№1

$f(x) = 3x^3 - 2x^2 + x + 2$

$g(x) = x^2 - x + 1$

1)
$$
\begin{array}{rl}
\overset{f}{3x^3 - 2x^2 + x + 2} & \Big|\ \overset{g}{x^2 - x + 1} \\
-\ 3x^3 - 3x^2 + 3x & \Big|\ 3x + 1 \quad q_1 \\
\hline
x^2 - 2x + 2 & \\
-\ x^2 - x + 1 & \\
\hline
-x + 1 \quad r_1 &
\end{array}
$$

2)
$$
\begin{array}{rl}
\overset{g}{x^2 - x + 1} & \Big|\ \overset{r_1}{-x + 1} \\
-\ x^2 - x & \Big|\ -x \quad q_2 \\
\hline
0 + 1 \quad r_2 &
\end{array}
$$

3)
$$
\begin{array}{rl}
\overset{r_1}{-x + 1} & \Big|\ \overset{r_2}{1} \\
-\ -x + 1 & \Big|\ -x + 1 \quad q_3 \\
\hline
0 \quad r_3 &
\end{array}
$$

$\Rightarrow$ НОД$(f, g) = 1$

• $r_2 = -f \cdot q_2 + g\,(1 + q_1 q_2)$

$1 = -f \cdot (-x) + g\,(1 + (3x+1)(-x))$

$$\boxed{1 = f\cdot x + g\left(1-3x^2-x\right)}$$

№2

$f(x) = x^4 - 2x^3 - x^2 - 2x + 1$

$g(x) = x^5 + x^4 - x^3 - 3x^2 - 3x - 1$

1) $f = \underset{q_1}{0}\cdot g + \underset{r_1}{f}$

2)
$$\begin{array}{l|l} \overset{g}{x^5+x^4-x^3-3x^2-3x-1} & \overset{r_1}{x^4-2x^3-x^2-2x+1} \\ \underline{x^5-2x^4-x^3-2x^2+x} & \overline{x+3} \quad q_2 \\ 3x^4+0x^3-x^2-4x-1 & \\ \underline{3x^4-6x^3-3x^2-6x+3} & \\ 6x^3+2x^2+2x-4 \quad r_2 & \end{array}$$

3)
$$\begin{array}{l|l} \overset{r_1}{x^4-2x^3-x^2-2x+1} & \overset{r_2}{6x^3+2x^2+2x-4} \\ \underline{x^4+\frac{x^3}{3}+\frac{x^2}{3}-\frac{2x}{3}} & \overline{\frac{1}{6}x-\frac{7}{18}} \quad q_3 \\ -\frac{7x^3}{3}-\frac{4x^2}{3}-\frac{4x}{3} & \\ \underline{-\frac{7x^3}{3}-\frac{7x^2}{9}-\frac{7x}{9}+\frac{14}{9}} & \\ -\frac{5x^2}{9}-\frac{5x}{9}-\frac{5}{9} \quad r_3 & \end{array}$$

$$\begin{array}{r|l} -6x^3+2x^2+2x-4 & -\frac{5x^2}{9}-\frac{5x}{9}-\frac{5}{9} \\ \hline \end{array}$$

4)

$$\begin{array}{l} \quad 6x^3+6x^2+6x \\ \hline \quad -4x^2-4x-4 \\ -\;-4x^2-4x-4 \\ \hline \qquad 0 \quad r_4 \end{array} \qquad -\frac{54x}{5}+\frac{36}{5} \quad q_4$$

r_2, r_3

$$\Rightarrow \text{НОД}(f,g) = \left(-\frac{5}{9}x^2-\frac{5}{9}x-\frac{5}{9}\right)\cdot\left(-\frac{9}{5}\right) =$$

$$= \boxed{x^2+x+1}$$

$$f = q_1 g + r_1 \quad \Rightarrow r_1 = f - q_1 g$$

$$g = q_2 r_1 + r_2 \Rightarrow r_2 = g - q_2 r_1$$

$$r_1 = q_3 r_2 + r_3 \quad \Rightarrow r_3 = r_1 - q_3 r_2 \Rightarrow$$

$$\Rightarrow r_3 = r_1 - q_3(g - q_2 r_1)$$

$$\Rightarrow r_3 = f - q_1 g - q_3(g - q_2(f - q_1 g))$$

$$r_3 = f - q_1 g - q_3 g + q_3 q_2 f - q_3 q_2 q_1 g$$

$$r_3 = f(1 + q_2 q_3) - g(q_1 + q_3 + q_1 q_2 q_3)$$

$$-\frac{5}{9}x^2 - \frac{5}{9}x - \frac{5}{9} = f\left(1 + (x+3)\left(\frac{1}{6}x - \frac{7}{18}\right)\right) -$$

