Quantum Least Square Solver

Problem :

Implement $|\beta\rangle = \frac{A^+|b\rangle}{\|A^+|b\rangle\|}$ using the AQC(exp) algorithm and the eigenstate filtering, where $A \in \mathbb{R}^{M\times N}$ and is not Hermitian. Also use AQC(P) and next solve the ordinary and weighted least square problem.

Solution :

A^+ is called the pseudo-inverse of A and has the form $A^+ = \left(A^T A\right)^{-1} A^T$ where A is our data-matrix. It is clear that A is also non-Hermitian. So, to get a Hermitian operator we need to increase the size of the Hilbert space. We define,

$$P = \sigma_+ \otimes A + \sigma_- \otimes A^\dagger = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \tag{1}$$

where $\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm \iota\sigma_y\right)$. As all the elements in the matrices are real $A^\dagger$ can also be written as A^T, but for sake of generality let's use $A^\dagger$

The top-left zero block of P is of dimension $M \times M$ and the bottom-right block is of dimension $N \times N$. So, P becomes a $(M+N)\times(M+N)$ Hermitian operator. Now, we need to consider the extended QLSP of the form

$$P\,|\chi\rangle = |B\rangle \tag{2}$$

in extended dimension $M+N$, where $|B\rangle = |0, b\rangle$. The solution $|\chi\rangle$ corresponds to, $|\chi\rangle = |1, x\rangle$, where $|x\rangle$ is a state such that $A\,|x\rangle = |b\rangle$

1 Applying AQC(exp):

The scheduling function of AQC(exp) looks like

$$f\left(s\right) = c_e^{-1} \int_0^s exp\left(-\frac{1}{s^{'}\left(1-s^{'}\right)}\right) ds^{'} \tag{3}$$

where c_e is a normalization constant such that $f(1) = 1$ and has the form,

$$c_e = \int_0^1 exp\left(-\frac{1}{s'(1-s')}\right) ds' \tag{4}$$

During the adiabatic evolution, our initial Hamiltonian will be H_0 and final Hamiltonian will be H_1. So, the evolution operator will be of the form,

$$H(s) = (1 - f(s))\, H_0 + f(s) H_1 \tag{5}$$

It should be noted that, if P is non-unitary, we can always use block encoding to make it unitary. More details about block encoding has been given in Appendix B.
Now, as H is not necessarily positive-definite, to overcome the indefiniteness, we need to enlarge the Hilbert space to dimension $4K$ (say, K=M+N), and so we will need two ancilla qubits to enlarge the matrix block. Now, we define

$$H_0 = \sigma_+ \otimes [(\sigma_z \otimes I_K)\, Q_{+,B}] + \sigma_- \otimes [Q_{+,B}\, (\sigma_z \otimes I_K)] \tag{6}$$

where $Q_{+,B} = I_{2K} - |+, B\rangle \langle +, B|$. The null space of H_0 is $\text{Null}(H_0) = \text{span}\{|0, -, B\rangle, |1, +, B\rangle\}$.
We also define

$$H_1 = \sigma_+ \otimes [(\sigma_x \otimes P)\, Q_{+,B}] + \sigma_- \otimes [Q_{+,B}\, (\sigma_x \otimes P)] \tag{7}$$

The null space of H_1 is $\text{Null}(H_1) = \text{span}\{|0, +, \chi\rangle, |1, +, B\rangle\}$. So, we will obtain our solution $|\chi\rangle$ if we can prepare the zero-eigenstate $|0, +, \chi\rangle$ of the Hamiltonian H_1.
For achieving this we start with the null-state $|0, -, B\rangle$ of the Hamiltonian H_0 and evolve adiabatically, according to the scheduling function $f(s)$ and arrive at the final zero-eigenstate $|0, +, \chi\rangle$.

It should be noted that AQC(exp) achieves an exponential speedup over RM and AQC(p) with respect to ϵ , thus is more suitable for preparing the solution of QLSP with high fidelity. Furthermore, the time scheduling of AQC(exp) is universal and AQC(exp) does not require any prior knowledge on the bound of κ.

Error and Complexity Analysis :

Let, the state we obtain after time T of adiabatic evolution be $|\psi_T(s)\rangle$ and the desired eigenpath is $|\widetilde{x}(s)\rangle$. The distance between the two density matrix formed by the states is given by

$$\| |\psi_T(s)\rangle \langle \psi_T(s)| - |\widetilde{x}(s)\rangle \langle \widetilde{x}(s)| \|_2 = \eta(s) \tag{8}$$

where, $1 - \eta^2(s)$ is the fidelity of obtaining the desired state, or in other words

$$|1 - \langle \psi_T(s)|\, P_0\, |\psi_T(s)\rangle| \leq \eta^2(s) \tag{9}$$

where, P_0 is the projector onto the null-space of $H(s)$. The detailed calculation of this relation between the fidelity and distance between the density matrix has been shown in Appendix A.

