Human Capital Growth in the Framework of Mean Field Games with Common Noise

Peilin Yang
(**Preliminary Idea**)

1 Introduction

A series of literature has explored the relationship between human capital and economic growth, including works by Becker [2009], Lucas Jr [1988], Barro and Lee [1994], Mankiw et al. [1992], Schultz [1961]. The problem of human capital accumulation with heterogeneity can be studied in the framework of mean field games Lasry and Lions [2007] Carmona et al. [2018]. This involves modeling a continuum of individuals whose wages depend not only on their own human capital, but also on the distribution of human capital in the population. This distribution dependency is important for understanding competition in the labor market and the ease or difficulty of improving human capital depending on proximity to the technological frontier.

In the first section of this paper, I present the setup of my model and derive a solution using classical methods, such as the Euler Equation. In the second section, I use mean field game partial differential equations to solve the problem in a different way and generalize the solution to a stochastic framework. Finally, I try to introduce the common noise to the accumulation process of human capital in the last part of the paper.

2 The model

2.1 Introduction

There is initially a working population of size 1 with a given distribution of human capital. Human capital will be denoted h and the distribution, at time t, will be referred to as $f_H(q,t)$. Agents' wage not only depends on the individual human capital but also on the scarcity of her specific human capital. In other words, the salary of a worker with human capital h is, at time t, given by:

$$w(h,t) = w(h, f_H(h,t)) \tag{1}$$

The logic behind this equation is to model a competition effect in the labor market: unskilled people have a small salary. The wage is set by firms. A standard production function for a representative firm would be

$$Y = A \int_h q_h^\alpha L_h^{1-\beta} \quad \alpha > 0, \beta \in (0;1]$$

Agents live forever, and they can improve their human capital with a cost depending on two factors. First, the cost, in monetary terms, is a function of human capital change and, second, it is also a function of the position of initial human capital in the distribution. More precisely, I will assume that the cost (in monetary terms) at time t is given by:

$$C\left(h, \frac{dh}{dt}, t\right) = C\left(\frac{dq}{dt}, 1 - F_H(h,t)\right) \tag{2}$$

where $F_H(h,t) = \int_h^\infty f_H(x,t)dx$ is the number of people in the population with a human capital greater than h. That is to say it is more costly for skilled workers to improve their human capital than for unskilled workers. This hypothesis is relevant since it is often more difficult to improve human capital for an individual in the right tail of the distribution since she is near the technological frontier.

2.2 The optimization problem

Suppose that people improve their human capital constantly. Human capital accumulation can be seen as the consequence of on-the-job training.

Each individual chooses her effort continuously to maximize her life-time earning.

$$\max_{(h_s), h_0 = h} \int_0^\infty \left[w\left(h_s, f_H\left(h_s, s\right)\right) - C\left(\frac{dq}{dt}\left(h_s, s\right), 1 - F_H(h_s, t)\right)\right] e^{-rs} ds \tag{3}$$

with a wealth constraint $\dot{s}_t \leq rs_t + \left(w(h,t) - C\left(h, \frac{dh}{dt}, t\right)\right) - c_t$. This gives a unique intertemporal constraint that is:

$$\int_0^\infty c_t e^{-rt} dt \leq s_0 + \int_0^\infty \left(w(h,t) - C\left(h, \frac{dh}{dt}, t\right)\right) e^{-rt} dt$$

r is exogenous.

2.3 Resolution

2.3.1 A specific setup

To solve the problem I need to specify the two functions w and C. My specification is the following:

$$w(h, f_H(h,t)) = \begin{cases} \bar{w} \frac{h^\alpha}{f_H(h,t)^\beta}, & \text{if } h \text{ is in the support of } f_H(t,\cdot) \\ 0 & \text{otherwise} \end{cases}$$

$$C\left(\frac{dq}{dt}, 1 - F_H(h,t)\right) = \frac{\bar{C}}{\varphi} \frac{\left(\frac{dq}{dt}\right)^\varphi}{(1 - F_H(h,t))^\delta}, \quad \forall q \text{ in the support of } f_H(t,\cdot)$$

where $\bar{w}$ and $\bar{C}$ are two constants and where α, β, δ and φ are fmy positive parameters subject to technical constraints that are: $\alpha + \beta = \varphi$, $\beta = \delta$ and I want typically φ to be strictly greater than 1 . This specification can be considered *adhoc* but in fact it must be regarded as quite general since I have two degrees of freedom to choose the parameters. An easier specification without any degree of freedom would have been to set $\alpha = \beta = \delta = 1$ and $\varphi = 2$ but I want to remain quite general.

