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Unit-1

A) Errors and Mathematical Review

(lesser important section) 1 Question for 5 marks may be asked from A section.

Error and Types of errors in numerical computing:-

Error → Error is defined as the measure of the estimated difference between the observed or calculated value of a quantity and its true value.

In any numerical analysis errors will arise during the calculation. Errors may come in various forms and sizes among them some are avoidable and some are not. To establish good numerical computing system we need to minimize the errors as per our requirement.

Types of Errors:

i) **True Error** → True error is denoted by E_t, is the difference between the true value and the approximate value.

i.e, True Error = True value (OR Exact value) − Approximate value

Eg. The derivative formula for function $f(x)$ at a particular value of x can be approximately calculated by,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

ii) **Relative Error** → Relative Error is denoted by E_r and is defined as the ratio between the true error and the true value.

i.e, $\text{Relative true Error} = \frac{\text{True Error}}{\text{True Value}}$

E.g. true value = 9.514
true error = −0.75061 then,

$$\text{Relative error } (E_r) = \frac{\text{true error}}{\text{true value}}$$
$$= \frac{-0.75061}{9.514}$$
$$= -0.078895$$

Relative error can also be represented in percentage, for e.g.

$$E_r = -0.078895 \times 100\%$$
$$= -7.8895\%$$

Absolute errors may also need to be calculated in such case as;

$$|E_r| = |-0.078895|$$
$$= 0.078895 \text{ OR } 7.8895\%$$

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iii) Approximate error → True errors can be calculated only if true values are known but mostly we don't know the true values. In such cases we want to find approximate value. When we are solving a problem numerically, we may only have access to approximate value. It is denoted by E_a and is defined as the difference between the ~~prest~~ present approximation and previous approximation.

i.e, Approximate Error (E_a) = Present approx. − previous approx.

E.g. The deviation of function f(x) at a particular value of x can be calculated approximately by, $f'(x) = \frac{f(x+h) - f(x)}{h}$.

iv) Relative Approximate Error (E_{ra}) → Relative approximate error is denoted by E_{ra} and is defined as the ratio between the approximate error and the present approximation.

$$\text{i.e, } E_{ra} = \frac{\text{Approximate Error}}{\text{Present approximation}}$$

E.g. from previous example;

approximate error = −0.38474

present approximation = 9.87799

Then, the relative approximate error is calculated as;

$$E_{ra} = \frac{-0.38474}{9.87799}$$

$$= -0.03894$$

Similarly as before this can also be represented in percentage as $E_{ra} = -0.03894 \times 100\%$

$$= -3.894\%$$

& Absolute relative approximate error may also need to be calculated as; $|E_{ra}| = |-0.03894|$

$$= 0.03894 \text{ OR } 3.894\%$$

Q1. For $f(x) = 7e^{0.5x}$ and $h = 0.3$. Find the approximate value of $f'(x)$, at $x=2$, the true value of $f'(x)$ and the true error.

Solution: Approximate value of derivative is calculated by formula,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Now, $x = 2$ & $h = 0.3$

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$$f'(x) \approx \frac{f(2+0.3) - f(2)}{0.3}$$

$$\approx \frac{f(2.3) - f(2)}{0.3}$$

$$\approx \frac{7.e^{0.5\times 2.3} - 7e^{0.5\times 2}}{0.3}$$

$$\approx 10.256$$

$\therefore f'(2) \approx 10.256$

Now,

Exact value of $f'(x)$ at $x=2$ can be calculated by using our knowledge of differential calculus;

$$f(x) = 7e^{0.5x}$$

Then,

$$f'(x) = 7\times 0.5\times e^{0.5x}$$

So, $f'(2) = 7\times 0.5\times e^{0.5\times 2}$

$\therefore$ true error = true value − approximate value

$$= 9.514 - 10.256$$

$$E_t = -0.75061.$$

Q2. For $f(x) = 7e^{0.5x}$ and $x=2$. Find the following;

a) $f'(2)$ using $h = 0.3$

b) $f'(2)$ using $h = 0.15$

c) Approximate error for the value of $f'(2)$.

