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The need for deeply understanding when algorithms work (or not) has never been greater.

BY TIM ROUGHGARDEN

Beyond Worst-Case Analysis

COMPARING DIFFERENT ALGORITHMS is hard. For almost any pair of algorithms and measure of algorithm performance like running time or solution quality, each algorithm will perform better than the other on some inputs.[a] For example, the insertion sort algorithm is faster than merge sort on already-sorted arrays but slower on many other inputs. When two algorithms have incomparable performance, how can we deem one of them "better than" the other?

Worst-case analysis is a specific modeling choice in the analysis of algorithms, where the overall performance of an algorithm is summarized by its worst performance on any input of a given size. The "better" algorithm is then the one with superior worst-case performance. Merge sort, with its worst-case asymptotic running time of $\Theta(n \log n)$ for arrays of length n, is better in this sense than insertion sort, which has a worst-case running time of $\Theta(n^2)$.

a In rare cases a problem admits an instance-optimal algorithm, which is as good as every other algorithm on every input, up to a constant factor.[23] For most problems, there is no instance-optimal algorithm, and there is no escaping the incomparability of different algorithms.

While crude, worst-case analysis can be tremendously useful, and it is the dominant paradigm for algorithm analysis in theoretical computer science. A good worst-case guarantee is the best-case scenario for an algorithm, certifying its general-purpose utility and absolving its users from understanding which inputs are relevant to their applications. Remarkably, for many fundamental computational problems, there are algorithms with excellent worst-case performance guarantees. The lion's share of an undergraduate algorithms course comprises algorithms that run in linear or near-linear time in the worst case.

For many problems a bit beyond the scope of an undergraduate course, however, the downside of worst-case analysis rears its ugly head. Here, I review three classical examples where worst-case analysis gives misleading or useless advice about how to solve a problem; further examples in modern machine learning are described later. These examples motivate the alternatives to worst-case analysis described in the article.[b]

The simplex method for linear programming. Perhaps the most famous failure of worst-case analysis concerns linear programming, the problem of optimizing a linear func-

b For many more examples, analysis frameworks, and applications, see the author's lecture notes.[36]

» key insights

- **Worse-case analysis takes a "Murphy's Law" approach to algorithm analysis, which is too crude to give meaningful algorithmic guidance for many important problems, including linear programming, clustering, caching, and neural network training.**
- **Research going "beyond worst-case analysis" articulates properties of realistic inputs, and proves rigorous and meaningful algorithmic guarantees for inputs with these properties.**
- **Much of the present and future research in the area is motivated by the unreasonable effectiveness of machine learning algorithms.**

COLLAGE BY ANDRIJ BORYS ASSOCIATES/SHUTTERSTOCK

tion subject to linear constraints (Figure 1). Dantzig's simplex method is an algorithm from the 1940s that solves linear programs using greedy local search on the vertices on the solution set boundary, and variants of it remain in wide use to this day. The enduring appeal of the simplex method stems from its consistently superb performance in practice. Its running time typically scales modestly with the input size, and it routinely solves linear programs with millions of decision variables and constraints. This robust empirical performance suggested the simplex method might well solve every linear program in a polynomial amount of time.

In 1972, Klee and Minty showed by example that there are contrived linear programs that force the simplex method to run in time exponential in the number of decision variables (for all of the common "pivot rules" for choosing the next vertex). This illustrates the first potential pitfall of worst-case analysis: overly pessimistic performance predictions that cannot be taken at face value. The running time of the simplex method is polynomial for all practical purposes, despite the exponential prediction of worst-case analysis.

To add insult to injury, the first worst-case polynomial-time algorithm for linear programming, the ellipsoid method, is not competitive with the simplex method in practice.[c]

c Interior-point methods, developed five years later, lead to algorithms that both run in worst-case polynomial time and are competitive with the simplex method in practice.

Figure 1. A two-dimensional linear programming problem.

