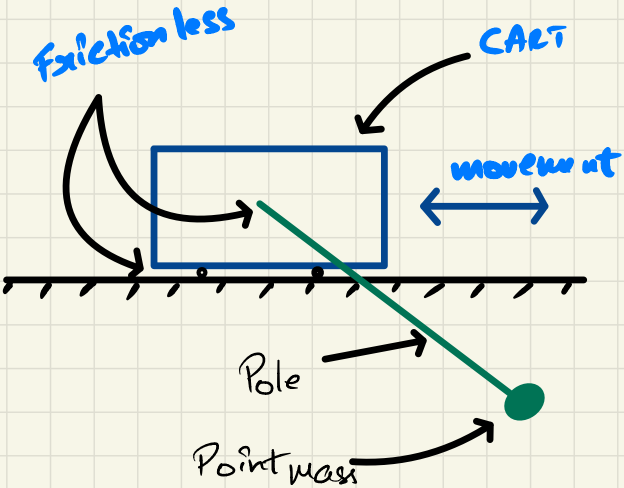
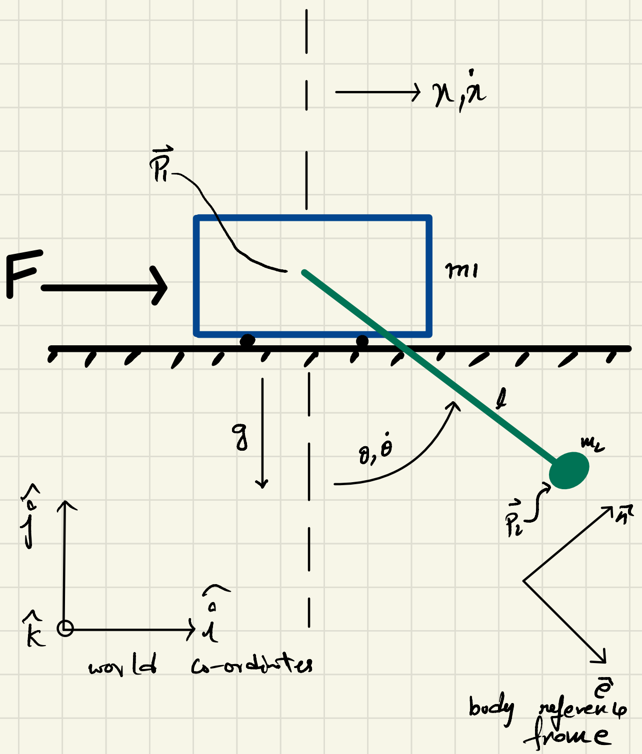
SES/FSE-598 Assignment 2

Cart-Pole Dynamics

System:





m_1 = Mass of the cart

m_2 = mass of the pole [we assume that the entire mass of the pole is concentrated on the edge ie point mass]

l = Length of the pole

$F \rightarrow$ force [Horizontal]

x = Position of the cart
$\dot{x}$ = velocity of the cart
} 1st DOF

θ = Angle of the pole
$\dot{\theta}$ = Angular velocity
} 2nd DOF

g = Acceleration due to gravity

There are two position vectors $\vec{p}_1$ & $\vec{p}_2$

$\vec{p}_1 \Rightarrow$ from the origin to the center of mass of the cart.

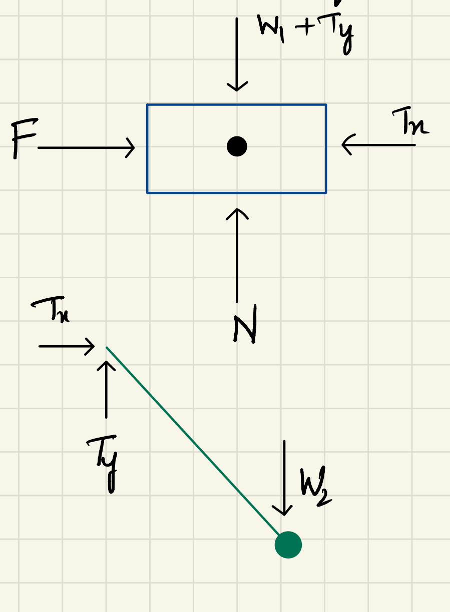
The pole is attached to this point.

