Shock Response of Fully Funded System: HANK Framework

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Abstract

China's pension system has been undergoing reformation these years. Fully funded pension is one of the direction that government explores to address the old-aging issues of China's economy. Our main goal is to model the fully funded system and predict the shock response of that. For the sake of idiosyncratic incoming has different marginal propensity consumption(MPC), we build a heterogeneous-agent New Keynesian(HANK) model, which can descript the relationship of incomes difference and pension assets allocation. We use an emprical model to verify the correlation to macro variables, and then construct the model to predict the effects of new pension system. As it turns out, technology shock has difference on these two regimes.

1 Introduction

Since the 1980s, along with the rapid aging of the population structure of Western developed countries, the existing pay-as-you-go pension systems in various countries have also faced a growing financial burden. In view of this problem, the more common view is :In an ageing society. This was not a problem in China in the past, but today these concerns are increasing, the rising old-age dependency ratio is transformed into a growing tax burden, and the rich pensions and medical benefits are crowded The public investment expenditure on infrastructure or education has a negative impact on capital accumulation and productivity growth. Acemoglu and Johnson, 2007 estimated the impact of life expectancy at birth on economic growth. They also found no evidence of positive effects. Pensions and retirement policies are very important for a country's welfare improvement and economic development. Most developed countries have begun to raise the retirement age or tighten the conditions for early retirement, reducing the pressure on social security taxes (OECD, 2007; OECD, 2009). Some scholars believe that postponing retirement can reduce the dependency ratio (the ratio of non-working-age population to working-age population) and relieve the pressure of pension payment to a certain extent, but aging is the trend of social development, and delaying the retirement age cannot Fundamentally solve the problem. Statistics of the World Social Security Research Data from the Chinese Academy of Social Sciences show that the basic pension insurance fund for urban enterprise employees will run into deficit in 2028 and be used up by 2035 when the enterprise benchmark level is 16%. Many media joked that this means "the descendants of the 1980s will not have the money to feed the

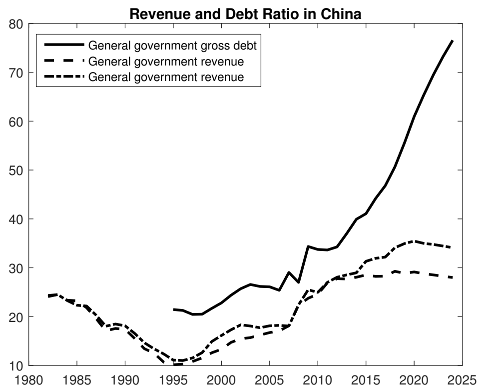

Figure 1: Revenue and Debt Ratio in China(1980-2025)

elderly." At the same time, China is entering an aging society, and the working-age population is decreasing. According to data from the National Bureau of Statistics, China's population aged 60 and over was 249 million in 2018, accounting for 17.9% of the total population. According to the 2015 prediction from the National Aging Office, by 2022, the elderly population over 60 will increase to 268 million, and this proportion will rise to 18.5%. By 2036, the proportion of the elderly population will further increase to 423 million, and the proportion will increase to 29.1%. By 2053, the proportion of the elderly population will reach a peak of 487 million, accounting for 34.8% of the national population. The World Social Security Research Center of the Chinese Academy of Social Sciences pointed out that the institutional pressure to support the payment of basic pension insurance for urban employees will continue to increase in the future. In short, retirees will be supported by nearly two contributors in 2019, but only one person will support one retiree around 2050. China's existing retirement system dates back to the 1950s. At that time, the average life expectancy of Chinese people was only 45 years old. Almost all the labor was high-intensity manual labor, mechanization and modernization. Therefore, we hope to improve the economic situation by changing backward retirement policies.

An effective policy tool is the fund accumulation system. The pay-as-you-go system will have a payment crisis, while the fund accumulation system does not have such problems. Under the guidance of this concept, a wave of reforms in the world that gradually replaced the pay-as-you-go system with a fund accumulation system has emerged. In particular, some countries represented by Chile have successfully implemented the pension insurance system from the current income. The pay-as-you-go system was transformed into a privatization of a complete fund system. The fund accumulation system completely got rid of the social security system's restrictive effect on economic development. The fund accumulation system put its national economy on a healthy track. The fund accumulation system differs from most other countries in that China's pension system reform in the 1990s faced two basic tasks from the beginning, namely, the "dual" system conversion problem: one is to cooperate with China's socialist market economy The reform of the system, the fund accumulation system established a socialized retirement protection system, and completed the transformation of pension insurance from enterprise to socialization;

the second is the transformation from the original single pay-as-you-go system to the multi-pillar pension insurance system. China's pension is mainly based on three pillars: basic pension, enterprise annuity and voluntary personal savings pension plan. The first pillar here is absolute in coverage and quantity. But China's pension system also faces huge problems: first, the pressure of aging; second, the pressure of aging. Second, coverage is still insufficient, especially the second and third pillars. Third, the rate of return on funds is insufficient. There are many articles on pension system reform. Fehr, Kallweit, and Kindermann, 2013 first proposed a mixed pension system, which includes pay-as-you-go and salary-related pensions. He, Ning, and Zhu, 2019 used the system to study China's labor supply in 2015. In the process of the fund accumulation system, in addition to paying attention to whether the Chinese pension insurance system can achieve financial sustainability, the fund accumulation system is also very Pay attention to a series of important issues closely related to pension insurance, such as: How will the transformation of the pension insurance system affect China's economic growth, and the reform of the original complete pay-as-you-go system and the introduction of the fund accumulation system will affect residents Does household savings have a "crowd-out effect" and how pension system reforms will change the way income is redistributed.

In order to analysis the effects of pension system reformation, we introduce a new Keynesian framework. It includes some main characteristic of typical scenarios: sticky price, wage. As Auclert, Rognlie, and Straub, 2018, for the sake of solve the weak wealth effect (Auclert and Rognlie, 2018), the labor supply is determined by labor union. Labor union will decide the labor supply for each kind of product in the intermediate market. Agents only choose optimal consumption and assets allocation, and then maximize the utility of whole life. We make the pension asset receive the interest rate of present value of R years later, and R is the retirement age of agents.

With respect the numerical solution, we use the Sequence-Jacobian algorithm mentioned in Auclert, Bardóczy, Rognlie, and Straub, 2019. It is an algorithm with high efficiency since it can avoid most of unnecessary computation in the process of finding equilibrium; it's only necessary for us to focus some key variables we want to find. The first part we explore the empirical evidence in China, namely the effect of macro variables on pension system. And we build a new Keynesian model to predict how fully funded system will affect the equilibrium state of the economy, including MIT shocks.

2 Empirical Evidence

In order to predict the effect of macro variables, we establish following specification:

$$\begin{aligned}\Delta \text{ Pension }_t = \beta_0 + \beta_1 r_t + \beta_2 Debt_t + \beta_3 Z_t + \beta_4 \Delta C_t \times r_t + \beta_5 \Delta Y_t \times r_t + \beta_6 \Delta C_t \times Debt_t \\ + \beta_7 \Delta Y_t \times Debt_t + \beta_8 \Delta Y_t \times Z_t + \beta_9 \Delta C_t \times Z_t + \epsilon_t\end{aligned}$$

r_t is the interest of the end of year. Debt$_t$ is the debt of government, which represents the investment and government purchasing. Z_t is the investment of technology, $\Delta C_t, \Delta Y_t$ are the change proportion of them. The reason why we choose them because that can represent the response of shock.

Since the national plan has great impact on Chinese economy and path of development. We also want to focus on the effect of nation plan and leadership alternate. By controlling time fixed effect, we choose to set leadership dummy variable D_i.

$$\Delta \text{ Pension }_{t,i} = \beta_0 + \beta_1 r_{t,i} + \beta_2 Debt_{t,i} + \beta_3 Z_{t,i} + \beta_4 \Delta C_{t,i} \times r_t + \beta_5 \Delta Y_{t,i} \times r_t + \beta_6 \Delta C_{t,i} \times Debt_t + \beta_7 \Delta Y_{t,i} \times Debt_t + \beta_8 \Delta Y_{t,i} \times Z_t + \beta_9 \Delta C_{t,i} \times Z_t + \beta_{10} D_i + \epsilon_{t,i}$$

	(1)	(2)	(3)	(4)
Δ Pension				
$Debt_t$	-0.00589**	-0.00630**	-0.00553*	-0.00469
	(-2.364)	(-2.252)	(-1.889)	(-1.520)
r_t	0.0920**	0.0921**	0.119**	0.108*
	(2.478)	(2.410)	(2.340)	(1.978)
Z_t	1.23e-05***	1.24e-05***	1.33e-05***	1.06e-05**
	(3.585)	(3.547)	(3.581)	(2.751)
$\Delta Y \times G$		-8.828	-9.718	-17.31*
		(-1.304)	(-1.541)	(-1.974)
$\Delta C \times G_t$		8.288	8.780	15.99*
		(1.312)	(1.541)	(2.051)
$\Delta Y \times r_t$			-2.217	-0.169
			(-0.578)	(-0.0434)
$\Delta C \times r_t$			0.648	-1.408
			(0.147)	(-0.311)
$\Delta Y \times Z_t$				4.933
				(1.526)
$\Delta C \times Z_t$				-6.885*
				(-1.760)
Constant	-0.0493	-0.0357	-0.143	-0.0729
	(-0.386)	(-0.268)	(-0.751)	(-0.329)
Time FE	NO	NO	NO	NO
R-squared	0.508	0.536	0.607	0.651
Ajusted R2	0.476	0.476	0.476	0.476

Robust t-statistics in parentheses
*** p<0.01, ** p<0.05, * p<0.1

	(1)	(2)	(3)	(4)
Δ Pension				
$Debt_t$	-0.00642**	-0.00704**	-0.00619	-0.00569
	(-2.106)	(-2.339)	(-1.730)	(-1.441)
r_t	0.117***	0.120***	0.138***	0.130**
	(3.206)	(3.358)	(3.014)	(2.645)
Z_t	4.98e-06	4.50e-06	6.19e-06	5.41e-06
	(1.345)	(1.236)	(1.577)	(1.291)
$\Delta Y \times G$		-12.01*	-11.55	-16.51
		(-1.738)	(-1.525)	(-1.554)
$\Delta C \times G_t$		11.43	10.73	15.41
		(1.575)	(1.342)	(1.423)
$\Delta Y \times r_t$			0.0693	0.936
			(0.0106)	(0.136)
$\Delta C \times r_t$			-1.203	-2.084
			(-0.168)	(-0.275)
$\Delta Y \times Z_t$				3.483
				(0.698)
$\Delta C \times Z_t$				-4.340
				(-0.805)
Constant	-0.0448	-0.0270	-0.119	-0.0819
	(-0.278)	(-0.171)	(-0.570)	(-0.368)
R-squared	0.570	0.629	0.660	0.674
Time FE	YES	YES	YES	YES
Ajusted R2	0.450	0.450	0.450	0.450

t-statistics in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Before the fixed effect is controlled, we can see that interest, technology investment and debt have significant impact on them. Government investments and debt manifest negative correlation, and interest rate and technology investment have positive effect on it. After controlling the time fixed effect, the technology investments turn into insignificant variable. Now we prove pension has directive correlation with macro economy. We will model the economy to predict the introduction of fully funded system in next part.

