Where is $R_{WS} = R_b$ if Drift Sets Particle Location?

If we were taking out particle surface density to be informed by observations, as in Powell et al. (2017), then our input would be $s\left(r\right)$, the particle location as a function of semi-major axis. In practice we will only measure the particle location at a few values of orbital separation. We can then fit these values with e.g. a power law. In this case we can infer the disk surface density as in Powell et al.:

$$\Sigma_g\left(r\right) = \frac{t_{\rm disk} v_0 \rho_s s}{r}$$

Then, assuming we're in the Epstein regime, that turbulence is negligible, and that $M \gg m$:

$$\begin{aligned}
R_{WS} &= \sqrt{\frac{GMt_s}{v_0}} \\
&= \sqrt{\frac{GM}{v_0}\frac{\rho_s}{\rho_g}\frac{s}{v_{th}}} \\
&= \sqrt{2\frac{GM}{v_0}\frac{\rho_s r s}{t_{\rm disk} v_0 \rho_s s}\frac{c_s}{v_{th}\Omega}} \\
&= \left(\frac{\pi}{2}\right)^{1/4}\frac{1}{c_s^2}\sqrt{\frac{GM}{t_{\rm disk}} r^3 \Omega} \\
&= \left(\frac{\pi}{2}\right)^{1/4}\frac{1}{c_s^2}\sqrt{\frac{q}{t_{\rm disk}} r^6 \Omega^5} \\
&= \left(\frac{\pi}{32}\right)^{1/4}\eta^{-1}\sqrt{\frac{q}{t_{\rm disk}\Omega}} r
\end{aligned}$$

So, setting $R_{WS} = R_b$ gives

$$\begin{aligned}
R_{ws} &= R_b \\
\left(\frac{\pi}{32}\right)^{1/4}\eta^{-1}\sqrt{\frac{q}{t_{\rm disk}\Omega}} r &= \frac{GM}{c_s^2} \\
M^{1/2} &= \left(\frac{\pi}{32}\right)^{1/4}\frac{c_s^2}{\eta}\frac{1}{G}\sqrt{\frac{r^2}{t_{\rm disk}\Omega M_*}} \\
M &= \left(\frac{\pi}{2}\right)^{1/2} r^4 \Omega^4 \left(\frac{M_*}{\Omega^2 r^3}\right)^2 \left(\frac{r^2}{t_{\rm disk}\Omega M_*}\right) \\
\frac{M}{M_*} &= \left(\frac{\pi}{2}\right)^{1/2} \left(t_{\rm disk}\Omega\right)^{-1}
\end{aligned}$$

What can we say about St given the relationship between Σ_g and s?

$$\begin{aligned}
\Sigma_g(r) &= \frac{t_{\rm disk} v_0 \rho_s s}{r} \\
St_{\rm Eps} &= \frac{\rho_s}{\rho_g}\frac{s}{v_{th}}\Omega \\
&= 2\rho_s \frac{c_s}{\Sigma_g}\frac{s}{v_{th}} \\
&= \sqrt{\frac{\pi}{2}}\frac{\rho_s s}{\Sigma_g} \\
St_{\rm Eps} =& \sqrt{\frac{\pi}{2}}\frac{r}{t_{\rm disk} v_0}
\end{aligned}$$

What we would we get with different prefactors?

$$\begin{aligned}
\Sigma_g(r) &= \frac{t_{\rm disk} v_0 \rho_s s}{r} \\
St_{\rm Eps} &= \frac{\rho_s}{\rho_g}\frac{s}{v_{th}}\Omega \\
&= \rho_s \frac{c_s}{\Sigma_g}\frac{s}{v_{th}} \\
&= \frac{\rho_s s}{\Sigma_g} \\
St_{\rm Eps} =& \frac{r}{t_{\rm disk} v_0}
\end{aligned}$$

Correct way with prefactors:

$$\dot{r} = 2v_0 St \Rightarrow \frac{r}{\dot{r}} = t_{\rm disk} \Rightarrow St = \frac{r}{2v_0 t_{\rm disk}}$$

So modeling the particle size in this manner implies there is a fixed value of St in the disk that depends only on disk age, semi-major axis, and sub-Keplerian velocity (essentially constant)? Seems a little odd, but intuitively makes some sense. This appears to imply that the “massive disk” part of Diana’s modeling makes no difference (approximately), since my pebble accretion modeling is basically only sensitive to St. This could be extremely interesting in its own right.

