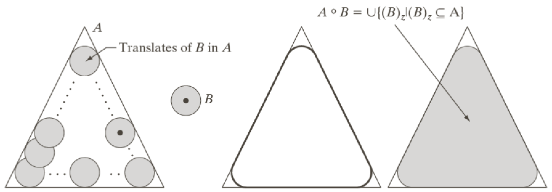
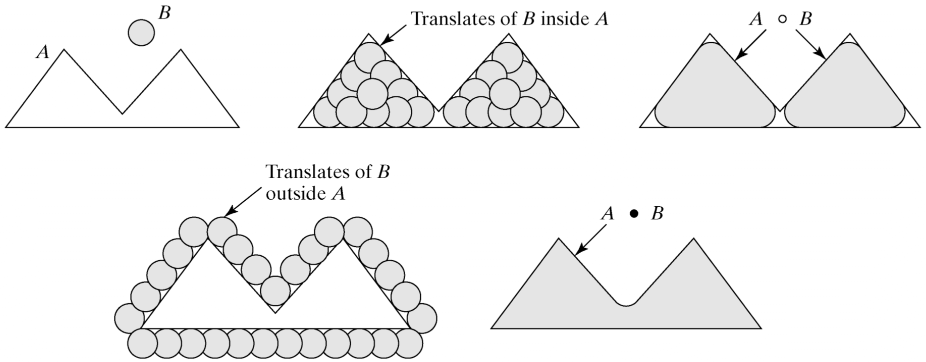
MORPHOLOGICAL IMAGE PROCESSING



Morphological Image Processing

- emphasis on shapes in the spatial domain
- set-theoretical approach to analysis of binary images (continuous and discrete, in any dimension)
- simple geometrical interpretation
- geometric intuition $\rightarrow$ algebra of image operations $\rightarrow$ algorithms
- can be extended to grey value images

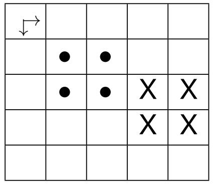
Basic idea: geometry



Sets

union	$\cup$
intersection	$\cap$
set inclusion	$\subseteq$
element of	$\in$
universal set	E
empty set	$\emptyset$
set difference	$X \backslash Y = \{x \in X : x \notin Y\}$
complement	$X^c = E \backslash X$

Set translation



A set X (black dots) and its shift X_h (crosses) with $h = (2, 1)$.

$$X_h = \{x + h : x \in X\},$$

is the translate of X over the vector $h \in E$.

Minkowski operations

Let $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$, $X_a = \{x + a : x \in X\}$ the translate of X over the vector $a \in E$.

Minkowski addition:

$$\boxed{\begin{aligned} X \oplus A &= \{x + a : x \in X, a \in A\} \\ &= \bigcup_{a \in A} X_a = \bigcup_{x \in X} A_x \end{aligned}}$$

Minkowski subtraction:

$$\boxed{X \ominus A = \bigcap_{a \in A} X_{-a}}$$

Dilation and erosion

Fix the structuring element A.

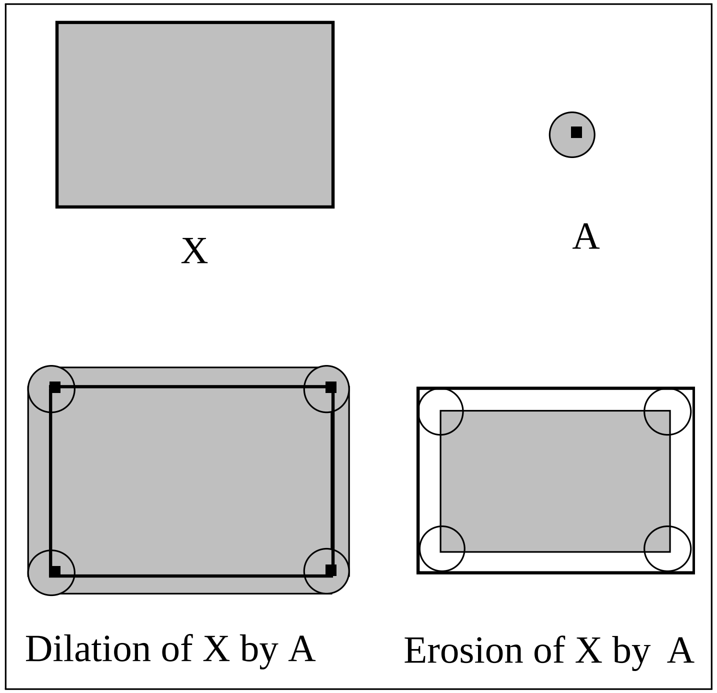
- Dilation by structuring element A:

