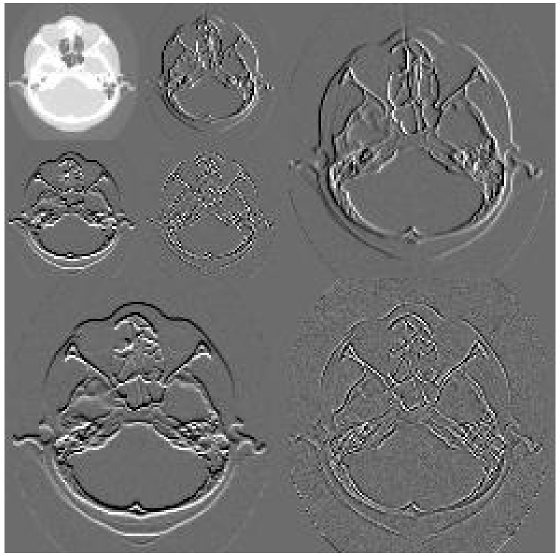
Wavelets and Multiresolution



Multiresolution Processing

$\boxed{\text{Image Pyramids}}$

Gaussian pyramid

Downsampling and Upsampling

- Downsampling by a factor of 2 (signal with even number of samples):

$$\downarrow_2 (a_0, a_1, a_2, \ldots, a_{2N-1}) = (a_0, a_2, a_4, \ldots, a_{2N-2})$$

- Upsampling by a factor of 2:

$$\uparrow_2 (a_0, a_1, a_2, \ldots, a_{N-1}) = (a_0, 0, a_1, 0, a_2, 0, \ldots, a_{N-1}, 0)$$

Image pyramid

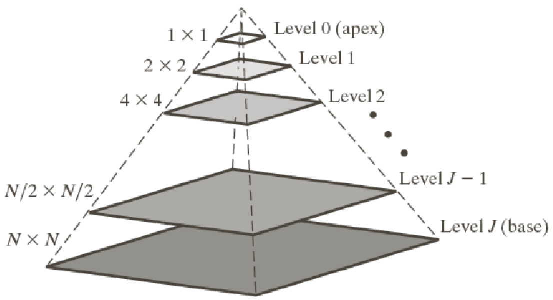


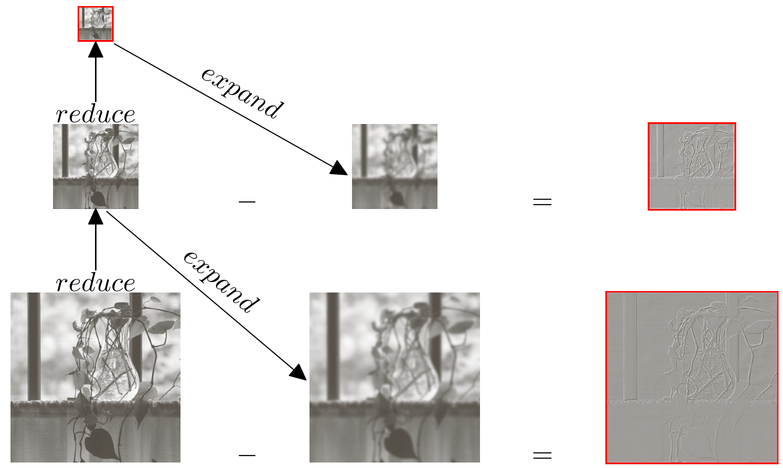
Image of size $N \times N$, with $N = 2^J$.

In a $J+1$-level pyramid, the total number of pixels is:

$$N^2\left(1+\frac{1}{4^1}+\frac{1}{4^2}+\cdots\frac{1}{4^J}\right) \leq \frac{4}{3}N^2$$

