Quantum Linear System Solver

Problem :

Implement $|\beta\rangle = \frac{A^{+}|b\rangle}{\||A^{+}|b\rangle\||}$ using the AQC(exp) algorithm and the eigenstate filtering, where $A \in \mathbb{R}^{M\times N}$ and is not Hermitian.

Solution :

A^{+} is called the pseudo-inverse of A and has the form $A^{+} = \left(A^{T}A\right)^{-1}A^{T}$ where A is our data-matrix. It is clear that A is also non-Hermitian. So, to get a Hermitian operator we need to increase the size of the Hilbert space. We define,

$$P = \sigma_{+} \otimes A + \sigma_{-} \otimes A^{\dagger} = \begin{pmatrix} 0 & A \\ A^{\dagger} & 0 \end{pmatrix} \tag{1}$$

where $\sigma_{\pm} = \frac{1}{2}\left(\sigma_x \pm \iota\sigma_y\right)$. As all the elements in the matrices are real $A^{\dagger}$ can also be written as A^{T}, but for sake of generality let's use $A^{\dagger}$

The top-left zero block of P is of dimension $M \times M$ and the bottom-right block is of dimension $N \times N$. So, P becomes a $(M+N)\times(M+N)$ Hermitian operator. Now, we need to consider the extended QLSP of the form

$$P|\chi\rangle = |B\rangle \tag{2}$$

in extended dimension $M+N$, where $|B\rangle = |0, b\rangle$. The solution $|\chi\rangle$ corresponds to,
$\chi = |1, x\rangle$, where $|x\rangle$ is a state such that $A|x\rangle = |b\rangle$

1 Applying AQC(exp):

The scheduling function of AQC(exp) looks like

$$f\left(s\right) = c_e^{-1} \int_0^s exp\left(-\frac{1}{s'\left(1-s'\right)}\right) ds' \tag{3}$$

where c_e is a normalization constant such that $f\left(1\right) = 1$ and has the form,

$$c_e = \int_0^1 exp\left(-\frac{1}{s'\left(1-s'\right)}\right) ds' \tag{4}$$

During the adiabatic evolution, our initial Hamiltonian will be H_0 and final Hamiltonian will be H_1. So, the evolution operator will be of the form,

$$H(s) = \left(1 - f(s)\right) H_0 + f(s) H_1 \tag{5}$$

It should be noted that, if P is non-unitary, we can always use block encoding to make it unitary.
Now, as H is not necessarily positive-definite, to overcome the indefiniteness, we need to enlarge the Hilbert space to dimension $4K$ (say, K=M+N), and so we will need two ancilla qubits to enlarge the matrix block. Now, we define

$$H_0 = \sigma_+ \otimes \left[\left(\sigma_z \otimes I_K\right) Q_{+,B}\right] + \sigma_- \otimes \left[Q_{+,B}\left(\sigma_z \otimes I_K\right)\right] \tag{6}$$

where $Q_{+,B} = I_{2K} - \left|+, B\right\rangle \left\langle +, B\right|$. The null space of H_0 is Null(H_0) = span$\{\left|0, -, B\right\rangle, \left|1, +, B\right\rangle\}$. We also define

$$H_1 = \sigma_+ \otimes \left[\left(\sigma_x \otimes P\right) Q_{+,B}\right] + \sigma_- \otimes \left[Q_{+,B}\left(\sigma_x \otimes P\right)\right] \tag{7}$$

The null space of H_1 is Null(H_1) = span$\{\left|0, +, \chi\right\rangle, \left|1, +, B\right\rangle\}$. So, we will obtain our solution $\left|\chi\right\rangle$ if we can prepare the zero-eigenstate $\left|0, +, \chi\right\rangle$ of the Hamiltonian H_1.
For achieving this we start with the null-state $\left|0, -, B\right\rangle$ of the Hamiltonian H_0 and evolve adiabatically, according to the scheduling function $f(s)$ and arrive at the final zero-eigenstate $\left|0, +, \chi\right\rangle$.

It should be noted that AQC(exp) achieves an exponential speedup over RM and AQC(p) with respect to ϵ , thus is more suitable for preparing the solution of QLSP with high fidelity. Furthermore, the time scheduling of AQC(exp) is universal and AQC(exp) does not require any prior knowledge on the bound of κ.

Error and Complexity Analysis :

The error in the solution of the extended QLSP of equation 2 is given by

$$\epsilon = c\log(\kappa)exp\left(\left(-c\frac{\kappa\log^2\kappa}{T}\right)^{\frac{-1}{4}}\right) \tag{8}$$

where c is a constant and T is the runtime.
We need three extra ancilla qubits, one for transforming A to a Hermitian operator and other two for the indefiniteness.
It can be easily derived from equation 8 that the time complexity of the algorithm for preparing the solution state to an error margin of ϵ is given by

$$T = O\left(\kappa\log^2(\kappa)\log^4\left(\frac{\log\kappa}{\epsilon}\right)\right) \tag{9}$$

We can obtain much better complexity in the condition number and error and obtain our desired state with enhanced precision using the techniques of *Quantum Signal Processing* discussed below.