$\vec{p}_2 \Rightarrow$ Origin to the tip of the pole.

The two vectors $\vec{p}_1$ & $\vec{p}_2$ live in the intertial reference frame with horizontal component $\hat{i}$ & vertical component $\hat{j}$

In the body reference frame there are two unit vectors $\hat{e}$ & $\hat{n}$.

Free Body Diagrams: [considering point mass]



$N \rightarrow$ Normal force acting on the cart

$F \rightarrow$ Horizontal Force.

$W_1 \rightarrow$ Weight of the cart ($m_1 \times g$)

$T_y, T_x \rightarrow$ components of the contact force.

$W_2 \rightarrow$ Weight of the Pole ($m_2 \times g$)

Newton - Euler Equations: [Dynamics]

Force balance on the Cart:

ⓐ $\underbrace{(F - T_x)}_{\text{horizontal Forces}}\hat{i} + \underbrace{(N - W_1 - T_y)}_{\text{Vertical Force}}\hat{j} = m_1 \ddot{\vec{P_1}}$

↳ acceleration

ⓑ Force balance on the pole:

$\underbrace{(T_x)}_{\text{horizontal}}\hat{i} + \underbrace{(T_y + W_2)}_{\text{Vertical}}\hat{j} = m_2 \ddot{\vec{P_2}}$

↳ acceleration

ⓒ Angular Momentum balance on the pole about pivot

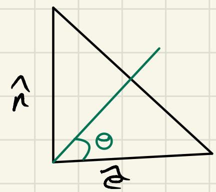
$\underbrace{(\vec{P_2} - \vec{P_1})}_{\text{relative position vector}} * \underbrace{(-W_2\hat{j})}_{\text{Force vector}} = \underbrace{(\vec{P_2} - \vec{P_1}) * (m_2 \ddot{\vec{P_2}})}_{\text{Rate of change of Angular Momentum.}}$

From the above diagram we can substitute the $\vec{P_1}$ & $\vec{P_2}$ values:-

1) $\underbrace{\vec{P_1} = x\hat{i}}_{\text{Position}}$; $\underbrace{\dot{\vec{P_1}} = \dot{x}\hat{i}}_{\text{velocity}}$; $\underbrace{\ddot{\vec{P_1}} = \ddot{x}\hat{i}}_{\text{acceleration}}$

a) $\vec{P_2} = \vec{P_1} + l\hat{e}$; $\dot{\vec{P_2}} = \dot{\vec{P_1}} + l\dot{\hat{e}}$; $\ddot{\vec{P_2}} = \ddot{P_2} + l\ddot{\hat{e}}$

From the figures we can compute the values of $\hat{e}$ & $\hat{n}$



c) $\hat{e} = \sin\theta\hat{i} - \cos\theta\hat{j}$

$\hat{n} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$\dot{\hat{e}} = \dot{\theta}\cos\theta\hat{i} + \dot{\theta}\sin\theta\hat{j} = \dot{\theta}\hat{n}$ — (i)

$\dot{\hat{n}} = -\dot{\theta}\sin\theta\hat{i} + \dot{\theta}\cos\theta\hat{j} = -\dot{\theta}\hat{e}$ — (ii)

d — $\ddot{\hat{e}} = \ddot{\theta}\hat{n} + \dot{\theta}\dot{\hat{n}} = \ddot{\theta}\hat{n} - \dot{\theta}^2\hat{e}$ — from (i) & (ii)

e — $\ddot{\hat{n}} = -\ddot{\theta}\hat{e} + \dot{\theta}\dot{\hat{e}} = -\ddot{\theta}\hat{e} + \dot{\theta}^2\hat{n}$ — from (i) & (ii)

from a

$F - T_x = m_1(\hat{i} \cdot \ddot{\vec{P}})$

$= m_1\ddot{x}$

— (f)

