Review: Supervised Learning

CS 6956: Deep Learning for NLP

THE UNIVERSITY OF UTAH

Previous lecture

- A broad overview of the class
  - Overview of why deep learning and NLP form an interesting intersection
  - Basically the syllabus of the class
- *Questions?*

Supervised learning, Binary classification

1. Linear classifiers
2. The Perceptron algorithm
3. Support vector machines and logistic regression
4. Learning as optimization
5. Loss functions for various tasks

Supervised learning: General setting

- Given: Training examples of the form <**x**, *f*(**x**)>
  - The function f is an unknown function

- The input **x** is represented in a *feature space*
  - Typically $\mathbf{x} \in \{0,1\}^n$ or $\mathbf{x} \in \Re^n$
- For a training example **x**, *f*(**x**) is called the *label*

- Goal: Find a good approximation for *f*
- Different kinds of problems
  - Binary classification: $f(\mathbf{x}) \in \{-1,1\}$
  - Multiclass classification: $f(\mathbf{x}) \in \{1, 2, 3, \cdots, K\}$
  - Regression: $f(\mathbf{x}) \in \Re$

Linear Classifiers

- Input is a n dimensional vector **x**
- Output is a label y ∈ {-1, 1}

For now

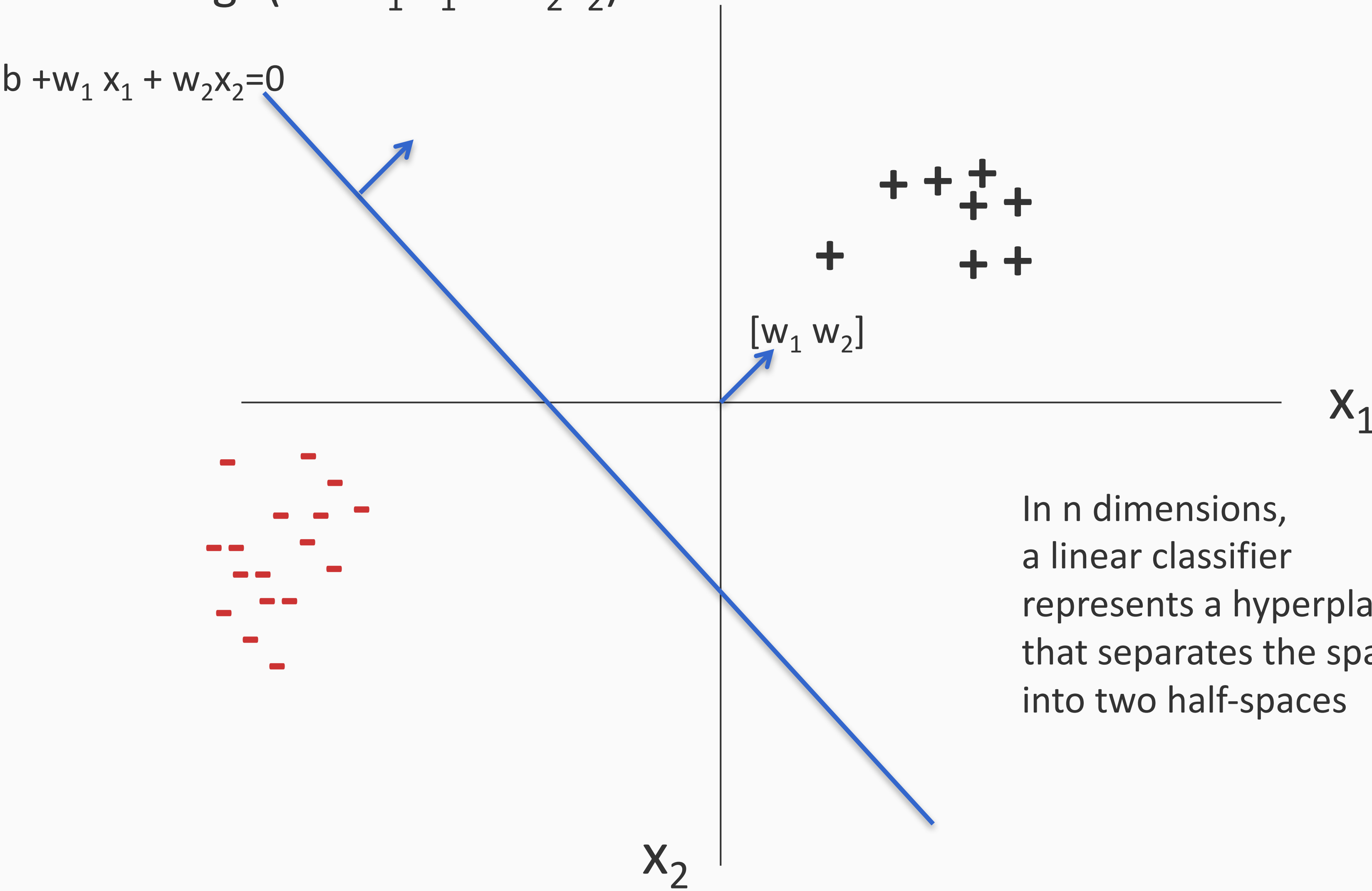
- Linear threshold units classify an example $\boldsymbol{x}$ using the classification rule

$$\mathrm{sgn}(b + \boldsymbol{w}^T\boldsymbol{x}) = \mathrm{sgn}(b + \sum_i w_i x_i)$$

  - $b + \boldsymbol{w}^T\boldsymbol{x} \geq 0 \Rightarrow$ Predict y = 1
  - $b + \boldsymbol{w}^T\boldsymbol{x} < 0 \Rightarrow$ Predict y = -1

