M1

1. $$-\left[\frac{\partial}{\partial x}\left(k_1(x,y)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_2(x,y)\frac{\partial u}{\partial y}\right)\right]=f(x,y)$$

$a\le x\le b,\ c\le y\le d,\ 0<c_{11}\le k_1(x,y)\le c_{12}\ \ 0<c_{21}\le k_2(x,y)\le c_{22}$

$u|_{x=a}=g_1(y)\qquad u|_{x=b}=g_2(y)$

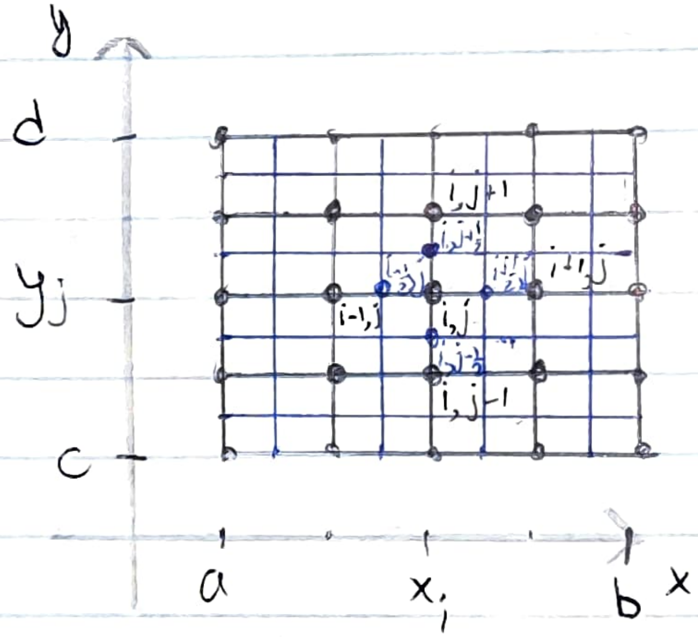
$u|_{y=c}=g_3(x)\qquad u|_{y=d}=g_3(x)$

N_x - число разб. $[a,b]$, N_y - число разб. $[c,d]$

$x_0<x_1<\dots<x_N,\ x_i\in[a,b],\ x_0=a,\ x_N=b\qquad y_0<y_1<\dots<y_N,\ y_i\in[c,d],\ y_0=c,\ y_N=d$

$$h_x=\frac{b-a}{N_x}\qquad h_y=\frac{c-d}{N_y}$$

$$x_{i-\frac12}=\frac{x_i+x_{i-1}}{2},\ i=1,2,\dots,N_x\qquad y_{j-\frac12}=\frac{y_j+y_{j-1}}{2},\ j=1,2,\dots,N_y$$



решение ищется в узлах осн. сетки

$u(x_i,y_j)=u_{i,j}$

$v(x_i,y_j)=v_{i,j}$

$u_{i,j}\approx v_{i,j}$

$$-\left[\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\frac{\partial}{\partial x}\left(k_1\frac{\partial u}{\partial x}\right)dxdy+\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\frac{\partial}{\partial y}\left(k_2\frac{\partial u}{\partial y}\right)dxdy\right]=\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}f\,dxdy$$

$$i=1,2,\dots,N_x-1,\quad j=1,2,\dots,N_y-1$$

$$-\left[\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}k_1(x_{i+\frac{1}{2}},y)\frac{\partial u}{\partial x}\Big|_{x=x_{i+\frac{1}{2}}}dy-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}k_1(x_{i-\frac{1}{2}},y)\frac{\partial u}{\partial x}\Big|_{x=x_{i-\frac{1}{2}}}dy+\right.$$

$$\left.+\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}k_2(x,y_{j+\frac{1}{2}})\frac{\partial u}{\partial y}\Big|_{y=y_{j+\frac{1}{2}}}dx-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}k_2(x,y_{j-\frac{1}{2}})\frac{\partial u}{\partial y}\Big|_{y=y_{j-\frac{1}{2}}}dx\right]=\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}f\,dxdy$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\varphi(x,y)dx\approx h_x\varphi(x_i,y)=h_x\varphi_i,\quad\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\varphi(x,y)dx\approx h_y\varphi(x,y_j)=h_y\varphi_j$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\varphi\,dxdy=h_xh_y\varphi$$