Solution:

The approximate exp^n for the derivative of function is;

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

a) for $x=2$ and $h=0.3$

$$f'(2) \approx \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5\times 2.3} - 7e^{0.5\times 2}}{0.3}$$

$$= 10.285$$

b) Simplarly $f'(2)$ at $x=2$ and $h=0.15$ is 9.8799.

c) The approximate error (E_a) is = Present approximation − Previous approximation

$$= 9.8799 - 10.265$$

$$= -0.38474$$

$$|E_a| = |-0.38474|$$

$$= 0.38474$$

lesser imp.

⊛. Numerical Computing, Process of Numerical Computing & Characteristics:

Numerical Computing → Numerical computing is an approach for solving complex mathematical problems using only simple arithmetic operations.

→ The approach involves formulation of mathematical models of physical situations that can be solved with arithmetic operations.

→ It requires development, analysis and use of algorithms.

→ Numerical Computing Methods deals with the following topics;

* finding the roots of equations.
* Solving system of linear algebraic equations.
* Interpolation and regression analysis.
* Numerical Integration.
* Numerical Differentiation etc.

Process of Numerical Computing:

→ Numerical Computing involves formulation of mathematical models of physical problems that can be solved by using basic mathematical operations.

→ The problem of ~~ne~~ numerical computing can be roughly divided into the following four phases.

a) Formulation of mathematical model.
b) Construction of approximate numerical method.
c) Implementation of method to obtain a solution
d) Validation of the solution.

Characteristics of Numerical Computing:

i) Accuracy → The results we obtain must be sufficiently accurate to serve the purpose for which mathematical model was build.

ii) Efficiency → A method that required less computing time and less programming effort and yet achieves the desired ~~ac~~ accuracy is always prefered.

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iii) Rate of convergence → Many numerical methods are based on the idea of an iterative process. This process involves generation of a sequence of approximations with the hope that the process will converge to the required solution. The certain method converge faster than others. Some methods may not converge at all. It is therefore important to test for convergence before a method is used.

iv) Numerical stability → In some cases errors tend to grow expotentially with disasterous computational result. A computing process that shows such exponential growth is said to be numerically unstable. We must choose methods that are not only fast but also stable.

⊛ Sources of Errors:

Errors in solving an engineering or science problem can arise due to several factors. First the errors may be in the modeling technique (i.e, A mathematical model). Second errors may arise from mistakes in programs themselves or in measurement of physical quantities. But in application of numerical methods, the main two errors to be focussed are;

i) Round off error:

→ A computer can only represent a number approximately.

→ Round off errors occur when a fixed number of digits are used to represent exact numbers.

→ For example, a number like $1/3$ may be represented as 0.33333 on PC. Then the round off error in this case is $\frac{1}{3} - 0.333333$ $= 0.00000033$.

→ There are other numbers like π and $\sqrt{2}$ that cannot be represented exactly.

ii) Truncation error:

→ Truncation error is defined as the error caused by truncating mathematical procedure.

→ For example, the Maclaurin series for e^x is given as;

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

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→ This series has infinite number of terms but when using this series to calculate e^x, only a finite number of terms can be used. For example, if one uses three terms to calculate e^x, then,

$$e^x \approx 1+x+\frac{x^2}{2!}.$$

Now the truncation error for such an approximation is,

$$\text{Truncation error} = e^x - \left(1+x+\frac{x^2}{2!}\right),$$

$$= \frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

⊛ Propagation of Errors:

→ If a calculation is made with numbers that are not exact, then the calculation itself will have an error.

→ Propagation of error is the effect of variables errors on the uncertainity of a function based on them.

→ When the variables are the values of experimental limitations which propagate to the combination of variables in the function.

Eg. Find the bounds for the propagation error in adding two numbers, if one is calculating $X+Y$ where, $X=1.5\pm0.05$ & $Y=3.4\pm0.04$

Soln

By looking at the numbers, the maximum possible value of X and Y are $X=1.55$ and $Y=3.44$.