$$\boxed{\delta_A(X) = X \oplus A}$$

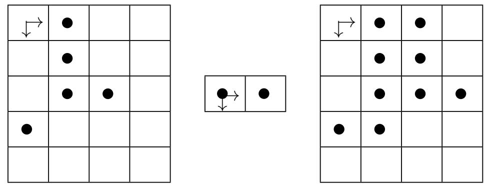
- Erosion by structuring element A:

$$\boxed{\varepsilon_A(X) = X \ominus A}$$

Dilation/erosion: continuous case

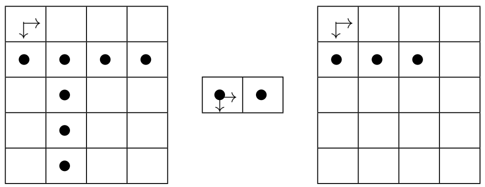


Dilation: discrete case



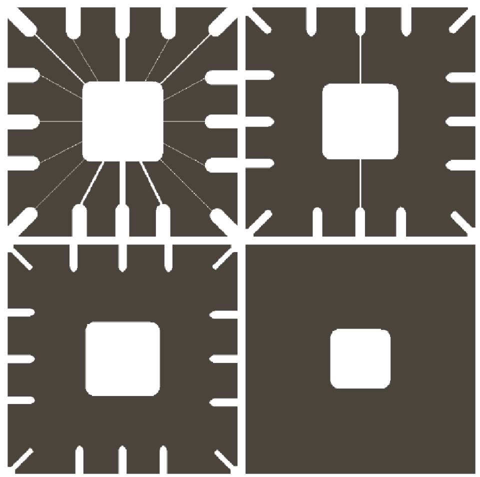
Left: binary image X. Middle: structuring element A. Right: dilation of X by A.

Erosion: discrete case



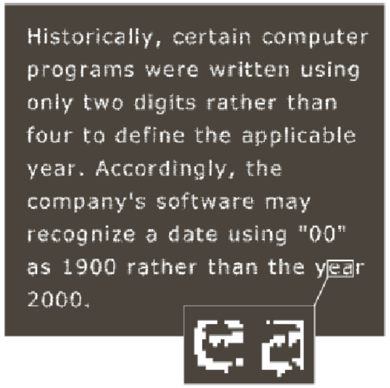
Left: binary image X. Middle: structuring element A. Right: erosion of X by A.

Example: Erosion

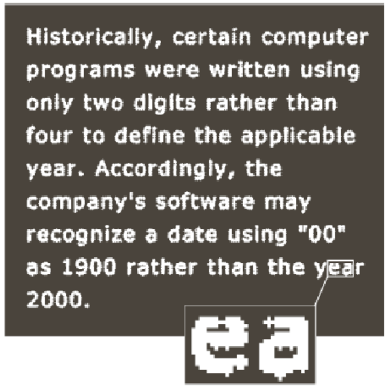


(a) Input. (b)-(d): erosions by structuring element (all 1s) of size 11×11, 15×15, 45×45.

Example: dilation



Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.



Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

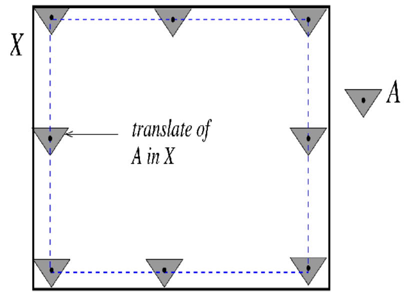
0	1	0
1	1	1
0	1	0

Left: input. Right: dilation of input by 'cross' structuring element.

