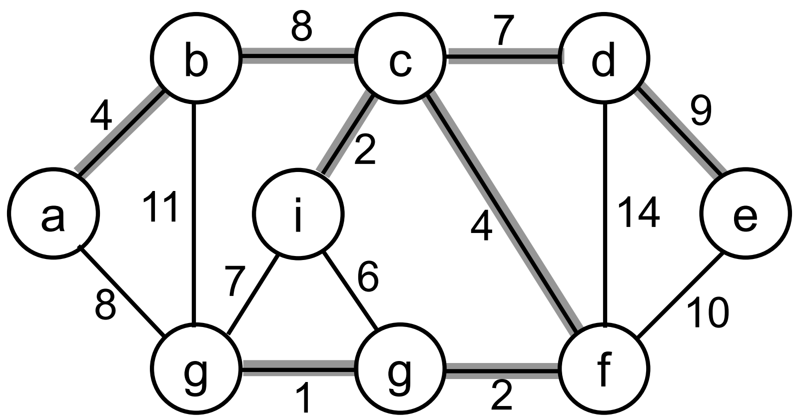
香港中文大學(深圳)
The Chinese University of Hong Kong

CSC3100 Data Structures Lecture 17: Graph minimum spanning tree

Yixiang Fang
School of Data Science (SDS)
The Chinese University of Hong Kong, Shenzhen

Minimum Spanning Trees

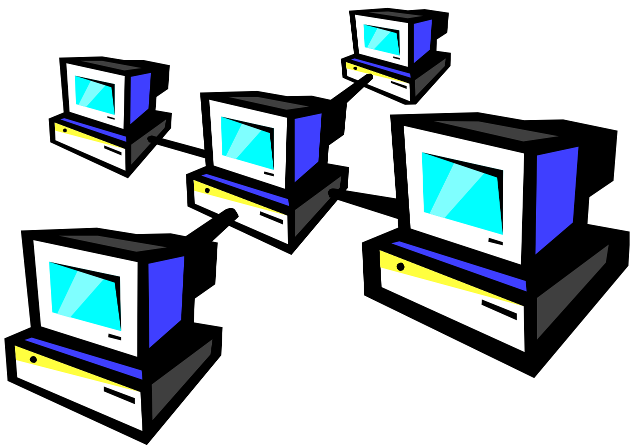
- Spanning Tree
  - A tree (i.e., connected, acyclic graph) which contains all the vertices of the graph
- Minimum Spanning Tree (MST)
  - Spanning tree with the **minimum sum of weights**

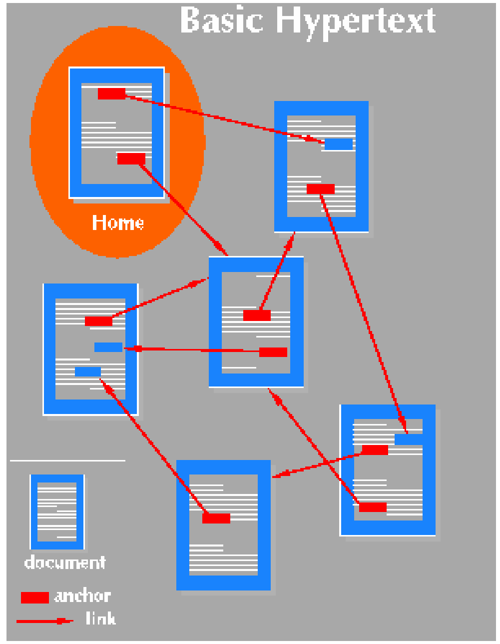


- Spanning forest
  - If a graph is not connected, then there is an MST for each connected component of the graph

Applications of MST

- Find the least expensive way to connect a set of houses, cities, terminals, computers, etc.

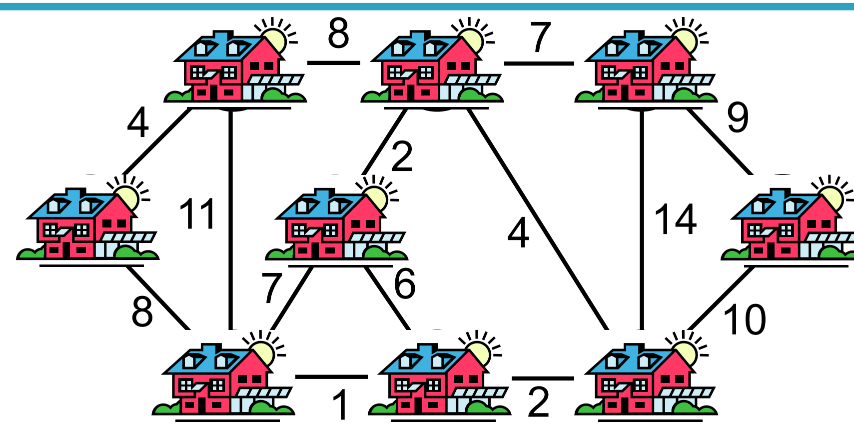




Example

Problem

- A town has a set of houses and a set of roads
- A road connects 2 and only 2 houses
- A road connecting houses u and v has a cost w(u, v)



Goal: Build enough (and no more) roads such that:

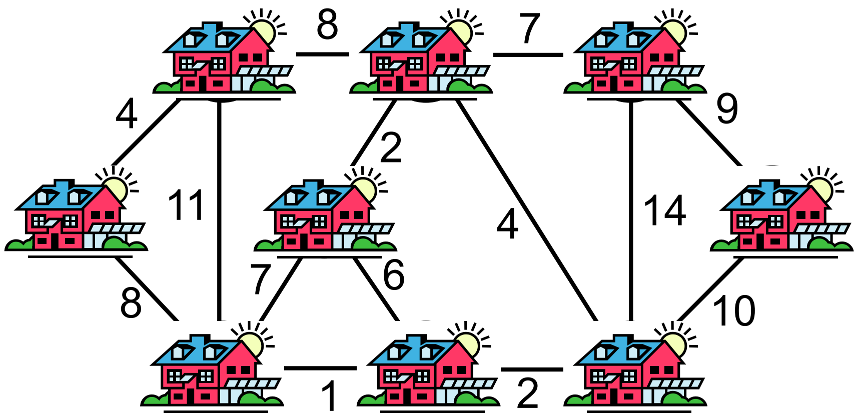
1. Everyone stays connected
   i.e., can reach every house from all other houses
2. Total cost is minimum

Minimum Spanning Trees (MSTs)

- A connected, undirected graph:
  - Vertices = houses, Edges = roads
- A **weight** $w(u, v)$ on each edge $(u, v) \in E$

Find $T \subseteq E$ such that:

1. T connects all vertices
2. $w(T) = \Sigma_{(u,v) \in T} w(u, v)$ is minimized



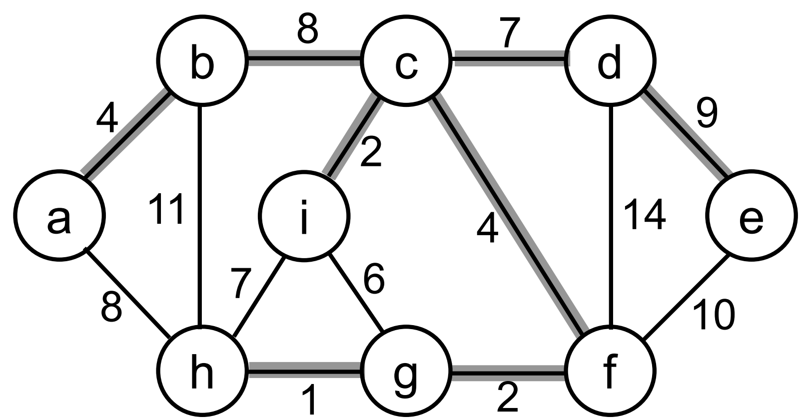
Properties of MSTs

- Minimum spanning tree is **not** unique

- MST has no cycles – see why:
  - We can take out an edge of a cycle, and still have the vertices connected while reducing the cost
- # of edges in a MST:
  - |V| - 1