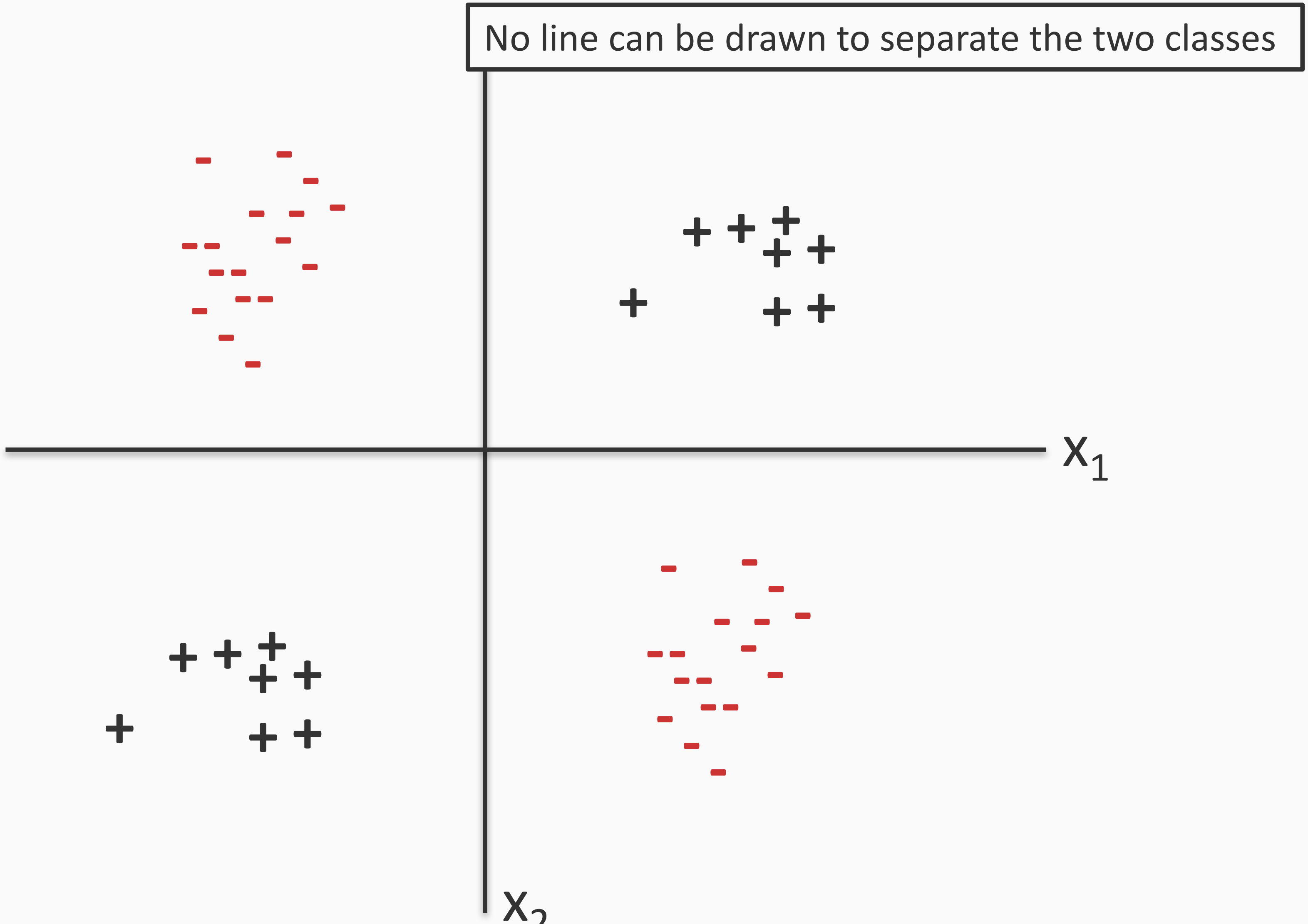
The geometry of a linear classifier

$\mathrm{sgn}(b + w_1 x_1 + w_2 x_2)$



In n dimensions,
a linear classifier
represents a hyperplane
that separates the space
into two half-spaces

XOR is not linearly separable



Even these functions can be *made* linear

These points are not separable in 1-dimension by a line

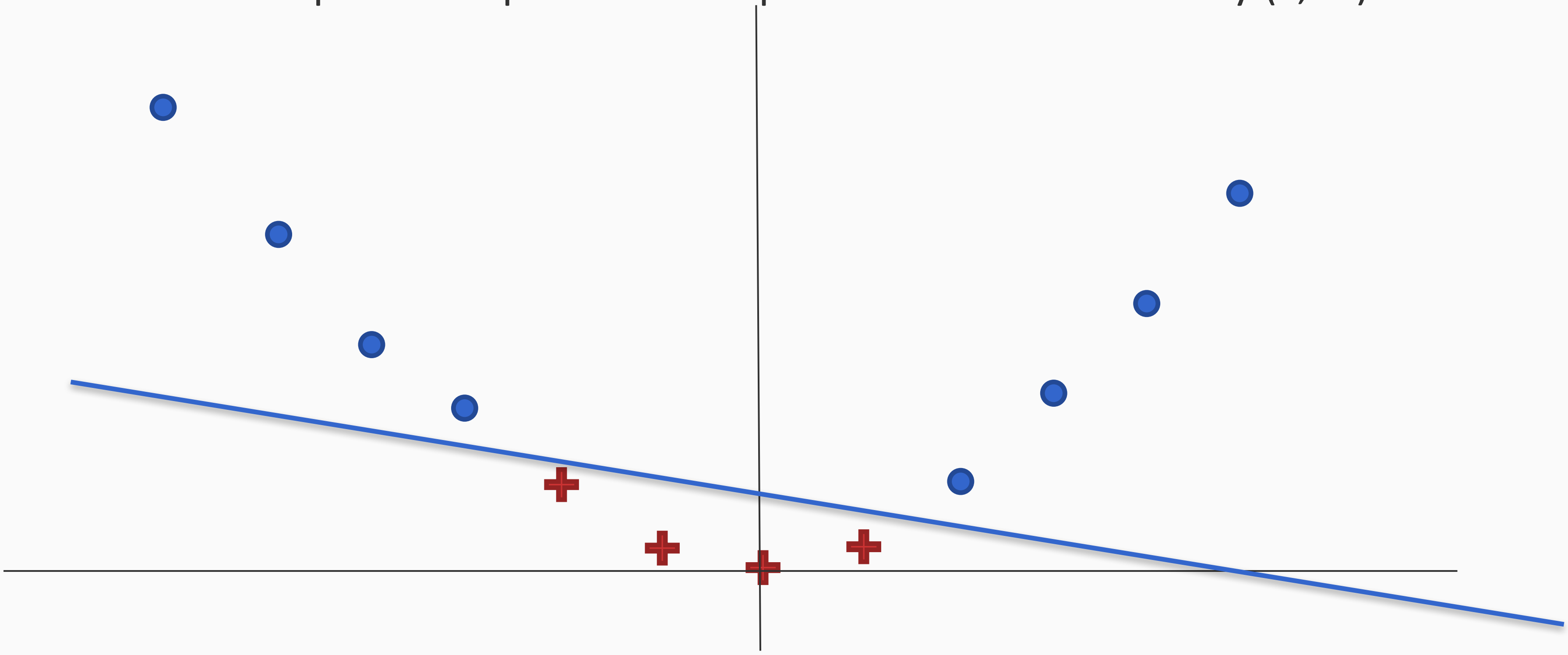
What is a one-dimensional line, by the way?

The trick: Change the representation

Even these functions can be *made* linear

The trick: Use feature *conjunctions*

Transform points: Represent each point x in 2 dimensions by (x, x^2)



Now the data is linearly separable in this space!

Linear classifiers are an expressive hypothesis class

- Many functions are linear
  - Conjunctions, disjunctions
  - At least m-of-n functions
- Often a good guess for a hypothesis space
  - *If we know a good feature representation*
- Some functions are not linear
  - The XOR function
  - Non-trivial Boolean functions

Where are we?