The minimum possible values of X and Y are $X=1.45$ and $Y=3.66$

Hence,

$$X+Y = 1.45+3.36$$
$$= 4.81 \text{ is the max value of } X+Y$$

Hence,

$$4.81 \le X+Y \le 4.99$$

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Review of Taylor theorem:

Taylor's theorem is given by,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}+h^3+\dots$$

provided all derivatives of $f(x)$ exist and are continuous between x and $x+h$.

This can also be written as: $f(h+x) = f(h) + f'(h)x + \frac{f''(h)}{2!}x^2 + \frac{f'''(h)}{3!}x^3 + \dots$

Putting $h=0$ we get Maclurian series as;

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Example: Find the value of $e^{0.25}$ using the first five terms of Maclaurian series.

Solution: The first five terms of Maclaurian series for e^x can be calculated as.

$$f(x) = e^x, \quad \Rightarrow f(0) = 1$$
$$f'(x) = e^x, \quad \Rightarrow f'(0) = 1$$
$$f''(x) = e^x, \quad \Rightarrow f''(0) = 1$$
$$f'''(x) = e^x, \quad \Rightarrow f'''(0) = 1$$
$$f''''(x) = e^x, \quad \Rightarrow f''''(0) = 1.$$

Thus, from Maclurian series,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + \dots$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{0.25} \approx 1 + 0.25 + \frac{0.25^2}{2!} + \frac{0.25^3}{3!} + \frac{0.25^4}{4!}$$

$$e^{0.25} \approx 1.2840$$

The exact value of $e^{0.25}$ upto 5 significant digits is 1.2840.

Note: If $f(x) = \sin x \Rightarrow f(0) = 0$

then, $f'(x)$ will be $\cos x \Rightarrow f'(0) = 1$

$f''(x)$ will be $-\sin x. \Rightarrow f''(0) = 0$

$f'''(x)$ will be $-\cos x \Rightarrow f'''(0) = -1$

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⊛. Mean Value Theorem (Review)

Suppose f is a continuous on $[a,b]$ and differentiable on (a,b). Draw the secant line joining the point $(a,f(a))$ to point $(b,f(b))$. Clearly there's a point c where $a<c<b$ such that the tangent line to the graph of f at $x=c$ is parallel to the secant line & has same slope. The slope of the secant line is $\frac{f(b)-f(a)}{b-a}$ and that of the tangent line is $f'(c)$.

