Economic Growth with Common Noise

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1 Introduction

The mean field games (MFG) framework has been applied to analyze optimal monetary and fiscal policy, see Kaplan et al. [2018] and Achdou et al. [2022]. As the revolutionary work of macro in recent years, Heterogeneous Agent New Keynesian (HANK) bridges the gap between inequality and monetary/fiscal policy shock. Inequality is the most crucial factor need to be accounted for. Although there are other perspectives, such as behavioral and network, inequality is the most direct factor that intertwines with policy. Consider a distribution of a group of people, defined by a distribution function, who are facing an expected shock. These shocks are referred to as ”common noise” in the MFG framework, as explained in Bilal [2023], which promotes an algorithm for solving MFG using the master equation method. Each individual will make a corresponding optimal decision, and the distribution will evolve over time. These dynamics correspond to the HJB and Kolmogorov equations, which compose the MFG systems that were first developed by mathematicians *Pierre-Louis Lions* and *Jean-Michel Lasry*.

Despite my mention of numerous applications of the MFG framework, it has yet to be implemented in the field of economic growth, one of most classical topic in macroeconomics. The “common shock” is prevalent in the economy. This proposal aims to explore the probability of introducing the MFG framework into growth, firm innovation and optimal tax mechanism design.

I would like to illustrate the concept of a “common shock” using the following sample equation as an example:

$$dh_t = \dot{h}\left(t, h_t\right) dt + \sigma h_t dW_t$$

This is the equation described the accumulation process of human capital. dW_t term is “common noise”, which can arise from various sources, such as a technology shock like ChatGPT, policy changes, or any other factors that affect the accumulation process of human capital for all individuals.

Common shocks, such as those arising from climate change, are also important in the literature on long-term growth, as demonstrated in research by Bilal and Rossi-Hansberg [2023], Moscona and Sastry [2023] and Dell et al. [2012]. For instance, climate change can have a global impact and serve as

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a common shock that affects the technology advancement of firms or geographical development.

$$dA_{it} = \dot{A_{it}}dt + \sigma A_{it} dW_t$$

In this proposal, I only use human capital as an example to explain ideas.

Literature Review for Human Capital Example A series of literature has explored the relationship between human capital and economic growth, including works by Becker [2009], Lucas Jr [1988], Barro and Lee [1994], Mankiw et al. [1992], Schultz [1961]. The problem of human capital accumulation with heterogeneity can be studied in the framework of mean field games Lasry and Lions [2007] and Carmona et al. [2018]. This involves modeling a continuum of individuals whose wages depend not only on their own human capital, but also on the distribution of human capital in the population. This distribution dependency is important for understanding competition in the labor market.

In the first section, I present the setup of my simple model. In the second section, I use MFG to solve the problem in a different way and generalize the solution to a stochastic framework. Finally, I try to introduce the common noise to the accumulation process of human capital as aformentioned.

2 Simple model

2.1 Introduction

There is initially a working population of size 1 with a given distribution of human capital. Human capital will be denoted h and the distribution, at time t, will be referred to as $f_H(h,t)$ and $\int_h f_H(h,t) = 1$.

The wage is set by firms. Standard production function for a representative firm would be

$$Y = A \int_h h^\alpha f_H(h,t)^{1-\beta} dh \quad \alpha > 0,\ \beta \in (0;1]$$

Thus, agents' wage not only depends on the individual human capital but also on the scarcity of her specific human capital. In other words, the salary of a worker with human capital h is, at time t, given by:

$$w(h,t) = A(1-\beta)\frac{h^\alpha}{f_H(h,t)^\beta} \tag{1}$$

The logic behind this equation is to model a competition effect in the labor market: unskilled people have a small salary.

As Mincer [1958] approach, agents live forever, and they can improve their human capital with a cost depending on two factors. First, the cost, in monetary terms, is a function of human capital change and, second, it is also a function of the position of initial human capital in the distribution. Precisely, I will assume that the cost (in monetary terms) at time t is given by:

$$C\left(\dot{h}, F_H(h,t)\right) \tag{2}$$

where $F_H(h,t)$ is the number of people in the population with a human capital *less* than h. That is to say it is more costly for skilled workers to improve their human capital than for unskilled workers. This hypothesis is relevant since it is often more difficult to improve human capital for an individual in the right tail of the distribution since she is near the technological frontier.

2.2 The optimization problem

Suppose that people improve their human capital constantly. Human capital accumulation can be seen as the consequence of on-the-job training. Each individual chooses her effort continuously to maximize her life-time earning.

$$\max_{(h_s),h_0=h} \int_0^\infty \left[w\left(h,s\right) - C\left(\dot{h}, F_H(h,s)\right)\right] e^{-rs} ds \tag{3}$$

with a wealth constraint $\dot{a}_t \leq ra_t + w\left(h,s\right) - C\left(\dot{h}, F_H(h,s)\right) - c_t$. This gives a unique intertemporal constraint that is:

$$\int_0^\infty c_t e^{-rt} dt \leq a_0 + \int_0^\infty \left(w\left(h,s\right) - C\left(\dot{h}, F_H(h,s)\right)\right) e^{-rt} dt$$

r is exogenous.

