香港中文大學(深圳)
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CSC3100 Data Structures
Lecture 3: Insertion sort, merge sort

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Outline

- Use array to solve the sorting problem

- Insertion sort
  - Recursion
  - Algorithm analysis

- Merge sort
  - Divide and conquer
  - Algorithm analysis

Paradigms of algorithm design

The sorting problem

- Input: a sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$

- Output: a permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of input such that $a'_1 \le a'_2 \le \ldots \le a'_n$
  - Stored in arrays
  - The numbers are referred as keys

- Many sorting algorithms

Expressing algorithms

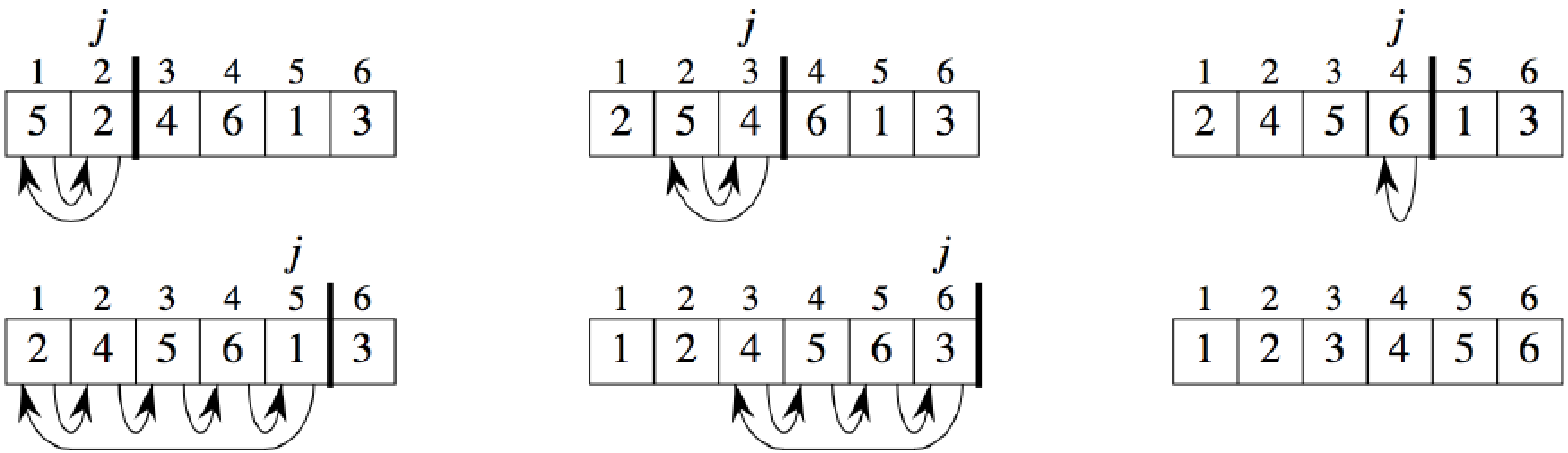
- Express algorithms in pseudocode
- Pseudocode is similar to C, C++, Python, and Java. Easy to understand
- Pseudocode contains English statements
- Analyze the complexity of pseudocode

Insertion sort

- A simple algorithm for a small number of elements

- Similar to sort a hand of playing card
  - Start with an empty left hand
  - Pick up one card and insert it into the correct position
  - To find the correct position, compare it with each of the cards in the hand, from right to left
  - The cards held in the left hand are sorted

Example of insertion sort



Insertion sort pseudocode

INSERTION-SORT(A)

for $j \leftarrow 2$ **to** n

    do $key \leftarrow A[j]$

        $\triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \,.\,.\, j-1]$.

        $i \leftarrow j - 1$

        while $i > 0$ and $A[i] > key$

            do $A[i+1] \leftarrow A[i]$

                $i \leftarrow i - 1$

        $A[i+1] \leftarrow key$

Correctness: loop invariant

- A property of the loop: loop invariant
  - Insertion sort: in each iteration, the array $A[1,\ldots,j-1]$ is sorted
- Help us prove the correctness of the algorithm
  - Initialization: true before the begin of loop
  - Maintenance: if true before an iteration, then also true after it
  - Termination: when the loop stops, use the invariant to show the algorithm is correct
- Similar to the mathematical induction

Correctness: loop invariant

INSERTION-SORT(A)

for $j \leftarrow 2$ **to** n Initialization

do $key \leftarrow A[j]$

M A I N T E N A N C E

$\triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.