1. Linear classifiers
2. The Perceptron algorithm
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The Perceptron algorithm

- Rosenblatt 1958
- The goal is to find a separating hyperplane
  - For separable data, guaranteed to find one
- An online algorithm
  - Processes one example at a time
- Several variants exist

The algorithm

Given a training set D = {($\mathbf{x}$,y)}, $x \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w} = 0 \in \Re^n$
2. For epoch = 1 … T:
   1. For each training example ($\mathbf{x}$, y) $\in$ D:
      1. Predict $y' = \text{sgn}(\mathbf{w}^T\mathbf{x})$
      2. If $y \neq y'$, update $\mathbf{w} \leftarrow \mathbf{w} + y\,\mathbf{x}$
3. Return **w**

Prediction: $\text{sgn}(\mathbf{w}^T\mathbf{x})$

The algorithm

Given a training set D = {($\mathbf{x}$,y)}

1. Initialize $\mathbf{w} = 0 \in \Re^n$
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    1. For each training example ($\mathbf{x}$, y) $\in$ D:
        1. Predict y' = sgn($\mathbf{w}^T\mathbf{x}$)
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3. Return $\mathbf{w}$

T is a hyperparameter to the algorithm

In practice, good to shuffle D before the inner loop

Update only on an error. Perceptron is an mistake-driven algorithm.

Prediction: sgn($\mathbf{w}^T\mathbf{x}$)

Convergence theorem

If there exist a set of weights that are consistent with the data (i.e. the data is linearly separable), the perceptron algorithm will converge.

– [Novikoff 1962]

Beyond the separable case

- The good news
  - Perceptron makes no assumption about data distribution
  - Even adversarial
  - After a fixed number of mistakes, you are done. Don’t even need to see any more data

- The bad news: Real world is not linearly separable
  - Can’t expect to *never* make mistakes again
  - What can we do: more features, try to be linearly separable if you can

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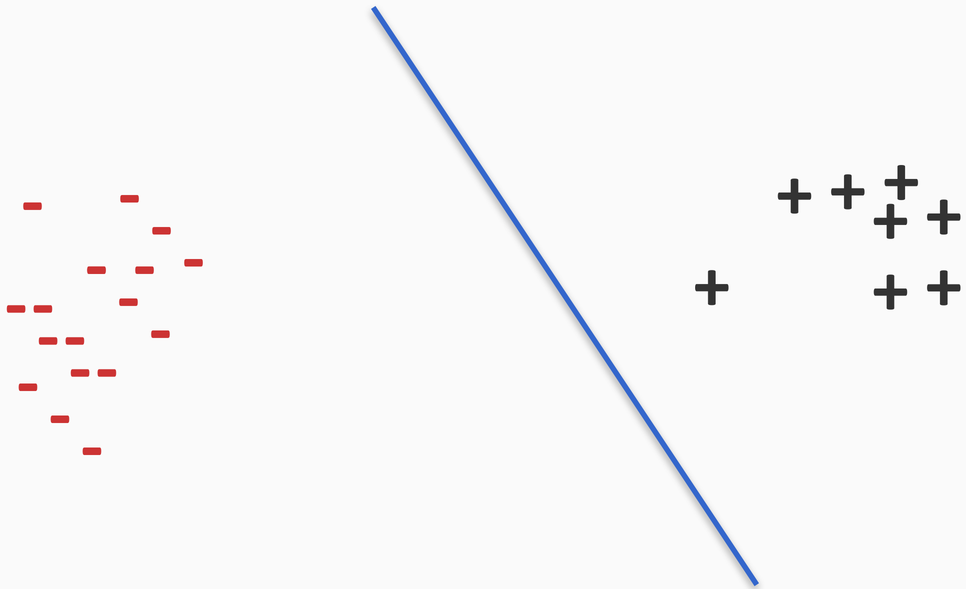
Margin

The margin of a hyperplane for a dataset is the distance between the hyperplane and the data point nearest to it.



Maximizing margin and minimizing loss

Find the linear separator that maximizes the margin

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_i \boxed{\max(0, 1 - y_i\mathbf{w}^T\mathbf{x}_i)}$$

Maximize margin

Penalty for the prediction:
The Hinge loss

SVM objective function

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_i \max(0, 1 - y_i\mathbf{w}^T\mathbf{x}_i)$$

Regularization term:

- Maximize the margin
- Imposes a preference over the hypothesis space and pushes for better generalization
- Can be replaced with other regularization terms which impose other preferences

Empirical Loss:

- Hinge loss
- Penalizes weight vectors that make mistakes
- Can be replaced with other loss functions which impose other preferences

SVM objective function

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_i \max(0, 1 - y_i\mathbf{w}^T\mathbf{x}_i)$$

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Empirical Loss:

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A hyper-parameter that controls the tradeoff between a large margin and a small hinge-loss

Regularized loss minimization: Logistic regression

- Learning: $\min_{f \in H} \ \mathrm{regularizer}(f) + C \sum_i L(y_i, f(\mathbf{x}_i))$
- With linear classifiers: $\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^T\mathbf{w} + C \sum_i L(y_i, \mathbf{x}_i, \mathbf{w})$
- SVM uses the hinge loss
- Another loss function: The logistic loss

$$L_{logistic}(y, \mathbf{x}, \mathbf{w}) = \log(1 + e^{-y\mathbf{w}^T\mathbf{x}})$$

The probabilistic interpretation

Suppose we believe that the labels are distributed as follows given the input:

$$\left.\begin{aligned} P(y=1|\mathbf{x},\mathbf{w}) &= \frac{e^{\mathbf{w}^T\mathbf{x}}}{1+e^{\mathbf{w}^T\mathbf{x}}} = \frac{1}{1+e^{-\mathbf{w}^T\mathbf{x}}} \\ P(y=-1|\mathbf{x},\mathbf{w}) &= \frac{1}{1+e^{\mathbf{w}^T\mathbf{x}}} \end{aligned}\right\} \quad P(y|\mathbf{x},\mathbf{w}) = \frac{1}{1+\exp\left(-y\mathbf{w}^T\mathbf{x}\right)}$$

Predict label = 1 if P(1 | **x**, **w**) > P(-1 | **x**, **w**)

– Equivalent to predicting 1 if $\mathbf{w}^T\mathbf{x} \geq 0$

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The log-likelihood of seeing a dataset D = {($\mathbf{x}_i$, y_i)} if the true weight vector was **w**:

