BEYOND MYOPIA: LEVERAGING HISTORICAL PATHS FOR SUPERIOR ASSET-LIABILITY MANAGEMENT

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Abstract. This paper addresses a novel continuous-time asset-liability management (ALM) model that integrates time delays, applying stochastic delay differential equations to both assets and liabilities influenced by delayed economic states. Departing from canonical approaches, we deploy the relative *asset-to-liability ratio* as the maximand nested in the expected CRRA utility function. This methodology allows for more effective cross-firm comparisons, especially between firms of different sizes. From a technical standpoint, we tackle the inherent non-Markovian optimal control challenges using a piecewise dynamic programming strategy. Our results contribute to the arena of non-Markovian local equilibrium theory by providing a closed-form solution to a specific category of optimal control problems. Moreover, our numerical example demonstrates that our strategy outperforms those based solely on current states.

Keywords: Asset-Liability Management, Time Delay, Piecewise Dynamic Programming Approach, Stochastic Delay Differential Equation, Optimal Strategy

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1. Introduction

Asset-liability management (ALM) is the cornerstone of modern financial risk management, which originates from Markowitz (1952) and a spectrum of papers established[1] on the foundation of mean-variance portfolio model. However, these models primarily focus on the asset side, and overlook the pivotal role of liabilities in optimizing portfolios. ALM addresses this crux by harmonizing assets and liabilities, with Sharpe and Tint (1990) introducing the "liability-hedging credit" concept to quantify return benefits within the mean-variance framework. Subsequently, ALM research has diversified to encompass various financial market models and optimization specifications[2]. Nevertheless, they typically operate against the backdrop of a first-order Markovian process, implying that optimal strategies rely solely on current information. Consequently, widespread non-Markovian phenomena with time delays in the real world cannot be understood well. Time delays are embodied in numerous economic entities[3]. Broadly, recent behavioral economics frameworks also debate on how time-delay myopia of agents affects their response to policy and aggregate state of the economy (Gabaix, 2020; Farhi and Gabaix, 2020; Angeletos and Huo, 2021).

However, the study of ALM under time delays has been neglected for a long time. In fact, in the operation of financial institutions, not only are their assets affected by time delays, but their liabilities behave as historical path depending. From the perspective of institutions supervision, the liability of an insurer's business, for example, regulatory requirements and actuarial practices often require insurers to index the price of products to 750-day moving averages of risk-free rates, delayed industry mortality tables and so forth, rather than the current market environment and economic performance per se. However, this is an uncharted field for traditional portfolio and ALM models based on efficient market hypotheses and Markovian state

[1] In model setting, Markowitz (1952) also brings on significant advancements in both discrete-time and continuous-time models (Merton, 1975; Li and Ng, 2000; Zhou and Yin, 2003; Blanchet, Chen and Zhou, 2022). These analysis lays down the basis for our model too.

[2] See Chen, Yang and Yin (2008), Zeng and Li (2011), Keel and Müller (1995), Zeng, Li, Zhu and Wu (2014), Wei and Wang (2017), Zhang, Wu, Li and Wiwatanapataphee (2017), Li, Shen and Zeng (2018), Chen, Huang and Li (2022), Li, Zhao and Chen (2023) for the literature of ALM in different financial market models and optimization criteria.

[3] Financial markets exhibit various time-dependent features, such as price momentum, long-run reversals, post-earnings announcement drifts, and long memory volatility highlighting slow reactions to information (Jegadeesh and Titman, 1993; De Bondt and Thaler, 1985; Bernard and Thomas, 1989). Macro indices resembling inflation, CPI, and GDP often incur publication delays due to the nature of data collection and calculations, which poses significant impacts on equity and bond returns (Fama, 1981; Ludvigson and Ng, 2009).

variables. Huang (1987) highlights the limitations of these models for investors exhibiting non-Markovian behavior, such as overextrapolation, disposition effects, and prospect preferences. Traditional dynamic programming methods (Merton, 1975) become inapplicable, rendering portfolio selection under time delays a largely practical issue. Thus, ALM decisions involve navigating both macro-level indexation lags and micro-level asset path dependencies, justifying the importance of a potential framework that addresses the time-delayed decision puzzle.

This paper aims to bridge this gap by utilizing continuous-time stochastic delay differential equations (SDDE), pioneered in the mathematical literature (Cont, Fournié et al., 2013). SDDE provides a natural framework for characterizing time delays and offer a unified analysis, with delay lengths incorporated as parameters within the system instead of requiring additional state variables. However, challenges remain: infinite initial values are needed for process specification, and no straightforward Itô's formula exists for SDDEs, hindering the analysis. Peng, Yang et al. (2009) introduces the anticipated backward stochastic differential equation (ABSDE) framework, but its computational complexity can be immense (He, Li and Li, 2018). Recently, Li and Liu (2018) develops a piecewise methodology for tackling these issues, providing a closed-form solution for optimal stochastic control problems involving delayed state variables. They introduce new path-induced state variables alongside existing ones, creating a sufficient statistic for dynamic portfolios. This enlightened various economic and financial applications, as demonstrated by Li (2019) who finds that incorporating indexation lags into optimal portfolios problem improving investors' welfare. Guo, Huang and Li (2020) further validates the benefits of time delays by showcasing how time-to-build enriches the CIR model's ability to replicate empirical facts. Akin to these researches, we also want to deploy this toolkit to solve time-dalayed ALM issues.

In this paper, we address the well-established practice of measuring ALM objectives using the Constant Relative Risk Aversion (CRRA) utility function, a common choice in both portfolio selection and ALM research[4]. To our best knowledge, all off-the-shelf ALM researches usually use the *surplus* (asset value minus liability value) as the wealth measure (e.g., Chen et al., 2008; Yuan and Mi, 2022). However, surplus can be skewed by company size, potentially hindering cross-firm comparisons. To overcome this limitation, we propose utilizing the *asset-to-liability ratio (ALR)* as the key metric[5]. The ALR offers two distinct advantages. Firstly, it facilitates

[4]e.g., Zeng et al. (2014),Li and Liu (2018), He, Li and Li (2018), Chen et al. (2022), among others.

[5]In Zou and Cadenillas (2014), Jin, Yang and Yin (2015), Zhang, Chen, Jin and Li (2020), Zhang et al. (2020) and Jin, Xu and Zou (2021), the *debt ratio* (liability to surplus) is treated as a control variable for the decision-makers. Differentiating from the debt ratio, the ALR aligns with

comparisons across companies of varying sizes by focusing on the relative value of assets and liabilities, as opposed to absolute differences. Secondly, the solvency[6] is regulated by limitations on liability levels. This alignment between our model and real-world practices enhances its practical relevance.

Technically, this paper builds upon two insightful studies with close ties to our work: A and Zeng (2022) and Yuan and Mi (2023). While A and Zeng (2022) also uses SDDEs to model the investor's wealth process, their focus remains quiet simply on the internal financial information delays. It digresses from incorporating both macroeconomic time delays or indexation lags faced against by companies, which is our more practical goal. Yuan and Mi (2023) focuses on a robust optimal ALM problem with delays for risk-averse investors. Their work employs a jump diffusion wealth process and aims to maximize the robust value, balancing the expected utility of a weighted terminal wealth with a penalty for model uncertainty. We take this a step further by exploring how time delays affect optimal ALM strategies.

This study makes several significant contributions to the field of asset-liability management. Firstly, we propose a novel non-Markovian ALM model of both asset and liability sides driven by time-delayed economic states. This model, built upon continuous-time SDDE, captures the influence of historical information on financial decisions. Secondly, we derived a closed-form solution for this path-dependent optimal control problem. This breakthrough is facilitated by employing the innovative piecewise dynamic programming approach of Li and Liu (2018). Thirdly, our analysis reveals a powerful result: incorporating both the current economic state and its historical path in portfolio strategies leads to demonstrably superior performance compared to solely relying on current information. This is convincingly illustrated by the *certainty equivalent wealth* index showcased in our numerical example.

The remainder of this paper is organized as follows. Section 2 delineates the financial market and the optimal specification we are interested in. We delve into the intricacies of the financial market and meticulously defines the optimal problem. In Section 3, we get the solution of the uncertain problems by using the piecewise dynamic programming approach. Section 4 provides a numerical example to study the economic value of time lags and Section 5 concludes this paper. In the appendix, we deliver the solution of ALM problem and the proofs of all the theorems and propositions in the main text.

target of insurance companies on expanding and managing risk through asset portfolios. Therefore, we formulate the decision-maker's utility optimization problem in terms of maximizing the ALR.

[6]Such as in the insurance industry, solvency index can be measured by the ALR.

2. Problem Formulation

Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space on which defined a standard 3-dimensional Brownian motion $B_t = (B_{A,t}, B_{L,t}, B_{X,t})'$. Define $\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}$. We denote $L^2_{\mathcal{F}}(0, T; \mathbb{R}^3)$ the set of all $\mathbb{R}^3$-valued, measurable stochastic processes $f(t)$ adapt to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $E\int_0^T |f(t)|^2 dt < +\infty$.

The economy is composed by three facets: financial market, companies making decision on asset-to-liability ratio, and exogenous state variables of the economy.