Growing an MST – Generic Approach

- Grow a set A of edges (initially empty)
- Incrementally add edges to A such that they would belong to an MST

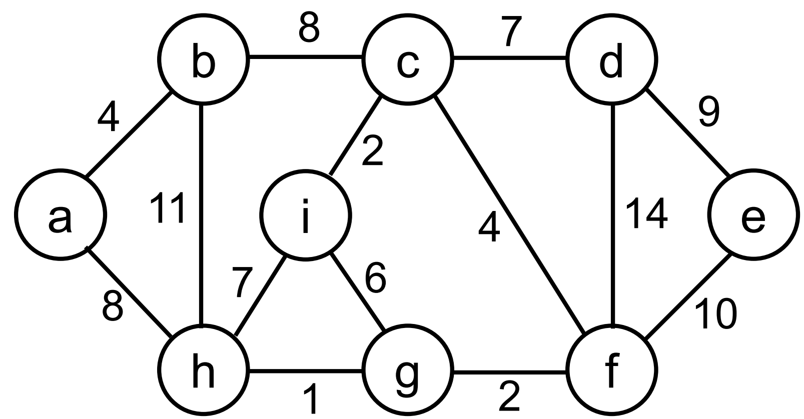


Idea: add only "safe" edges

- An edge (u, v) is **safe** for A, if and only if $A \cup \{(u, v)\}$ is also a subset of **some** MST

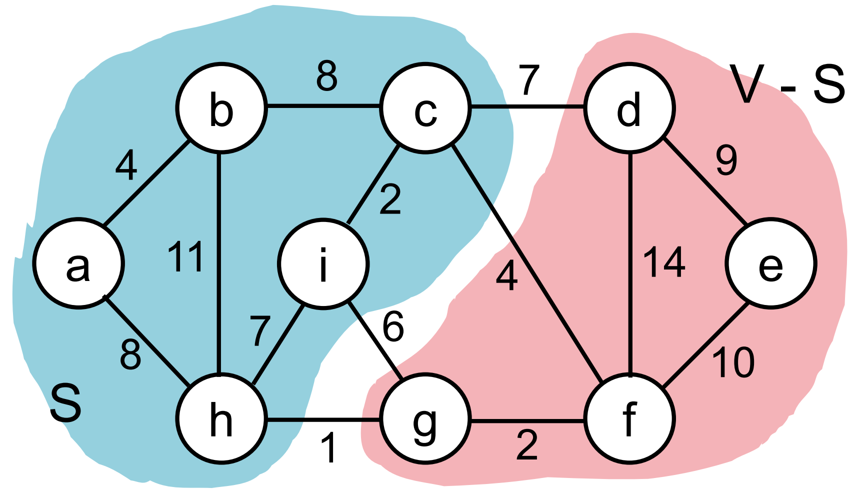
Generic MST algorithm

1. $A \leftarrow \varnothing$
2. **while** A is not a spanning tree
3. **do** find an edge (u, v) that is safe for A
4. $A \leftarrow A \cup \{(u, v)\}$
5. **return** A



- How do we find safe edges?

Finding Safe Edges

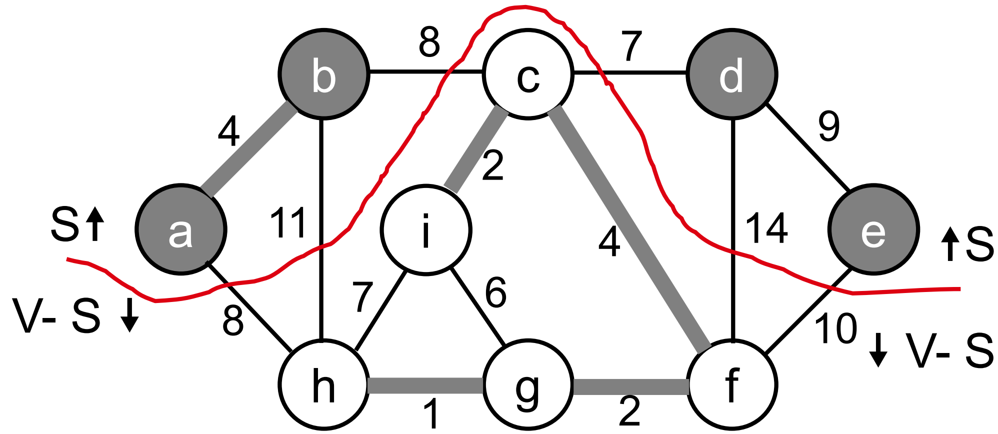


- Let's look at edge (h, g)
  - Is it safe for A initially?

- Yes. Why?
  - Let $S \subset V$ be any set of vertices that includes h but not g (so that g is in V - S)
  - In any MST, there has to be one edge (at least) that connects S with V - S
  - Why not choose the edge with **minimum weight** (h,g)?

Definitions

- A **cut** (S, V - S)
is a partition of vertices
into disjoint sets S and V - S



- An edge **crosses** the cut
(S, V - S) if one endpoint is in S
and the other in V – S

- A cut **respects** a set A of edges ⇔ no edge in A crosses the cut

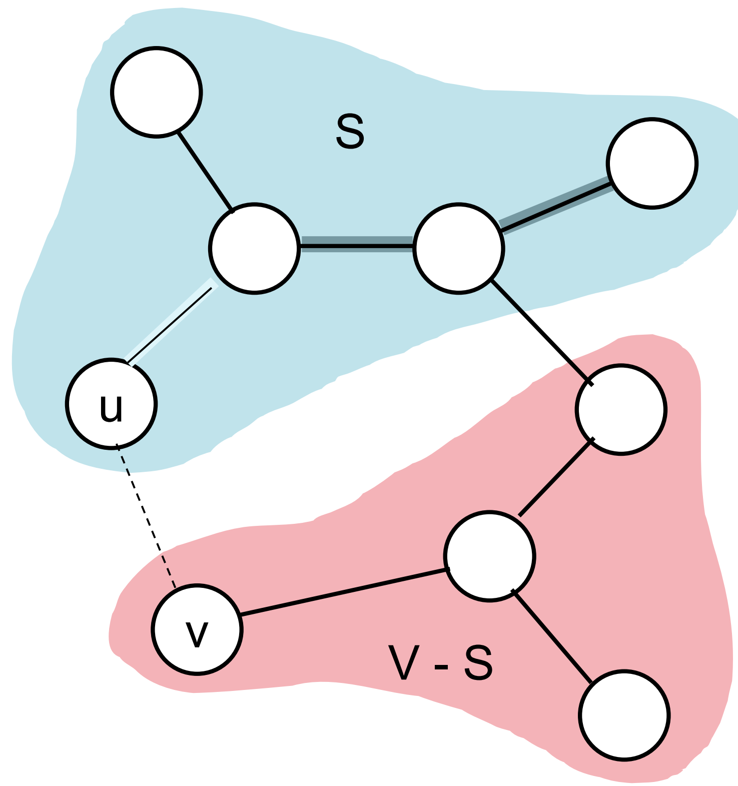
- An edge is a **light edge** crossing a cut
⇔ its weight is minimum over all edges crossing the cut
  - Note that for a given cut, there can be > 1 light edges crossing it

Theorem

- Let A be a subset of some MST (i.e., T), (S, V - S) be a **cut** that respects A, and (u, v) be a **light edge** crossing (S, V-S). Then (u, v) is safe for A.