2.3 Resolution of simple model

2.3.1 A specific setup

To solve the problem I need to specify the two functions w and C. My specification is the following:

$$w(h,t) = \begin{cases} A(1-\beta)\frac{h^\alpha}{f_H(h,t)^\beta}, & \text{if } h \text{ is in the support of density function} f_H(t,\cdot) \\ 0 & \text{otherwise} \end{cases}$$

$$C\left(\dot{h}, F_H(h,t)\right) = \frac{\bar{C}}{\varphi} \frac{\dot{h}^\varphi}{\left(1 - F_H(h,t)\right)^\delta}, \quad \forall h \text{ in the support of } f_H(t,\cdot)$$

where $\bar{C}$ is the constant and where α, β, δ and φ are positive parameters subject to technical constraints that are: $\alpha + \beta = \varphi$, $\beta = \delta$ and $\varphi > 1$.

Now, I assume that the initial distribution of human capital is a Pareto distribution. I can use a normalization and assume that the minimal point of the initial distribution is 1. The Pareto coefficient of the initial distribution is denoted k, so that:

$$f_H(h,0) = k\frac{1}{h^{k+1}}\mathbf{1}_{h\geq 1}$$

This Pareto distribution is central in the study of economic inequalities.

The optimal path has to satisfy the following Euler equation:

$$\partial_h w_t - \partial_h C = -\frac{d\left[\partial_{\dot{h}} C\right]}{dt} + r\partial_{\dot{h}} C \tag{4}$$

3 Stochastic framework

3.1 The mean field game partial differential equations

The problem involves indeed the probability distribution function and the tail function of the human capital across the population.
Let's first introduce the Bellman function of the problem:

$$V(h,t) = \max_{(h_s),h_t=h} \int_t^\infty \left[w\left(h,s\right) - C\left(\dot{h}, F_H(h,s)\right)\right] e^{-r(s-t)} ds \tag{5}$$

my optimization problem can be represented by the MFG system in what follows:

$$\dot{V} + \max_{\dot{h}} \left(\frac{\partial V}{\partial h}\dot{h} + w(h,s) - C\left(\dot{h}, F_H(h,s)\right)\right) - rV = 0 \quad \text{(HJB)} \tag{6}$$

$$\partial_t f_H(h,t) + \partial_h \left(\dot{h} f_H(h,t)\right) = 0 \quad \text{(Kolmogorov)} \tag{7}$$

In the aforementioned special case, the two equations can be written as:

$$A(1-\beta)\frac{h^\alpha}{f_H(h,t)^\beta} + \frac{\varphi-1}{\varphi}\frac{1}{\bar{C}^{\frac{1}{\varphi-1}}}(1-F_H(h,t))(h,t)^{\frac{\beta}{\varphi-1}}\left(\partial_h V\right)^{\frac{\varphi}{\varphi-1}} + \partial_t V - rV = 0$$

$$\partial_t f_H(h,t) + \partial_h \left(\left(\frac{(1-F_H(h,t))^\beta}{\bar{C}} \partial_h V(h,t) \right)^{\frac{1}{\varphi-1}} f_H(h,t) \right) = 0$$

and the optimal control is given by:

$$\frac{dh}{dt}(h,t) = \left(\frac{(1-F_H(h,t)^\beta}{\bar{C}} \partial_h V(h,t) \right)^{\frac{1}{\varphi-1}}$$

3.2 The model with common noise

So far, my model was completely deterministic whereas most applications of mean field games are in a random setting. It's possible to introduce randomness through a common noise on the evolution of the human capital.

More elaborately, I can replace the dynamics for h by a stochastic one:

$$dh_t = \dot{h}\left(t, h_t\right) dt + \sigma h_t dW_t \tag{8}$$

where W is a noise common to all agents.
Consider the same Bellman function V.

$$V(h,t) = \max_{(h_s),h_t=h} \int_t^\infty \left[w\left(h,s\right) - C\left(\dot{h}, F_H(h,s)\right)\right] e^{-r(s-t)} ds \tag{9}$$

The HJB equation corresponding to the above optimization problem can be written in the following differential form:

$$\max_{\dot{h}} \left[w\left(h,s\right) - C\left(\dot{h}, F_H(h,s)\right) + \dot{h}\partial_h V + \frac{\sigma^2}{2} h^2 \partial^2_{hh} V \right] = rV + \partial_t V$$

The optimal control function is given by the same expression as in the deterministic case:

$$\frac{dh}{dt}(h,t) = \left(\frac{(1-F_H(h,t))^\beta}{\bar{C}}\partial_h V(h,t)\right)^{\frac{1}{\varphi-1}}$$

4 Optimal Taxation Framework

Following two framework are the corresponding social planner problem.