$$k_1(x_{i-\frac{1}{2}},y_j)\frac{\partial u}{\partial x}\Big|_{\substack{x=x_{i-\frac{1}{2}}\\ y=y_j}}\approx k_1(x_{i-\frac{1}{2}},y_j)\frac{v_{i,j}-v_{i-1,j}}{h_x}$$

$$k_2(x_i,y_{j+\frac{1}{2}})\frac{\partial u}{\partial y}\Big|_{\substack{x=x_i\\ y_j=y_{j-\frac{1}{2}}}}\approx k_2(x_i,y_{j-\frac{1}{2}})\frac{v_{i,j}-v_{i,j-1}}{h_y}$$

$$-\left[h_yk_1(x_{i+\frac{1}{2}},y_j)\frac{v_{i+1,j}-v_{i,j}}{h_x}-h_yk_1(x_{i-\frac{1}{2}},y_j)\frac{v_{i,j}-v_{i-1,j}}{h_x}+\right.$$

$$\left.+h_xk_2(x_i,y_{j+\frac{1}{2}})\frac{v_{i,j+1}-v_{i,j}}{h_y}-h_xk_2(x_i,y_{j-\frac{1}{2}})\frac{v_{i,j}-v_{i,j-1}}{h_y}\right]=h_xh_yf_{ij}$$

$$i=1,2,\dots,N_x-1\quad j=1,2,\dots,N_y-1$$

$v_{i,j} = g_1(y_j) \quad i=0, j=0,1,\dots,N_y \qquad v_{i,j} = g_2(y_j), i = N_x, j = 0,1,\dots,N_y$

$v_{i,j} = g_3(x_i), i=0,1,\dots,N_x, j=0 \qquad v_{i,j} = g_4(x_i), i=0,1,\dots,N_x, j = N_y$

$N = (N_x+1)(N_y+1)$

$$2.\ \xi_{i,j} = h_x h_x f_{i,j} + h_y k_1(x_{i+\frac{1}{2}}, y_j)\frac{u_{i+1,j}-u_{i,j}}{h_x} - h_y k_1(x_{i-\frac{1}{2}}, y_j)\frac{u_{i,j}-u_{i-1,j}}{h_x}$$

$$+ h_x k_2(x_i, y_{j+\frac{1}{2}})\frac{u_{i,j+1}-u_{i,j}}{h_y} - h_x k_2(x_i, y_{j-\frac{1}{2}})\frac{u_{i,j}-u_{i,j-1}}{h_x}$$

$$u_{i+1,j} = u(x_i+h_x, y_j) = u_{i,j} + h_x\frac{\partial u_{i,j}}{\partial x} + \frac{h_x^2}{2}\frac{\partial^2 u_{i,j}}{\partial x^2} + \frac{h_x^3}{6}\frac{\partial^3 u_{i,j}}{\partial x^3} + \frac{h_x^4}{24}\frac{\partial^4 u_{i,j}}{\partial x^4} + O(h_x^5)$$

$$k_{1\,i+\frac{1}{2},j} = k_1(x_i+\frac{h_x}{2}, y_j) = k_{1\,i,j} + \frac{h_x}{2}\frac{\partial k_{1\,i,j}}{\partial x} + \frac{h_x^2}{8}\frac{\partial^2 k_{1\,i,j}}{\partial x^2} + \frac{h_x^3}{48}\frac{\partial^3 k_{1\,i,j}}{\partial x^3} + O(h_x^4)$$

$$u_{i-1,j} = u(x_i-h_x, y_j) = u_{i,j} - h_x\frac{\partial u_{i,j}}{\partial x} + \frac{h_x^2}{2}\frac{\partial^2 u_{i,j}}{\partial x^2} - \frac{h_x^3}{6}\frac{\partial^3 u_{i,j}}{\partial x^3} + \frac{h_x^4}{24}\frac{\partial^4 u_{i,j}}{\partial x^4} + O(h_x^5)$$

$$k_{1\,i-\frac{1}{2},j} = k_1(x_i-\frac{h_x}{2}, y_j) = k_{1\,i,j} - \frac{h_x}{2}\frac{\partial k_{1\,i,j}}{\partial x} + \frac{h_x^2}{8}\frac{\partial^2 k_{1\,i,j}}{\partial x^2} - \frac{h_x^3}{48}\frac{\partial^3 k_{1\,i,j}}{\partial x^3} + O(h_x^4)$$