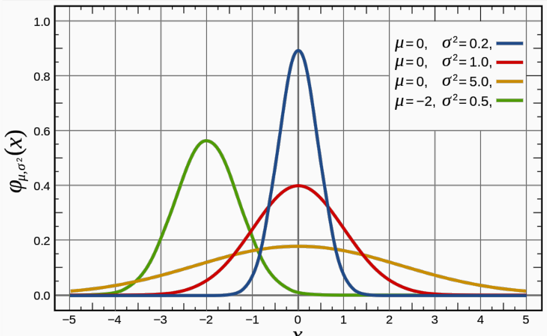
$$\log P(D|\mathbf{w}) = -\sum_i \log\left(1+\exp(-y\mathbf{w}^T\mathbf{x}\right)$$

Prior distribution over the weight vectors

A prior balances the tradeoff between the likelihood of the data and existing belief about the parameters

- Suppose each weight w_i is drawn independently from the normal distribution centered at zero with variance σ^2
  - Bias towards smaller weights

$$P(w_i) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{w_i^2}{2\sigma^2}\right)$$



Source: Wikipedia

- Probability of the entire weight vector:

$$\log P(\mathbf{w}) = -\frac{1}{2\sigma^2}\mathbf{w}^T\mathbf{w} + \text{constant terms}$$

Regularized logistic regression

What is the probability of a weight vector **w** being the true one for a dataset D = {<$\mathbf{x}_i$, y_i>}?

$$P(\boldsymbol{w} \mid D) \propto P(\boldsymbol{w}, D) = P(\ D \mid \boldsymbol{w}\ )P(\ \boldsymbol{w})$$

Regularized logistic regression

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Learning: Find weight vector by maximizing the posterior distribution P(**w** | D)

$$\log P(D, \mathbf{w}) = -\frac{1}{2\sigma^2}\mathbf{w}^T\mathbf{w} - \sum_i \log\left(1 + \exp(-y\mathbf{w}^T\mathbf{x}\right)$$

Once again, regularized loss minimization! This is the Bayesian interpretation of regularization

Regularized loss minimization

Learning objective for both SVM & logistic regression:

"loss over training data + regularizer"

- Different loss functions
  - Hinge loss vs. logistic loss
- Same regularizer, but different interpretation
  - Margin vs prior
- Hyper-parameter controls tradeoff between the loss and regularizer
- Other regularizers/loss functions also possible

Questions?

Where are we?

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Learning as loss minimization

- Collect some annotated data. More is generally better
- Pick a hypothesis class (also called model)
  - Eg: linear classifiers, deep neural networks
- Also, decide on how to impose a preference over hypotheses
  - Regularizer, but with neural networks, dropout is common
- Choose a loss function that penalizes incorrect preditions
  - Eg: negative log-likelihood, hinge loss
- Minimize the expected loss
  - Eg: Set derivative to zero, but more often procedural, usually gradient-based

Learning as loss minimization

- The setup
  - Examples $\mathbf{x}$ drawn from a fixed, unknown distribution D
  - Hidden oracle classifier f labels examples
  - We wish to find a hypothesis h that mimics f

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  - Examples $\mathbf{x}$ drawn from a fixed, unknown distribution D
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- The ideal situation
  - Define a function L that penalizes bad hypotheses
  - **Learning:** Pick a function h $\in$ H to minimize expected loss

$$\min_{h \in H} E_{\mathbf{x} \sim D} \left[L\left(h(\mathbf{x}), f\left(\mathbf{x}\right)\right)\right]$$

But distribution D is unknown

Learning as loss minimization

- The setup
  - Examples $\mathbf{x}$ drawn from a fixed, unknown distribution D
  - Hidden oracle classifier f labels examples
  - We wish to find a hypothesis h that mimics f
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$$\min_{h \in H} E_{\mathbf{x} \sim D} \left[L\left(h(\mathbf{x}), f(\mathbf{x}) \right) \right]$$

But distribution D is unknown

- Instead, minimize *empirical loss* on the training set

$$\min_{h \in H} \frac{1}{m} \sum_i L(h(x_i), y_i)$$

Empirical loss minimization

Learning = minimize *empirical loss* on the training set

$$\min_{h \in H} \frac{1}{m} \sum_i L(h(x_i), y_i)$$

Is there a problem here?

Overfitting!

We need something that biases the learner towards simpler hypotheses

- Achieved using a regularizer, which penalizes complex hypotheses
- Capacity control for better generalization

Regularized loss minimization

- Learning: $\min_{h \in \mathrm{H}} \mathrm{regularizer(w)} + \mathrm{C}\frac{1}{m}\sum_i L(h(x_i), y_i)$

- With L2 regularization: $\min_{w} \frac{1}{2}\, w^T w + C \sum_i L(F(x_i, w), y_i)$

Regularized loss minimization

- Learning: $\min_{h \in \mathrm{H}} \mathrm{regularizer(w)} + \mathrm{C}\frac{1}{m}\sum_i L(h(x_i), y_i)$
- With L2 regularization: $\min_{w} \frac{1}{2} w^T w + C\sum_i L(F(x_i, w), y_i)$
- What is a loss function?
  – Loss functions should penalize mistakes
  – We are minimizing average loss over the training data

How do we train in such a regime?

- Suppose we have a predictor F that maps inputs x to labels as F(x, w)
  - Here w are the parameters that define the function
  - Say F is a differentiable function

- How do we use a labeled training set to learn the weights i.e. solve this minimization problem?

$$\min_{w} \sum_{i} L(F(x_i, w), y_i)$$

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$$\min_{w} \sum_{i} L(F(x_i, w), y_i)$$

- We could compute the gradient of F and descend along that direction to minimize the loss

$$\min_{\boldsymbol{w}} \sum_i L(F(x_i, w), y_i)$$

Stochastic gradient descent

Given a training set S = $\{(\mathbf{x}_i, \mathrm{y}_i)\}$, $\mathbf{x} \in \Re^{\mathrm{d}}$

1. Initialize parameters w
2. For epoch = 1 … T:

3. Return w

$$\min_{\boldsymbol{w}} \sum_i L(F(x_i, w), y_i)$$

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Given a training set S = $\{(\mathbf{x}_i, \mathrm{y}_i)\}$, $\mathbf{x} \in \Re^d$

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    2. For each training example $(\mathbf{x}_i, \mathrm{y}_i) \in$ S:
        - Treat this example as the entire dataset
          Compute the gradient of the loss $\nabla L(F(\boldsymbol{x}_i, \boldsymbol{w}), y_i)$

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        - Update: $\boldsymbol{w} \leftarrow \boldsymbol{w} - \gamma_t \nabla L(F(\boldsymbol{x}_i, \boldsymbol{w}), y_i))$
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γ_t: learning rate, many tweaks possible

$$\min_{w} \sum_{i} L(F(x_i, w), y_i)$$

Stochastic gradient descent

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If the objective is not convex, initialization can be important

γ_t: learning rate, many tweaks possible

In practice…

- There are many variants of this idea

- We will encounter named learning algorithms as we go on
  - AdaGrad, Adam

- But the key components are the same. We need to…
  1. …sample a tiny subset of the data at each step
  2. …compute the gradient of the loss using this subset
  3. …take a step in the negative direction of the gradient

Where are we?