2.1. **Financial Market.**

The simplest market only contains two assets, bonds and risky asset, are traded continuously. Bonds' price P_t is subjected to the following ordinary differential equation (ODE):

$$\begin{cases} dP_t = rP_tZ_tdt, \quad t \in [0, T] \\ P_0 = p > 0, \end{cases} \tag{2.1}$$

where risk-free interest rate $r \geq 0$. The price of bond is affected by the delayed economic state Z_t as we discussed in aforementioned part. The other one, risky asset whose price S_t is affected by the delayed economic state concurrently and satisfies the following system of stochastic delay differential equation (SDDE):

$$\begin{cases} dS_t = \mu S_tZ_tdt + \sigma_S S_t\sqrt{Z_t}dB_{A,t}, \quad t \in [0, T] \\ S_0 = s > 0, \end{cases} \tag{2.2}$$

where $\mu > 0$ is the appreciation rate and $\sigma_S > 0$ is the volatility of the risky asset. The price of the risky asset evolves according to the standard geometric Brownian motion driven by a standard Wiener process $B_{A,t}$. Total asset of company is A_t. Assuming that the trading of shares happens continuously and that transaction cost and consumption are vanishing, whereafter the asset obeys the process in what follows:

$$\begin{cases} dA_t = (A_tr + \mu U_t)Z_tdt + \sigma_S U_t\sqrt{Z_t}dB_{A,t}, \quad t \in [0, T] \\ A_0 = a > 0, \end{cases} \tag{2.3}$$

where U_t is a risk asset portfolio of the company.

Meanwhile, as the mirror of assets, the liability L_t is set as an exogenous process[7] following a SDDE, which is also affected by the delayed economic state variables:

$$\begin{cases} dL_t = \beta L_t Z_t dt + \sigma_L L_t \sqrt{Z_t} dB_{L,t}, \quad t \in [0,T] \\ L_0 = l > 0, \end{cases} \tag{2.4}$$

where $\beta > 0$ and $\sigma_L > 0$ are appreciation rate and volatility of the liability. As the counterpart of the asset, the price also evolves with the standard geometric Brownian motion driven by $B_{L,t}$. We assume that the two processes $\{B_{A,t}, 0 < t < T\}$ and $\{B_{L,t}, 0 < t < T\}$ are correlated with coefficient[8] $\rho_{AL} \in [-1, 1]$, i.e.,

$$dB_{A,t} dB_{L,t} = \rho_{AL} dt. \tag{2.5}$$

Asset Allocation. ALM determines how insurance companies control long-term liability streams. We regard the *asset-to-liability ratio (ALR)* as a relative financial indicator for measuring ALM goals. Compared to canonical approaches[9], ALR offers several advantages. First, it facilitates comparison and evaluation of ALM effectiveness across different scenarios, regardless of absolute size or current business situation (Chen et al., 2022). Comparatively speaking, the absolute value of the *surplus* tends to be the focal index[10] of decision-maker to maximize utility. However, this metric presents limitations owing to it is context-dependent and vary across insurance companies and their developmental stages. Real-world decision-makers often prioritize the relative value of assets compared to liabilities, so surplus index is unfounded.

This paper nests the ALR into optimization problem in following way: Define ALR by

$$Y_t = \frac{A_t}{L_t}. \tag{2.6}$$

Let π_t be the fraction of asset invested in the risky asset at time t. We call the process $\pi := \{\pi_t, 0 \le t \le T\}$ an ALM strategy, denote by Y_t^π, the ALR of the agent at time t under the strategy π. Therefore, we can obtain the path of change in the ALR Y_t during the ALM management process under the asset allocation strategy π:

[7]For insurers, unlike the management of the asset side where the company takes the initiative, the liability side is often not susceptible to active management by the company. The size and structure of their liabilities are often affected by the market environment, the level of sales staff, the regulatory measures and other factors, so we temporarily consider their liabilities to be exogenous.

[8]This assumption is embodied in widespread phenomena such as the fact that the value of assets vary either in the same direction or inversely with the state of the economy (e.g., CPI, GDP, etc.).

[9]See Chen et al. (2008), Zeng et al. (2014), etc.

[10]The difference between asset and liability values ($Y = A - L$).

$$\begin{cases} dY_t = \left[r + \pi_t(\mu - r) - \beta - \pi_t \rho_{AL}\sigma_S\sigma_L + \sigma_L^2\right] Z_t Y_t dt \\ \qquad + \pi_t \sigma_S \sqrt{Z_t} Y_t dB_{A,t} - \sigma_L \sqrt{Z_t} Y_t dB_{L,t}, \quad t \in [0, T] \\ Y(0) = y. \end{cases} \tag{2.7}$$

State Process. State process is set for closing the financial system. Assume that the state variable Z_t is measured by an exponentially decayed weighted average of historical state variables over a time interval $[t - \tau, t]$ with time delay τ, namely,

$$Z_t = \frac{\kappa}{1 - e^{-\kappa\tau}} \int_{t-\tau}^{t} e^{-\kappa(t-s)} X_s ds, \tag{2.8}$$

where $\kappa > 0$ measures the decaying rate of the weights on the historical state variables[11]. This formulation captures the persistence of past economic conditions, such as indexation lags, within a new state variable, Z_t, that aggregates the states X_s for $t - \tau < s < t$ within the delay period. For the underlying state process X_t, we adopt the following model:

$$dX_t = (a_0 + a_1 X_t + a_2 Z_t)dt + \sigma_X \sqrt{Z_t} dB_{X,t}, \tag{2.9}$$

where a_0, a_1, a_2 and σ_X are constants, and X_t is driven by Wiener process $B_{X,t}$.

The instantaneous correlations between the changes in the economic state variable and the changes in assets and liabilities are characterized by the coefficients ρ_{AX} and ρ_{LX}, respectively. These coefficients, both falling within the range $[-1, 1]$, reflect the degree of linear dependence between economic fluctuations and asset/liability dynamics, i.e., $dB_{A,t}dB_{X,t} = \rho_{AX}dt$ and $dB_{L,t}dB_{X,t} = \rho_{LX}dt$.

Remark 2.1. (2.8) reveals the implicit belief of trend followers within the state trend construction. They assign greater weight to more recent state variables, assuming they hold more relevant information for predicting future state movements. Consequently, the exponential decay with rate κ reflects a diminishing influence of historical states on the current trend. Notably, in the extreme case where $\kappa \to 0$, the state trend Z_t simplifies to the standard moving average, emphasizing the prioritization of recent observations, that is,

$$Z_t = \frac{1}{\tau} \int_{t-\tau}^{t} X_s ds. \tag{2.10}$$

When $\kappa \to \infty$, all the weights go to the current state so that $y_t \to x_t$(He, Li, Wei and Zheng, 2009).

[11]This type of setting can also by seen Li and Liu (2018) and Guo et al. (2020).

In general, for $0 < \kappa < \infty$, (2.8) can be expressed as a differential equation with time delay τ:

$$dZ_t = \frac{\kappa}{1 - e^{-\kappa\tau}} \left[X_t - e^{-\kappa\tau} X_{t-\tau} - (1 - e^{-\kappa\tau}) Z_t\right] dt. \tag{2.11}$$

2.2. **Optimization.** In comparison to traditional maximization utility framework, e.g. consumption, cash follow or investment revenue, our decision makers seek an admissible strategy that optimizes the company's financial well-being embodying in ALM. Previous discussion has justified the rationale of ALM. We exert Constant Relative Risk Aversion (CRRA) utility function to nest the maximand, which is a common choice in portfolio management and ALM applications[12], videlicet,

$$U(t, Y) = \int_0^T e^{-\delta t} \frac{Y_t^{1-\gamma}}{1-\gamma},$$

where $\delta > 0$ is the discount rate and γ denotes the coefficient of relative risk aversion. Meanwhile, canonical optimization issues should also account for the terminal value of the integrate decision period. Presume we put weight $(1 - \alpha)$ on the end T. ALM is finally casted into a constrained stochastic optimization problem:

$$\begin{cases} \max\limits_{\pi} E \left[\alpha \int_0^T e^{-\delta t} \dfrac{Y_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha) e^{-\delta T} \dfrac{Y_T^{1-\gamma}}{1-\gamma}\right], \\ \text{subject to} \begin{cases} \text{Equation(2.7)}, \\ (Y(\cdot), \pi(\cdot)) \ \textit{admissible}, \end{cases} \end{cases}$$

Hence, it induces the value function

$$J(t, y) = \max_{\pi} E_{t,y} \left[\alpha \int_0^T e^{-\delta t} \frac{Y_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha) e^{-\delta T} \frac{Y_T^{1-\gamma}}{1-\gamma}\right], \tag{2.12}$$

where $E_{t,y}[\cdot] = E[\cdot | Y_t^{\pi} = y]$.

3. Solution to the Uncertain Problem

Technically, on the one hand, the asset and liability sides studied in this paper are jointly driven by the current state variables and their historical paths, which is fundamentally different from the traditional Markovian ALM decision problem. On the other hand, Itô's Lemma serves as the prerequisite find the optimal ALM decision, but Itô's Lemma can only be applied to a class of stochastic differential systems driven by the current state variables, and there is no Itô's Lemma that can be applied to time-delayed problems. To address this challenge, we propose a novel method inspired by Li and Liu (2018) and Guo et al. (2020), which aims to obtain closed-form solutions for optimal ALM strategies with arbitrary lag length τ. Based

[12]See He et al. (2018), Li, Zeng and Yang (2018) and Li (2019) for reference.

on the current state and its historical path within the current period, we aim to construct a sufficient statistic for the next period[13].