$$- g\left(\frac{1}{6}x - \frac{7}{18}\right)$$

$$x^2+x+1 = -\frac{9}{5}f\left(\frac{1}{6}x^2+\frac{1}{9}x-\frac{1}{6}\right)+\frac{9}{5}g\left(\frac{1}{6}x-\frac{7}{18}\right)$$

$$\boxed{x^2+x+1 = f(-0{,}3x^2-\tfrac{1}{5}x+0{,}3)+g(0{,}3x-0{,}7)}$$

№3

$f(x) = x^5+x+1$, $g(x) = x^4+x^3+1$

над F_2

1)
$$\begin{array}{l|l} x^5+x+1 \ (f) & x^4+x^3+1 \ (g) \\ \underline{-\ x^5+x^4+x} & \overline{x+1} \ (q_1) \\ x^4 \qquad +1 & \\ \underline{-\ x^4+x^3+1} & \\ x^3 \ (r_1) & \end{array}$$

2)
$$\begin{array}{l|l} x^4+x^3+1 \ (g) & x^3 \ (r_1) \\ \underline{-\ x^4+x^3} & \overline{x+1} \ (q_2) \\ 1 \ (r_2) & \end{array}$$

3)
$$\begin{array}{l|l} x^3 \ (r_1) & 1 \ (r_2) \\ \underline{-\ x^3} & \overline{x^3} \ (q_3) \\ 0 \ (r_3) & \end{array}$$

$\Rightarrow$ НОД$(f,g) = 1$

$$1 = -f\cdot(x+1)+g\big(1+(x+1)(x+1)\big)$$

$$\Rightarrow 1 = f(-x-1) + g(x^2+2x+2)$$

$$\boxed{1 = f(x+1) + g(x^2)}$$

№4

$f(x) = x^5 + x^3 + x$; $g(x) = x^4 + x + 1$; над F_2.

1) f: $x^5 + x^3 + x$ ÷ g: $x^4 + x + 1$

$$\begin{array}{l|l} x^5 + x^3 + x & x^4 + x + 1 \\ \hline -\,x^5 + x^2 + x & x \quad (q_1) \\ \hline x^3 + x^2 \quad (r_1) & \end{array}$$

2) g: $x^4 + x + 1$ ÷ r_1: $x^3 + x^2$

$$\begin{array}{l|l} x^4 + x + 1 & x^3 + x^2 \\ \hline -\,x^4 + x^3 & x + 1 \quad (q_2) \\ \hline x^3 + x + 1 & \\ -\,x^3 + x^2 & \\ \hline x^2 + x + 1 \quad (r_2) & \end{array}$$

3) r_1: $x^3 + x^2$ ÷ r_2: $x^2 + x + 1$

$$\begin{array}{l|l} x^3 + x^2 & x^2 + x + 1 \\ \hline -\,x^3 + x^2 + x & x \quad (q_3) \\ \hline x \quad (r_3) & \end{array}$$

4)
$$\begin{array}{r|l} x^2+x+1 & x \\ \cline{2-2} -\;x^2+x & x+1 \\ \hline 1 & \end{array}$$

r_2, r_3, q_4, r_4

5)
$$\begin{array}{r|l} x & 1 \\ \cline{2-2} -\;x & x \\ \hline 0 & \end{array} \Rightarrow \text{НОД}(f,g) = 1$$

r_3, r_4, q_5, r_5

$$\begin{cases} f = q_1 g + r_1 \Rightarrow r_1 = f - q_1 g \\ g = q_2 r_1 + r_2 \Rightarrow r_2 = g - q_2 r_1 = g - q_2(f - q_1 g) = \\ \qquad = f q_2 + g(1 + q_1 q_2) \\ r_1 = q_3 r_2 + r_3 \Rightarrow r_3 = r_1 - q_3 r_2 \\ r_2 = q_4 r_3 + r_4 \Rightarrow r_4 = r_2 - q_4 r_3 = r_2 - q_4(r_1 - q_3 r_2) \circleddash \\ r_3 = q_5 r_4 \end{cases}$$

