Conformal Inference for Continuous Treatment Effect

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Our goal is to generate prediction intervals with certain coverage for heterogeneous treatment effect when treatment is continuous. Generally, we will use conformal prediction to get them. We first review the relating literature by topic, then we show some basic simulation results we already have.

1 Literature Review

Conformal Inference Conformal prediction (or inference) (Vovk et al. [2005], Lei et al. [2018]) is a method for uncertainty quantification. It's goal is to give an interval $\hat{C}(X)$, such that for a *i.i.d* test data pair (X_{n+1}, Y_{n+1}), we have

$$\mathbb{P}(Y_{n+1} \in \hat{C}(X_{n+1})) \geq 1 - \alpha$$

Original conformal prediction requires exchangeability, which is violated when there are distribution shifts. To solve this issue, Tibshirani et al. [2019] develop weighted conformal algorithm, and Romano et al. [2019] purpose conformal quantile regression to further simplify the process. Lei and Candès [2021] first introduce conformal prediction to causal inference framework. They show that covariate shift can be represented by propensity scores in causal settings with binary treatment. Zhang et al. [2023] use conformal inference on off-policy evaluation for multiple treatments. Foffano et al. [2023] base on their work to use conformal off-policy evaluation for markov decision process. As a result, the potential of this method is gradually being realized. The key procedure of this approach is the quantile adjustment after we get first step point estimation with either machine learning algorithm like Neural Networks or basic regressions.

Continuous Treatment Effect Continuous-treatment effect (CTE)/Dose Response Function has been explored in traditional Non-parametric/Semi-parametric framework Kennedy et al. [2017] and Galvao and Wang [2015]. The main area of practical problem concerning CTE is off-policy evaluation. Recently, some scholars have investigated how to integrate the framework of conformal inference with multiple treatment effects, as discussed in Zhang et al. [2023] and Taufiq et al. [2022]. Differently, off-policy evaluation has a known propensity score $e(x, 1)$ but we need to estimate $\hat{e}(x, 1)$ in causal inference framework. These papers propose a sampling method to approximate the evaluation policy (propensity score in the causal inference framework). Additionally, CTE has been explored in the balanced ML method presented in Kazemi and Ester [2022]. This paper

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proposes an encoder method to construct Z from X to address the confounding of treatment. Furthermore, it offers a possible alternative to the traditional propensity score.

Generalized Propensity Score The concept of propensity score was initially proposed by Hirano and Imbens [2004] and has since become a mature field with the development of numerous algorithms, such as the one proposed by Wu et al. [2022], which uses matching to use neighboring points to get an estimation. Tu [2019] compare several different machine learning algorithms for the precision of estimating GPS. In the continuous treatment setting, one problem we did not meet before is the sampling issue. Thus we might want to add t into the estimated functional form, as S-Learner did as in Künzel et al. [2019].

2 Preliminary Results

Problem Our current target is to do get a individual level prediction interval $\hat{C}_i(t, X)$, *s.t.*

$$P(\theta_i(t) - \theta_i(t_0) \in \hat{C}_i(t, X)) \geq 1 - \alpha$$

Some points to note for this problem:

- How to select samples? This time we focus on CTE, we can almost impossiblely get samples s.t $T = t$.
- Above problems certainly heavily rely on exchangeability of data, it has been solved in 0-1 framework. Now we need to construct a new ***Generalized Conformity Score*** to weighted samples belong to $[t - \delta, t + \delta]$

Naive Algorithm We only modified the Lei and Candès [2021] nested algorithm slightly for presentation. The differences only manifest in the **sampling method** and **GPS** used.

An initial algorithm was proposed to solve this problem; however, the coverage was not satisfactory, see example Figure 1 and 2.

We use this data generating process:

$$Y_i = X_{1i} + 0.5 \times X_{2i} + T_i + \epsilon_i$$

where $T_i \in [0, 6]$. All RHS variables are generated randomly.

