CHAPTER 7 PROBLEMS

7.1. Find a rectangle of maximum perimeter that can be inscribed in a circle of unit radius given by

$$g(x, y) = x^2 + y^2 - 1 = 0$$

Check the eigenvalues for sufficient conditions.

Consider point c with coordinates (x, y) as shown in Figure 7.1.

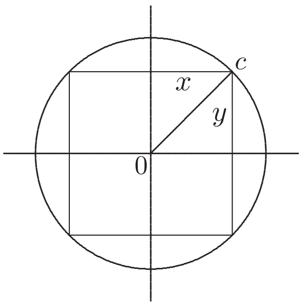

FIGURE 7.1
Constraint function of Problem 7.1.

The problem is to find the minimum value of the function

$$f(x, y) = 4(x + y)$$

subject to the equality constraint

$$g(x, y) = x^2 + y^2 - 1 = 0$$

Forming the Lagrangian function, we obtain

$$\mathcal{L} = 4(x + y) + \lambda(x^2 + y^2 - 1)$$

The resulting necessary conditions for constrained local maxima of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial x} = 4 + 2\lambda x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 4 + 2\lambda y = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

From the first two conditions, we obtain $x = y$. Substituting for y in the third condition results in

$$2x^2 = 1$$

or

$$x = y = 0.707 \qquad \text{and} \quad \lambda = -\frac{4}{2(0.707)} = -2.828$$

and the perimeter is

$$p = 4(x + y) = 4(0.707 + 0.707) = 5.656$$

To see if the perimeter is a maximum, we evaluate the second derivatives and form the Hessian matrix

$$H = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix} = \begin{bmatrix} -5.656 & 0 \\ 0 & -5.656 \end{bmatrix}$$

The second partial derivatives are negative, thus p is a maximum.

7.2. Find the minimum of the function

$$f(x, y) = x^2 + 2y^2$$

subject to the equality constraint

$$g(x, y) = x + 2y + 4 = 0$$

Check for the sufficient conditions.

Forming the Lagrangian function, we obtain

$$\mathcal{L} = x^2 + 2y^2 + \lambda(x + 2y + 4)$$

The resulting necessary conditions for constrained local minima of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 4y + 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + 2y + 4 = 0$$

From the first two conditions, we obtain $x = y$. Substituting for y in the third condition results in

$$3x = -4$$

or

$$x = y = -\frac{4}{3} \qquad \text{and} \quad \lambda = -2x = \frac{8}{3}$$

and the minimum distance is

$$f(x, y) = x^2 + 2y^2 = \left(\frac{-4}{3}\right)^2 + 2\left(\frac{-4}{3}\right)^2 = 5.333$$

To see if this distance is a minimum, we evaluate the second derivatives and form the Hessian matrix

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

The second partial derivatives are positive, thus the minimum distance to origin is at point $(\hat{x}, \hat{y}) = (-\frac{4}{3}, -\frac{4}{3})$.

7.3. Use the Lagrangian multiplier method for solving constrained parameter optimization problems to determine an isosceles triangle of maximum area that may be inscribed in a circle of radius 1.

Consider point c with coordinates $(x, 1 + y)$ as shown in Figure 7.2. The problem

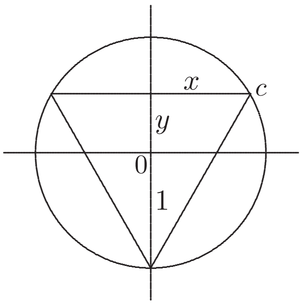

FIGURE 7.2
Constraint function of Problem 7.3.

is to maximize the area of the triangle given by

$$f(x, y) = x(1 + y)$$

subject to the equality constraint

$$g(x, y) = x^2 + y^2 - 1 = 0$$

The minimum distance from the cost function is 17, located at point $(1, 4)$, and the maximum distance is 50.22 located at point $(5.333, -4.666)$.

7.7. The fuel-cost functions in \$/h for two 800 MW thermal plants are given by

$$C_1 = 400 + 6.0P_1 + 0.004P_1^2$$
$$C_2 = 500 + \quad \beta P_2 + \qquad \gamma P_2^2$$

where P_1 and P_2 are in MW.