4.1 Human Capital

This framework is designed to explore optimal taxation and economic growth in a heterogeneous-agent environment. In this context, a, h represent assets and human capital, respectively.
The initial distribution of individuals is represented by $f_0(a,h)$. The final goods department determines the values of r_t, w_t, and the technology used for production is denoted by $Y_t = H_t K_t^\alpha L^{1-\alpha}$, where labor supply is inflexible or fixed. H_t is aggregate human capital.

$$v(a,h) = \int_0^\infty e^{-\rho t} u\left(c_t\right) dt$$

$$da_t = \left(h_t w_t + r_t a_t - c_t\right) dt$$

$$dh_t = \mu\left(h_t\right) dt + \sigma\left(h_t\right) dW_t$$

Social planner problem

$$SWF = \int_A G(v(a,h)) dF(a,h)$$

subject to equations

$$\rho v\left(a_t,h_t\right) - \partial_t v\left(a_t,h_t\right) = \max_c \left\{ u\left(c_t\right) + \left(w_t h_t + \left(1-\mathcal{T}_K\right) r_t a_t - c_t + D\right) \partial_a v\left(a_t,h_t\right) + \mu\left(h_t\right) \partial_h v + \frac{1}{2}\sigma^2(h) \partial_{hh}\left(a_t,h_t\right) \right\}$$

$$\partial_t f = -\partial_a\left(\left(w_t h_t + \left(1-\mathcal{T}_K\right) r_t a_t - c_t + D\right) f\right) - \partial_h\left(\mu\left(h_t\right) f(a,h)\right) + \frac{1}{2}\partial_{hh}\left(\sigma^2(h) f(a,h)\right)$$

$$\int \mathcal{T}_K r_t a_t dF(a,h) = D$$

$$H_t = \int h_t dF(a,h)$$

$$r_t = H_t \alpha \left(\int a dF(a,h)\right)^{\alpha-1} - \delta$$

$$w_t = H_t(1-\alpha)\left(\int a dF(a,h)\right)^{\alpha}$$

4.2 Climate Change

This is designed for explaining how innovation can temper climate change and utilize taxes to solve externalities.
There are n-type countries with varying sensitivity $s = \{s_0, \ldots, s_n\}$ to climate change (as expressed by temperature). Sensitivity is embodied on term $\sigma^2\left(T_t, s_i\right)$.

$\{k_t, A_t^E\}$ denotes the capital and technology used for addressing climate change (reducing carbon emissions). i_t represents the investment made to enhance A_t^E.

$$v\left(k_t, A_t^E, s\right) = \int_0^\infty e^{-\rho t} u\left(c_t\right) dt$$
$$dk_t = \left(A_t k_t - c_t - i_t\right) dt + \sigma\left(T_t, s_i\right) k_t dW_t$$
$$dA_t^E = i_t dt$$

Countries have incentives to produce more without investing in technology for resolve climate change. In this game, the temperature path is unpredictable for individual countries. The emission function $e\left(\int A_t k_t, A_t^E\right)$ is associated with the global total production and technology that can reduce carbon emissions.

$$dT_t = e\left(\int A_t k_t, A_t^E\right) dt + \sigma_T dW_t$$

Social Planner Problem For addressing externalities, if there is a global institution that utilizes taxes to promote the advancement of technology A_t^E, the patent becomes public and there is no heterogeneity in that aspect. The tax τ_t replaces the investment i_t, and the total research fund available is denoted as $\mathcal{R}_t$.

$$SWF = \int_A G\left(v\left(k_t, s\right)\right) dF\left(k_t, s\right)$$

subject to equations

$$\rho v\left(k_t, s\right) - \partial_t v\left(k_t, s\right) = \max_c \left\{u\left(c_t\right) + \left(A_t k_t - c_t - \tau_t\right) \partial_k v\left(k_t, s\right) + \frac{1}{2}\sigma^2\left(T_t, s_i\right) \partial_{kk}\left(k_t, s\right)\right\}$$
$$\partial_t f = -\partial_k\left(\left(A_t k_t - c_t - \tau_t\right) f\left(k_t, s\right)\right) + \frac{1}{2}\partial_{kk}\left(\sigma^2\left(T_t, s_i\right) f\left(k_t, s\right)\right)$$
$$\int \tau_t\left(k_t, s\right) dF\left(k_t, s\right) = \mathcal{R}_t$$
$$dT_t = e\left(\int A_t k_t, \mathcal{R}_t\right) dt + \sigma_T dW_t$$

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