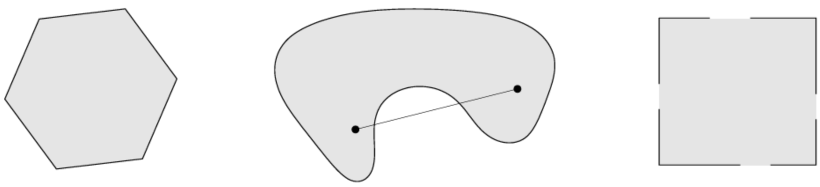

**Figure 3:** Illustration of a convex set (line segments stay inside).

## 3.2 Examples and Counter-Examples

- **Convex:**
  - Hyperplanes ($a^T x = b$) and Halfspaces ($a^T x \leq b$).
  - Norm balls ($\|x\| \leq r$). Proof uses the triangle inequality.
  - Polyhedra ($Ax \leq b$).
- **Not Convex:**
  - The Unit Circle Boundary ($\{x \mid \|x\|_2 = 1\}$). *Reason:* Take two points on the circle. The midpoint is inside the circle, not on the boundary.
  - A ”Donut” or Annulus.

**Definition 3.3** (Convex hull / polytopes)**.** *For points $v_1, \ldots, v_m \subset \mathbb{R}^n$, their convex hull is*

$$\operatorname{conv}\{v_1, \ldots, v_m\} = \left\{ \sum_{i=1}^{m} \lambda_i v_i \;\middle|\; \lambda_i \geq 0,\; \sum_i \lambda_i = 1 \right\}.$$

*This is convex by definition (closed under convex combinations).*

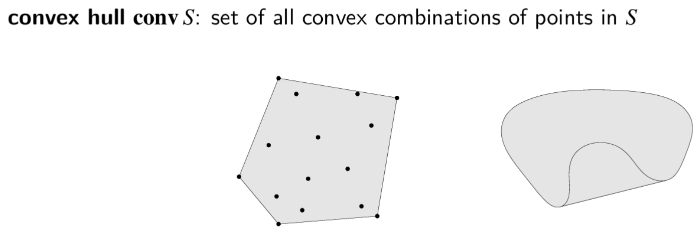


**Figure 4:** Example of a convex hull in $\mathbb{R}^2$.

**Example 3.1** (Norm balls (classic convex sets))**.** •

- *$\ell_1$-ball: $\{x \mid \|x\|_1 \leq r\}$ is convex.*
- *$\ell_2$-ball: $\{x \mid \|x\|_2 \leq r\}$ is convex.*
- *$\ell_\infty$-ball: $\{x \mid \|x\|_\infty \leq r\}$ is convex.*

*Each can be shown convex via the triangle inequality. Moreover, $\ell_1$- and $\ell_\infty$-balls can be written as finite intersections of halfspaces, while the $\ell_2$-ball can be written as an intersection of (possibly infinitely many) supporting halfspaces.*

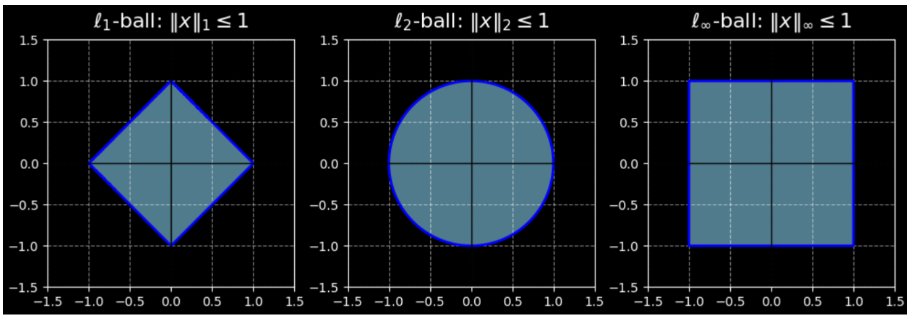


**Figure 5:** Comparison of different norm constraints in 2D space.(a) $\ell_1$-ball: $\|x\|_1 \leq 1$(b) $\ell_2$-ball: $\|x\|_2 \leq 1$(c) $\ell_\infty$-ball: $\|x\|_\infty \leq 1$

**Definition 3.4** (Ellipsoid)**.** *An ellipsoid in $\mathbb{R}^n$ can be written as*

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \;\middle|\; (x - x_c)^T P^{-1} (x - x_c) \leq 1 \right\},$$

