Systems and Signals: Lecture Notes 1

zqy1018

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Contents

1 Basic Definitions of Signals

1.1 Definition of Signals

Both in electronics and computer science, a signal is something *whose variations represent coded information*.

Examples in real life: voltages, currents, images, etc.

Definition. The mathematical representation of a **signal** is a *function of one or more independent variables*. Usually we use $x : I \to X$ to represent a signal, where I can be seen as the index set.

In this course, we will focus on *1-D signals of one independent variable*. $I \subset \mathbb{R}$, and usually $X \subset \mathbb{R}$ or $\mathbb{C}$. The independent variable is often referred to as "time".

1.2 Two Kinds of Signals

There are two kinds of signals.

- **Continuous-time (CT) signals**: $I \subset \mathbb{R}$, often $I = \mathbb{R}$. Called **analog signal** if X is also continuum. Notation: $x(t)$.

- **Discrete-time (DT) signals**: $I \subset \mathbb{Z}$, often $I = \mathbb{Z}$. Called **digital signal** if X is also discrete. Notation: $x[n]$.

$x(t)$ can be either explained as a signal or a function value. It depends on the context.

We can transform a CT signal to a DT signal by **sampling**.

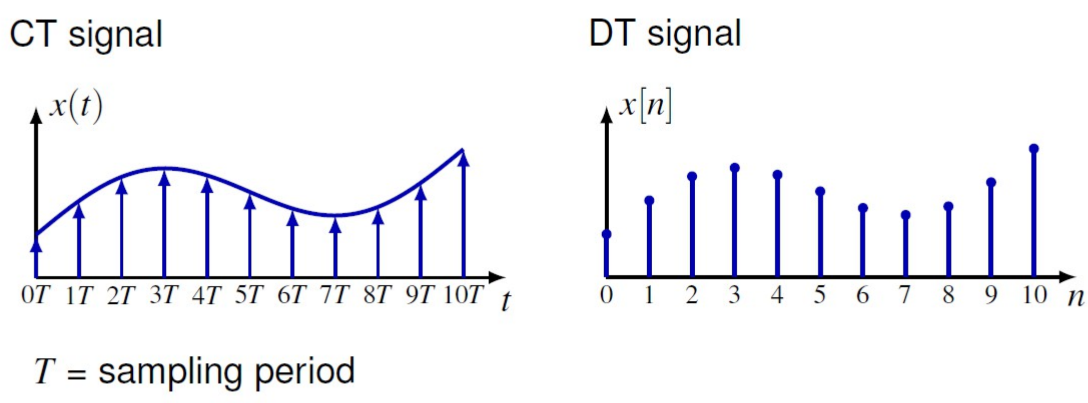


Figure 1: Here T is the **sampling period**. Obviously the sampled data contains no information about T.

We can also reconstruct a CT signal from a DT signal.

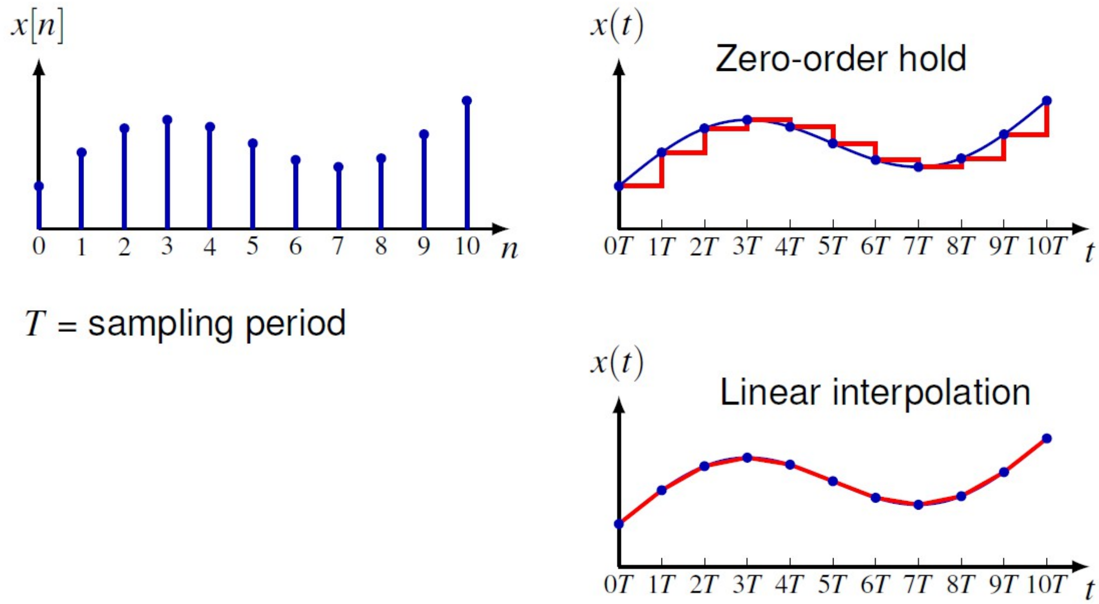


Figure 2: Two ways of reconstruction. Different T yields different reconstructed signals.