Geometrical interpretation of erosion

The erosion of X by structuring element A is the set of points h such that A translated over h fits in X:

$$\boxed{X \ominus A = \{h \in E : A_h \subseteq X\}}$$



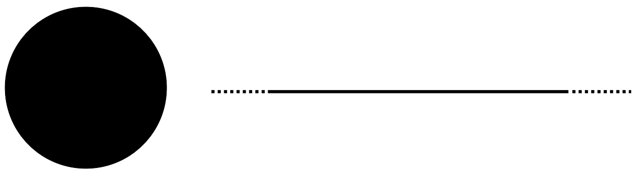
Erosion of a set by itself

Let $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$. Let D be a disc. The erosion of D by the structuring element D equals the origin $(0,0)$:

$$\boxed{D \ominus D = \{h \in E : D_h \subseteq D\} = (0,0)}$$

Let L be an infinite horizontal line. The erosion of L by the structuring element L equals itself:

$$\boxed{L \ominus L = \{h \in E : L_h \subseteq L\} = L}$$

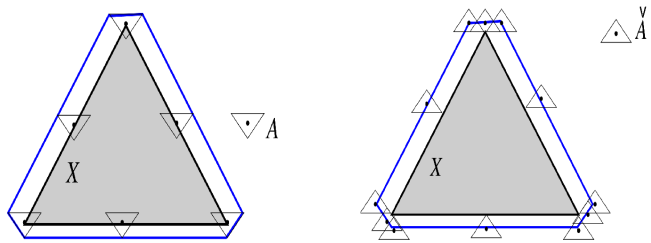


Geometrical interpretation of dilation

The dilation of X by structuring element A is the set of points h such that $\check{A}$ translated over h hits X:

$$\boxed{X \oplus A = \{h \in E : \check{A}_h \cap X \neq \emptyset\},}$$

where $\check{A} = \{-a : a \in A\}$ is the reflection of A.



Geometrical interpretation: Proof

$h \in X \oplus A$
$\iff$ { definition $\oplus$ }
$h \in \bigcup_{a \in A} X_a$
$\iff$ { set theory }
$\exists a \in A : h \in X_a$
$\iff$ { set theory }
$\exists a \in A : h - a \in X$
$\iff$ { definition $\check{A}$ }
$\exists a' \in \check{A} : h + a' \in X$
$\iff$ { definition intersection }
$\{h + a' : a' \in \check{A}\} \cap X \neq \emptyset$
$\iff$ { definition shift }
$(\check{A})_h \cap X \neq \emptyset$

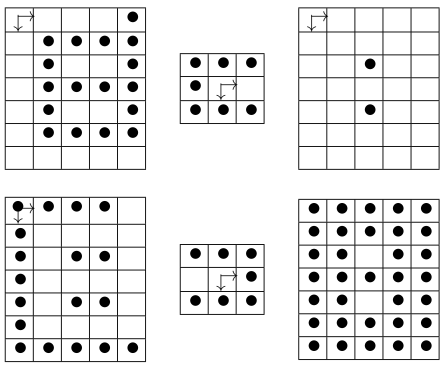
Duality w.r.t. set-complementation

Let X^c denotes the complement of the set X. Then:

$$\boxed{X \oplus A = (X^c \ominus \check{A})^c}$$

In words: dilating an image by A gives the same result as eroding the background by $\check{A}$ and taking the complement.

Duality: example



Top. left: binary image X; Middle: structuring element A; Right: erosion of X by A. Bottom. left: binary image X^c; Middle: reflected structuring element $\check{A}$; Right: dilation of X^c by $\check{A}$.

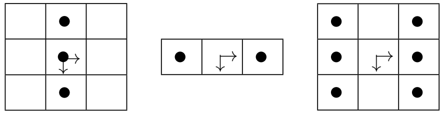
Choice of the origin

If the structuring element does contain the origin, i.e., $(0,0) \in A$, then:

- $X \oplus A \supseteq X$: the dilation of X is bigger than X.
  Proof:
  $X \oplus A = \bigcup_{a \in A} X_a = X \bigcup \left(\bigcup_{a \in A \setminus \{(0,0)\}} X_a \right) \supseteq X$
- $X \ominus A \subseteq X$: the erosion of X is smaller than X.
  Proof:
  Analogous.