*where $x_c \in \mathbb{R}^n$ is the center and $P \in \mathbb{S}^n_{++}$ is symmetric positive definite.*

**Remark 3.1** (Ball as an affine image of the unit ball)**.** *Let $r > 0$ and define $B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$. Then*

$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \leq 1\}.$$

***Proof.*** *If $x \in B(x_c, r)$, set $u = (x - x_c)/r$. Then $\|u\|_2 \leq 1$ and $x = x_c + ru$. Conversely, if*

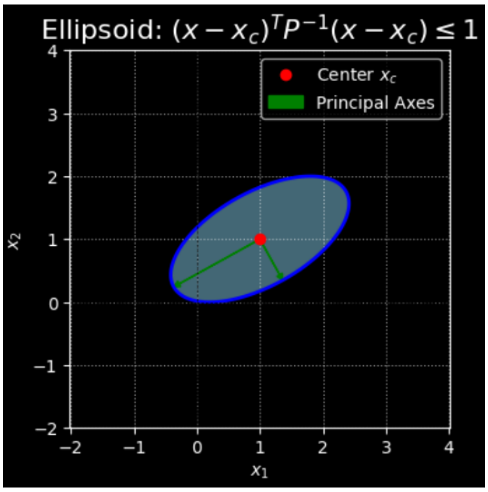


**Figure 6:** An ellipsoid region defined by the quadratic inequality $(x - x_c)^T P^{-1}(x - x_c) \leq 1$. The shaded area represents the convex set derived from matrix $P$ and center $x_c$.

*$x = x_c + ru$ with $\|u\|_2 \leq 1$, then $\|x - x_c\|_2 = r\|u\|_2 \leq r$.*

**Exercise 3.1** (Ellipsoid in "axis-length" form)**.** ***Question.***

*Consider the ellipsoid*

$$E = \left\{x \in \mathbb{R}^2 \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\right\},$$

*with $x_c = (1, 2)$ and $P = \mathrm{diag}(4, 1)$. Rewrite $E$ in axis-length form and express it as an affine image of the unit ball.*

***Answer.*** *Since $P^{-1} = \mathrm{diag}(1/4, 1)$,*

$$\frac{(x_1 - 1)^2}{4} + (x_2 - 2)^2 \leq 1.$$

*Thus the semi-axis lengths are $\sqrt{4} = 2$ along $x_1$ and $\sqrt{1} = 1$ along $x_2$.*

*One valid $A$ satisfying $AA^T = P$ is $A = \mathrm{diag}(2, 1)$, hence*

$$E = \{x_c + Au \mid \|u\|_2 \leq 1\}.$$

**Exercise 3.2** (Ellipsoid membership test)**.** ***Question.***

*Let $x_c = (0, 0)$, $P = \mathrm{diag}(9, 4)$,*

$$E = \{x \mid x^T P^{-1} x \leq 1\},$$

*and $x = (2, 1)$. Decide whether $x \in E$.*

***Answer.*** *Since $P^{-1} = \mathrm{diag}(1/9, 1/4)$,*

$$x^T P^{-1} x = \frac{2^2}{9} + \frac{1^2}{4} = \frac{4}{9} + \frac{1}{4} = \frac{16}{36} + \frac{9}{36} = \frac{25}{36} < 1.$$

*So $x \in E$.*

**Exercise 3.3** (Ellipsoid = affine image of the unit ball)**.** ***Question.***

*Show that for $P \in \mathbb{S}^n_{++}$,*

$$E = \left\{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\right\} = \{x_c + Au \mid \|u\|_2 \le 1\}$$

*for some nonsingular $A$ satisfying $AA^T = P$.*

***Answer.*** *Since $P \succ 0$, it has a Cholesky factorization $P = LL^T$ with $L$ nonsingular. Let $A = L$. Then $AA^T = P$.*

*($\subseteq$) Take any $x \in E$. Define $u = A^{-1}(x - x_c)$ (well-defined since $A$ is nonsingular). Then*

$$(x - x_c)^T P^{-1}(x - x_c) = (Au)^T (AA^T)^{-1}(Au) = (Au)^T (A^{-T}A^{-1})(Au) = u^T u = \|u\|_2^2 \le 1.$$