$$u_{i,j+1} = u(x_i, y_j+h_y) = u_{i,j} + h_y\frac{\partial u_{i,j}}{\partial y} + \frac{h_y^2}{2}\frac{\partial^2 u_{i,j}}{\partial y^2} + \frac{h_y^3}{6}\frac{\partial^3 u_{i,j}}{\partial y^3} + \frac{h_y^4}{24}\frac{\partial^4 u_{i,j}}{\partial y^4} + O(h_y^5)$$

$$k_{2\,i,j+\frac{1}{2}} = k_2(x_i, y_j+\frac{h_y}{2}) = k_{2\,i,j} + \frac{h_y}{2}\frac{\partial k_{2\,i,j}}{\partial y} + \frac{h_y^2}{8}\frac{\partial^2 k_{2\,i,j}}{\partial y^2} + \frac{h_y^3}{48}\frac{\partial^3 k_{2\,i,j}}{\partial y^3} + O(h_y^4)$$

$$u_{i,j-1} = u(x_i, y_j-h_y) = u_{i,j} - h_y\frac{\partial u_{i,j}}{\partial y} + \frac{h_y^2}{2}\frac{\partial^2 u_{i,j}}{\partial y^2} - \frac{h_y^3}{6}\frac{\partial^3 u_{i,j}}{\partial y^3} + \frac{h_y^4}{24}\frac{\partial^4 u_{i,j}}{\partial y^4} + O(h_y^5)$$

$$k_{2\,i,j-\frac{1}{2}} = k_2(x_i, y_j-\frac{h_y}{2}) = k_{2\,i,j} - \frac{h_y}{2}\frac{\partial k_{2\,i,j}}{\partial y} + \frac{h_y^2}{8}\frac{\partial^2 k_{2\,i,j}}{\partial y^2} - \frac{h_y^3}{48}\frac{\partial^3 k_{2\,i,j}}{\partial y^3} + O(h_y^4)$$

$$k_1\frac{\partial^2 u}{\partial x^2} + \frac{\partial k_1}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(k_1\frac{\partial u}{\partial x}\right); \quad k_2\frac{\partial^2 u}{\partial y^2} + \frac{\partial k_2}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(k_2\frac{\partial u}{\partial y}\right)$$

$$\xi_{i,j} = h_x h_y f_{i,j} + h_y\Big[h_x\Big(\frac{\partial}{\partial x}\Big(k_1\frac{\partial u}{\partial x}\Big)\Big)_{i,j} + h_x^3\Big(\frac{1}{12}k_1\frac{\partial^4 u}{\partial x^4} + \frac{1}{6}\frac{\partial k_1}{\partial x}\frac{\partial^3 u}{\partial x^3} +$$

$$+\frac{1}{8}\frac{\partial^2 k_1}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial x^3}\frac{\partial u}{\partial x}\Big)_{i,j} + O(h_x^4)\Big] + h_x\Big[h_y\Big(\frac{\partial}{\partial y}\Big(k_2\frac{\partial u}{\partial y}\Big)\Big)_{i,j} + h_y^3\Big(\frac{1}{12}k_2\frac{\partial^4 u}{\partial y^4}$$

$$+\frac{1}{6}\frac{\partial k_2}{\partial y}\frac{\partial^3 u}{\partial y^3} + \frac{1}{24}\frac{\partial^3 k_2}{\partial y^3}\frac{\partial u}{\partial y} + \frac{1}{8}\frac{\partial^2 k_2}{\partial y^2}\frac{\partial^2 u}{\partial y^2}\Big)_{i,j} + O(h_y^4)\Big]$$

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{h_x h_y}$$

$$\tilde{\xi}_{i,j} = \Big[f + \frac{\partial}{\partial x}\Big(k_1\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(k_2\frac{\partial u}{\partial y}\Big)\Big]_{i,j} + h_x^2\Big[\frac{1}{12}k_1\frac{\partial^4 u}{\partial x^4} + \frac{1}{6}\frac{\partial k_1}{\partial x}\frac{\partial^3 u}{\partial x^3} +$$

$$+\frac{1}{8}\frac{\partial^2 k_1}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial x^3}\frac{\partial u}{\partial x}\Big]_{i,j} + O(h_x^3) + h_y^2\Big[\frac{1}{12}k_2\frac{\partial^4 u}{\partial y^4} + \frac{1}{6}\frac{\partial k_2}{\partial y}\frac{\partial^3 u}{\partial y^3} +$$

$$+\frac{1}{8}\frac{\partial^2 k_2}{\partial y^2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial y^3}\frac{\partial u}{\partial y}\Big]_{i,j} + O(h_y^3)$$