From b

$T_x = m_2(\hat{i} \cdot \ddot{\vec{P_2}})$

$\Rightarrow m_2(\ddot{x} + l\hat{i} \cdot (\ddot{\theta}\hat{n} - \dot{\theta}^2\hat{e}))$

↳ from d & (a)

$\Rightarrow T_x = m_2[\ddot{x} + l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)]$

— (g)

From (c) $\times \hat{k}$ (z axis)

$\Rightarrow \hat{k} \cdot \left\{ (\vec{p}_2 - \vec{p}_1) \times (-w_2 \hat{j}) = (\vec{p}_2 - \vec{p}_1) \times (m_2 \ddot{\vec{p}}_2) \right\}$

$\Rightarrow$ From (1) & (2) substituting $\vec{p}_1$ & $\vec{p}_2$ values:

$\Rightarrow \hat{k} \cdot \left\{ (l\hat{e}) \times (-m_2 g \hat{j}) = (l\hat{e}) \times [m_2 (\ddot{x}\hat{i} + l\ddot{\hat{e}})] \right\}$

$\Rightarrow -\cancel{m_2} g \cancel{l} \sin\theta = \cancel{m_2}\cancel{l} (\ddot{x}\cos\theta + l\ddot{\theta})$

$\Rightarrow -g \sin\theta = \ddot{x}\cos\theta + l\ddot{\theta}$ — (h)

from (f), (g)

$$F - m_1 \ddot{x} = m_2 [\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2 \sin\theta]$$

$$\Rightarrow F = (m_1 + m_2)\ddot{x} + m_2 l \ddot{\theta}\cos\theta - m_2 l \dot{\theta}^2 \sin\theta$$

from (h)

$-g\sin\theta = \ddot{x}\cos\theta + l\ddot{\theta}$

$\Rightarrow -g\theta = \ddot{x} + l\ddot{\theta}$

$\Rightarrow \underline{\ddot{x} = -g\theta + l\ddot{\theta}}$

$$F = (m_1+m_2)(-g\theta + l\ddot{\theta}) + m_2 l \ddot{\theta} \quad -0$$

$$\Rightarrow \quad F = -m_1 g\theta - m_2 g\theta + m_1 l\ddot{\theta} + m_2 l \ddot{\theta} + m_2 l \ddot{\theta}$$

$$\Rightarrow \quad F = 2 m_2 l \ddot{\theta} + m_1 l \ddot{\theta} - (m_1+m_2) g\theta$$

$$= \ddot{\theta}\,(2m_2 l + m_1 l) - (m_1+m_2) g\theta$$

$$\boxed{\ddot{\theta} = \frac{F + (m_1+m_2) g\theta}{(2m_2+m_1)\, l}}$$

$$\ddot{\theta} = \frac{F}{(2m_2+m_1)\, l} + \frac{(m_1+m_2)\, g\theta}{(2m_2+m_1)\, l}$$

from (h):

$$-g\theta = \ddot{x} + l\ddot{\theta}$$

$$\ddot{\theta} = \frac{-g\theta - \ddot{x}}{l} \quad \Rightarrow \quad -\frac{(g\theta + \ddot{x})}{l}$$

$$F = (m_1+m_2)\ddot{x} + m_2 l \left(-\frac{(g\theta + \ddot{x})}{l}\right)$$

$$= (m_1+m_2)\ddot{x} - m_2 \cancel{l}\, \frac{(g\theta + \ddot{x})}{\cancel{l}}$$

$$\Rightarrow (m_1+m_2)\ddot{x} - m_2(g\theta) + m_2\ddot{x}$$

$$F \;\Rightarrow\; m_1\ddot{x} + 2m_2\ddot{x} - m_2 g\theta$$

$$= \ddot{x}(m_1+2m_2) - m_2 g\theta$$

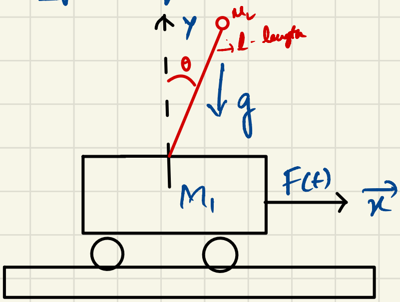
$$\boxed{\ddot{x} = \frac{F + m_2 g\theta}{(m_1+2m_2)}}$$