*So $\|u\|_2 \le 1$ and $x = x_c + Au$.*

*($\supseteq$) Conversely, if $x = x_c + Au$ with $\|u\|_2 \le 1$, the same computation gives*

$$(x - x_c)^T P^{-1}(x - x_c) = \|u\|_2^2 \le 1,$$

*so $x \in E$.*

*Thus the two representations are equivalent.*

## 3.3 Convex Cones

**Definition 3.5** (Conic Combination). *Given points $x_1, \ldots, x_k \in \mathbb{R}^n$, a point $x$ is a **conic combination** if*

$$x = \sum_{i=1}^{k} \theta_i x_i, \qquad \theta_i \ge 0.$$

**Definition 3.6** (Convex Cone). *A set $K \subseteq \mathbb{R}^n$ is a **convex cone** if it is closed under conic combinations:*

$$x_1, \ldots, x_k \in K,\ \theta_i \ge 0\ \Rightarrow\ \sum_{i=1}^{k} \theta_i x_i \in K.$$

*Equivalently, $K$ is a **convex cone** iff it is closed under nonnegative scaling and addition:*

$$x \in K,\ \alpha \ge 0 \Rightarrow \alpha x \in K, \qquad x, y \in K \Rightarrow x + y \in K.$$

**Definition 3.7** (Hyperplane and Halfspace). *A **hyperplane** is $\{x \mid a^T x = b\}$ and a **halfspace** is $\{x \mid a^T x \le b\}$, for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.*

**Exercise 3.4** (Distance and projection onto a hyperplane). ***Question.***

*Let $H = \{x \mid a^T x = b\}$ with $a = (2, 1)$ and $b = 3$. For the point $p = (1, 4)$, compute the distance $\mathrm{dist}(p, H)$ and the orthogonal projection $\Pi_H(p)$.*

***Answer.*** *The signed residual is $a^T p - b = (2 \cdot 1 + 1 \cdot 4) - 3 = 3$. The distance is*

$$\mathrm{dist}(p, H) = \frac{|a^T p - b|}{\|a\|_2} = \frac{3}{\sqrt{5}}.$$

*The projection is*

$$\Pi_H(p) = p - \frac{a^T p - b}{\|a\|_2^2} a = (1,4) - \frac{3}{5}(2,1) = \left(-\frac{1}{5}, \frac{17}{5}\right).$$

**Exercise 3.5** (A separating hyperplane for two points)**.** ***Question.***

*Let $p = (1,0)$ and $q = (0,1)$. Find a hyperplane that separates $p$ and $q$.*

***Answer.*** *The hyperplane*

$$H = \{x \mid (1,-1)^T x = 0\} \quad (i.e.,\ x_1 = x_2)$$

*separates them because*

$$(1,-1)^T p = 1 > 0, \qquad (1,-1)^T q = -1 < 0.$$

*So $p$ and $q$ lie in opposite open halfspaces.*

**Definition 3.8** (Norm Cone)**.** *Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, the associated* ***norm cone*** *is*

$$K_{\|\cdot\|} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \le t\}.$$

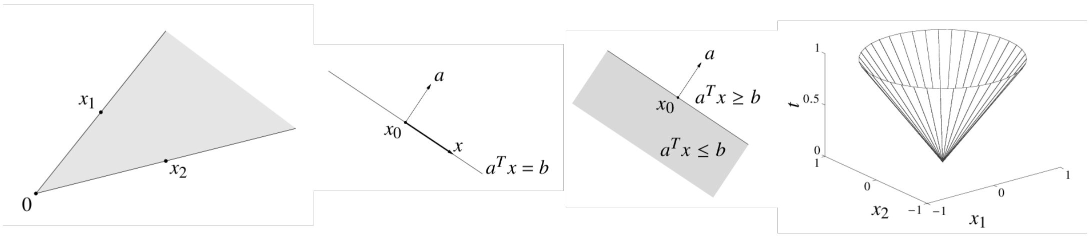


**Figure 7:** Convex Cones

**Definition 3.9** (Polyhedron)**.** *A* ***polyhedron*** *is a set of the form*

$$\{x \in \mathbb{R}^n \mid Ax \le b,\ Cx = d\},$$

*where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{p\times n}$, and $d \in \mathbb{R}^p$. Here $Ax \le b$ denotes a* ***componentwise inequality****, i.e., $(Ax)_i \le b_i$ for all $i = 1, \ldots, m$.*