Now, the distance between our final state and desired state for *AQC(exp)* can be bounded from above by

$$\| \left|\psi_T\left(1\right)\right\rangle \left\langle\psi_T\left(1\right)\right| - \left|0,+,\chi\right\rangle \left\langle 0,+,\chi\right| \|_2 \leq C\log(\kappa) exp\left(\left(-C\frac{\kappa\log^2\kappa}{T}\right)^{\frac{-1}{4}}\right) \tag{10}$$

where, C is a constant independent of s, Δ, T. The detailed calculation has been shown in Appendix C.
It can be easily derived from equation 10 that the run time of *AQC(exp)* for preparing the solution state to an error margin of ϵ is given by

$$T = O\left(\kappa\log^2\left(\kappa\right)\log^4\left(\frac{\log\kappa}{\epsilon}\right)\right) \tag{11}$$

We need three extra ancilla qubits, one for transforming A to a Hermitian operator and other two for addressing the issue of indefiniteness.

It can be seen that we obtain an optimal complexity in condition-number upto logarithmic terms, and also an almost optimal complexity in terms of the error bound.

2 Applying AQC(P) :

The scheduling function of AQC(P) looks like

$$\dot{f}\left(s\right) = c_p\Delta_*^p f\left(s\right) \tag{12}$$

where, $c_p = \int_0^1 \Delta_*^{-p}\left(u\right)du$ and $\Delta_*\left(f\right)$ is the lower bound of the spectral gap $\Delta\left(f\right)$, and for our case is given by

$$\Delta\left(f\right) \geq \Delta_*\left(f\right) \geq \frac{1-f+f/\kappa}{\sqrt{2}} \tag{13}$$

When, $1 < p \leq 2$ equation 10 can be explicitly solved as

$$f\left(s\right) = \frac{\kappa}{1-\kappa}\left[1-\left(1+s\left(\kappa^{p-1}-1\right)\right)^{\frac{1}{1-p}}\right] \tag{14}$$

Now, keeping $H\left(s\right), H_0, H_1$ same as in equation $5, 6, 7$ respectively, we apply the Hamiltonian $H\left(s\right)$, according to this scheduling function and obtain our desired state.

Error and Complexity Analysis :

Let, the state we obtain after time T of adiabatic evolution be $\left|\psi_T\left(s\right)\right\rangle$ and the desired eigenpath is $\left|\widetilde{x}\left(s\right)\right\rangle$. The distance between the two density matrix formed by the states is given by

$$\| \left|\psi_T\left(s\right)\right\rangle \left\langle\psi_T\left(s\right)\right| - \left|\widetilde{x}\left(s\right)\right\rangle \left\langle\widetilde{x}\left(s\right)\right| \|_2 = \eta\left(s\right) \tag{15}$$

where, $1-\eta^2(s)$ is the fidelity of obtaining the desired state, or in other words

$$|1-\langle\psi_T(s)|P_0|\psi_T(s)\rangle| \leq \eta^2(s) \tag{16}$$

where, P_0 is the projector onto the null-space of $H(s)$. The detailed calculation of this relation between the fidelity and distance between the density matrix has been shown in Appendix A.
Now, the distance between our final state and desired state for any choice of $1 < p < 2$ can be bounded from above by

$$\| |\psi_T(1)\rangle\langle\psi_T(1)| - |0,+,\chi\rangle\langle 0,+,\chi| \|_2 \leq C\kappa/T \tag{17}$$

where, C is a constant independent of s, Δ, T.
Therefore, in order to prepare a ϵ-approximation of the solution of the given QLSP, it is sufficient to choose the runtime $T = O(\kappa/\epsilon)$.
Now, for the limiting case, i.e. for $p = 1, 2$. the distance between the density matrices can be bounded by

$$\| |\psi_T(1)\rangle\langle\psi_T(1)| - |0,+,\chi\rangle\langle 0,+,\chi| \|_2 \leq C\frac{\kappa\log(\kappa)}{T} \tag{18}$$

In that case, the run-time $T = O\left(\frac{\kappa\log\kappa}{\epsilon}\right)$.

The number of extra ancilla qubits need is same as in the technique of *AQC(exp)*.

It can be clearly seen that, we obtain the optimal speed-up in terms of κ. To also obtain an optimal speed-up in terms of ϵ we can augment *AQC(P)* with the technique of *Quantum Signal Processing* and get an exponential speed-up in terms of the error bound.