To better delineate optimal strategies for long-standing horizons, we coin the following variables sequence at equidistant intervals of τ:

$$\begin{aligned} &X_t^{(n)} = X_{t-(n-1)\tau}, \quad Z_t^{(n)} = Z_{t-(n-1)\tau}, \\ &B_{A,t}^{(n)} = B_{A,t-(n-1)\tau}, \quad B_{L,t}^{(n)} = B_{L,t-(n-1)\tau}, \quad B_{X,t}^{(n)} = B_{X,t-(n-1)\tau}, \quad n \geq 1. \end{aligned} \tag{3.1}$$

Especially, we have $X_t^{(1)} = X_t$, $Z_t^{(1)} = Z_t$, $B_{A,t}^{(1)} = B_{A,t}$, $B_{L,t}^{(1)} = B_{L,t}$ and $B_{X,t}^{(1)} = B_{X,t}$ when $n = 1$.

Case 1: $T \leq \tau$. Holding $T \leq \tau$, the state variables Y, X, and Z can constitute a sufficient statistic for the value function[14] and hence fully span ALM strategies. Hereafter, the agent's value function is of the form $J = J(t, Y, X, Z)$ and can be solved from the following Hamilton-Jacobi-Bellman (HJB) equation (Merton, 1975):

$$\begin{aligned} \Psi = \sup_{\pi} \Bigg\{ &\frac{\partial J}{\partial t} + \frac{\partial J}{\partial Y}\left[r + \pi_t(\mu - r) - \beta - \pi_t\rho_{AL}\sigma_S\sigma_L + \sigma_L^2\right] Y_t Z_t \\ &+ \frac{\partial J}{\partial Z}\frac{\kappa}{1-e^{-\kappa\tau}}\left[X_t - e^{-\kappa\tau}X_{t-\tau} - (1-e^{-\kappa\tau})Z_t\right] \\ &+ \frac{\partial J}{\partial x}[a_0 + a_1 X_t + a_2 Z_t] + \frac{1}{2}\frac{\partial^2 J}{\partial Y^2}Y_t^2 Z_t[\pi_t^2\sigma_S^2 + \sigma_L^2 - 2\pi_t\sigma_S\sigma_L\rho_{AL}] + \frac{1}{2}\frac{\partial^2 J}{\partial x^2}[\sigma_X^2 Z_t] \\ &+ \frac{\partial^2 J}{\partial Y \partial x}Y_t\left[\pi_t\rho_{AX}\sigma_S\sigma_X Z_t - \rho_{LX}\sigma_L\sigma_X Z_t\right] + \alpha e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}\Bigg\}, \end{aligned} \tag{3.2}$$

with the boundary condition $V(T, Y_T) = (1-\alpha)e^{-\delta T}\frac{Y_T^{1-\gamma}}{1-\gamma}$. Next proposition portrays the exact analytical natures of HJB equations, and the proof of Proposition 3.1 is provided in Appendix A.2.

Proposition 3.1. *For $0 < T \leq \tau$, the optimal ALM strategy under time delays is given by*

$$\pi_t^* = \frac{\mu - r - \rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AX}\sigma_S\sigma_X\frac{\partial \ln f}{\partial X_t}}{\gamma\sigma_S^2}, \tag{3.3}$$

and the value function is given by

$$J(t, Y, X, Z) = e^{-\beta t}\frac{Y_t^{1-\gamma}}{1-\gamma}f^{\gamma}(t, X, Z).$$

[13]This is achieved by introducing continuous-time SDDEs, which provide a unified framework for analyzing different time-delay scenarios, eliminating the need for an additional state variable as seen in discrete-time models.

[14]See Li and Liu (2018) for a proof for how to construct the "sufficient statistics" for value function.

The f is given by

$$f(t,X,Z) = \alpha^{\frac{1}{\theta}} \int_t^T \hat{f}(u,X,Z)du + (1-\alpha)^{\frac{1}{\theta}} \hat{f}(t,X,Z),$$

where

$$\hat{f}(t,X,Z) = \exp\left\{ D_{1,t}^{(1)} X_t + D_{2,t}^{(1)} Z_t + D_{3,t}^{(1)} \right\}, \tag{3.4}$$

and $D_{i,t}^{(1)}$ are governed by ODEs:

$$\begin{aligned}
&\dot{D}_{1,t}^{(1)} + D_{1,t}^{(1)} a_1 + D_{2,t}^{(1)} \frac{\kappa}{1-e^{-\kappa\tau}} = 0, \\
&\dot{D}_{2,t}^{(1)} + \frac{1-\gamma}{\gamma}(M_1 + D_{1,t}^{(1)} M_2) + D_{1,t}^{(1)} a_2 - D_{2,t}^{(1)} \kappa - \frac{1}{2}(1-\gamma)(M_3 + D_{1,t}^{(1)} M_4 - \rho_{AX}^2 \sigma_X^2 (D_{1,t}^{(1)})^2) \\
&+ \frac{1}{2}\gamma\sigma_X^2 (D_{1,t}^{(1)})^2 + (1-\gamma) M_5 D_{1,t}^{(1)} = 0 \\
&\dot{D}_{3,t}^{(1)} + D_{1,t}^{(1)} a_0 - \frac{\delta}{\gamma} - D_{2,t}^{(1)} \frac{\kappa}{1-e^{-\kappa\tau}} e^{-\kappa\tau} x_{t-\tau} = 0,
\end{aligned} \tag{3.5}$$

Remark 3.2. The integration of historical economic trajectories alongside current states is paramount in ALM decision-making processes. Two distinct mechanisms by which the historical paths of the state variable X have material impacts on optimal ALM strategies:

1. *State variable Z_t*: This variable, as defined in (2.8), inherently incorporates historical information by elegantly representing an exponentially decaying weighted average of past states. This integration paradigm reflects the underlying belief that the more recent states, the greater degree of predictive strength they have for the future market movements.

2. *Coefficient $D_3^{(1)}$*: This coefficient results from the integration of the historical values of $X_{t-\tau}$ in (3.5) and directly captures the intertwined effects of time lags, which ensures the optimal ALM strategy derived from (3.3) takes the full spectrum of market dynamics that have unfolded over the time interval $[t-\tau, t]$ into account, rather than myopically relying on the instantaneous state of the system.

In stark contrast to typical ALM models that disregard time delays, the optimal decision presented in (3.3) is characterized by the term $\frac{\partial \ln f}{\partial X_t}$, which inherently incorporates historical economic information. Unlike the ALM optimal decision that ignore time delays (see Appendix B), $\frac{\partial \ln f}{\partial X_t}$ contains historical information about the state of the economy, e.g., the insurance company considers information-lagged industry life tables, CPI, inflation, etc., when performing asset-liability management. While this term previously appears in non-time-delay models, their underlying ODEs conspicuously lack the historical context terms Z_t and $D_3^{(1)}$. This fundamental distinction accentuates the central role of Z_t and $D_3^{(1)}$ in carefully incorporating historical context into ALM decisions, leading to more comprehensive and scientific underpinning strategies.

Besides, Proposition 3.1 further supports foregoing conclusion by clearly demonstrates that time delays exert their profound influence on both optimal investment strategies and value functions through the locuses of Z_t and $D_3^{(1)}$, the conduits for time lag information.

As the epilogue of this part, Corollary 3.3 depicts the corresponding strategy of a specific complexion holding $\alpha = 0$.

Corollary 3.3. *In particular, for a special class of ALM problems that focus only on the target value at the terminal time T, the fraction of management goal $\alpha = 0$, and*

$$f(t, X, Z) = \exp\left\{D_{1,t}^{(1)} X_t + D_{2,t}^{(1)} Z_t + D_{3,t}^{(1)}\right\},$$

and the optimal portfolio weight is given by

$$\pi_t^* = \frac{\mu - r - \rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AX}\sigma_S\sigma_X D_{1,t}^{(1)}}{\gamma\sigma_S^2}.$$

Case 2: $\tau < T \leq 2\tau$. For $T > \tau$, asset allocation is out of control by (3.3) due to $X_{t-\tau}$ in (3.5) is not adapted to the filtration $\mathcal{F}_t$. New variables $X_t^{(2)}$ and $Z_t^{(2)}$ defined in (3.1) are introduced to handle this problem whence $\tau < T \leq 2\tau$. Hence, the optimal problem (2.2) is slightly modified:

$$\max_\pi E\left[\alpha\int_0^T e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}dt + (1-\alpha)e^{-\delta T}\frac{Y_T^{1-\gamma}}{1-\gamma}\right] \tag{3.6}$$

where

$$\begin{aligned}
dY_t &= \left[r + \pi_t(\mu - r) - \beta - \pi_t\rho_{AL}\sigma_S\sigma_L + \sigma_L^2\right] Z_t^{(1)} Y_t dt + \pi_t\sigma_S\sqrt{Z_t^{(1)}} Y_t dB_{A,t}^{(1)} \\
&\quad - \sigma_L\sqrt{Z_t^{(1)}} Y_t dB_{L,t}^{(1)}, \\
dX_t^{(1)} &= (a_0 + a_1 X_t^{(1)} + a_2 Z_t^{(1)})dt + \sigma_X\sqrt{Z_t^{(1)}} dB_{X,t}^{(1)}, \\
dX_t^{(2)} &= (a_0 + a_1 X_t^{(2)} + a_2 Z_t^{(2)})dt + \sigma_X\sqrt{Z_t^{(2)}} dB_{X,t}^{(2)}, \\
dZ_t^{(1)} &= \frac{\kappa}{1 - e^{-\kappa\tau}}\left[X_t^{(1)} - e^{-\kappa\tau}X_t^{(2)} - (1 - e^{-\kappa\tau})Z_t^{(1)}\right] dt, \\
dZ_t^{(2)} &= \frac{\kappa}{1 - e^{-\kappa\tau}}\left[X_t^{(2)} - e^{-\kappa\tau}X_{t-\tau}^{(2)} - (1 - e^{-\kappa\tau})Z_t^{(2)}\right] dt,
\end{aligned} \tag{3.7}$$