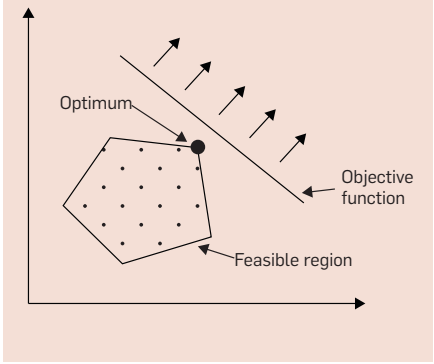
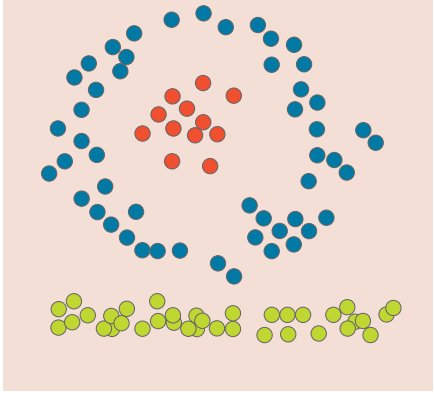


Figure 2. One possible way to group data points into three clusters.



Taken at face value, worst-case analysis recommends the ellipsoid method over the empirically superior simplex method. One framework for narrowing the gap between these theoretical predictions and empirical observations is *smoothed analysis*, discussed later in this article.

Clustering and *NP*-hard optimization problems. Clustering is a form of unsupervised learning (finding patterns in unlabeled data), where the informal goal is to partition a set of points into "coherent groups" (Figure 2). One popular way to coax this goal into a well-defined computational problem is to posit a numerical objective function over clusterings of the point set, and then seek the clustering with the best objective function value. For example, the goal could be to choose k cluster centers to minimize the sum of the distances between points and their nearest centers (the k-median objective) or the sum of the squared such distances (the k-means objective). Almost all natural optimization problems that are defined over clusterings are *NP*-hard.

In practice, clustering is not viewed as a particularly difficult problem. Lightweight clustering algorithms, like Lloyd's algorithm for k-means and its variants, regularly return the intuitively "correct" clusterings of real-world point sets. How can we reconcile the worst-case intractability of clustering problems with the empirical success of relatively simple algorithms?[d]

One possible explanation is that *clustering is hard only when it doesn't matter*.[18] For example, if the difficult instances of an *NP*-hard clustering problem look like a bunch of random unstructured points, who cares? The common use case for a clustering algorithm is for points that represent images, or documents, or proteins, or some other objects where a "meaningful clustering" is likely to exist. Could instances with a meaningful clustering be easier than worst-case instances? This article surveys recent theoretical developments that support an affirmative answer.

Cache replacement policies. Consider a system with a small fast memory (the cache) and a big slow memory. Data is organized into blocks called *pages*, with up to k different pages fitting in the cache at once. A page request results in either a cache hit (if the page is already in the cache) or a cache miss (if not). On a cache miss, the requested page must be brought into the cache. If the cache is already full, then some page in it must be evicted. A cache policy is an algorithm for making these eviction decisions. Any systems textbook will recommend aspiring to the least recently used (LRU) policy, which evicts the page whose most recent reference is furthest in the past. The same textbook will explain why: real-world page request sequences tend to exhibit locality of reference, meaning that recently requested pages are likely to be requested again soon. The LRU policy uses the recent past as a prediction for the near future. Empirically, it typically suffers fewer cache misses than competing policies like first-in first-out (FIFO).

Sleator and Tarjan[37] founded the area of *online algorithms*, which are algorithms that must process their input as it arrives over time (like cache policies). One of their first observations was that worst-case analysis, straightforwardly applied, provides no useful insights about the performance of different cache replacement policies. For every deterministic policy and cache size k, there is a pathological page request sequence that triggers a page fault rate of 100%, even though the optimal clairvoyant replacement policy (known as Bélády's algorithm) would have a page fault rate of at most $(1/k)$%. This observation is troublesome both for its absurdly pessimistic performance prediction and for its failure to differentiate between competing replacement policies (like LRU vs. FIFO). One solution, discussed next, is to choose an appropriately fine-grained parameterization of the input space and to assess and compare algorithms using parameterized guarantees.

Models of Typical Instances

Maybe we shouldn't be surprised that worst-case analysis fails to advocate LRU over FIFO. The empirical superiority of LRU is due to the special structure in real-world page request sequences—locality of reference—and traditional worst-case analysis provides no vocabulary to speak about this structure.[e] This is what work on "beyond worst-case analysis" is all about: *articulating properties of "real-world" inputs, and proving rigorous and meaningful algorithmic guarantees for inputs with these properties.*

Research in the area has both a scientific dimension, where the goal is to develop transparent mathemati-

d More generally, optimization problems are more likely to be NP-hard than not. In many cases, even computing an approximately optimal solution is an NP-hard problem (see Trevisan[36] for example). Whenever an efficient algorithm for such a problem performs better on real-world instances than (worst-case) complexity theory would suggest, there's an opportunity for a refined and more accurate theoretical analysis.

e If worst-case analysis has an implicit model of data, then it's the "Murphy's Law" data model, where the instance to be solved is an adversarially selected function of the chosen algorithm. Outside of cryptographic applications, this is a rather paranoid and incoherent way to think about a computational problem.

cal models that explain empirically observed phenomena about algorithm performance, and an engineering dimension, where the goals are to provide accurate guidance about which algorithm to use for a problem and to design new algorithms that perform particularly well on the relevant inputs.