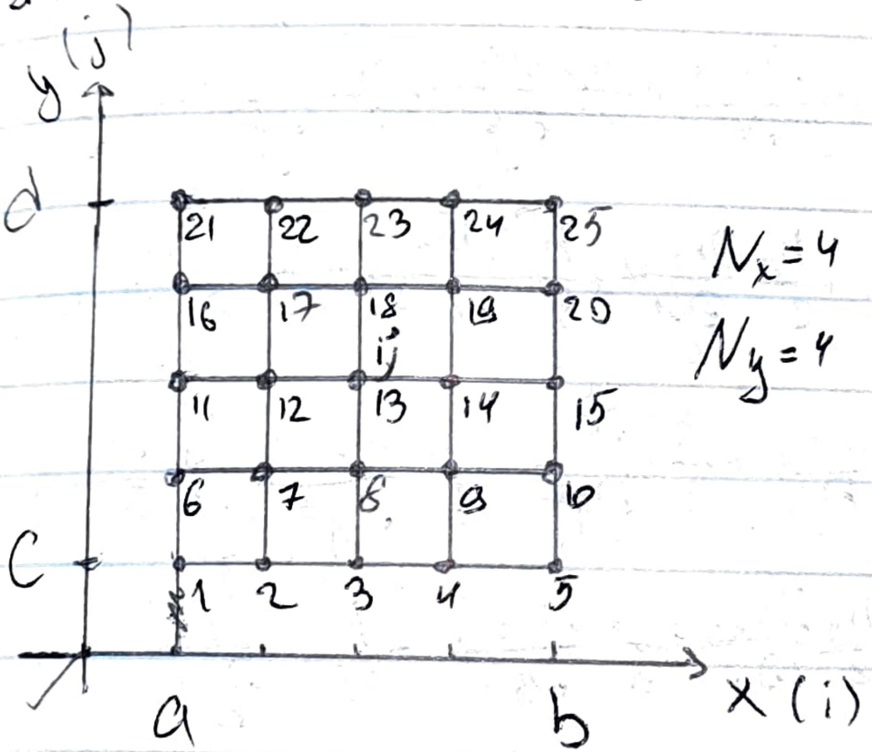
$p_x = 2 - 0 = 2$ $\qquad$ $p_y = 2 - 0 = 2$

$$\varphi_x = \frac{1}{12}k_1\frac{\partial^4 u}{\partial x^4} + \frac{1}{6}\frac{\partial k_1}{\partial x}\frac{\partial^3 u}{\partial x^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial x^3}\frac{\partial u}{\partial x}$$

$$\varphi_y = \frac{1}{12}k_2\frac{\partial^4 u}{\partial y^4} + \frac{1}{6}\frac{\partial k_2}{\partial y}\frac{\partial^3 u}{\partial y^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial y^2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial y^3}\frac{\partial u}{\partial y}$$

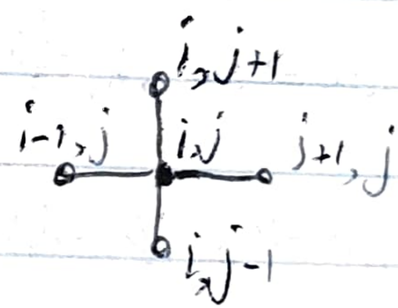
В граничных случаях получаются точные решения

3. $-\frac{h_x}{h_y} k_2(x_i, y_{j-\frac{1}{2}}) v_{i,j-1} - \frac{h_y}{h_x} k_1(x_{i-\frac{1}{2}}, y_j) v_{i-1,j} + \left[\frac{h_x}{h_y} k_2(x_i, y_{j-\frac{1}{2}}) + \frac{h_x}{h_y} k_2(x_i, y_{j+\frac{1}{2}}) + \frac{h_y}{h_x} k_1(x_{i-\frac{1}{2}}, y_j) + \frac{h_y}{h_x} k_1(x_{i+\frac{1}{2}}, y_j)\right] v_{i,j} - \frac{h_y}{h_x} k_1(x_{i+\frac{1}{2}}, y_j) v_{i+1,j} - \frac{h_x}{h_y} k_2(x_i, y_{j+\frac{1}{2}}) v_{i,j+1} = h_x h_y f_{ij}$



