TOPOLOGY IN AND VIA LOGIC 2026
TUTORIAL 1

Basic Set Theory

Exercise 1. *The following results are used often in topology: Let X, Y be sets, $f : X \to Y$ a function, $S \subseteq X$, $\{S_i : i \in I\} \subseteq \mathcal{P}(X)$, $T \subseteq Y$ and $\{T_j : j \in J\} \subseteq \mathcal{P}(Y)$. Then*

(1) $f[\bigcup_{i\in I} S_i] = \bigcup_{i\in I} f[S_i]$.

(2) $f[\bigcap_{i\in I} S_i] \subseteq \bigcap_{i\in I} f[S_i]$.

(3) $f^{-1}[\bigcup_{j\in J} T_j] = \bigcup_{j\in J} f^{-1}[T_j]$.

(4) $f^{-1}[\bigcap_{j\in J} T_j] = \bigcap_{j\in J} f^{-1}[T_j]$.

(5) $f[S] \cap T = f[S \cap f^{-1}[T]]$.

Furthermore, if $f[\bigcap_{i\in I} S_i] = \bigcap_{i\in I} f[S_i]$ if f is injective. Prove them.

Basic Topology

Exercise 2. *Recall that the* Euclidean topology τ_{Euc} *on* $\mathbb{R}$ *is defined as follows:*

$$\text{for all } U \subseteq \mathbb{R},\ U \in \tau_{Euc} \text{ if and only if } \forall z \in U \exists x, y \in U (z \in (x, y) \subseteq U).$$

Verify that $(\mathbb{R}, \tau_{Euc})$ is a topological space.

Exercise 3. *Recall that the* Cantor set *is defined to be the set 2^ω of all binary sequences of length ω. Let $2^{<\omega}$ denote the set of all finite binary sequences. For all $s \in 2^{<\omega}$ and $t \in 2^\omega \cup 2^{<\omega}$, we write $s \lhd t$ if $t{\restriction}\mathsf{dom}(s) = s$. Intuitively, $s \lhd t$ means that s is an initial subsequence of t. For each $s \in 2^{<\omega}$, we define the set $C(s)$ by*

$$C(s) = \{t \in 2^\omega : s \lhd t\}.$$

Let $B = \{C(s) : s \in 2^{<\omega}\}$. Verify that there is a unique topology τ_{Can} on the Cantor set for which B is a basis.

The topological space $(2^\omega, \tau_{Can})$ is called the Cantor space.

Exercise 4. *Let X, Y be topological spaces.*

(1) *Show that the product topology is the coarsest topology on the set $X \times Y$ such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.*

(2) *Show that for any other topological space Z, if there exist continuous functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, then there exists a unique continuous function $f_1 \times f_2 :$*

$Z \to X \times Y$ *making the following diagram commute*

$$\begin{tikzcd}
& Z \arrow[dl, "f_1"'] \arrow[d, dashed, "f_1 \times f_2"] \arrow[dr, "f_2"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

(3) *Show that this defines the product topology up to homeomorphism: whenever a topological space A together with two continuous functions $\pi_{A,X} : A \to X$ and $\pi_{A,Y} : A \to Y$ satisfy the condition in (2), then there exists a homeomorphism between A and $X \times Y$.* Hint: Given topological spaces X, Y, a continuous map $f : X \to Y$ is a homeomorphism if and only if there is a continuous map $g : Y \to X$ such that $fg = id_Y$ and $gf = id_X$.

Closure, Interior and Neighbourhoods

Definition 1. *Let (X, τ) be a topological space. We say that a set $U \in \mathcal{P}(X)$ is closed if its complement is open, i.e., if $(X \setminus U) \in \tau$.*

Exercise 5. *Let (X, τ) be a topological space and $S \subseteq X$. Show that the following hold:*

(1) *There exists an open set $int(S)$ such that (i) $int(S) \subseteq S$; and (ii) for all open set U, $U \subseteq S$ implies $U \subseteq int(S)$.*

(2) *There exists an closed set $cl(S)$ such that (i) $S \subseteq cl(S)$; and (ii) for all closed set U, $S \subseteq U$ implies $cl(S) \subseteq U$.*

Definition 2. *The sets $int(S)$ and $cl(S)$ in Exercise 5 are called the* interior *and the* closure *of S, respectively. Moreover, we see that a set S is closed if $S = cl(S)$, and open if $S = int(S)$. The operators*

$$int : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto int(S)$$

and

$$cl : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto cl(S)$$

are called the topological interior *and* topological closure*, respectively.*

Exercise 6. *Let (X, τ) be a topological space and $A, B \subseteq X$. Prove the following statements:*

(1) $A \subseteq cl(A)$ *and* $int(A) \subseteq A$.

(2) $cl(cl(A)) = cl(A)$ *and* $int(int(A)) = int(A)$.

(3) $cl(A) = X \setminus (int(X \setminus A))$ *and* $int(A) = X \setminus (cl(X \setminus A))$.

(4) $cl(A) \cup cl(B) = cl(A \cup B)$ *and* $int(A) \cap int(B) = int(A \cap B)$.

(5) *If* $A \subseteq B$, *then* $cl(A) \subseteq cl(B)$ *and* $int(A) \subseteq int(B)$.

Is $cl(A) \cap cl(B) = cl(A \cap B)$ or $int(A) \cup int(B) = int(A \cup B)$ true in general? Prove your answer.

Definition 3. *Given a topological space (X, τ) and a point $x \in X$, we say that $V \in \mathcal{P}(X)$ is a* neighbourhood of x *if there is an open set U such that $x \in U \subseteq V$.*

Moreover, observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x if and only if $x \in V$ and V is open.[1]

Let $N(x)$ denote the set of all open neighbourhoods of x, i.e., $N(x) = \{U \in \tau : x \in U\}$.

Exercise 7. *Suppose (X, τ) is a topological space and $S \subseteq X$. Then for all $x \in X$, the following are equivalent:*

- *x is in the closure of S, i.e., $x \in cl(S)$.*
- *All open neighbourhoods U of x have non-empty intersection with S, i.e.,*
$$\forall U \in N(x)(U \cap S \neq \varnothing).$$

There is a proof of this proposition in the note, but try to prove it yourself first :)

[1] In the literature, you will sometimes find that a neighbourhood is already required to be open. We do not adopt that convention, but speak of 'open neighbourhoods' when needed.