2 Applying Quantum Signal Processing :

Let's assume we pre-process the entries of the data matrix A such that $a_{ij} \leq 1$, where a_{ij} is the element of i^{th} row and j^{th} column of A. Let's also assume that A is α-sparse.
Then, we can have a $(\alpha, m, 0)$ block encoding U_{H_1} of the Hamiltonian H_1. Let P_{λ_n} be the projector into the λ_n-eigenspace, in our case $\lambda_n = 0$ and so our projection space is zero-eigenspace of H_1, i.e. the null-space.
Now, we have to concoct such a projector. For this, suppose we have a polynomial P such that $P(0) = 1$ and $|P(x)| = 0$ for $x \in D_{\Delta/2\alpha}$, where $D_{1/\kappa} = [-1, -1/\kappa] \cup [1/\kappa, 1]$. Then

$$P\left(\left(H_1 - \lambda_n I\right)/2\alpha\right) \approx P_{\lambda_n}. \tag{10}$$

Now, for this problem we will use the following 2ℓ degree polynomial.

$$\mathbf{R}_{\ell}\left(x;\Delta\right)=\frac{\mathbf{T}_{\ell}\left(-1+2\frac{x^2-\Delta^2}{1-\Delta^2}\right)}{\mathbf{T}_{\ell}\left(-1+2\frac{-\Delta^2}{1-\Delta^2}\right)} \tag{11}$$

Now, R is an even polynomial. Now we can apply the polynomial to $H_1-\lambda_n I$ using the techniques of Quantum Signal Processing and get rid of the unwanted components. Let's define

$$\widetilde{H_1}=\frac{H_1-\lambda_n I}{\alpha+|\lambda_n|} \tag{12}$$

For our problem, $\lambda_n=0$, so the operator becomes

$$\widetilde{H_1}=\frac{H_1}{\alpha} \tag{13}$$

and we also define

$$\widetilde{\Delta}=\frac{\Delta}{2\alpha} \tag{14}$$

where, $\Delta(f)\geq\Delta_*(f)$ and $\Delta_*(f)=1-s+\frac{f}{\kappa}$, f is the value of the scheduling function and at the end of applying AQC(exp) $f=1$ and so the value of Δ is lower-bounded by $\frac{1}{\kappa}$.
Now, as we know

$$|R_{\ell}\left(x;\Delta\right)|\leq 2e^{-\sqrt{2}\ell\Delta} \tag{15}$$

for $x\in D_{\Delta}$.
Therefore, we have

$$\|R\left(\widetilde{H_1};\widetilde{\Delta}\right)-P_{\lambda_n}\|\leq 2e^{-\sqrt{2}\ell\widetilde{\Delta}} \tag{16}$$

Now we can create $(\alpha, m+1, \varepsilon)$ block encoding of $R\left(\widetilde{H_1};\widetilde{\Delta}\right)$ and let it be $U_{\widetilde{H_1}}$.

The state obtained after application of AQC(exp) is

$$|\psi_0\rangle=(1-\epsilon)\,|0,+,\chi\rangle+\epsilon\,|\bot\rangle \tag{17}$$

We, are interested in obtaining the state $|0,+,\chi\rangle$ that is in the null-space of H_1, with high fidelity and so we apply the operator $U_{\widetilde{H_1}}$ to the state $|\psi_0\rangle$.

To obtain the state $|0, +, \chi\rangle$ with constant probability, we need to repeat the step $O\left(\frac{1}{1-\epsilon}\right)$ times with out using amplitude amplification and $O\left(\frac{1}{\sqrt{1-\epsilon}}\right)$ times using amplitude amplification.

Thus, after getting rid of the two ancilla qubits corresponding to $|0\rangle$ and $|+\rangle$, we get our desired state $|\chi\rangle$. From this state $|\chi\rangle$, we can easily obtain out state $|x\rangle$ by performing a Z-basis measurement on the ancilla and keeping the state corresponding to $|1\rangle$.

Error and Complexity Analysis :

Suppose, the cost of implementing the unitary $U_{\widetilde{H_1}}$ once is T. Assuming that we are using amplitude amplification, we need to run the block encoding $O\left(\frac{1}{\sqrt{1-\epsilon}}\right)$ times. So, the cost is $O\left(\frac{T}{\sqrt{1-\epsilon}}\right)$.

Now, if U_{H_1} is the $(\alpha, m, 0)$ block encoding of H_1, then we can obtain a $(1, m+1, \varepsilon)$ block encoding of $U_{\widetilde{H_1}}$ using $O\left((\alpha/\Delta)\, log\,(1/\varepsilon)\right)$ applications of controlled- U_{H_1} and it's conjugate and $O\left((m\alpha/\Delta)\, log\,(1/\varepsilon)\right)$ other primitive gates.
Therefore, neglecting the cost of using other primitive gates, the overall complexity of filtering the desired state, given the block encoding U_{H_1} is $O\left((T\alpha/c_n\Delta)\, log\,(1/\varepsilon)\right)$.

Now, as H_1 is a Hermitian operator, we can obtain the block encoding U_{H_1} using only two extra ancilla qubits and obtain the block encoding $U_{\widetilde{H_1}}$ from it using only one extra ancilla qubit. So, we need only three extra ancilla qubit if we don't use amplitude amplification and only four if we use amplitude amplification for this task.

In this manner, we can obtain the desired state $|\beta\rangle$