$N_x = 4$

$N_y = 4$



$$i = 0, 1, \dots, N_x; \; j = 0, 1, \dots, N_y$$

$$(i, j) \to m = Lj + i + 1, \quad L = N_x + 1$$

$$v_{i,j-1} \to w_{m-L}$$

$$v_{i-1,j} \to w_{m-1}$$

$$v_{i,j} \to w_m$$

$$v_{i+1,j} \to w_{m+1}$$

$$v_{i,j+1} \to w_{m+L}$$

$$j = 1, \dots, N_y - 1; \; i = 1, 2, \dots, N_x - 1$$

$$a_m w_{m-L} + b_m w_{m-1} + c_m w_m + d_m w_{m+1} + e_m w_{m+L} = g_m$$

$$a_m = -\frac{h_x}{h_y} K_2(x_i, y_{j-\frac{1}{2}}) \qquad b_m = -\frac{h_{\cancel{x}y}}{h_{\cancel{y}x}} K_1(x_{i-\frac{1}{2}}, y_j)$$

$$c_m = \frac{h_x}{h_y} K_2(x_i, y_{j-\frac{1}{2}}) + \frac{h_x}{h_y} K_2(x_i, y_{j+\frac{1}{2}}) + \frac{h_y}{h_x} K_1(x_{i-\frac{1}{2}}, y_j) + \frac{h_y}{h_x} K_2(x_{i+\frac{1}{2}}, y_j)$$

$$d_m = -\frac{h_y}{h_x} K_1(x_{i+\frac{1}{2}}, y_j) \qquad e_m = -\frac{h_x}{h_y} K_2(x_i, y_{j+\frac{1}{2}})$$

$$g_m = h_x h_y f_{ij}$$

$i = 0;\ j = 0, 1, \ldots, N_y$ $\quad c_m w_m = g_m \quad c_m = 1,\ g_m = g_1(y_j)$