2 Basic Transformation on Signals

It is common to perform some transformations on the time axis. Here we use operators (functions manipulating functions) to describe them.

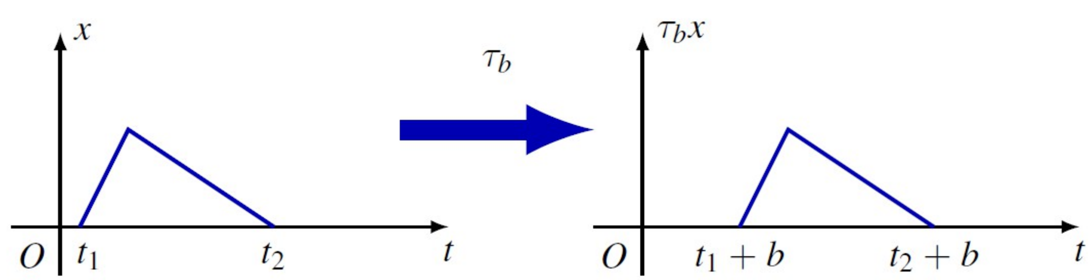
We will discuss two views of these transformations: signal view (how the graph changes) and time view (how a time changes).

2.1 Time Shift

Time shift (Translation) operator $\tau_b : x \mapsto \tau_b x$.

- CT signal: $(\tau_b x)(t) = x(t - b), b \in \mathbb{R}$.
- DT signal: $(\tau_b x)[n] = x[n - b], b \in \mathbb{Z}$.

For example:



The two views:

- Signal view: The y-axis moves left by b (or the signal *delays* by b). If $b < 0$, then it moves right by $-b$ (or the signal *advances* by $-b$).

- Time view: $t \mapsto t + b$.

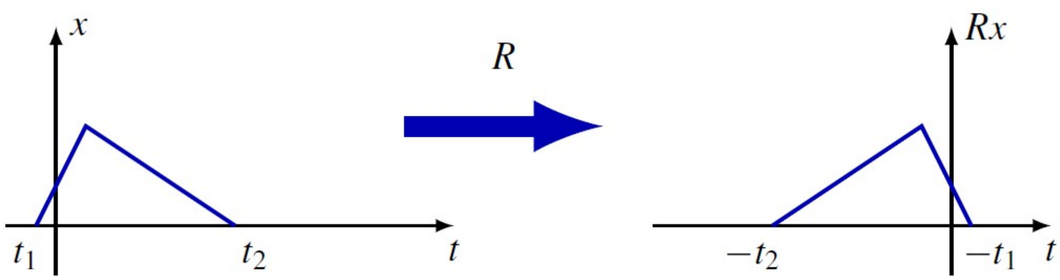
2.2 Time Reversal

Time reversal (Reflection) operator $R : x \mapsto Rx$.

- CT signal: $(Rx)(t) = x(-t)$.

- DT signal: $(Rx)[n] = x[-n]$.

For example:



The two views:

- Signal view: The graph reverses by the y-axis.

- Time view: $t \mapsto -t$.

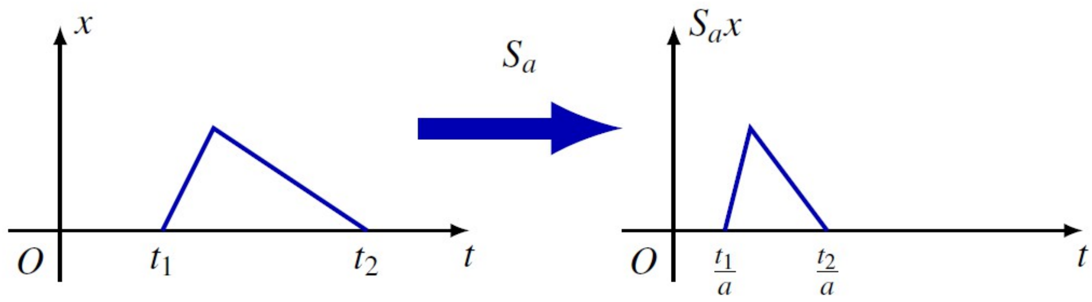
2.3 Time Scaling

Time scaling operator $S_a : x \mapsto S_a x$.