(a) The incremental cost of power λ is \$8/MWh when the total power demand is 550 MW. Neglecting losses, determine the optimal generation of each plant.
(b) The incremental cost of power λ is \$10/MWh when the total power demand is 1300 MW. Neglecting losses, determine the optimal generation of each plant.
(c) From the results of (a) and (b) find the fuel-cost coefficients β and γ of the second plant.

$$\frac{dC_1}{dP_1} = 6 + 0.008P_1 = \lambda$$
$$\frac{dC_2}{dP_2} = \beta + 2\gamma P_2 = \lambda$$

(a) For $\lambda = 8$, and $P_D = 550$ MW, we have

$$P_1 = \frac{8-6}{0.008} = 250 \text{ MW}$$
$$P_2 = P_D - P_1 = 500 - 250 = 300 \text{ MW}$$

(b) For $\lambda = 10$, and $P_D = 1300$ MW, we have

$$P_1 = \frac{10-6}{0.008} = 500 \text{ MW}$$
$$P_2 = P_D - P_1 = 1300 - 500 = 800 \text{ MW}$$

(c) The incremental cost of power for plant 2 are given by

$$\beta + 2\gamma(300) = 8$$
$$\beta + 2\gamma(800) = 10$$

Solving the above equations, we find $\beta = 6.8$, and $\gamma = 0.002$

7.8. The fuel-cost functions in \$/h for three thermal plants are given by

$$C_1 = 350 + 7.20P_1 + 0.0040P_1^2$$
$$C_2 = 500 + 7.30P_2 + 0.0025P_2^2$$
$$C_3 = 600 + 6.74P_3 + 0.0030P_3^2$$

where P_1, P_2, and P_3 are in MW. The governors are set such that generators share the load equally. Neglecting line losses and generator limits, find the total cost in \$/h when the total load is

(i) $P_D = 450$ MW
(ii) $P_D = 745$ MW
(iii) $P_D = 1335$ MW

(i) For $P_D = 450$ MW, $P_1 = P_2 = P_3 = \frac{450}{3} = 150$ MW. The total fuel cost is

$$C_t = 350 + 7.20(150) + 0.004(150)^2 + 500 + 7.3(150) + 0.0025(150)^2 + 600 + 6.74(150) + 0.003(150)^2 = 4,849.75 \ \$/\mathrm{h}$$

(ii) For $P_D = 745$ MW, $P_1 = P_2 = P_3 = \frac{745}{3}$ MW. The total fuel cost is

$$C_t = 350 + 7.20\left(\frac{745}{3}\right) + 0.004\left(\frac{745}{3}\right)^2 + 500 + 7.3\left(\frac{745}{3}\right) + 0.0025\left(\frac{745}{3}\right)^2 + 600 + 6.74\left(\frac{745}{3}\right) + 0.003\left(\frac{745}{3}\right)^2 = 7,310.46 \ \$/\mathrm{h}$$

(iii) For $P_D = 1335$ MW, $P_1 = P_2 = P_3 = 445$ MW. The total fuel cost is

$$C_t = 350 + 7.20(445) + 0.004(445)^2 + 500 + 7.3(445) + 0.0025(445)^2 + 600 + 6.74(445) + 0.003(445)^2 = 12,783.04 \ \$/\mathrm{h}$$

7.9 Neglecting line losses and generator limits, determine the optimal scheduling of generation for each loading condition in Problem 7.8
(a) by analytical technique, using (7.33) and (7.31).
(b) using Iterative method. Start with an initial estimate of $\lambda = 7.5$ \$/MWh.
(c) find the savings in \$/h for each case compared to the costs in Problem 7.8 when the generators shared load equally.
Use the **dispatch** program to check your results.