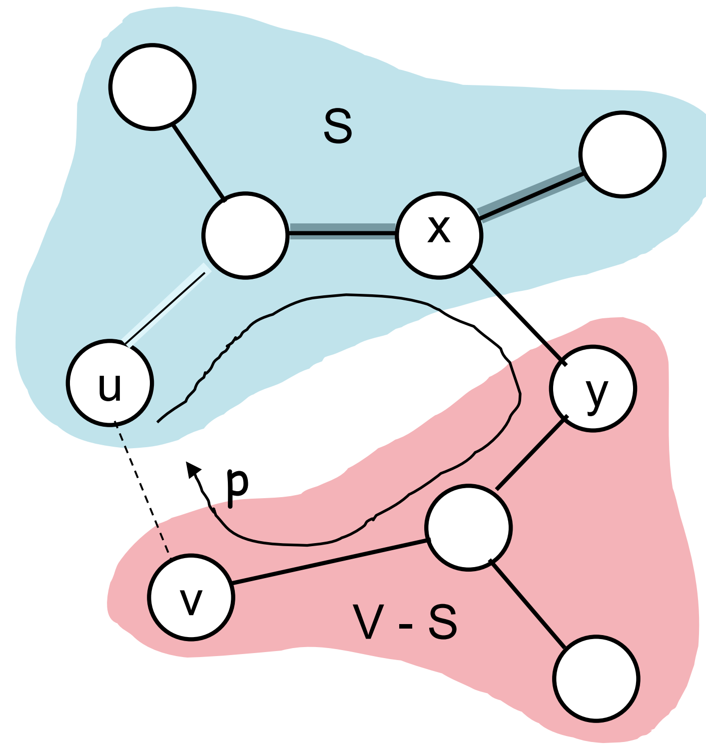
Proof:

- Let T be an MST that includes A
  - edges in A are shaded
- Case1: If T includes (u,v), then it would be safe for A
- Case2: Suppose T does not include the edge (u, v)
  - **Idea**: construct another MST T' that includes A + {(u, v)}



Theorem - Proof

- T contains a unique path p between u and v



- Path p must cross the cut (S, V - S) at least once: let (x, y) be that edge
- Let's remove (x,y) ⇒ breaks T into two components
- Adding (u, v) reconnects the components

  $T' = T - \{(x, y)\} + \{(u, v)\}$

Theorem - Proof (cont.)

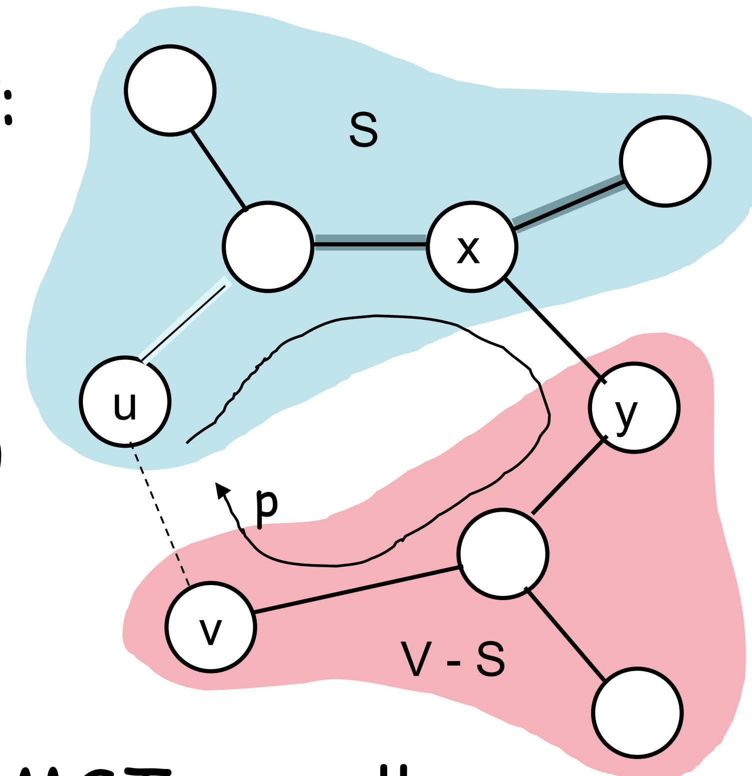
T' = T - {(x, y)} + {(u, v)}

Have to show that T' is an MST:

- (u, v) is a light edge
  $\Rightarrow$ w(u, v) $\leq$ w(x, y)

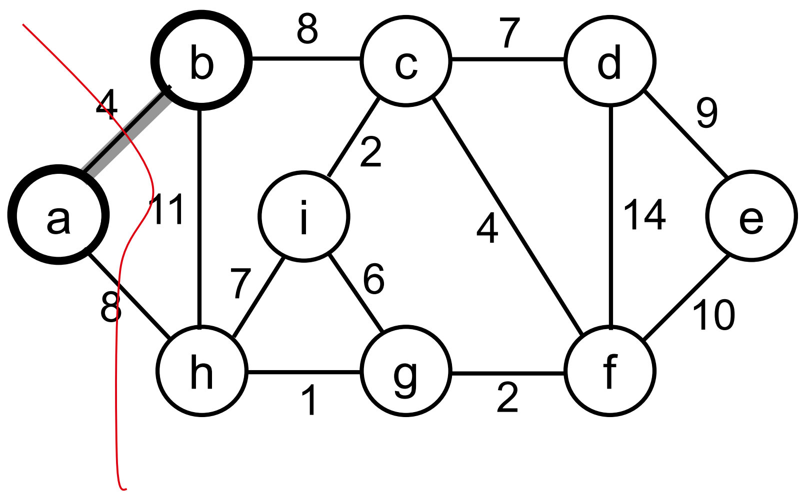
- w(T ') = w(T) - w(x, y) + w(u, v)
  $\leq$ w(T)

- Since T is a spanning tree
  w(T) $\leq$ w(T ') $\Rightarrow$ T' must be an MST as well



Prim's Algorithm

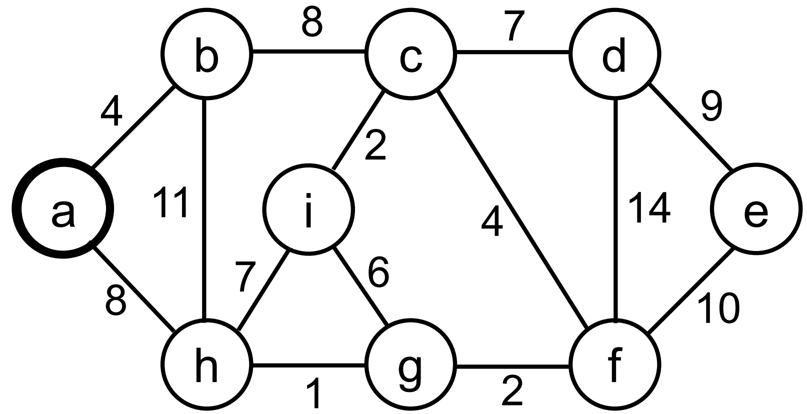
- The edges in set A always <u>form a single tree</u>
- Starts from an arbitrary "root": $V_A = \{a\}$
- At each step:
  - Find a light edge crossing $(V_A, V - V_A)$
  - Add this edge to A
  - Repeat until the tree spans all vertices



How to Find Light Edges Quickly?