3 The Model Economy

3.1 Demographic

Time is discrete and runs from $t = 0$ to ∞. The economy is populated by a unit mass of agents, or households, who face no aggregate uncertainty, but may face idiosyncratic uncertainty. Agents vary in their idiosyncratic ability state e, which follows a Markov process with fixed transition matrix Π. We assume that the mass of agents in idiosyncratic state e is always equal to $\pi(e)$, the probability of e in the stationary distribution of Π. The average ability level is normalized to be one, so that $\sum_e \pi(e)e = 1$. If agents are permanently different, Π is the identity matrix and π the initial distribution over e.

3.2 Earnings

The family faces certain budget constraints. Every household's income comes from wages, capital returns and fully funded pension. Wage income is regarding with ability e_t, wage w_t, and government proportion tax τ_t. Liquid asset a_t can get interest rate r_t^a. In addition, each term can borrow capital ≥ 0, which is a constraint for households.

We introduce the fully funded system in this specification. In order to simply our analysis of pension system, we assume the pension system can be gotten back early, but the drawback is with liquid cost $\Psi(b_t,)$ and the present value interest rate $\frac{1+r_t^b}{(1+r_t)^R}$.

As a result, the budget constraint can be represented by:

$$c_t + a_t + \frac{b_t}{(1+r_t)^{R-1}} = (1-\tau_t)\, w_t n_t e_t + (1+r_t^a)\, a_{t-1} + \frac{(1+r_t^b)\, b_{t-1}}{(1+r_t)^R} + tr_t - \Psi(b_t, a_t) \tag{1}$$

As Kaplan, Moll, and Violante 2018, the adjustment cost function is specified as

$$\Psi(b_t,) = \chi_0 |b_t| + \chi_1 \left| \frac{b_t}{\max\{a_t, \underline{a_t}\}} \right|^{\chi_2} \max\{a_t, \underline{a}\}, \quad \chi_2 > 1, \underline{a} > 0 \tag{2}$$

In order to describe the illiquid cost between the transferring between liquid assets and pension assets we set the function $\Psi(b_t,)$. The transaction cost has two components that play different roles: the linear component generates an inaction region in households' optimal deposit policies due to some households the marginal gain from depositing or withdrawing the first dollar is smaller than the marginal cost of transacting χ_0. The convex component ensures that deposit rates are finite, $|dt| < \infty$ and hence household's holdings of assets never jump. Finally, scaling the convex term by illiquid assets a above some threshold a delivers the desirable property that marginal costs $\Psi(b_t,)$ are homogeneous of degree zero in the deposit rate d/a so that the marginal cost of transacting depends on the fraction of illiquid assets transacted, rather than the raw size of the transaction. The threshold > 0 guarantees that costs remain finite for individuals with $= 0$.

3.3 Preferences

In period t, agent i enjoys the consumption of a generic consumption good c_{it} and gets disutility from working n_{it} hours, leading to a time- 0 utility of

$$\mathbb{E}\left[\sum_{t\geq 0} \beta^t \left\{u\left(c_{it}\right) - v\left(n_{it}\right)\right\}\right]$$

The utility function is separable felicity function; agents enjoy the consumption of a generic consumption good c_t and get disutility from working hours n_t.

$$u\left(c_t, n_t\right) = \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{n_t^{1+v}}{1+v} \tag{3}$$

Therefore, the dynamic programming problem for agents is

$$V_t\left(e_t, b_{t-1}, a_{t-1}\right) = \max_{c_t, b_t, a_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{n_t^{1+v}}{1+v} + \beta E_t V_{t+1}\left(e_{t+1}, b_t, a_t\right) \right\} \tag{4}$$

This specification has three state variables e_t, b_{t-1}, a_{t-1}, where e_t is individual productivity following a Markov process as in the other models. All in all, the household block takes as inputs the sequences of interest rates $\{r_s^a, r_s^b\}$, wage per efficiency units $\{w_s\}$, labor tax rate $\{\tau_s\}$ and labor demand $\{N_s\}$ as inputs. The relevant outputs are illiquid asset demand, liquid asset demand, productivity-weighted marginal utility, consumption, and portfolio adjustment costs. Similar to İmrohoroğlu and Zhao 2020, the state transition follows a Markov chain, which represents the transition between 3 states, namely health and disease.

$$\Pi(\eta, \eta^{'}) = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \tag{5}$$

p_{11} denotes the transition of state from health to health p_{12} denotes from health to sick, and so on. In real life, this shows that the individual will decline in work efficiency due to certain factors, such as diseases.

As discrete variable, e_t will not improve the complexity of this problem. The only thing we should focus is the continuous variables a_t and b_t. We will explain the algorithm to solve this double state variable dynamic programming problem.

3.4 Production

In order to describe the sticky prices, we now assume a standard two-tier production structure. Intermediate goods are produced by a mass one of identical monopolistically competitive firms, whose shares are traded and owned by households. All firms have the same production technology, now assumed to be Cobb-Douglas in labor and capital.

$$F\left(k_{t-1}, n_t\right) = k_{t-1}^{\alpha} n_t^{1-\alpha} \tag{6}$$

Final goods firms aggregate intermediate goods with a constant elasticity of substitution $\mu/(\mu - 1) > 1$. Capital is subject to quadratic capital adjustment costs, so that the costs arising from choosing capital stocks k_{t-1} and k_t in any period t are given by $\phi(x)$,

$$\phi(x) = x - (1-\delta) + \frac{1}{2\delta\epsilon_I}(x-1)^2 \tag{7}$$

where $\delta > 0$ denotes depreciation and $\epsilon_I > 0$ is the sensitivity of net investment to Tobin's Q. Finally, any firm chooses a price p_t in period t subject to Rotemberg, 1982 adjustment costs $\xi(p_t, p_{t-1})$ where $\kappa_p > 0.24$.

$$\xi\left(p_t, p_{t-1}\right) = \frac{1}{2\kappa^p}\left(\frac{p_t - p_{t-1}}{p_{t-1}}\right)^2 \tag{8}$$

Intermediate goods producer maximize the value by adjust the sequence p_t, k_t, n_t. Subject to the final goods market requirement:

$$F(k_{t-1}, n_t) = Y_t\left(\frac{p_t}{P_t}\right)^{\frac{-\mu}{\mu-1}} \tag{9}$$

Since all these firms are identical, then we have

$$k_t = K_t \quad n_t = N_t \quad p_t = P_t \tag{10}$$

in equilibrium.

3.5 Labor Market

We assume that each household provides a continuous differentiated labor service, each represented by a union. Unions set hours and wages to maximize the average utility of their members and make their consumption-savings including other identical union decisions. Changing the nominal wage incurs quadratic adjustment costs. The programming problem of union k is

Pretax labor income is subject to a log-linear retention function as in Heathcote, Storesletten, and Violante, 2017. This retention function is indexed to real wages, so that if P_t is the nominal price of consumption goods, W_t is the nominal wage per unit of ability, and e_{it} is the agent's current ability, real after-tax income is

$$z_{it} \equiv \tau_t \left(\frac{W_t}{P_t} e_{it} n_{it} \right)^{1-\lambda}$$

$$\begin{aligned} U_t\left(w_{kt-1}\right) = \max_{n_{kt}, w_{kt}} \int \left[u\left(c_{it}\right) - v\left(n_{kt}\right)\right] dD_t - \frac{\mu_w}{1-\mu_w} \frac{1}{2\kappa_w} \left[\log\left(1+\pi_{kt}^w\right)\right]^2 N_t + \beta U_{t+1}\left(w_{kt}\right) \\ \text{s.t. } n_{kt} = \left(\frac{w_{kt}}{w_t}\right)^{-\frac{\mu w}{\mu w - 1}} N_t \end{aligned} \tag{11}$$

where π_{kt}^w is wage inflation

$$\pi_{kt}^w = \left(1+\pi_t\right) \frac{w_{kt}}{w_{kt-1}} - 1 \tag{12}$$

Same as the setup of that in Auclert, Rognlie, and Straub, 2018 and, as shown there, leads to a wage Phillips curve of the form

$$\log\left(1+\pi_t^w\right) = \kappa_w \left[\varphi N_t^{1+v} - \mu_w \left(1-\tau_t\right) w_t N_t \mathcal{U}_t\right] + \beta \log\left(1+\pi_{t+1}^w\right) \tag{13}$$

For convenience later on, let's introduce the following shorthand for wage adjustment costs

$$\psi_t^w = \frac{\mu_w}{1-\mu_w} \frac{1}{2\kappa_w} \left[\log\left(1+\pi_t^w\right)\right]^2 N_t \tag{14}$$

3.6 Monetary Policy

The monetary policy is given by Taylor's rule

$$i_t = r^* t + \phi_m \pi_t + \phi_y (Y_t - Y_{ss}) \tag{15}$$

where the coefficient ϕ_m and ϕ_y are exogenous and certain and r^* is the steady state interest rate, Y_{ss} is the steady state output and $Y_t - Y_{ss}$ represents the deviation of the output relative to the normal of that. It also meet the Fisher's equation

$$1 + r_t = \frac{1 + i_{t-1}}{1 + \pi_t} \tag{16}$$

3.7 Government's Budget

The only market clearing conditions not implied implicitly by notation are that of goods and assets. Differentiating from Auclert, Rognlie, and Straub, 2018, we introduce government transferring target to balance the budget of government.

$$\begin{aligned} Y_t &= \mathcal{C}_t + G_t + I_t + \mathcal{P}_t + \omega \mathcal{B}_t + \psi_t^p + \psi_t^w + \mathcal{TR}_t \\ \mathcal{A}_t + \mathcal{B}_t &= p_t + B^g \end{aligned} \tag{17}$$

4 Stationary Equilibrium

When the economy tends to be stable, the individual's behavior is consistent with the overall economic behavior: the production sector maximizes profits, the individual makes optimal planning for consumption and labor, and the commodity market clears. For heterogeneous agent model, we usually adopt the definition of density function to describe the dynamic transfer of economy through the transfer of density function. Therefore, we set D_t as a cross-sectional measure of the time t of the economy, which describes the distribution relationship between the number of economic individuals and asset holdings.

