Д39

№1

$$a_n = \ln(n+1) \Rightarrow \boxed{a_n = \ln(e^{a_{n-1}}+1)} \quad a_0 = 0$$

$\Downarrow$ Ответ

$$a_{n-1} = \ln(n) \Rightarrow n = e^{a_{n-1}}$$

b) $$a_n = \frac{1}{(n+1)(n+2)}$$

$$\Delta a_n = a_{n+1} - a_n = \frac{1}{(n+2)(n+3)} - \frac{1}{(n+1)(n+2)} =$$

$$= \frac{(n+1)-(n+3)}{(n+1)(n+2)(n+3)} = \frac{-2}{(n+1)(n+2)(n+3)}$$

$$\Delta^2 a_n = \Delta a_{n+1} - \Delta a_n = \frac{-2}{(n+2)(n+3)(n+4)} - \frac{-2}{(n+1)(n+2)(n+3)} =$$

$$= \frac{-2(n+1)+2(n+4)}{(n+1)(n+2)(n+3)(n+4)} = \frac{6}{(n+1)(n+2)(n+3)(n+4)} =$$

$$= 6 \cdot \underbrace{\frac{1}{(n+1)(n+2)}}_{a_n} \underbrace{\frac{1}{(n+3)(n+4)}}_{a_{n+2}}$$

$$\Delta^2 a_n = \Delta a_{n+1} - \Delta a_n = a_{n+2} - a_{n+1} - a_{n+1} + a_n =$$

$$= a_{n+2} - 2a_{n+1} + a_n$$

$$\Rightarrow a_{n+2} - 2a_{n+1} + a_n = 6a_n a_{n+2}$$

$$\Rightarrow a_{n+2} = \frac{2a_{n+1} - a_n}{1 - 6a_n}$$

$$\Downarrow$$

$$\boxed{a_n = \frac{2a_{n-1} - a_{n-2}}{1 - 6a_{n-2}}}$$

$a_0 = 1/2$

$a_1 = 1/6$

$a_2 = 1/12$

$a_3 = 1/20$

Ответ:

c) $a_n = n + \sqrt{n}$

$\Downarrow$

$n + \sqrt{n} - a_n = 0$

$D = 1 + 4a_n \Rightarrow \sqrt{n} = \dfrac{-1 \pm \sqrt{1+4a_n}}{2}$

$\dfrac{-1-\sqrt{1+4a_n}}{2} < 0 \Rightarrow \sqrt{n} = \dfrac{-1+\sqrt{1+4a_n}}{2}$

$\Downarrow$

$$n = \left(\frac{-1+\sqrt{1+4a_n}}{2}\right)^2 =$$

$$= \frac{1 - 2\sqrt{1+4a_n} + 1 + 4a_n}{4} = \frac{1+2a_n - \sqrt{1+4a_n}}{2}$$

$$\Rightarrow n-1 = \frac{1+2a_{n-1} - \sqrt{1+4a_{n-1}}}{2}$$

$$\Rightarrow n = 1 + \frac{1+2a_{n-1} - \sqrt{1+4a_{n-1}}}{2} = \frac{3+2a_{n-1} - \sqrt{1+4a_{n-1}}}{2}$$

$$\Rightarrow a_n = n + \sqrt{n} = \frac{3+2a_{n-1} - \sqrt{1+4a_{n-1}}}{2} + \sqrt{\frac{3+2a_{n-1} - \sqrt{1+4a_{n-1}}}{2}}$$

$a_0 = 0$

Ответ:

№2

$$P(n) = \alpha n^m + a_{m-1} n^{m-1} + \ldots + a_1 n + a_0$$

Очевидно, что $\forall t \ \Delta^t P(n) = a_{k_t} n^k + \ldots + a_{1_t} n + a_{0_t}$ (некоторая линейная комбинация многочленов, которой является $\Delta^t P(n)$ по формуле конечных разностей, является многочленом).

Пусть $\Delta^{t-1} P(n) = b_k x^k + \ldots + b_1 x + b_0$

$$\Delta^{t-1} P(n+1) = b_k (x+1)^k + \ldots + b_1 (x+1) + b_0$$

$$\Rightarrow \Delta^t P(n) = \Delta^{t-1} P(n+1) - \Delta^{t-1} P(n) =$$

$$\cancel{= (b_k x^k + b_{k-1} x^{k-1} + \ldots + b_1 x + b_0) -}$$

$$= (b_k (x+1)^k + b_{k-1} (x+1)^{k-1} + \ldots + b_1 (x+1) + b_0) -$$
$$- (b_k x^k + b_{k-1} x^{k-1} + \ldots + b_1 x + b_0)$$

$\Rightarrow$ в $\Delta^t P(n)$:

при x^k: $b_k x^k - b_k x^k = 0$

при x^{k-1}:

$$b_k \cdot x^{k-1} \cdot (1)^1 \cdot C_k^1 + b_{k-1} x^{k-1} \cdot (1)^0 \cdot C_k^0 -$$
$$- b_{k-1} x^{k-1}$$ — из формулы разложения $(x+1)^k$

$$\parallel$$

$b_k C_k^1 x^{k-1} \neq 0 \Rightarrow b_k C_k^1$ — старший коэфф.

$$в\ \Delta^0 P(n):\ \alpha - \text{старш. коэфф.}$$

$$\Downarrow$$

$$в\ \Delta^1 P(n):\ m\cdot\alpha\ \ (C_m^1\cdot\alpha)$$

$$\Downarrow$$

$$в\ \Delta^t P(n):\ \alpha\cdot\prod_{i=0}^{t-1}(m-i) =$$

$$= \alpha\cdot m\cdot(m-1)\cdot\ldots\cdot(m-t+1)$$

На каждом шаге степень многочлена уменьшается на 1, а коэффициент умножается на старый старший показатель степени

$$\Delta^m n^m \Leftrightarrow P(n) = n^m \Rightarrow \alpha = 1$$

$$\Rightarrow \text{Старш. коэфф.: } \prod_{i=0}^{m-1}(m-i) =$$

$$= m(m-1)\ldots(m-m+1) =$$

$$= m(m-1)\cdot\ldots\cdot 2\cdot 1 = m!$$

Ответ: 1) $\alpha\prod_{i=0}^{t-1}(m-i)$

2) $m!$

№3

$$\sum_{k=0}^{m}(-1)^k C_m^k (m-k)^m = m!$$

Пусть последовательность a задана так:

(*) $a_{n-k} = (n-k)^m$; $a_n = n^m$

Тогда $\forall n$ $\Delta^m a_n = m!$ по задаче 2.

$\Rightarrow \Delta^m a_0 = m!$ (1)

Но по формуле конечных разностей,

$$\Delta^m a_n = \sum_{k=0}^{m} (-1)^k C_m^k \cdot a_{n+m-k}$$

При $n=0$: $\Delta^m a_0 = \sum_{k=0}^{m} (-1)^k C_m^k \underbrace{a_{m-k}}_{\overset{\text{по } (*)}{=} (m-k)^m}$

$$\Rightarrow \Delta^m a_0 = \sum_{k=0}^{m} (-1)^k C_m^k (m-k)^m \quad (2)$$

$\Rightarrow$ по (1) и (2): $\Delta^m a_0 = \underbrace{m! = \sum_{k=0}^{m} (-1)^k C_m^k (m-k)^m}_{\text{ч.т.д.}}$

№4.

$$\Delta \frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{a_{n+1}b_n - b_{n+1}a_n}{b_n b_{n+1}} =$$

$$= \frac{a_{n+1}b_n \overbrace{- a_n b_n + a_n b_n}^{+0} - a_n b_{n+1}}{b_n b_{n+1}} =$$

$$= \frac{b_n(a_{n+1}-a_n) - a_n(b_{n+1}-b_n)}{b_n b_{n+1}} = \frac{b_n \Delta a_n - a_n \Delta b_n}{b_n b_{n+1}}$$

№5

$$\Delta \frac{\overset{a_n}{n^2}}{\underset{b_n}{(-3)^n}} = \frac{(-3)^n \Delta n^2 - n^2 \Delta (-3)^n}{(-3)^n (-3)^{n+1}} =$$

по №4

$$= \frac{(-3)^n\left((n+1)^2 - n^2\right) - n^2\left((-3)^{n+1} - (-3)^n\right)}{(-3)^n (-3)^{n+1}} =$$

$$= \frac{(-3)^n (2n+1) - n^2\left((-3)^n(-3-1)\right)}{(-3)^{2n+1}} =$$

$$= \frac{(-3)^n (2n+1) + 4n^2 \cdot (-3)^n}{(-3)^{2n+1}} =$$

$$= \frac{(-3)^n (4n^2 + 2n + 1)}{(-3)^{2n+1}} = \boxed{\frac{4n^2+2n+1}{(-3)^{n+1}}}$$

Ответ: ↗

№6

$$\Delta \cos(\alpha n + \beta) = \cos\left(\alpha(n+1) + \beta\right) - \cos(\alpha n + \beta) =$$

$$= -2 \sin\left(\frac{\alpha(n+1) + \beta + \alpha n + \beta}{2}\right) \sin\left(\frac{\alpha(n+1) + \beta - (\alpha n + \beta)}{2}\right) =$$

$$= -2 \sin\left(\frac{2\alpha n + \alpha + 2\beta}{2}\right) \sin\left(\frac{\alpha}{2}\right) =$$

$$= \boxed{-2\sin\left(\alpha n+\beta+\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)}$$

Ответ:

№7.