Applying Quantum Signal Processing :

Let's assume we pre-process the entries of the data matrix A such that $a_{ij} \leq 1$, where a_{ij} is the element of i^{th} row and j^{th} column of A. Let's also assume that A is α-sparse.
Then, we can have a block encoding U_{H_1} of the Hamiltonian H_1. Let P_{λ_n} be the projector into the λ_n-eigenspace, in our case $\lambda_n = 0$ and so our projection space is zero-eigenspace of H_1, i.e. the null-space.
Now, we have to concoct such a projector. For this, suppose we have a polynomial P such that $P(0) = 1$ and $|P(x)| = 0$ for $x \in D_{\Delta/2\alpha}$, where $D_{1/\kappa} = [-1, -1/\kappa] \cup [1/\kappa, 1]$. Then

$$P\left(\left(H_1 - \lambda_n I\right)/2\alpha\right) \approx P_{\lambda_n}. \tag{19}$$

Now, for this problem we will use the following 2ℓ degree polynomial.

$$\mathbf{R}_\ell(x;\Delta) = \frac{\mathbf{T}_\ell\left(-1+2\frac{x^2-\Delta^2}{1-\Delta^2}\right)}{\mathbf{T}_\ell\left(-1+2\frac{-\Delta^2}{1-\Delta^2}\right)} \tag{20}$$

Now, R is an even polynomial. Now we can apply the polynomial to $H_1 - \lambda_n I$ using the techniques of Quantum Signal Processing and get rid of the unwanted components. Let's define

$$\widetilde{H_1} = \frac{H_1 - \lambda_n I}{\alpha + |\lambda_n|} \tag{21}$$

For our problem, $\lambda_n = 0$, so the operator becomes

$$\widetilde{H_1} = \frac{H_1}{\alpha} \tag{22}$$

and we also define

$$\widetilde{\Delta} = \frac{\Delta}{2\alpha} \tag{23}$$

where, $\Delta(f) \geq \Delta_*(f)$ and $\Delta_*(f) = 1 - s + \frac{f}{\kappa}$, f is the value of the scheduling function and at the end of applying AQC(exp) $f = 1$ and so the value of Δ is lower-bounded by $\frac{1}{\kappa}$.
Now, as we know

$$|R_\ell(x;\Delta)| \leq 2e^{-\sqrt{2}\ell\Delta} \tag{24}$$

for $x \in D_\Delta$.
Therefore, we have

$$\|R\left(\widetilde{H_1};\widetilde{\Delta}\right) - P_{\lambda_n}\| \leq 2e^{-\sqrt{2}\ell\widetilde{\Delta}} \tag{25}$$

Now we can create $(\alpha, m+1, \varepsilon)$ block encoding of $R\left(\widetilde{H_1};\widetilde{\Delta}\right)$ and let it be $U_{\widetilde{H_1}}$.

The state obtained after application of AQC(exp) is

$$|\psi_0\rangle = (1-\epsilon)|0,+,\chi\rangle + \epsilon|\perp\rangle \tag{26}$$

We, are interested in obtaining the state $|0,+,\chi\rangle$ that is in the null-space of H_1, with high fidelity and so we apply the operator $U_{\widetilde{H_1}}$ to the state $|\psi_0\rangle$. To obtain the state $|0,+,\chi\rangle$ with constant probability, we need to repeat the step $O\left(\frac{1}{1-\epsilon}\right)$ times with out using amplitude amplification and $O\left(\frac{1}{\sqrt{1-\epsilon}}\right)$ times using amplitude amplification.

Thus, after getting rid of the two ancilla qubits corresponding to $|0\rangle$ and $|+\rangle$, we get our desired state $|\chi\rangle$. From this state $|\chi\rangle$, we can easily obtain out state $|x\rangle$ by performing a Z-basis measurement on the ancilla and keeping the state corresponding to $|1\rangle$.

Error and Complexity Analysis :

Suppose, the cost of implementing the unitary $U_{\widetilde{H_1}}$ once is T. Assuming that we are using amplitude amplification, we need to run the block encoding $O\left(\frac{1}{\sqrt{1-\epsilon}}\right)$ times. So, the cost is $O\left(\frac{T}{\sqrt{1-\epsilon}}\right)$.

Now, if U_{H_1} is the $(\alpha, m, 0)$ block encoding of H_1, then we can obtain a $(1, m+1, \varepsilon)$ block encoding of $U_{\widetilde{H_1}}$ using $O\left((\alpha/\Delta)\, log\,(1/\varepsilon)\right)$ applications of controlled- U_{H_1} and it's conjugate

and $O\left((m\alpha/\Delta)\, log\,(1/\varepsilon)\right)$ other primitive gates.
Therefore, neglecting the cost of using other primitive gates, the overall complexity of filtering the desired state, given the block encoding U_{H_1} is $O\left((T\alpha/c_n\Delta)\, log\,(1/\varepsilon)\right)$. It should be mentioned that T itself has a linear dependence on κ, and Δ can be lower bounded by $\frac{1}{\kappa}$. So, clearly we can see that the overall complexity has a dependence on κ^2 and not κ. To obtain a linear dependence, we need to use complex subroutines such as VTAA.

Now, as H_1 is a Hermitian operator, we can obtain the block encoding U_{H_1} using only two extra ancilla qubits and obtain the block encoding $U_{\widetilde{H_1}}$ from it using only one extra ancilla qubit. So, we need only three extra ancilla qubit if we don't use amplitude amplification and only four if we use amplitude amplification for this task.

