

Perfectly Secret Encryption

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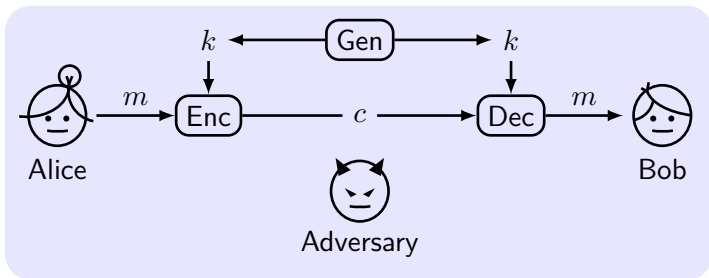
FIT - University of Transport and Communication

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- 1 Definitions and Basic Properties
- 2 The One-Time Pad (Vernam's Cipher)
- 3 Limitations of Perfect Secrecy
- 4 Shannon's Theorem
- 5 Eavesdropping Indistinguishability

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Recall The Syntax of Encryption



- $k \in \mathcal{K}, m \in \mathcal{M}, c \in \mathcal{C}$.
- $k \leftarrow \text{Gen}, c := \text{Enc}_k(m), m := \text{Dec}_k(c)$.
- **Encryption scheme**: $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$.
- **Random Variable**: K, M, C for key, plaintext, ciphertext.
- **Probability**: $\Pr[K = k], \Pr[M = m], \Pr[C = c]$.
- What's the basic correctness requirement?

Definition of 'Perfect Secrecy'

Intuition: An adversary knows the probability distribution over \mathcal{M} . c should have no effect on the knowledge of the adversary; the *a posteriori* likelihood that some m was sent should be no different from the *a priori* probability that m would be sent.

Definition 1

Π over \mathcal{M} is **perfectly secret** if for every probability distribution over \mathcal{M} , $\forall m \in \mathcal{M}$ and $\forall c \in \mathcal{C}$ for which $\Pr[C = c] > 0$:

$$\Pr[M = m|C = c] = \Pr[M = m].$$

Simplify: non-zero probabilities for $\forall m \in \mathcal{M}$ and $\forall c \in \mathcal{C}$.

Is the below scheme perfectly secret?

For $\mathcal{M} = \mathcal{K} = \{0, 1\}$, $\text{Enc}_k(m) = m \oplus k$.

Perfect Secrecy On One Bit

XORing one bit is perfectly secret.

Let $\Pr[M = 1] = p$ and $\Pr[M = 0] = 1 - p$. Let us consider a case that $M = 1$ and $C = 1$.

$$\begin{aligned}\Pr[M = 1|C = 1] &= \Pr[C = 1|M = 1] \cdot \Pr[M = 1] / \Pr[C = 1] \\ &= \frac{\Pr[K = 1 \oplus 1] \cdot p}{\Pr[C = 1|M = 1] \cdot \Pr[M = 1] + \Pr[C = 1|M = 0] \cdot \Pr[M = 0]} \\ &= \frac{1/2 \cdot p}{1/2 \cdot p + 1/2 \cdot (1 - p)} = p = \Pr[M = 1]\end{aligned}$$

We can do the same for other cases.

Note that $\Pr[M = 1|C = 1] \neq \Pr[M = 1, C = 1] = \Pr[C = 1|M = 1] \cdot \Pr[M = 1] = 1/2 \cdot p$.

An Equivalent Formulation

Lemma 2

Π over \mathcal{M} is perfectly secret \iff for every probability distribution over \mathcal{M} , $\forall m \in \mathcal{M}$ and $\forall c \in \mathcal{C}$:

$$\Pr[C = c|M = m] = \Pr[C = c].$$

Proof.

