LIM - Assignment 2

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Exercise 3.1 (Schilling)

(i) $A_1, A_2, \dots, A_N \in \mathcal{A} \implies \bigcap_{i=1}^N A_i \in \mathcal{A}$

Assuming that $A_1, A_2, \dots, A_N \in \mathcal{A}$, then also have that $A_1^c, A_2^c, \dots, A_N^c \in \mathcal{A}$ by definition 3.1 in [1]. Using the same definition we can also conclude that $\bigcup_{i=1}^N A_i^c \in \mathcal{A}$. Using De Morgans laws we have that $\bigcup_{i=1}^N A_i^c = \left(\bigcap_{i=1}^N A_i\right)^c$. Again by using the definition of a σ-algebra we have that $\left(\bigcap_{i=1}^N A_i\right)^c \in \mathcal{A} \implies \left(\left(\bigcap_{i=1}^N A_i\right)^c\right)^c = \bigcap_{i=1}^N A_i \in \mathcal{A}$ which is exactly what we wanted. ■

(ii) $A \in \mathcal{A} \iff A^c \in \mathcal{A}$

The fact that $A \in \mathcal{A} \implies A^c \in \mathcal{A}$ is given by definition 3.1 in [1]. The other direction is given by the fact that $A^c \in \mathcal{A} \implies A^{cc} \in \mathcal{A}$. But $A^{cc} = A$ so we have that $A \in \mathcal{A}$. ■

(iii) $A, B \in \mathcal{A} \implies A \setminus B, A \triangle B \in \mathcal{A}$

If $A, B \in \mathcal{A}$ then $B^c \in \mathcal{A}$ by definition and combining that with **(i)** we have that $A \cap B^c \in \mathcal{A}$. Which is what we wanted since $A \cap B^c = A \setminus B$. ■

Using this we can then also conclude that $B \setminus A \in \mathcal{A}$ by the same argument. As a σ-algebra is closed under countable unions by definition, we can then say that since $(A \setminus B), (B \setminus A) \in \mathcal{A}$ we must also have that $(A \setminus B) \cup (B \setminus A) = A \triangle B \in \mathcal{A}$. ■

Exercise S2.3.

(i) $\sigma(G_1) = \mathcal{P}(X)$

We have that $\sigma(G_1) \subseteq \mathcal{P}(X)$ by the definition of a σ-algebra, so we need only to show that $\mathcal{P}(X) \subseteq \sigma(G_1)$. Let $S \in \mathcal{P}(X)$ and $S \neq \emptyset$, then we have that $S = \bigcup_{x \in S}\{x\}$ and since $\{x\} \in G_1$ for all $x \in X$ we also have that $\{x\} \in \sigma(G_1)$ and so is their union as $\sigma(G_1)$ is a σ-algebra. Thus $S \in \sigma(G_1)$, and we have now shown that $\mathcal{P}(X) = \sigma(G_1)$. ■

(ii) $\sigma(G_2) = \mathcal{P}(X)$

Yet again we can argue that $\sigma(G_2) \subseteq \mathcal{P}(X)$ by the definition of a σ-algebra. So to show that $\mathcal{P}(X) \subseteq \sigma(G_2)$, we show it by showing that $G_1 \subseteq \sigma(G_2)$ which would imply that

$$\mathcal{P}(X) = \sigma(G_1) \subseteq \sigma(\sigma(G_2)) = \sigma(G_2)$$

which we have from Remark 3.5 in [1], and by definition of $\sigma(G_2)$ being the smallest σ-algebra containing G_2.

So to show $G_1 \subseteq \sigma(G_2)$, we simply observe that $G_2 \subseteq \sigma(G_2)$ and $X \in \sigma(G_2)$ and so is the union/intersection/difference of any of the sets in G_2 and X. We have the following

$$\{1\} = \{1,2\} \setminus \{2,3\}$$

$$\{2\} = \{1,2\} \cap \{2,3\}$$

$$\{3\} = \{2,3\} \setminus \{1,2\}$$

$$\{4\} = X \setminus (\{1,2\} \cup \{2,3\})$$

As each set on the right hand side is in $\sigma(G_2)$, we must then have that each set on the left hand side is also in $\sigma(G_2)$. This concludes that $G_1 \subseteq \sigma(G_2)$ and we have shown that $\mathcal{P}(X) = \sigma(G_2)$.

■

[1] Rene Schilling "Measures, Integrals and Martingales", second edition, Cambridge University Press, ISBN: 978-1-316-62024-3 (paperback)