$B_{X,t}^{(1)}$ and $B_{X,t}^{(2)}$ are mutually independent standard Brownian motions. The optimal ALM strategy is still in the form of (3.3), but the $\hat{f}$ in (A.14) of value function change to

$$\hat{f}(t,X,Z) = \exp\left\{ D_{1,t}^{(2)} X_t^{(1)} + D_{2,t}^{(2)} Z_t^{(1)} + D_{3,t}^{(2)} X_t^{(2)} + D_{4,t}^{(2)} Z_t^{(2)} + D_{5,t}^{(2)} \right\}, \tag{3.8}$$

where $D_{i,t}^{(2)}$ are governed by ODEs:

$$\begin{aligned}
&\dot{D}_{1,t}^{(2)} + D_{2,t}^{(2)} \frac{\kappa}{1-e^{-\kappa\tau}} + D_{1,t}^{(2)} a_1 = 0, \\
&\dot{D}_{3,t}^{(2)} + D_{4,t}^{(2)} \frac{\kappa}{1-e^{-\kappa\tau}} - D_{2,t}^{(2)} \frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}} + D_{3,t}^{(2)} a_1 = 0, \\
&\dot{D}_{2,t}^{(2)} + \frac{1-\gamma}{\gamma} M_1 + \frac{1-\gamma}{\gamma} M_2 D_{1,t}^{(2)} - D_{2,t}^{(2)} \kappa + D_{1,t}^{(2)} a_2 - \frac{1}{2}(1-\gamma) M_3 - \frac{1}{2}(1-\gamma) D_{1,t}^{(2)} M_4 \\
&\qquad + \frac{1}{2}(1-\gamma)\sigma_X^2 \rho_{Ax}^2 \left(D_{1,t}^{(2)}\right)^2 + \frac{1}{2}\gamma \left(D_{1,t}^{(2)}\right)^2 \sigma_X^2 + (1-\gamma) D_{1,t}^{(2)} M_5 = 0, \\
&\dot{D}_{4}^{(2)} - D_{4,t}^{(2)} \kappa + D_{3,t}^{(2)} a_2 + \frac{1}{2}\gamma \left(D_{3,t}^{(2)}\right)^2 \sigma_X^2 = 0, \\
&\dot{D}_{5,t}^{(2)} - D_{4,t}^{(2)} \frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}} X_{t-\tau}^{(2)} + D_{1,t}^{(2)} a_0 + D_{3,t}^{(2)} a_0 - \frac{\delta}{\gamma} = 0.
\end{aligned} \tag{3.9}$$

The proofs of $\tau < T \le 2\tau$ and other cases of $T > \tau$ can be found in Appendix A.3.

Case 3: $(n-1)\tau < T \le n\tau$

Inheriting the above method, we can derive the optimal ALM strategy for any horizon. The general results are summarized in the following proposition. Condition on $(n-1)\tau < T \le n\tau$, $n = 1, 2, \ldots$ the optimization target becomes:

$$\max_{\pi} E\left[\alpha \int_0^T e^{-\delta t} \frac{Y_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha) e^{-\delta T} \frac{Y_T^{1-\gamma}}{1-\gamma} \right] \tag{3.10}$$

where

$$\begin{aligned}
dY_t &= \left[r + \pi_t(\mu - r) - \beta - \pi_t \rho_{AL} \sigma_S \sigma_L + \sigma_L^2 \right] Z_t^{(1)} Y_t dt + \pi_t \sigma_S \sqrt{Z_t^{(1)}} Y_t dB_{A,t}^{(1)} \\
&\qquad - \sigma_L \sqrt{Z_t^{(1)}} Y_t dB_{L,t}^{(1)}, \\
dX_t^{(i)} &= (a_0 + a_1 X_t^{(i)} + a_2 Z_t^{(i)}) dt + \sigma_X \sqrt{Z_t^{(i)}} dB_{X,t}^{(i)}, \qquad 1 \le i \le n \\
dZ_t^{(i)} &= \frac{\kappa}{1-e^{-\kappa\tau}} \left[X_t^{(i)} - e^{-\kappa\tau} X_t^{(i+1)} - (1-e^{-\kappa\tau}) Z_t^{(i)} \right] dt, \qquad 1 \le i < n \\
dZ_t^{(n)} &= \frac{\kappa}{1-e^{-\kappa\tau}} \left[X_t^{(n)} - e^{-\kappa\tau} X_{t-\tau}^{(n)} - (1-e^{-\kappa\tau}) Z_t^{(n)} \right] dt,
\end{aligned} \tag{3.11}$$

and $dB_{A,t}^{(1)} dB_{L,t}^{(1)} = \rho_{AL} dt$, $dB_{A,t}^{(1)} dB_{X,t}^{(1)} = \rho_{AX} dt$ and $dB_{L,t}^{(1)} dB_{X,t}^{(1)} = \rho_{LX} dt$. $B_{X,t}^{(i)}$ and $B_{X,t}^{(j)}$ $(i \ne j)$ are mutually independent standard Brownian motions.

Akin to what we provided in Corollary 3.3, the optimum approached by deploying the strategy provided in next proposition:

Proposition 3.4. *For $(n-1)\tau < T \leq n\tau$, the optimal ALM strategy under time delays is given by*

$$\pi_t^* = \frac{\mu - r - \rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AX}\sigma_S\sigma_X\frac{\partial \ln f}{\partial X_t^{(1)}}}{\gamma\sigma_S^2}, \tag{3.12}$$

and the value function is given by

$$J(t,Y,X,Z) = e^{-\beta t}\frac{Y_t^{1-\gamma}}{1-\gamma}f^\gamma(t,X,Z).$$

The f is given by

$$f(t,X,Z) = \alpha^{\frac{1}{\theta}}\int_t^T \hat{f}(u,X,Z)du + (1-\alpha)^{\frac{1}{\theta}}\hat{f}(t,X,Z),$$

where

$$\hat{f}(t,X,Z) = \exp\left\{D_{1,t}^{(n)}X_t^{(1)} + D_{2,t}^{(n)}Z_t^{(1)} + \cdots + D_{2n-1,t}^{(n)}X_t^{(n)} + D_{2n,t}^{(n)}Z_t^{(n)} + D_{2n+1,t}^{(n)}\right\}, \tag{3.13}$$

where $D_{i,t}^{(n)}$ are governed by ODEs:

$$\begin{aligned}
&\dot{D}_{1,t}^{(n)} + D_{2,t}^{(n)}\frac{\kappa}{1-e^{-\kappa\tau}} + D_{1,t}^{(n)}a_1 = 0,\\
&\dot{D}_{2i-1,t}^{(n)} + D_{2i,t}^{(n)}\frac{\kappa}{1-e^{-\kappa\tau}} - D_{2i-2,t}^{(n)}\frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}} + D_{2i-1,t}^{(n)}a_1 = 0, \quad 1 < i \leq n\\
&\dot{D}_{2,t}^{(n)} + \frac{1-\gamma}{\gamma}M_1 + \frac{1-\gamma}{\gamma}M_2 D_{1,t}^{(n)} - D_{2,t}^{(n)}\kappa + D_{1,t}^{(n)}a_2 - \frac{1}{2}(1-\gamma)M_3 - \frac{1}{2}(1-\gamma)D_{1,t}^{(n)}M_4\\
&\qquad + \frac{1}{2}(1-\gamma)\sigma_X^2\rho_{Ax}^2\left(D_{1,t}^{(n)}\right)^2 + \frac{1}{2}\gamma\left(D_{1,t}^{(n)}\right)^2\sigma_X^2 + (1-\gamma)D_{1,t}^{(n)}M_5 = 0,\\
&\dot{D}_{2i}^{(n)} - D_{2i,t}^{(n)}\kappa + D_{2i-1,t}^{(n)}a_2 + \frac{1}{2}\gamma\left(D_{2i-1,t}^{(n)}\right)^2\sigma_X^2 = 0, \quad 1 < i \leq n\\
&\dot{D}_{2n+1,t}^{(n)} - D_{2n,t}^{(n)}\frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}X_{t-\tau}^{(n)} + \sum_{i=1}^n D_{2i-1,t}^{(n)}a_0 - \frac{\delta}{\gamma} = 0.
\end{aligned} \tag{3.14}$$

The proof of Proposition 3.4 are given in Appendix A.3.