(a) (i) For $P_D = 450$ MW, from (7.33), λ is found to be

$$\lambda = \frac{450 + \frac{7.2}{0.008} + \frac{7.3}{0.005} + \frac{6.74}{0.006}}{\frac{1}{0.008} + \frac{1}{0.005} + \frac{1}{0.006}} = \frac{450 + 3483.333}{491.666} = 8.0 \ \$/\mathrm{MWh}$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{8.0 - 7.2}{2(0.004)} = 100$$
$$P_2 = \frac{8.0 - 7.3}{2(0.0025)} = 140$$
$$P_3 = \frac{8.0 - 6.74}{2(0.003)} = 210$$

(a) (ii) For $P_D = 745$ MW, from (7.33), λ is found to be

$$\lambda = \frac{745 + 3483.333}{491.666} = 8.6 \ \ \$/\text{MWh}$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{8.6 - 7.2}{2(0.004)} = 175$$
$$P_2 = \frac{8.6 - 7.3}{2(0.0025)} = 260$$
$$P_3 = \frac{8.6 - 6.74}{2(0.003)} = 310$$

(a) (iii) For $P_D = 1335$ MW, from (7.33), λ is found to be

$$\lambda = \frac{1335 + 3483.333}{491.666} = 9.8 \ \ \$/\text{MWh}$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{9.8 - 7.2}{2(0.004)} = 325$$
$$P_2 = \frac{9.8 - 7.3}{2(0.0025)} = 500$$
$$P_3 = \frac{9.8 - 6.74}{2(0.003)} = 510$$

(b) For the numerical solution using the gradient method, assume the initial value of $\lambda^{(1)} = 7.5$. From coordination equations, given by (7.31), P_1, P_2, and P_3 are

$$P_1^{(1)} = \frac{7.5 - 7.2}{2(0.004)} = 37.5000$$
$$P_2^{(1)} = \frac{7.5 - 7.3}{2(0.0025)} = 40.0000$$
$$P_3^{(1)} = \frac{7.5 - 6.74}{2(0.003)} = 126.6666$$

(i) $P_D = 450$ MW, the error ΔP from (7.39) is

$$\Delta P^{(1)} = 450 - (37.5 + 40 + 126.6666) = 245.8333$$

From (7.37)

$$\Delta\lambda^{(1)} = \frac{245.8333}{491.6666} = 0.5$$

Therefore, the new value of λ is

$$\lambda^{(2)} = 7.5 + 0.5 = 8.0$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{8.0 - 7.2}{2(0.004)} = 100$$
$$P_2^{(2)} = \frac{8.0 - 7.3}{2(0.0025)} = 140$$
$$P_3^{(2)} = \frac{8.0 - 6.74}{2(0.003)} = 210$$

and

$$\Delta P^{(2)} = 450 - (100 + 140 + 210) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations.

(ii) $P_D = 745$ MW, the error ΔP from (7.39) is

$$\Delta P^{(1)} = 745 - (37.5 + 40 + 126.6666) = 540.8333$$

From (7.37)

$$\Delta\lambda^{(1)} = \frac{540.8333}{491.6666} = 1.1$$

Therefore, the new value of λ is

$$\lambda^{(2)} = 7.5 + 1.1 = 8.6$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{8.6 - 7.2}{2(0.004)} = 175$$
$$P_2^{(2)} = \frac{8.6 - 7.3}{2(0.0025)} = 260$$
$$P_3^{(2)} = \frac{8.6 - 6.74}{2(0.003)} = 310$$

and

$$\Delta P^{(2)} = 745 - (175 + 260 + 130) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations.

(iii) $P_D = 1335$ MW, the error ΔP from (7.39) is

$$\Delta P^{(1)} = 1335 - (37.5 + 40 + 126.6666) = 1130.8333$$

From (7.37)

$$\Delta\lambda^{(1)} = \frac{1130.8333}{491.6666} = 2.3$$

Therefore, the new value of λ is

$$\lambda^{(2)} = 7.5 + 2.3 = 9.8$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{9.8 - 7.2}{2(0.004)} = 325$$
$$P_2^{(2)} = \frac{9.8 - 7.3}{2(0.0025)} = 500$$
$$P_3^{(2)} = \frac{9.8 - 6.74}{2(0.003)} = 510$$

and

$$\Delta P^{(2)} = 1335 - (325 + 500 + 510) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations.