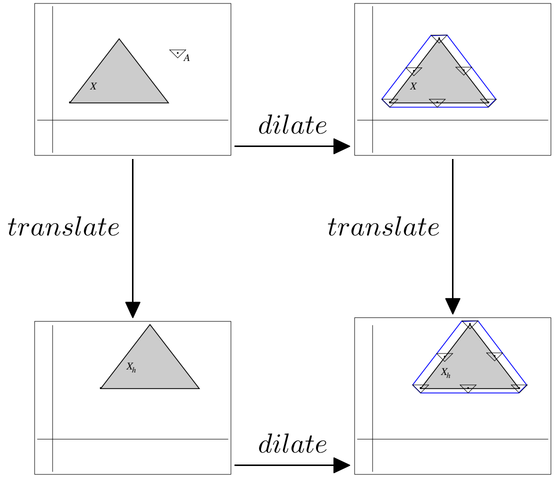
Dilation: effect of the origin

If the structuring element does not contain the origin, then $X \oplus A$ may have zero intersection with X.



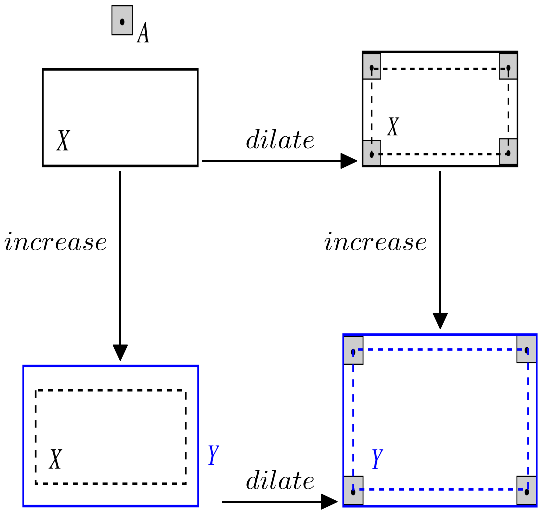
Left: binary image X. Middle: structuring element A. Right: dilation of X by A.

Translation invariance



$$(X \oplus A)_h = X_h \oplus A$$

Dilation/erosion is increasing

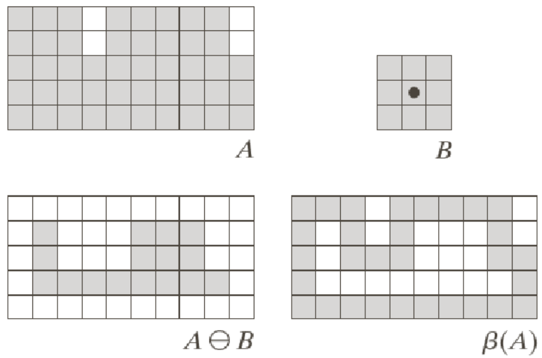


$$X \subseteq Y \implies X \oplus A \subseteq Y \oplus A$$

Boundary extraction

Let B be a structuring element containing the origin. The morphological boundary of A is defined by:

$$\boxed{\beta(A) = A \setminus (A \ominus B)}$$

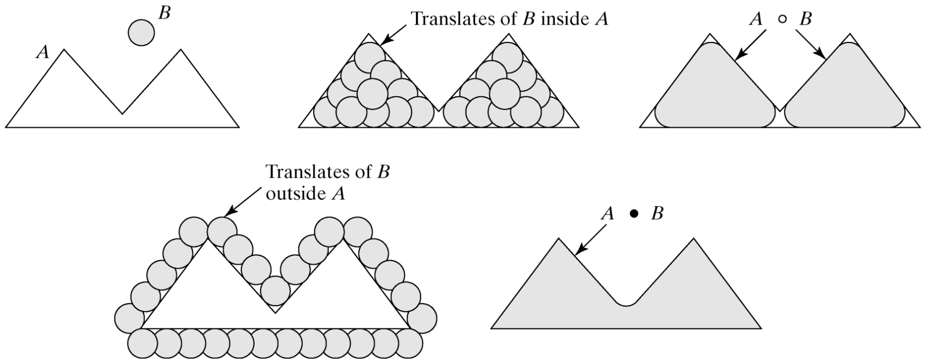


Boundary extraction

Algebraic properties

$$X \oplus A = A \oplus X \qquad \text{commutativity}$$
$$(X \oplus A) \oplus B = X \oplus (A \oplus B) \qquad \text{associativity}$$
$$(X \ominus A) \ominus B = X \ominus (A \oplus B) \qquad \text{iteration}$$
$$(X \cup Y) \oplus A = (X \oplus A) \cup (Y \oplus A) \qquad \text{distributivity}$$
$$(X \cap Y) \ominus A = (X \ominus A) \cap (Y \ominus A) \qquad \text{distributivity}$$
$$X \ominus (A \cup B) = (X \ominus A) \cap (X \ominus B)$$
$$(X \oplus A)_h = X_h \oplus A \qquad \text{translation invariance}$$
$$(X \ominus A)_h = X_h \ominus A \qquad \text{translation invariance}$$
$$X \subseteq Y \implies X \oplus A \subseteq Y \oplus A \qquad \text{increasing}$$
$$X \subseteq Y \implies X \ominus A \subseteq Y \ominus A \qquad \text{increasing}$$

Opening and closing: geometry



Opening and closing

- The opening γ_A is an erosion followed by a dilation:

$$\boxed{\gamma_A(X) = X \circ A := (X \ominus A) \oplus A}$$

  That is, $\gamma_A = \delta_A \varepsilon_A$.

- The closing ϕ_A is a dilation followed by an erosion.

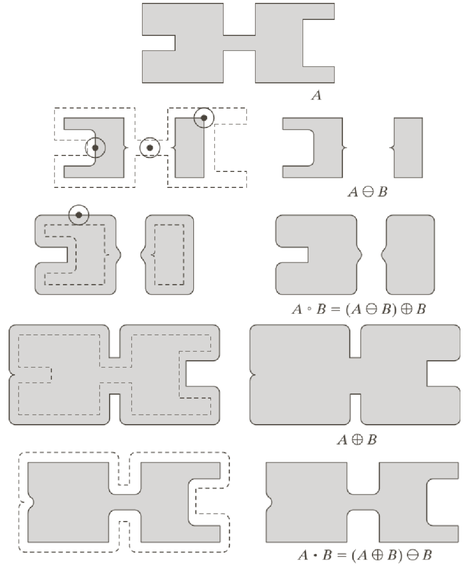
$$\boxed{\phi_A(X) = X \bullet A := (X \oplus A) \ominus A}$$

  That is, $\phi_A = \varepsilon_A \delta_A$.

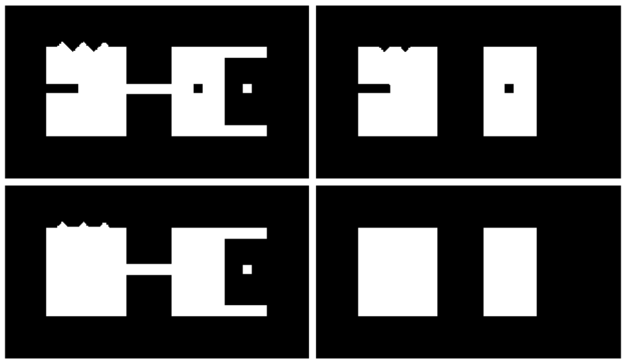
Opening and closing

- An opening smooths contours, cuts narrow bridges, removes small islands and sharp corners.
- A closing fills narrow channels and small holes.

Opening and closing: example



Opening and closing: example

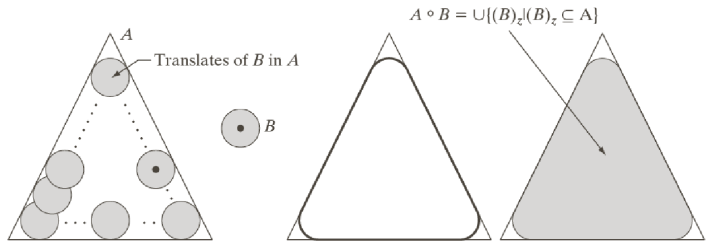


(a) Original. (b) Opening. (c) Closing. (d) Closing of (b).
The structuring element is a square.