$$\boxed{\text{So, } f'(c) = \frac{f(b)-f(a)}{b-a}}$$

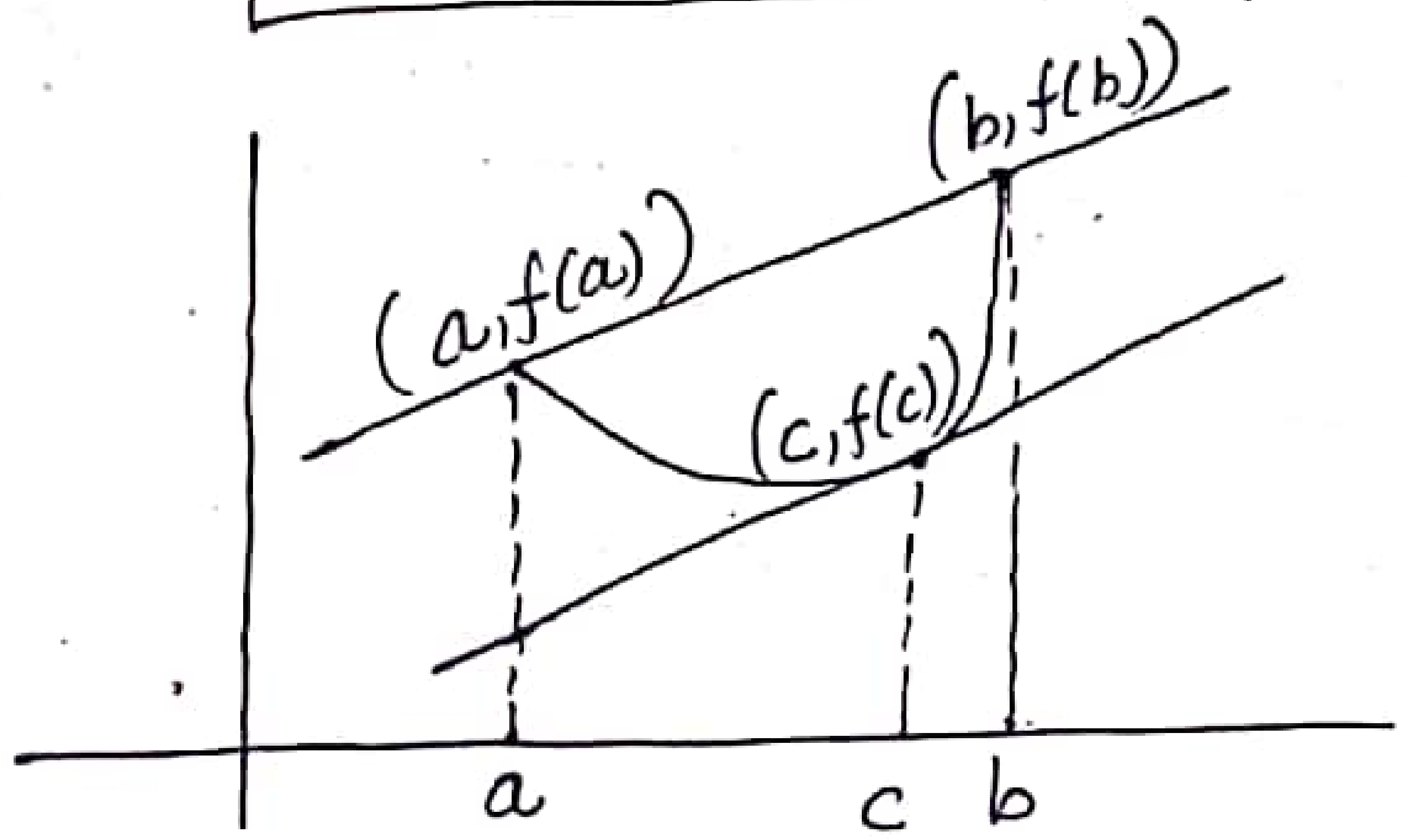

fig. Graphical Interpretation of Mean Value Theorem.

Example: Consider $f(x) = x^3 + 3x^2$ on the interval $-5 \leq x \leq 1$.

Since f is a polynomial, f is continuous on $-5 \leq x \leq 1$ and differentiable on $-5 < x < 1$.

$$\frac{f(1) - f(-5)}{1-(-5)} = \frac{4+50}{6} = 9.$$

So, we should be able to find a number c between -5 and 1 such that $f'(c) = 9$.

$$f'(x) = 3x^2 + 6x, \text{ So, } f'(c) = 3c^2 + 6c.$$

Now, we set $f'(c)$ equal to 9 and solve for c, we get.

$$3c^2 + 6c = 9$$
$$\Rightarrow c^2 + 2c = 3$$
$$\Rightarrow c^2 + 2c - 3 = 0$$
$$\Rightarrow c = -3 \text{ or } c = 1.$$

$c=1$ is not in the interval $-5<x<1$, but $c=-3$ is. Therefore $c=-3$ is a number satisfying the conclusion of MVT.

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Algorithm Analysis:

Complexity analysis of an algorithm is very hard if we try to analyze exact. We know that complexity of an algorithm is the mathematical function of the size of the input. So if we analyze the algorithm in terms of bound (upper and lower) then it would be eaiser. For this purpose we need the concept of following asympotatic notations.

i) **Big Oh (O) notation** → When we have only asymptotic upper bound then we use O notation. A function $f(x) = O(g(x))$ [we read it as $f(x)$ is bigh oh of $g(x)$] iff there exists two constants c and k such that for all $x >= k$, $f(x) <= c * g(x)$. This relation says that $g(x)$ is upper bound of $f(x)$.