4. Economic implications

In this section, we investigate the economic value of mitigating long-term risks in the backdrops of time delays or information lags, by analyzing the certainty equivalent wealth (CEW) of the optimal ALM strategy. We define the CEW of an ALM strategy for investors as the initial wealth level that makes the utility indifferent between two scenarios: (1) receiving the CEW with certainty at the terminal time T, and (2) starting with a unit ALR and deploying the chosen ALM strategy for investment until time T (Li, 2019).

To deepen our understanding of the relationship between CEW and time delay, we explore a more general scenario. Consider an insurance company bound by regulatory and accounting standards. Their investment decisions rely on data with inherent lags, such as the 750-day moving average of risk-free rate, industrial mortality table which is published delay or historical GDP/CPI figures. However, ALM results must be presented publicly at specific intervals, like quarterly, semi-annually, or annually. This means that, in general, ALM decision makers are more concerned about the ALR at the terminal time T ($\alpha = 0$), and their optimal decisions will have a time horizon T that is shorter than the delayed time horizon τ.

Denote CEW_τ as the CEW of the optimal strategy (3.3) with an indexation lag of τ satisfying

$$\frac{(Y_0 CEW_\tau)^{1-\gamma}}{1-\gamma} = \mathbb{E}_0\left[\frac{Y_T^{1-\gamma}}{1-\gamma}\right]. \tag{4.1}$$

Plugging (A.5) into it, the CEW is given by

$$CEW_\tau = \exp\left\{\frac{\gamma}{1-\gamma}\left(D_{1,0}^{(1)}X_0 + D_{2,0}^{(1)}Z_0 + D_{3,0}^{(1)}\right)\right\}, \tag{4.2}$$

where $D_{i,t}$ for $i = 1, 2, 3$ depend on the history path depicted by (3.5).

Notice that the optimal investment strategy (3.3) depends on the entire historical path of state variables. Different paths lead to different optimal ALM strategies and consequently different CEW[15]. To disentangle the relationship between the time delay and the ALR, we use the average CEW based on 1000 paths of state process generated from (2.9).

Table 4.1. The parameters for the numerical example

δ	a_0	a_1	a_2	κ	σ_X	μ	σ_S
0.025	6.97	-1.67	-1.17	0.3	0.35	0.035	0.07
β	σ_L	γ	r	ρ_{AL}	ρ_{AX}	ρ_{LX}	T
0.04	0.1	2	0.03	0.5	0.5	0.5	12

Some of the parameter settings refer to Guo et al. (2020), Chen et al. (2022) and Li (2019).

Fig. 4.1 illustrates the significant utility gained by incorporating time delays in ALM decisions. The figure displays the ratio of the average CEW of the optimal ALM strategy with time delay to the CEW of the strategy without time delay. This ratio is plotted against various time lags ranging from 12 to 60 months, with a fixed ALM terminal time of 12 months.

Consistent with our intuition, the y-axis values consistently remain above 1, highlighting the positive economic value gained after pondering time delays for ALM.

[15]See Li (2019) and Li and Liu (2022).

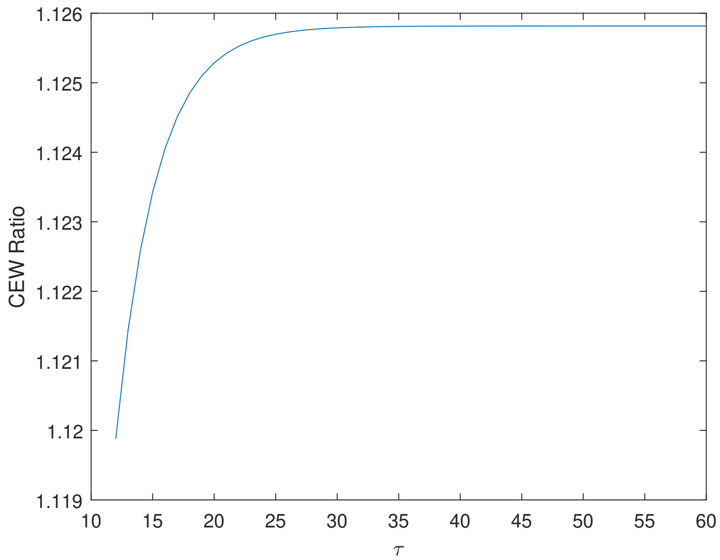

FIGURE 4.1. The average CEW ratio of optimal ALM strategy to the CEW generated by ALM strategy ignoring the time dalay from 12 to 60 months

Notably, with a 12-month lag, the CEW ratio reaches approximately 1.12, as evident in the figure. This can be converted to a 12% increase in the ALR by simply accounting for the time delay.

Furthermore, the upward trend of the curve in Fig. 4.1 aligns with our anticipation: greater time lag corresponds to higher CEW. The figure visually explains this relationship, demonstrating an 60bps increase in CEW as the lag extends from 12 months to 60 months. This implied a 12.6% potential increase in the final ALR if the decision-maker utilizes historical information from the past five years, compared to exclusively leaning on current economic data.

This graphical evidence confirms the crucial role of time delays in optimizing ALM strategies. In view of historical information, decision-makers can make more informed investment decisions, ultimately leading to a more robust and financially dominated asset-to-liability position.

5. Conclusion

This paper presents a time-delay theoretical model for asset-liability management. Resting on a system of continuous-time stochastic delay differential equations, our model uniquely captures the internal nexus between the asset and liability. Both sides are dynamically driven by past and present economic conditions. We depart from conventional approaches by focusing on the relative asset-to-liability ratio as the decision objective rather than the absolute surplus. This is not only in accordance with decision-makers' priorities, but also facilitates comparisons across firms of different sizes.

Using the cutting-edge piecewise dynamic programming approach of Li and Liu (2018), we obtain a closed-form solution of this path-dependent, non-Markovian optimal control problem. Our analysis conclusively shows that incorporating both the current and historical economic landscape into optimal portfolio strategies significantly outperforms the one relying on current information alone. This comparison between optimal ALM strategy and others, embodied by deterministic certainty-equivalent wealth measures, underscores the critical role of historical paths in investment performance.

The progenies of this paper are twofold. First, it pushes the boundaries of ALM theory by initially integrating of macroeconomic and price delays into the workhorse model. Second, it advances the field of non-Markovian local equilibrium theory by demonstrating the feasibility of piecewise dynamic programming to solve a time-delay-based ALM model analytically. These advancements pave the way for further leveraging of time-delay effects in both assets and liabilities, enlightening future research in related terrains.

Appendix A. Proofs

A.1. **Proof of** (2.7)**.**

$$\begin{aligned}dY = d\left(\frac{A}{L}\right) &= \frac{dA}{L} - \frac{AdL}{L^2} + \frac{1}{2}\left[-2\frac{dAdL}{L^2} + 2\frac{A(dL)^2}{L^3}\right] \\ &= \frac{A}{L}\left\{[r + \pi(\mu - r)]Z_t dt + \pi\sigma_S\sqrt{Z_t}dB_{A,t}\right\} - \frac{A}{L}\left\{\beta Z_t dt + \sigma_L\sqrt{Z_t}dB_{L,t}\right\} \\ &\quad - \frac{A}{L}\left\{[r + \pi(\mu - r)]Z_t dt + \pi\sigma_S\sqrt{Z_t}dB_{A,t}\right\}\left\{\beta Z_t dt + \sigma_L\sqrt{Z_t}dB_{L,t}\right\} \\ &\quad + \frac{A}{L}\left\{\beta Z_t dt + \sigma_L\sqrt{Z_t}dB_{L,t}\right\}^2 \\ &= Y\left[r + \pi(\mu - r) - \beta - \pi\rho_{AL}\sigma_S\sigma_L + \sigma_L^2\right]Z_t dt + Y\sqrt{Z_t}\left[\pi\sigma_S dB_{A,t} - \sigma_L dB_{L,t}\right].\end{aligned} \tag{A.1}$$

A.2. **Proof of Proposition 3.1.**

Definition A.1. A strategy $\pi : [0, T] \times \Omega \to \mathbb{R}^3$ is called an admissible strategy if π satisfies:
(i) $\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; R^3)$,
(ii) $(Y(\cdot), \pi(\cdot))$ satisfies (2.7),
(iii) $\|\pi(t, Y(t))\| \leq \kappa_1(1 + |Y(t)|^2)$,
(iv) $\|\pi(t, Y_1(t)) - \pi(t, Y_2(t)\| \leq \kappa_2(|Y_1(t) - Y_2(t)|)$,
where κ_1 and κ_2 are positive constants, $Y_1(t)$ and $Y_2(t)$ are continuous measurable functions. We denote $\pi(t, y)$ the admissible strategy set with initial wealth y and $\mathbf{\Pi}$ the set of all admissible strategies.