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Standard loss functions

We need to think about the problem we have at hand

Is it a…

1. Binary classification problem?
2. Regression problem?
3. Multi-class classification problem?
4. Or something else?

Each case is naturally paired with a different loss function

The ideal case for binary classification: The 0-1 loss

Penalize classification mistakes between true label y and prediction y'

$$L_{0-1}(y, y') = \begin{cases} 1 & \text{if } y \neq y', \\ 0 & \text{if } y = y'. \end{cases}$$

More generally, suppose we have a prediction function of the form $sgn(F(x, w))$

- Note that F need not be linear

$$L_{0-1}(y, y') = \begin{cases} 1 & \text{if } yF(x, w) \leq 0, \\ 0 & \text{if } yF(x, w) > 0. \end{cases}$$

Minimizing 0-1 loss is intractable. Need surrogates

$$\min_{h \in H} \text{regularizer(w)} + \mathrm{C}\frac{1}{m}\sum_i L(F(x_i, w), y_i)$$

The loss function zoo

For binary classification

Many loss functions exist

Perceptron	$L_{Perceptron}(y, x, w) = \max(0, -yF(x, w))$
Hinge (SVM)	$L_{Hinge}(y, x, w) = \max(0, 1 - yF(x, w))$
Exponential (Adaboost)	$L_{Exponential}(y, x, w) = e^{-yF(x,w)}$
Logistic loss	$L_{Logistic}(y, x, w) = \log(1 + e^{-yF(x,w))})$

$$\min_{h \in \mathrm{H}} \text{regularizer}(\mathrm{w}) + \mathrm{C}\frac{1}{m}\sum_i L(F(x_i, w), y_i)$$

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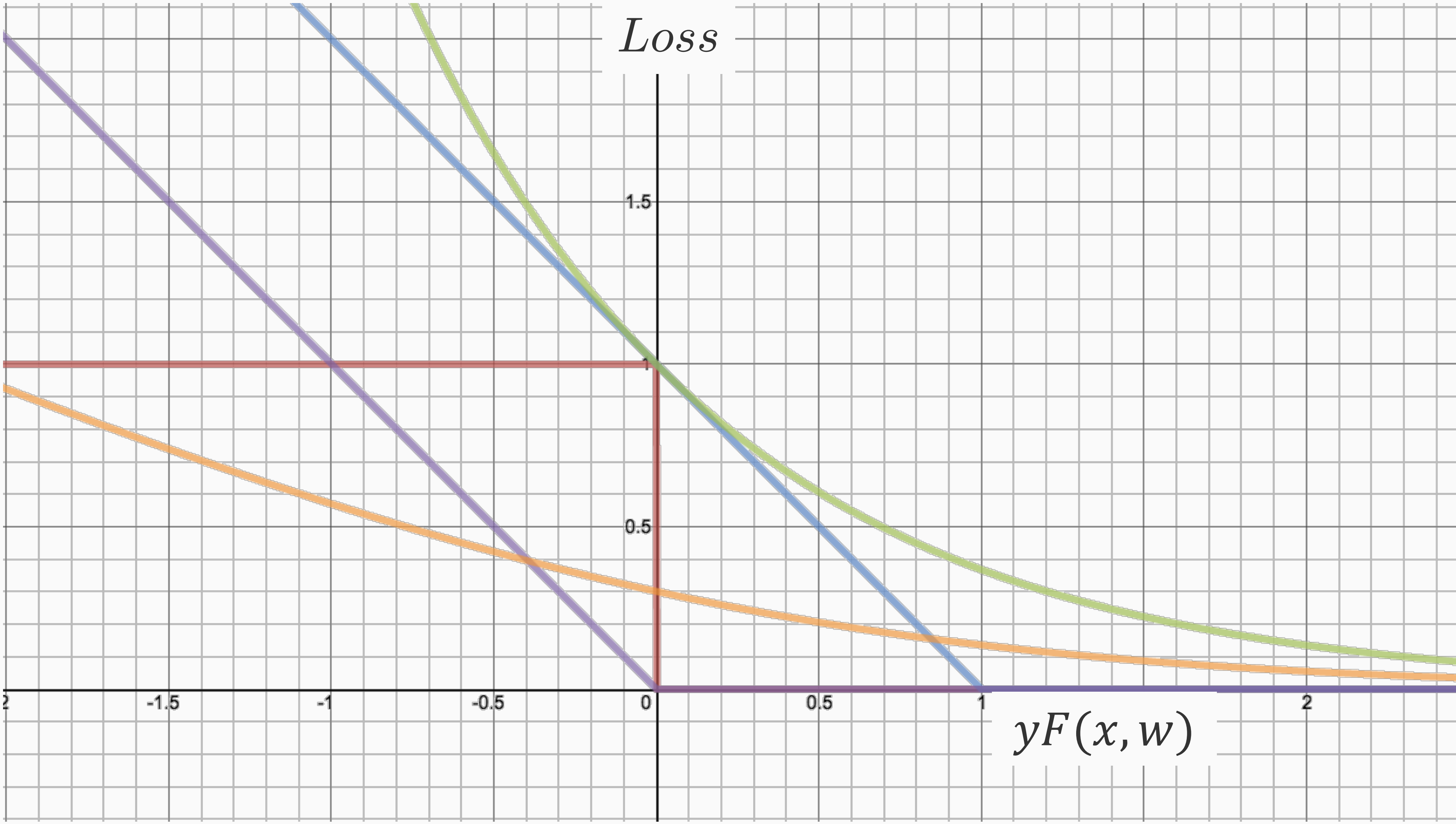