**Definition 3.10** (Positive Semidefinite (PSD) Cone)**.** *Let $\mathbb{S}^n$ be the set of symmetric $n \times n$ matrices. The* ***positive semidefinite cone*** *is*

$$\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \succeq 0\},$$

*where $X \succeq 0$ means $z^T X z \ge 0$ for all $z \in \mathbb{R}^n$.*

**Exercise 3.6** (A $2 \times 2$ PSD/PD check)**.** ***Question.***

*Let*

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

*Decide whether $X$ is PSD or PD.*

***Answer.*** *For* $2 \times 2$ *symmetric matrices,* $X \succ 0$ *iff the leading principal minors are positive:*

$$X_{11} = 2 > 0, \qquad \det(X) = 2 \cdot 1 - 1^2 = 1 > 0.$$

*Hence* $X$ *is positive definite (and therefore also PSD).*

**Exercise 3.7** (PSD cone $\Leftrightarrow$ rotated SOC for $2 \times 2$)**.** ***Question.***

*Explain how a* $2 \times 2$ *PSD constraint can be written as a rotated second-order cone constraint.*

***Answer.*** *For*

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2,$$

$X \succeq 0$ *is equivalent to*

$$x \geq 0, \qquad z \geq 0, \qquad xz - y^2 \geq 0.$$

*The inequality* $xz \geq y^2$ *matches a* ***rotated second-order cone***

$$\{(u, v, w) \mid 2uv \geq \|w\|_2^2, \ u \geq 0, \ v \geq 0\}.$$

*Setting* $u = x$*,* $v = z$*, and* $w = \sqrt{2}\, y$ *gives* $2xz \geq (\sqrt{2}y)^2 \iff xz \geq y^2$*.*

**Definition 3.11** (Second-Order Cone (SOC))**.** *The* ***second-order cone*** *(Lorentz cone) is*

$$\mathcal{Q}^{n+1} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 \leq t\}.$$

**Example 3.2** (Probability simplex (affine + inequalities))**.**

$$\Delta^n = \{x \in \mathbb{R}^n \mid x \geq 0, \ \mathbf{1}^\top x = 1\}$$

*is convex because it is an intersection of convex sets: halfspaces* $x_i \geq 0$ *and an affine hyperplane* $\mathbf{1}^\top x = 1$*.*

**Example 3.3** (PSD cone)**.** *The positive semidefinite cone*

$$\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \succeq 0\}$$

*is convex: if* $X, Y \succeq 0$ *then* $\theta X + (1 - \theta) Y \succeq 0$ *for* $\theta \in [0, 1]$*.*

**Example 3.4** (Spectrahedron)**.** *A spectrahedron (LMI feasible set)*

$$\{x \in \mathbb{R}^d \mid A_0 + \sum_{i=1}^{d} x_i A_i \succeq 0\}$$

*is convex (affine map + PSD constraint).*

**Example 3.5** (Two important nonconvex sets (contrast))**.**

- *Sphere:* $\{x \mid \|x\|_2 = 1\}$ *is* ***not*** *convex (midpoint of two points on the sphere lies inside).*
- *Union of two disjoint convex sets (e.g., two separated disks) is generally* ***not*** *convex (line*

