# Preparing Eigenstate of a Hamiltonian

## Problem :

Suppose we have a Hamiltonian 'H' fulfilling the following properties:
1. H has a dimension $2n-1$ and $\|H\| = 1$
2. All it's eigenvalues are symmetric around 0 i.e., $\lambda_i = -\lambda_{2n-i}$ for $i \in \{1, 2, ....., 2n-1\}$, implying $\lambda_n = 0$.
Let, $|v_j\rangle$ be the eigenstates of the Hamiltonian H with eigenvalues $\lambda_j$. The 0-eigenvalue $\lambda_n$ is separated from the rest by a gap $\Delta$.
Starting from some initial state $|\psi_0\rangle = \sum_{l=1}^{2n-1} c_l |v_l\rangle$, prepare a state $|G\rangle$ such that $\||G\rangle - |v_n\rangle\| \leq \varepsilon$.

## Solution using QSP:

Suppose we have a $(\alpha, m, 0)$ block encoding $U_H$ of the Hamiltonian H. Let $P_{\lambda_n}$ be the projector into the $\lambda_n$-eigenspace of H, i.e. the null-space.
Now, we have to concoct such a projector. For this, suppose we have a polynomial $P$ such that $P(0) = 1$ and $|P(x)| = 0$ for $x \in D_{\Delta/2\alpha}$, where $D_{1/\kappa} = [-1, -1/\kappa] \cup [1/\kappa, 1]$. Then

$$P\left(\left(H - \lambda_n I\right)/2\alpha\right) \approx P_{\lambda_n}. \tag{1}$$

Now, for this problem we will use the following $2\ell$ degree polynomial.

$$\mathbf{R}_\ell\left(x; \Delta\right) = \frac{\mathbf{T}_\ell\left(-1 + 2\frac{x^2 - \Delta^2}{1 - \Delta^2}\right)}{\mathbf{T}_\ell\left(-1 + 2\frac{-\Delta^2}{1-\Delta^2}\right)} \tag{2}$$

Now, $R$ is an even polynomial. Now we can apply the polynomial to $H - \lambda_n I$ using the techniques of Quantum Signal Processing and get rid of the unwanted components. Let's define

$$\widetilde{H} = \frac{H - \lambda_n I}{\alpha + |\lambda_n|} \tag{3}$$

For our problem, $\lambda_n = 0$, so the operator becomes

$$\widetilde{H} = \frac{H}{\alpha} \tag{4}$$

and we also define

$$\widetilde{\Delta} = \frac{\Delta}{2\alpha} \tag{5}$$

Now, as we know

$$|R_\ell\left(x; \Delta\right)| \leq 2e^{-\sqrt{2}\ell\Delta} \tag{6}$$

for $x \in D_\Delta$.
Therefore, we have

$$\|R\left(\widetilde{H}; \widetilde{\Delta}\right) - P_{\lambda_n}\| \leq 2e^{-\sqrt{2}\ell\widetilde{\Delta}} \tag{7}$$

Now we can create $(\alpha, m+1, \varepsilon)$ block encoding of $R\left(\widetilde{H}; \widetilde{\Delta}\right)$ and let it be $U_{\widetilde{H}}$. Now we can apply this block encoding to the given state $|\psi_0\rangle$ and the probability of getting all $0's$ while measuring the ancilla qubits is atleast $c_n^2$. So we need to run the block encoding $O\left(1/c_n^2\right)$ times and can cut it down to $O\left(1/c_n\right)$ using amplitude amplification for the desired eigenstate, in our case $|v_n\rangle$.

# Cost and Complexity Analysis of the Solution using QSP :

**Number of qubits :** If the Hamiltonian $H$ works on $n$ qubits and has a $(\alpha, m, 0)$ block encoding, then using the minmax polynomial of equation 2, we need only 1 more ancilla qubit for the Quantum Signal Processing method. So, the total number of qubits needed is $n + m + 1$. If we use amplitude amplification we will need one extra ancilla qubit pushing the qubit count to

$n + m + 2$.

**Complexity of the algorithm :** Suppose, the cost of implementing the unitary $U_{\widetilde{H}}$ once is $T$. Assuming that we are using amplitude amplification, we need to run the block encoding $O\left(1/c_n\right)$ times. So, the cost is $O\left(T/c_n\right)$.

Now, if $U_H$ is the $(\alpha, m, 0)$ block encoding of $H$, then we can obtain a $(1, m+1, \epsilon)$ block encoding of $U_{\widetilde{H}}$ using $O\left(\left(\alpha/\Delta\right) log\left(1/\epsilon\right)\right)$ applications of controlled- $U_H$ and it's conjugate and $O\left(\left(m\alpha/\Delta\right) log\left(1/\epsilon\right)\right)$ other primitive gates.
Therefore, neglecting the cost of using other primitive gates, the overall complexity of filtering the desired state, given the block encoding $U_H$ is $O\left(\left(T\alpha/c_n\Delta\right) log\left(1/\epsilon\right)\right)$.