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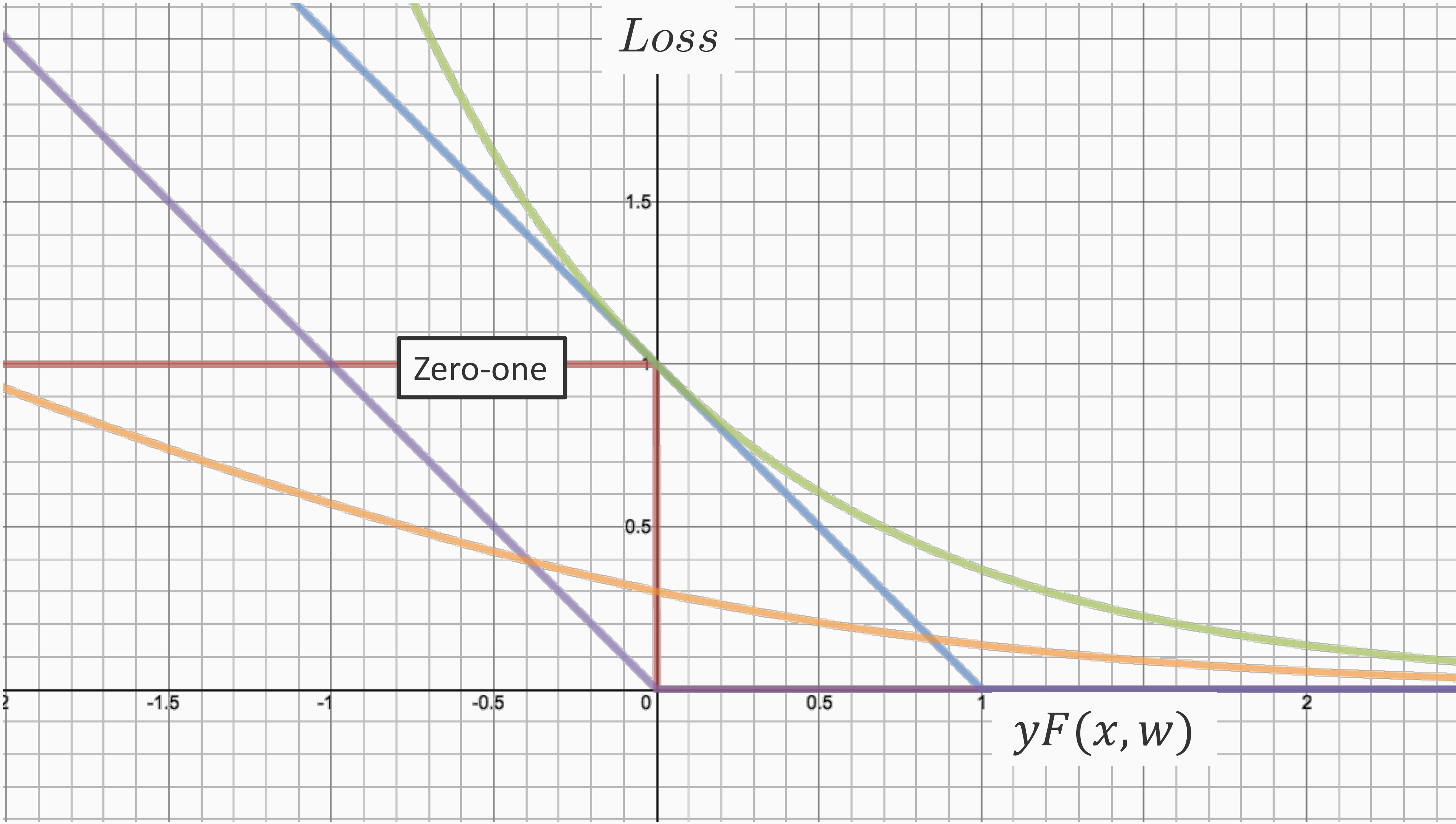
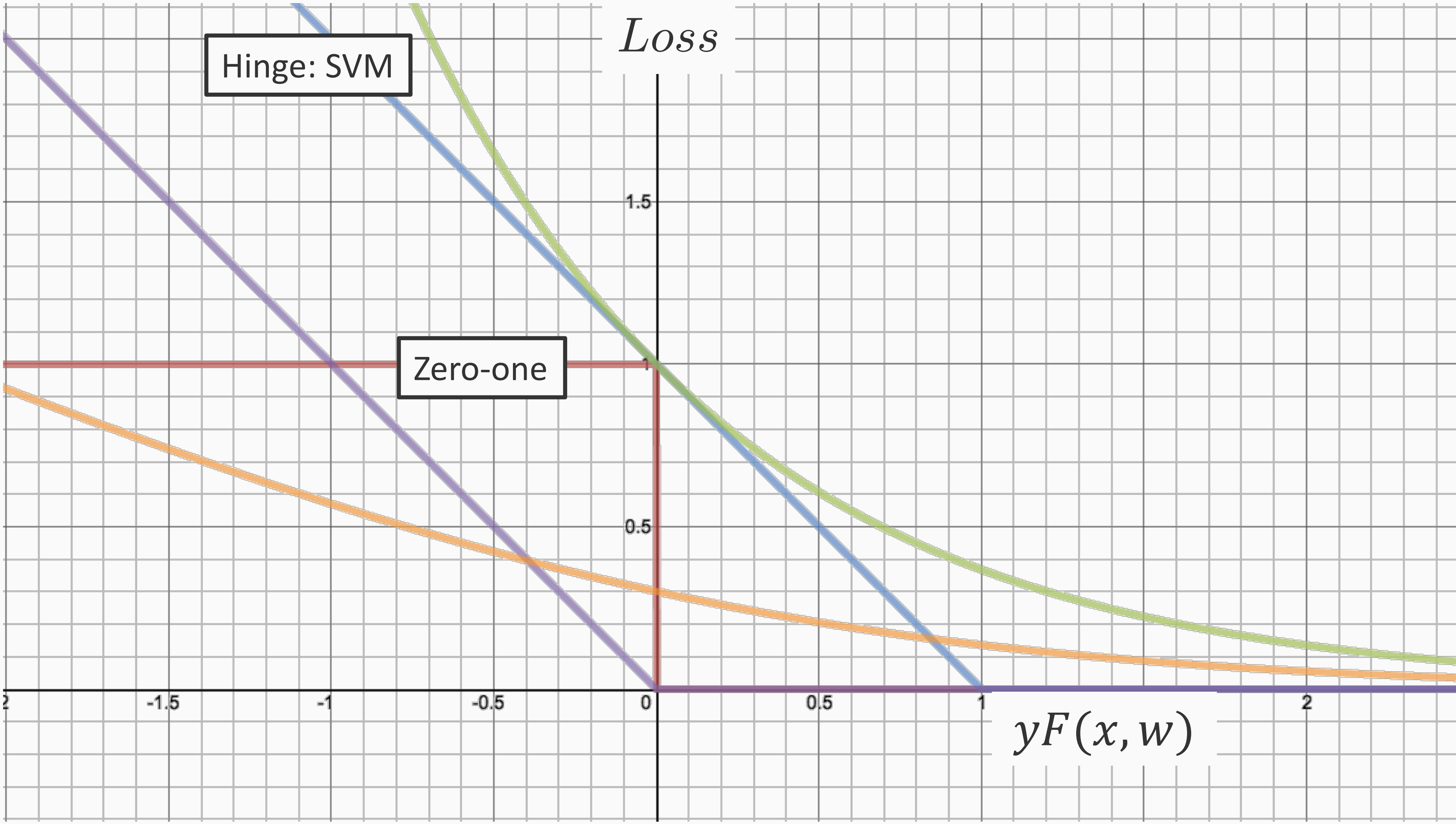