To estimate treatment effect $Y_i(t) - Y_i(0), t > 0$

Step I. data splitting

1: Split the data into two folds $\mathcal{Z}_1$ and $\mathcal{Z}_2$

2: Estimate general propensity score $\hat{e}(x, t)$ on $\mathcal{Z}_1$

Step II. counterfactual inference on $\mathcal{Z}_2$

For $i \in Z_2$ with $T_i = t$. Group samples $\left\{Y_i, X_i, Y_t^{obs}\right\}_{i=1}^{n_t}$ and $\left\{Y_i, X_i, Y_0^{obs}\right\}_{i=1}^{n_t}$

1: Compute $\left[\widehat{Y}_t^L(X_i, 0), \widehat{Y}_t^R(X_i, 0)\right]$ by using CQR on Z_1 with level α and $w(x, 0)$ (unbiased estimation with general propensity score $E[Y(0)] = E\left[Y_0^{obs} w(x, 0)\right]$)

2: Compute $\widehat{C}_t = \left[Y_i(X_i, t) - \widehat{Y}_t^R(X_i, 0), Y_i(X_i, t) - \widehat{Y}_t^L(X_i, 0)\right]$

Now reverse the order.

1: Compute $\left[\widehat{Y}_t^L\left(X_i,t\right),\widehat{Y}_t^R\left(X_i,t\right)\right]$ by using CQR on z_1 with level α and $w(x,t)$ (unbiased estimation with general propensity score $E[Y(t)]=E\left[Y_t^{obs}w(x,t)\right]$)

2: Compute $\widehat{C}_\iota = \left[\widehat{Y}_t^L\left(X_i,t\right)-Y_i(0),\widehat{Y}_t^R\left(X_i,t\right)-Y_i(0)\right]$ These two sets compose the set $\Gamma = (X_i,C_i)$ and $C_i=\left[C_i^L,C_i^R\right]$

Step III Exact version of ITE on the testing point

Input: level γ, data $\mathcal{Z}=(X_i,C_i)_{i\in\mathcal{I}}$ where $C_i=\left[C_i^L,C_i^R\right]$, testing point x, functions $\hat{m}^L(x;\mathcal{D}),\hat{m}^R(x;\mathcal{D})$ to fit the conditional mean/median of C^L,C^R

Procedure:

1: Split $\mathcal{Z}$ into a training fold $\mathcal{Z}_{\text{tr}}\triangleq(X_i,C_i)_{i\in\mathcal{I}_{\text{tr}}}$ and a calibration fold $\mathcal{Z}_{\text{ca}}\triangleq(X_i,C_i)_{i\in\mathcal{I}_{\text{ca}}}$

2: For each $i\in\mathcal{I}_{\text{ca}}$, compute score $V_i=\max\left\{\hat{m}^L\left(X_i;\mathcal{Z}_{\text{tr}}\right)-C_i^L,C_i^R-\hat{m}^R\left(X_i;\mathcal{Z}_{\text{tr}}\right)\right\}$

3: Compute η as the $(1-\gamma)\left(1+1/\left|\mathcal{Z}_{\text{ca}}\right|\right)$ quantile of the empirical distribution of $\{V_i:i\in\mathcal{I}_{\text{ca}}\}$

Output: $\hat{\mathcal{C}}(x)=\left[\hat{m}^L\left(x;\mathcal{Z}_{\text{tr}}\right)-\eta,\hat{m}^R\left(x;\mathcal{Z}_{\text{tr}}\right)+\eta\right]$