$$\circleddash\; f q_2 + g(1+q_1q_2) + q_4\big(f + q_1 g + q_3(f q_2 + g(1+q_1q_2))\big) =$$

$$= \underline{f q_2} + g(1+q_1q_2) + \underline{f q_4} + q_1 q_4 g + q_4 q_3(\underline{f q_2} + g(1+q_1q_2)) =$$

$$= f(q_2 + q_4 + q_2 q_3 q_4) + g(1 + q_1q_2 + q_1q_4 + q_3q_4 + q_1q_2q_3q_4)$$

$$\Rightarrow 1 = f\big(\underbrace{x+1+x+1}_{=0} + (x+1)^2 x\big) + g\big(1 + \overbrace{x(x+1) + x(x+1)}^{=0} + x(x+1) + x^2(x+1)^2\big)$$

$$1 = f\left(x(x^2+\overset{0}{\cancel{2x}}+1)\right) + g\left(1+x^2+x+x^2(x^2+\overset{0}{\cancel{2x}}+1)\right)$$

$$1 = f(x^3+x) + g(x^2+x+1+x^4+x^2)$$

$$= 0$$

$$\boxed{1 = f(x^3+x) + g(x^4+x+1)}$$

№5.

$f(x) = 3x^3+3x^2+x+2$; $g(x) = 2x^2+x+2$; над F_5.

1)

$$\begin{array}{r|l} \overset{f}{3x^3+3x^2+x+2} & \overset{g}{2x^2+x+2} \\ -\;3x^3+4x^2+3x & \overline{4x+2} \quad q_1 \\ \hline 4x^2+3x+2 & \\ -\;4x^2+2x+4 & \\ \hline x+3 \quad r_1 & \end{array}$$

2) $$\begin{array}{l|l} 2x^2 + x + 2 \; (g) & x + 3 \; (r_1) \\ -(2x^2 + x) & 2x \; (q_2) \\ \hline \quad 2 \; (r_2) & \end{array}$$

3) $$\begin{array}{l|l} x + 3 \; (r_1) & 2 \; (r_2) \\ -x & 3x + 4 \; (q_3) \\ \hline 0 + 3 & \\ -3 & \\ \hline \quad 0 \; (r_3) & \end{array}$$

$$\Rightarrow mcd(f, g) = 1 \quad (= 2 \cdot 3)$$

$$1 = 3\left(-f \cdot (2x) + g\left(1 + (4x+2)(2x)\right)\right)$$

$$1 = f \cdot (-6x) + g\left(3 + \underline{3}(4x+2)(\underline{2x})\right)$$

$$1 = f(4x) + g\left(3 + x(4x+2)\right)$$

$$\boxed{1 = f \cdot (4x) + g\left(4x^2 + 2x + 3\right)}$$

№6

$f(x) = 3x^3 + 5x^2 + x + 2$, $g(x) = x^2 + 6x + 1$; над F_2

1)
$$\begin{array}{r|l} 3x^3 + 5x^2 + x + 2 \; (f) & x^2 + 6x + 1 \; (g) \\ -\;3x^3 + 4x^2 + 3x & 3x + 1 \; (q_1) \\ \hline x^2 + 5x + 2 & \\ -\;x^2 + 6x + 1 & \\ \hline 6x + 1 \; (r_1) & \end{array}$$

2)
$$\begin{array}{r|l} x^2 + 6x + 1 \; (g) & 6x + 1 \; (r_1) \\ -\;x^2 + 6x & 6x \; (q_2) \\ \hline 1 \; (r_2) & \end{array}$$

3)
$$\begin{array}{r|l} 6x + 1 \; (r_1) & 1 \; (r_2) \\ -\;6x + 1 & 6x + 1 \; (q_3) \\ \hline 0 \; (r_3) & \end{array}$$

$\Rightarrow$ НОД$(f, g) = 1$

$$1 = -f \cdot (6x) + g\big(1 + (3x+1)(6x)\big)$$

$$1 = f(x) + g(1 + 4x^2 + 6x)$$

$$\boxed{1 = f \cdot x + g \cdot (4x^2 + 6x + 1)}$$