Laplacian pyramid

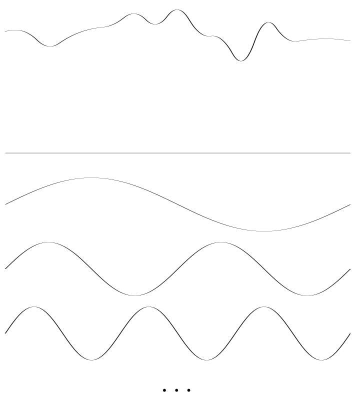
Approximations Predictions Residuals



Wavelets

The Mathematical Microscope

1-D Fourier analysis



Weighted sum of basic signals

coefficient c_0

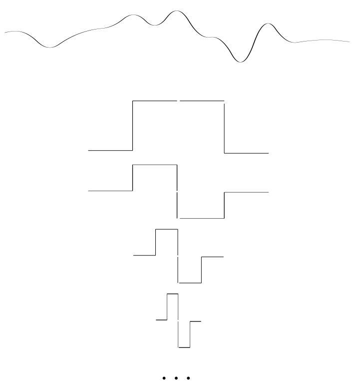
coefficient c_1

coefficient c_2

coefficient c_3

...

1-D Wavelets: Haar wavelet



Weighted sum of basic signals

approximation coefficient c_3

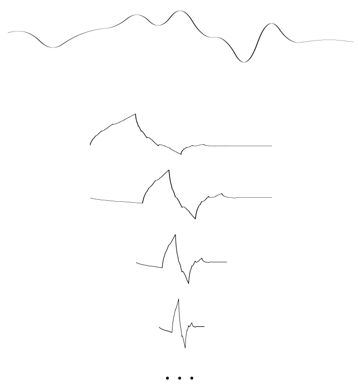
detail coefficient d_3

detail coefficient d_2

detail coefficient d_1

$\dots$

1-D Wavelets: Daubechies wavelet



Weighted sum of basic signals

approximation coefficient c_3

detail coefficient d_3

detail coefficient d_2

detail coefficient d_1

$\dots$

Simple example: decomposition

Signal decomposition:

$$c_0 = (9, 1, 2, 0)$$

$$c_1 = (\overbrace{9, 1}^{\text{mean}}, \overbrace{2, 0}^{\text{mean}}) = (5, 1) \qquad \text{approximation}$$

$$d_1 = (\overbrace{9, 1}^{\text{diff}}, \overbrace{2, 0}^{\text{diff}}) = (4, 1) \qquad \text{detail}$$

$$c_2 = (\overbrace{5, 1}^{\text{mean}}) = (3) \qquad \text{approximation}$$

$$d_2 = (\overbrace{5, 1}^{\text{diff}}) = (2) \qquad \text{detail}$$

$\text{mean}(a, b) = (a + b)/2$. $\text{diff}(a, b) = (a - b)/2$.

Simple example: reconstruction

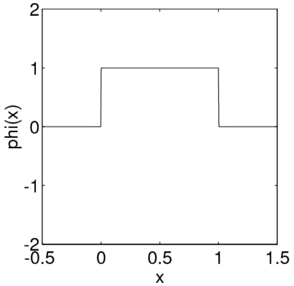
Signal reconstruction.
From the coefficients in $c_2 = (3), d_2 = (2), d_1 = (4, 1)$ we get:

$$c_0 = \begin{pmatrix} 9 \\ 1 \\ 2 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Haar wavelets

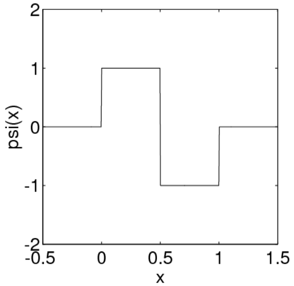
Haar scaling function

$$\phi(x) = \begin{cases} 1 & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



Haar wavelet

$$\psi(x) = \begin{cases} 1 & -\frac{1}{2} \le x < 0 \\ -1 & 0 \le x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

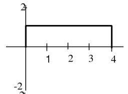


Wavelet representation

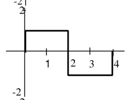
Orthonormal basis functions:

$$\phi_{j,k}(x) = 2^{-j/2}\phi(2^{-j}x - k) \quad \text{scaling functions}$$
$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k) \quad \text{wavelet functions}$$

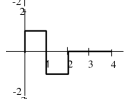
$\phi_{2,0}(x) \leftrightarrow \frac{1}{2}(1,1,1,1)$