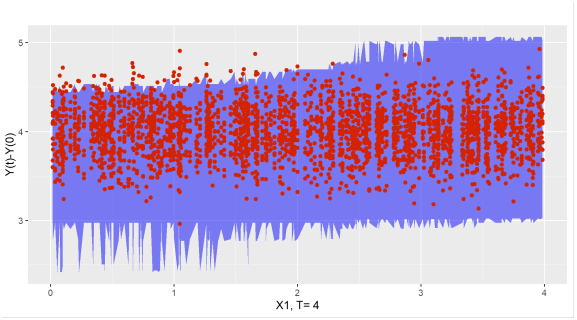

Figure 1: $T=4$

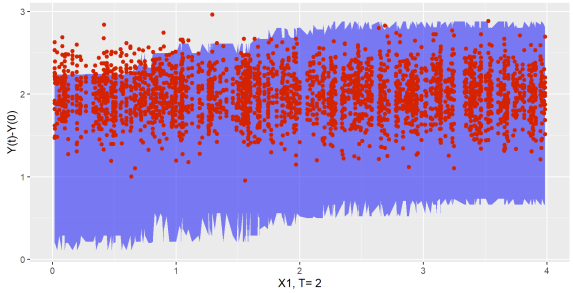

Figure 2: $T=2$

3 How to improve the naive approach?

Generalized Conformity Score The first problem need to solve is to modify the weighting strategy. $\tilde{t}\neq t$, therefore it doesn't follow exchangeability if we only use weight on test point t. We need to get a new Generalized Conformity Score $g_i^w(x,\tilde{t},t)$ like this:

$$S_i=\max\{\hat{q_{low}}(X_i,\tilde{t})-Y_i,Y_i-\hat{q_{high}}(X_i,\tilde{t})\}$$

and the new empirical distribution

$$\sum_i g_i^w(x,\tilde{t},t)\delta_{S_i}$$

we have not thought about the specific formula of $g_i^w(x,\tilde{t},t)$, but it is possibly related to the distance between $|\tilde{t}-t|\leq\bar{\delta}$, $\bar{\delta}$ is the upper bound of band width for sampling.

Encoding Method Kazemi and Ester [2022] promotes a method that transfer $X\to Z$ to make $Y\perp T|Z$ to resolve confoundness. We can possibly recast the algorithm without using any "score".

Step I. encoder

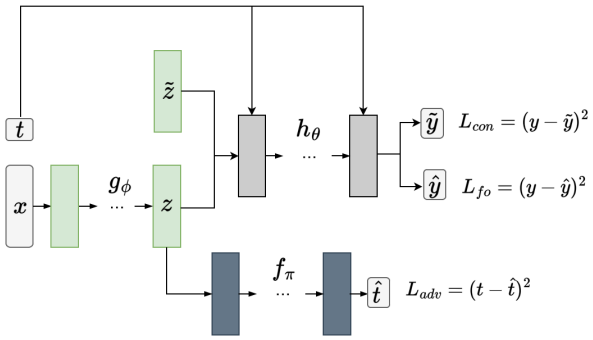

Figure 2: The architecture of ACFR network.

Figure 3: From Kazemi and Ester [2022]

I is mutual information that can capture non-linear dependency between two variables. The latent representation Z extracted via a parametric encoder $g_\phi(x)$ is assumed to be causally dependent to covariate X, and to be conditionally independent of treatment T and outcome Y given X.

$$\min_{\phi,\theta} \quad I(Z, Y|T; \phi) - \gamma_1 I(Z, T; \phi)$$

this step generates $g_\phi(x)$, and $h_\theta(z, t)$. At treatment level t_e where there are no samples, we want to use this h_θ to generate samples, and use conformality score quantiles to adjust the prediction interval to reach certain marginal coverage level.

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Appendix

1 δ_t Method Splitted Conformal

The incentive for using the δ_t method is driven by the nature of inferring continuous treatment effects on t, which relies on the characteristics of the data around t. The difficulty in this procedure arises from the fact that, given the dataset $\mathcal{Z}$, we cannot group data from the exact treatment t. To overcome this challenge, one possible approach is to use a point estimate of the weight $\hat{w}(t,x)$ to represent data generated from the distribution $(X_t, Y_t) \sim P_{X|T\in[t-\delta_t,t+\delta_t]} \times P_{Y|X}$.

Algorithm 1 describes the entire procedure for estimating the interval of the Individual Treatment Effect (ITE) on treatment t. Figures 1 and 2 provide examples of this estimation. In Section 2, I present the proof of its asymptotic nature.