$\Sigma\, n^4$, a_n

$a_0=0$	$\Delta a_0=1$	$\Delta^2 a_0=14$	$\Delta^3 a_0=36$	$\Delta^4 a_0=24$
$a_1=1$	$\Delta a_1=15$	$\Delta^2 a_1=50$	$\Delta^3 a_1=60$	$\Delta^4 a_1=24$
$a_2=16$	$\Delta a_2=65$	$\Delta^2 a_2=110$	$\Delta^3 a_2=84$	
$a_3=81$	$\Delta a_3=175$	$\Delta^2 a_3=194$		
$a_4=256$	$\Delta a_4=369$			
$a_5=625$				$\Delta^5 a_0=0$

Аналогично задаче 2 нетрудно заметить, $\cancel{\Delta^5 a_i = 0\ \forall i}$

что $\forall i\ \ \Delta^5 a_i = 0 \Rightarrow \forall t \geq 5\ \ \Delta^t a_0 = 0$

по доказанному, $\sum a_n = a_0 n + \frac{\Delta a_0}{2} n^{(2)} + \frac{\Delta^2 a_0}{3!} n^{(3)} + \ldots$

$$\Rightarrow \sum n^4 = \frac{1}{2}\cdot n(n-1) + \frac{14}{3!} n(n-1)(n-2) + \frac{36}{4!} n(n-1)(n-2)(n-3) + \frac{24}{5!} n(n-1)(n-2)(n-3)(n-4) + \sum_{k=6}^{\infty} \frac{0}{k!} n^{(k)}$$

Ответ:
$$= \boxed{\frac{n(n-1)}{2} + \frac{7n(n-1)(n-2)}{3} + \frac{3n(n-1)(n-2)(n-3)}{2} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5}}$$

№8.

a) $a_n = \frac{1}{(n+1)(n+3)}$

$$\sum a_n = \sum \frac{1}{(n+1)(n+3)} = \sum \left(\frac{1}{n+1} - \frac{1}{n+3}\right) \cdot \frac{1}{2} =$$

$$= \frac{1}{2} \sum \frac{1}{n+1} - \frac{1}{2} \sum \frac{1}{n+3} =$$

$$= \frac{1}{2}\left(\frac{1}{0+1} + \frac{1}{1+1} + \ldots + \frac{1}{(n-1)+1}\right) -$$

$$- \frac{1}{2}\left(\frac{1}{0+3} + \ldots + \frac{1}{(n-1)+3}\right) =$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) - \frac{1}{2}\left(\frac{1}{3} + \ldots + \frac{1}{n+2}\right) =$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \ldots + \cancel{\frac{1}{n}} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \ldots - \cancel{\frac{1}{n}} - \frac{1}{n+1} - \frac{1}{n+2}\right) =$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) =$$

$$= \frac{3}{4} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) =$$

$$= \frac{3}{4} - \frac{1}{2}\left(\frac{n+2+n+1}{(n+1)(n+2)}\right) =$$

$$= \boxed{\frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}}$$

Ответ.

$$\lim_{n\to\infty} \sum a_n = \lim_{n\to\infty}\left(\frac{3}{4} - \frac{2n+3}{2n^2+6n+4}\right) = \boxed{3/4}$$

b) $a_n = \frac{n^2}{(-3)^n}$

• Заметим, что $\Delta \frac{-3/4}{(-3)^n} = \frac{-3/4}{(-3)^{n+1}} - \frac{-3/4}{(-3)^n} =$

$$= \frac{-\frac{3}{4} + \frac{3}{4}\cdot(-3)}{(-3)^{n+1}} = \frac{-12}{4(-3)^{n+1}} = \frac{1}{(-3)^n}$$

$$\Rightarrow a_n = n^2 \cdot \Delta \frac{-3/4}{(-3)^n}$$

$$\Rightarrow \sum a_n = \sum n^2 \cdot \Delta \frac{-3/4}{(-3)^n} =$$

$$= \left(n^2 \cdot \frac{-3/4}{(-3)^n} - 0\right) - \sum \frac{-3/4}{(-3)^{n+1}} \underbrace{\Delta n^2}_{(n+1)^2 - n^2 = 2n+1} =$$

$$= \frac{n^2}{4\cdot(-3)^{n-1}} - \sum \frac{2n+1}{4\cdot(-3)^n} =$$

$$= \frac{n^2}{4\cdot(-3)^{n-1}} - \frac{1}{2}\sum \frac{n}{(-3)^n} - \frac{1}{4}\sum \frac{1}{(-3)^n} \ominus$$