Use a priority queue Q:

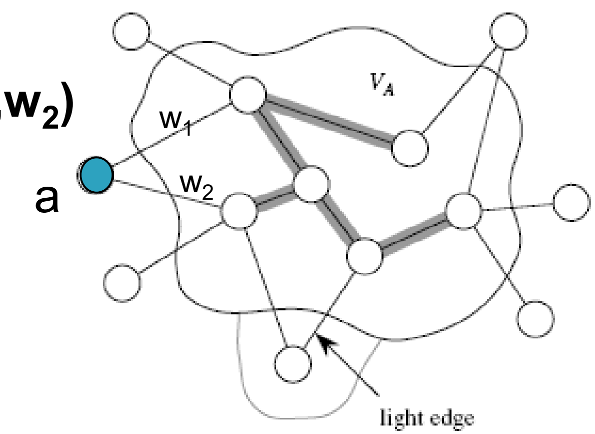
- Contains vertices not yet included in the tree, i.e., ($V - V_A$)
  - $V_A = \{a\}$, $Q = \{b, c, d, e, f, g, h, i\}$



- We associate a key with each vertex v:

  key[v] = minimum weight of any edge (u, v) connecting v to V_A

Key[a]=min(w_1,w_2)

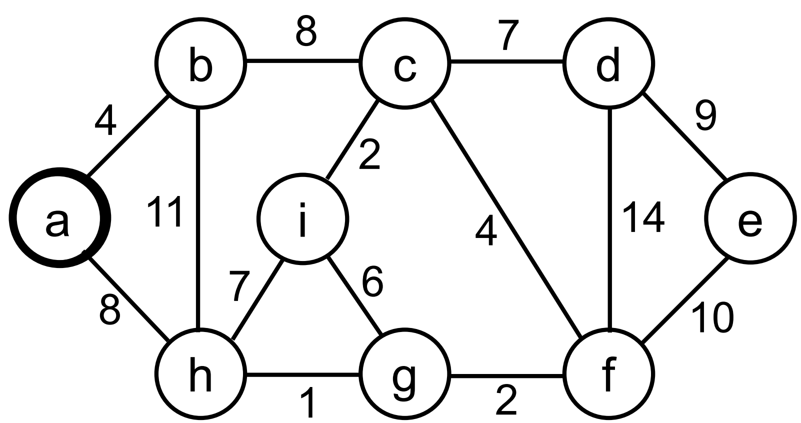


How to Find Light Edges Quickly? (cont.)

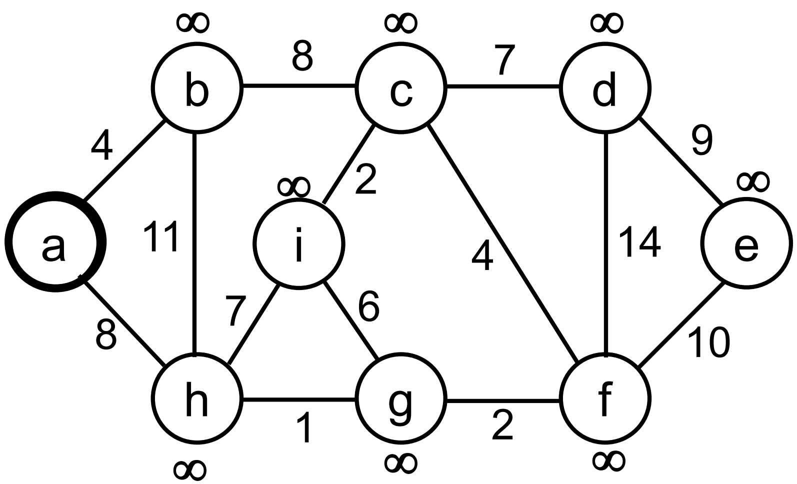
- After adding a new node to V_A we update the weights of all the nodes adjacent to it

  e.g., after adding a to the tree, k[b]=4 and k[h]=8

- Key of v is ∞ if v is not adjacent to any vertices in V_A



Example

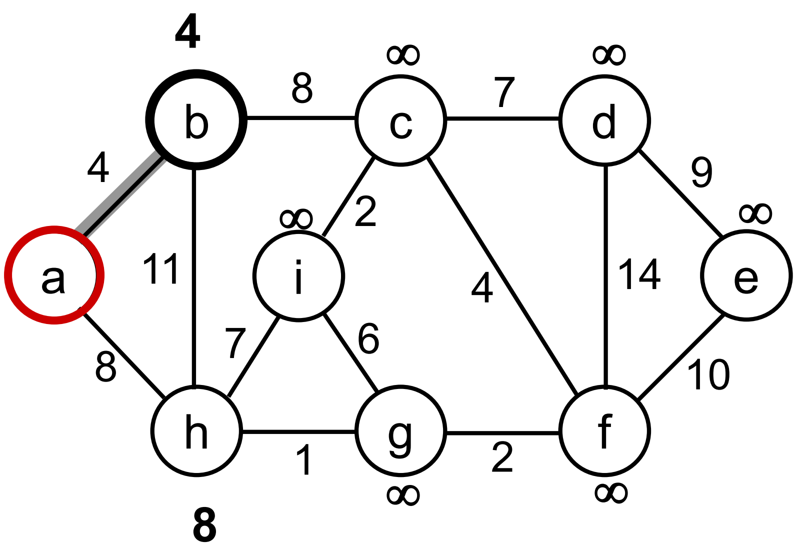


0 ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Q = {a, b, c, d, e, f, g, h, i}

