Topological Semantics for Epistemic Logic

Revisiting Justified Belief

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Outline

A gap in traditional epistemic logic

Classical starting point (Hintikka 1962)

Mainstream approach, based on relational possible world semantics. Conceptually simple, technically mature, well-suited for axiomatization.

The Key Gap

It lacks ingredients to talk about the **evidential nature** of justified belief.

Consequence

To capture *justified belief* as *belief supported by evidence*, we need a semantic structure that represents **evidence/justification** explicitly, beyond mere accessibility between worlds.

van Benthem & Pacuit's Evidence Model

- **Key shift:** Instead of encoding an agent's epistemic state only by an accessibility relation, represent **evidence** explicitly.
- **Evidence as information:** A piece of evidence is a "test/observation" that rules out some worlds.

Combining Evidence

Multiple pieces of evidence can be aggregated; intuitively, we keep only the worlds that survive all of the evidence we take seriously.

The Intuition behind Belief

Evidence-based belief (Intuition)

van Benthem and Pacuit define belief by looking at **maximally informative coherent** evidence states, and taking as believed what holds across all such "best" evidence completions.

The Bug

Maximal coherent evidence may require *infinitely many* constraints. "Pooling everything at once" can lead to an unintuitive bug—this motivates the paper's revised belief operator later on.

Evidence Model

Definition (Evidence Model)

An **evidence model** is a triple $M = (X, E_0, V)$ where:

- X: Set of possible worlds.
- $E_0 \subseteq \mathcal{P}(X)$: Basic evidence pieces (not necessarily true).
- V: Valuation of propositional atoms.

Body of Evidence

Given an evidence model $M = (X, E_0, V)$, how do we represent a collection of evidence that "makes sense" together?

Definition (Body of Evidence)

A subset $F \subseteq E_0$ is called a **Body of Evidence** if it satisfies the **Finite Intersection Property (FIP)**:

$$\forall F' \subseteq_{\text{fin}} F \quad \left(F' \neq \emptyset \implies \bigcap F' \neq \emptyset\right).$$

Intuition

A body of evidence represents a coherent argumentative position.

- It does not require that *all* evidence in F is simultaneously true.
- It only requires that no **finite** combination of pieces leads to a contradiction.

Finite vs. General Bodies

Let $\mathcal{F}$ be the family of all bodies of evidence. We distinguish two types:

1. Finite Bodies ($\mathcal{F}^{\text{fin}}$)

$$\mathcal{F}^{\text{fin}} := \{F \subseteq_{\text{fin}} E_0 \mid \bigcap F \neq \emptyset\}$$

- For finite sets, "finite consistency" implies "global consistency".
- If you combine finitely many consistent pieces, the result is a non-empty set.

2. General (Infinite) Bodies

If F is infinite, having the Finite Intersection Property **does NOT** guarantee $\bigcap F \neq \emptyset$.

- It is possible that every finite part is consistent, but the whole body collapses to $\emptyset$.
- **Foreshadowing:** This is exactly where the old operator Bel (based on maximal infinite bodies) crashes!

Evidential Support & Maximal Bodies

Support & Strength

- **Support:** A body F supports a proposition $P \subseteq X$ iff $\bigcap F \subseteq P$.
- **Strength order:** $F \subseteq F'$ means F' is (at least) as strong as F. Stronger bodies support more propositions.

Definition (Maximal Body)

A body $F \in \mathcal{F}$ is **maximal** iff it has no proper extension in $\mathcal{F}$:

$$\mathrm{Max}_{\subseteq}\mathcal{F} \;=\; \Big\{ F \in \mathcal{F} \;\Big|\; \forall F' \in \mathcal{F}\,(F \subseteq F' \Rightarrow F = F') \Big\}.$$

Note on Zorn's Lemma

Every $F \in \mathcal{F}$ can be extended to some maximal body in $\mathrm{Max}_{\subseteq}\mathcal{F}$. *This maximality step is exactly where the bug comes from.*

Combined Evidence & Evidential Topology

Definition (Combined Evidence & Topology)

A set $e \subseteq X$ is combined evidence iff $e = \bigcap F$ for some $F \in \mathcal{F}^{\mathrm{fin}}$. Let

$$E = \{\bigcap F \mid F \in \mathcal{F}^{\mathrm{fin}}\}.$$

Then E forms a **basis** of a topology τ_E on X, called the **evidential topology**.