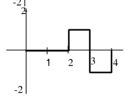
$\psi_{2,0}(x) \leftrightarrow \frac{1}{2}(1,1,-1,-1)$



$\psi_{1,0}(x) \leftrightarrow \frac{1}{\sqrt{2}}(1,-1,0,0)$

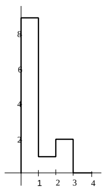


$\psi_{1,1}(x) \leftrightarrow \frac{1}{\sqrt{2}}(0,0,1,-1)$



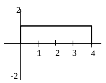
Wavelet representation

$$\boxed{f = c_2(0)\phi_{2,0} + d_2(0)\psi_{2,0} + d_1(0)\psi_{1,0} + d_1(1)\psi_{1,1}}$$

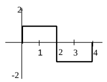


=

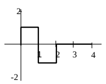
3 x



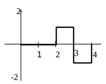
+ 2 x



+ 4 x



+ 1 x



$c_2 = (3), d_2 = (2), d_1 = (4,1)$

Decomposition: general form

We can write the operations in general form:

- For c_j: First filter c_{j-1} with a low pass filter $h_\phi = \frac{1}{2}(1,1)$, then downsample by a factor of 2.
- For d_j: First filter c_{j-1} with a band pass filter $h_\psi = \frac{1}{2}(1,-1)$, then downsample by a factor of 2.

$$\boxed{\begin{aligned} c_j &= \mathbf{H}c_{j-1} = \downarrow_2 (h_\phi * c_{j-1}) \\ d_j &= \mathbf{G}c_{j-1} = \downarrow_2 (h_\psi * c_{j-1}) \end{aligned}}$$

Fast wavelet transform

Input: signal c_0 of length $N = 2^K$ $(J \leq K)$.
Output: signals $c_J, d_J, d_{J-1}, \ldots, d_1$.

$$\begin{tikzcd}
c_0 \arrow[r, "\mathbf{H}_N"] \arrow[dr, "\mathbf{G}_N"'] & c_1 \arrow[r, "\mathbf{H}_{N/2}"] \arrow[dr, "\mathbf{G}_{N/2}"'] & c_2 & \cdots \arrow[r, "\mathbf{H}_{N/2^J}"] \arrow[dr, "\mathbf{G}_{N/2^J}"'] & c_J \\
& d_1 & d_2 & & d_J
\end{tikzcd}$$

Mallat's pyramid algorithm for decomposition.

Reconstruction: general form

We can write the operations in general form:

- Upsample c_j by a factor of 2, then filter the result with a low pass filter
$\tilde{h}_\phi = (1, 1)$.
- Upsample d_j by a factor of 2, then filter the result with a band pass filter
$\tilde{h}_\psi = (1, -1)$.
- Add the results.

$$\boxed{c_{j-1} = \tilde{h}_\phi * (\uparrow_2 c_j) + \tilde{h}_\psi * (\uparrow_2 d_j)}$$

Fast inverse wavelet transform

Input: signals $c_J, d_J, d_{J-1}, \ldots, d_1$.
Output: signal c_0 of length $N = 2^K$.

$$\begin{tikzcd}
c_J \arrow[r, "\mathbf{H}^T_{N/2^J}"] & c_{J-1} \arrow[r, "\mathbf{H}^T_{N/2^{J-1}}"] & \cdots & c_1 \arrow[r, "\mathbf{H}^T_{N/2}"] & c_0 \\
d_J \arrow[ur, "\mathbf{G}^T_{N/2^J}"'] & d_{J-1} \arrow[ur, "\mathbf{G}^T_{N/2^{J-1}}"'] & & d_1 \arrow[ur, "\mathbf{G}^T_{N/2}"'] &
\end{tikzcd}$$

Wavelet pyramid

Signal of size N, with $N = 2^K$. In a J-level pyramid $(J \leq K)$, the total number of points is:

$$N \left(\overbrace{\frac{1}{2^1}}^{d_1} + \overbrace{\frac{1}{2^2}}^{d_2} + \cdots \overbrace{\frac{1}{2^J}}^{d_J} + \overbrace{\frac{1}{2^J}}^{c_J} \right) = N$$

The wavelet decomposition is non-redundant: a signal of size N is mapped to a signal of size N.