ii) **Big Omega (Ω) notation** → Big omega gives asymptotic lower bound. A function $f(x) = \Omega(g(x))$ [read as $f(x)$ is big omega of $g(x)$] iff there exists two positive constants c and k such that for all $x >= k$, $c * g(x) <= f(x)$. This relation says that $g(x)$ is lower bound of $f(x)$.

iii) **Big Theta (Θ) notation** → When we need asympotically tight bound then we use Big-Θ. A function $f(x) = \Theta(g(x))$ [read as $f(x)$ is big theta of $g(x)$] iff there exists three positive constants c1, c2 and k such that for all $x >= k$, $c1 * g(x) <= f(x) <= c2 * g(x)$. This relation shows that $f(x)$ is order of $g(x)$.

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B). Solving Non-Linear Equations: (This onwards important)

Any function of one variable which does not graph as a straight line in two dimensions or any function of two variables which does not graph as a plane in three dimensions can be said to be non-linear.

Types of non-linear equation:

i) Algebraic equation → An eqn of type $y = f(x)$ is said to be algebraic if it can be expressed in the form;

$$f_n x_n + f_{n-1} x_{n-1} + f_{n-2} x_{n-2} + \ldots + f_1 x_1 + f_0 = 0$$

E.g.

$$3x + 5y - 21 = 0 \text{ (linear)}$$
$$2x + 5xy - 25 = 0 \text{ (non-linear)}$$
$$x^3 + xy - 3y^3 = 0 \text{ (non-linear)}$$

ii) Polynomial equations → Polynomial equations are a simple class of algebraic equations that are represented as follows:-

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0.$$

e.g. $5x^5 + x^3 + 2x^2 + 5 = 0.$

iii) Transcendental equation → A non-algebraic equation is called transcendental equation. These include, trignometric, exponential and logarithmic function.

Eg.

$$2\sin x - x = 0$$
$$x - e^{1/x} = 0$$
$$\log x^2 - 1 = 0 \text{ etc.}$$

Methods of solving non-linear equations:-

There are number of ways to find roots of non-linear equations. They include.

Not imp:
- i) Direct analytical method.
- ii) Graphical method
- iii) Trial and Error Method

Imp: iv) Iterative methods [Bisection method, Newton Raphson method, Secant method & Fixed-point method].

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Direct Analytical Method:

In this method equations are solved by using derived formulas. We just need to apply the value of input parameters in the equation.

Eg. Solution of algebraic equation $ax^2+bx+c=0$ is given by the formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Graphical Method:

This is the simplest method to determine the root of an equation $f(x)=0$. The procedure is quite straight forward as;

→ Plot the function $f(x)$

→ Observe when it crosses the x-axis, this point represent the value for which $f(x)=0$.

Trial & Error Method:

→ In this method we make a series of guesses for x & then evaluate $f(x)$ at that x. If it is close to zero, it is one of the approximate root of given non-linear equation. Otherwise we make another guess for x and repeat the same process again, until the x for which $f(x)=0$ is found.

Iterative Methods;

1) Bisection Method:

This method is also known as half-interval method. It is based on the fact that if $f(x)$ is real and continious in the interval $a<x<b$ and $f(a)$ and $f(b)$ are of opposite sign. i.e, $f(a).f(b)<0$. Then there is at least one real root in the interval between a & b.

Let $x_1=a$ & $x_2=b$. Let us define another point x_m to be the mid-point between a & b. i.e, $x_m = \frac{x_1+x_2}{2}$

Now there exist the following conditions;-

* If $f(x_m)=0$, we have a root at x_m.
* If $f(x_1).f(x_m)<0$, then there is a root between x_1 & x_m.
* If $f(x_2).f(x_m)<0$, then there is a root between x_2 & x_m.