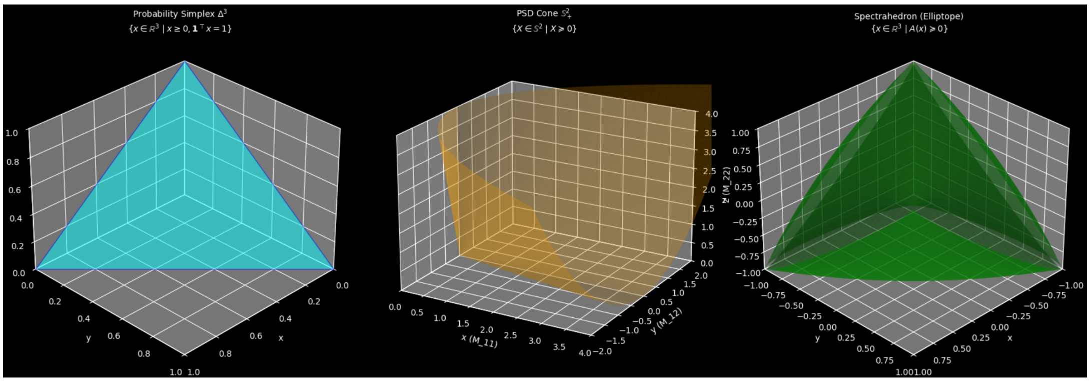


**Figure 8:** Geometric visualizations of fundamental convex sets in $\mathbb{R}^3$. (Left) The probability simplex $\Delta^3$, defined as the intersection of the non-negative orthant and the affine hyperplane $\mathbf{1}^\top x = 1$. It represents the feasible region for standard linear programming. (Center) The positive semidefinite (PSD) cone $\mathbb{S}^2_+$, visualized via the boundary surface $xz = y^2$. This cone is the central object in semidefinite programming.(Right) A 3D spectrahedron known as the Elliptope, representing the set of $3 \times 3$ correlation matrices. It illustrates a convex set formed by the intersection of the PSD cone and an affine subspace (LMI constraints).

*segment can pass through the gap).*

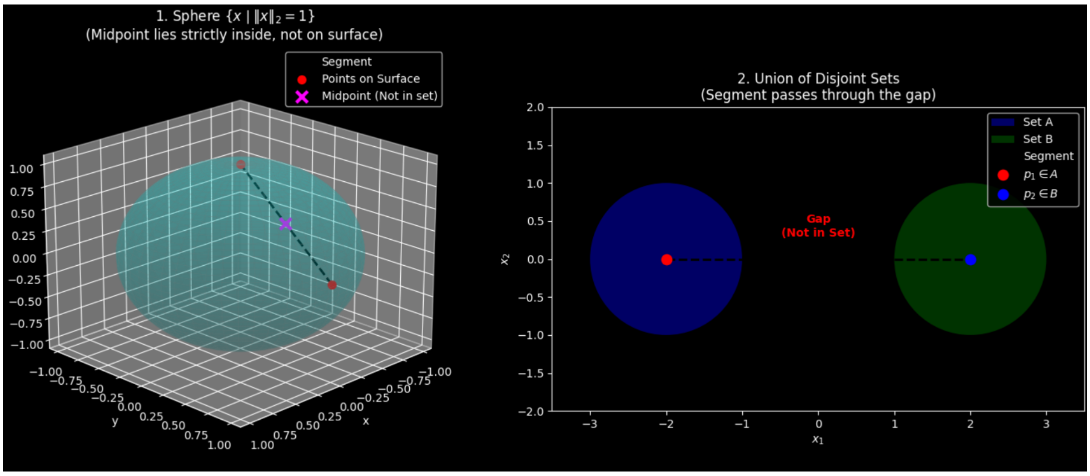


**Figure 9:** Geometric illustration of non-convex sets.(Left) The unit sphere defined by the equality constraint $\{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$. It is non-convex because the line segment connecting two distinct points on the surface traverses the interior, which is not part of the set.(Right) The union of two disjoint convex sets. Although each disk is individually convex, their union is not, as the convex hull includes the gap between them.

## 3.4 Calculus of Convex Sets (Operations)

How do we build complex convex sets from simple ones?

### 3.4.1 1. Intersection

**Theorem 3.1.** *The intersection of any number (even infinite) of convex sets is convex.*

*Proof Sketch:* If $x, y \in \bigcap C_i$, then $x, y \in C_i$ for all $i$. Since each $C_i$ is convex, the segment is in each $C_i$, hence in the intersection.

#### 3.4.2 2. Affine Functions

Let $f(x) = Ax + b$.

- **Image:** If $C$ is convex, $f(C)$ is convex.
- **Inverse Image:** If $D$ is convex, $f^{-1}(D) = \{x \mid f(x) \in D\}$ is convex.

*Application (P-C5):* A polyhedron $\{x \mid Ax \leq b\}$ is the inverse image of the non-positive quadrant (convex) under an affine mapping.

# 4 Module 3: Generalized Inequalities & Cones

This section addresses the advanced topics $\rightarrow$ Spectrahedra and PSD cones.

## 4.1 Positive Semidefinite (PSD) Cone

Let $\mathbb{S}^n$ be the set of symmetric $n \times n$ matrices.

$$\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \succeq 0\}$$

where $X \succeq 0$ means $z^T X z \geq 0$ for all vectors $z$.

**Why is this convex?** Let $A, B \succeq 0$ and $\theta \in [0, 1]$. Consider $C = \theta A + (1 - \theta)B$. For any vector $z$:

$$z^T C z = \theta(z^T A z) + (1 - \theta)(z^T B z).$$

Since $z^T A z \geq 0$, $z^T B z \geq 0$, and $\theta \geq 0$, the sum is $\geq 0$. Thus $C \succeq 0$.