Algorithm 1: δ_t-nested approach for interval estimates of ITE

Input: Data $\mathcal{Z} = (X_i, Y_i, T_i)$, set interval length δ_t. To estimate treatment effect $Y_i(t) - Y_i(0), t > 0$

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Step II. counterfactual inference on $\mathcal{Z}_2$

For $i \in \mathcal{Z}_2$ with $T_i = t$. Group samples in $[t-\delta_t, t+\delta_t]$ and $[0,\delta_t]$: $\left\{Y_i, X_i, Y_t^{obs}\right\}_{i=1}^{n_t}$ and $\left\{Y_i, X_i, Y_0^{obs}\right\}_{i=1}^{n_t}$

1: Compute $\left[\widehat{Y}_\iota^L(X_i,0), \widehat{Y}_\iota^R(X_i,0)\right]$ by using CQR on Z_1 with level α and $w(x,0)$

2: Compute $\widehat{C}_\iota = \left[Y_i(X_i,t) - \widehat{Y}_\iota^R(X_i,0), Y_i(X_i,t) - \widehat{Y}_\iota^L(X_i,0)\right]$

Now reverse the order.

1: Compute $\left[\widehat{Y}_\iota^L(X_i,t), \widehat{Y}_\iota^R(X_i,t)\right]$ by using CQR on z_1 with level α and $w(x,t)$ (unbiased estimation with general propensity score $E[Y(t)] = E\left[Y_t^{obs} w(x,t)\right]$)

2: Compute $\widehat{C}_\iota = \left[\widehat{Y}_\iota^L(X_i,t) - Y_i(0), \widehat{Y}_\iota^R(X_i,t) - Y_i(0)\right]$ These two sets compose the set $\Gamma = (X_i, C_i)$ and $C_i = \left[C_i^L, C_i^R\right]$

Step III Exact version of ITE on the testing point

Input: level γ, data $\mathcal{Z} = (X_i, C_i)_{i\in\mathcal{I}}$ where $C_i = \left[C_i^L, C_i^R\right]$, testing point x, functions $\hat{m}^L(x;\mathcal{D}), \hat{m}^R(x;\mathcal{D})$ to fit the conditional mean/median of C^L, C^R

Procedure:

1: Split $\mathcal{Z}$ into a training fold $\mathcal{Z}_{\text{tr}} \triangleq (X_i, C_i)_{i \in \mathcal{I}_{\text{tr}}}$ and a calibration fold $\mathcal{Z}_{\text{ca}} \triangleq (X_i, C_i)_{i \in \mathcal{I}_{\text{ca}}}$

2: For each $i \in \mathcal{I}_{\text{ca}}$, compute score $V_i = \max\left\{\hat{m}^L\left(X_i; \mathcal{Z}_{\text{tr}}\right) - C_i^L, C_i^R - \hat{m}^R\left(X_i; \mathcal{Z}_{\text{tr}}\right)\right\}$

3: Compute η as the $(1-\gamma)\left(1 + 1/\left|\mathcal{Z}_{\text{ca}}\right|\right)$ quantile of the empirical distribution of $\left\{V_i : i \in \mathcal{I}_{\text{ca}}\right\}$ Output: $\hat{\mathcal{C}}(x) = \left[\hat{m}^L\left(x; \mathcal{Z}_{\text{tr}}\right) - \eta, \hat{m}^R\left(x; \mathcal{Z}_{\text{tr}}\right) + \eta\right]$

Output: $\widehat{C}_j(x) = [\hat{q}_{j,lo}(x; \mathcal{Z}_{j,tr}) - \eta(x), \hat{q}_{j,hi}(x; \mathcal{Z}_{j,tr}) + \eta(x)]$, $j = 1, 2 \ldots m$