Figure 2 illustrates the DAG for our third example: a two-asset HANK model with household side similar to Kaplan and Violante, 2018. For households, the model features liquid and illiquid assets with convex adjustment costs of portfolio adjustment. On the supply side, it features wage as well as price rigidities, as well as capital with quadratic adjustment costs. Hence investment follows the standard q theory equations. [25]

Monetary policy follows a standard interest rate rule. The government levies a distortionary labor income tax to finance its debt and its expenditure on the final good. We assume a balanced budget. Some government bonds are held by households directly (along with firm equity) as illiquid assets, and the rest are transformed into liquid assets by a competitive financial intermediary. This liquidity transformation incurs a proportional cost, which determines the equilibrium spread between liquid and illiquid assets in all periods along perfect-foresight paths.

Competitive equilibrium includes individual decision variables $\{\tilde{c}_t^s, \tilde{k}_t^s, \tilde{l}_t^s, \tilde{d}_t^s\}$, firms' plans for production $\{K, N\}$, factor prices w_t, r_t.

1. Individual efficiency impact satisfies the transition rule of Markov chain $\Pi(e, e^{'})$.

2. In a stable state, agents meet budget constraints.

3. In this case, individual dynamic programming lifelong asset changes, consumption, and labor supply, this behavior is based on state variables e. Finally, when the economy reaches a steady state, the optimal strategy does not depend on time, but only on the current state. We can get the optimal policy function $a'(e, b, a)$, $c(e, b, a)$, $n(e, b, a)$.

4. The firm's optimal factor price sequence In the case of efficient market, the company's profit is zero.

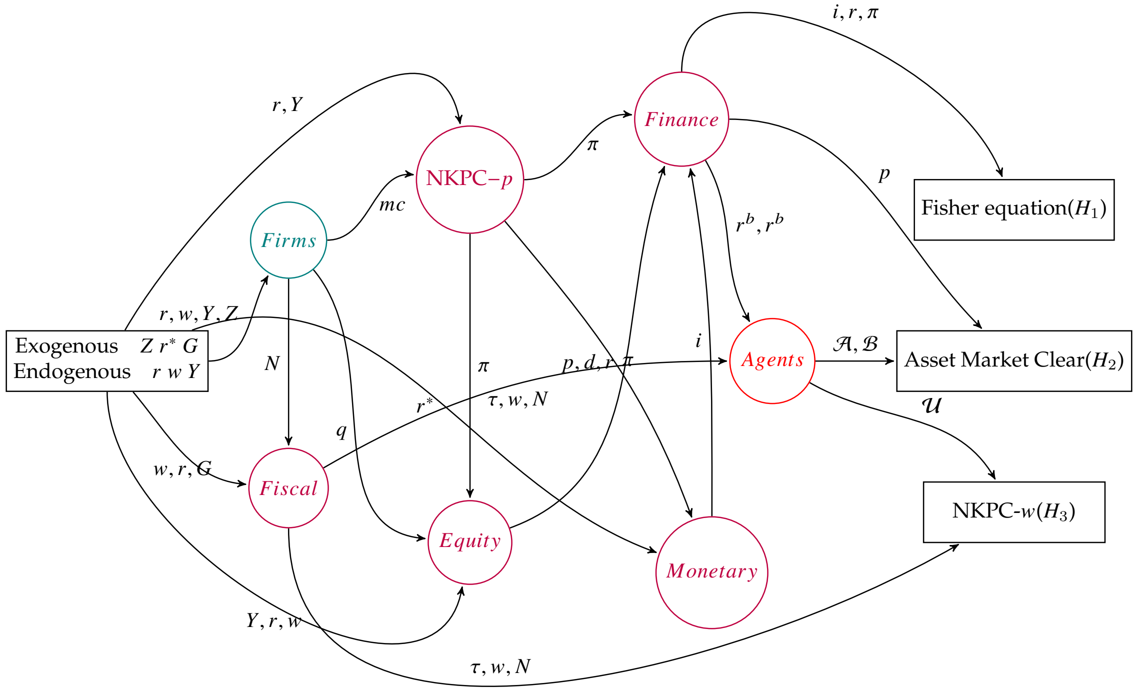

Figure 2: DAG representation of HANK economy

5. According to the density function $D_t(e,b,a)$, we can find the variables of the whole economy, which is the sum of the individual economy.

$$\mathcal{A}_t\left(\left\{r_s^a, r_s^b, w_s, \tau_s, N_s\right\}\right) = \int a dD_t(e,b,a) \tag{18}$$

$$\mathcal{B}_t\left(\left\{r_s^a, r_s^b, w_s, \tau_s, N_s\right\}\right) = \int b_t(e,b,a) dD_t(e,b,a) \tag{19}$$

$$\mathcal{U}_t\left(\left\{r_s^a, r_s^b, w_s, \tau_s, N_s\right\}\right) = \int e \cdot u(c_t(e,b,a), n_t(e,b,a))^{-\sigma} dD_t(e,b,a) \tag{20}$$

$$\mathcal{C}_t\left(\left\{r_s^a, r_s^b, w_s, \tau_s, N_s\right\}\right) = \int c_t(e,b,a) dD_t(e,b,a) \tag{21}$$

$$\mathcal{TR}_t\left(\left\{r_s^a, r_s^b, w_s, \tau_s, N_s\right\}\right) = \int tr_t(e,b,a) dD_t(e,b,a) \tag{22}$$

$$\mathcal{P}_t\left(\left\{r_s^a, r_s^b, w_s, \tau_s, N_s\right\}\right) = \int \Phi\left(a_t(e,b,a), a\right) dD_t(e,b,a) \tag{23}$$

The last two are required only for checking the omitted goods market clearing condition

6. The government budget is balanced. Transfer payment tr_t is used to balance the budget.

$$\begin{aligned} Y_t &= \mathcal{C}_t + G_t + I_t + \mathcal{P}_t + \omega \mathcal{B}_t + \psi_t^p + \psi_t^w + \mathcal{TR}_t \\ \mathcal{A}_t + \mathcal{B}_t &= p_t + B^g \end{aligned} \tag{24}$$

5 SHADE Model and Sequence-Space Jacobian Algorithm

Auclert, Bardóczy, Rognlie, and Straub, 2019 promoted an algorithm to solve these problems. The issue is so-called Sequence space Heterogeneous-Agent Dynamic Equilibrium(SHADE). These models include most heterogeneous agent neoclassical and New Keynesian models.

We can view this procedure as a mapping from exogenous shocks and unknowns $\left\{\mathbf{X}^i\right\}_{i\in\mathcal{Z}\cup\mathcal{U}}$ to targets $\left\{\mathbf{X}^o\right\}_{o\in\mathcal{H}}$. We write this mapping in more condensed form as $\mathbf{H}(\mathbf{U},\mathbf{Z})$, where $\mathbf{U}$ is defined as the stacked vector of unknown sequences $\left\{\mathbf{X}^i\right\}_{i\in\mathcal{U}}$, $\mathbf{Z}$ is defined as the stacked vector of exogenous sequences $\left\{\mathbf{X}^i\right\}_{i\in\mathcal{Z}}$, and $\mathbf{H}(\mathbf{U},\mathbf{Z})$ itself is the implied stacked vector of targets $\left\{\mathbf{X}^i\right\}_{i\in\mathcal{H}}$ since the procedure satisfies $\mathbf{X}^o = h^o\left(\left\{\mathbf{X}^i\right\}_{i\in\mathcal{I}_b}\right)$ by construction, equilibrium is then equivalent to

$$\mathbf{H}(\mathbf{U},\mathbf{Z}) = 0 \tag{25}$$

As Auclert, Bardóczy, Rognlie, and Straub, 2019, the DAG makes it easy to visualize some of the dependencies embedded in the model: for instance, the dividends from firms are distributed to households (according to a certain rule), so the output d of the firm block is an input to the HA block. Similarly, the real interest rate r affects the taxes required for the government to achieve its balanced-budget target, so r is an input to the fiscal block, which has an output τ that is an input to the HA block. Monetary policy follows a standard interest rate rule. The government levies a distortionary labor income tax to finance its debt and its expenditure on the final good. We assume a balanced budget. Some government bonds are held by households directly(along with firm equity)as illiquid assets, and the rest are transformed into liquid assets by a competitive financial intermediary. This liquidity transformation incurs aproportional cost, which determines the equilibrium spread between liquid and illiquid assets in all periods along perfect-foresight paths.

6 Calibration

In this paper, in order to adapt to the development of China's economy, our calibration strategy is to select commonly used parameters in the literature.