$V_A = \varnothing$

Extract-MIN(Q) $\Rightarrow$ a



key [b] = 4 π [b] = a

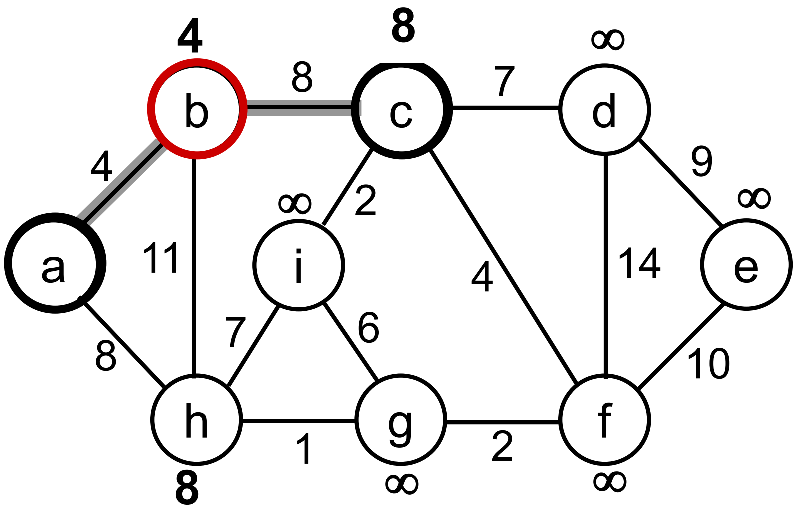
key [h] = 8 π [h] = a

4 ∞ ∞ ∞ ∞ ∞ **8** ∞

Q = {b, c, d, e, f, g, h, i} V_A = {a}

Extract-MIN(Q) $\Rightarrow$ b

Example



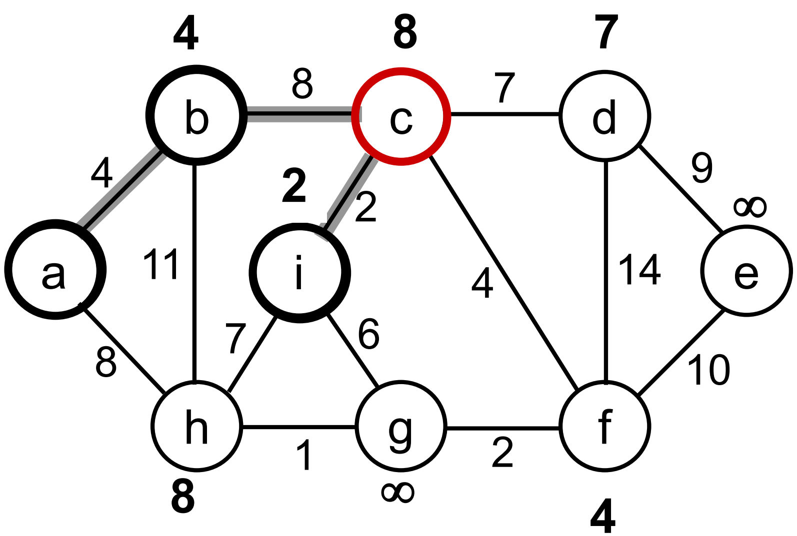
key [c] = 8 π [c] = b

key [h] = 8 π [h] = a - unchanged

8 ∞ ∞ ∞ ∞ 8 ∞

Q = {c, d, e, f, g, h, i} V_A = {a, b}

Extract-MIN(Q) ⇒ c



key [d] = 7 π [d] = c

key [f] = 4 π [f] = c

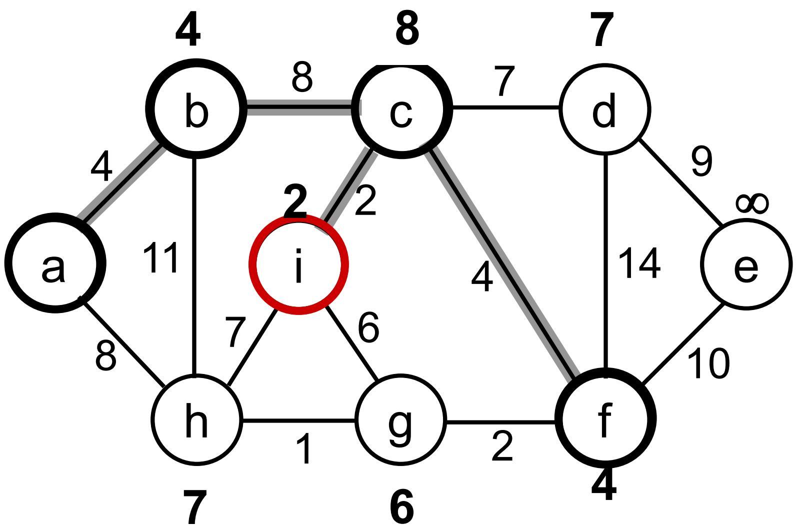
key [i] = 2 π [i] = c

7 ∞ 4 ∞ 8 2

Q = {d, e, f, g, h, i} V_A = {a, b, c}

Extract-MIN(Q) ⇒ i

Example



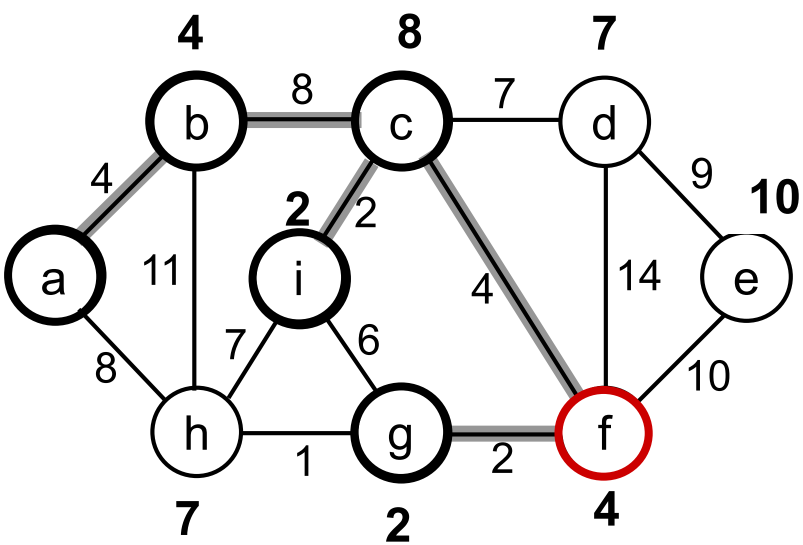
key [h] = 7 π [h] = i

key [g] = 6 π [g] = i

7 ∞ 4 6 8

Q = {d, e, f, g, h} V_A = {a, b, c, i}

Extract-MIN(Q) $\Rightarrow$ f



key [g] = 2 π [g] = f

key [d] = 7 π [d] = c unchanged

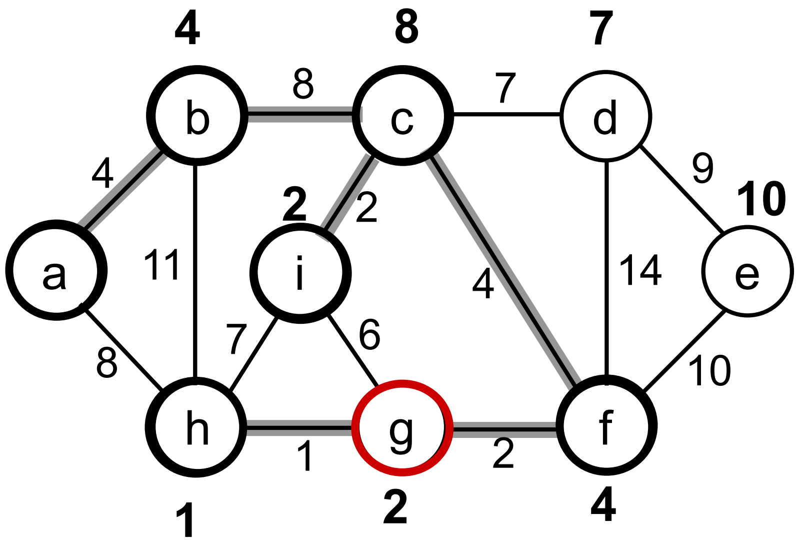
key [e] = 10 π [e] = f

7 10 2 8

Q = {d, e, g, h} V_A = {a, b, c, i, f}

Extract-MIN(Q) $\Rightarrow$ g

Example

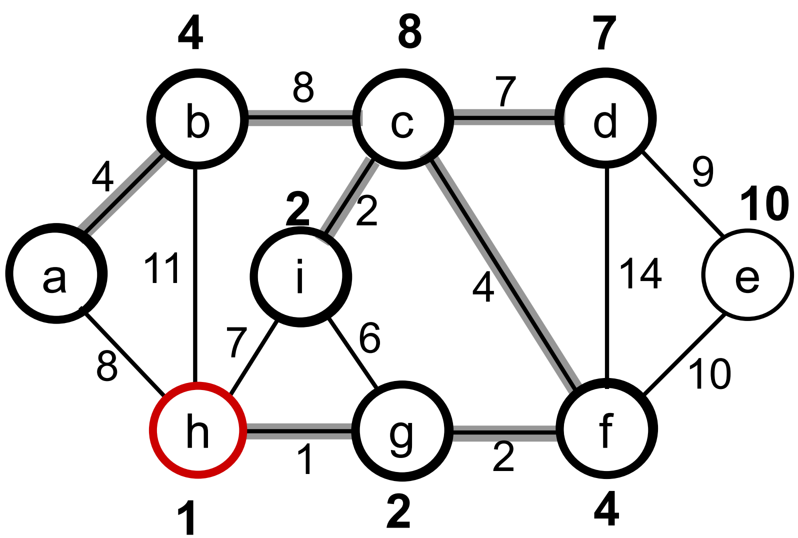


key [h] = 1 π [h] = g

7 10 1

Q = {d, e, h} V_A = {a, b, c, i, f, g}

Extract-MIN(Q) $\Rightarrow$ h

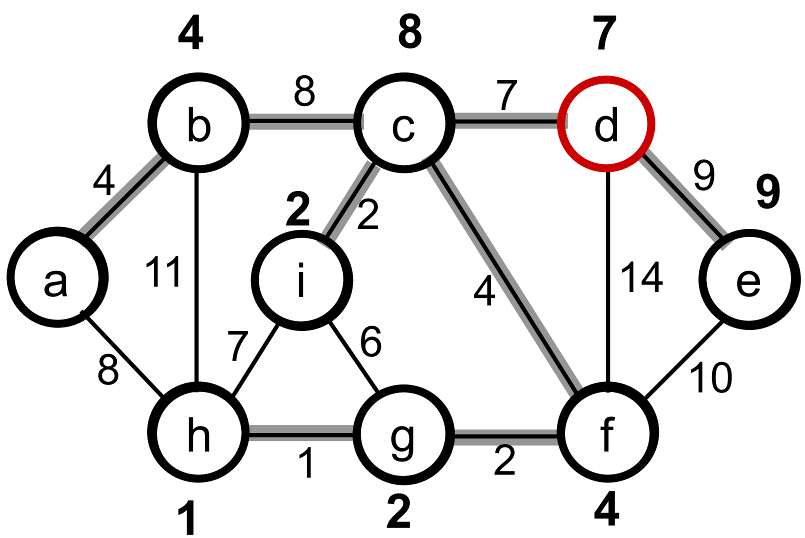


7 10

Q = {d, e} V_A = {a, b, c, i, f, g, h}

Extract-MIN(Q) $\Rightarrow$ d

Example



key [e] = 9 π [e] = d

9

Q = {e} V_A = {a, b, c, i, f, g, h, d}

Extract-MIN(Q) $\Rightarrow$ e

Q = $\varnothing$ V_A = {a, b, c, i, f, g, h, d, e}

PRIM(V, E, w, r)