Opening: geometrical interpretation

$$\boxed{A \circ B = \bigcup_{z \in E} \{B_z : B_z \subseteq A\}}$$

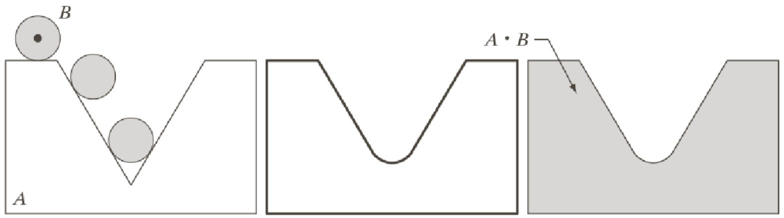
The opening of the set A by structuring element B is the union of all the translates of B which are included in the set A.



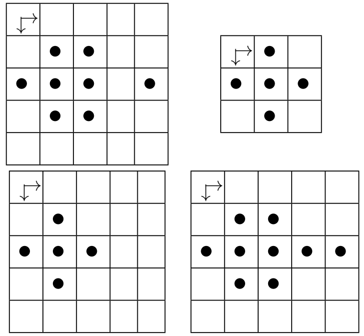
Closing: geometrical interpretation

Duality: $(A^c \circ B)^c = A \bullet \check{B}$.

The closing of the set A by structuring element B is the complement of the opening of the complement of A by B.

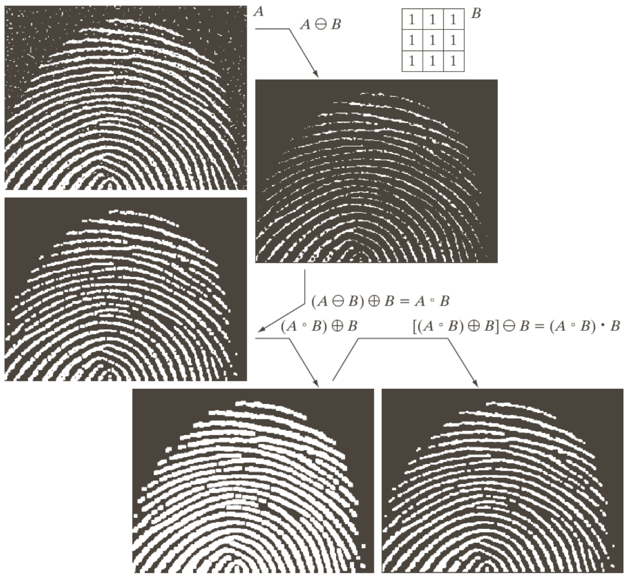


Opening & Closing: discrete case



Upper left: binary image X. Upper right: structuring element A. Lower left: opening of X by A. Lower right: closing of X by A.

Example



Properties of Openings and Closings

- The opening γ_A is:

1. increasing: $X \subseteq Y \implies \gamma_A(X) \subseteq \gamma_A(Y)$
2. idempotent: $\gamma_A(\gamma_A(X)) = \gamma_A(X)$
3. anti-extensive: for every $X,\ \gamma_A(X) \subseteq X$

- The closing ϕ_A is:

1. increasing: $X \subseteq Y \implies \phi_A(X) \subseteq \phi_A(Y)$
2. idempotent: $\phi_A(\phi_A(X)) = \phi_A(X)$
3. extensive: for every $X,\ \phi_A(X) \supseteq X$

Idempotence of the opening

$$
\begin{aligned}
&\gamma_A(\gamma_A(X))\\
\subseteq &\{\ \gamma_A \text{ is anti-extensive }\}\\
&\gamma_A(X)
\end{aligned}
$$

$$
\begin{aligned}
&\gamma_A(\gamma_A(X))\\
= &\{\ \gamma_A = \delta_A\varepsilon_A\ \}\\
&\delta_A\varepsilon_A\delta_A\varepsilon_A(X)\\
= &\{\ \phi_A = \varepsilon_A\delta_A\ \}\\
&\delta_A\phi_A\varepsilon_A(X)\\
\supseteq &\{\ \phi_A \text{ is extensive, } \delta_A \text{ is increasing }\ \}\\
&\delta_A\varepsilon_A(X)\\
= &\{\ \gamma_A = \delta_A\varepsilon_A\ \}\\
&\gamma_A(X)
\end{aligned}
$$