In this manner, we can obtain the desired state $|\beta\rangle$

3 Gate-base Implementation of AQC :

To carry out our AQC using a gate-based Quantum Computer, we use the time dependent Hamiltonian simulation method based on truncated Dyson series. The input model of this scheme is based on a block-encoding named HAM-T.
We need to use the block-encodings of H_0 and H_1 that are U_{H_0} and U_{H_1} respectively. Our construction of HAM-T satisfies

$$\left(\langle s|\left\langle 0^{l+1+n_0}\right|\otimes I\otimes\left\langle 0^{n_1+1}\right|\right) HAM-T\left(|s\rangle\left|0^{l+1+n_0}\right\rangle\otimes I\otimes\left|0^{n_1+1}\right\rangle\right)=H\left(f\left(s\right)\right)/d \quad (27)$$

for any $s\in S:=\{j/2^l : j=0,1,....,2^l-1\}$
We also need to unitaries namely V_1 and V_2. They are given by

$$V_1=\Sigma_{s\in S}\,|s\rangle\langle s|\otimes\frac{1}{\sqrt{1-s+ds}}\begin{pmatrix}\sqrt{1-s} & -\sqrt{ds}\\ \sqrt{ds} & \sqrt{1-s}\end{pmatrix} \quad (28)$$

$$V_2=\Sigma_{s\in S}\,|s\rangle\langle s|\otimes\begin{pmatrix}\frac{\alpha(s)}{d} & -\sqrt{1-\left(\frac{\alpha(s)}{d}\right)^2}\\ \sqrt{1-\left(\frac{\alpha(s)}{d}\right)^2} & \frac{\alpha(s)}{d}\end{pmatrix} \quad (29)$$

where, $\alpha\left(s\right)=1-s+ds$.
To compute the scheduling function needed in out time optimal AQC and the unitaries V_1 and V_2, we need an unitary U_f in the unitary HAM-T given by

$$U_f\,|s\rangle\,|z\rangle=|s\rangle\,|z\oplus f\left(s\right)\rangle \quad (30)$$

V_1 is used for preparing the linear combination $\left(1-f\left(s\right)\right)U_{H_0}+f\left(s\right)U_{H_1}$. It should be noted that with out V_2 the $(d,l+n_0+n_1+2,0)$ block-encoding of $\Sigma_s\,|s\rangle\langle s|\otimes H\left(s\right)$ will become $\left(\alpha\left(s\right),l+n_0+n_1+2,0\right)$ and so it means that V_2 makes the normalizing factor time-independent, as is required for the input model.
If P is positive-definite then $n_0=1$, but if it is Hermitian-indefinite then we need to take

$n_0 = 2$, and $n_1 = n + 4$ for both the cases.

The Cost and Complexity of the Implementation :

The number of queries to HAM-T is $O\left(d\kappa \frac{\log(d\kappa)}{\log\log(d\kappa)}\right)$. The number of qubits needed is $O\left(n + \log\left(d\kappa\right)\right)$ and the number of other primitive gates need is $O\left(d\kappa\left(n + \log\left(d\kappa\right)\right)\frac{\log(d\kappa)}{\log\log(d\kappa)}\right)$.

Table 1: Computational cost of $AQC\left(p\right)$ and $AQC\left(exp\right)$ via a time-dependent Hamiltonian simulation using truncated Dyson expansion.

Quantity	$AQC\left(P\right)$	$AQC\left(exp\right)$
Queries	$\widetilde{O}\left(d\kappa/\epsilon \log\left(d\kappa/\epsilon\right)\right)$	$\widetilde{O}\left(d\kappa poly\left(\log\left(d\kappa/\epsilon\right)\right)\right)$
Qubits	$\widetilde{O}\left(n + \log\left(d\kappa/\epsilon\right)\right)$	$\widetilde{O}\left(n + \log\left(d\kappa/\epsilon\right)\right)$
Primitive gates	$\widetilde{O}\left(nd\kappa/\epsilon \log\left(d\kappa/\epsilon\right)\right)$	$\widetilde{O}\left(nd\kappa poly\left(\log\left(d\kappa/\epsilon\right)\right)\right)$

4 Solving Ordinary Least Square Problem

We can easily reformulate the linear system solvers that we discussed earlier, to solve the ordinary least square problem. The problem state can be formulated like following :
Suppose we have a $M \times N$ data matrix X that holds all the independent values, and the corresponding dependent values are held in the vector $|y\rangle$. We need to find a vector $|\beta\rangle$ that holds all the parameters such that,

$$X\left|\beta\right\rangle = \left|y\right\rangle \tag{31}$$

Clearly, $|\beta\rangle$ is $N \times 1$ dimensional, and $|y\rangle$ is $M \times 1$ dimensional. For this problem, let's assume $M \geq N$. X is non-Hermitian and not positive-definite in general. So, the problem boils down to find a state $|\beta\rangle$ such that $|\beta\rangle = \frac{X^{-1}|y\rangle}{\|X^{-1}|y\rangle\|}$, or in general case when X is non-invertible, $|\beta\rangle = \frac{X^{+}|y\rangle}{\|X^{+}|y\rangle\|}$.