• $\sum \frac{1}{(-3)^n}$ – геом. прогрессия с $b_1 = 1$, $q = \frac{1}{-3}$

$$\Rightarrow \sum \left(\frac{1}{-3}\right)^n = \frac{b_1(1-q^n)}{1-q} = \frac{1\left(1 - \frac{1}{(-3)^n}\right)}{1 - \left(\frac{1}{-3}\right)} =$$

$$= \frac{\frac{(-3)^n - 1}{(-3)^n}}{4/3} = \frac{((-3)^n - 1)\cdot 3}{4\cdot(-3)^n} = \frac{(-3)^n - 1}{-4\cdot(-3)^{n-1}} = \frac{1-(-3)^n}{4\cdot(-3)^{n-1}}$$

$$\bullet \sum \frac{n}{(-3)^n} = \sum n \cdot \Delta \frac{-3/4}{(-3)^n} =$$

$$= \left(n \cdot \frac{-3/4}{(-3)^n} - 0 \right) - \sum \frac{-3/4}{(-3)^{n+1}} \underbrace{\Delta n}_{=1} =$$

$$= \frac{-3n}{4 \cdot (-3)^n} - \frac{1}{4} \sum \frac{1}{(-3)^n} =$$

$$= \frac{n}{4 \cdot (-3)^{n-1}} - \frac{1}{4} \cdot \frac{1-(-3)^n}{4 \cdot (-3)^{n-1}} =$$

$$= \frac{4n - 1 + (-3)^n}{16 \cdot (-3)^{n-1}}$$

$$= \frac{n^2}{4 \cdot (-3)^{n-1}} - \frac{1}{2} \cdot \frac{4n-1+(-3)^n}{16 \cdot (-3)^{n-1}} - \frac{1}{4} \cdot \frac{1-(-3)^n}{4 \cdot (-3)^{n-1}} =$$

$$= \frac{8n^2 - 4n + 1 - (-3)^n - 2 + 2(-3)^n}{32 \cdot (-3)^{n-1}} =$$

$$= \boxed{\frac{8n^2 - 4n - 1 + (-3)^n}{32 \cdot (-3)^{n-1}}}$$

Ответ:

$$\lim_{n \to \infty} \sum a_n = \lim_{n \to \infty} \frac{8n^2 - 4n - 1 + (-3)^n}{32 \cdot (-3)^{n-1}} =$$

$$= \lim_{n\to\infty} \cancel{\frac{8n^2}{32\cdot(-3)^{n-1}}}^{\to 0} + \lim_{n\to\infty} \cancel{\frac{-4n-1}{(-3)^{n-1}\cdot 32}}^{\to 0} + \lim_{n\to\infty} \frac{(-3)^n}{32\cdot(-3)^{n-1}} =$$

$$= \lim_{n\to\infty} \frac{-3}{32} = \boxed{-3/32}$$

(I способ) №8

c) $\sum \cos(\alpha n+\beta) = \sum \dfrac{e^{i(\alpha n+\beta)}+e^{-i(\alpha n+\beta)}}{2} =$

по формуле Эйлера

$= \dfrac{1}{2}\left(\sum e^{i\alpha n+\beta i} + \sum e^{-i\alpha n-\beta i}\right) =$

• $\sum e^{i\alpha n+\beta i} = e^{\beta i} + e^{\beta i+\alpha i} + e^{\beta i+2\alpha i} + \ldots + e^{\beta i+(n-1)\alpha i}$ – геометрическая прогрессия с $b_1 = e^{\beta i}$; $q = e^{\alpha i}$

$$\Rightarrow \sum e^{i\alpha n+\beta i} = \frac{e^{\beta i}\left(1-\left(e^{i\alpha}\right)^n\right)}{1-e^{i\alpha}} =$$

$$= \frac{e^{\beta i}\left(1-e^{i\alpha n}\right)}{1-e^{i\alpha}}$$

$$\sum e^{-i\alpha n-\beta i} = \frac{e^{-\beta i}\left(1-\left(e^{-i\alpha}\right)^n\right)}{1-e^{-i\alpha}} =$$

$$= \frac{e^{-\beta i}\left(1-e^{-i\alpha n}\right)}{1-e^{-i\alpha}}$$

$$= \boxed{\frac{1}{2}\left(\frac{e^{\beta i}\left(1-e^{i\alpha n}\right)}{1-e^{i\alpha}} + \frac{e^{-\beta i}\left(1-e^{-i\alpha n}\right)}{1-e^{-i\alpha}}\right)}$$

Ответ:

№8с. (II способ)

$$\sum \cos(\alpha n+\beta) = \sum \cos(\alpha n+\beta)\cdot \frac{2\sin\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}} =$$

$$= \frac{1}{2\sin\frac{\alpha}{2}} \cdot \sum 2\cos(\alpha n+\beta)\sin\frac{\alpha}{2} =$$