Lemma A.2. *Let w be a function in $C^{1,2}([-\tau, T) \times \mathbb{R}^n) \cap C^0([-\tau, T) \times \mathbb{R}^n)$, and satisfy a quadratic growth condition, i.e., there exists a constant C, such that,*

$$|W(t, y)| \leq C(1 + |y|^2), \quad \forall (t, y) \in [-\tau, T] \times \mathbb{R}^n.$$

Define

$$\begin{aligned}\mathcal{L}^\pi W(t, Y_t) =& \frac{\partial J}{\partial t} + \frac{\partial J}{\partial Y_t}\left[r + \pi_t(\mu - r) - \beta - \pi_t\rho_{AL}\sigma_S\sigma_L + \sigma_L^2\right]Y_t Z_t \\ &+ \frac{\partial J}{\partial Z_t}\frac{\kappa}{1 - e^{-\kappa\tau}}\left[X_t - e^{-\kappa\tau}X_{t-\tau} - (1 - e^{-\kappa\tau})Z_t\right] \\ &+ \frac{\partial J}{\partial X_t}[a_0 + a_1 X_t + a_2 Z_t] + \frac{1}{2}\frac{\partial^2 J}{\partial Y_t^2}Y_t^2 Z_t[\pi_t^2\sigma_S^2 + \sigma_L^2 - 2\pi_t\sigma_S\sigma_L\rho_{AL}] \\ &+ \frac{1}{2}\frac{\partial^2 J}{\partial X_t^2}[\sigma_X^2 Z_t] + \frac{\partial^2 J}{\partial Y_t \partial X_t}Y_t\left[\pi_t\rho_{AX}\sigma_S\sigma_X Z_t - \rho_{LX}\sigma_L\sigma_X Z_t\right].\end{aligned} \tag{A.2}$$

Theorem A.3 (Verification Theorem[16]). *Let $W(t,y) \in C^{1,2}([-\tau,T) \times \mathbb{R}^n)$ and a optimal control $\pi^* \in \mathbf{\Pi}$, $\pi^*(t) = \pi^*(t, Y^{\pi^*}(t))$ such that*

(i) *for any $\pi \in [0,T] \times \mathbb{R}$, $U(t, Y(t,\pi)) + \mathcal{L}W \leq 0$;*
(ii) *$U(t, Y(t,\pi)) + \mathcal{L}W = 0$;*
(iii) *for all $\pi \in \mathbf{\Pi}$, $\lim_{t \to T^-} W(t, Y^\pi(t)) = U(Y^\pi(T))$;*
(iv) *$\{W(s, Y^\pi(s)\}_{s \in \Im}$ and $\{U(s, Y^\pi(s))\}_{\tau \in \Im}$ are uniformly integrable, where $\Im$ denotes the set of stopping times $s \leq T$.*

Then, $W(t,Y) = J(t,Y)$ and π^* is an optimal control.

Using the first order condition to π_t

$$\begin{aligned}\frac{\partial \Psi}{\partial \pi_t} = \frac{\partial J}{\partial Y_t}[\mu - r - \rho_{AL}\sigma_S\sigma_L]Y_t Z_t + \pi_t \frac{\partial^2 J}{\partial Y_t^2} Y_t^2 Z_t \sigma_S^2 - \frac{\partial^2 J}{\partial Y_t^2} Y_t^2 Z_t \rho_{AL}\sigma_S\sigma_L \\ + \frac{\partial^2 J}{\partial Y \partial X} Y \rho_{AX}\sigma_S\sigma_X Z_t = 0,\end{aligned} \tag{A.3}$$

we can get the optimal investment strategy

$$\pi_t^* = -\frac{\frac{\partial J}{\partial Y_t}[\mu - r - \rho\sigma_S\sigma_L] - \frac{\partial^2 J}{\partial Y_t^2} Y_t \rho_{AL}\sigma_S\sigma_L + \frac{\partial^2 J}{\partial Y_t \partial X_t}\rho_{AX}\sigma_S\sigma_X}{\frac{\partial^2 J}{\partial Y^2} Y_t \sigma_S^2}. \tag{A.4}$$

We conjecture that the value function has the following form,

$$J(t, Y_t, X_t, Z_t) = e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma} f^\gamma(t, X_t, Z_t), \tag{A.5}$$

with boundary condition $f(T, Y_T, X_T, Z_T) = (1-\alpha)$. Then, the value function J follows,

$$\begin{aligned}
\frac{\partial J}{\partial t} &= -\delta e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma} f^\gamma + e^{-\delta t}\gamma \frac{Y_t^{1-\gamma}}{1-\gamma} f^{\gamma-1} f_t', \\
\frac{\partial J}{\partial Y_t} &= e^{-\delta t} Y_t^{-\gamma} f^\gamma, \\
\frac{\partial J}{\partial X_t} &= e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}\gamma f^{\gamma-1} f_X', \\
\frac{\partial J}{\partial Z_t} &= e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}\gamma f^{\gamma-1} f_Z', \\
\frac{\partial^2 J}{\partial Y_t^2} &= -\gamma e^{-\delta t} Y_t^{-\gamma-1} f^\gamma, \\
\frac{\partial^2 J}{\partial X_t^2} &= e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}\gamma((\gamma-1) f^{\gamma-2}(f_X')^2 + f^{\gamma-1} f_{XX}''), \\
\frac{\partial^2 J}{\partial Y_t \partial X_t} &= \gamma e^{-\delta t} Y_t^{-\gamma} f^{\gamma-1} f_X'.
\end{aligned} \tag{A.6}$$

[16]The proof refers to the Chapter 3.5 in Pham (2009).

Thus,

$$\begin{aligned}\pi_t^* &= \frac{\mu - r - \rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AX}\sigma_S\sigma_X\frac{f_X'}{f}}{\gamma\sigma_S^2} \\ &\equiv \pi_1^* + \rho_{AX}\frac{\sigma_X}{\sigma_S}\frac{f_X'}{f},\end{aligned} \tag{A.7}$$

where $\pi_1^* = (\mu - r - \rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AL}\sigma_S\sigma_L)/(\gamma\sigma_S^2)$.

Put Equation (A.7) back into (3.2), we obtain

$$\begin{aligned}&-\delta f^\gamma + \frac{\partial f}{\partial t}\gamma f^{\gamma-1} + (1-\gamma)f^\gamma\left[r - \beta + \sigma_L^2 + \pi_1^*(\mu - r - \rho_{AL}\sigma_S\sigma_L)\right]Z_t + (1-\gamma)f^\gamma Z_t \\ &\left[\rho_{AX}\frac{\sigma_X}{\sigma_S}(\mu - r - \rho_{AL}\sigma_S\sigma_L)\frac{\frac{\partial f}{\partial X}}{f}\right] + \frac{\partial f}{\partial X}\gamma f^{\gamma-1}(a_0 + a_1X_t + a_2Z_t) + \frac{\partial f}{\partial Z}\gamma f^{\gamma-1}\frac{\kappa}{1-e^{-\kappa\tau}}X_t \\ &-\frac{\partial f}{\partial Z}\gamma f^{\gamma-1}\frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}X_{t-\tau} - \frac{\partial f}{\partial Z}\gamma f^{\gamma-1}\kappa Z_t - \frac{1}{2}\gamma(1-\gamma)f^\gamma Z_t[\sigma_L^2 + \pi_1^{*2}\sigma_S^2 - 2\pi_1^*\sigma_S\sigma_L\rho_{AL} \\ &+ (2\pi_1^*\rho_{AX}\sigma_X\sigma_S - 2\rho_{AX}\rho_{AL}\sigma_X\sigma_L)\frac{\frac{\partial f}{\partial X}}{f} + \rho_{AX}^2\sigma_X^2\left(\frac{\frac{\partial f}{\partial X}}{f}\right)^2] + \frac{1}{2}\gamma\sigma_X^2 Z_t(\gamma-1)f^{\gamma-2}\left(\frac{\partial f}{\partial X}\right)^2 \\ &+ \frac{1}{2}\gamma\sigma_X^2 Z_t f^{\gamma-1}\frac{\partial^2 f}{\partial X^2} + \gamma(1-\gamma)f^{\gamma-1}\frac{\partial f}{\partial X}Z_t[\pi_1^*\rho_{AX}\sigma_S\sigma_X - \rho_{LX}\sigma_L\sigma_X] \\ &+ \gamma(1-\gamma)f^{\gamma-2}\left(\frac{\partial f}{\partial X}\right)^2 Z_t\rho_{AX}^2\sigma_X^2 + \alpha = 0\end{aligned} \tag{A.8}$$

For simple, let

$$\begin{aligned}M_1 &\equiv r - \beta + \sigma_L^2 + \pi_1^*(\mu - r - \rho_{AL}\sigma_S\sigma_L), \\ M_2 &\equiv \rho_{AX}\frac{\sigma_X}{\sigma_S}(\mu - r - \rho_{AL}\sigma_S\sigma_L), \\ M_3 &\equiv \sigma_L^2 + \pi_1^{*2}\sigma_S^2 - 2\pi_1^*\sigma_S\sigma_L\rho_{AL}, \\ M_4 &\equiv 2\pi_1^*\rho_{AX}\sigma_X\sigma_S - 2\rho_{AX}\rho_{AL}\sigma_X\sigma_L \\ M_5 &\equiv \pi_1^*\rho_{AX}\sigma_S\sigma_X - \rho_{LX}\sigma_L\sigma_X.\end{aligned} \tag{A.9}$$

Lemma A.4 (Liu, 2006)**.** *Suppose that*

$$\frac{\partial \hat{f}}{\partial t} + \mathcal{L}\hat{f} = 0, \tag{A.10}$$

and $\hat{f}(T, X) = 1$. *Then the function* f *defined by*

$$f(t, X) = \alpha^{\frac{1}{\theta}}\int_t^T \hat{f}(u, X)du + (1-\alpha)^{\frac{1}{\theta}}\hat{f}(t, X), \tag{A.11}$$

satisfies

$$\frac{\partial f}{\partial t} + \mathcal{L}f + \alpha^{\frac{1}{\theta}} = 0, \tag{A.12}$$

and $f(T,X)=(1-\alpha)^{\frac{1}{\theta}}$.