## 4.2 Linear Matrix Inequalities (LMIs)

A constraint of the form:

$$A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0$$

defines a convex set called a **Spectrahedron**. This is convex because it is the inverse image of the PSD cone $\mathbb{S}^n_+$ under the affine map $f(x) = A_0 + \sum x_i A_i$.

# 5 Module 4: Problem Set (Affine Sets)

**Exercise 5.1** (P-A1). *Determine whether* $C = \{x \in \mathbb{R}^2 \mid x_1 - 2x_2 = 3\}$ *is affine. If yes, write it as* $x_0 + \mathrm{span}(v)$.

**Exercise 5.2** (P-A2). *Let* $x_1 = (0, 0)$ *and* $x_2 = (2, 1)$. *Compute* $x = \theta x_1 + (1 - \theta)x_2$ *for* $\theta = -1, 0, 1/2, 2$. *Interpret where each point lies.*

**Exercise 5.3** (P-A3). *Is the unit circle* $S = \{x \in \mathbb{R}^2 \mid \|x\|_2 = 1\}$ *affine?*
*Give a short proof/counterexample.*

**Exercise 5.4** (P-A4). *Prove that the intersection of two affine sets is affine. Give a counterexample showing that the union of two affine sets need not be affine.*

**Exercise 5.5** (P-A5). *Suppose* $A \in \mathbb{R}^{m\times n}$ *has rank* $r$ *and* $\{x \mid Ax = b\} \neq \emptyset$. *Show that the solution set is an affine set of dimension* $n - r$.

**Exercise 5.6** (P-A6). *Given* $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, $p_3 = (0, 0, 1)$ *in* $\mathbb{R}^3$, *compute* $\mathrm{aff}\{p_1, p_2, p_3\}$ *and express it in the form* $Ax = b$.

**Exercise 5.7** (P-A7 (Key theorem)). *Prove: A set* $C \subset \mathbb{R}^n$ *is affine iff for any finite collection* $x_1, \ldots, x_k \in C$ *and scalars* $\theta_1, \ldots, \theta_k$ *with* $\sum_i \theta_i = 1$, *we have* $\sum_i \theta_i x_i \in C$.

**Exercise 5.8** (P-A8 (Projection onto an affine set)). *Let* $C = \{x \in \mathbb{R}^n \mid Ax = b\}$ *with* $A$ *full row-rank. Derive the Euclidean projection formula*

$$\Pi_C(y) = \arg\min_x \|x - y\|_2^2 \quad \text{s.t.} \quad Ax = b$$

*and implement it.*

# 6 Module 5: Problem Set (Convex Sets)

**Exercise 6.1** (P-C1)**.** *Check whether each set is convex: (i)* $\{x \in \mathbb{R}^2 \mid x_1 + x_2 \le 1\}$*, (ii)* $\{x \mid \|x\|_2 = 1\}$*, (iii)* $\{x \mid \|x\|_2 \le 1\}$*.*

**Exercise 6.2** (P-C2)**.** *Given* $x_1 = (0,0)$*,* $x_2 = (2,1)$*, compute the segment points* $x = \theta x_1 + (1-\theta)x_2$ *for* $\theta = 0, 0.25, 0.5, 0.75, 1$*.*

**Exercise 6.3** (P-C3)**.** *Is the set* $C = \{(x,y) \in \mathbb{R}^2 \mid y \ge x^2\}$ *convex? Justify.*

**Exercise 6.4** (P-C4)**.** *Prove that the intersection of any family of convex sets is convex.*

**Exercise 6.5** (P-C5)**.** *Let* $C = \{x \mid Ax \le b\}$*. Prove* $C$ *is convex. Then give a geometric interpretation (polyhedron).*

**Exercise 6.6** (P-C6 (convex hull computation))**.** *For points* $v_1 = (0,0)$*,* $v_2 = (2,0)$*,* $v_3 = (2,2)$*,* $v_4 = (0,1)$*, describe* $\mathrm{conv}\{v_1, v_2, v_3, v_4\}$ *and sketch it.*

**Exercise 6.7** (P-C7 (affine image/preimage))**.** *Prove: If* $C$ *is convex and* $f(x) = Ax + b$ *is affine, then* $f(C)$ *is convex. Also show that* $f^{-1}(D)$ *is convex whenever* $D$ *is convex.*