$i \leftarrow j - 1$

while $i > 0$ and $A[i] > key$

do $A[i+1] \leftarrow A[i]$

$i \leftarrow i - 1$

$A[i+1] \leftarrow key$

Endfor

Termination

Loop invariant: insertion sort

- Proof:
  - Initialization: true before the begin of loop
    <u>Only one element A[1]</u>
  - Maintenance: true before an iteration and after it.
    <u>A[j] is in the correct position j' ⇔ A[j'-1]<= A[j']<= A[j'+1]</u>
  - Termination: when the loop stops, use the loop invariant to show the algorithm is correct
    <u>j = n when loop stops, A[1,…,j-1] is sorted</u>

Algorithm analysis: running time

- Random-access machine (RAM) model
  - Sequential and no concurrent operations
  - Operations taking a constant amount of time:
    - E.g., arithmetic, data movement, conditions, function all, etc.

How to analyze running time?

- Random-access machine (RAM) model
  - Sequential and no concurrent operations
  - Operations taking a constant amount of time:
    - E.g., arithmetic, data movement, conditions, function all, etc.
- For a given input, the time cost can be measured by the number of primitive operations (steps) executed
- Each line of pseudocode is composed of some numbers of operations and therefore requires a constant amount of time
  - One line may take a different amount of time than another

Analysis of insertion sort

INSERTION-SORT(A)	*cost*	*times*
for $j \leftarrow 2$ **to** n	c_1	n
do $key \leftarrow A[j]$	c_2	$n-1$
▷ Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0	$n-1$
$i \leftarrow j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
do $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^{n}(t_j - 1)$
$i \leftarrow i-1$	c_7	$\sum_{j=2}^{n}(t_j - 1)$
$A[i+1] \leftarrow key$	c_8	$n-1$

Analysis of insertion sort

- The running time of insertion sort

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) \\ &\quad + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) \, . \end{aligned}$$

- $T(n)$ depends on n and t_j

Analysis of insertion sort

- Best case: the array is sorted
  => t_j=1

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\ &= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) \,. \end{aligned}$$

- Worse case: the array is in reverse order
  => $t_j = j$

$$\sum_{j=2}^{n} j = \left(\sum_{j=1}^{n} j\right) - 1, \text{ it equals } \frac{n(n+1)}{2} - 1$$

Analysis of insertion sort

- Worse case (con't)

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) \\
&\quad + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\
&= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right) n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right) n \\
&\quad - (c_2 + c_4 + c_5 + c_8)\ .
\end{aligned}
$$

Can express $T(n)$ as $an^2 + bn + c$ for constants a, b, c (that again depend on statement costs) $\Rightarrow T(n)$ is a *quadratic function* of n.

Analysis of insertion sort

- Concentrate on the worst-case running time

- Reasons:
  - Give a guaranteed upper bound for any input
  - For some algorithms, the worst case occurs often
    - For example, search for absent items

- Why not analyze the average case?
  - Because it is often as bad as the worst case

Average case of Insertion sort

- On average, A[j] is less than half of A[1,…,j-1]
  => $t_j = j/2$

- The average case is about half of the worse case but still a quadratic of *n*

- Note: when comparing the complexity, we only keep the higher-order term
  - For example: n^2 vs $1000n+10000$

Recursion

- What is recursion?
  - self-reference
  - recursive function: based upon itself
  - Solution of the whole problem is composed of solutions of sub-problems

$$f(x) = 2f(x-1) + x^2$$

```
public int f( int x ){
        if ( x == 0 )
                return 0;
        else
                return 2 * f( x - 1 ) + x*x;
}
```

Recursion

- Characteristics of a recursive definition
  - It has a stopping point (base case)
  - It recursively evaluates an expression involving a variable *n* from a higher value to a lower value of *n*
  - Base case must be reached