• $2\cos x \sin y = \sin(x+y) - \sin(x-y)$

$$= \frac{1}{2\sin\alpha/2} \cdot \sum \sin\left(\alpha n+\beta+\frac{\alpha}{2}\right) - \sin\left(\alpha n+\beta-\frac{\alpha}{2}\right) =$$

$$= \frac{1}{2\sin\frac{\alpha}{2}}\left(\sum \sin\left(\frac{\alpha(2n+1)}{2}+\beta\right) - \sum \sin\left(\frac{\alpha(2n-1)}{2}+\beta\right)\right) =$$

n	0	1		$n-2$	$n-1$
$\sin\frac{\alpha(2n+1)}{2}+\beta$	$\sin\beta+\frac{\alpha}{2}$	$\sin\beta+\frac{3\alpha}{2}$	...	$\sin\left(\frac{\alpha(2n-3)}{2}+\beta\right)$	$\sin\left(\frac{\alpha(2n-1)}{2}+\beta\right)$
$\sin\frac{\alpha(2n-1)}{2}+\beta$	$\sin\beta-\frac{\alpha}{2}$	$\sin\beta+\frac{\alpha}{2}$	...	$\sin\left(\frac{\alpha(2n-5)}{2}+\beta\right)$	$\sin\left(\frac{\alpha(2n-3)}{2}+\beta\right)$

$$= \frac{1}{2\sin\alpha/2}\left(\sin\left(\frac{\alpha(2n-1)}{2}+\beta\right) - \sin\left(\beta-\frac{\alpha}{2}\right)\right) =$$

$$= \frac{2\cos\left(\frac{2n\alpha-\alpha+2\beta+2\beta-\alpha}{2}\right)\sin\left(\frac{2n\alpha-\alpha+2\beta-2\beta+\alpha}{2}\right)}{2\sin\alpha/2} =$$

$$= \frac{\cos\left(\frac{2n\alpha - 2\alpha + 4\beta}{4}\right) \sin\left(\frac{2n\alpha}{4}\right)}{\sin \alpha/2} =$$

$$\cancel{= \frac{\cos(n\alpha - \alpha + 2\beta) \sin(n\alpha)}{\sin \frac{\alpha}{2}}}$$

$$= \frac{\cos((n\alpha - \alpha + 2\beta)/2) \sin n\alpha/2}{\sin \alpha/2} =$$

$$= \boxed{\frac{\cos\left(\frac{\alpha(n-1)}{2} + \beta\right) \sin\left(\frac{n\alpha}{2}\right)}{\sin \frac{\alpha}{2}}}$$

Если $\sin \frac{\alpha}{2} = 0$, то
если $\alpha = 2\pi k$, то
$\cos(\alpha n + \beta) = \cos\beta \Rightarrow$
$\Rightarrow \sum \cos(\alpha n + \beta) =$
$= n \cos\beta$

Ответ:

№9

$$\Delta a_n^3 = a_{n+1}^3 - a_n^3 =$$

$$= (a_{n+1} - a_n)(a_{n+1}^2 + a_{n+1}a_n + a_n^2) =$$

$$= \Delta a_n (a_{n+1}^2 - 2a_{n+1}a_n + a_n^2 + 3a_{n+1}a_n) =$$

$$= \Delta a_n ((a_{n+1} - a_n)^2 + 3a_{n+1}a_n) =$$

$* \ \Delta a_n = a_{n+1} - a_n \Rightarrow a_{n+1} = a_n + \Delta a_n$

$$= \Delta a_n ((\Delta a_n)^2 + 3a_n(a_n + \Delta a_n)) =$$

$$= (\Delta a_n)^3 + 3a_n^2 \Delta a_n + 3a_n(\Delta a_n)^2 + a_n^3 - a_n^3 =$$

$$= \boxed{(a_n + \Delta a_n)^3 - a_n^3}$$

Ответ: ↑

№10

$$3\sum n^5 = 4\left(\sum n\right)^3 - \left(\sum n\right)^2$$

$\bullet \ \sum n = 0 + 1 + \ldots + (n-1) = \frac{n(n-1)}{2}$

$$\Rightarrow 4\left(\sum n\right)^3 - \left(\sum n\right)^2 = 4\left(\frac{n(n-1)}{2}\right)^3 - \left(\frac{n(n-1)}{2}\right)^2 =$$

$$= \frac{\cancel{4}^{2} n^3(n-1)^3}{4\cdot\cancel{2}} - \frac{n^2(n-1)^2}{4} = \frac{2n^3(n-1)^3 - n^2(n-1)^2}{4} =$$

$$= \frac{n^2(n-1)^2\left(2n(n-1)-1\right)}{4} = \frac{n^2(n-1)^2(2n^2-2n-1)}{4}$$