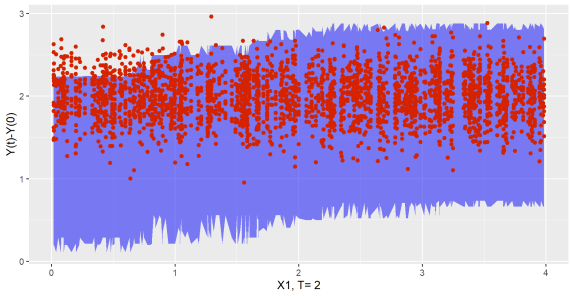

Figure 1: $T = 2$

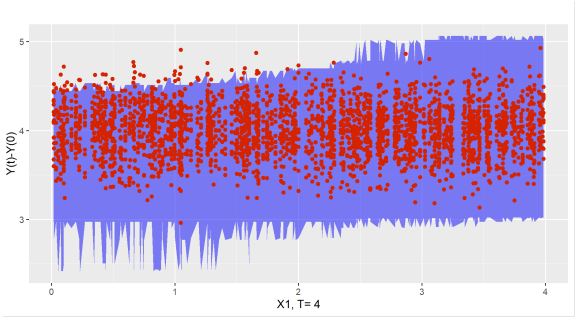

Figure 2: $T = 4$

2 Proof

Proposition 1. *There are two groups of samples* $(X_i, Y_i) \overset{i.i.d.}{\sim} (X, Y) \sim P_{X|T=t} \times P_{Y|X}$; $(X_i, Y_i) \overset{i.i.d.}{\sim} (X, Y) \sim P_{X|T\in[t-\delta_t, t+\delta_t]} \times P_{Y|X}$ *where* $\delta_t > 0$ *and* Q_X *be another distribution on the domain of* X. *Set* $N = |\mathcal{Z}_{tr}|$ *and* $n = |\mathcal{Z}_{ca}|$. *Further, let* $\hat{q}_{\beta,N}(x) = \hat{q}_{\beta,N}\left(x; \mathcal{Z}_{tr}\right)$ *be an estimate of the* β*-th conditional quantile* $q_\beta(x)$ *of* $Y \mid X = x$, $\hat{w}_N(x) = \hat{w}_N\left(x; \mathcal{Z}_{tr}\right)$ *be an estimate of* $w(x) = \left(\frac{dQ_X}{dP_{X|T\in[t-\delta_t, t+\delta_t]}}\right)(x)$, *and* $\hat{C}_{N,n}(x)$ *be the conformal interval resulting from splitted conformal algorithm. Assume* $E\left[\hat{w}_N(x) \mid \mathcal{Z}_{tr}\right] < \infty$, *where* E_{δ_t} *denotes expectation over* $X \sim P_{X|T\in[t-\delta_t, t+\delta_t]}$ *and* E_t *denotes expectation over* $X \sim P_{X|T=t}$. *If redefine* $E\left[\hat{w}_N(x) \mid \mathcal{Z}_{tr}\right] = 1$ *we have*

$$P_{Q_X \times P_{Y|X}}\left(Y_{n+1} \in \hat{C}\left(X_{n+1}\right)\right) \geq 1 - \alpha - \frac{1}{2} E_{\delta_t} |\hat{w}(x) - w(x)|$$

If for all ϵ_{δ_t} *there exists a* δ_t *such that* $\left|\frac{dP_{X|T\in[t-\delta_t, t+\delta_t]}}{dP_{X|T=t}}(x) - 1\right| \leq \epsilon_{\delta_t}$ *then*

$$P_{Q_X \times P_{Y|X}}\left(Y_{n+1} \in \hat{C}\left(X_{n+1}\right)\right) \geq 1 - \alpha - \frac{\epsilon_{\delta_t}}{2} E_t |\hat{w}(x) - w(x)|$$

Proof. According to Lei and Candès [2021] Theorem 3, Let $(X_{n+1}, Y_{n+1}) \sim Q_X \times P_{Y|X}$ denote samples generated from $d\tilde{Q}_X = \hat{w}(x) dP_{X|T\in[t-\delta_t, t+\delta_t]}(x)$ and $\left(\tilde{X}_{n+1}, \tilde{Y}_{n+1}\right) \sim \tilde{Q}_X \times P_{Y|X}$ from $dQ_X = w(x) dP_{X|T\in[t-\delta_t, t+\delta_t]}(x)$. Theorem 3 implies $P_{Q_X \times P_{Y|X}}\left(Y_{n+1} \in \hat{C}\left(X_{n+1}\right)\right) \geq 1 - \alpha - \frac{1}{2} E_{\delta_t} |\hat{w}(x) - w(x)|$

