OBB: Open Ball & Beam Mathematics

Contents

1 Model

The dynamics of the ball-and-beam are modeled as follows, according to a simplified development of [1–3]. Define the model constants

Quantity	Expression
Gravitational acceleration	g
Damping coefficient	d

Define the model coordinates

Quantity	Expression
Position of the ball along the beam	p
Beam angle	θ
Weight force	$W = mg$
Damping force	$D = d\dot{p}$

Define the system quantities

Quantity	Expression
State	$x = [p\ \dot{p}]^\intercal$
Input	$u = \sin(\theta)$
Output	$y = p$

A basic model[1] can be derived using Newtonian or Lagrangian mechanics:

$$\left(m + \frac{J}{R^2}\right)\ddot{p} = -W\sin\theta - D \tag{1}$$

$$= -mg\sin\theta - d\dot{p} \tag{2}$$

Define the scaling factor

$$c = \frac{m + \frac{J}{R^2}}{m} = 1 + \frac{J}{mR^2}, \tag{3}$$

which has the effect of reducing the effect of gravity on the acceleration of the ball depending on the rotational inertia of the ball. Assuming a solid spherical ball,

$$c = 1 + \frac{\frac{2}{5}mR^2}{mR^2} = 1 + \frac{2}{5} = 1.4 \tag{4}$$

Using the scaling constant,

$$mc\ddot{p} = -mg\sin\theta - d\dot{p} \tag{5}$$

Rearranging,

$$\ddot{p} = -\frac{g}{c}\sin\theta - \frac{d}{mc}\dot{p} \tag{6}$$

Using the system quantities, we obtain the linear continuous-time model

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{d}{mc} \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{g}{c} \end{bmatrix} u, \tag{7}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x. \tag{8}$$

[1] These dynamics ignore certain higher-order terms, see [3].

1.1 Discretization

TODO discretization to obtain a discrete-time from the continuous-time one

1.2 Disturbance modeling

TODO introducing additive process and measurement noise we obtain the stochastic state-input-output model

1.3 Integral control

TODO state augmentation for input difference penalization

1.4 Control input difference penalization

TODO state augmentation for integral control

1.5 Linearization of servo actuation and sensor readings

Due to the mechanical linkage and possibly other sources, there is a nonlinear relation between the PWM signal given to the servo motor and the angle of the beam.

Due to variations in the reflected light from the ball as it travels away from the ToF sensor, there is a nonlinear relation between the sensor reading and the actual physical distance of the ball from the sensor.

These are both dealt with by fitting polynomials to collected data using least-squares. This is part of the calibration procedure.

The nonlinear transform to convert from an action u to a raw PWM signal for the servo is given by

$$\mathrm{PWM} = \mathrm{poly}(u) \tag{9}$$

The nonlinear transform to convert from a raw sensor reading to an observation y is given by

$$y = \mathrm{poly}(\mathrm{reading}) \tag{10}$$

In particular, degree-5 polynomials are used.

2 Control

2.1 Proportional-integral-derivative (PID)

TODO

2.2 Linear-quadratic regulator (LQR)

TODO

2.3 Model-predictive control (MPC)

TODO

3 State estimation

3.1 Exponential filtering

TODO

3.2 Optimal linear-quadratic estimation (LQE) / Kalman filtering (KF)

TODO

3.3 Extended Kalman filter (EKF)

TODO

3.4 Unscented Kalman filter (UKF)

TODO

3.5 Particle filter (PF)

TODO

References

[1] D. Morin, *The Lagrangian method*, p. 218–280. Cambridge University Press, 2008.

[2] C. G. Bolívar-Vincenty and G. Beauchamp-Báez, "Modelling the ball-and-beam system from Newtonian mechanics and from Lagrange methods," in *Twelfth LACCEI Latin American and Caribbean Conference for Engineering and Technology*, vol. 22, p. 24, 2014.

[3] J. Hauser, S. Sastry, and P. Kokotovic, "Nonlinear control via approximate input-output linearization: the ball and beam example," *IEEE Transactions on Automatic Control*, vol. 37, no. 3, pp. 392–398, 1992.