Now, I assume that the initial distribution of human capital is a Pareto distribution. I can use a normalization and assume that the minimal point of the initial distribution is 1. The Pareto coefficient of the initial distribution is denoted k, so that:

$$f_H(h,0) = k\frac{1}{h^{k+1}}1_{h\geq 1}$$

This Pareto distribution is central in the study of economic inequalities and will be stable in my model in the sense that my solution involves Pareto distributions of human capital at all time.

The optimal path has to satisfy the following Euler equation:

$$\partial_h \tilde{w}_t - \partial_h \tilde{C} = -\frac{d}{dt}\left[\partial_{\dot{h}} \tilde{C}\right] + r\partial_{\dot{h}} \tilde{C} \tag{4}$$

3 Stochastic framework

3.1 The mean field game partial differential equations

The problem involves indeed the probability distribution function and the tail function of the human capital across the population and the mean field games partial differential equations are in a way far more relevant to solve the problem.

Let's first introduce the Bellman function of the problem:

$$J(h,t) = \max_{(h_s), h_t = h} \int_t^{\infty} \left[w\left(h_s, m\left(s, h_s\right)\right) - C\left(\frac{dh}{dt}\left(s, h_s\right), 1 - F_H\left(s, h_s\right)\right)\right] e^{-r(s-t)} ds \tag{5}$$

my optimization problem can be represented by the two following PDEs:

$$w(q, f_H(h,t)) + \partial_t J + \max_{\frac{dh}{dt}}\left(\frac{dh}{dt}\partial_h J - C\right) - rJ = 0 \quad \text{(HJB)} \tag{6}$$

$$\partial_t f_H(h,t) + \partial_h\left(\frac{dh}{dt}(h,t) f_H(h,t)\right) = 0 \quad \text{(Kolmogorov)} \tag{7}$$

In the special case I solved, the two equations can be written as:

$$C\frac{h^{\alpha}}{f_H(h,t)^{\beta}} + \frac{\varphi - 1}{\varphi}\frac{1}{E^{\frac{1}{\varphi-1}}}\bar{F}(h,t)^{\frac{\beta}{\varphi-1}}\left(\partial_h J\right)^{\frac{\varphi}{\varphi-1}} + \partial_t J - rJ = 0$$

$$\partial_t f_H(h,t) + \partial_h\left(\left(\frac{\bar{F}(h,t)^{\beta}}{E}\partial_h J(h,t)\right)^{\frac{1}{\varphi-1}} f_H(h,t)\right) = 0$$

and the optimal control is given by:

$$\frac{dh}{dt}(h,t) = \left(\frac{\bar{F}(h,t)^{\beta}}{E}\partial_h J(h,t)\right)^{\frac{1}{\varphi-1}}$$

3.2 The model with common noise

So far, my model was completely deterministic whereas most applications of mean field games are in a random setting. Here, it is not natural to introduce a specific noise for each individual if I want to keep explicit solutions with f_H being a Pareto distribution (because of the threshold). However, it's possible to introduce randomness through a common noise on the evolution of the human capital.

More precisely, I can replace the dynamics for h by a stochastic one:

$$dh_t = \dot{h}\left(t, h_t\right) dt + \sigma h_t dW_t \tag{8}$$

where W is a noise common to all agents. This specification seems complicated since all the functions J, m and $\bar{F}$ are now random variables. However, the intuitions developed above can be applied *mutatis mutandis* and my specification is robust and deep enough to be generalized to a complex stochastic framework.

First, consider the Bellman function J. I can see that the expression for J was in fact a function of q and of the lower bound h^m of the q 's so that it's more natural in general to define here $J = J\left(t, q, h^m\right)$ as:

$$\max_{(h_s)_{s>t}, h_t = q, h_t^m = h^m} \mathbb{E}\left[\int_t^\infty \left[C\frac{h^\alpha}{f_H(h,t)^\beta} - \frac{E}{\varphi}\frac{\frac{dh}{dt}(h,t)^\varphi}{\bar{F}(h,t)^\beta}\right] e^{-r(s-t)} ds \mid \mathcal{F}_t\right]$$

The HJB equation corresponding to the above optimization problem can be written in the following differential form:

$$\max_{\dot{h}} C\frac{h^\alpha}{f_H(h,t)^\beta} - \frac{E}{\varphi}\frac{a^\varphi}{\bar{F}(h,t)^\beta} - rJ$$
$$+\partial_t J + a\partial_h J + \frac{\sigma^2}{2}h^2\partial^2_{hh}J + a'\partial_{h^m}J + \frac{\sigma^2}{2}q^{m2}\partial^2_{h^m h^m}J + \sigma^2 q h^m \partial^2_{hh^m}J = 0$$

The optimal control function is given by the same expression as in the deterministic case:

$$\frac{dh}{dt}(h,t) = \left(\frac{\bar{F}(h,t)^\beta}{E}\partial_h J(h,t)\right)^{\frac{1}{\varphi-1}}$$

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