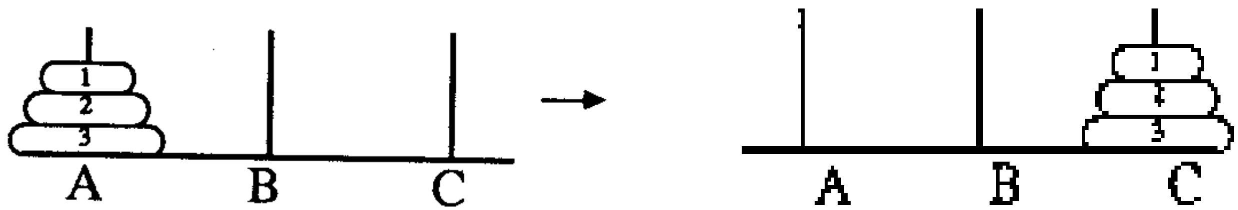
```
public static int bad (int N)
{
  if (N == 0)
       return 0;
  else
       return bad (N / 3 + 1) + N - 1;
}
```

Recursion: insertion sort

- Base Case: If array size is 1 or smaller, return
- Recursively sort first n-1 elements
- Insert last element at its correct position in sorted array

Recursion: Tower of Hanoi

- Problem:
  - It consists of three rods and a number of disks of different diameters, which can slide onto any rod



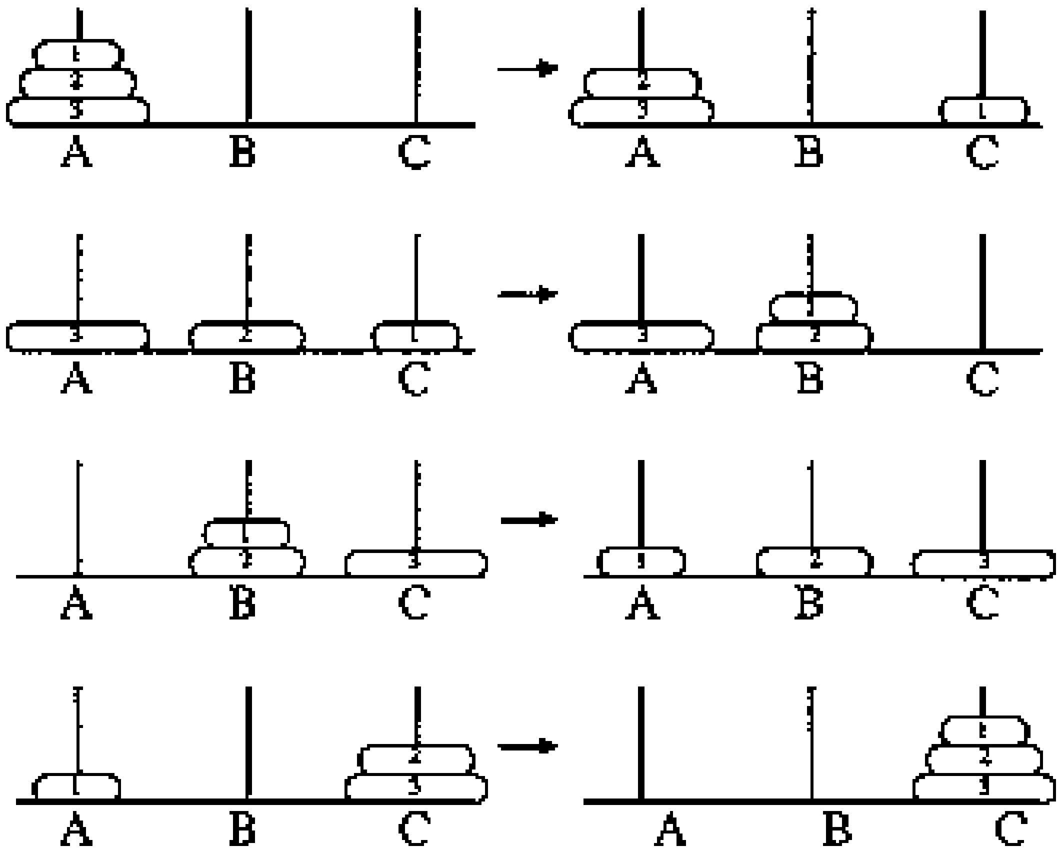
- Constraints:

(1) only one disk can be moved at a time, and

(2) at no time may a disk be placed on top of a smaller disk

Recursion: Tower of Hanoi

- N=3



Recursion: Tower of Hanoi

- Solution
  - If n = 1, move the single disk from A to C and stop;
  - Otherwise, move the top n-1 disks from A to B, using C as auxiliary,
  - Move the remaining disk from A to C,
  - Move the n-1 disks from B to C, using A as auxiliary

Recursion: Tower of Hanoi

```
void towers (int n, char frompeg, char topeg, char auxpeg)
{   /* If only one disk, make the move and return */
   if (n == 1)
         printf ("\n%s%c%s%c", "move disk 1 from peg ",
                    frompeg, " to peg ", topeg);
   else
   {      /* move top n-1 disks from A to B, C as auxiliary*/
          towers (n-1, frompeg, auxpeg, topeg);
          /* move remaining disk from A to C */
           printf ("\n%s%d%s%c%s%c", "move disk ",
                               n," from peg ", frompeg, " to peg ", topeg);
         /* move n-1 disks from B to C, A as auxiliary */
         towers (n-1, auxpeg, topeg, frompeg);
   }
}
```

Alternative sorting algorithm

- Many ways to sort
- Insertion sort is incremental: having sorted A[1,…,j-1], place A[j] correctly, so that A[1,…,j] is sorted.
- Another common approach: **divide and conquer**

Divide and conquer

- **Divide** the problem into a number of subproblems
- **Conquer** the subproblems by solving them recursively (further divide if not small enough)
  - **Base case**: If the subproblems are small enough, may solve them by brute force
- **Combine** the subproblem solutions to give a solution to the original problem

Merge sort

- A sorting algorithm based on divide and conquer
- Its worst-case running time has a lower order of growth rate than insertion sort
- Each subproblem is to sort a subarray A[p,...,r].
  - p=1, r=n at the start and changes during splitting

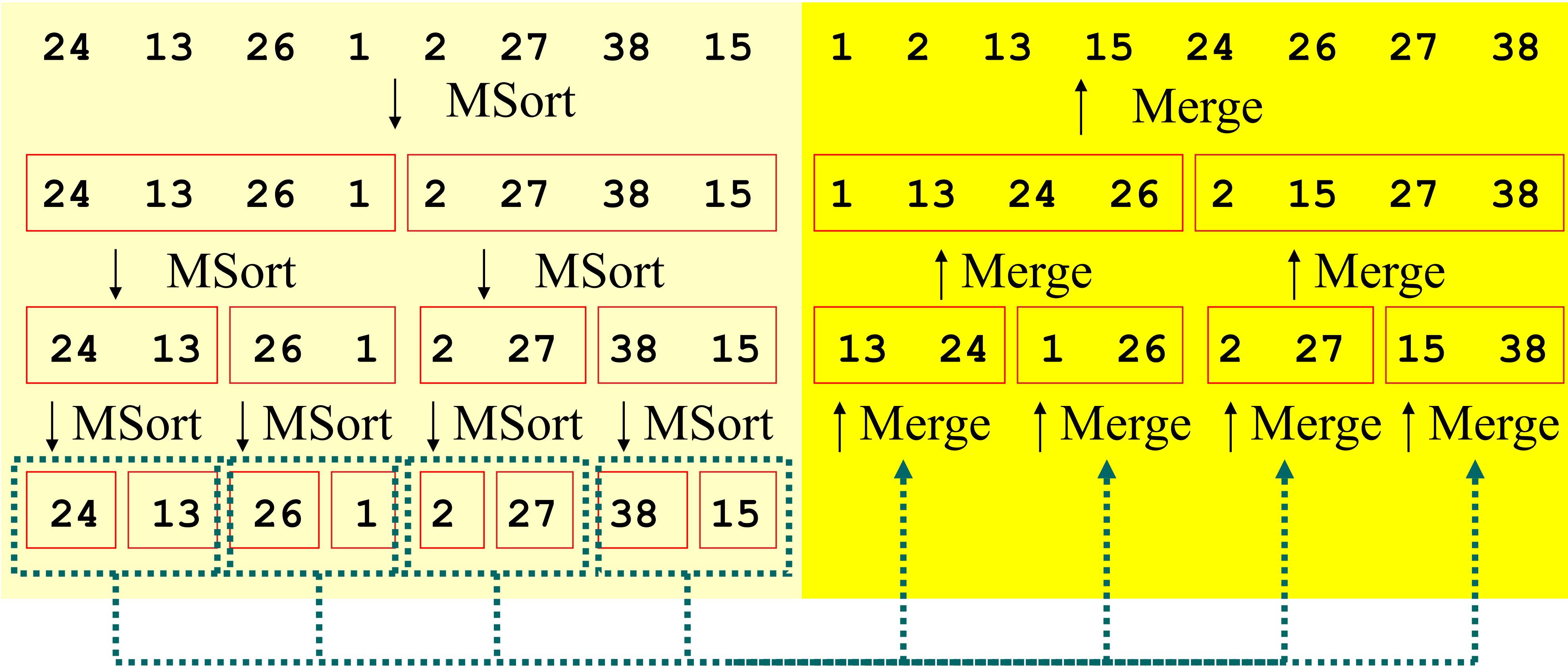
To sort A[p,…,r]