Theorem 1 (Quantum OLS solver using AQC): Given data matrix X with condition number κ, and the vector $|y\rangle$, we can implement the quantum ordinary least square solver, i.e. find $|\beta\rangle$ with error bound ϵ in complexity $O\left(\kappa \log^2\left(\kappa\right) \log^4\left(\frac{\log\kappa}{\epsilon}\right) polylog\left(MN\right)\right)$ that can be written as $O\left(\kappa \log^2\left(\kappa\right) polylog\left(\frac{MN\log\kappa}{\epsilon}\right)\right)$, using AQC(exp), and in complexity $O\left(\frac{\kappa\log\kappa}{\epsilon} polylog\left(MN\right)\right)$ using AQC(p).

Now, this can be proved by the methods already discussed in the previous sections. We need to just replace A by X in *equation 1*, and redefine $|\chi\rangle$ as $|\chi\rangle = |1, \beta\rangle$ and $|B\rangle$ as $|B\rangle = |0, y\rangle$. Rest of the operations can be done according to the steps mentioned in the $AQC(p)$ and $AQC(exp)$ section above. The corresponding complexities given in the theorem can also be proved by going through the previous steps.

5 Solving Weighted Least Square Problem

It may be that some samples are of higher quality than others. So, we can assign higher weights to the better samples and lower weights to the worse samples. Suppose we have the weights as the entries of the diagonal weight matrix W, given as block-encoding M. We can obtain a (α, m, μ) block-encoding of $W^{-1/2}$ denoted as U, from the block-encoding M in complexity $(\alpha\kappa_W T_M polylog(1/\mu))$, where κ_W is the condition number of W, and T_M is the cost of implementing M.
Let, (β, n, δ) block-encoding of X be given as V. So, $(I_a \otimes U)(I_b \otimes V)$ is a $(\alpha\beta, m+n, \alpha\delta+\beta\mu)$ block-encoding of $W^{-1/2}X$. I_a and I_b should be seen as acting on the ancilla qubits of V and U respectively.
Clearly, $\|X^+\| \leq \kappa_X$ and $\|W^{-1/2}\| \leq \sqrt{\kappa_W}$, where κ_X, κ_W are the condition number of X, W respectively. Therefore the dependence on the condition number of $\left(W^{-1/2}X\right)^+$ can be given by $O\left(\kappa_X\sqrt{\kappa_W}\right)$. Therefore the overall dependence on condition number for obtaining $\left(W^{-1/2}X\right)^+$, given M and V is given by $O\left(\kappa_X\kappa_W^{3/2}\right)$.

Theorem 2 (Quantum WLS solver using AQC): Let $A = W^{-1/2}X$ such that $\|A^+\| \leq \kappa$, where $\kappa = \kappa_X\kappa_W^{3/2}$, $X \in \mathbb{R}^{M\times N}$ and $W \in \mathbb{R}^{M\times M}$. Given a state $W^{-1/2}|y\rangle$, we can implement the quantum weighted least square solver, i.e. find β with error bound ϵ in complexity $O\left(\kappa\log^2(\kappa)\, polylog\left(\frac{MN\log\kappa}{\epsilon}\right)\right)$, using AQC(exp), and in complexity $O\left(\frac{\kappa\log\kappa}{\epsilon} polylog(MN)\right)$ using AQC(p).

We can now apply similar steps as we did for ordinary least square for solving this. We need to use the same A of *theorem 2* in *equation 1*, and redefine $|\chi\rangle$ as $|\chi\rangle = |1, \beta\rangle$ and $|B\rangle$ as $|B\rangle = \left|0, W^{-1/2}|y\rangle\right\rangle$. Following these steps , we will obtain our answer for weighted least square problem. The proof for the corresponding complexities can also be obtained from those steps.

Appendix A

In this section we show the relation between the error in distance between density between density matrix distance and the error in fidelity that are used in the proof of AQC(p) and AQC(exp). As per the notation of the main text let, $|\widetilde{x}(s)\rangle$ be the desired eigenpath of $H(f(s))$ corresponding to the 0-eigenvalue and $Null(H(f(s))) = \{|\widetilde{x}(s)\rangle, |\bar{B}\rangle\}$, where $|\bar{B}\rangle = |1, B\rangle$. $P_0(s)$ is a projector onto $Null(H(f(s)))$.

Assume that

$$|1 - \langle\psi_T(s)| P_0 |\psi_T(s)\rangle| \leq \eta^2(s) \tag{32}$$

Then the fidelity can be bounded from below by $1 - \eta^2(s)$, and the $2 - norm$ error in the distance between desired and obtained density matrix can be bounded from above by $\eta(s)$.

