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Lucas's Tree Model

$$E\sum_{t=0}^{\infty}\beta^t u(c_t)$$

$$c_t + \pi_{t+1}p_t \le \pi_t p_t + \pi_t y_t$$
$$y_{t+1} = G(y_t, \xi_{t+1})$$

Dynamic Programming

$$v(\pi, y) = \max_{c,\pi'}\left\{u(c) + \beta\int v\big(\pi', G(y,z)\big)\phi(dz)\right\}$$

We can invoke the fact that utility is increasing to claim equality and hence eliminate the constraint, obtaining

$$c(\pi, y) = \pi\big(y + p(y)\big) - \pi' p(y)$$
$$v(\pi, y) = \max_{\pi'}\left\{u\big[\pi\big(y + p(y)\big) - \pi' p(y)\big] + \beta\int v\big(\pi', G(y,z)\big)\phi(dz)\right\}$$

What we need to do now is determine equilibrium prices.

It seems that to obtain these, we will have to

1. Solve this two-dimensional dynamic programming problem for the optimal policy.
2. Impose equilibrium constraints.
3. Solve out for the price function $p(y)$ directly.

However, as Lucas showed, there is a related but more straightforward way to do this.

FOC

$$-u'(c)p(y) + \beta\int \frac{\partial v}{\partial \pi'}\phi(dz) = 0$$
$$\frac{\partial v}{\partial \pi} = u'(c)[y + p(y)]$$

Equilibrium

$$c_t = y_t$$

$$p(y) = \beta\int \frac{u'\big(G(y_t, \xi_{t+1})\big)}{u'(y_t)}\big[G(y_t, \xi_{t+1}) + p\big(G(y_t, \xi_{t+1})\big)\big]\phi(dz)$$

i.e.

$$p_t = E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}(y_{t+1} + p_{t+1})\right]$$

Lucas Indirect Method to solve functional equation

$$f(y) \coloneqq u'p$$

$$f(y) = h(y) + \beta \int f[G(y,z)]\phi(dz)$$

Operator

$$(Tf)(y) = h(y) + \beta \int f[G(y,z)]\phi(dz)$$

Prove:

$$\begin{aligned}
|\boldsymbol{T}f(y) - \boldsymbol{T}g(y)| &= \left|\beta \int f[\boldsymbol{G}(y,z)]\phi(dz) - \beta \int g[\boldsymbol{G}(y,z)]\phi(dz)\right| \\
&\leq \beta \int |f[\boldsymbol{G}(y,z)] - g[\boldsymbol{G}(y,z)]|\, \phi(dz) \\
&\leq \beta \int \|f - g\| \phi(dz) \\
&= \beta \|f - g\|
\end{aligned}$$