$$\ddot{x} = \frac{F}{(m_1+2m_2)} + \frac{m_2 g\theta}{(m_1+2m_2)}$$

$$Ax + Bu = \dot{X}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{m_1 g}{m_1+2m_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(m_1+m_2)g}{(2m_2+m_1)l} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m_1+2m_2} \\ 0 \\ \frac{1}{(2m_2+m_1)l} \end{bmatrix}}_{B} F$$

Legrangian [Taking the mass as uniform]



Potential Energy:

$$V = \frac{1}{2} m_2 2 \dot{x} \frac{l}{2} \dot{\theta} \cos\theta$$

Kinetic Energy:

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 \left(\frac{l}{2} \dot{\theta} \cos\theta\right)^2$$

$$L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[\dot{x}^2 + \left(l/_2 \cos\theta \dot{\theta}\right)^2 + 2\, l/_2 \cos\theta \dot{\theta} \dot{x}\right)$$

$$+ \frac{1}{2} m_2 \left(l/_2 \sin\theta \dot{\theta}\right)^2 + \frac{1}{2} I \dot{\theta}^2 - m_2 g\, l/_2 \cos\theta$$

$$\Rightarrow \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 \frac{l^2}{4} \dot{\theta}^2 + \frac{1}{2} m_2 l \cos\theta \dot{\theta} \dot{x} + \frac{1}{2} I \dot{\theta}^2$$

$$- m_2 g \frac{l}{2} \cos\theta$$

$$F = (m_1+m_2)\ddot{x} + \frac{1}{2} m l \cos\theta \ddot{\theta} - \frac{1}{2} m_2 l \sin\theta \dot{\theta}^2$$

$$\frac{\partial\left(\frac{\partial L}{\partial \dot{\theta}}\right)}{\partial t} - \frac{\partial L}{\partial \theta} = 0$$

$$0 = \frac{m_2 l^2}{4}\ddot{\theta} + \frac{1}{2} m_2 l \cos\theta \ddot{x} - \frac{1}{2} m_2 l \sin\theta \dot{\theta}\dot{x}$$
$$+ I\ddot{\theta} + m_2 g l/_2 \sin\theta$$

Linearizing at $\theta = 0$, $\cos\theta = 1$, $\sin\theta \simeq \theta$, $\sin\theta\dot{\theta} \simeq 0$

$$(m_1+m_2)\ddot{x} + \frac{1}{2} m_2 l \ddot{\theta} = F$$

$$\Rightarrow \frac{1}{4} m_2 l^2 \ddot{\theta} + \frac{1}{2} m_2 l \ddot{x} + I\ddot{\theta} + \frac{1}{2} m_2 g l \theta = 0$$

$$\ddot{x} = \frac{m_2^2 l^2 g\theta + F l^2 m_2 + 4FI}{m_1 \cdot m_2 l^2 + 4m_1 I + 4m_2 I}$$

$$\ddot{\theta} = \frac{-2lm_2(F + g m_1\theta + m_2 g\theta)}{m_1 m_2 l^2 + 4m_1 I + 4m_2 I}$$

System Equations:

$$
\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-m_2^2 l^2 g}{m_1 m_2 l^2 + 4(m_1+m_2)I} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-2 l m_2 (m_1 g + m_2 g)}{m_1 m_2 l^2 + 4(m_1+m_2)I} & 0 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}} +
$$

$$
\overset{B}{\begin{bmatrix} 0 \\ \dfrac{4I + m_2 l^2}{m_1 m_2 l^2 + 4(m_1+m_2)I} \\ 0 \\ \dfrac{-2 m_2 l}{m_1 m_2 l^2 + 4(m_1+m_2)I} \end{bmatrix}} \overset{u}{F}
$$