Now, note that $|\bar{B}\rangle$ lies in the null-space of both H_0 and H_1. So, $H(f(s))|\bar{B}\rangle = 0$ and thus $\frac{d}{ds}\langle\bar{B}|\psi_T(s)\rangle = 0$ along with the initial condition $\langle\bar{B}|\psi_T(0)\rangle = 0$ proves that $\langle\bar{B}|\psi_T(s)\rangle = 0$ for the whole time period. Since $P_0(s) = |\widetilde{x}(s)\rangle\langle\widetilde{x}(s)| + |\bar{B}\rangle\langle\bar{B}|$, so $P_0|\psi_T(s)\rangle = |\widetilde{x}(s)\rangle\langle\widetilde{x}(s)|\psi_T(s)\rangle$. Therefore we have

$$|1 - \langle\psi_T(s)| P_0 |\psi_T(s)\rangle| = |1 - \langle\psi_T(s)|\widetilde{x}(s)\rangle\langle\widetilde{x}(s)|\psi_T(s)\rangle| = 1 - |\langle\psi_T(s)|\widetilde{x}(s)\rangle|^2 \tag{33}$$

Now, let $M = |\psi_T(s)\rangle\langle\psi_T(s)| - |\widetilde{x}(s)\rangle\langle\widetilde{x}(s)|$. Note that $\|M\|_2^2 = \lambda_{max}\left(M^\dagger M\right)$. Now

$$M^\dagger M = |\psi_T(s)\rangle\langle\psi_T(s)| + |\widetilde{x}(s)\rangle\langle\widetilde{x}(s)| - \langle\psi_T(s)|\widetilde{x}(s)\rangle|\psi_T(s)\rangle\langle\widetilde{x}(s)| - \langle\widetilde{x}(s)|\psi_T(s)\rangle|\widetilde{x}(s)\rangle\langle\psi_T(s)| \tag{34}$$

Clearly, for any state $|y\rangle \in span\{|\psi_T(s)\rangle, |\widetilde{x}(s)\rangle\}^\perp$, $M^\dagger M|y\rangle = 0$. Therefore

$$M^\dagger M|\psi_T(s)\rangle = \left(1 - |\langle\psi_T(s)|\widetilde{x}(s)\rangle|^2\right)|\psi_T(s)\rangle \tag{35}$$

$$M^\dagger M|\widetilde{x}(s)\rangle = \left(1 - |\langle\psi_T(s)|\widetilde{x}(s)\rangle|^2\right)|\widetilde{x}(s)\rangle \tag{36}$$

And so, $\|M\|_2^2 = \lambda_{max}\left(M^\dagger M\right) = 1 - |\langle\psi_T(s)|\widetilde{x}(s)\rangle|^2$.

Appendix B

The modified data matrix P may or may not be an unitary. So, in this section we describe the way of constructing the block-encoding of the Hamiltonians H_0 and H_1 so that they can be easily applied in the gate-based implementation of our AQC. But first let's start from scratch. We will start by assuming that we are given the original data matrix A in form of a block encoding and build everything up from there.
Assume we have access to our data matrix A in form of (α, a, ϵ) block-encoding U, i.e.

$$\|A - \alpha\left(\langle 0|^{\otimes a} \otimes I\right) U \left(|0\rangle^{\otimes a} \otimes I\right)\| \leq \epsilon \tag{37}$$

Let's assume, A is a n-qubit operator and we have access to the controlled version of the unitary U. Let cU denote the $(a + n + 1)$ qubit $controlled - U$ operator that acts on the last

n qubits and is controlled by $(a+1)\,st$ qubits. Then $cU^{\dagger}\left(I_a \otimes X \otimes I_n\right) cU$ is a $(\alpha, a+1, \epsilon)$ block-encoding of P (it is defined in equation 1).
Since H_0 itself is unitary, therefore we can naturally obtain a $(1,1,0)$ block-encoding of H_0. Let's denote the block-encoding as U_{H_0}.
Now, we need to construct a block-encoding for H_1 where it is given by, $H_1 = \sigma_+ \otimes \left[\left(\sigma_x \otimes P\right) Q_{+,B}\right] + \sigma_- \otimes \left[Q_{+,B}\left(\sigma_x \otimes P\right)\right]$

Observe that

$$H_1 = \begin{pmatrix} I & 0 \\ 0 & Q_{+,B} \end{pmatrix} \begin{pmatrix} 0 & \sigma_x \otimes P \\ \sigma_x \otimes P & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q_{+,B} \end{pmatrix} \tag{38}$$

From block-encoding of $Q_{+,B}$, we can construct the block-encoding of $controlled - Q_{+,B}$ by replacing all gates with their analogous controlled version. Now, if we have a $(d,b,0)$ block-encoding of P, we can have a $(d,b,0)$ block-encoding of $\sigma_x \otimes P$. As we have a $(d,b,0)$ block-encoding of $\sigma_x \otimes P$, we can have a $(d,b,0)$ block-encoding of $\sigma_x \otimes (\sigma_x \otimes P)$, i.e. the operator in the middle of equation 37. For our problem, $b = a+1$. Then we can use the result of product of block encoding to obtain a $(d, b+2, 0)$ block-encoding of H_1 and let's call it U_{H_1}.