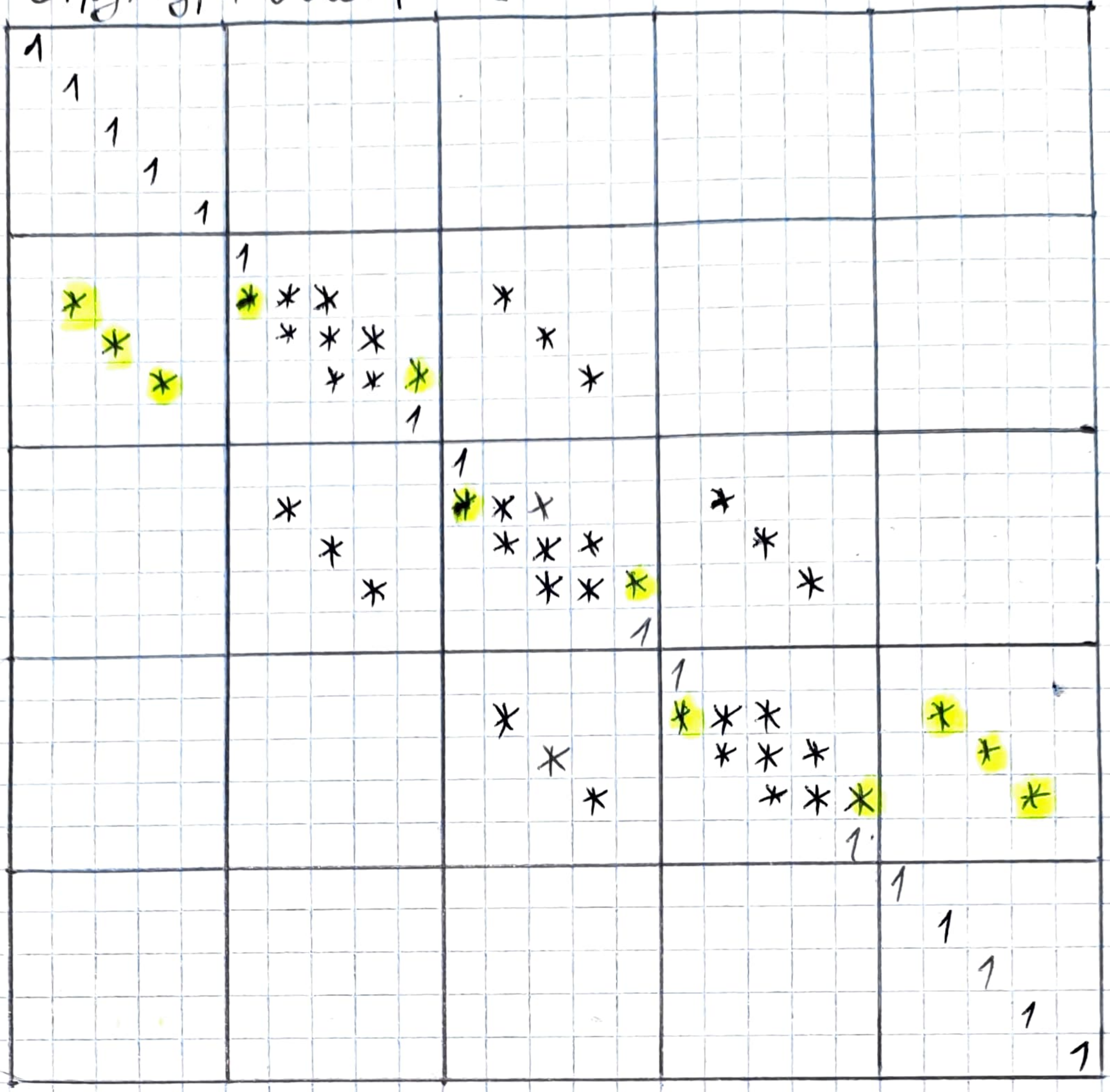
$i = N_x;\ j = 0, 1, \ldots, N_y$ $\quad c_m w_m = g_m \quad c_m = 1,\ g_m = g_2(y_j)$

$i = 0, 1, \ldots, N_x;\ j = 0$ $\quad c_m w_m = g_m \quad c_m = 1,\ g_m = g_3(x_i$

$i = 0, 1, \ldots, N_x;\ j = N_y$ $\quad c_m w_m = g_m \quad c_m = 1,\ g_m = g_4(x_i)$

$Aw = g \qquad A \in \mathbb{R}^{N \times N},\ w, g \in \mathbb{R}^N,\ N = (N_x + 1)(N_y + 1)$