Intuition

Finite intersections model "bundling finitely many reasons"; the generated topology collects all evidence-based *arguments* (open sets).

The Induced Topological Model

Definition (Topological Evidence Model)

A **topological evidence model** (or **topo-e-model**) is a tuple

$$M = (X, E_0, \tau, V)$$

where:

- (X, E_0, τ) is an evidence model.
- $\tau = \tau_E$ is the topology generated by the family of combined evidence E(or equivalently, by the family of basic evidence sets E_0).

Bel: Belief from Maximal Bodies

van Benthem–Pacuit Belief (Bel)

$$\mathrm{Bel}(P) \text{ holds iff } \forall F \in \mathrm{Max}_{\subseteq}\mathcal{F},\ \bigcap F \subseteq P.$$

Reading: In all strongest consistent evidence states, P must follow.

Why Maximality Can Break Consistency

Finite Consistency vs. Infinite Inconsistency

1. Evidence is combined **finitely**: If $F \in \mathcal{F}$, then every *finite* subfamily of F is consistent by construction.
2. But maximal bodies can be **infinite**: It may happen that every finite subfamily intersects nontrivially, but the full intersection $\bigcap F$ is empty.

The Crash

Then a maximal F becomes "inconsistent as a whole", and Bel can collapse into doxastic inconsistency:

$$\mathrm{Bel}(\perp) \text{ becomes true!}$$

The Paper's Fix: A Weakened Belief Operator B

Idea

Avoid infinitary intersections when defining belief. Rely on "sufficiently strong" finite evidence.

Definition (Topological Belief B)

For $P \subseteq X$:

$$\mathrm{B}(P) \text{ holds iff } \forall F \in \mathcal{F}^{\mathrm{fin}}\, \exists F' \in \mathcal{F}^{\mathrm{fin}} \big(F \subseteq F' \ \wedge \ \bigcap F' \subseteq P\big).$$

Reading

Every finite evidence situation can be **strengthened (still finitely)** to one that supports P.

Immediate Payoff

Why B is better

- B is a **global** notion: depends only on the evidence structure (not the actual world).
- B never forces infinite intersections, so it avoids the "maximality" bug.
- **Consistency:** In particular, $\mathrm{B}(\perp)$ is never true.

Example 2: $\mathrm{B} \not\Rightarrow \mathrm{Bel}$

Model M

Let $X = \mathbb{N} \cup \{\spadesuit\}$. $E_0 = \{[i, \infty) \cup \{\spadesuit\} \mid i \in \mathbb{N}\} \cup \{\{n\} \mid n \in \mathbb{N}\}$.

Analysis

Key fact: there is a maximal body $F = \{e_i \mid i \in \mathbb{N}\}$ with $\bigcap F = \{\spadesuit\}$.

- $\mathrm{Bel}(\mathbb{N})$ **fails**: indeed, the only proposition believed by Bel here is $X = \mathbb{N} \cup \{\spadesuit\}$.
- $\mathrm{B}(\mathbb{N})$ **holds**: any finite F can be strengthened by adding some singleton $\{m\}$ to force intersection $\subseteq \mathbb{N}$.

Example 2: $\mathrm{Bel} \not\Rightarrow \mathrm{B}$

Model M' (Even if Bel is consistent)

$X = \mathbb{N} \cup \{\spadesuit\}$, but $E'_0 = \{[n, \infty) \cup \{\spadesuit\} \mid n \in \mathbb{N}\}$.

Analysis

- Maximal body is E'_0 itself, $\bigcap E'_0 = \{\spadesuit\}$.
- So $\mathrm{Bel}(\{\spadesuit\})$ **holds**.
- However, for every finite body $F \in \mathcal{F}^{\mathrm{fin}}$, we have $\{\spadesuit\} \not\subseteq \bigcap F$.
- Thus, $\mathrm{B}(\{\spadesuit\})$ **fails**.

Takeaway

Bel and B are **not comparable** in general.