Appendix C

Consider the adiabatic evolution

$$\frac{1}{T}\iota\partial_s \left|\psi_T(s)\right\rangle = H\left(f(s)\right)\left|\psi_T(s)\right\rangle, \quad \left|\psi_T(s)\right\rangle = \left|\widetilde{B}\right\rangle \tag{39}$$

where $0 \le s \le 1$, and T is the runtime of AQC. The quantum adiabatic theorem states that for any $0 \le s \le 1$,

$$\left|1 - \left\langle\psi_T(s)\right| P_0 \left|\psi_T(s)\right\rangle\right| \le \eta^2(s) \tag{40}$$

where

$$\eta(s) = C\left\{\frac{\|H^{(1)}(0)\|_2}{T\Delta^2(0)} + \frac{\|H^{(1)}(s)\|_2}{T\Delta^2(f(s))} + \frac{1}{T}\int_0^s \left(\frac{\|H^{(2)}\left(s'\right)\|_2}{\Delta^2\left(f\left(s'\right)\right)} + \frac{\|H^{(1)}\left(s'\right)\|_2^2}{\Delta^3\left(f\left(s'\right)\right)}\right) ds'\right\} \tag{41}$$

where $H^{(k)}(s) = \frac{d^k}{ds^k} H\left(f(s)\right), k = 1,2$. C is a constant independent of s, Δ, T. In Appendix A, we have shown how equation 39 and equation 15 are related.
Now, equation 17 and 18 can be proved in the following manner. Note that we let $\Delta_*(f) = (1 - f + f/\kappa)/\sqrt{2}$ as this Δ_* serves as the lower bound to the spectral gap. Let's first calculate the derivatives of $H\left(f(s)\right)$. Note, $H\left(f(s)\right) = (1-f)H_0 + fH_1$.

$$H^{(1)}(s) = \frac{d}{ds} H\left(f(s)\right) = \frac{dH(f(s))}{df}\frac{df(s)}{ds} = \left(H_1 - H_0\right) c_p \Delta_*^p f(s) \quad ,$$

and

$$H^{(2)}(s) = \frac{d}{ds} H^{(2)}(s) = \frac{d}{ds}\left(H_1 - H_0\right) c_p \Delta_*^p f(s)$$

$$= (H_1 - H_0)\, c_p p \Delta_*^{p-1} f(s) \frac{d\Delta_* f(s)}{ds} \frac{df(s)}{ds}$$

$$= \frac{1}{\sqrt{2}} (-1 + 1/\kappa)\, c_p^2 p \Delta_*^{2p-1} f(s)$$

Then, the first two terms of $\eta(s)$ in equation 40 can be written as

$$\frac{\|H^{(1)}(0)\|_2}{T\Delta^2(0)} + \frac{\|H^{(1)}(s)\|_2}{T\Delta^2(f(s))} \leq \frac{\|H^{(1)}(0)\|_2}{T\Delta_*^2(0)} + \frac{\|H^{(1)}(s)\|_2}{T\Delta_*^2(f(s))}$$
$$\leq \frac{\|(H_1-H_0)c_p\Delta_*^p(f(0))\|_2}{T\Delta_*^2(0)} + \frac{\|(H_1-H_0)c_p\Delta_*^p(f(0))\|_2}{T\Delta_*^2(f(s))}$$
$$\leq \frac{C}{T} \left(c_p \Delta_*^{p-2}(0)\right) + c_p \Delta_*^{p-2}\left(\left(f(s)\right)\right)$$

where C is a constant independent of Δ, T, s. Now, let's introduce a change of variable in the following manner,

$$u = f\left(s^{/}\right), du = \frac{d}{ds^{/}} f\left(s^{/}\right) ds^{/} = c_p \Delta_*^p \left(f\left(s^{/}\right)\right) ds^{/}$$

So, the last two terms of equation 40 become,

$$\frac{1}{T}\int_0^s \frac{\|H^{(2)}\|_2}{\Delta^2} ds^{/} \leq \frac{1}{T}\int_0^s \frac{\|H^{(2)}\|_2}{\Delta_*^2} ds^{/}$$
$$= \frac{1}{T}\int_0^s \frac{\|\frac{1}{\sqrt{2}}(-1+1/\kappa)(H_1-H_0)c_p^2 p\Delta_*^{(2p-1)}\left(f\left(s^{/}\right)\right)\|_2}{\Delta_*^2\left(f\left(s^{/}\right)\right)} ds^{/}$$
$$= \frac{1}{T}\int_0^{f(s)} \frac{\|\frac{1}{\sqrt{2}}(-1+1/\kappa)(H_1-H_0)c_p^2 p\Delta_*^{(2p-1)}\left(f\left(s^{/}\right)\right)\|_2}{\Delta_*^2(u)} \frac{du}{c_p\Delta_*^p(u)}$$
$$\leq \frac{C}{T}\left((1-1/\kappa)\, c_p \int_0^{f(s)} \Delta_*^{p-3}(u)\, du\right)$$
$$\leq \frac{C}{T}\left((1-1/\kappa)\, c_p \int_0^{1} \Delta_*^{p-3}(u)\, du\right)$$