The loss function zoo

$$\min_{h \in \mathrm{H}} \mathrm{regularizer(w)} + \mathrm{C}\frac{1}{m}\sum_i L(F(x_i, w), y_i)$$

$$L_{Hinge}(y, x, w) = \max(0, 1 - yF(x, w))$$



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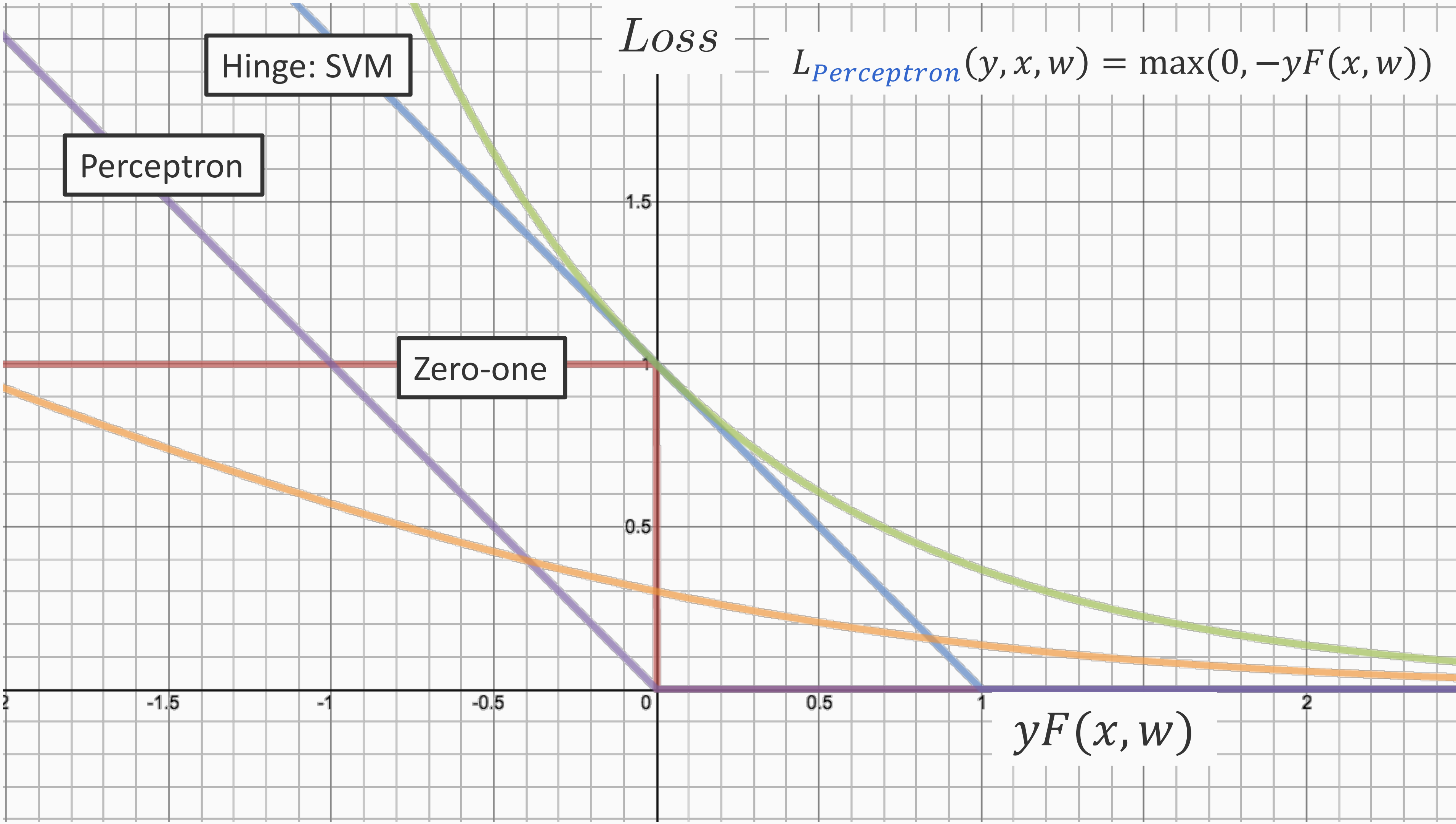

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The loss function zoo

$$L_{Hinge}(y, x, w) = \max(0, 1 - yF(x, w))$$

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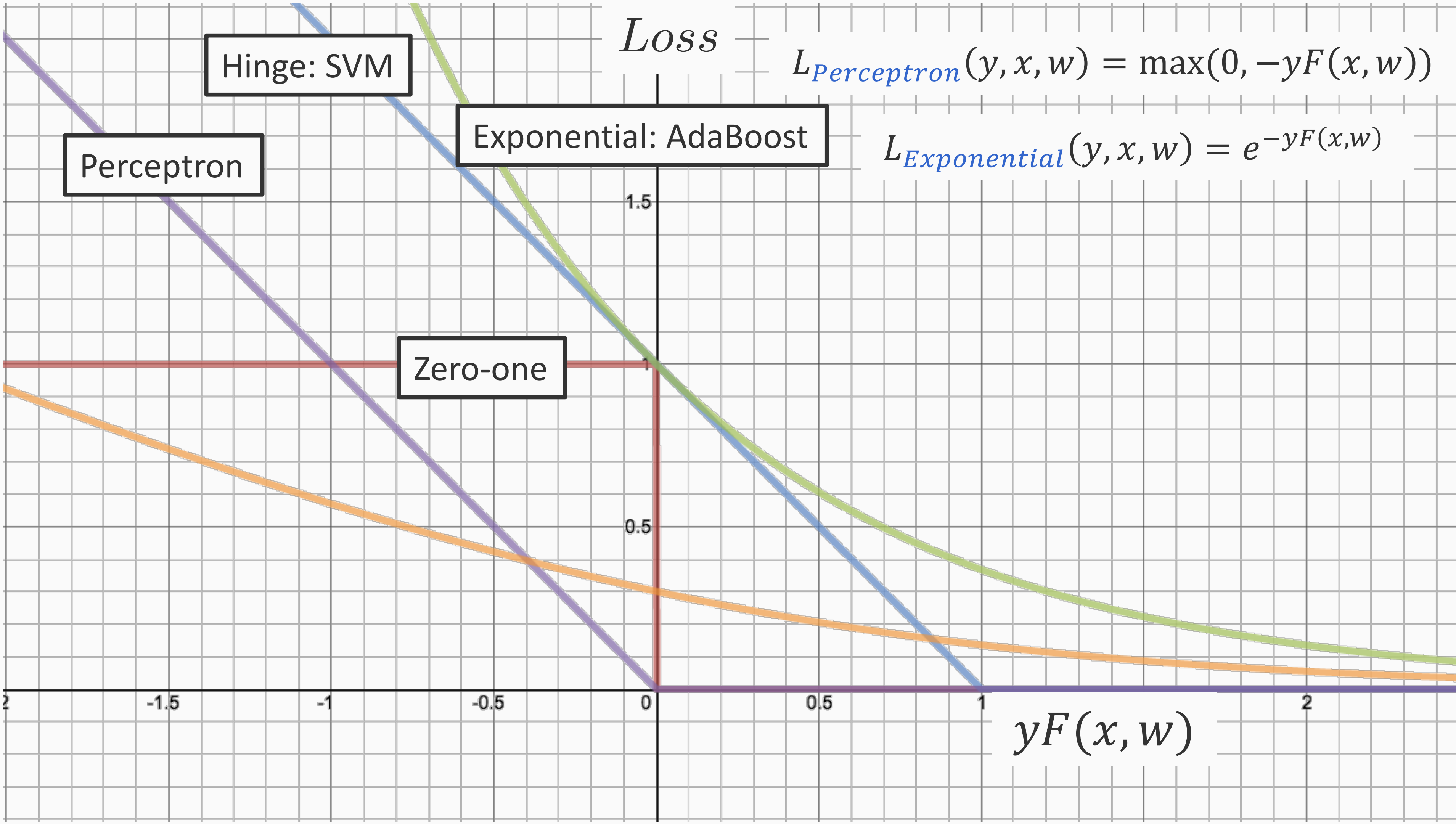