One exemplary result in beyond worst-case analysis is due to Albers et al.,[2] for the online paging problem described in the introduction. The key idea is to parameterize page request sequences according to how much locality of reference they exhibit, and then prove parameterized worst-case guarantees. Refining worst-case analysis in this way leads to dramatically more informative results.[f]

Locality of reference is quantified via the size of the working set of a page request sequence. Formally, for a function $f : \mathbb{N} \to \mathbb{N}$, we say that a request sequence *conforms to* f if, in every window of w consecutive page requests, at most $f(w)$ distinct pages are requested. For example, the identity function $f(w) = w$ imposes no restrictions on the page request sequence. A sequence can only conform to a sublinear function like $f(w) = [\sqrt{w}]$ or $f(w) = [1 + \log_2 w]$ if it exhibits locality of reference.[g]

The following worst-case guarantee is parameterized by a number $\alpha_f(k)$, between 0 and 1, that we discuss shortly; recall that k denotes the cache size. It assumes that the function f is "concave" in the sense that the number of inputs with value x under f (that is, $|f^{-1}(x)|$) is nondecreasing in x.

Theorem 1 (Albers et al.[2])
(a) For every f and k and every deterministic cache replacement policy, the worst-case page fault rate (over sequences that conform to f) is at least $\alpha_f(k)$.

(a) For every f and k and every sequence that conforms to f, the page fault rate of the LRU policy is at most $\alpha_f(k)$.

(b) There exists a choice of f and k, and a page request sequence that conforms to f, such that the page fault rate of the FIFO policy is strictly larger than $\alpha_f(k)$.

Parts (a) and (*b*) prove the worst-case optimality of the LRU policy in a strong sense, f-by-f and k-by-k. Part (c) differentiates LRU from FIFO, as the latter is suboptimal for some (in fact, many) choices of f and k.

The guarantees in Theorem 1 are so good that they are meaningful even when taken at face value—for sublinear f's, $\alpha_f(k)$ goes to 0 reasonably quickly with k. For example, if $f(w) = [\sqrt{w}]$, then $\alpha_f(k)$ scales with $1/\sqrt{k}$. Thus, with a cache size of 10,000, the page fault rate is always at most 1%. If $f(w) = [1 + \log_2 w]$, then $\alpha_f(k)$ goes to 0 even faster with k, roughly as $k/2^k$.[h]

Stable Instances

Are point sets with meaningful clusterings easier to cluster than worst-case point sets? Here, we describe one way to define a "meaningful clustering," due to Bilu and Linial;[12] for others, see Ackerman and Ben-David,[1] Balcan et al.,[9] Daniely et al.,[18] Kumar and Kannan,[29] and Ostrovsky et al.[34]

The maximum cut problem. Suppose you have a bunch of data points representing images of cats and images of dogs, and you would like to automatically discover these two groups. One approach is to reduce this task to the *maximum cut* problem, where the goal is to partition the vertices V of a graph G with edges E and nonnegative edge weights into two groups, while maximizing the total weight of the edges that have one endpoint in each group. The reduction forms a complete graph G, with vertices corresponding to the data points, and assigns a weight w_e to each edge e indicating how dissimilar its endpoints are. The maximum cut of G is a 2-clustering that tends to put dissimilar pairs of points in different clusters.