Using Lemma A.4, we obtain

$$f(t,X_t)=\alpha^{\frac{1}{\theta}}\int_t^T \hat{f}(u,X_t)du+(1-\alpha)^{\frac{1}{\theta}}\hat{f}(t,X_t),$$

where $\hat{f}$ satisfies

$$\begin{aligned}
&-\frac{\delta}{\gamma}\hat{f}+\frac{\partial\hat{f}}{\partial t}+\frac{1-\gamma}{\gamma}\hat{f}M_1Z_t+\frac{1-\gamma}{\gamma}Z_tM_2\frac{\partial\hat{f}}{\partial X_t}+\frac{\partial\hat{f}}{\partial X_t}(a_0+a_1X_t+a_2Z_t)\\
&+\frac{\partial\hat{f}}{\partial Z_t}\frac{\kappa}{1-e^{-\kappa\tau}}X_t-\frac{\partial\hat{f}}{\partial Z_t}\frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}X_{t-\tau}-\frac{\partial\hat{f}}{\partial Z_t}\kappa Z_t-\frac{1}{2}(1-\gamma)\hat{f}Z_t\left[M_3+M_4\frac{\frac{\partial\hat{f}}{\partial X_t}}{\hat{f}}\right.\\
&\left.+\rho_{AX}^2\sigma_X^2\left(\frac{\frac{\partial\hat{f}}{\partial X_t}}{\hat{f}}\right)^2\right]+\frac{1}{2}\sigma_X^2Z_t(\gamma-1)\hat{f}^{-1}\left(\frac{\partial\hat{f}}{\partial X_t}\right)^2+\frac{1}{2}\sigma_X^2Z_t\frac{\partial^2 f}{\partial(X_t)^2}+(1-\gamma)\frac{\partial\hat{f}}{\partial X_t}Z_tM_5\\
&+(1-\gamma)\hat{f}^{-1}\left(\frac{\partial\hat{f}}{\partial X_t}\right)^2Z_t\rho_{AX}^2\sigma_X^2=0.
\end{aligned}\tag{A.13}$$

$\hat{f}$ is conjectured to have the form

$$\hat{f}(t,x)=\exp\left\{D_{1,t}^{(1)}X_t+D_{2,t}^{(1)}Z_t+D_{3,t}^{(1)}\right\}.\tag{A.14}$$

So,

$$\begin{aligned}
\frac{\partial\hat{f}}{\partial t}&=\hat{f}(\dot{D}_{1,t}^{(1)}X_t+\dot{D}_{2,t}^{(1)}Z_t+\dot{D}_{3,t}^{(1)}),\\
\frac{\partial\hat{f}}{\partial X_t}&=\hat{f}(D_{1,t}^{(1)}),\\
\frac{\partial\hat{f}}{\partial Z_t}&=\hat{f}(D_{2,t}^{(1)}),\\
\frac{\partial^2\hat{f}}{\partial(X_t)^2}&=\hat{f}(D_{1,t}^{(1)})^2.
\end{aligned}\tag{A.15}$$

Using this structure, we can obtain

$$\begin{aligned}
&-\frac{\delta}{\gamma}+\dot{D}_{1,t}^{(1)}X_t+\dot{D}_{2,t}^{(1)}Z_t+\dot{D}_{3,t}^{(1)}+\frac{1-\gamma}{\gamma}M_1Z_t+\frac{1-\gamma}{\gamma}Z_tM_2D_{1,t}^{(1)}+D_{1,t}^{(1)}(a_0+a_1X_t+a_2Z_t)\\
&+D_{2,t}^{(1)}\frac{\kappa}{1-e^{-\kappa\tau}}X_t-D_{2,t}^{(1)}\frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}X_{t-\tau}-D_{2,t}^{(1)}\kappa Z_t-\frac{1}{2}(1-\gamma)Z_t\left[M_3+M_4D_{1,t}^{(1)}+\rho_{AX}^2\sigma_X^2(D_{1,t}^{(1)})^2\right]\\
&+\frac{1}{2}\sigma_X^2Z_t\gamma(D_{1,t}^{(1)})^2+(1-\gamma)D_{1,t}^{(1)}Z_tM_5+(1-\gamma)(D_{1,t}^{(1)})^2Z_t\rho_{AX}^2\sigma_X^2=0,
\end{aligned}\tag{A.16}$$

and following ODEs:

$$\dot{D}_{1,t}^{(1)} + D_{1,t}^{(1)} a_1 + D_{2,t}^{(1)} \frac{\kappa}{1-e^{-\kappa\tau}} = 0,$$

$$\dot{D}_{2,t}^{(1)} + \frac{1-\gamma}{\gamma}(M_1 + D_{1,t}^{(1)} M_2) + D_{1,t}^{(1)} a_2 - D_{2,t}^{(1)} \kappa - \frac{1}{2}(1-\gamma)(M_3 + D_{1,t}^{(1)} M_4 - \rho_{AX}^2 \sigma_X^2 (D_{1,t}^{(1)})^2)$$

$$+ \frac{1}{2}\gamma \sigma_X^2 (D_{1,t}^{(1)})^2 + (1-\gamma) M_5 D_{1,t}^{(1)} = 0$$

$$\dot{D}_{3,t}^{(1)} + D_{1,t}^{(1)} a_0 - \frac{\delta}{\gamma} - D_{2,t}^{(1)} \frac{\kappa}{1-e^{-\kappa\tau}} e^{-\kappa\tau} x_{t-\tau} = 0,$$

with terminal conditions $D_{1,t}^{(1)} = D_{2,t}^{(1)} = D_{3,t}^{(1)} = 0$.

A.3. **Proof of Proposition 3.4.** For $(n-1)\tau < T \le n\tau$, original state variables and their historical paths cannot constitute sufficient statistics for the optimal ALM strategy, (3.1) give the new variables to solve the problem. In this case, the goal function changes to (2.2), and the state variables becomes

$$\begin{aligned}
dX_t^{(i)} &= (a_0 + a_1 X_t^{(i)} + a_2 Z_t^{(i)})dt + \sigma_X \sqrt{Z_t^{(i)}} dB_{X,t}^{(i)}, \qquad 1 \le i \le n \\
dZ_t^{(i)} &= \frac{\kappa}{1-e^{-\kappa\tau}} \left[X_t^{(i)} - e^{-\kappa\tau} X_t^{(i+1)} - (1-e^{-\kappa\tau}) Z_t^{(i)} \right] dt, \qquad 1 \le i < n \\
dZ_t^{(n)} &= \frac{\kappa}{1-e^{-\kappa\tau}} \left[X_t^{(n)} - e^{-\kappa\tau} X_{t-\tau}^{(n)} - (1-e^{-\kappa\tau}) Z_t^{(n)} \right] dt,
\end{aligned} \tag{A.17}$$

The wealth process are only depend on the current $X_t^{(1)}$ and $Z_t^{(1)}$,

$$\begin{aligned}
dY_t = &\left[r + \pi_t(\mu - r) - \beta - \pi_t \rho_{AL} \sigma_S \sigma_L + \sigma_L^2 \right] Z_t^{(1)} Y_t dt + \pi_t \sigma_S \sqrt{Z_t^{(1)}} Y_t dB_{A,t} \\
&- \sigma_L \sqrt{Z_t^{(1)}} Y_t dB_{L,t}^{(1)}.
\end{aligned} \tag{A.18}$$

Then, the HJB function satisfies

$$\begin{aligned}
\Psi = \sup_{\pi} \Bigg\{ &\frac{\partial J}{\partial t} + \frac{\partial J}{\partial Y} \left[r + \pi_t(\mu - r) - \beta - \pi_t \rho_{AL} \sigma_S \sigma_L + \sigma_L^2 \right] Y_t Z_t^{(1)} \\
&+ \sum_{i=1}^{n-1} \frac{\partial J}{\partial Z^{(i)}} \frac{\kappa}{1-e^{-\kappa\tau}} \left[X_t^{(i)} - e^{-\kappa\tau} X_t^{(i+1)} - (1-e^{-\kappa\tau}) Z_t^{(i)} \right] \\
&+ \frac{\partial J}{\partial Z^{(n)}} \frac{\kappa}{1-e^{-\kappa\tau}} \left[X_t^{(n)} - e^{-\kappa\tau} X_{t-\tau}^{(n)} - (1-e^{-\kappa\tau}) Z_t^{(n)} \right] \\
&+ \sum_{i=1}^{n} \frac{\partial J}{\partial x^{(i)}} [a_0 + a_1 X_t^{(i)} + a_2 Z_t^{(i)}] + \frac{1}{2} \frac{\partial^2 J}{\partial Y^2} Y^2 Z_t^{(1)} [\pi_t^2 \sigma_S^2 + \sigma_L^2 - 2\pi_t \sigma_S \sigma_L \rho_{AL}] \\
&+ \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 J}{\partial (x^{(i)})^2} [\sigma_X^2 Z_t^{(i)}] + \frac{\partial^2 J}{\partial Y \partial X^{(1)}} Y_t Z_t^{(1)} \left[\pi_t \rho_{AX} \sigma_S \sigma_X - \rho_{LX} \sigma_L \sigma_X \right] + \alpha e^{-\delta t} \frac{Y_t^{1-\gamma}}{1-\gamma} \Bigg\},
\end{aligned} \tag{A.19}$$

with the boundary condition $V(Y_T,T)=(1-\alpha)e^{-\delta T}\frac{Y_T^{1-\gamma}}{1-\gamma}$. We conjecture the value function $J(t,Y,X,Z)$ in the form of

$$J(t,Y,X,Z)=e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}f^{\gamma}(t,Y_t,X_t^{(1)},Z_t^{(1)},\cdots,X_t^{(n)},Z_t^{(n)}). \tag{A.20}$$

