Planning Problem

A planner chooses an allocation $\{\vec{C}, \vec{K}\}$ to maximize (1) subject to (4).

Let $\vec{\mu} = \{\mu_0, \ldots, \mu_T\}$ be a sequence of nonnegative **Lagrange multipliers**.

To find an optimal allocation, form a Lagrangian

$$\mathcal{L}(\vec{C}, \vec{K}, \vec{\mu}) = \sum_{t=0}^{T} \beta^t \left\{ u(C_t) + \mu_t \left(F(K_t, 1) + (1-\delta)K_t - C_t - K_{t+1} \right) \right\}$$

and then pose the following min-max problem:

$$\min_{\vec{\mu}} \max_{\vec{C}, \vec{K}} \mathcal{L}(\vec{C}, \vec{K}, \vec{\mu}) \tag{5}$$

- **Extremization** means maximization with respect to $\vec{C}, \vec{K}$ and minimization with respect to $\vec{\mu}$.
- Our problem satisfies conditions that assure that required second-order conditions are satisfied at an allocation that satisfies the first-order conditions that we are about to compute.

Before computing first-order conditions, we present some handy formulas.

First-order necessary conditions

We now compute **first order necessary conditions** for extremization of the Lagrangian:

$$C_t: \qquad u'(C_t) - \mu_t = 0 \qquad \text{for all} \quad t = 0, 1, \ldots, T \tag{7}$$

$$K_t: \qquad \beta \mu_t \left[(1-\delta) + f'(K_t) \right] - \mu_{t-1} = 0 \qquad \text{for all} \quad t = 1, 2, \ldots, T \tag{8}$$

$$\mu_t: \qquad F(K_t, 1) + (1-\delta)K_t - C_t - K_{t+1} = 0 \qquad \text{for all} \quad t = 0, 1, \ldots, T \tag{9}$$

$$K_{T+1}: \qquad -\mu_T \le 0, \ \ \le 0 \text{ if } K_{T+1} = 0; \ \ = 0 \text{ if } K_{T+1} > 0 \tag{10}$$

In computing (9) we recognize that of K_t appears in both the time t and time $t-1$ feasibility constraints.

(10) comes from differentiating with respect to K_{T+1} and applying the following **Karush-Kuhn-Tucker condition** (KKT) (see Karush-Kuhn-Tucker conditions):

$$\mu_T K_{T+1} = 0 \tag{11}$$

Combining (7) and (8) gives

$$u'(C_t) \left[(1-\delta) + f'(K_t) \right] - u'(C_{t-1}) = 0 \quad \text{for all } t = 1, 2, \ldots, T+1$$

which can be rearranged to become

$$u'\left(C_{t+1}\right)\left[(1-\delta)+f'\left(K_{t+1}\right)\right]=u'\left(C_t\right)\quad \text{for all } t=0,1,\ldots,T \tag{12}$$

Applying the inverse of the utility function on both sides of the above equation gives

$$C_{t+1}=u'^{-1}\left(\left(\frac{\beta}{u'(C_t)}[f'(K_{t+1})+(1-\delta)]\right)^{-1}\right)$$

which for our utility function (2) becomes the consumption **Euler equation**

$$\begin{aligned} C_{t+1} &= \left(\beta C_t^{\gamma}[f'(K_{t+1})+(1-\delta)]\right)^{1/\gamma} \\ &= C_t\left(\beta[f'(K_{t+1})+(1-\delta)]\right)^{1/\gamma} \end{aligned}$$

Below we define a `jitclass` that stores parameters and functions that define our economy.