instantly. Furthermore,

$$\begin{aligned} d_{TV}\left(\tilde{Q}_X, Q_X\right) &= \frac{1}{2}\int \left|\hat{w}(x)dP_{X|T\in[t-\delta_t,t+\delta_t]}(x) - w(x)dP_{X|T\in[t-\delta_t,t+\delta_t]}(x)\right| \\ &= \frac{1}{2}E_{\delta_t}|\hat{w}(x) - w(x)| \\ &= \frac{1}{2}\int |\hat{w}(x) - w(x)|\frac{dP_{X|T\in[t-\delta_t,t+\delta_t]}(x)dP_{X|T=t}}{dP_{X|T=t}} \\ &\leq \frac{\epsilon_{\delta_t}}{2}\int |\hat{w}(x) - w(x)|dP_{X|T=t} = \frac{\epsilon_{\delta_t}}{2}E_t|\hat{w}(x) - w(x)| \end{aligned}$$

Therefore

$$P_{Q_X\times P_{Y|X}}\left(Y_{n+1}\in\hat{C}\left(X_{n+1}\right)\right) \geq 1-\alpha-\frac{\epsilon_{\delta_t}}{2}E_t|\hat{w}(x) - w(x)|$$

This preposition implies that the estimation accuracy for samples from distribution $(X_{n+1}, Y_{n+1}) \sim Q_X \times P_{Y|X}$ where $dQ_X = w(x)dP_{X|T=t}(x)$ is determined by distance of distribution ϵ_{δ_t}. ■

Proposition 2. *Assume further that (1) $\alpha_{hi} - \alpha_{lo} = 1-\alpha$; (2) There exist $r, b_1, b_2 > 0$ such that $P(Y = y \mid X = x) \in [b_1, b_2]$ uniformly over all (x, y) for $y \in [q_{\alpha_{lo}}(x) - r, q_{\alpha_{lo}}(x) + r] \cup [q_{\alpha_{hi}}(x) - r, q_{\alpha_{hi}}(x) + r]$ (3) $P_{X\sim Q_X}(w(X) < \infty) = 1$, and there exist $\delta, M > 0$ such that $\left(E_{\delta_t}\left[\hat{w}_N(X)^{1+\delta}\right]\right)^{\frac{1}{1+\delta}} \leq M$; (4) There exist $k, \ell > 0$ such that $\lim_{N\to\infty} E_{\delta_t}\left[\hat{w}_N(X)H_N^k(X)\right] = \lim_{N\to\infty} E_{\delta_t}\left[w(X)H_N^l(X)\right] = 0$ where*

$$H_N(x) = \max\left\{\left|\hat{q}_{\alpha_{lo},N}(x) - q_{\alpha_{lo}}(x)\right|, \left|\widehat{q}_{\alpha_{hi},N}(x) - q_{\alpha_{hi}}(x)\right|\right\}$$

Then there is a constant B_1 that only depends on $r, b_1, b_2, \delta, M, k, \ell$ such that

$$\begin{aligned} &P_{(X,Y)\sim Q_X\times P_{Y|X}}\left(Y\in\hat{C}_{N,n}(X)\right) \\ &\qquad \geq 1-\alpha \\ &\qquad - B_1\left\{\frac{(\log n)^{\frac{1+\delta'}{2(2+\delta')}}}{n^{\frac{\delta'}{2+\delta'}}} + E_t\left[\hat{w}_N(X)H_N^k(X)\right]^{\frac{1}{2+k}}(1+\epsilon_{\delta_t})^{\frac{1}{2+k}}\right. \\ &\qquad \left. + E_t\left[w(X)H_N^\ell(X)\right]^{\frac{1}{1+\ell}}(1+\epsilon_{\delta_t})^{\frac{1}{2+\ell}}\right\} \end{aligned} \tag{1}$$