By calculated as Appendix A.2, we can get the optimal ALM strategy follows (3.12). Using the Lemma A.4, we conjecture $\hat{f}$ as (3.13), and the HJB function change to

$$\begin{aligned}
&-\frac{\delta}{\gamma}+\sum_{i=1}^{n}\dot{D}_{2i-1,t}^{(n)}X_t^{(i)}+\sum_{i=1}^{n}\dot{D}_{2i,t}^{(n)}Z_t^{(i)}+\dot{D}_{2n+1,t}^{(n)}+\frac{1-\gamma}{\gamma}M_1Z_t^{(1)}+\frac{1-\gamma}{\gamma}M_2D_{1,t}^{(n)}Z_t^{(1)}\\
&\sum_{i=1}^{n-1}D_{2i,t}^{(n)}\frac{\kappa}{1-e^{-\kappa\tau}}X_t^{(i)}-\sum_{i=1}^{n-1}D_{2i,t}^{(n)}\frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}X_t^{(i+1)}-\sum_{i=1}^{n-1}D_{2i,t}^{(n)}\kappa Z_t^{(i)}+D_{2n,t}^{(n)}\frac{\kappa}{1-e^{-\kappa\tau}}X_t^{(n)}\\
&-D_{2n,t}^{(n)}\frac{\kappa e^{-\kappa\tau}}{1-e^{-\kappa\tau}}X_{t-\tau}^{(n)}-D_{2n,t}^{(n)}\kappa Z_t^{(n)}+\sum_{i=1}^{n}D_{2i-1,t}^{(n)}a_0+\sum_{i=1}^{n}D_{2i-1,t}^{(n)}a_1X_t^{(i)}+\sum_{i=1}^{n}D_{2i-1,t}^{(n)}a_2Z_t^{()}\\
&-\frac{1}{2}(1-\gamma)M_3Z_t^{(1)}-\frac{1}{2}(1-\gamma)M_4Z_t^{(1)}D_{1,t}^{(n)}+\frac{1}{2}(1-\gamma)\sigma_X^2\rho_{AX}^2(D_{1,t}^{(n)})^2Z_t^{(1)}\\
&+\frac{1}{2}\gamma\sum_{i=1}^{n}(D_{2i-1,t}^{(n)})^2\sigma_X^2Z_t^{(i)}+(1-\gamma)D_{1,t}^{(n)}M_5Z_t^{(1)}=0.
\end{aligned} \tag{A.21}$$

To solve the HJB function (A.21), we can obtain the ODEs follows (3.14).

Appendix B. ALM without Time Delay

If the market is Markovian, that is, all the information is contained in the recent economic state. Then, the state process degenerates to

$$dX_t=(a_0+aX_t)dt+\sigma_X\sqrt{X_t}dB_{X,t}, \tag{B.1}$$

where $a=a_1+a_2$ in (2.9). The ALR Y_t is only affected by the current state X_t, that is,

$$\begin{cases} dY_t=Y_t\left[r+\pi_t(\mu-r)-\beta-\pi_t\rho_{AL}\sigma_S\sigma_L+\sigma_L^2\right]X_tdt\\ \qquad\quad+Y_t[\pi_t\sigma_S\sqrt{X_t}dB_{A,t}-\sigma_L\sqrt{X_t}dB_{L,t}],\quad t\in[0,T]\\ Y_0=y.\end{cases} \tag{B.2}$$

With the same maximum problem (2.2), the HJB equation changes to

$$\begin{aligned}
\Psi=\sup_{\pi}\Bigg\{&\frac{\partial J}{\partial t}+\frac{\partial J}{\partial Y}\left[r+\pi_t(\mu-r)-\beta-\pi_t\rho_{AL}\sigma_S\sigma_L+\sigma_L^2\right]Y_tX_t\\
&+\frac{\partial J}{\partial x}[a_0+aX_t]+\frac{1}{2}\frac{\partial^2 J}{\partial Y^2}Y_t^2X_t[\pi^2\sigma_S^2+\sigma_L^2-2\pi\sigma_S\sigma_L\rho_{AL}]+\frac{1}{2}\frac{\partial^2 J}{\partial x^2}[\sigma_X^2X_t]\\
&+\frac{\partial^2 J}{\partial Y\partial x}Y_t\left[\pi\rho_{AX}\sigma_S\sigma_XX_t-\rho_{LX}\sigma_L\sigma_XX_t\right]+\alpha e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}\Bigg\},
\end{aligned} \tag{B.3}$$

with the boundary condition $V(T, Y_T) = (1-\alpha)e^{-\delta T}\frac{Y_T^{1-\gamma}}{1-\gamma}$.

Proposition B.1. *The optimal ALM strategy without time delays is given by*

$$\pi_t^* = \frac{\mu - r - \rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AL}\sigma_S\sigma_L + \gamma\rho_{AX}\sigma_S\sigma_X\frac{\partial \ln g}{\partial X}}{\gamma\sigma_S^2}, \tag{B.4}$$

and the value function is given by

$$J(t, Y, X) = e^{-\delta t}\frac{Y_t^{1-\gamma}}{1-\gamma}g^{\gamma}(t, X),$$

where

$$g(t, X, Z) = \alpha^{\frac{1}{\gamma}}\int_t^T \hat{g}(u, X)du + (1-\alpha)^{\frac{1}{\gamma}}\hat{g}(t, X),$$

and

$$\hat{g}(t, X, Z) = \exp\left\{E_{1,t}^{(1)}X + E_{2,t}^{(1)}\right\}.$$

$E_{i,t}$ are governed by following ODEs

$$\begin{aligned}
&\dot{E}_{1,t} + \frac{1-\gamma}{\gamma}(M_1 + E_{1,t}M_2) + E_{1,t}a - \frac{1}{2}(1-\gamma)(M_3 + E_{1,t}M_4 - \rho_{AX}^2\sigma_X^2(E_{1,t})^2) \\
&\quad + \frac{1}{2}\gamma\sigma_X^2(E_{1,t})^2 + (1-\gamma)M_5E_{1,t} = 0, \\
&\dot{E}_{2,t} + E_{1,t}a_0 - \frac{\delta}{\gamma} = 0,
\end{aligned} \tag{B.5}$$

with terminal conditions $E_{1,t} = E_{2,t} = 0$, where M_i ($i = 1, \ldots, 5$) are given in (A.9).

Appendix C. Further analysis of Economic implications

Analyzing the impact of the delay parameter τ on CEW, we present Figure C.1. The figure illustrates the variation of CEW with τ for two scenarios: $\tau < T \leq 2\tau$ and $T \leq \tau$, using the parameters from Table 4.1. As expected from our previous analysis, CEW remains demonstrably greater than 1 across both scenarios, exhibiting a consistent upward trend throughout. This observation aligns with our core conclusion. However, it's crucial to acknowledge a notable difference in the rate of increase between the two sections. The section with $\tau < T \leq 2\tau$ witnesses a significantly steeper incline compared to $T \leq \tau$. This disparity might be attributed to the introduction of additional parameters and state variables within the computation for the $\tau < T \leq 2\tau$ section. Notwithstanding this observed rate difference, the overall upward trend in CEW across both sections underscores the validity of our key takeaway: incorporating the historical path of state variables substantially enhances Asset-Liability Management outcomes.

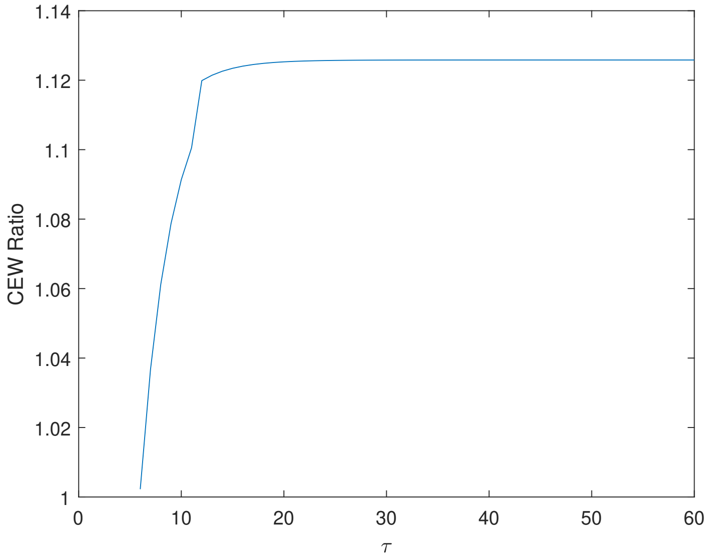

FIGURE C.1. The average CEW ratio of optimal ALM strategy to the CEW of ALM strategy ignoring the time dalay from 6 to 60 months

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