and in the same manner,

$$\frac{1}{T}\int_0^s \frac{\|H^{(1)}\|_2^2}{\Delta^3} ds^{/} \leq \frac{1}{T}\int_0^s \frac{\|H^{(1)}\|_2^2}{\Delta_*^3} ds^{/}$$
$$= \frac{1}{T}\int_0^s \frac{\|(H_1-H_0)c_p\Delta_*^p\left(f\left(s^{/}\right)\right)\|_2^2}{\Delta_*^3\left(f\left(s^{/}\right)\right)} ds^{/}$$
$$= \frac{1}{T}\int_0^s \frac{\|(H_1-H_0)c_p\Delta_*^p(u)\|_2^2}{\Delta_*^3(u)} \frac{du}{c_p\Delta_*^p(u)}$$
$$\leq \frac{C}{T}\left(c_p \int_0^{f(s)} \Delta_*^{p-3}(u)\, du\right)$$
$$\leq \frac{C}{T}\left(c_p \int_0^{1} \Delta_*^{p-3}(u)\, du\right)$$

Substituting all these terms in the expression of $\eta(s)$, we get an upper bound as

$$\eta(s) \leq \frac{C}{T}\left\{\left(c_p\Delta_*^{p-2}(0)+c_p\Delta_*^{p-2}(1)\right)+\left(\left(1-1/\kappa\right)c_p\int_0^1\Delta_*^{p-3}(u)\,du\right)+\left(c_p\int_0^1\Delta_*^{p-3}(u)\,du\right)\right\}$$
$$=\frac{C}{T}\left\{2^{-(p-2)/2}\left(c_p+c_p\kappa^{2-p}\right)+\left(\left(1-1/\kappa\right)c_p\int_0^1\Delta_*^{p-3}(u)\,du\right)+\left(c_p\int_0^1\Delta_*^{p-3}(u)\,du\right)\right\}.$$

Finally, as for $1<p<2$,

$$c_p=\int_0^1\Delta_*^{-p}(u)\,du=\frac{2^{p/2}}{p-1}\frac{\kappa}{\kappa-1}\left(\kappa^{p-1}-1\right)$$

and

$$\int_0^1\Delta_*^{p-3}(u)=\frac{2^{-(p-3)/2}}{2-p}\frac{\kappa}{\kappa-1}\left(\kappa^{2-p}-1\right)$$

then we will have,

$$\eta(s)\leq\frac{C}{T}\left\{\frac{\kappa}{\kappa-1}\left(\kappa^{p-1}-1\right)+\frac{\kappa}{\kappa-1}\left(\kappa^{2-p}-1\right)+\frac{\kappa}{\kappa-1}\left(\kappa^{p-1}-1\right)\left(\kappa^{2-p}-1\right)+\left(\frac{\kappa}{\kappa-1}\right)^2\left(\kappa^{p-1}-1\right)\left(\kappa^{2-p}-1\right)\right\}$$

The leading term of this bound for η is $O(\kappa/T)$ in this case when $1<p<2$.
Now, let's limit ourselves to the case when $p=1,2$. We can write the expression for $\eta(s)$ as following

$$\eta(s)\leq\frac{C}{T}\left\{\left(c_p\Delta_*^{p-2}(0)+c_p\Delta_*^{p-2}(1)\right)+\left(\left(1-1/\kappa\right)c_p\int_0^1\Delta_*^{p-3}(u)\,du\right)+\left(c_p\int_0^1\Delta_*^{p-3}(u)\,du\right)\right\}$$
$$=\frac{C}{T}\left\{2^{-(p-2)/2}\left(c_p+c_p\kappa^{2-p}\right)+\left(1-1/\kappa\right)c_pc_{3-p}+c_pc_{3-p}\right\}$$

It can be shown by calculation that

$$c_1=\int_0^1\Delta_*^{-1}(u)\,du=\sqrt{2}\frac{\kappa}{1-\kappa}\log(\kappa)$$

and

$$c_2=\int_0^1\Delta_*^{-2}(u)\,du=2\frac{\kappa}{1-\kappa}\left(\kappa-1\right)$$

Therefore, when $p=1,2$ we have

$$\eta(s)\leq\frac{C}{T}\left\{2^{-(2-p)/2}\left(c_p+c_p\kappa^{2-p}\right)+\left(1-1/\kappa\right)c_1c_2+c_1c_2\right\}\leq C\frac{\kappa\log(\kappa)}{T}.$$