Total time: $O(V\lg V + E\lg V) = O(E\lg V)$

```
Q ← ∅
for each u ∈ V
    do key[u] ← ∞
        π[u] ← NIL
        INSERT(Q, u)
DECREASE-KEY(Q, r, 0)        ▸ key[r] ← 0
while Q ≠ ∅
    do u ← EXTRACT-MIN(Q)
        for each v ∈ Adj[u]
            do if v ∈ Q and w(u, v) < key[v]
                then π[v] ← u
                    DECREASE-KEY(Q, v, w(u, v))
```

$O(V)$ if Q is implemented as a min-heap

$O(\lg V)$

Executed |V| times

Takes $O(\lg V)$

Min-heap operations: $O(V\lg V)$

Executed O(E) times total

Constant

Takes $O(\lg V)$

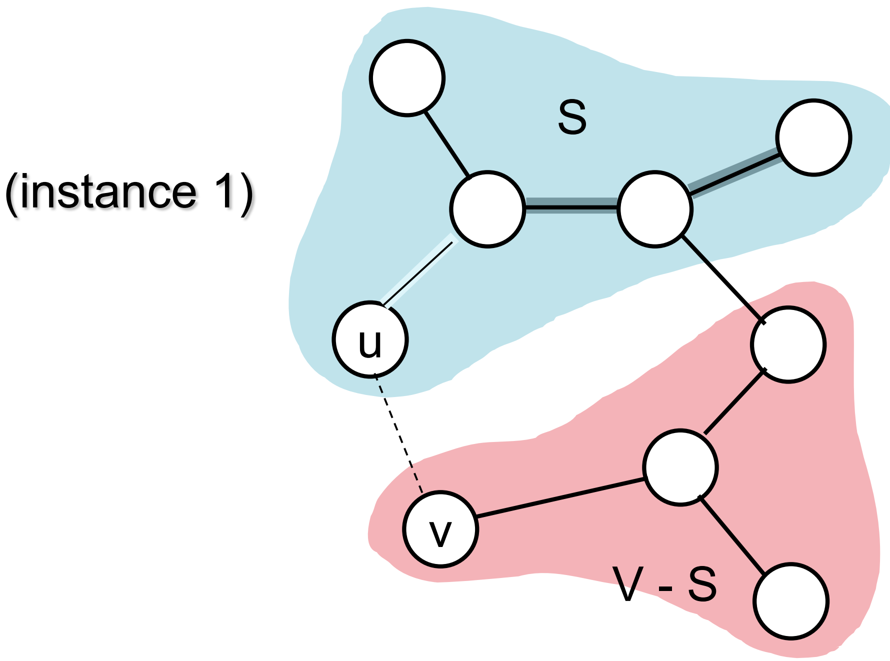
$O(E\lg V)$

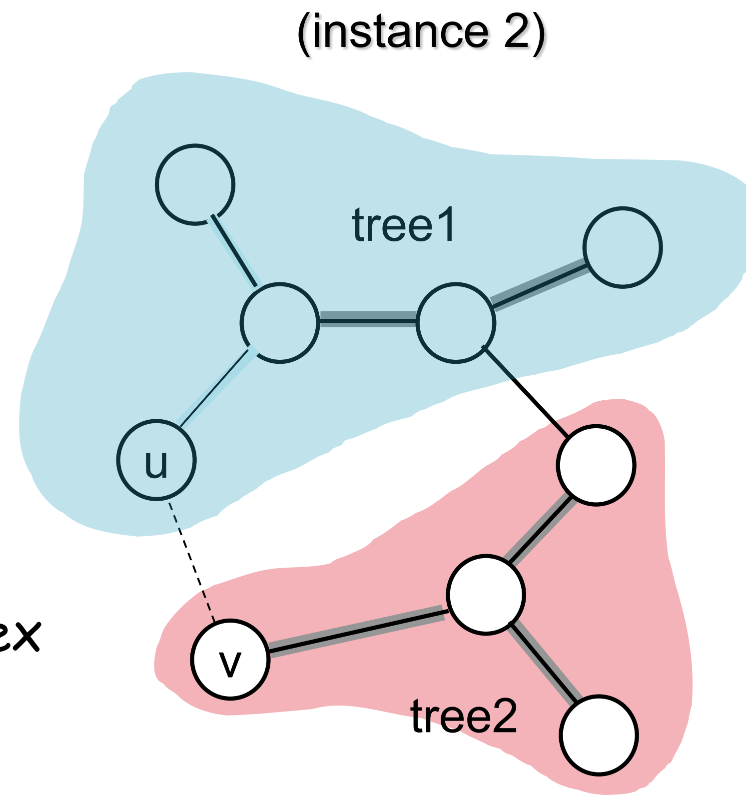
Prim's Algorithm

- Prim's algorithm is a **"greedy"** algorithm
  - Greedy algorithms find solutions based on a sequence of choices which are **"locally"** optimal at each step

- Nevertheless, Prim's greedy strategy produces a globally optimum solution!
  - See proof in previous slides

A different instance of the generic approach

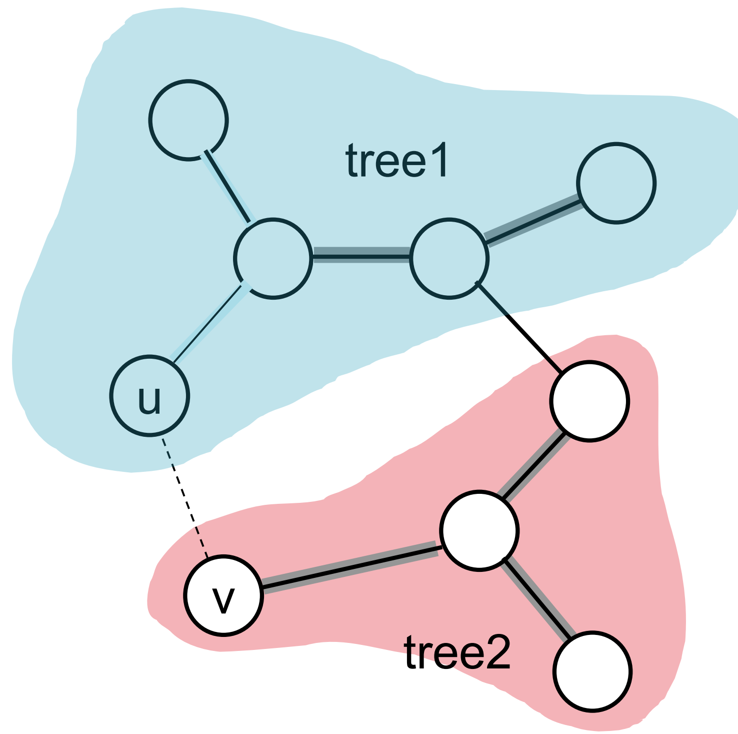




- *A* is a forest containing connected components
  - Initially, each component is a single vertex
- Any safe edge merges two of these components into one
  - Each component is a tree

Kruskal's Algorithm

- How is it different from Prim's algorithm?
  - Prim's algorithm grows one tree all the time
  - Kruskal's algorithm grows multiple trees (i.e., a forest) at the same time
  - Trees are merged together using **safe** edges