6.1 Labor Wage

First, we calibrate the efficiency data. The first aspect is the random impact of income, which is given by $log(\mu_j) = \theta log(\mu_{j-1}) + v_j$. Based on the research of He, Ning, and Zhu, 2019, we take $\theta = 0.86$ and variance $\sigma_v^2 = 0.06$. Then we use the technology of Imrohoroglu, Imrohoroglu, and Joines, 1998, now we discretize the process into Markov chain

$$\Pi(e, e^{'}) = \begin{bmatrix} 0.9259 & 0.0741 & 0 \\ 0.235 & 0.953 & 0.0235 \\ 0 & 0.0741 & 0.9259 \end{bmatrix} \tag{26}$$

and the value $\mu = \{0.36, 1.0, 2.7\}$.

Also, we calibrate the age-specific labor efficiencies e_t^s. Based on He, Ning, and Zhu, 2019, they use the data in CHNS. They used CHNS to obtain data on the average working hours per worker per week, which were calculated on the basis of two questions. " C5: For how many days in a week, on the average, did you work ?" and "C6: For how many hours in a day, on the average, did you work?" We choose to use the method of Ludwig(2012): regress the data on the third-order polynomials, which are given in the following specifications.

$$\log e_s = \eta_0 + \eta_1 j + \eta_2 j^2 + \eta_3 j^3 + \epsilon_s \tag{27}$$

Here, e_j is age-specific productivity and ϵ_j is residual. And the coefficient vector represented by $\hat{\eta} = [\hat{\eta_1}, \hat{\eta_2}, \hat{\eta_3}]'$ determines the polynomial slope estimated according to the actual working time data.

7 Shocks and Response

7.1 Monetary Policy Shock

As we all know, if the central bank deviates from its system's Taylor rule, that is, to raise the nominal interest rate, then due to price stickiness, it also changes the real interest rate, and so

Table 1: Calibration

Parameter		Value	Target
β	Discount factor	0.971	$r = 0.03$
σ	Inverse IES	2	
χ_0	Portfolio adj. cost pivot	0.25	
χ_1	Portfolio adj. cost scale	6.41	
χ_2	Portfolio adj. cost curvature	2	
$\underline{b}$	Borrowing constraint	0	
ρ_e	Autocorrelation of earnings	0.966	
σ_e	Cross-sectional std of log earnings	0.92	
φ	Disutility of labor	2.073	$N = 1$
ν	Inverse Frisch elasticity	1	
μ_w	Steady state wage markup	1.1	
κ_w	Slope of wage Phillips curve	0.1	
Z	TFP	0.468	$Y = 1$
α	Capital share	0.33	$K = 10Y$
μ_p	Steady-state markup	1.015	$\mathcal{A} + \mathcal{B} = 16Y$
δ	Depreciation	0.02	
κ_p	Slope of price Phillips curve	0.1	
ω	liquidity premium	0.005	
τ	Labor tax	0.356	
G	Government spending	0.2	
B^g	Bond supply	2.8	
ϕ	Taylor rule coefficient	1.5	
ϕ_y	Taylor rule coefficient on output	0	
n_e	Points in Markov chain for \$e\$	3	
n_b	Points on liquid asset grid	50	
n_a	Points on illiquid asset grid	70	

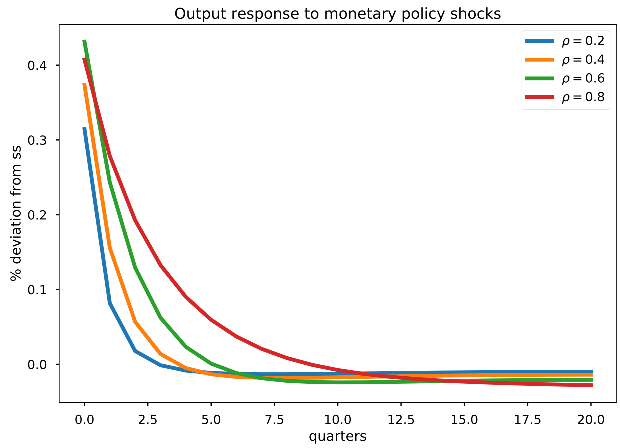

Figure 3: Output response to monetary policy shocks

can influence the performance of the real economy (Figure 3). Strict monetary policy conditions forced companies to postpone investment, and households began to accumulate more savings or reduce their credit stock. Due to the reduction in total demand, companies lower prices to avoid (reduce) loss of profits. Due to price stickiness and price index, the nominal adjustment is gradual, and inflation requires more than four quarters. The central bank observed that total demand was shrinking and lowered the nominal interest rate back to its steady state level, and the entire economy stabilized at its initial steady state. Aging makes a significant difference in the output gap response (GDP deviates from its flexible price equilibrium level). This is also reflected in the fact that young people and old people have different reactions. In particular, in the old society, retired agents had more savings, and workers also assumed more debt over longer lives. In addition, in young societies, currency restrictions have created incentives to postpone consumption and increase savings. However, in the old society, the impact of raising interest rates was even more asymmetric: the elderly could interpret the shock as extra income, so they increased consumption, while young people faced higher credit costs and reduced consumption. Since aging has also changed the relative size of similar groups of people, in the grey society, the total consumption and output gap have decreased by a small margin. Therefore, we find that aging will reallocate asset positions between generations, which reduces the effectiveness of monetary policy-defined as the rate of inflation falls significantly under the same currency deflation, and the total demand for interest rate changes less elastic. As Figure 3 points out that a negative interest shock will galvanize the short-term investment of firms and lead to the increasing of production. And the response is increasing with the size of the shocking.

Consumption of the economy(Figure 4) is in tune with the response of production but the degree is weak than production. But the short-term welfare(Figure 5) present value showcase the trend of decreasing, it prove that it costs the future value of welfare. The key variable we want to focus on in this paper is the change of asset $\mathcal{B}_t$ as Figure 6. The fully funded asset goes up with the negative shock of liquid asset interest. It shows the substitute affect of pension and liquid

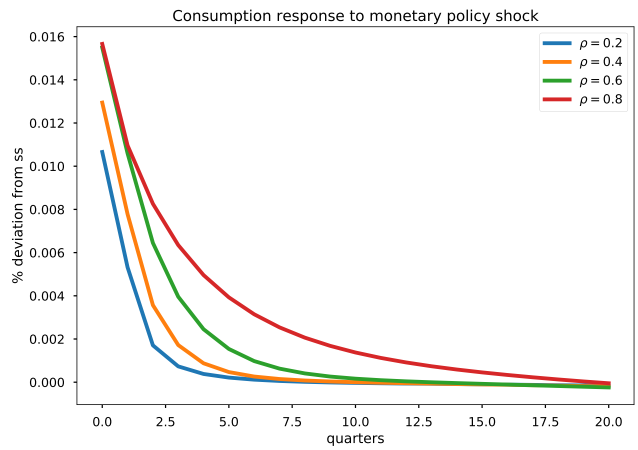

Figure 4: Consumption response to monetary policy shock

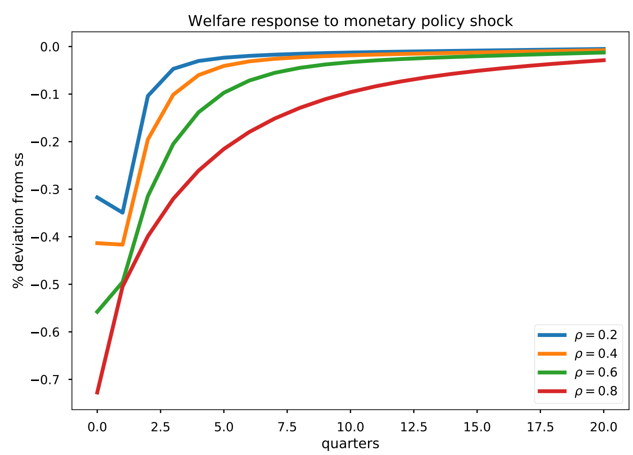

Figure 5: Welfare response to monetary policy shock

asset.

7.2 Fiscal Policy Shock

The government increases public expenditure by one percent of steady-state GDP (Figure 7). As a result, firms increase production to satisfy the extra demand. A higher level of production requires more labor, so firms increase wages to attract more workers. To offset the increase in production costs and the loss in profits, price-setting firms increase their prices. The central bank launches a tightening cycle, and holds the interest rate elevated until the demand-side inflationary pressure disappears, and inflation goes back to its original steady-state level. Due to the increase in wages, young households consume more. Old households consume less because of the increasing interest rate and credit demand (from the young households). Later on, though, the higher interest rates push young consumption down and old consumption up. Aging also

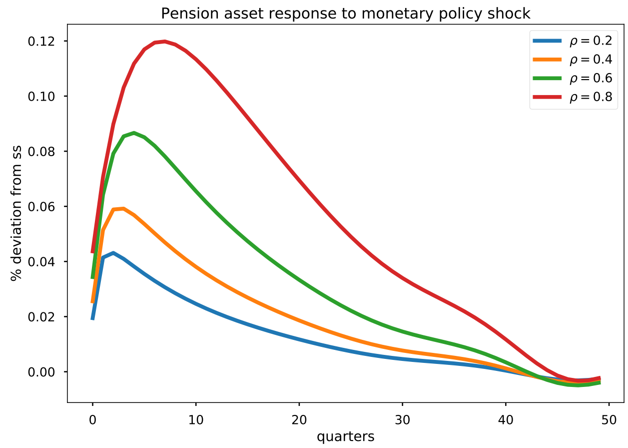

Figure 6: Pension asset response to monetary policy shock

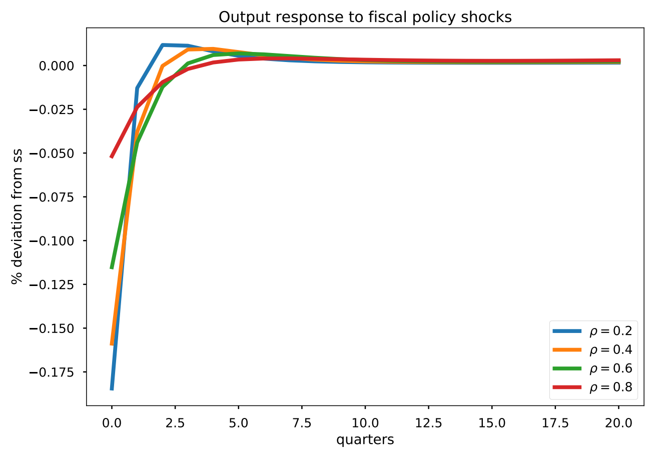

Figure 7: Changes of Debt under Different Tax Rates(2025-2100, Short Term)

changes the relative size of cohorts, and decreases the available labor force in the economy. In an old society the labor supply is more inelastic (the Frisch-elasticity is lower). Hence, firms are forced to increase wages more than in a young society to attract the required labor force. This additional increase in wages amplifies the increase in marginal costs and inflation, too. As a consequence, in an old society the central bank needs to raise the policy rate by a larger amount than in a young society. A stronger monetary policy reaction in a more gray society forces the young to give up more consumption.