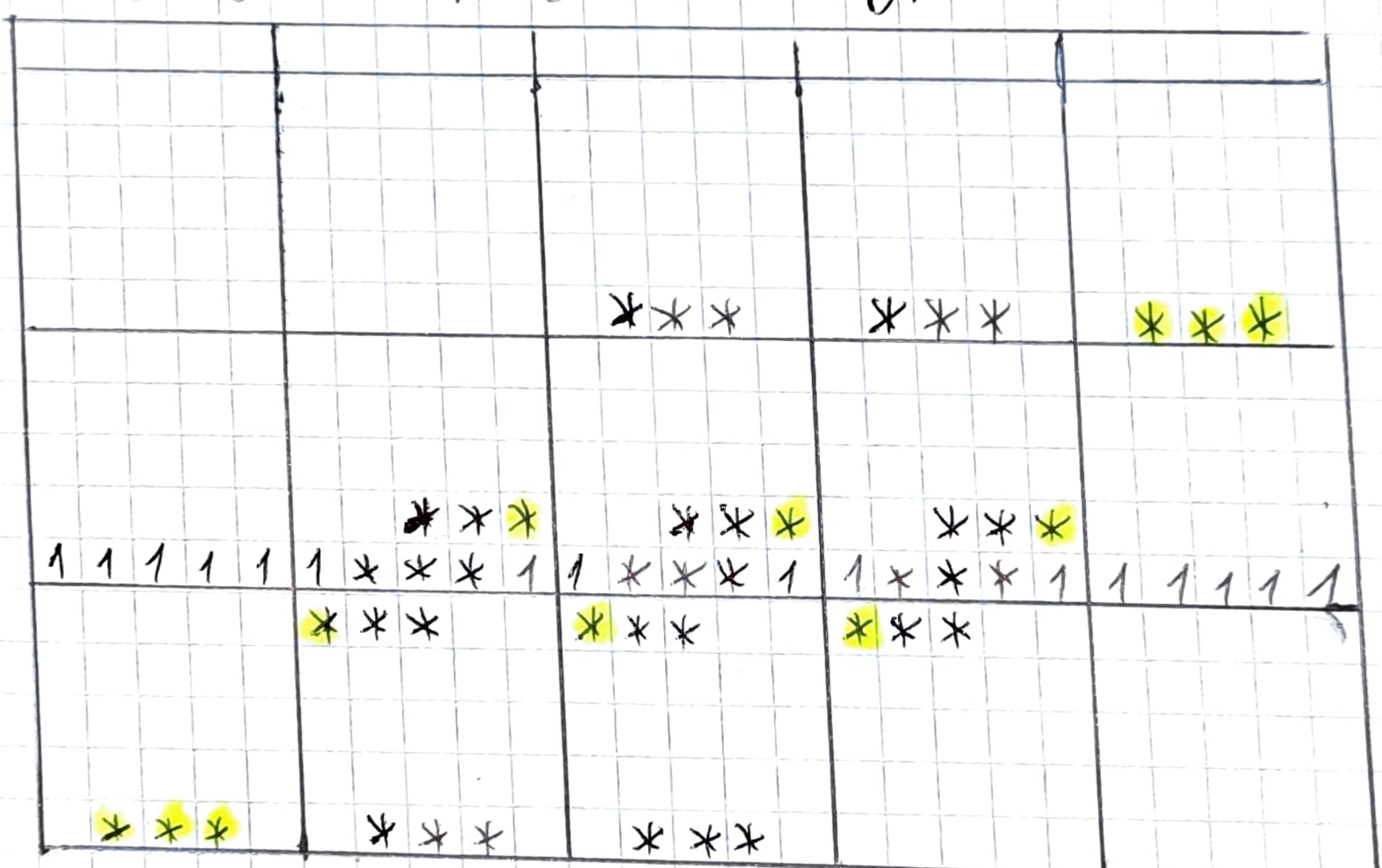
Структура матрицы алг. системы



Все граничные значения отмечены жёлтым. Если их подставить, получится симметричная матрица

Можно понизить размерность, подставив все известные значения. Также матрица станет симметричной

Структура матрицы системы ур-й



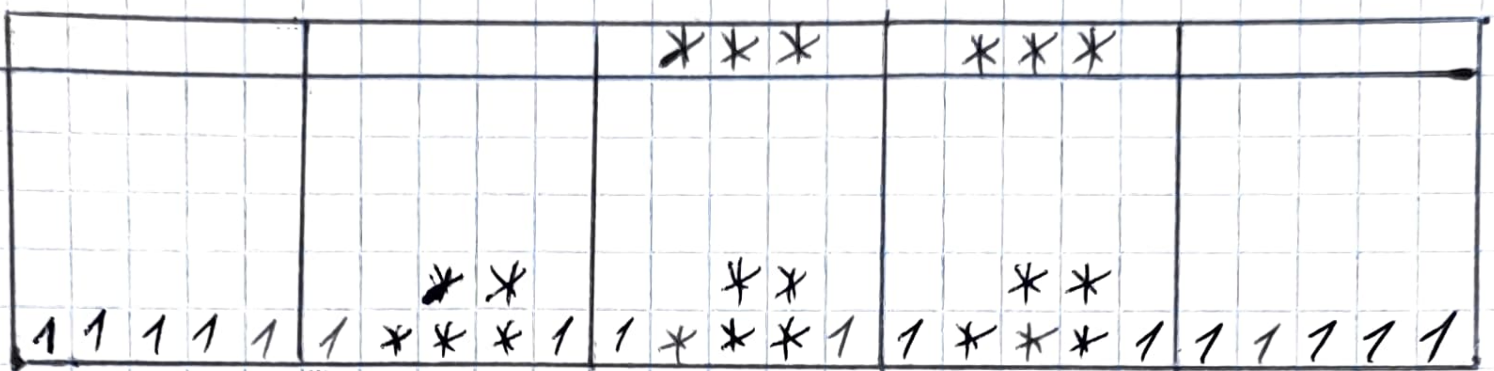
$m_L = L$, $m_U = L$, $L = N_x + 1$, $ABD(m_L + m_U + 1 + m_L, N)$

$PA = LU$

Структура матр. в сист. ур-й для сим. матр.