Пусть $a_n = 3\sum n^5$

$$b_n = 4\left(\sum n\right)^3 - \left(\sum n\right)^2 = \frac{n^2(n-1)^2(2n^2-2n-1)}{4}$$

$$\Delta a_n = \Delta\left(3\sum n^5\right) = 3\,\Delta\sum n^5 = 3n^5$$

$$\Delta b_n = \frac{(n+1)^2(n)^2\left(2(n+1)^2-2(n+1)-1\right)}{4} - \frac{n^2(n-1)^2(2n^2-2n-1)}{4} =$$

$$= \frac{(n+1)^2 n^2(2n^2+2n-1) - n^2(n-1)^2(2n^2-2n-1)}{4} =$$

$$= \frac{n^2\left((n^2+2n+1)(2n^2+2n-1) - (n^2-2n+1)(2n^2-2n-1)\right)}{4} =$$

$$= \frac{n^2}{4}\left(\cancel{2n^4}+2n^3-\cancel{n^2}+4n^3+\cancel{4n^2}-\cancel{2n}+\cancel{2n^2}+\cancel{2n}-\cancel{1}-\cancel{2n^4}+2n^3+\cancel{n^2}+4n^3-\cancel{4n^2}-\cancel{2n}-\cancel{2n^2}+\cancel{2n}+\cancel{1}\right) =$$

$$= \frac{n^2}{4}\cdot 12n^3 = 3n^5$$

$a_0 = 0$; $b_0 = 0$, то есть $a_0 = b_0$ и $\Delta a_n = \Delta b_n\ \forall n$

$\Rightarrow a_n = b_n$, ч.т.д.

№11

$$(a+b)^{(n)} = \sum_{k=0}^{n} C_n^k a^{(k)} b^{(n-k)}$$

1) $n=0 \Rightarrow (a+b)^{(0)} = C_0^0 a^{(0)} b^{(0)}$

$1 = 1 \Rightarrow$ верно.

2) Пусть верно при $n \in N$. Докажем, что верно при $n+1$.

$$(a+b)^{(n+1)} = (a+b)^{(n)}(a+b-(n+1)+1) =$$

$$= \underbrace{\left(\sum_{k=0}^{n} C_n^k a^{(k)} b^{(n-k)}\right)}_{\substack{\text{Так как} \\ \text{верно для } n}} \cdot (a+b-n) =$$

$$= \sum_{k=0}^{n} C_n^k a^{(k)} b^{(n-k)} \left(a + b - n \overbrace{-k+k}^{=0}\right) =$$

$$= \sum_{k=0}^{n} C_n^k a^{(k)} b^{(n-k)} (b-n+k) + C_n^k a^{(k)} (a-k) b^{(n-k)} =$$

$$= \sum_{k=0}^{n} \underbrace{\overbrace{C_n^k a^{(k)} b^{(n-k+1)}}^{u_k} + \overbrace{C_n^k a^{(k+1)} b^{(n-k)}}^{v_k}}_{s_k} =$$

$$= u_0 + v_0 + \ldots + u_k + v_k =$$

$$= \underbrace{\left(C_n^0 a^{(0)} b^{(n+1)}\right)}_{=u_0} + \left(\sum_{k=1}^{n} \underbrace{C_n^k a^{(k)} b^{(n+1-k)}}_{u_k} + \underbrace{C_n^{k-1} a^{(k)} b^{(n+1-k)}}_{v_{k-1}}\right) +$$

$$+ \underbrace{C_n^n a^{(n+1)} b^{(0)}}_{v_k} =$$

Учитывая, что
$C_n^0 = C_{n+1}^0 = 1$
$C_n^n = C_{n+1}^{n+1} = 1$

$$= C_{n+1}^0 a^{(0)} b^{(n+1)} + \sum_{k=1}^{n} \left(C_n^k + C_n^{k-1}\right) a^{(k)} b^{(n+1-k)} +$$

$$+ C_{n+1}^{n+1} a^{(n+1)} b^{(0)} =$$

$$\cdot\ C_n^k + C_n^{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} =$$

$$= \frac{n!(n-k+1) + n!\,k}{k!(n-k+1)!} = \frac{n!(n-\cancel{k}+1+\cancel{k})}{k!(n-k+1)!} =$$

$$= \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = C_{n+1}^k$$

$$= C_{n+1}^0 a^{(0)} b^{(n+1)} + \sum_{k=1}^{n} C_{n+1}^k a^{(k)} b^{(n+1-k)} +$$

$$+ C_{n+1}^{n+1} a^{(n+1)} b^{(0)} =$$

$$= \sum_{k=0}^{n+1} C_{n+1}^k a^{(k)} b^{(n+1-k)}$$

$\Rightarrow$ формула верна для $n+1$

$\Rightarrow$ по принципу математической индукции, формула верна при всех $n \in \mathbb{N}$.

ч.т.д.