Reconciliation: Proposition 1

Definition (Maximally Compact (MC))

An evidence model M is *maximally compact* iff

$$\forall F \in \mathcal{F}\ \exists F_0 \in \mathcal{F}^{\mathrm{fin}}\ \left(\bigcap F = \bigcap F_0\right).$$

(Infinite bodies reduce to finite bodies content-wise.)

Theorem (Proposition 1)

If M is maximally compact, then for all $P \subseteq X$:

$$\mathrm{Bel}(P) \iff \mathrm{B}(P).$$

What B looks like topologically

Topological Characterization

In the evidential topology (X, τ_E):

$$\mathrm{B}(P) \iff P \text{ contains an } \textbf{open dense} \text{ set } U.$$

Also equivalent to: $\mathrm{Int}(P)$ is dense, or $X \setminus P$ is nowhere dense.

Intuition

Believed propositions are true in "almost all" epistemically possible worlds, where "almost all" is spelled out topologically.

Features and Limitations

Features

- Always consistent (avoids Bel bug).
- Agrees with Bel on maximally compact models.
- Clean topological meaning (open dense justification).

Limitations

- B is **global**: true everywhere or nowhere (depends only on evidence).
- Diverges from "strongest-evidence" intuitions on infinite evidence.

Syntax

Syntax (The Full Language $\mathcal{L}$)

Following the grammar in Section 6.1, the full language is defined recursively:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid E_0\varphi \mid E\varphi \mid \Box_0\varphi \mid \Box\varphi \mid \mathrm{B}\varphi \mid K\varphi \mid [\forall]\varphi$$

Focus: The Belief Fragment $\mathcal{L}_B$

As mentioned in the text, we focus on the fragment $\mathcal{L}_B$ having the **belief modality** B as the only modality.
However, to define B rigorously, we need the underlying topological structure provided by the other operators

Semantics(selected)

Semantics

Given a topo-e-model $\mathfrak{M} = (X, E_0, \tau, V)$, we extend the valuation map V to an interpretation map $[\![\cdot]\!]^{\mathfrak{M}} : \mathcal{L} \to \mathcal{P}(X)$ recursively as follows:

- $[\![p]\!]^{\mathfrak{M}} = V(p)$
- $[\![\neg\varphi]\!]^{\mathfrak{M}} = X \setminus [\![\varphi]\!]^{\mathfrak{M}}$
- $[\![\varphi \wedge \psi]\!]^{\mathfrak{M}} = [\![\varphi]\!]^{\mathfrak{M}} \cap [\![\psi]\!]^{\mathfrak{M}}$
- $[\![\Box\varphi]\!]^{\mathfrak{M}} = \{x \in X \mid \exists U \in \tau(x \in U \text{ and } U \subseteq [\![\varphi]\!]^{\mathfrak{M}})\}$
- $[\![B\varphi]\!]^{\mathfrak{M}} = \{x \in X \mid \exists U \in \tau(U \subseteq [\![\varphi]\!]^{\mathfrak{M}} \text{ and } Cl(U) = X)\}$
- $[\![[\forall]\varphi]\!]^{\mathfrak{M}} = \{x \in X \mid [\![\varphi]\!]^{\mathfrak{M}} = X\}$

Logic of Belief: $KD45_B$

Definability

$$\mathrm{B}\varphi \leftrightarrow [\forall]\Diamond\Box\varphi$$

Theorem (Chapter 6 Result)

Let L_B be the language with only modality B.

- *$KD45_B$ is sound and complete w.r.t. the class of all topo-models.*
- *$KD45_B$ has the* **finite model property**.

Interpretation

Even though the motivation came from infinitary evidence models, the resulting belief logic is a familiar and well-behaved doxastic system.

Takeaways

1. **Evidence Models:** Capture fallible/conflicting evidence structuredly.
2. **The Bug:** Bel (maximal bodies) fails in infinite settings (inconsistency).
3. **The Fix:** B quantifies over **finite** bodies and "finite strengthening".
4. **Comparison:** Bel and B are incomparable in general, but coincide on Maximally Compact models.
5. **Topology:** $\mathrm{B}(P)$ iff P has an open dense justification.
6. **Logic:** $\mathsf{KD45}_B$ (sound, complete, FMP).

Thanks!