Pension assets(Figure 10) are facilitated by the decreasing of government purchasing. The demand of liquid asset declines and the interest rate of that go along with that trend, which means the requirement of pension assets will exacerbate due to that.

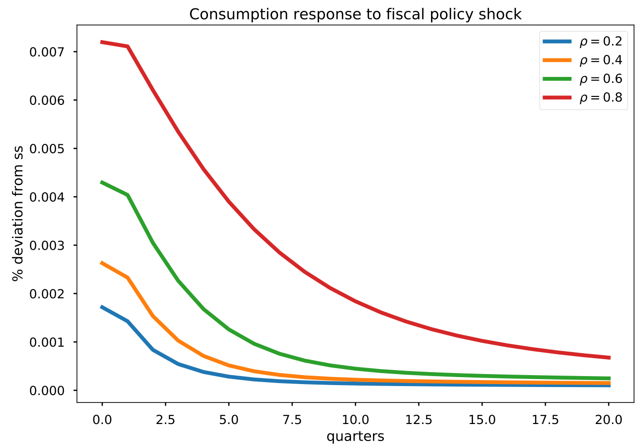

Figure 8: Changes of Debt Ratio under Different Tax Rates(2025-2100, Short Term)

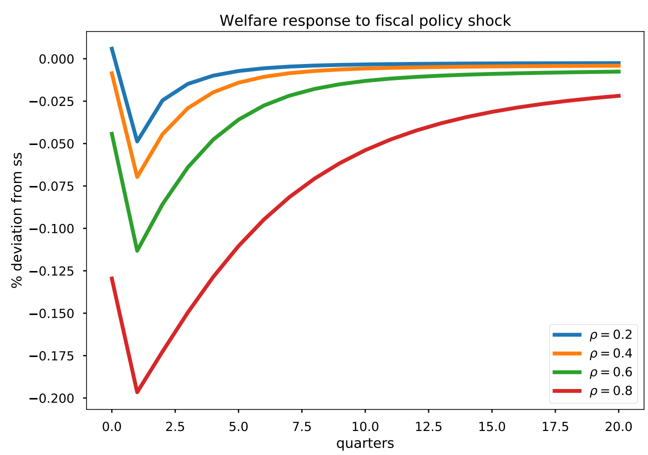

Figure 9: Changes of Debt Ratio under Different Tax Rates(2025-2100, Short Term)

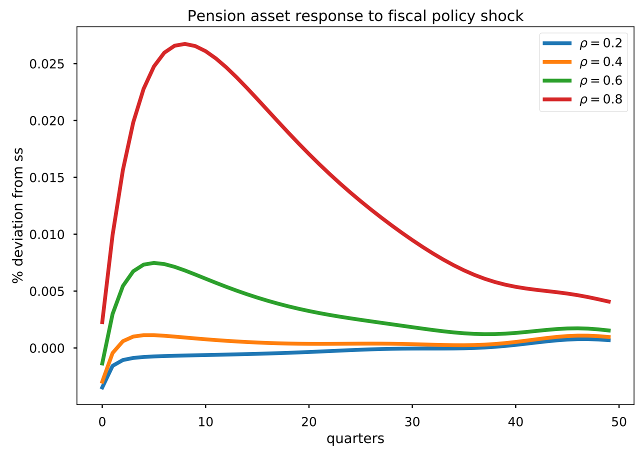

Figure 10: Pension asset response to fiscal policy shock

7.3 Technology Shock

In the article Guerrieri, Lorenzoni, Straub, and Werning, 2020 introduced a concept that might be accurately described as ”supply creating its own excess demand”. In other words, a negative supply shock leads to a shortage of demand, resulting in more output and employment contraction than supply shock itself. We call supply shocks with these characteristics Keynesian supply shocks. Temporary negative supply shocks (such as those caused by a pandemic) will reduce production and employment. The decline in supply shocks may be frightening, but it is an effective response to a certain extent, because output and employment are bound to decline. Some people question whether any demand stimulus measures can justify a response to a supply shock in essence. They believe that the economic response should be based solely on social protection. Others argue that negative shocks may cost more output than efficiency. Gourinchas and Pierre-Olivier, 2020, for example, has proposed a height measurement aimed at ”flattening the curve”.

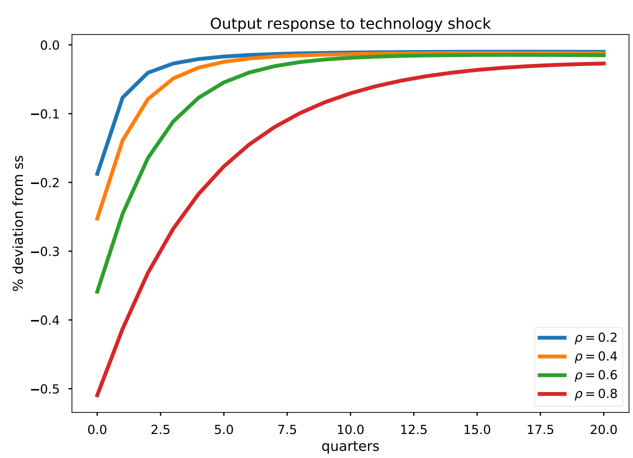

Figure 11: Consumption response to technology shock

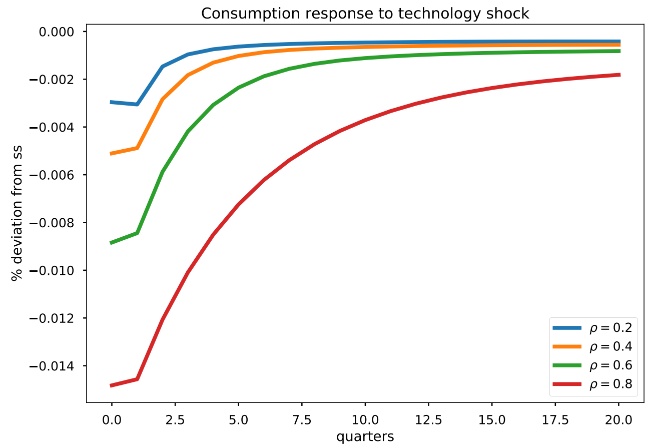

Figure 12: Output response to technology shock

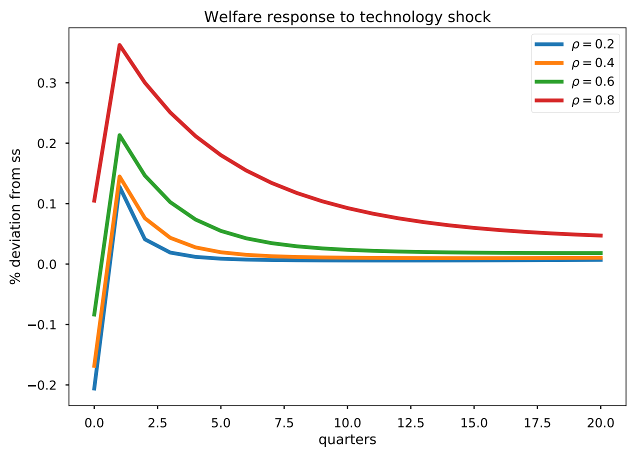

Figure 13: Welfare response to technology shock

They present a theory of Keynesian supply shocks: supply shocks that trigger changes in aggregate demand larger than the shocks themselves. We argue that the economic shocks associated to the COVID-19 epidemic—shutdowns, layoffs, and firm exits—may have this feature. In one-sector economies supply shocks are never Keynesian. We show that this is a general result that extend to economies with incomplete markets and liquidity constrained consumers. In economies with multiple sectors Keynesian supply shocks are possible, under some conditions. A 50% shock that hits all sectors is not the same as a 100% shock that hits half the economy. Incomplete markets make the conditions for Keynesian supply shocks more likely to be met. Firm exit and job destruction can amplify the initial effect, aggravating the recession. We discuss the effects of various policies. Standard fiscal stimulus can be less effective than usual because the fact that some sectors are shut down mutes the Keynesian multiplier feedback. Monetary policy, as long as it is unimpeded by the zero lower bound, can have magnified effects, by preventing firm exits. Turning to optimal policy, closing down contact-intensive sectors and providing full

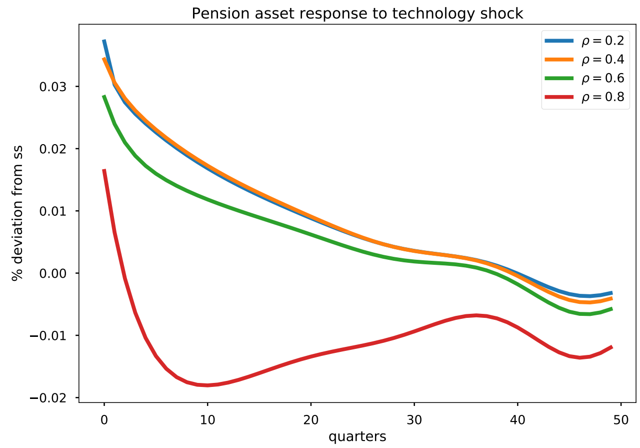

Figure 14: Pension asset response to technology shock

insurance payments to affected workers can achieve the first-best allocation, despite the lower per-dollar potency of fiscal policy.