- CT signal: $(S_a x)(t) = x(at), a \in \mathbb{R}^+$.

- DT signal: $(S_a x)[n] = x[an], a \in \mathbb{Z}^+$.

For example:



The two views:

- Signal view: If $\frac{1}{a} > 1$, the graph is stretched. If $\frac{1}{a} < 1$, the graph is compressed.
- Time view: $t \mapsto \frac{t}{a}$.

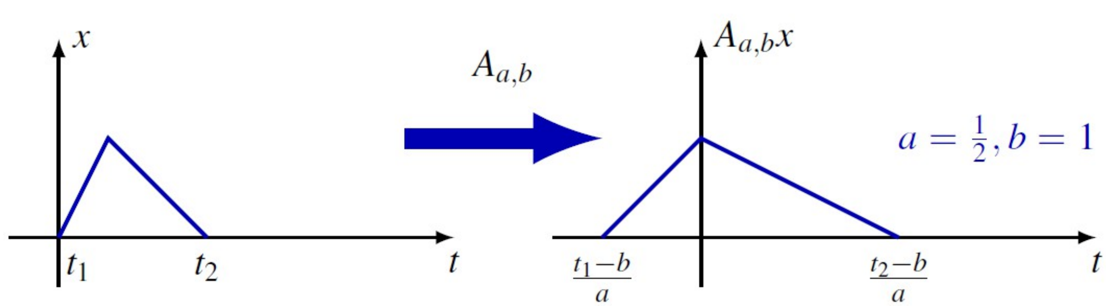
Note that for CT signals, S_a can be undo by $S_{\frac{1}{a}}$, if $a \neq 0$. But for DT signals, if $a > 1$, then the scaling can be seen as **down sampling**, which is irreversible.

2.4 Affine Transformation

Affine transformation $A_{a,b} : x \mapsto A_{a,b}x$.

- CT signal: $(A_{a,b}x)(t) = x(at + b), a \in \mathbb{R}\setminus\{0\}, b \in \mathbb{R}$.
- DT signal: $(A_{a,b}x)[n] = x[an + b], a \in \mathbb{Z}\setminus\{0\}, b \in \mathbb{Z}$.

For example:



It can be decomposed as product of shift, reversal and scaling. For example:

$$a > 0 : A_{a,b} = S_a \circ \tau_{-b}$$

$$a < 0 : A_{a,b} = S_{|a|} \circ R \circ \tau_{-b}$$

where $(f \circ g)(x) = g(f(x))$.

The approaches are not unique, but *doing time shift first* will be easier. And in [OWN], “**first shift, then scale, finally reverse**” is recommended.

Here we just show the time view (the graph view is complicated): $t \mapsto \frac{t-b}{a}$.

2.5 Useful Identities

Here we show some useful identities about the operators. It is easy to prove them, so we skip the proof.

Theorem 1. $\forall a \neq 0, b \in \mathbb{R}, S_a \circ \tau_{-b} = \tau_{-\frac{b}{a}} \circ S_a$.

Theorem 2. $\forall b \in \mathbb{R}, R \circ \tau_{-b} = \tau_b \circ R$.

Theorem 3. $\forall a \neq 0, S_a \circ R = R \circ S_a$.

3 Signal Energy and Power

3.1 Formulas

We can define the **energy** and **power** of a signal as those of a 1Ω resistor. That is, regard $x(t)$ as the voltage and do calculation.

Instantaneous power $\quad p(t) = |v(t)|^2$

Energy over $[t_1, t_2]$ $\quad E(t_1, t_2) = \int_{t_1}^{t_2} |v(t)|^2 dt$

Average power over $[t_1, t_2]$ $\quad P(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |v(t)|^2 dt$

Total energy

$$E(x) = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad E(x) = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Average power

$$P(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad P(x) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Figure 3: Basic definitions of signal energy and power.

3.2 Classify the Signals by E and P

We can divide all the signals into three classes.

1. **Finite-energy signal**: $E(x) < +\infty$. These signals are all in L_2 space.

2. **Finite-power signal**: $E(x) = +\infty, P(x) < +\infty$, like $x(t) = \sin t$.

3. **Infinite-power signal**: $P(x) = +\infty$, like $x(t) = t$.

Note. $E(x) < +\infty \implies P(x) = 0$, and $P(x) > 0 \implies E(x) = +\infty$. But the converse may not hold. For example, $x(t) = t^{-\frac{1}{2}}(t > 1)$. $P(x) = 0$ but $E(x) = +\infty$.