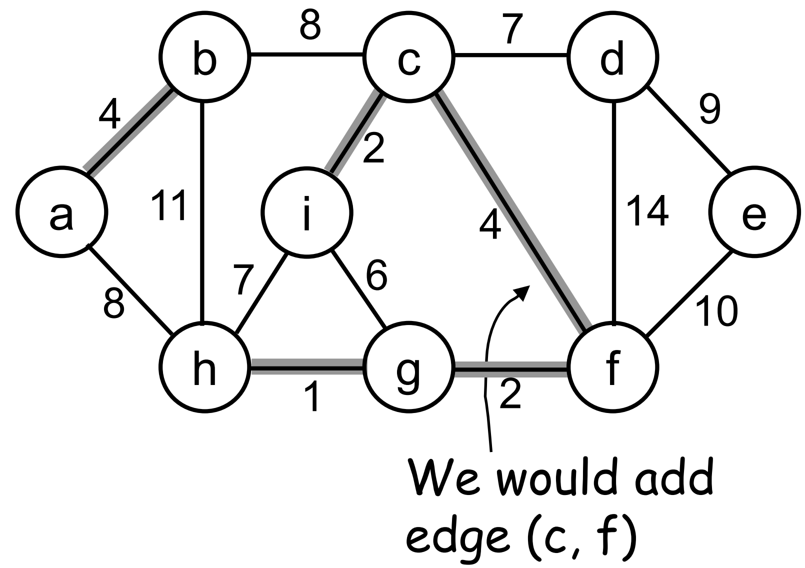


Kruskal's Algorithm

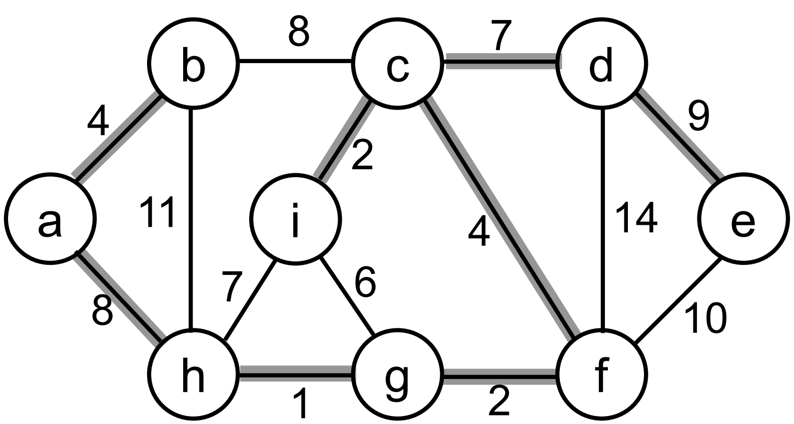
- Start with each vertex being its own component

- Repeatedly merge two components into one by choosing the **light** edge that connects them

- Which components to consider at each iteration?
  - Scan the set of edges in monotonically increasing order by weight



Example



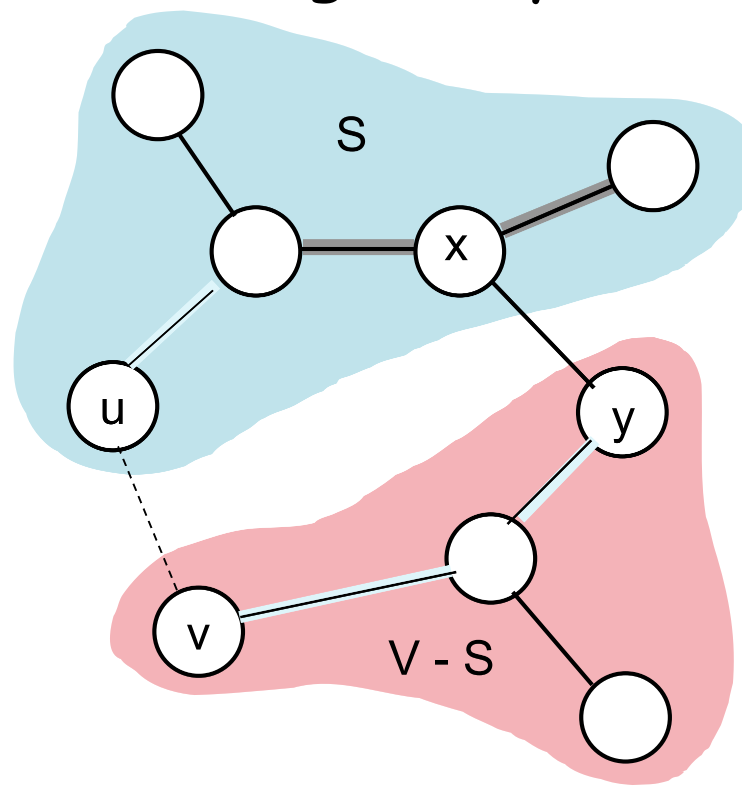
1: (h, g)
2: (c, i), (g, f)
4: (a, b), (c, f)
6: (i, g)
7: (c, d), (i, h)
8: (a, h), (b, c)
9: (d, e)
10: (e, f)
11: (b, h)
14: (d, f)

{a}, {b}, {c}, {d}, {e}, {f}, {g}, {h}, {i}

1. Add (h, g) {g, h}, {a}, {b}, {c}, {d}, {e}, {f}, {i}
2. Add (c, i) {g, h}, {c, i}, {a}, {b}, {d}, {e}, {f}
3. Add (g, f) {g, h, f}, {c, i}, {a}, {b}, {d}, {e}
4. Add (a, b) {g, h, f}, {c, i}, {a, b}, {d}, {e}
5. Add (c, f) {g, h, f, c, i}, {a, b}, {d}, {e}
6. Ignore (i, g) {g, h, f, c, i}, {a, b}, {d}, {e}
7. Add (c, d) {g, h, f, c, i, d}, {a, b}, {e}
8. Ignore (i, h) {g, h, f, c, i, d}, {a, b}, {e}
9. Add (a, h) {g, h, f, c, i, d, a, b}, {e}
10. Ignore (b, c) {g, h, f, c, i, d, a, b}, {e}
11. Add (d, e) {g, h, f, c, i, d, a, b, e}
12. Ignore (e, f) {g, h, f, c, i, d, a, b, e}
13. Ignore (b, h) {g, h, f, c, i, d, a, b, e}
14. Ignore (d, f) {g, h, f, c, i, d, a, b, e}

Kruskal's Algorithm

- Kruskal's algorithm is a **"greedy"** algorithm
- Kruskal's greedy strategy produces a globally optimum solution
- Proof for generic approach applies to Kruskal's algorithm too



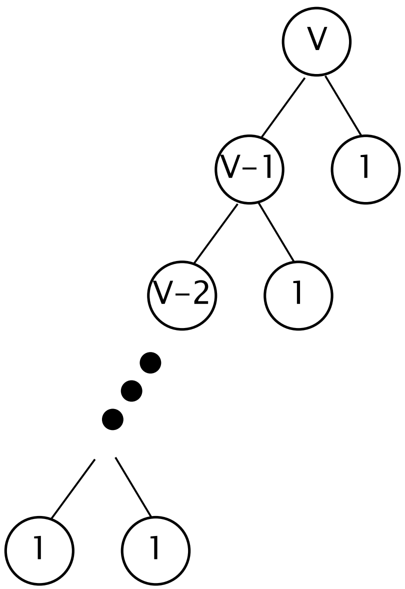
Kruskal(V, E, R)