- Divide it into two subarrays A[p,…,q] and A[q+1,…,r], where q is the middle point
- Conquer by recursively sorting the two subarrays A[p,…,q] and A[q+1,…,r]
- Merge the two sorted subarrays A[p,…,q] and A[q+1,…,r]

Merge sort pseudocode

```
MERGE-SORT(A, p, r)
if p < r                              ▷ Check for base case
   then q ← ⌊(p + r)/2⌋               ▷ Divide
        MERGE-SORT(A, p, q)           ▷ Conquer
        MERGE-SORT(A, q + 1, r)       ▷ Conquer
        MERGE(A, p, q, r)             ▷ Combine
```

Example: $n = 8$



24 13 26 1 2 27 38 15
MSort
24 13 26 1
2 27 38 15
MSort
MSort
24 13
26 1
2 27
38 15
MSort
MSort
MSort
MSort
24
13
26
1
2
27
38
15
1 2 13 15 24 26 27 38
Merge
1 13 24 26
2 15 27 38
Merge
Merge
13 24
1 26
2 27
15 38
Merge
Merge
Merge
Merge

Merge

- Merge ordered subarray A[p,…,q] and ordered subarray A[q+1,…,r]

- How to efficiently implement it?
  - Think of two piles of cards.
  - Each pile is sorted and placed face-up on a table with the smallest cards on top.
  - We will merge them into a single sorted pile.
  - Basic idea
    - Choose the smaller of the two top cards
    - Remove it from its pile
    - Repeat

Merge: example

1 13 24 26 2 15 27 38

↑ ↑ ↑

Aptr *Bptr* *Cptr*

1 13 24 26 2 15 27 38 1

↑ ↑ ↑

Aptr *Bptr* *Cptr*

1 13 24 26 2 15 27 38 1 2

↑ ↑ ↑

Aptr *Bptr* *Cptr*

Merge: example

1 13 24 26   2 15 27 38   1 2 13

↑ ↑ ↑

Aptr   *Bptr*   *Cptr*

1 13 24 26   2 15 27 38   1 2 13 15

↑ ↑ ↑

Aptr   *Bptr*   *Cptr*

1 13 24 26   2 15 27 38   1 2 13 15 24

↑ ↑ ↑

Aptr   *Bptr*   *Cptr*

Implementation of merge sort

```
public static void mergeSort(int[] a) {
    int[] tmpArray = new int[a.length];
    mergeSort(a, tmpArray, 0, a.length - 1);
}

private static void mergeSort(int[] a, int[] tmpArray, int left, int right) {
    if (left < right) {
            int center = (left + right) / 2;
            mergeSort(a, tmpArray, left, center);
            mergeSort(a, tmpArray, center + 1, right);
            merge(a, tmpArray, left, center + 1, right);
    }
}
```

Implementation of merge sort

```
private static void merge(int[] a, int[] tmpArray, int leftPos, int rightPos, int rightEnd){
    int leftEnd = rightPos – 1, tmpPos = leftPos;
    int numElements = rightEnd - leftPos + 1;

    while (leftPos <= leftEnd && rightPos <= rightEnd)
            if (a[leftPos] <= a[rightPos])
                        tmpArray[tmpPos++] = a[leftPos++];
            else
                        tmpArray[tmpPos++] = a[rightPos++];

    while (leftPos <= leftEnd)
            tmpArray[tmpPos++] = a[leftPos++];

    while (rightPos <= rightEnd)
            tmpArray[tmpPos++] = a[rightPos++];

    for (int i = 0; i < numElements; i++, rightEnd--)
            a[rightEnd] = tmpArray[rightEnd];
}
```