4 Periodicity

4.1 Definitions

Definition. For CT signals, $T \in \mathbb{R}$ is a **period** iff $x(t+T) = x(t), \forall t \in \mathbb{R}$. $T_0 = \min\{T > 0, T \text{ is a period}\}$ is the **fundamental period** of $x(t)$. $N \in \mathbb{Z}$ and $N_0 \in \mathbb{Z}^+$ can be similarly defined for DT signals.

Note. If T is a period, then mT is also a period for $m \in \mathbb{Z}$.

There are some slight differences between the CT case and DT case.

- A constant CT signal $x(t) = c$ has no well-defined fundamental period. But all constant DT signals have $N_0 = 1$, since $\mathbb{Z}^+$ is well-ordered.

- For the same function, the fundamental period in CT may be different from that in DT. For example: $\sin(\frac{4\pi}{3}t)(T_0 = \frac{3}{2})$ and $\sin(\frac{4\pi}{3}n)(N_0 = 2)$.

4.2 Common Periods and Sinusoids

Theorem 4. $x_1(t)$ has the period $T_1 > 0$, and $x_2(t)$ has the period $T_2 > 0$. If $\frac{T_1}{T_2} \in \mathbb{Q}$ then $x(t) = x_1(t) + x_2(t)$ is periodic.

Proof. Let $\frac{T_1}{T_2} = \frac{m_2}{m_1}, m_i \in \mathbb{Z}^+$. Then $T = m_1 T_1 = m_2 T_2$ is the period of $x(t)$. □

In some special cases, we can apply it on $x(t) = x_1(t)x_2(t)$. For example, $x_1(t)$ and $x_2(t)$ are sinusoids.

Theorem 5. For $A_i \neq 0$, $\omega_i > 0$, $x(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$ is periodic **iff** $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$. If we let $\frac{\omega_1}{\omega_2} = \frac{m_1}{m_2}, m_i \in \mathbb{Z}^+, \gcd(m_1, m_2) = 1$, then $T_0 = \frac{2\pi m_1}{\omega_1} = \frac{2\pi m_2}{\omega_2}$ is the fundamental period of $x(t)$.

Proof. We prove the necessity by contradiction. Suppose $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$. $\square$

Theorem 6. $x[n] = \sin(\omega n + \phi)$ is periodic **iff** $\exists \frac{k}{N} \in \mathbb{Q}, w = \frac{2\pi k}{N}$.

Proof. We only prove the " $\implies$ " part. Suppose N is a period of $x[n]$. Then:

$$x[n+N] = x[n] \implies \cos(\omega n + \phi + \frac{\omega N}{2}) \sin(\frac{\omega N}{2}) = 0$$

which implies $\cos(\omega n + \phi + \frac{\omega N}{2}) = 0$ or $\sin(\frac{\omega N}{2}) = 0$.

1. $\sin(\frac{\omega N}{2}) = 0 \implies \exists k \in \mathbb{Z}, \frac{\omega N}{2} = k\pi \implies w = \frac{2\pi k}{N}$.
2. $\cos(\omega n + \phi + \frac{\omega N}{2}) = 0 \implies \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, \omega n + \phi + \frac{\omega N}{2} = \frac{\pi}{2} + k\pi$. By choosing $n = 0$ and $n = 1$, we have $\omega = (k_{(n=1)} - k_{(n=0)})\pi \in \pi\mathbb{Z} \subset 2\pi\mathbb{Q}$.

So $\exists \frac{k}{N} \in \mathbb{Q}, w = \frac{2\pi k}{N}$. $\square$

Corollary. For $x[n] = \sin(2\pi \frac{k}{N} n + \phi), k \in \mathbb{Z}, N \in \mathbb{Z}^+, N \geq 3, \gcd(N, k) = 1$, the fundamental period $N_0 = N$.

Proof. By Theorem 6, $\frac{2\pi k}{N} \notin \mathbb{Z} \implies \frac{2\pi k}{N} \in \frac{2\pi}{N_0}\mathbb{Z} \implies \frac{kN_0}{N} \in \mathbb{Z}$. Since $\gcd(N, k) = 1$, $N|N_0$. Again by Theorem 6, N is a period $\implies N_0|N$. So $N_0 = N$. $\square$

Note. If $N = 2$ and $\sin\phi = 0$, then $N_0 = 1$.