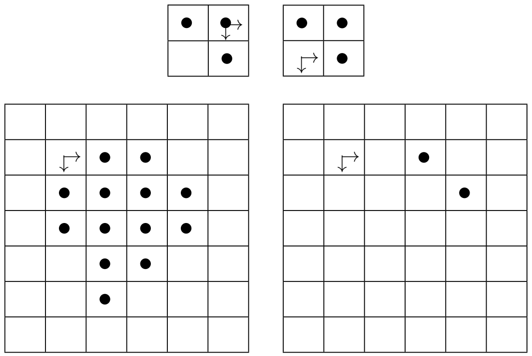
So we have shown that $\gamma_A(\gamma_A(X)) \subseteq \gamma_A(X)$ and also that $\gamma_A(\gamma_A(X)) \supseteq \gamma_A(X)$.
Conclusion: $\gamma_A(\gamma_A(X)) = \gamma_A(X)$.

Hit-or-miss transform

$$
\begin{aligned}
X \circledast (B_1, B_2) &= \{h \in E : (B_1)_h \subseteq X, (B_2)_h \subseteq X^c\}\\
&= (X \ominus B_1) \cap (X^c \ominus B_2)
\end{aligned}
$$

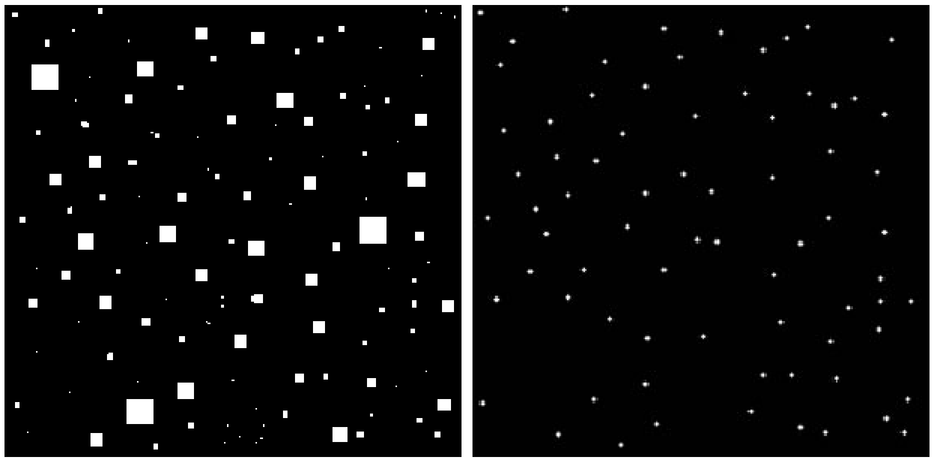
In words, the hit-or-miss transform of X by (B_1, B_2) is the collection of points h such that the shifted set $(B_1)_h$ fits into X and the shifted set $(B_2)_h$ fits into the complement of X.

Hit-or-miss transform: example



Extracting north-east corner pixels.
Upper left: structuring element B_1. Upper right: structuring element B_2. Lower left: binary set X. Lower right: hit-or-miss transform of X.

Hit-or-miss transform



Extracting corner pixels.

Set laws

Commutativity

$$X \cap Y = Y \cap X \quad X \cup Y = Y \cup X$$

Distributivity

$$\begin{aligned} X \cap (Y \cup Z) &= (X \cap Y) \cup (X \cap Z) \\ X \cup (Y \cap Z) &= (X \cup Y) \cap (X \cup Z) \end{aligned}$$

Associativity

$$\begin{aligned} X \cup (Y \cup Z) &= (X \cup Y) \cup Z \\ X \cap (Y \cap Z) &= (X \cap Y) \cap Z \end{aligned}$$

Set laws

De Morgan's Laws

$$\begin{aligned} (X \cup Y)^c &= X^c \cap Y^c \\ (X \cap Y)^c &= X^c \cup Y^c \end{aligned}$$

Minimax Theorem

$$\bigcap_i \left(\bigcup_j X_{ij} \right) \supseteq \bigcup_j \left(\bigcap_i X_{ij} \right)$$