8 HANK Estimation

It will be time-consuming if we use traditional algorithm to estimate the exogenous shock of HANK model, but it will be efficient in Sequence-Jacobian framework. The reason is the time complexity is $O(N)$ but it is $O(N^2)$ in directive Bayesian estimation. As Auclert, Bardóczy, Rognlie, and Straub, 2019, we set the impulse responses computed so far correspond to the $MA(\infty)$ representation of the model with aggregate shocks. We have get the blocks(individual policy function, aggregate variable and price), mapping stochastic processes $\{\tilde{\mathbf{X}}_t\}$ into one another, rather than sequences $\{\mathbf{X}_t\}$. Our results can be summarized as follows.

Assume that exogenous shocks in the model are given by a set of $MA(\infty)$ processes

$$d\tilde{Z}_t^z = \sum_{s=0}^{\infty} m_s^z \epsilon_{t-s}^z$$

where there are as many processes as there are exogenous shocks $z \in \mathcal{Z}$. Here, $\{\epsilon_t^z\}$ are mutually iid standard normally distributed innovations, and $\{m_s^z\}$ are the MA coefficients of the shock to z We denote by $\mathbf{m}^z$ the column vector that results from stacking the m_s^z 's. For any output $o \in \mathcal{O}$ of the stochastic SHADE model, consider the $MA(\infty)$ representation of the equilibrium sequence $\{d\tilde{X}_t^o\}$

$$d\tilde{X}_t^o = \sum_{s=0}^{\infty} \sum_{z \in \mathcal{Z}} m_s^{o,z} \epsilon_{t-s}^z$$

Given shock $z \in \mathcal{Z}$ and assume that all shock innovations are zero, except for $\epsilon_t^z = 1$ in some period t. In that case, the expected path of $d\tilde{Z}_t^z$ going forward is given by

$$\mathbb{E}_t \left[d\tilde{Z}_{t+s}^z \right] = m_s^z$$

Hence, this expected path corresponds to a shock $\mathbf{m}^z = d\mathbf{Z}^z$ to z in the perfect-foresight model. In other words, the impulse response of output o is given by $\mathbf{G}^{0,z} d\mathbf{Z}^z = \mathbf{G}^{0,z}\mathbf{m}^z$.

Therefore the MA coefficients $\mathbf{m}^{0,z} \equiv (m_s^{o,z})$ of output o in response to shock z can be obtained by solving the associated perfect foresight SHADE model, that is, for any $o \in \mathcal{O}, z \in \mathcal{Z}$ we have:

$$\mathbf{m}^{0,z} = \mathbf{G}^{0,z}\mathbf{m}^z$$

where $G^{0,z}$ is the general equilibrium Jacobian.

We first compute all second moments from the impulse responses, and then compute the likelihood function from these second moments.

The first step consists of computing the model's autocovariance function. Let $\widehat{\mathcal{O}} \subset \mathcal{O}$ be the set of outputs whose second moments we would like to characterize, and denote by $d\tilde{\mathbf{X}}_t = \left(d\tilde{X}_t^o\right)_{o\in\widehat{\mathcal{O}}}$ the vector-valued stochastic process of all outputs in $\widehat{\mathcal{O}}$. Similarly, let $\mathbf{m}_t^{\widehat{\mathcal{O}},\mathcal{Z}} = (m_t^{o,z})_{o\in\tilde{\mathcal{O}},z\in\mathcal{Z}}$ be the stacked $|\widehat{\mathcal{O}}| \times n_z$ matrix of MA coefficients of $d\tilde{\mathbf{X}}_t$. Then, the autocovariances of $d\tilde{\mathbf{X}}_t$ are given by

$$\mathrm{Cov}\left(d\tilde{\mathbf{X}}_t, d\tilde{\mathbf{X}}_{t'}\right) = \sum_{s=0}^{T-(t'-t)} \left[\mathbf{m}_s^{\widehat{\mathcal{O}},\mathcal{Z}}\right]\left[\mathbf{m}_{s+t'-t}^{\widehat{\mathcal{O}}\mathcal{Z}}\right]'$$

The second step to estimation is to evaluate the likelihood function. Let

$$d\tilde{\mathbf{X}}_t^{obs} = Bd\tilde{\mathbf{X}}_t + \mathbf{u}_t$$

denote the vector of n_{obs} observables whose likelihood we would like to determine. Here $\{\mathbf{u}_t\}$ is iid normal with mean 0 covariance matrix $\Sigma_{u,}$ and B is a $n_{obs} \times |\widehat{\mathcal{O}}|$ matrix. since $d\tilde{\mathbf{X}}_t^{obs}$ is a linear combination of the ϵ_t^z and $\mathbf{u}_t$ terms, it has a multivariate normal distribution. Moreover, its second moments are a simple linear transformation of those of $d\tilde{\mathbf{X}}_t$

$$\mathrm{Cov}\left(d\tilde{\mathbf{X}}_t^{obs}, d\tilde{\mathbf{X}}_{t'}^{obs}\right) = 1_{t=t'} \cdot \Sigma_{\mathbf{u}} + B\,\mathrm{Cov}\left(d\tilde{\mathbf{X}}_t, d\tilde{\mathbf{X}}_{t'}\right)B'$$

We stack these covariances into a large symmetric $n_{\mathrm{obs}}T_{\mathrm{obs}} \times n_{\mathrm{obs}}T_{\mathrm{obs}}$ matrix $\mathbf{V}$, where T_{obs} is the number of time periods in our data. The log-likelihood function is then the conventional log multivariate density. Dropping the constant term, it can be expressed as a function of the observed data $d\tilde{\mathbf{X}}^{obs} = \left(d\tilde{\mathbf{X}}_t^{obs}\right)$ (stacked as $n_{obs}T_{obs}$ -dimensional vector) as

$$\mathcal{L} = -\frac{1}{2}\log\det\mathbf{V} - \frac{1}{2}\left[d\tilde{\mathbf{X}}^{obs}\right]'\mathbf{V}^{-1}\left[d\tilde{\mathbf{X}}^{obs}\right]$$

The efficiency of the maximum likelihood function has been proved by Auclert, Bardóczy, Rognlie, and Straub, 2019.

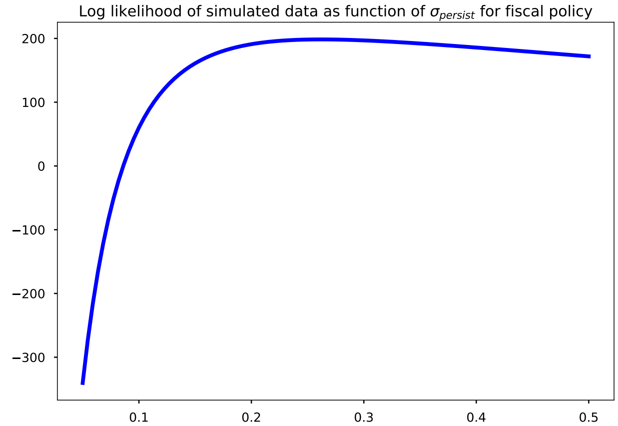

Figure 16: Log likelihood of simulated data as function of $\sigma_{persist}$ for fiscal policy

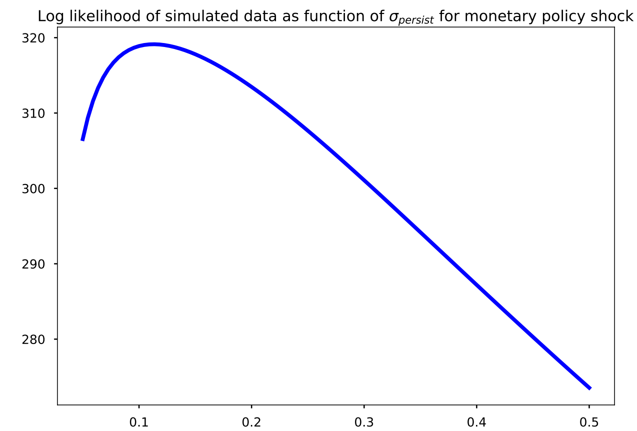

Figure 15: Log likelihood of simulated data as function of $\sigma_{persist}$ for monetary policy shock

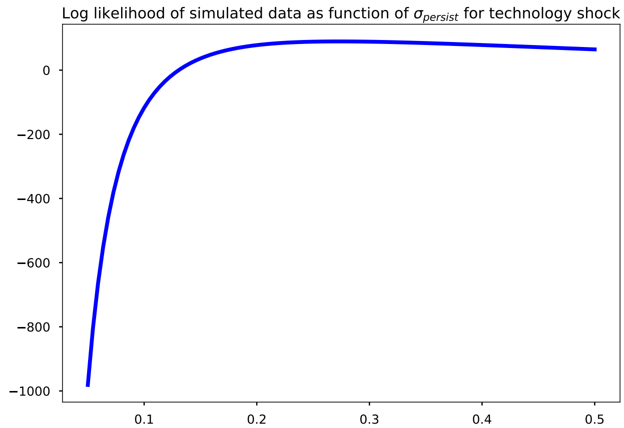

Figure 17: Log likelihood of simulated data as function of $\sigma_{persist}$ for technology shock

9 MIT Shocks

What if we consider perfect foresight path of the economy? Many researches have focused on MIT shocks(Boppart, Krusell, and Mitman, 2018,Kydland and Prescott, 1982). We first point out that before the researchers conducted groundbreaking work, the researchers used analytical methods to characterize the perfect expected equilibrium caused by the "MIT shock." The MIT shock created by Thomas Sargent refers to an unpredictable shock without a shock to the steady state balance of the economy. In other words, in this economy, no shocks are expected, but now there will be shocks. Then, assuming that no more shocks will occur, the analysis focuses on understanding the equilibrium transition that occurs along the perfect prediction path. Therefore, the described process seems to be difficult to coincide with rational expectations.