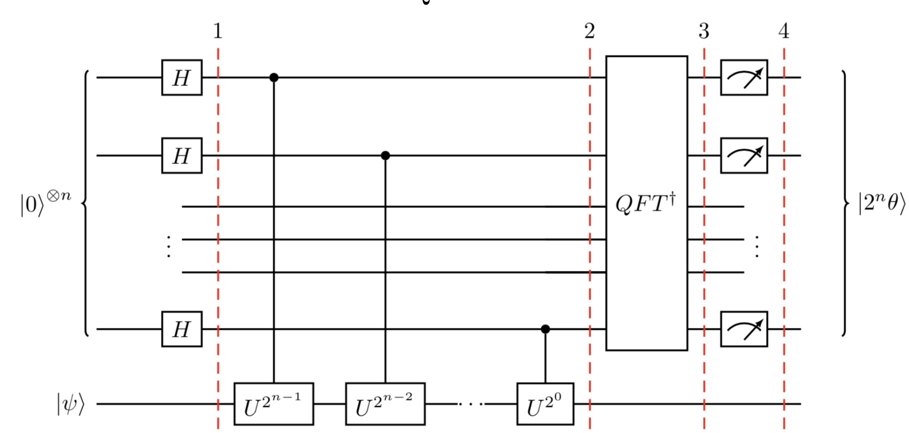
# Solution using QPE:

We are given the state $|\psi_0\rangle$ that is a superposition in the eigenbasis of the Hamiltonian $H$. Now, suppose we have access to the controlled unitary $U$ and

$$U = e^{-\iota H} \tag{8}$$

Now, if we want to find the eigenstate of $H$ having eigenvalue 0 we need to look for the eigenstate of $U$, that has an eigenvalue of 1. So, we need to do Quantum Phase Estimation using $U$ and look for the eigenvalue 1.

Below we have the circuit for Quantum Phase Estimation :



Now, as we don't have any particular eigenstate we will use the given superposition of eigenstates $|\psi_0\rangle$ as the target state and apply the operators $U$. As

$$|\psi_0\rangle = \sum_{l=1}^{2n-1} c_l |v_l\rangle \tag{9}$$

Therefore, after applying the QPE circuit, with a rough probability of $O\left(1/c_n^2\right)$, we will obtain an eigenstate close to 1, if all other parameters are properly taken into account. So, if we measure for the eigenvalue 1 in the control register, we will get the corresponding eigenstate $|v_n\rangle$ in the target register, which is our desired eigenstate.

# Cost and Complexity Analysis of the Solution using QPE :

**Number of qubits :** For simplicity, let's assume we are given an eigenstate $|\psi\rangle$ of the operator $U$ and it's corresponding eigenvalue is $\omega$. Suppose after applying the QPE sub-routine we obtain an estimate $\widetilde{\omega}$ of the eigenvalue $\omega$. Assume we obtain $\widetilde{\omega}$ with a probability at least $1 - \frac{1}{2^m}$ where the error term is $\frac{1}{2^m}$ and also $|\widetilde{\omega} - \omega| \leq \frac{1}{2^r}$, i.e. the spectral gap measurable is bounded by $\frac{1}{2^r}$.
To meet the above conditions, it suffices to use a total of $m + r + 1$ control-qubits.
Now for our problem, we need to filter out the state $|v_n\rangle$ with an error upper-bounded by $\epsilon$ and spectral gap lower-bounded by $\Delta$. Therefore, the number of control-qubits need is $t = log\left(1/\Delta\epsilon\right) + 1$.

Therefore, assuming that $U$ acts on $n$ qubits, the total number of qubits need is $log\left(1/\Delta\epsilon\right) + n + 1$.

**Complexity of the algorithm :** Let's assume, we can apply each unitary $U$ in time $T\left(U\right)$. As we have $t$ number of control-qubits, we need to apply $U$, $O\left(2^t\right)$ times. So, the complexity of applying the $U$'s is

$$O\left(2^t T\left(U\right)\right) = O\left(\widehat{T}\right) \tag{10}$$

Now, as we obtain our desired state with a probability $O\left(1/c_n^2\right)$, we need to run the QSP sub-routine $O\left(1/c_n^2\right)$ times to obtain our desired eigenstate

with a constant probability.
Therefore, the overall complexity of this algorithm to filter the desired state of the Hamiltonian $H$, neglecting the cost of applying the Hadamard and QFT gates, is $O\left(\widehat{T}/c_n^2\right)$.