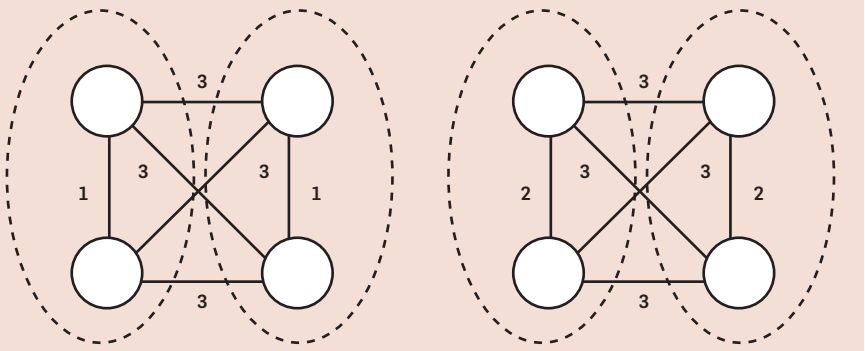
There are many ways to quantify "dissimilarity" between images, and different definitions might give different optimal 2-clusterings of the data points. One would hope that, for a range of reasonable measures of dissimilarity, the maximum cut in the example above would have all cats on one side and all dogs on the other. In other words, the maximum cut should be invariant under minor changes to the specification of the edge weights (Figure 3).

Definition 2 (Bilu and Linial[12]). An instance $G = (V, E, w)$ of the maximum cut problem is γ-*perturbation stable* if, for all ways of multiplying the weight w_e of each edge e by a factor $a_e \in [1, \gamma]$, the optimal solution remains the same.

A perturbation-stable instance has a "clearly optimal" solution—a uniqueness assumption on steroids—thus formalizing the idea of a "meaningful clustering." In machine learning parlance, perturbation stability can be viewed as a type of "large margin" assumption.

The maximum cut problem is *NP*-hard in general. But what about the special case of γ-perturbation-stable in-

Figure 3. In a perturbation-stable maximum cut instance, the optimal solution is invariant under small perturbations to the edges' weights.



f Parameterized guarantees are common in the analysis of algorithms. For example, the field of parameterized algorithms and complexity has developed a rich theory around parameterized running time bounds (see the book by Cygan et al.[16]). Theorem 1 employs an unusually fine-grained and problem-specific parameterization, and in exchange obtains unusually accurate and meaningful results.

g The notation $[x]$ means the number x, rounded up to the nearest integer.

h See Albers et al.[2] for the precise closed-form formula for $\alpha_f(k)$ in general.

stances? As γ increases, fewer and fewer instances qualify as γ-perturbation stable. Is there a sharp stability threshold—a value of γ where the maximum cut problem switches from *NP*-hard to polynomial-time solvable?

Makarychev et al.[30] largely resolved this question. On the positive side, they showed that if γ is at least a slowly growing function of the number of vertices n, then the maximum cut problem can be solved in polynomial time for all γ-perturbation stable instances.[i] Makarychev et al. use techniques from the field of metric embeddings to show that, in such instances, the unique optimal solution of a certain semidefinite programming relaxation corresponds precisely to the maximum cut.[j] Semidefinite programs are convex programs, and can be solved to arbitrary precision in polynomial time. There is also evidence that the maximum cut cannot be recovered in polynomial time in γ-perturbation-stable instances for much smaller values of γ.[30]

Other clustering problems. Bilu and Linial[12] defined γ-perturbation-stable instances specifically for the maximum cut problem, but the definition makes sense more generally for any optimization problem with a linear objective function. The study of γ-perturbation-stable instances has been particularly fruitful for *NP*-hard clustering problems in metric spaces, where interpoint distances are required to satisfy the triangle inequality. Many such problems, including the k-means, k-median, and k-center problems, are polynomial-time solvable already in 2-perturbation-stable instances.[5,10] The algorithm in Angelidakis et al.,[5] like its precursor in Awasthi et al.,[8] is inspired by the well known single-linkage clustering algorithm. It computes a minimum spanning tree (where edge weights are the interpoint distances) and uses dynamic programming to optimally remove k - 1 edges to define k clusters. To the extent that we are comfortable identifying "instances with a meaningful clustering" with 2-perturbation-stable instances, these results give a precise sense in which clustering is hard only when it doesn't matter.[k]

The unreasonable effectiveness of modern machine learning algorithms has thrown down the gauntlet to algorithms researchers, and there is perhaps no other problem domain with a more urgent need for the beyond worst-case approach.

Overcoming *NP*-hardness. Polynomial-time algorithms for γ-perturbation-stable instances continue the age-old tradition of identifying "islands of tractability," meaning polynomial-time solvable special cases of *NP*-hard problems. Two aspects of these results diverge from a majority of 20th century research on tractable special cases. First, perturbation-stability is not an easy condition to check, in contrast to a restriction like graph planarity or Horn-satisfiability. Instead, the assumption is justified with a plausible narrative about why "real-world instances" might satisfy it, at least approximately. Second, in most work going beyond worst-case analysis, the goal is to study general-purpose algorithms, which are well defined on all inputs, and use the assumed instance structure only in the algorithm analysis (and not explicitly in its design). The hope is the algorithm continues to perform well on many instances not covered by its formal guarantee. The results here for mathematical programming relaxations and single-linkage-based algorithms are good examples of this paradigm.