where $\delta' = \min\{\delta, 1\}$. Furthermore, for any $\beta \in (0,1)$, there is a constant B_2 that only depends on $r, b_1, b_2, \delta, M, k, \ell, \beta$ such that, with probability at least $1-\beta$

$$\begin{aligned} &P_{(X,Y)\sim Q_X\times P_{Y|X}}\left(Y\in\hat{C}_{N,n}(X)\mid X\right) \\ &\qquad \geq 1-\alpha \\ &\qquad - B_2\left\{\frac{(\log n)^{\frac{1+\delta'}{2(2+\delta')}}}{n^{\frac{\delta'}{2+\delta'}}} + E_t\left[\hat{w}_N(X)H_N^k(X)\right]^{\frac{1}{2+k}}(1+\epsilon_{\delta_t})^{\frac{1}{2+k}}\right. \\ &\qquad \left. + E_t\left[w(X)H_N^\ell(X)\right]^{\frac{1}{\ell}}(1+\epsilon_{\delta_t})^{\frac{1}{\ell}}\right\} \end{aligned} \tag{2}$$

Proof. Based on Theorem 4 in Lei and Candès [2021], the only difference is we add an extra condition for the distance of distribution $\left|\frac{dP_{X|T\in[t-\delta_t,t+\delta_t]}}{dP_{X|T=t}}(x) - 1\right| \leq \epsilon_{\delta_t}$. This condition implies that

$$\frac{E_{\delta_t}\left[\hat{w}_N(X)H_N^k(X)\right]^{\frac{1}{2+k}}}{E_t\left[\hat{w}_N(X)H_N^k(X)\right]^{\frac{1}{2+k}}} = \left(\frac{\int_X \hat{w}_N(X)H_N^k(X)\frac{dP_{\delta_t}}{dP_t}dP_t}{\int_X \hat{w}_N(X)H_N^k(X)dP_t}\right)^{\frac{1}{2+k}} \in \left((1-\epsilon_{\delta_t})^{\frac{1}{2+k}}, (1+\epsilon_{\delta_t})^{\frac{1}{2+k}}\right)$$

$$\frac{E_{\delta_t}\left[w(X)H_N^\ell(X)\right]^{\frac{1}{1+\ell}}}{E_t\left[w(X)H_N^\ell(X)\right]^{\frac{1}{1+\ell}}} = \left(\frac{\int_X w(X)H_N^\ell(X)\frac{dP_{\delta_t}}{dP_t}dP_t}{\int_X w(X)H_N^\ell(X)dP_t}\right)^{\frac{1}{1+\ell}} \in \left((1-\epsilon_{\delta_t})^{\frac{1}{1+\ell}}, (1+\epsilon_{\delta_t})^{\frac{1}{1+\ell}}\right)$$

These two conditions deliver equation (1) and (2). ■

Proposition 3. *The estimation of weight for ATE should be*

$$\hat{w}(x,t) = \frac{\frac{1}{\hat{\pi}(t|x)}}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]}$$

And the theoretical weight is

$$w(x,t) = \frac{\frac{1}{\int_{t-\delta_t}^{t+\delta_t}\pi(t'|x)dt'}}{E\left[\frac{1}{\int_{t-\delta_t}^{t+\delta_t}\pi(t'|x)dt'} \mid \mathcal{Z}_{tr}\right]}$$

Assume $E\left[\frac{1}{\hat{\pi}(t|x)} - \frac{1}{\hat{\pi}(t'|x)}\right] \to 0$ *(general propensity score is continuous) for all* $t' \in [t-\delta_t, t+\delta_t]$ *when* $\delta_t \to 0$. *Then* $\lim_{N\to\infty}|\hat{w}(x,t) - w(x,t)| = 0$ *, under SUTVA and the strong ignorability assumption,*

$$\lim_{N,n\to\infty} P_{(X,Y(t))\sim Q_X \times P_{Y(1)|X}}\left(Y(t) \in \hat{C}_{N,n}(X)\right) > 1-\alpha$$