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The loss function zoo

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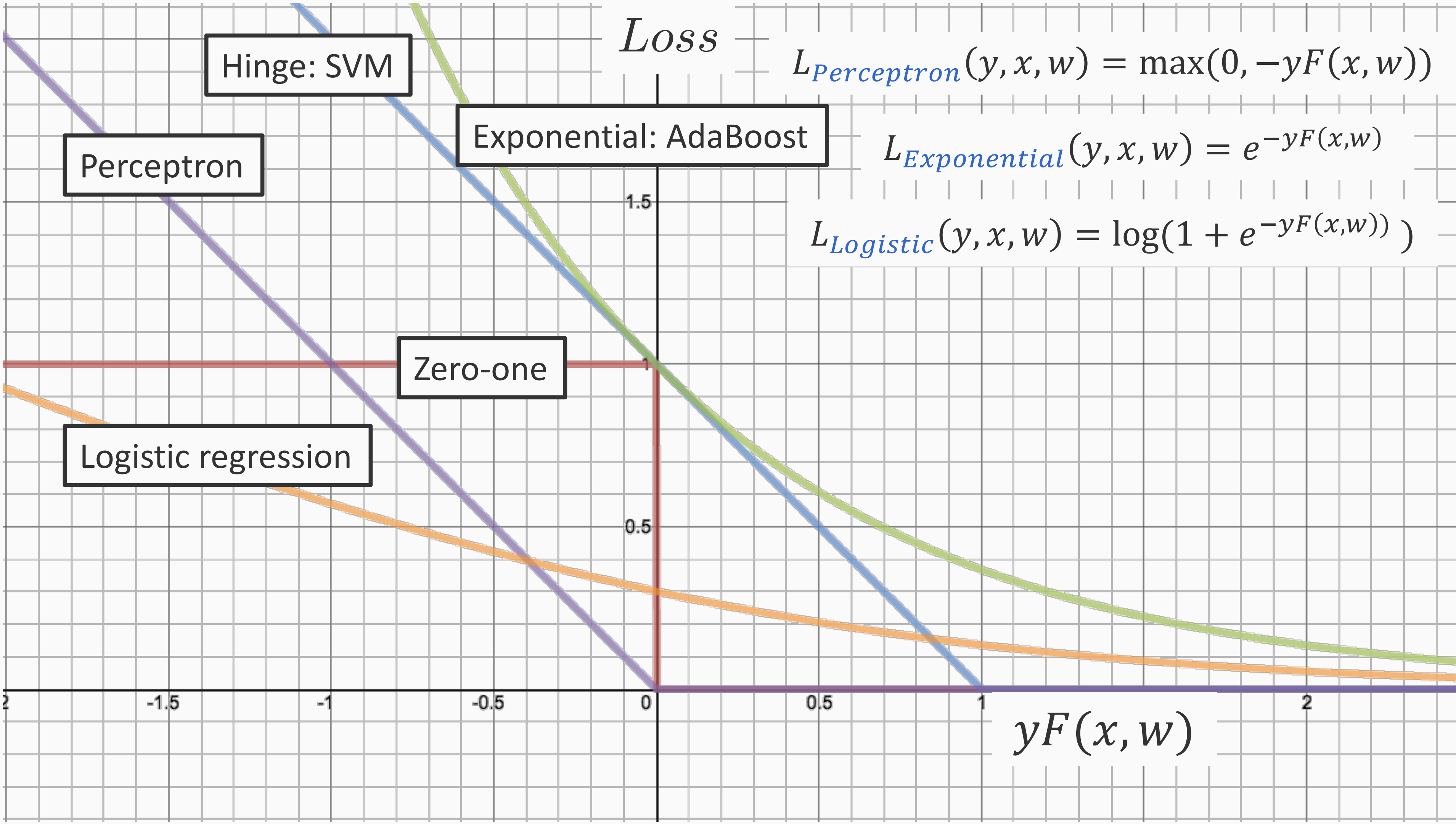
$$L_{Perceptron}(y, x, w) = \max(0, -yF(x, w))$$

$$L_{Exponential}(y, x, w) = e^{-yF(x,w)}$$

$$L_{Logistic}(y, x, w) = \log(1 + e^{-yF(x,w)})$$



What if we have a regression task

- Real valued outputs
  - That is, our model is a function F(x, w) that maps inputs x to a real number
  - Parameterized by w
  - The ground truth y is also a real number
- A natural loss function for this situation is the squared loss

$$L(x, y, w) = \left(y - F(x, w)\right)^2$$

What if we have more than two labels?

What is multiclass classification?

- An input can belong to one of K classes

- Training data: Input associated with class label (a number from 1 to K)
- Prediction: Given a new input, predict the class label

Each input belongs to exactly one class. Not more, not less.

- Otherwise, the problem is not multiclass classification
- If an input can be assigned multiple labels (think tags for emails rather than folders), it is called *multi-label classification*

Example applications: Images

– *Input*: hand-written character; *Output*: which character?

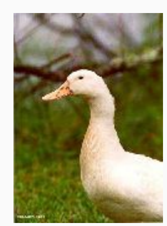
all map to the letter A

– *Input*: a photograph of an object; *Output*: which of a set of categories of objects is it?

  • Eg: the Caltech 256 dataset

Car tire

Car tire



Duck

laptop

Example applications: Language

- *Input*: a news article
- *Output*: Which section of the newspaper should be be in

- *Input*: an email
- *Output*: which folder should an email be placed into

- *Input*: an audio command given to a car
- *Output*: which of a set of actions should be executed

Coming up…

- Multi-class classification
  - Modeling multiple classes
  - Natural loss functions for this situation

- The computation graph abstraction