(c)(i) For $P_1 = 100$ MW, $P_2 = 140$ MW, and $P_3 = 210$ MW, the total fuel cost is

$$\begin{aligned} C_t = {} & 350 + 7.20(100) + 0.004(100)^2 + 500 + 7.3(140) + 0.0025(140)^2 + \\ & 600 + 6.74(210) + 0.003(210)^2 = 4,828.70 \ \ \$/\text{h} \end{aligned}$$

Compared to Problem 7.8 (i), when the generators shared load equally, the saving is 4,849.75 - 4,828,70 = 21.05 $/h.

(c)(ii) For $P_1 = 175$ MW, $P_2 = 260$ MW, and $P_3 = 310$ MW, the total fuel cost is

$$\begin{aligned} C_t = {} & 350 + 7.20(175) + 0.004(175)^2 + 500 + 7.3(260) + 0.0025(260)^2 + \\ & 600 + 6.74(310) + 0.003(310)^2 = 7,277.20 \ \ \$/\text{h} \end{aligned}$$

Compared to Problem 7.8 (ii), when the generators shared load equally, the saving is 7,310.46 - 7,277.20 = 33.26 $/h.

(c)(iii) For $P_1 = 325$ MW, $P_2 = 500$ MW, and $P_3 = 510$ MW, the total fuel cost is

$$C_t = 350 + 7.20(325) + 0.004(325)^2 + 500 + 7.3(500) + 0.0025(500)^2 + 600 + 6.74(510) + 0.003(510)^2 = 12,705.20 \ \ \$/\text{h}$$

Compared to Problem 7.8 (iii), when the generators shared load equally, the saving is 12,783.04 - 12,705.20 = 77.84 $/h.

To check the results we use the following commands

```
cost = [350  7.2   0.004
        500  7.3   0.0025
        600  6.74  0.003];
disp('(i) Pdt = 450 MW')
Pdt = 450;
dispatch
gencost
disp('(ii) Pdt = 745 MW')
Pdt = 745;
dispatch
gencost
disp('(iii) Pdt = 1335 MW')
Pdt = 1335;
dispatch
gencost
```

The result is

```
(i) Pdt = 450 MW
Incremental cost of delivered power (system lambda)=8.0 $/MWh
Optimal Dispatch of Generation:

  100.0000
  140.0000
  210.0000
Total generation cost =    4828.70 $/h

(ii) Pdt = 745 MW
Incremental cost of delivered power (system lambda)=8.6 $/MWh
Optimal Dispatch of Generation:

  175.0000
  260.0000
  310.0000
Total generation cost =    7277.20 $/h

(iii) Pdt = 1335 MW
```

```
Incremental cost of delivered power (system lambda)=9.80 $/MWh
Optimal Dispatch of Generation:

  325.0000
  500.0000
  510.0000
Total generation cost =   12705.20 $/h
```

7.10. Repeat Problem 7.9 (a) and (b), but this time consider the following generator limits (in MW)

$$
\begin{aligned}
122 &\le P_1 \le 400 \\
260 &\le P_2 \le 600 \\
50 &\le P_3 \le 445
\end{aligned}
$$

Use the **dispatch** program to check your results.

In Problem 7.9, in part (a) (i), the optimal dispatch are $P_1 = 100$ MW, $P_2 = 140$ MW, and $P_3 = 210$ MW. Since P_1 and P_2 are less that their lower limit, these plants are pegged at their lower limits. That is, $P_1 = 122$, and $P_2 = 260$ MW. Therefore, $P_3 = 450 - (122 + 260) = 68$ MW.

In Problem 7.9, in part (a) (ii), the optimal dispatch are $P_1 = 175$ MW, $P_2 = 260$ MW, and $P_3 = 310$ MW. which are within the plants generation limits.

In Problem 7.9, in part (a) (iii), the optimal dispatch are $P_1 = 325$ MW, $P_2 = 500$ MW, and $P_3 = 510$ MW. Since P_3 exceed its upper limit, this plant is pegged at $P_2 = 445$. Therefore, a load of $1335 - 445 = 890$ MW must be shared between plants 1 and 2, with equal incremental fuel cost give by

$$
\begin{aligned}
\lambda &= \frac{890 + \frac{7.2}{0.008} + \frac{7.3}{0.005}}{+\frac{1}{0.008} + \frac{1}{0.005}} \\
&= \frac{890 + 2360}{325} = 10 \ \ \$/\text{MWh}
\end{aligned}
$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$
\begin{aligned}
P_1 &= \frac{10 - 7.2}{2(0.004)} = 350 \\
P_2 &= \frac{10 - 7.3}{2(0.0025)} = 540
\end{aligned}
$$

Since P_1 and P_2 are within their limits the above result is the optimal dispatch.

