Computing Equilibrium

We formulate a linear Markov perfect equilibrium as follows.

Player i employs linear decision rules $u_{it} = -F_{it} x_t$, where F_{it} is a $k_i \times n$ matrix.

A Markov perfect equilibrium is a pair of sequences $\{F_{1t}, F_{2t}\}$ over $t = t_0, \ldots, t_1 - 1$ such that

- $\{F_{1t}\}$ solves player 1's problem, taking $\{F_{2t}\}$ as given, and
- $\{F_{2t}\}$ solves player 2's problem, taking $\{F_{1t}\}$ as given

If we take $u_{2t} = -F_{2t} x_t$ and substitute it into (6) and (7), then player 1's problem becomes minimization of

$$\sum_{t=t_0}^{t_1-1} \beta^{t-t_0} \left\{ x_t' \Pi_{1t} x_t + u_{1t}' Q_1 u_{1t} + 2 u_{1t}' \Gamma_{1t} x_t \right\} \tag{8}$$

subject to

$$x_{t+1} = \Lambda_{1t} x_t + B_1 u_{1t}, \tag{9}$$

where

- $\Lambda_{it} := A - B_{-i} F_{-it}$
- $\Pi_{it} := R_i + F_{-it}' S_i F_{-it}$
- $\Gamma_{it} := W_i' - M_i' F_{-it}$

This is an LQ dynamic programming problem that can be solved by working backwards.

Decision rules that solve this problem are

$$F_{1t} = (Q_1 + \beta B_1' P_{1t+1} B_1)^{-1} (\beta B_1' P_{1t+1} \Lambda_{1t} + \Gamma_{1t}) \tag{10}$$

where P_{1t} solves the matrix Riccati difference equation

$$P_{1t} = \Pi_{1t} - (\beta B_1' P_{1t+1} \Lambda_{1t} + \Gamma_{1t})' (Q_1 + \beta B_1' P_{1t+1} B_1)^{-1} (\beta B_1' P_{1t+1} \Lambda_{1t} + \Gamma_{1t}) + \beta \Lambda_{1t}' P_{1t+1} \Lambda_{1t} \tag{11}$$

Similarly, decision rules that solve player 2's problem are

$$F_{2t} = (Q_2 + \beta B_2' P_{2t+1} B_2)^{-1} (\beta B_2' P_{2t+1} \Lambda_{2t} + \Gamma_{2t}) \tag{12}$$

where P_{2t} solves

$$P_{2t} = \Pi_{2t} - (\beta B_2' P_{2t+1} \Lambda_{2t} + \Gamma_{2t})' (Q_2 + \beta B_2' P_{2t+1} B_2)^{-1} (\beta B_2' P_{2t+1} \Lambda_{2t} + \Gamma_{2t}) + \beta \Lambda_{2t}' P_{2t+1} \Lambda_{2t} \tag{13}$$

Here, in all cases $t = t_0, \ldots, t_1 - 1$ and the terminal conditions are $P_{it_1} = 0$.

The solution procedure is to use equations (10), (11), (12), and (13), and "work backwards" from time $t_1 - 1$.

Since we're working backward, P_{1t+1} and P_{2t+1} are taken as given at each stage.

Moreover, since

- some terms on the right-hand side of (10) contain F_{2t}
- some terms on the right-hand side of (12) contain F_{1t}

we need to solve these $k_1 + k_2$ equations simultaneously.