<table>
<tr><td colspan="4">c_0</td></tr>
<tr><td>c_3</td><td>d_3</td><td>d_2</td><td>d_1</td></tr>
</table>

Haar wavelet filters

It is convenient to normalize the filters:

$$\text{Decomposition low pass filter}: h_\phi = \frac{1}{\sqrt{2}}(1,1)$$

$$\text{Decomposition band pass filter}: h_\psi = \frac{1}{\sqrt{2}}(1,-1)$$

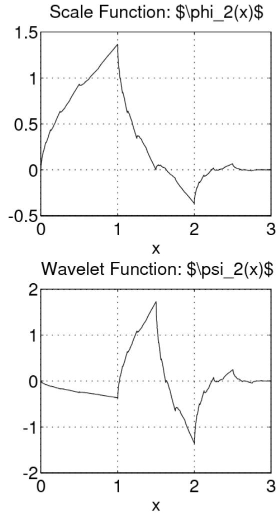
$$\text{Reconstruction low pass filter}: \tilde{h}_\phi = \frac{1}{\sqrt{2}}(1,1)$$

$$\text{Reconstruction band pass filter}: \tilde{h}_\psi = \frac{1}{\sqrt{2}}(1,-1)$$

Now the decomposition and reconstruction filters are the same: $h_\phi = \tilde{h}_\phi$, $h_\psi = \tilde{h}_\psi$.

This is an example of an orthonormal wavelet basis.

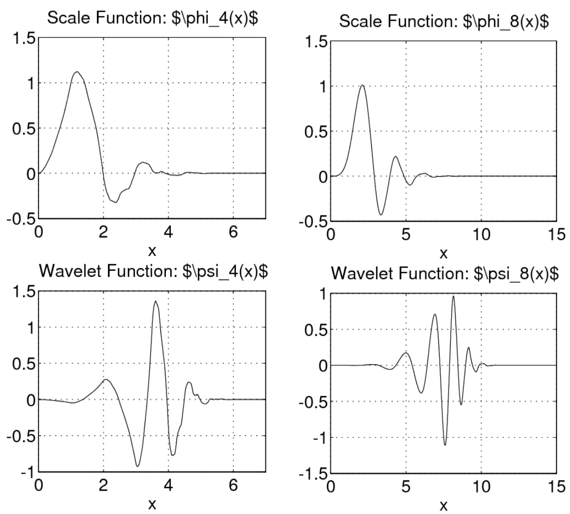
Daubechies D_4 wavelet



4-tap filter:

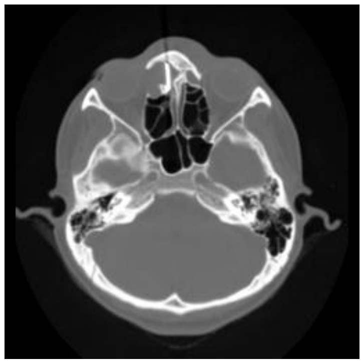
$$\begin{aligned} h_\phi(0) &= \frac{1+\sqrt{3}}{4\sqrt{2}} \\ h_\phi(1) &= \frac{3+\sqrt{3}}{4\sqrt{2}} \\ h_\phi(2) &= \frac{3-\sqrt{3}}{4\sqrt{2}} \\ h_\phi(3) &= \frac{1-\sqrt{3}}{4\sqrt{2}} \end{aligned}$$

More Daubechies wavelets

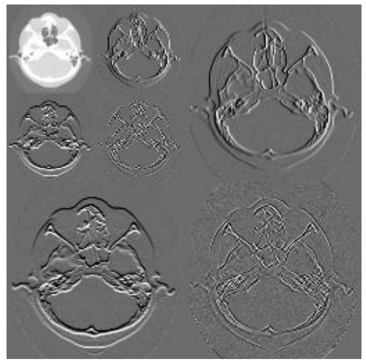


8-tap 16-tap

2-D Wavelet analysis



CT image



decomposition (Haar wavelet)

university of groningen

Local histograms