Analogy with sparse recovery. There are compelling parallels between the recent research on clustering in stable instances and slightly older results in a field of applied mathematics known as *sparse recovery*, where the goal is to reverse engineer a "sparse" object from a small number of clues about it. A common theme in both areas is identifying relatively weak conditions under which a tractable mathematical programming relaxation of an *NP*-hard problem is guaranteed to be exact, meaning the original problem and its relaxation have the same optimal solution.

For example, a canonical problem in sparse recovery is *compressive sensing*, where the goal is to recover

i Specifically, $\gamma = \Omega(\sqrt{\log n} \log \log n)$.

j In general, the optimal solution of a linear or semidefinite programming relaxation of an *NP*-hard problem is a "fractional solution" that does not correspond to a feasible solution to the original problem.

k A relaxed and more realistic version of perturbation-stability allows small perturbations to make small changes to the optimal solution. Many of the results mentioned in this section can be extended to instances meeting this relaxed condition, with a polynomial-time algorithm guaranteed to recover a solution that closely resembles the optimal one.[5,9,30]

an unknown sparse signal (a vector of length n) from a small number m of linear measurements of it. Equivalently, given an $m \times n$ measurement matrix A with $m \ll n$ and the measurement results $b = Az$, the problem is to figure out the signal z. This problem has several important applications, for example in medical imaging. If z can be arbitrary, then the problem is hopeless: since $m < n$, the linear system $Ax = b$ is underdetermined and has an infinite number of solutions (of which z is only one). But many real-world signals are (approximately) k-sparse in a suitable basis for small k, meaning that (almost) all of the mass is concentrated on k coordinates.[l] The main results in compressive sensing show that, under appropriate assumptions on A, the problem can be solved efficiently even when m is only modestly bigger than k (and much smaller than n).[15,20] One way to prove these results is to formulate a linear programming relaxation of the (*NP*-hard) problem of computing the sparsest solution to $Ax = b$, and then show this relaxation is exact.

Planted and Semi-Random Models

Our next genre of models is also inspired by the idea that interesting instances of a problem should have "clearly optimal" solutions, but differs from the stability conditions in assuming a generative model—a specific distribution over inputs. The goal is to design an algorithm that, with high probability over the assumed input distribution, computes an optimal solution in polynomial time.

The planted clique problem. In the *maximum clique* problem, the input is an undirected graph $G = (V, E)$, and the goal is to identify the largest subset of vertices that are mutually adjacent. This problem is *NP*-hard, even to approximate by any reasonable factor. Is it easy when there is a particularly prominent clique to be found?

Jerrum[27] suggested the following generative model: There is a fixed set V of n vertices. First, each possible edge (u, v) is included independently with 50% probability. This is also known as an Erdös-Renyi random graph with edge density ½. Second, for a parameter $k \in \{1, 2, \ldots, n\}$, a subset $Q \subseteq V$ of k vertices is chosen uniformly at random, and all remaining edges with both endpoints in Q are added to the graph (thus making Q a k-clique).

How big does k need to be before Q becomes visible to a polynomial-time algorithm? The state of the art is a spectral algorithm of Alon et al.,[3] which recovers the planted clique Q with high probability provided k is at least a constant times $\sqrt{n}$. Recent work suggests that efficient algorithms cannot recover Q for significantly smaller values of k.[11]

