$$v(x) = \max_{y,u}\{-y'Py - u'Qu\}$$

$$s.t. \quad y = Ax + Bu$$

$$L = -y'Py - u'Qu + \lambda'(Ax + Bu - y)$$

FOC

$$\frac{\partial L}{\partial y} = -2Py - \lambda = 0$$

$$\lambda = -2Py$$

$$\frac{\partial L}{\partial u} = -2Qu - B'\lambda = 0$$

$$Qu + B'Py = 0$$

$$Qu + B'P(Ax + Bu) = 0$$

$$(Q + B'PB)u + B'PAx = 0$$

$$u = -(Q + B'PB)^{-1}B'PAx = -Sx$$

Optimal Value

$$v(x) = \max_{y,u}\{-y'Py - u'Qu\}$$

$$v(x) = \max_{y,u}\{-(Ax + Bu)'P(Ax + Bu) - u'Qu\}$$

$$v(x) = -x'A'PAx - 2u'B'PAx - u'(Q + B'PB)u$$