№12.

а) $a_0 = 3,\ a_1 = 4$

$$a_{n+2} = 4a_{n+1} + 5a_n$$

$$a_{n+2} - 4a_{n+1} - 5a_n = 0$$

$$x^2 - 4x - 5 = 0$$ – хар. уравнение

$$D = 16 + 4 \cdot 5 = 36$$

$$x = \frac{4 \pm 6}{2} = \left[\begin{array}{l} 5 \\ -1 \end{array}\right.$$

$$\Rightarrow a_n = C_0 \cdot (-1)^n + C_1 \cdot 5^n$$

$$\begin{cases} a_0 = C_0 + C_1 = 3 \\ a_1 = -C_0 + 5C_1 = 4 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 5 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 6 & 7 \end{array}\right) \longrightarrow$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 11/6 \\ 0 & 1 & 7/6 \end{array}\right) \Rightarrow C_0 = 11/6;\ C_1 = 7/6$$

$$\Rightarrow a_n = \boxed{\frac{11}{6}(-1)^n + \frac{7}{6}\cdot 5^n}$$

Ответ:

b) $a_{n+2} = -4a_{n+1} - 5a_n$

$a_{n+2} + 4a_{n+1} + 5a_n = 0$

$x^2 + 4x + 5 = 0$ – хар. уравнение

$D = 16 - 20 = -4$

$$x = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

$$\Rightarrow a_n = C_0(-2-i)^n + C_1(-2+i)^n$$

$a_0 = 3 = C_0 + C_1$

$a_1 = 4 = C_0(-2-i) + C_1(-2+i)$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ -2-i & -2+i & 4 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -2-i & -2-i & -6-3i \\ -2-i & -2+i & 4 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2i & 10+3i \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & \frac{3}{2}-5i \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{2}+5i \\ 0 & 1 & \frac{3}{2}-5i \end{array}\right)$$

$\Rightarrow C_0 = \frac{3}{2} + 5i, \; C_1 = \frac{3}{2} - 5i$

$\Rightarrow \cancel{\forall n} \; a_n = \boxed{\left(\frac{3}{2}+5i\right)(-2-i)^n + \left(\frac{3}{2}-5i\right)(-2+i)^n}$

Ответ:

c) $a_{n+2} = 10a_{n+1} - 25a_n$

$a_{n+2} - 10a_{n+1} + 25a_n = 0$

$x^2 - 10x + 25 = 0$ - хар. уравнение

~~$D = 100 - 100$~~

$(x-5)^2 = 0$

$\Rightarrow a_n = (C_0 + C_1 n) \cdot 5^n$

$a_0 = 3 = (C_0 + C_1 \cdot 0) \cdot 5^0 = C_0$

$a_1 = 4 = (C_0 + C_1) \cdot 5 = 5C_0 + 5C_1 \Rightarrow \begin{cases} C_0 = 3 \\ C_1 = -\frac{11}{5} \end{cases}$

$\Rightarrow a_n = \left(3 - \frac{11}{5}n\right) \cdot 5^n = \boxed{3 \cdot 5^n - 11n \cdot 5^{n-1}}$

Ответ

№13

$$F_n^2 + F_{n+1}^2 = F_{2n+1}$$

Известно, что $F_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

Пусть $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$. $\Rightarrow F_n = \dfrac{x_1^n - x_2^n}{\sqrt{5}}$

$$\Rightarrow F_n^2 + F_{n+1}^2 = \left(\frac{x_1^n - x_2^n}{\sqrt{5}}\right)^2 + \left(\frac{x_1^{n+1} - x_2^{n+1}}{\sqrt{5}}\right)^2 =$$

$$= \frac{x_1^{2n} - 2x_1^n x_2^n + x_2^{2n}}{5} + \frac{x_1^{2n+2} - 2x_1^{n+1} x_2^{n+1} + x_2^{2n+2}}{5} =$$

$$= \frac{x_1^{2n}(1+x_1^2)+x_2^{2n}(1+x_2^2)-2x_1^n x_2^n(1+x_1x_2)}{5} =$$

$$\cdot\ 1+x_1^2 = 1+\left(\frac{1+\sqrt5}{2}\right)^2 =$$

$$= 1+\frac{1+2\sqrt5+5}{4} = \frac{5+2\sqrt5+5}{2} =$$

$$= \frac{10+2\sqrt5}{4} = \frac{5+\sqrt5}{4}$$

$$\cdot\ 1+x_2^2 = 1+\left(\frac{1-\sqrt5}{2}\right)^2 = 1+\frac{1-2\sqrt5+5}{4} = \frac{5-2\sqrt5+5}{4} =$$