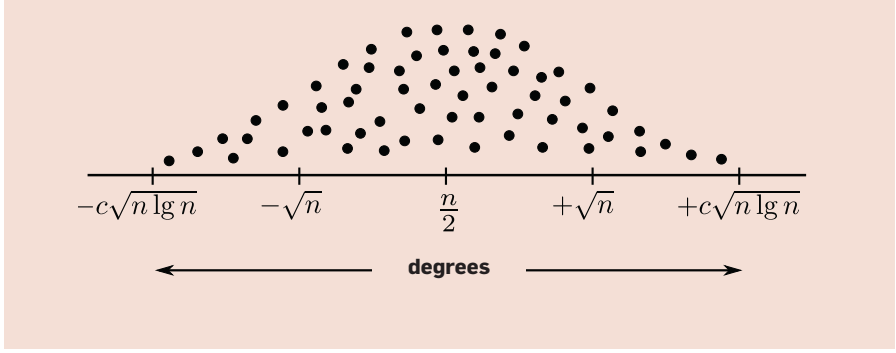
An unsatisfying algorithm. The algorithm of Alon et al.[3] is theoretically interesting and plausibly useful. But if we take k to be just a bit bigger, at least a constant times $\sqrt{n \log n}$, then there is an uninteresting and useless algorithm that recovers the planted clique with high probability: return the k vertices with the largest degrees. To see why this algorithm works, think first about the sampled Erdös-Renyi random graph, before the clique Q is planted. The expected degree of each vertex is $\approx n/2$, with standard deviation $\approx\sqrt{n}/2$. Textbook large deviation inequalities show that, with high probability, the degree of every vertex is within $\approx\sqrt{\ln n}$ standard deviations of its expectation (Figure 4). Planting a clique Q of size $a\sqrt{n \log n}$, for a sufficiently large constant a, then boosts the degrees of all of the clique vertices enough that they catapult past the degrees of all of the non-clique vertices.

What went wrong? The same thing that often goes wrong with pure average-case analysis—the solution is brittle and overly tailored to a specific distributional assumption. How can we change the input model to encourage the design of algorithms with more robust guarantees? Can we find a sweet spot between average-case and worst-case analysis?

Semi-random models. Blum and Spencer[13] proposed studying *semi-random* models, where nature and an adversary collaborate to produce an input. In many such models, nature first samples an input from a specific distribution (like the probabilistic planted clique model noted here), which is then modified by the adversary before being presented as an input to an algorithm. It is important to restrict the adversary's power, so that it cannot simply throw out nature's starting point and replace it with a worst-case instance. Feige and Killian[24] suggested studying *monotone adversaries*, which can only modify the input by making the optimal solution "more obviously optimal." For example, in the semi-random version of the planted clique problem, a monotone adversary is only allowed to remove edges that are not in the planted clique Q—it cannot remove edges from Q or add edges outside Q.

Semi-random models with a monotone adversary may initially seem no harder than the planted models that they generalize. But let's return to the planted clique model with $k = \Omega(\sqrt{n \log n})$, where the "top-$k$ degrees" algorithm succeeds with high probability when there is no adversary. A monotone adversary can easily foil this algorithm in the semi-random planted clique model, by removing edges between clique

Figure 4. Degree distribution of an Erdős–Rényi graph with edge density ½, before planting the k-clique Q. If $k = \Omega(\sqrt{n \lg n})$, then the planted clique will consist of the k vertices with the highest degrees.



l For example, audio signals are typically approximately sparse in the Fourier basis, images in the wavelet basis.

and non-clique vertices to decrease the degrees of the former back down to $\approx n/2$. Thus the semi-random model forces us to develop smarter, more robust algorithms.[m]

For the semi-random planted clique model, Feige and Krauthgamer[24] gave a polynomial-time algorithm that recovers the clique with high probability provided $k = \Omega(\sqrt{n})$. The spectral algorithm by Alon et al.[3] achieved this guarantee only in the standard planted clique model, and it does not provide any strong guarantees for the semi-random model. The algorithm of Feige and Krauthgamer[24] instead uses a semidefinite programming relaxation of the problem. Their analysis shows that this relaxation is exact with high probability in the standard planted clique model (provided $k = \Omega(\sqrt{n})$), and uses the monotonicity properties of optimal mathematical programming solutions to argue this exactness cannot be sabotaged by any monotone adversary.

Smoothed Analysis

Smoothed analysis is another example of a semi-random model, now with the order of operations reversed: an adversary goes first and chooses an arbitrary input, which is then perturbed slightly by nature. Smoothed analysis can be applied to any problem where "small perturbations" make sense, including most problems with real-valued inputs. It can be applied to any measure of algorithm performance, but has proven most effective for running time analyses.

Like other semi-random models, smoothed analysis has the benefit of potentially escaping worst-case inputs (especially if they are "isolated"), while avoiding overfitting a solution to a specific distributional assumption. There is also a plausible narrative about why real-world inputs are captured by this framework: whatever problem you would like to solve, there are inevitable inaccuracies in its formulation (from measurement error, uncertainty, and so on).