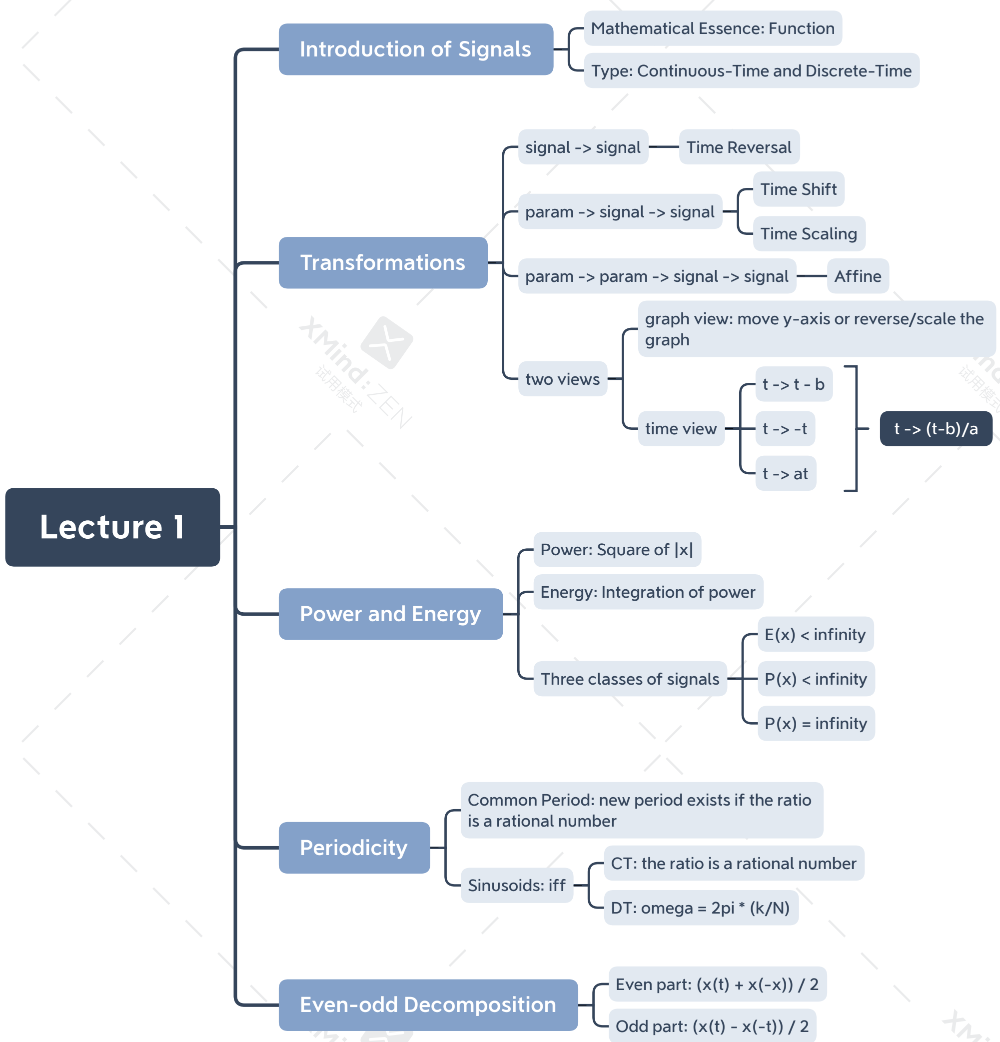
5 Even-odd Decomposition

We know that any function can be represented as the sum of an even function and an odd function.

Definition. The **even part** of x is $x_e = \mathcal{E}v(x) = \frac{x+Rx}{2}$. The **odd part** of x is $x_o = \mathcal{O}d(x) = \frac{x-Rx}{2}$.

Theorem 7. $\forall x, x = \mathcal{E}v(x) + \mathcal{O}d(x)$. If x is even, then $x = \mathcal{E}v(x)$. If x is odd, then $x = \mathcal{O}d(x)$.

6 Review


Lecture 1
Introduction of Signals
Mathematical Essence: Function
Type: Continuous-Time and Discrete-Time
Transformations
signal -> signal
Time Reversal
param -> signal -> signal
Time Shift
Time Scaling
param -> param -> signal -> signal
Affine
two views
graph view: move y-axis or reverse/scale the graph
time view
t -> t - b
t -> -t
t -> at
t -> (t-b)/a
Power and Energy
Power: Square of |x|
Energy: Integration of power
Three classes of signals
E(x) < infinity
P(x) < infinity
P(x) = infinity
Periodicity
Common Period: new period exists if the ratio is a rational number
Sinusoids: iff
CT: the ratio is a rational number
DT: omega = 2pi * (k/N)
Even-odd Decomposition
Even part: (x(t) + x(-x)) / 2
Odd part: (x(t) - x(-t)) / 2

Acknowledgment

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