Assume vertices are 1, 2, …, n, and E >= V

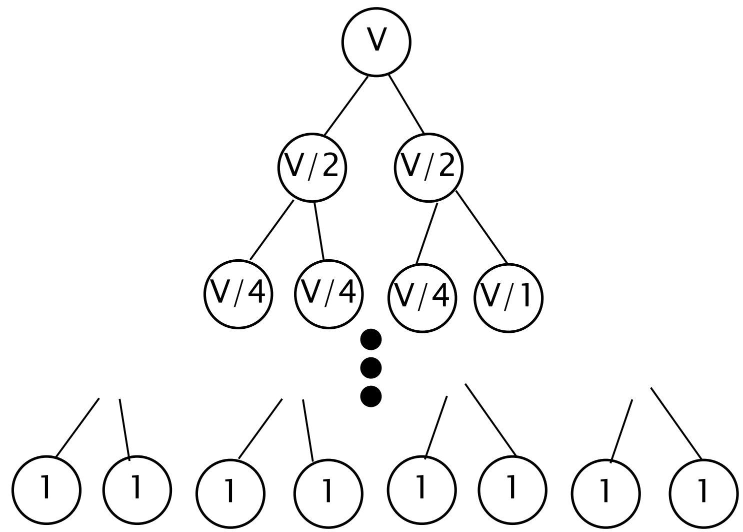
```
Sort all the edges  ←  O(ElogE)
for each v ∈ V
    label[v] = v                          O(V)
    setArray[v] = {v}
for each edge (u, v) ∈ E
    uL = label[u], vL = label[v]
    if uL == vL  continue                 O(E)
    R.add((u, v))
    if setArray[uL].size >= setArray[vL].size
        for each vertex w ∈ setArray[vL]
            label[w] = uL                 O(setArray[vL].size)
            setArray[uL].add(w)
    else                                                            ???
        for each vertex w ∈ setArray[uL]
            label[w] = vL                 O(setArray[uL].size)
            setArray[vL].add(w)
Output R
```

What's the total times of changing the labels for all the vertices?

- Case 1
  - Height: V-1
  - O(V)



- Case 2
  - Height: lgV
  - O(VlgV)



Consider a specific vertex v: if v's label is changed, then the updated set setArray[vL] will be at least twice larger than the original set setArray[vL]. Hence, the number of times for changing labels for v is at most O(lgV).

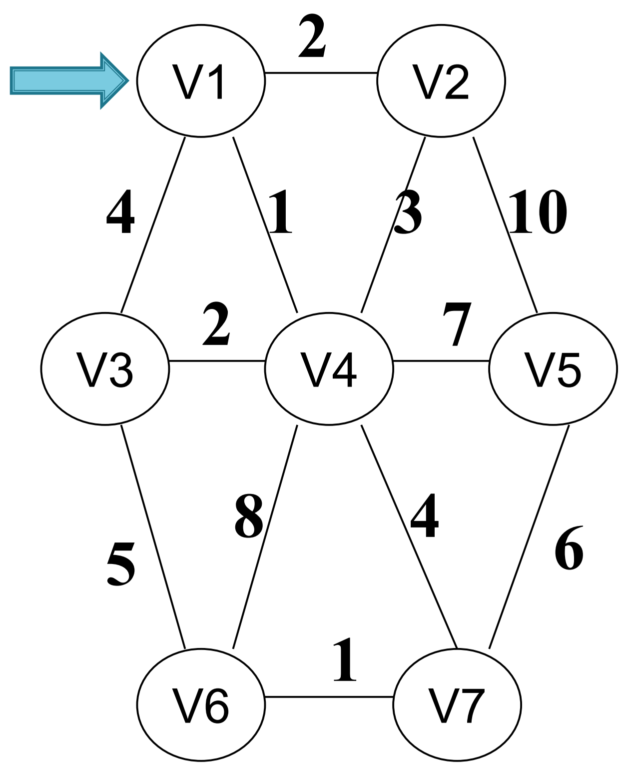
Kruskal(V, E, R)

Assume vertices are 1, 2, …, n, and E >= V

```
Sort all the edges  ←  O(ElogE)
for each v ∈ V
    label[v] = v                                O(V)
    setArray[v] = {v}
for each edge (u, v) ∈ E
    uL = label[u], vL = label[v]
    if uL == vL  continue                       O(E)
    R.add((u, v))
    if setArray[uL].size >= setArray[vL].size                 O(ElgE)
        for each vertex w ∈ setArray[vL]
            label[w] = uL
            setArray[uL].add(w)
    else                                        O(VlogV)
        for each vertex w ∈ setArray[uL]
            label[w] = vL
            setArray[vL].add(w)
Output R
```

Exercise 1: Find an MST

- Use Prim's algorithm
- Use Kruskal's algorithm



Exercise 2

Compare Prim's algorithm with and Kruskal's algorithm assuming:

(a) sparse graphs:

In this case, $E=O(V)$

Kruskal: $O(E\lg E)=O(V\lg V)$

Prim: $O(E\lg V)=O(V\lg V)$

(b) dense graphs:

In this case, $E=O(V^2)$

Kruskal: $O(E\lg E)=O(V^2\lg V^2)=O(2V^2\lg V)=O(V^2\lg V)$

Prim: $O(E\lg V)=O(V^2\lg V)$

Exercise 3

- Suppose that some of the weights in a connected graph G are negative. Will Prim's algorithm still work? What about Kruskal's algorithm? Justify your answers.
  - Yes, both algorithms will work with negative weights
  - Review the proof of the generic approach; there is no assumption in the proof about the weights being positive

Exercise 4

- Analyze Prim's algorithm assuming:

(a) an adjacency-list representation of G

(b) an adjacency-matrix representation of G

PRIM(V, E, w, r)

Total time: O(VlgV + ElgV) = O(ElgV)

```
Q ← ∅
for each u ∈ V
    do key[u] ← ∞
       π[u] ← NIL
       INSERT(Q, u)
DECREASE-KEY(Q, r, 0)          ▸ key[r] ← 0
while Q ≠ ∅
    do u ← EXTRACT-MIN(Q)
       for each v ∈ Adj[u]
           do if v ∈ Q and w(u, v) < key[v]
                 then π[v] ← u
                      DECREASE-KEY(Q, v, w(u, v))
```

- Lines 1–5: O(V) if Q is implemented as a min-heap
- DECREASE-KEY(Q, r, 0): O(lgV)
- while Q ≠ ∅: Executed |V| times
- EXTRACT-MIN(Q): Takes O(lgV)
- Min-heap operations: O(VlgV)
- for each v ∈ Adj[u]: Executed O(E) times
- if v ∈ Q and w(u, v) < key[v]: Constant
- DECREASE-KEY(Q, v, w(u, v)): Takes O(lgV)
- Lines 9–12: O(ElgV)

PRIM(V, E, w, r)

Total time: O(VlgV + ElgV+V²) = O(ElgV+V²)

```
Q ← ∅
for each u ∈ V
    do key[u] ← ∞
       π[u] ← NIL
       INSERT(Q, u)                       O(V) if Q is implemented as a min-heap
DECREASE-KEY(Q, r, 0)     ▸ key[r] ← 0  ←——  O(lgV)
while Q ≠ ∅               ←——  Executed |V| times    Min-heap
    do u ← EXTRACT-MIN(Q) ←——  Takes O(lgV)          operations: O(VlgV)
       for (j=0; j<|V|; j++)  ←——  Executed O(V²) times total
           if (A[u][j]=1)  ←——  Constant
               if v ∈ Q and w(u, v) < key[v]
                   then π[v] ← u           ←——  Takes O(lgV)    O(ElgV)
                        DECREASE-KEY(Q, v, w(u, v))
```

Exercise 5

- Find an algorithm for the "maximum" spanning tree. That is, given an undirected weighted graph G, find a spanning tree of G of maximum cost. Prove the correctness of your algorithm.
  - Consider choosing the "heaviest" edge (i.e., the edge associated with the largest weight) in a cut. The generic proof can be modified easily to show that this approach will work.
  - Alternatively, multiply the weights by -1 and apply either Prim's or Kruskal's algorithms without any modification at all!

Prim vs. Kruskal

- One begins with an edge while the other starts with a node
- One selects the next edge in order while the other does it from one node to another node
- One works on both connected and disconnected graphs while the other is mainly for connected graph
- One is good for sparse graph and the other is for dense graph
  - Depending on the implementation of data structure

Recommended Reading

- Reading materials
  - Textbook Chapter 23

- Next lecture
  - Shortest paths, Chapters 24&25