The simplex method. Spielman and Teng[38] developed the smoothed analysis framework with the specific goal of proving that bad inputs for the simplex method are exceedingly rare. Average case analyses of the simplex method from the 1980s (for example, Borgwardt[14]) provide evidence for this thesis, but smoothed analysis provides more robust support for it.

The perturbation model in Spielman and Teng[38] is: independently for each entry of the constraint matrix and right-hand side of the linear program, add a Gaussian (that is, normal) random variable with mean 0 and standard deviation σ.[n] The parameter σ interpolates between worst-case analysis (when $\sigma = 0$) and pure average-case analysis (as $\sigma \to \infty$, the perturbation drowns out the original linear program). The main result states that the expected running time of the simplex method is polynomial as long as typical perturbations have magnitude at least an inverse polynomial function of the input size (which is small!).

Theorem 3 (Spielman and Teng[38]) For every initial linear program, in expectation over the perturbation to the program, the running time of the simplex method is polynomial in the input size and in $1/\sigma$.

The running time blow-up as $\sigma \to 0$ is necessary because the worst-case running time of the simplex method is exponential. Several researchers have devised simpler analyses and better polynomial running times, most recently Dadush and Huiberts.[17] All of these analyses are for a specific pivot rule, the "shadow pivot rule." The idea is to project the high-dimensional feasible region of a linear program onto a plane (the "shadow") and run the simplex method there. The hard part of proving Theorem 3 is showing that, with high probability over nature's perturbations, the perturbed instance is well-conditioned in the sense that each step of the simplex method makes significant progress traversing the boundary of the shadow.

Local search. A local search algorithm for an optimization problem maintains a feasible solution, and iteratively improves that solution via "local moves" for as long as possible, terminating with a locally optimal solution. Local search heuristics are ubiquitous in practice, in many different application domains. Many such heuristics have an exponential worst-case running time, despite always terminating quickly in practice (typically within a sub-quadratic number of iterations). Resolving this disparity is right in the wheelhouse of smoothed analysis. For example, Lloyd's algorithm for the k-means problem can require an exponential number of iterations to converge in the worst case, but needs only an expected polynomial number of iterations in the smoothed case (see Arthur et al.[7] and the references therein).[o]

Much remains to be done, however. For a concrete challenge problem, let's revisit the maximum cut problem. The input is an undirected graph $G = (V, E)$ with edge weights, and the goal is to partition V into two groups to maximize the total weight of the edges with one endpoint in each group. Consider a local search algorithm that modifies the current solution by moving a single vertex from one side to the other (known as the "flip neighborhood"), and performs such moves as long as they increase the sum of the weights of the edges crossing the cut. In the worst case, this local search algorithm can require an exponential number of iterations to converge. What about in the smoothed analysis model, where a small random perturbation is added

m The extensively studied "stochastic block model" generalizes the planted clique model (for example, see Moore[32]), and is another fruitful playground for semi-random models. Here, the vertices of a graph are partitioned into groups, and the probability that an edge is present is a function of the groups that contain its endpoints. The responsibility of an algorithm in this model is to recover the (unknown) vertex partition. This goal becomes provably strictly harder in the presence of a monotone adversary.[31]

n This perturbation results in a dense constraint matrix even if the original one was sparse, and for this reason Theorem 3 is not fully satisfactory. Extending this result to sparsity-preserving perturbations is an important open question.

o An orthogonal issue with local search heuristics is the possibility of outputting a locally optimal solution that is much worse than a globally optimal one. Here, the gap between theory and practice is not as embarrassing—for many problems, local search algorithms really can produce pretty lousy solutions. For this reason, one generally invokes a local search algorithm many times with different starting points and returns the best of all of the locally optimal solutions found.

to each edge's weight? The natural conjecture is that local search should terminate in a polynomial number of iterations, with high probability over the perturbation. This conjecture has been proved for graphs with maximum degree $O(\log n)$[21] and for the complete graph;[4] for general graphs, the state-of-the-art is a quasi-polynomial-time guarantee (meaning $n^{O(\log n)}$ iterations).[22]

More ambitiously, it is tempting to speculate that for every natural local search problem, local search terminates in a polynomial number of iterations in the smoothed analysis model (with high probability). Such a result would be a huge success story for smoothed analysis and beyond worst-case analysis more generally.