The Kydland-Prescott paper developed a non-linear model without market frictions and, hence, could focus on solving a planning problem. This problem was approached using "linearization" around the steady state; Kydland and Prescott first solved for the steady state, then provided a quadratic (second-order Taylor expansion) approximation of the objective function expressed in terms of the vector of choice variables- around that steady state, along with a linear set of (or linearized) constraints. Thus, they obtained a linear-quadratic problem, which was well known to have an exact solution in the form of a quadratic value function along with linear policy rules.

A key element in the approach pioneered by Kydland and Prescott was the use of a recursive system, which fits value-function methods well. The key objects sought were thus functions of a state vector of exogenous and endogenous variables. To be concrete, one would for example in the very simplest case of a stochastic growth model with optimal saving and no adjustment costs find output, y, to be a function G of the capital stock, k, and TFP, $z: \quad y = G(k,z)$. Similarly, the law of motion for the endogenous state would be written using a function H as $k' = H(k,z)$.

Use the parameters we estimate above, we can get the response of MIT shock. The sensitivity

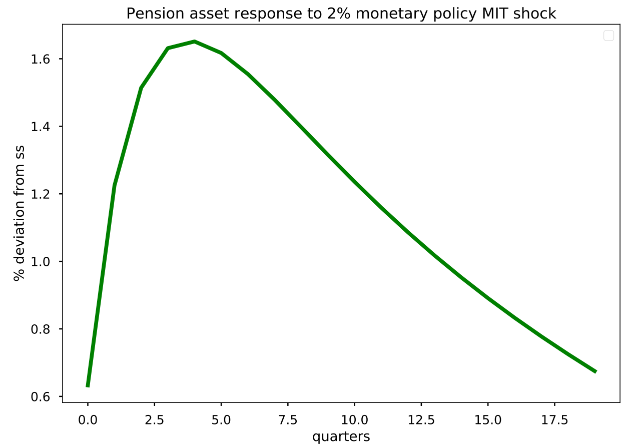

Figure 18: Pension asset response to 2% monetary policy MIT shock

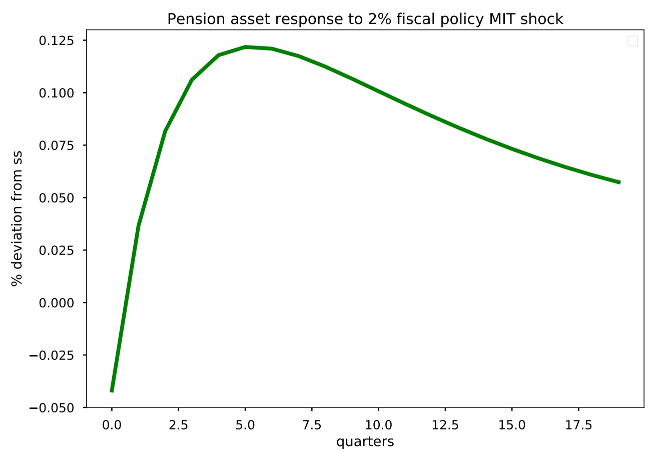

Figure 19: Pension asset response to 2% fiscal policy MIT shock

of shock responses to fiscal, monetary, technology shocks are idiosyncratic (Figure 18, Figure 19, Figure 20). Monetary policy is the most effective way in these 3 tools, but it decades fast meanwhile. Pension assets weakly response to the shock of fiscal and technology shock, but the affect is relatively long-term. And the impacting of technology shock is instant and simultaneous.

10 Conclusion

We build a HANK model with sticky price and wage and perfect foresight. The fully funded system in this system has different response to exogenous shocks. Fiscal policy has an negative effects on pension system, discerning that from baseline scenario. Our model gives an insight to the topic that the impact of fully funded system will have on economy.

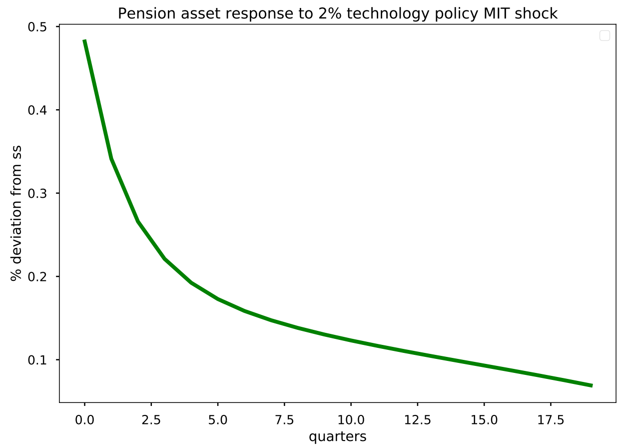

Figure 20: Pension asset response to 2% technology policy MIT shock

References

Daron Acemoglu and Simon Johnson. Disease and Development: The Effect of Life Expectancy on Economic Growth. *Journal of Political Economy*, 115(6):925–985, December 2007.

Adrien Auclert and Matthew Rognlie. Inequality and aggregate demand. Working Paper 24280, National Bureau of Economic Research, February 2018.

Adrien Auclert, Matthew Rognlie, and Ludwig Straub. The intertemporal keynesian cross. Working Paper 25020, National Bureau of Economic Research, 2018.

Adrien Auclert, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub. Using the sequence-space jacobian to solve and estimate heterogeneous-agent models. Working Paper 26123, National Bureau of Economic Research, 2019.

Timo Boppart, Per Krusell, and Kurt Mitman. Exploiting mit shocks in heterogeneous-agent economies: the impulse response as a numerical derivative. *Journal of Economic Dynamics and Control*, 89:68 – 92, 2018. ISSN 0165-1889. Fed St. Louis-JEDC-SCG-SNB-UniBern Conference, titled: “Fiscal and Monetary Policies”.

Christopher D. Carroll. The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters*, 91(3):312–320, 2006.

Hans Fehr, Manuel Kallweit, and Fabian Kindermann. Reforming Family Taxation in Germany: Labor Supply vs. Insurance Effects. SOEPpapers on Multidisciplinary Panel Data Research 613, DIW Berlin, The German Socio-Economic Panel (SOEP), 2013.

Gourinchas and Pierre-Olivier. Flattening the pandemic and recession curves. Working Paper 26918, National Bureau of Economic Research, 2020.

Veronica Guerrieri, Guido Lorenzoni, Ludwig Straub, and Iván Werning. Macroeconomic implications of covid-19: Can negative supply shocks cause demand shortages? Working Paper 26918, National Bureau of Economic Research, April 2020.

Hui He, Lei Ning, and Dongming Zhu. The impact of rapid aging and pension reform on savings and the labor supply. *IMF Working Papers*, 19:1, 01 2019.

Jonathan Heathcote, Kjetil Storesletten, and Giovanni L. Violante. Optimal Tax Progressivity: An Analytical Framework*. *The Quarterly Journal of Economics*, 132(4):1693–1754, 06 2017. ISSN 0033-5533.

Ayse Imrohoroglu, Selahattin Imrohoroglu, and Douglas H Joines. The Effect of Tax-Favored Retirement Accounts on Capital Accumulation. *American Economic Review*, 88(4):749–768, September 1998.

Greg Kaplan and Giovanni L. Violante. Microeconomic heterogeneity and macroeconomic shocks. *Journal of Economic Perspectives*, 32(3):167–94, 2018.

Greg Kaplan, Benjamin Moll, and Giovanni L. Violante. Monetary policy according to hank. *American Economic Review*, 108(3):697–743, 2018.

Finn E. Kydland and Edward C. Prescott. Time to build and aggregate fluctuations. *Econometrica*, 50(6):1345–1370, 1982. ISSN 00129682, 14680262.

Julio J. Rotemberg. Monopolistic price adjustment and aggregate output. *The Review of Economic Studies*, 49(4):517–531, 1982.

Ayşe İmrohoroğlu and Kai Zhao. Household saving, financial constraints, and the current account in china. *International Economic Review*, 61(1):71–103, 2020.

A Optimal Problem and FOC

A.1 Household

The optimization problem for household is

$$\mathcal{L} = \max_{c_t,b_t,a_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{n_t^{1+v}}{1+v} + \beta E_t V_{t+1}\left(e_{t+1}, b_t, a_t\right) \right\}$$
$$+\lambda_t \left[-c_t - a_t - \frac{b_t}{(1+r_t)^R} + (1-\tau_t)\, w_t n_t e_t + \left(1+r_t^a\right) a_{t-1} + \frac{\left(1+r_t^b\right) b_{t-1}}{(1+r_t)^R} + tr_t - \Phi\left(a_t, a_{t-1}\right) \right] \tag{28}$$

And the FOC for control variable c_t, n_t is:

$$\frac{\partial \mathcal{L}}{\partial c_t} = c_t^{-\sigma} - \lambda_t = 0 \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = \varphi n_t^v - \lambda_t\left(1-\tau_t\right) w_t e_t = 0 \tag{30}$$

Combine these two specifications

$$\varphi n_t^v = c_t^{-\sigma}\left(1-\tau_t\right) w_t e_t \tag{31}$$

$$n_t = \left(\frac{c_t^{-\sigma}\left(1-\tau_t\right) w_t e_t}{\varphi} \right)^{\frac{1}{v}} \tag{32}$$

We can find the key structure for these problems: we can always transform the muti-variables' dynamic programming problem into univariate of that. In our problem, the state variables are $\{a_t, b_t, e_t\}$ and control variables $\{c_t, n_t\}$. Using Lagrange method we can convert it into one control variable $\{c_t\}$. Therefore, we need to solve the dynamic programming problem by giving the discrete space of state variable and then we can get the policy function on this discrete space.

For dynamic programming problems, finite-term dynamic programming and infinite-term dynamic programming are different. In contrast, infinite dynamic programming can be regarded as a homogeneous problem, because there is no difference between different time and infinite time. However, for a finite terms problem, this is not the same, different time will start to make the problem is not homogeneous. So we can only start planning from the last phase.

When calculating multivariate optimization problems, we can express all decision variables in terms of state variables. If we use budget constraints instead, we determine the interval and search for the optimal value in a certain range.