\Leftarrow : Multiplying both sides by $\Pr[M = m]/\Pr[C = c]$, then use Bayes' Theorem.¹

$$\begin{aligned}\Pr[C = c|M = m] \cdot \Pr[M = m]/\Pr[C = c] &= \Pr[M = m] \\ \Pr[M = m|C = c] \cdot \Pr[C = c]/\Pr[C = c] &= \Pr[M = m|C = c]\end{aligned}$$

\Rightarrow : Multiplying both sides by $\Pr[C = c]/\Pr[M = m]$, then use Bayes' Theorem. □

¹If $\Pr[B] \neq 0$ then $\Pr[A|B] \cdot \Pr[B] = \Pr[B|A] \cdot \Pr[A]$

Perfect Indistinguishability

Lemma 3

Π over \mathcal{M} is perfectly secret \iff for every probability distribution over \mathcal{M} , $\forall m_0, m_1 \in \mathcal{M}$ and $\forall c \in \mathcal{C}$:

$$\Pr[C = c | M = m_0] = \Pr[C = c | M = m_1].$$

Proof.

\Rightarrow : By Lemma 2: $\Pr[C = c | M = m] = \Pr[C = c]$.

\Leftarrow : $p \stackrel{\text{def}}{=} \Pr[C = c | M = m_0]$.

$$\begin{aligned}\Pr[C = c] &= \sum_{m \in \mathcal{M}} \Pr[C = c | M = m] \cdot \Pr[M = m] \\ &= \sum_{m \in \mathcal{M}} p \cdot \Pr[M = m] = p = \Pr[C = c | M = m_0].\end{aligned}$$



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One-Time Pad (Vernam's Cipher)

- $\mathcal{M} = \mathcal{K} = \mathcal{C} = \{0, 1\}^\ell$.
- Gen chooses a k randomly with probability exactly $2^{-\ell}$.
- $c := \text{Enc}_k(m) = k \oplus m$.
- $m := \text{Dec}_k(c) = k \oplus c$.

Theorem 4

The one-time pad encryption scheme is perfectly-secret.

Proof.

$$\begin{aligned}\Pr[C = c | M = m] &= \Pr[M \oplus K = c | M = m] \\ &= \Pr[m \oplus K = c] = \Pr[K = m \oplus c] = 2^{-\ell}.\end{aligned}$$

Then Lemma 3: $\Pr[C = c | M = m_0] = \Pr[C = c | M = m_1]$. □

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Limitations of OTP and Perfect Secrecy

Key k is as long as m , difficult to store and share k .

Theorem 5

Let Π be perfectly-secret over \mathcal{M} , and let \mathcal{K} be determined by Gen. Then $|\mathcal{K}| \geq |\mathcal{M}|$.

Proof.

Assume $|\mathcal{K}| < |\mathcal{M}|$. $\mathcal{M}(c) \stackrel{\text{def}}{=} \{\hat{m} | \hat{m} = \text{Dec}_k(c) \text{ for some } \hat{k} \in \mathcal{K}\}$. Since for one k , there is at most one m such that $m = \text{Dec}_k(c)$, $|\mathcal{M}(c)| \leq |\mathcal{K}| < |\mathcal{M}|$. So $\exists m' \notin \mathcal{M}(c)$. Then

$$\Pr[M = m' | C = c] = 0 \neq \Pr[M = m']$$

and so not perfectly secret. □

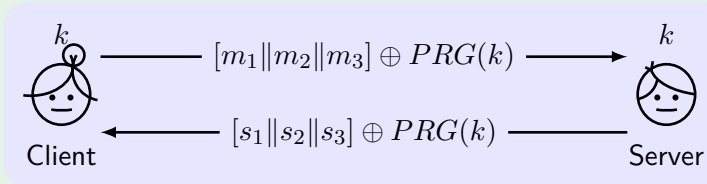
Two Time Pad: Real World Cases

Only used once for the same key, otherwise

$$c \oplus c' = (m \oplus k) \oplus (m' \oplus k) = m \oplus m'.$$

Learn m from $m \oplus m'$ due to the redundancy of language.

MS-PPTP (Win NT)



Improvement: use two keys for C-to-S and S-to-C separately.