If assumptions in Proposition 2 holds then for any $\epsilon > 0$

$$\lim_{N,n\to\infty} P_{X\sim Q_X}\left(P\left(Y(t) \in \hat{C}_{N,n}(X) \mid X\right) \le 1-\alpha-\epsilon\right) = 0$$

Proof. What we need to prove is that weight function is enough close when $\delta_t \to 0$.

$$\begin{aligned}
&\lim_{N\to\infty} E\left|\frac{\frac{1}{\hat{\pi}(t|x)}}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} - \frac{\frac{1}{\int_{t-\delta_t}^{t+\delta_t}\pi(t'|x)dt'}}{E\left[\frac{1}{\int_{t-\delta_t}^{t+\delta_t}\pi(t'|x)dt'}\right]}\right| \\
&= \lim_{N\to\infty} E\left|\frac{\frac{1}{\hat{\pi}(t|x)}}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} - \frac{\frac{1}{\pi(t'|x)}}{E\left[\frac{1}{\pi(t'|x)}\right]}\right| \quad \text{(Integral Mean Value Theorem)} \\
&= \lim_{N\to\infty} E\left|\frac{\frac{1}{\hat{\pi}(t|x)}}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} - \frac{\frac{1}{\pi(t'|x)}}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} + \frac{\frac{1}{\pi(t'|x)}}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} - \frac{\frac{1}{\pi(t'|x)}}{E\left[\frac{1}{\pi(t'|x)}\right]}\right| \\
&\le \limsup_{N\to\infty} \frac{1}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} E\left|\frac{1}{\hat{\pi}(t \mid x)} - \frac{1}{\pi(t' \mid x)}\right| \\
&+ \limsup_{N\to\infty} E\left[\frac{1}{\pi(t' \mid x)}\right] E\left|\frac{1}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} - \frac{1}{E\left[\frac{1}{\pi(t'|x)}\right]}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{N\to\infty} \left| \frac{1}{\hat{\pi}(t \mid x)} - \frac{1}{\pi\left(t' \mid x\right)} \right| + \limsup_{N\to\infty} E\left[\frac{1}{\pi\left(t' \mid x\right)}\right] E\left| \frac{1}{E\left[\frac{1}{\hat{\pi}(t|x)} \mid \mathcal{Z}_{tr}\right]} - \frac{1}{E\left[\frac{1}{\pi(t'|x)}\right]} \right| \\
&\leq \limsup_{N\to\infty} \left| \frac{1}{\hat{\pi}(t \mid x)} - \frac{1}{\pi\left(t' \mid x\right)} \right| + E\left[\frac{1}{\pi\left(t' \mid x\right)}\right] E\left| E\left[\frac{1}{\hat{\pi}(t \mid x)} - \frac{1}{\hat{\pi}\left(t' \mid x\right)} \mid \mathcal{Z}_{tr}\right] \right| \\
&= \left(1 + E\left[\frac{1}{\pi\left(t' \mid x\right)}\right]\right) \lim_{N\to\infty} E\left[\frac{1}{\hat{\pi}(t \mid x)} - \frac{1}{\hat{\pi}\left(t' \mid x\right)}\right]
\end{aligned}
$$

according to the assumption $E\left[\frac{1}{\hat{\pi}(t|x)} - \frac{1}{\hat{\pi}(t'|x)}\right] \to 0$ for all $t' \in [t - \delta_t, t + \delta_t]$ when $\delta_t \to 0$, proposition 3 holds. ■

3 Next Step

The issue with Algorithm 1 lies in the challenge of determining an appropriate value for δ_t since it depends on the sparsity of data at a specific value t. Recent non-parametric analyses of continuous treatment effects, such as the work by Huling et al. [2023], have discussed various strategies to address confounding and model misspecification problems. Inspired by similar research, one possible approach is to modify the weighting function as follows:

$$w(x, t, t') = f(x, t, |t' - t|)$$

This modification considers the distance from point t' to the target value t. It provides a potential solution for avoiding the need to select a specific δ_t and offers improved consistency.

References

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