The algorithm is a variant of the endogenous grid point method of Carroll, 2006 that we developed for this two-asset problem. The key trick is, whenever the household is partially constrained, to include Lagrange multipliers in the backward iteration. We also exploit the fact that, endogenously, the constraint on the illiquid asset will never be binding unless the constraint on the liquid asset is also binding(otherwise, a simple variation will improve utility)—and if both are binding, then the policy is trivial. Overall, we start from a guess for the (discretized) partials of the value function and iterate backward until convergence. Throughout, we will use (z', b', a') to refer to tomorrow's grid and (z, b, a) today's grid.

1. **Initial guess** Guess $V_a(z', b', a')$ and $V_b(z', b', a')$

2. **Common** $z' \to z$. By definition

$$W_b(z, b', a') = \beta\Pi V_b(z', b', a')$$
$$W_a(z, b', a') = \beta\Pi V_a(z', b', a')$$

3. **Unconstrained** $a' \to a$. Assuming that no constraints bind, $\lambda_t = \mu_t = 0$, and the FOCs become

$$u'(c) = W_b\left(z, b', a'\right)$$
$$u'(c)\left[1 + \Phi_1\left(a', a\right)\right] = W_a\left(z, b', a'\right)$$

Combine these to get

$$0 = F\left(z, b', a, a'\right) \equiv \frac{W_a(z, b', a')}{W_b(z, b', a')} - 1 - \Phi_1\left(a', a\right)$$

which characterizes $a'(z, b', a)$. Use this to map $W_b(z, b', a')$ into $W_b(z, b', a)$ by interpolation, then compute consumption as

$$c\left(z, b', a\right) = W_b\left(z, b', a\right)^{-\frac{1}{\sigma}}$$

4. **Unconstrained** $\underline{b'} \to b$. Now using $a'(z, b', a)$ and $c(z, b', a)$ from the previous step, use the budget constraint to obtain

$$b\left(z, b', a\right) = \frac{c(z, b', a) + a'(z, b', a) + \frac{b'}{(1+r_t)^{R-1}} - (1 + r^a)\, a + \Psi\left(a'(z, b', a), a\right) - (1 - \tau_t)\, w_t n_t e_t + tr_t}{(1 + r^b)^R}$$

We invert this function via interpolation to get $b'(z, b, a)$. The same interpolation weights can be used to do $a'(z, b', a) \to a'(z, b, a)$

5. **Liquidity constrained** $a' \to a$. This branch is analogous to the unconstrained case. Assuming that the liquidity constraint is binding, $\lambda_t > 0$, and (89) and (90) become

$$u'(c) = \lambda + W_b\left(z, 0, a'\right)$$
$$u'(c)\left[1 + \Phi_1\left(a', a\right)\right] = W_a\left(z, 0, a'\right)$$

To help with scaling, let us define $\kappa \equiv \lambda / W_b(z, 0, a')$ and rewrite the first equation as

$$u'(c) = (1 + \kappa) W_b\left(z, 0, a'\right)$$

Divide and rearrange to get

$$0 = F\left(z, \kappa, a, a'\right) \equiv \frac{1}{1+\kappa}\frac{W_a(z, 0, a')}{W_b(z, 0, a')} - 1 - \Phi_1\left(a', a\right)$$

We solve this for $a'(z, \kappa, a)$, and compute consumption as

$$c(z, \kappa, a) = \left[(1 + \kappa) W_b(z, \kappa, a)\right]^{-\frac{1}{\sigma}}$$

6. **Liquidity constrained** $\underline{x} \to \underline{b}$. Now using $a'(z,x,a)$ and $c(z,\kappa,a)$ from the previous step, use the budget constraint to obtain

$$b(z,\kappa,a) = \frac{c(z,\kappa,a) + a'(z,\kappa,a) + \underline{b} - (1+r^a)\,a + \Phi\left(a'(z,\kappa,a),a\right) - z}{1+r^b}$$

We invert this function via interpolation to get $\kappa(z,b,a)$. The same interpolation weights can be used to map $a'(z,\kappa,a)$ into $a'(z,b,a)$. We already know that $b'(z,b,a) = \underline{b}$

7. **Update guesses** The final $b'(z,b,a)$ is the element-wise maximum of its unconstrained and liquidity-constrained counterparts. Replace the unconstrained $a'(z,b,a)$ with constrained one at the exact same points. Compute consumption from the budget constraint as

$$c(z,b,a) = z + (1+r^a)\,a + \left(1+r^b\right) b - \Phi\left(a'(z,b,a),a\right) - a'(z,b,a) - b'(z,b,a)$$

Finally use the envelope conditions (91) and (92) to update the guesses

$$\begin{aligned} V_b(z,b,a) &= \left(1+r^b\right) c(z,b,a)^{-\sigma} \\ V_a(z,b,a) &= \left[1+r^a - \Phi_2\left(a'(z,b,a),a\right)\right] c(z,b,a)^{-\sigma} \end{aligned}$$

Go back to step 2, repeat until convergence.

A.2 Firms

The firm's optimization problem is

$$\begin{aligned} J_t\left(k_{t-1}\right) = \max_{p_t,k_t,n_t} \Big\{ &\tfrac{p_t}{P_t} F\left(k_{t-1},n_t\right) - \tfrac{W_t}{P_t} n_t - \xi\left(1-\delta+\tfrac{\iota_t}{k_{t-1}}\right) k_{t-1} - \xi\left(p_t,p_{t-1}\right) Y_t \\ &+ \tfrac{1}{1+r_t} J_{t+1}\left(k_t\right) \Big\} \end{aligned} \tag{33}$$

subject to

$$\frac{p_t}{P_t} Y_t = \left(\frac{F\left(K_{t-1},L_t\right)}{Y_t}\right)^{\frac{1}{\mu}-1} Y_t \tag{34}$$

$$k_t = (1-\delta)k_{t-1} + \iota_t \tag{35}$$

Lagrange equation for this problem:

$$\begin{aligned} \mathcal{L} = \max_{p_t,k_t,n_t} \Big\{ &\tfrac{p_t}{P_t} F\left(k_{t-1},n_t\right) - \tfrac{W_t}{P_t} n_t - \phi\left(1-\delta+\tfrac{\iota_t}{k_{t-1}}\right) k_{t-1} - \xi\left(p_t,p_{t-1}\right) Y_t + \tfrac{1}{1+r_t} J_{t+1}\left(k_t\right) \\ &-\lambda_t \left[\tfrac{p_t}{P_t} Y_t - \left(\tfrac{F\left(K_{t-1},L_t\right)}{Y_t}\right)^{\frac{1}{\mu}-1} Y_t\right] - Q_t\left[k_t - (1-\delta)k_{t-1} - \iota_t\right] \Big\} \end{aligned} \tag{36}$$

FOC is

$$\frac{\partial \mathcal{L}}{\partial p_t} = \frac{1}{P_t} F\left(k_{t-1},n_t\right) - \frac{\partial \xi\left(p_t,p_{t-1}\right)}{\partial p_t} Y_t - \lambda_t \frac{Y_t}{P_t} + \frac{1}{1+r_t} \frac{\partial J_{t+1}\left(k_t\right)}{\partial p_t} = 0 \tag{37}$$

$$\frac{\partial \mathcal{L}}{\partial k_t} = \frac{1}{1+r_t} \frac{\partial J_{t+1}\left(k_t\right)}{\partial k_t} - Q_t = 0 \tag{38}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = \frac{p_t}{P_t} \frac{\partial F\left(k_{t-1},n_t\right)}{\partial n_t} - \frac{W_t}{P_t} = 0 \tag{39}$$

$$\frac{\partial \xi\left(p_{t}, p_{t-1}\right)}{\partial p_{t}}=\left[\frac{1}{2 \kappa^{p}}\left(\frac{p_{t}-p_{t-1}}{p_{t-1}}\right)^{2}\right]^{\prime}=\frac{1}{p_{t-1}} \frac{1}{\kappa^{p}} \frac{p_{t}-p_{t-1}}{p_{t-1}} \tag{40}$$

$$\frac{\partial J_{t+1}\left(k_{t}\right)}{\partial p_{t}}=-\frac{\partial \xi\left(p_{t+1}, p_{t}\right)}{\partial p_{t}}=\left[\frac{1}{2 \kappa^{p}}\left(\frac{p_{t+1}-p_{t}}{p_{t}}\right)^{2}\right]^{\prime}=-\frac{1}{\kappa^{p}} \frac{p_{t+1}}{p_{t}^{2}}\left(\frac{p_{t+1}-p_{t}}{p_{t}}\right) \tag{41}$$

$$\frac{\partial \mathcal{L}}{\partial p_{t}}=\frac{1}{P_{t}} F\left(k_{t-1}, n_{t}\right)-\frac{1}{p_{t-1}} \frac{1}{\kappa^{p}} \frac{p_{t}-p_{t-1}}{p_{t-1}} Y_{t}-\lambda_{t} \frac{Y_{t}}{P_{t}}-\frac{1}{1+r_{t}} \frac{1}{\kappa^{p}}\left(\frac{p_{t+1}-p_{t}}{p_{t}}\right)\left(-\frac{p_{t+1}}{p_{t}^{2}}\right) Y_{t+1}=0 \tag{42}$$

$$\left(1-\lambda_{t}\right) \frac{1}{P_{t}} Y_{t}=\frac{1}{\kappa^{p}} Y_{t}\left(\frac{p_{t}-p_{t-1}}{p_{t-1}}\right) \frac{1}{p_{t-1}}+\frac{1}{1+r_{t}} \frac{1}{\kappa^{p}}\left(\frac{p_{t+1}-p_{t}}{p_{t}}\right)\left(-\frac{p_{t+1}}{p_{t}^{2}}\right) Y_{t+1} \tag{43}$$

$$\frac{p_{t+1}-p_{t}}{p_{t}}=\pi_{t}^{p} \tag{44}$$

This setup generates a nonlinear Phillips curve for price inflation

$$\pi_{t}(1+\pi_{t})=\kappa^{p} m c_{t}+\frac{1}{1+r_{t}} \frac{Y_{t+1}}{Y_{t}} \pi_{t+1}(1+\pi_{t}) \tag{45}$$