Analyzing merge sort

- Suppose N is a power of 2

$$T(n) = \begin{cases} c & \text{if } n = 1\,, \\ 2T(n/2) + cn & \text{if } n > 1\,. \end{cases}$$

$$T(1) = C$$

$$T(N) = 2\,T(N/2) + CN$$

$$\frac{T(N)}{N} = \frac{T(N/2)}{N/2} + C = \ldots = \frac{T(1)}{1} + C\log N$$

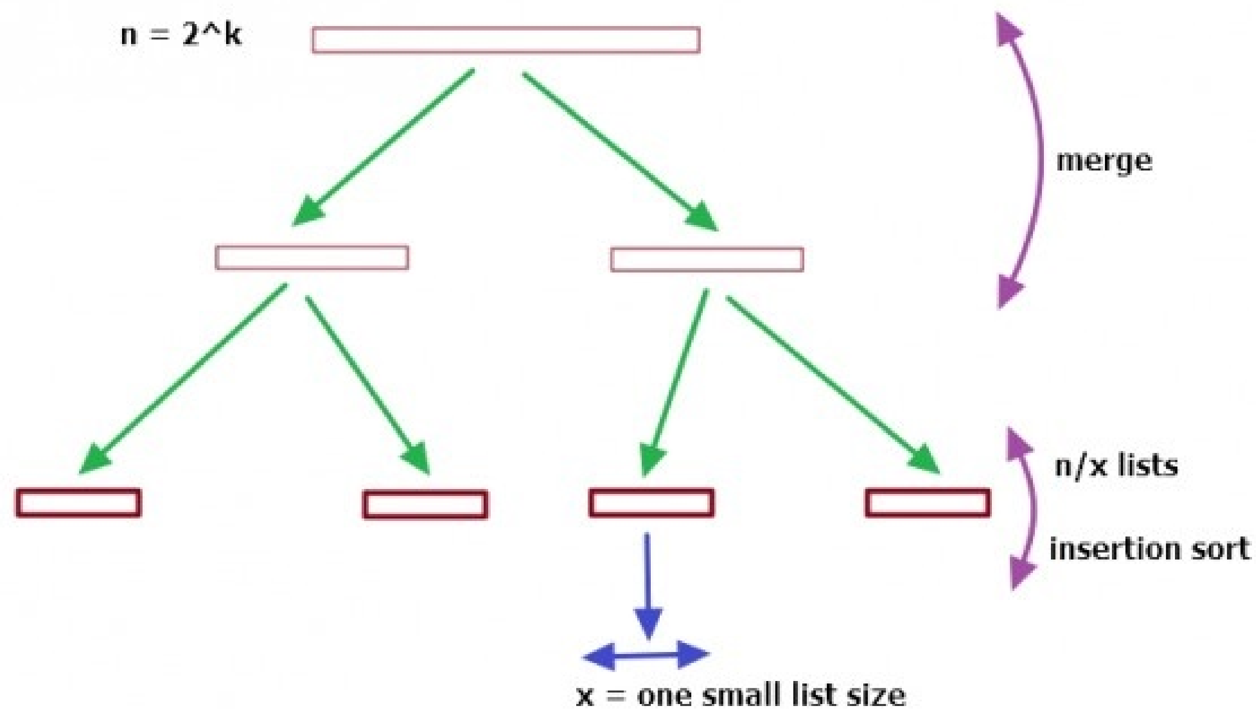
$$T(N) = CN\log N + CN = O(N\log N)$$

Compare with insertion sort

- Compared to insertion sort (worst-case time is a quadratic of n), merge sort is faster
- On small inputs, insertion sort may be faster, but for large enough inputs, merge sort will always be faster
- What is your thinking now?

Combining these two

- Exercise: Implement a hybrid sorting algorithm combining merge sort and selection sort



Recommended reading

- Reading this week
  - Chapter 2, textbook

- Next lecture:
  - Complexity analysis