Can we use this for $R_{\rm shear} = R_b$?

$$\begin{aligned}
R_{\rm shear} &= R_b \\
R_H (3St)^{1/3} &= \frac{GM}{c_s^2} \\
R_H\left(3\sqrt{\frac{\pi}{2}}\frac{r}{t_{\rm disk}v_0}\right)^{1/3} &= \frac{3R_H^3\Omega^2}{c_s^2} \\
3^{1/3}(\pi/2)^{1/6}\left(\frac{r}{t_{\rm disk}v_0}\right)^{1/3} &= \frac{v_H^2}{c_s^2} \\
\frac{v_k^2}{c_s^2}\left(\frac{M_p}{3M_*}\right)^{2/3} &= 3^{1/3}(\pi/2)^{1/6}\left(\frac{r}{t_{\rm disk}v_0}\right)^{1/3} \\
\left(\frac{M_p}{3M_*}\right)^{2/3} &= 2\times 3^{1/3}\times(\pi/2)^{1/6}\left(\frac{r\,v_0^2}{t_{\rm disk}v_k^3}\right)^{1/3} \\
\frac{M_p}{M_*} &= 3\left[2\times 3^{1/3}\times(\pi/2)^{1/6}\right]^{3/2}\left(\frac{r\,v_0^2}{t_{\rm disk}v_k^3}\right)^{1/2}
\end{aligned}$$

Gross. Mathematica appears to agree, and gives the (slightly) more simple expression:

$$\frac{M_p}{M_*} = (32\pi)^{1/4}\left(\frac{v_0}{v_k}\right)(t_{\rm disk}\Omega)^{-1/2}$$

From plots, it appears that the $R_{WS} = R_b$ limit sets the upper mass. What sets the lower mass?

Since we're at low mass, I would expect the $St = 12\,(v_H/v_{gas})^3$ limit to hold for $KE = W$. Does combining this with the above value for St imply a lower mass limit?

$$\begin{aligned}
St &< 12\frac{v_H^3}{v_{gas}^3} \\
\sqrt{\frac{\pi}{2}}\frac{r}{t_{\rm disk}v_0} &< 12\frac{v_H^3}{v_{gas}^3} \\
\frac{M_p}{3M_*}r^3\Omega^3 &> \frac{v_{gas}^3}{12}\sqrt{\frac{\pi}{2}}\frac{r}{t_{\rm disk}v_0} \\
\frac{M_p}{M_*} &> \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{v_{gas}^3}{v_0 t_{\rm disk} r^2\Omega^3}\right)
\end{aligned}$$

If we're laminar, then $v_{gas} = v_0$?

$$\frac{M_p}{M_*} > \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{v_0^2}{t_{\rm disk} r^2\Omega^3}\right)$$

Since it appears from plots that $St < 1$, this also means that if we reach the mass such that $v_{pg} > v_{enc}$ occurs before the St limit is reached, then accretion will immediately kick in, since this essentially raises the upper limit to $St = 4\sqrt{3}$. From Paper I, this mass is set by

$$\frac{v_H}{v_{gas}} = 48^{-1/3}$$

or

$$\frac{M_p}{M_*} > \frac{1}{16}\frac{v_{gas}^3}{v_k^3}$$

So in general, we require

$$\frac{M_p}{M_*} > \min\left[\frac{1}{16}\frac{v_{gas}^3}{v_k^3}, \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{v_0^2}{t_{\rm disk} r^2\Omega^3}\right)\right]$$

How does factor of 2π come into Mie Scattering law in conversation between λ and s?

Outline paper with tentative results to see what minimum amount of work needs to be done to publish.

Lingering Questions:

1 Why does the value of St increase with r in Diana's Model but *decrease* in the LJ14 paper?

In Diana's model it's easy to see from the analytic expression:

$$St = \frac{r}{2v_0 t_{\rm disk}}$$

And its simply because the drift timescale is longer farther out. What happens if we're more careful about the integration?

$$\dot{r} = \frac{dr}{dt} \Rightarrow dt = \frac{dr}{\dot{r}}$$
$$t_{\rm drift} = \int\limits_{r_{\rm edge}}^{r} -\frac{1}{2v_0\tau_s} dr'$$

Assume $\tau_s = \tau_{s,0} \left(r/r_0 \right)^{\alpha}$:

$$\begin{aligned}
t_{\rm drift} &= \int\limits_{r_{\rm edge}}^{r} -\frac{1}{2v_0\tau_{s,0}} \left(\frac{r_0}{r'}\right)^{\alpha} dr' \\
&= \frac{r_0^{\alpha}}{2v_0\tau_{s,0}} \int\limits_{r}^{r_{\rm edge}} r'^{-\alpha} dr' \\
&= \frac{r_0^{\alpha}}{2v_0\tau_{s,0}} \frac{1}{1-\alpha} \left[r_{\rm edge}^{-\alpha+1} - r^{-\alpha+1} \right] \\
&= \frac{1}{1-\alpha} \frac{1}{2v_0} \left[\tau_{s,0}^{-1} \left(\frac{r_{\rm edge}}{r_0} \right)^{-\alpha} \frac{r_{\rm edge}}{2v_0} - \tau_{s,0}^{-1} \left(\frac{r}{r_0} \right)^{-\alpha} \frac{r}{2v_0} \right] \\
&= \frac{1}{1-\alpha} \frac{1}{2v_0} \left[\frac{r_{\rm edge}}{\tau_{s,\rm edge}} - \frac{r}{\tau_s} \right]
\end{aligned}$$

This doesn't seem to imply what I want...

2 How Important is the earlier, fragmentation dominated stage of growth? The drift dominated stage only applies later, but what exactly is later? Can we model the fragmentation dominated stage?