**Exercise 6.8** (P-C11 (Ellipsoid membership test))**.** *Let* $x_c = (0,0)$*,* $P = \mathrm{diag}(9,4)$*. Decide whether* $x = (2,1)$ *lies in the ellipsoid*

$$E = \left\{x \mid x^T P^{-1} x \le 1\right\}.$$

***Hint.*** *Compute* $2^2/9 + 1^2/4$*.*

**Exercise 6.9** (P-C12 (Ellipsoid as affine image of unit ball))**.** *Prove that for* $P \in \mathbb{S}^n_{++}$*, the ellipsoid*

$$E = \left\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\right\}$$

*can be written as*

$$E = \{x_c + Au \mid \|u\|_2 \le 1\}$$

*for some nonsingular* $A$ *satisfying* $AA^T = P$*.* ***Hint.*** *Use Cholesky factorization* $P = LL^T$ *or* $P^{1/2}$*.*

**Exercise 6.10** (P-C13 (A weighted $\ell_\infty$ norm))**.** *Define* $\|x\| = \max\{|x_1|,\ 2|x_2|\}$ *on* $\mathbb{R}^2$*.*

*(a) Show this is a norm.*

*(b) Sketch (or describe) the unit ball* $\{x \mid \|x\| \le 1\}$*.*

***Hint.*** *A* max *of absolute homogeneous terms is often a norm; the unit ball becomes a rectangle.*

**Exercise 6.11** (P-C14 (SOC reformulation))**.** *Rewrite the constraint*

$$\|Ax + b\|_2 \le c^T x + d$$

*as a second-order cone constraint, and state the additional condition needed for convexity/feasibility.* ***Hint.*** *SOC form is $\|y\|_2 \le t$. Make sure $t = c^T x + d \ge 0$.*

**Exercise 6.12** (P-C15 (Vertices of a polyhedron))**.** *Find all vertices of the polyhedron*

$$P = \left\{x \in \mathbb{R}^2 \mid x_1 \ge 0,\ x_2 \ge 0,\ x_1 + x_2 \le 3,\ x_1 \le 2\right\}.$$

*Then sketch it.* ***Hint.*** *Enumerate intersections of pairs of active constraints and check feasibility.*

**Exercise 6.13** (P-C16 (Boundedness via recession cone))**.** *Boundedness via recession cone. Let $P = \{x \mid Ax \le b\}$. The recession cone is*

$$\operatorname{rec}(P) = \{d \mid Ad \le 0\}.$$

*Prove: $P$ is bounded iff $\operatorname{rec}(P) = \{0\}$.*

*Apply this to decide bounded/unbounded for*

$$P = \left\{x \in \mathbb{R}^2 \mid x_2 \ge 0,\ x_1 - x_2 \le 1\right\}.$$

***Hint.*** *If a nonzero $d \in \operatorname{rec}(P)$, then $(x + \lambda d \in P)$ for all $\lambda \ge 0$.*

**Exercise 6.14** (P-C17 (PSD vs. PD checks))**.** *Determine whether each matrix is PSD / PD:*

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}.$$

***Hint.*** *For $2 \times 2$, PSD iff $x \ge 0$, $z \ge 0$, $xz - y^2 \ge 0$. PD iff $x > 0$ and $\det > 0$.*

**Exercise 6.15** (P-C18 (Schur complement LMI))**.** *Schur complement LMI (classic SDP bridge). Let $P \in \mathbb{S}^n_{++}$. Show that the ellipsoid constraint*

$$x^T P^{-1} x \le 1$$

*is equivalent to the linear matrix inequality (LMI)*

$$\begin{bmatrix} P & x \\ x^T & 1 \end{bmatrix} \succeq 0.$$

***Hint.*** *Use the Schur complement: for $P \succ 0$,*

$$\begin{bmatrix} P & x \\ x^T & 1 \end{bmatrix} \succeq 0 \iff 1 - x^T P^{-1} x \ge 0.$$

# 7 Useful Problem Set

- **Book problems 2.1–2.4 (p. 60):** Definition of convexity
- **Book problems 2.5–2.6 (p. 60):** Examples
- **Book problem 2.12 (p. 61):** Examples
- **Book problem 2.12 (a)–(e) (p. 61):** Examples
- **Book problem 2.15 (a)–(g) (p. 61):** Examples