On Machine Learning

Much of the present and future of research going beyond worst-case analysis is motivated by advances in machine learning.[p] The unreasonable effectiveness of modern machine learning algorithms has thrown down the gauntlet to algorithms researchers, and there is perhaps no other problem domain with a more urgent need for the beyond worst-case approach.

To illustrate some of the challenges, consider a canonical supervised learning problem, where a learning algorithm is given a dataset of object-label pairs and the goal is to produce a classifier that accurately predicts the label of as-yet-unseen objects (for example, whether or not an image contains a cat). Over the past decade, aided by massive datasets and computational power, deep neural networks have achieved impressive levels of performance across a range of prediction tasks.[25] Their empirical success flies in the face of conventional wisdom in multiple ways. First, most neural network training algorithms use first-order methods (that is, variants of gradient descent) to solve nonconvex optimization problems that had been written off as computationally intractable. Why do these algorithms so often converge quickly to a local optimum, or even to a global optimum?[q] Second, modern neural networks are typically over-parameterized, meaning that the number of free parameters (weights and biases) is considerably larger than the size of the training dataset. Over-parameterized models are vulnerable to large generalization error (that is, overfitting), but state-of- the-art neural networks generalize shockingly well.[40] How can we explain this? The answer likely hinges on special properties of both real-world datasets and the optimization algorithms used for neural network training (principally stochastic gradient descent).[r]

There are compelling parallels between the recent research on clustering in stable instances and slightly older results in a field of applied mathematics known as *sparse recovery*, where the goal is to reverse engineer a "sparse" object from a small number of clues around it.

Another interesting case study, this time in unsupervised learning, concerns topic modeling. The goal here is to process a large unlabeled corpus of documents and produce a list of meaningful topics and an assignment of each document to a mixture of topics. One computationally efficient approach to the problem is to use a singular value decomposition subroutine to factor the term-document matrix into two matrices, one that describes which words belong to which topics, and one indicating the topic mixture of each document.[35] This approach can lead to negative entries in the matrix factors, which hinders interpretability. Restricting the matrix factors to be nonnegative yields a problem that is *NP*-hard in the worst case, but Arora et al.[6] gave a practical factorization algorithm for topic modeling that runs in polynomial time under a reasonable assumption about the data. Their assumption states that each topic has at least one "anchor word," the presence of which strongly indicates that the document is at least partly about that topic (such as the word "Durant" for the topic "basketball"). Formally articulating this property of data was an essential step in the development of their algorithm.

The beyond worst-case viewpoint can also contribute to machine learning by "stress-testing" the existing theory

p Arguably, even the overarching goal of research in beyond worst-case analysis—determining the best algorithm for an application-specific special case of a problem—is fundamentally a machine learning problem.[26]

q See Jin et al.[28] and the references therein for recent progress on this question.

r See Neyshabur[33] and the references therein for recent developments in this direction.

and providing a road map for more robust guarantees. While work in beyond worst-case analysis makes strong assumptions relative to the norm in theoretical computer science, these assumptions are usually weaker than the norm in statistical machine learning. Research in the latter field often resembles average-case analysis, for example when data points are modeled as independent and identically distributed samples from some (possibly parametric) distribution. The semi-random models described earlier in this article are role models in blending adversarial and average-case modeling to encourage the design of algorithms with robustly good performance. Recent progress in computationally efficient robust statistics shares much of the same spirit.[19]

Conclusion

With algorithms, silver bullets are few and far between. No one design technique leads to good algorithms for all computational problems. Nor is any single analysis framework—worst-case analysis or otherwise—suitable for all occasions. A typical algorithms course teaches several paradigms for algorithm *design*, along with guidance about when to use each of them; the field of beyond worst-case analysis holds the promise of a comparably diverse toolbox for algorithm *analysis*.

Even at the level of a specific problem, there is generally no magical, always-optimal algorithm—the best algorithm for the job depends on the instances of the problem most relevant to the specific application. Research in beyond worst-case analysis acknowledges this fact while retaining the emphasis on robust guarantees that is central to worst-case analysis. The goal of work in this area is to develop novel methods for articulating the relevant instances of a problem, thereby enabling rigorous explanations of the empirical performance of known algorithms, and also guiding the design of new algorithms optimized for the instances that matter.

With algorithms increasingly dominating our world, the need to understand when and why they work has never been greater. The field of beyond worst-case analysis has already produced several striking results, but there remain many unexplained gaps between the theoretical and empirical performance of widely used algorithms. With so many opportunities for consequential research, I suspect the best work in the area is yet to come.

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