$m_u = L$, $L_* = N_x + 1$, $N = (N_x+1)(N_y+1)$

$ABD(m_u+1, N)$ $A = A^T$, $\lambda(A) > 0$, $A = LL^T$



Структура матр. сист. ур-й

$$AV = F, \quad A \in \mathbb{R}^{N\times N}, \quad N = (N_x+1)(N_y+1), \quad V, F \in \mathbb{R}^N$$

$$A = \begin{bmatrix} C_0 & B_0 & & & & & \\ A_1 & C_1 & B_1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & A_j & C_j & B_j & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & A_{N_y-1} & C_{N_y-1} & B_{N_y-1} \\ & & & & & A_{N_y} & C_{N_y} \end{bmatrix}, \quad V = \begin{bmatrix} V_0 \\ V_1 \\ \dots \\ V_j \\ \dots \\ V_{N_y-1} \\ V_{N_y} \end{bmatrix}, \quad F = \begin{bmatrix} F_0 \\ F_1 \\ \dots \\ F_j \\ \dots \\ F_{N_y-1} \\ F_{N_y} \end{bmatrix}$$

$$A_j = \begin{bmatrix} * & & & & \\ & * & & & \\ & & \dots & & \\ & & & * & \\ & & & & * \end{bmatrix}, \quad C_j = \begin{bmatrix} * & * & & & \\ * & * & * & & \\ & \cdot & \cdot & \cdot & \\ & & * & * & * \\ & & & * & * \end{bmatrix}, \quad B_j = \begin{bmatrix} * & & & & \\ & * & & & \\ & & \dots & & \\ & & & * & \\ & & & & * \end{bmatrix}$$

$$A_j, C_j, B_j \in \mathbb{R}^{(N_x+1)\times(N_x+1)}$$

$$= \begin{bmatrix} V_{0,j} \\ V_{1,j} \\ \dots \\ V_{N_x,j} \end{bmatrix}, \quad F_j = \begin{bmatrix} f_{0,j} \\ f_{1,j} \\ \dots \\ F_{N_x,j} \end{bmatrix} \qquad V_j, F_j \in \mathbb{R}^{N_x+1}$$

Метод матр. прогонки

$$\begin{cases} C_j V_j + B_j V_{j+1} = F_j & , j = 0 \\ A_j V_{j-1} + C_j V_j + B_j V_{j+1} = F_j & , j = 1, 2, \dots, N_y - 1 \\ A_j V_{j-1} + C_j V_j = F_j & , j = N_x \end{cases}$$

$$V_j = \alpha_{j+1} V_{j+1} + \beta_{j+1}, \quad j = 0, 1, \dots, N_y - 1$$

α_j и β_j - прогоночные коэф., $\alpha_j \in \mathbb{R}^{(N_x+1)\times(N_x+1)}$

$\beta_j \in \mathbb{R}^{N_x+1}$

$$\alpha_1 = -C_0^{-1} B_0, \quad \beta_1 = C_0^{-1} F_0$$

$j = 1, 2, \dots, N_y - 1$

$$\alpha_{j+1} = -(A_j \alpha_j + C_j)^{-1} B_j$$

$$\beta_{j+1} = (A_j \alpha_j + C_j)^{-1} (F_j - A_j B_j)$$

$$V_{N_y} = (A_{N_y} \alpha_{N_y} + C_{N_y})^{-1} (F_{N_y} - A_{N_y} \beta_{N_y})$$

$$j = N_y - 1, N_y - 2, \ldots, 1, 0$$

$$V_j = \alpha_{j+1} V_{j+1} + \beta_{j+1}$$

$\det C_j \neq 0$ $\quad \| C_0^{-1} B_0 \| \leq 1, \| C_{N_y}^{-1} B_{N_y} \| \leq 1, \| C_j^{-1} A_j \| + \| C_j^{-1} B_j \| \leq$

$\| \alpha_j \| \leq 1$ $\quad \det (A_j \alpha_j + C_j) \neq 0$