$$= \frac{5-\sqrt5}{2}$$

$$\cdot\ x_1x_2 = \left(\frac{1+\sqrt5}{2}\right)\left(\frac{1-\sqrt5}{2}\right) = \frac{(1+\sqrt5)(1-\sqrt5)}{4} = \frac{-4}{4} = -1$$

$$= \frac{x_1^{2n}\cdot\frac{5+\sqrt5}{2}+x_2^{2n}\cdot\frac{5-\sqrt5}{2}-\cancel{2x_1^nx_2^n(1-1)}^{\,0}}{5} =$$

$$= \frac{x_1^{2n}(5+\sqrt5)+x_2^{2n}(5-\sqrt5)}{10} =$$

$$= \frac{\sqrt5\cdot x_1^{2n}\cdot\overset{x_1}{\frac{1+\sqrt5}{2}}+x_2^{2n}\cdot\overset{-x_2}{\frac{\sqrt5-1}{2}}\cdot\sqrt5}{5} =$$

$$= \frac{\left(x_1^{2n+1}-x_2^{2n+1}\right)\sqrt5}{\sqrt5\cdot\sqrt5} = \frac{x_1^{2n+1}-x_2^{2n+1}}{\sqrt5} = F_{2n+1}$$

2.T.9.

№14

$$a_n = n\Delta a_n + (\Delta a_n)^2$$

$$\Downarrow$$

$$\Delta a_n = \Delta\left(n\Delta a_n + (\Delta a_n)^2\right)$$

$$\Delta a_n = \Delta(n\Delta a_n) + \Delta\left((\Delta a_n)^2\right)$$

$$\Delta a_n = n\Delta^2 a_n + \Delta a_{n+1}\Delta n + \Delta\left((a_{n+1}-a_n)^2\right)$$

$$\Delta a_n = n\Delta^2 a_n + \Delta a_{n+1} + \Delta\left(a_{n+1}^2 - 2a_n a_{n+1} + a_n^2\right) =$$

$$\Delta a_n = n\Delta^2 a_n + \Delta a_{n+1} + \left(a_{n+2}^2 - 2a_{n+1}a_{n+2} + \cancel{a_{n+1}^2} - \cancel{a_{n+1}^2} + 2a_n a_{n+1} - a_n^2\right)$$

$$\Delta a_n = n\Delta^2 a_n + \Delta a_{n+1} + (a_{n+2} - a_n)(a_{n+2} + a_n) - 2a_{n+1}(a_{n+2} - a_n)$$

$$\Delta a_n = n\Delta^2 a_n + \Delta a_{n+1} + (a_{n+2} - a_n)\underbrace{(a_{n+2} - 2a_{n+1} + a_n)}_{\Delta^2 a_n}$$

$$\underbrace{\Delta a_n - \Delta a_{n+1}}_{-\Delta^2 a_n} = \Delta^2 a_n\left(n + a_{n+2} - a_n\right)$$

$$\Rightarrow \Delta^2 a_n\left(a_{n+2} - a_n + n + 1\right) = 0$$

$$\left[\begin{array}{l} \Delta^2 a_n = 0 \Leftrightarrow a_{n+2} - 2a_{n+1} + a_n = 0 \\ a_{n+2} - a_n + n + 1 = 0 \end{array}\right.$$

$$\Rightarrow \begin{cases} a_{n+2} = 2a_{n+1} - a_n \quad (1) \\ \\ a_{n+2} = a_n - n - 1 \quad (2) \end{cases}$$

(1) $x^2 - 2x + 1 = 0$

$D = 4 - 4 = 0$

$x = \frac{2}{2} = 1 \Rightarrow a_n = C_0 + C_1 n$

(2) 1. $a_{n+2} - a_n = 0$

$x^2 - 1 = 0 \Rightarrow x = \pm 1$

$\Rightarrow a_n = C_0 + C_1 \cdot (-1)^n$ — общее решение $a_{n+2} - a_n = 0$

2. Найдём частное решение $a_{n+2} = a_n - n - 1$.

Предположим, $a_n = C_3 n^2$

$$\Rightarrow C_3 (n+2)^2 = C_3 n^2 - n + 1$$

$$\cancel{C_3 n^2} + 4nC_3 + 4C_3 = \cancel{C_3 n^2} - n + 1$$

$$n(4C_3 + 1) + (4C_3 + 1) = 0$$

$$\Rightarrow 4C_3 + 1 = 0 \Rightarrow C_3 = -1/4$$

$$\Rightarrow a_n = -\frac{1}{4} n^2$$

$\Rightarrow a_{n+2} = a_n - n - 1$

$\Updownarrow$

$$a_n = C_0 + C_1 \cdot (-1)^n - \frac{1}{4} n^2$$

Ответ: условию удовлетворяют последовательности вида

$a_n = C_0 + C_1 n$ или

$a_n = C_0 + C_1 \cdot (-1)^n - 1/4\, n^2$