(b) For part (iii), The iterative method is as follows

In Problem 7.9 (b) part (iii) starting with an initial value of $\lambda^{(1)} = 7.5$, the optimal dispatch was obtained in two iterations as $P_1 = 325$ MW, $P_2 = 500$ MW, and $P_3 = 510$ MW, with $\lambda = 9.8$\$/MWh. Since P_3 exceed its upper limit, this plant is pegged at $P_3 = 445$ and is kept constant at this value. Thus, the new imbalance in power is

$$\Delta P^{(2)} = 1335 - (325 + 500 + 445) = 65$$

From (7.37)

$$\Delta\lambda^{(2)} = \frac{65}{\frac{1}{2(0.004)} + \frac{1}{2(0.0025)}} = \frac{65}{325} = 0.2$$

Therefore, the new value of λ is

$$\lambda^{(3)} = 9.8 + 0.2 = 10$$

For the third iteration, we have

$$P_1^{(3)} = \frac{10 - 7.2}{2(0.004)} = 350$$
$$P_2^{(3)} = \frac{10 - 7.3}{2(0.0025)} = 540$$
$$P_3^{(3)} = 445$$

and

$$\Delta P^{(3)} = 1335 - (350 + 540 + 445) = 0.0$$

$\Delta P^{(3)} = 0$, and the equality constraint is met and P_1 and P_2 are within their limits.

The following commands can be used to obtain the optimal dispatch of generation including generator limits.

To check the results, we use the following commands

```
cost = [350  7.2   0.004
        500  7.3   0.0025
        600  6.74  0.003];

mwlimits =[ 122 400
            260 600
            50 445]
%Pdt = 450;
```

```
%Pdt = 745;
Pdt = 1335;
dispatch
gencost
```

The result is

```
Incremental cost of delivered power (system lambda) = 10.0 $/MWh
Optimal Dispatch of Generation:

   350
   540
   445

Total generation cost =   12724.38 $/h
```

7.11. The fuel-cost function in \$/h of two thermal plants are

$$C_1 = 320 + 6.2P_1 + 0.004P_1^2$$
$$C_2 = 200 + 6.0P_2 + 0.003P_2^2$$

where P_1 and P_2 are in MW. Plant outputs are subject to the following limits (in MW)

$$50 \le P_1 \le 250$$
$$50 \le P_2 \le 350$$

The per-unit system real power loss with generation expressed in per unit on a 100-MVA base is given by

$$P_{L(pu)} = 0.0125P_{1(pu)}^2 + 0.00625P_{2(pu)}^2$$

The total load is 412.35 MW. Determine the optimal dispatch of generation. Start with an initial estimate of $\lambda = 7$ \$/MWh. Use the **dispatch** program to check your results.

In the cost function P_i is expressed in MW. Therefore, the real power loss in terms of MW generation is

$$P_L = \left[0.0125\left(\frac{P_1}{100}\right)^2 + 0.00625\left(\frac{P_2}{100}\right)^2\right] \times 100 \ \text{ MW}$$
$$= 0.000125P_1^2 + 0.0000625P_2^2 \ \text{ MW}$$

For the numerical solution using the gradient method, assume the initial value of $\lambda^{(1)} = 7.0$. From coordination equations, given by (7.70), $P_1^{(1)}$, and $P_2^{(1)}$ are

$$P_1^{(1)} = \frac{7.0 - 6.2}{2(0.004 + 7.0 \times 0.000125)} = 82.05128 \quad \text{MW}$$
$$P_2^{(1)} = \frac{7.0 - 6.0}{2(0.003 + 7.0 \times 0.0000625)} = 145.4545 \ \text{ MW}$$