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Shannon's Theorem

Theorem 6

For $|\mathcal{M}| = |\mathcal{K}| = |\mathcal{C}|$, Π is perfectly secret \iff

- 1 Every $k \in \mathcal{K}$ is chosen with probability $1/|\mathcal{K}|$ by Gen.
- 2 $\forall m \in \mathcal{M}$ and $\forall c \in \mathcal{C}$, \exists unique $k \in \mathcal{K}$: $c := \text{Enc}_k(m)$.

Proof.

\Leftarrow : $\Pr[C = c|M = m] = 1/|\mathcal{K}|$, use Lemma 3.

\Rightarrow (2): At least one k , otherwise $\Pr[C = c|M = m] = 0$;
at most one k , because $\{\text{Enc}_k(m)\}_{k \in \mathcal{K}} = \mathcal{C}$ and $|\mathcal{K}| = |\mathcal{C}|$.

\Rightarrow (1): k_i is such that $\text{Enc}_{k_i}(m_i) = c$.

$$\begin{aligned}\Pr[M = m_i] &= \Pr[M = m_i|C = c] \\ &= (\Pr[C = c|M = m_i] \cdot \Pr[M = m_i]) / \Pr[C = c] \\ &= (\Pr[K = k_i] \cdot \Pr[M = m_i]) / \Pr[C = c],\end{aligned}$$

so $\Pr[K = k_i] = \Pr[C = c] = 1/|\mathcal{K}|$. □

Application of Shannon's Theorem

Is the below scheme perfectly secret?

Let $\mathcal{M} = \mathcal{C} = \mathcal{K} = \{0, 1, 2, \dots, 255\}$

$\text{Enc}_k(m) = m + k \pmod{256}$

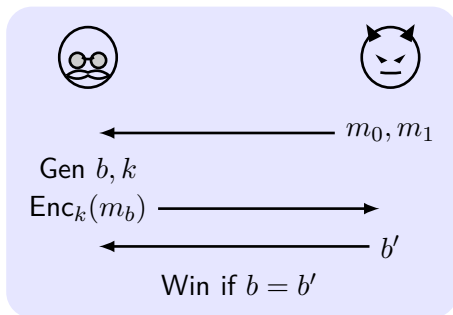
$\text{Dec}_k(c) = c - k \pmod{256}$

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Eavesdropping Indistinguishability Experiment

$\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}}$ denote a **private-key** encryption experiment for a given Π over \mathcal{M} and an **eavesdropping** adversary \mathcal{A} .

- 1 \mathcal{A} outputs a pair of messages $m_0, m_1 \in \mathcal{M}$.
- 2 $k \leftarrow \text{Gen}$, a random bit $b \leftarrow \{0, 1\}$ is chosen. Then $c \leftarrow \text{Enc}_k(m_b)$ is given to \mathcal{A} .
- 3 \mathcal{A} outputs a bit b'
- 4 If $b' = b$, \mathcal{A} succeeded $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}} = 1$, otherwise 0.



Definition 7

Π over \mathcal{M} is **perfectly secret** if for every \mathcal{A} it holds that

$$\Pr[\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}} = 1] = \frac{1}{2}.$$

Which in the below schemes are perfectly secret?

- $\text{Enc}_{k,k'}(m) = \text{OTP}_k(m) \parallel \text{OTP}_{k'}(m)$
- $\text{Enc}_k(m) = \text{reverse}(\text{OTP}_k(m))$
- $\text{Enc}_k(m) = \text{OTP}_k(m) \parallel k$
- $\text{Enc}_k(m) = \text{OTP}_k(m) \parallel \text{OTP}_k(m)$
- $\text{Enc}_k(m) = \text{OTP}_{0^n}(m)$
- $\text{Enc}_k(m) = \text{OTP}_k(m) \parallel \text{LSB}(m)$

- Perfect secrecy = Perfect indistinguishability = Adversarial indistinguishability
- Perfect secrecy is attainable. The One-Time Pad (Vernam's cipher)
- Shannon's theorem