By testing the sign of the function at mid-point, we decide which part of the interval contains the root. This can be illustrated in the figure as follows;-

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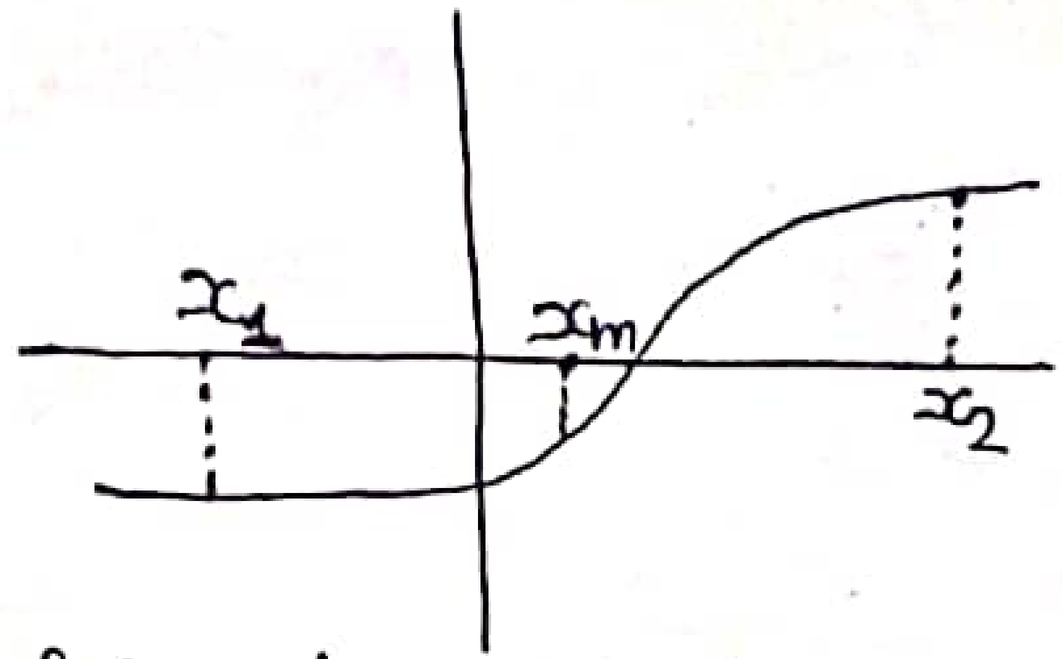

Advantages of Bisection Method:

i) The bisection method is always convergent since the method brackets the roots.

ii) As iterations are conducted, the interval gets halved. So one can guarantee the decrease in error in the solution of the equation.

Drawbacks of Bisection Method:

i) The convergence of bisection method is slow as it is simply based on halving the interval.

(Write algorithm before this I have written few pages later)

ii) If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.

Q. Using the bisection method solve $x^2 - 4\cos x = 0$, correct up to 2 significant figures.

Solution:- Let $f(x) = x^2 - 4\cos x$.

Assume initial guess as $x_l = 1$ and $x_u = 2$.

Iteration 1

$x_l = 1$ $\qquad$ $x_u = 2$ $\qquad$ $\left(\because x_m = \dfrac{x_l + x_u}{2}\right)$

$f(x_l) = -1.161$ $\qquad$ $f(x_u) = 5.664$

$x_m = 1.5$ $\qquad$ $f(x_m) = 1.97$

where x_l and x_u are initial guess.

since, $f(x_l) \times f(x_m) < 0$

so, $x_u = x_m = 1.5$

Error $= 0.500$ $\qquad$ $\left(\because \text{Error} = \left|\dfrac{x_l - x_u}{x_u}\right|\right)$

Iteration 2

$x_l = 1$ $\qquad$ $x_u = 1.5$

$f(x_l) = -1.161$ $\qquad$ $f(x_u) = 1.97$

$x_m = 1.25$ $\qquad$ $f(x_m) = 0.301$

Since, $f(x_l) \times f(x_m) < 0$

so, $x_u = x_m = 1.25$

Error $= 0.333$

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Iteration 3

$x_l = 1$ $\quad$ $x_u = 1.25$

$f(x_l) = -1.161$ $\quad$ $f(x_u) = 0.301$

$x_m = 1.125$ $\quad$ $f x_m = -0.459$

Since, $f(x_u) \times f(x_m) < 0$

$x_l = x_m = 1.125$

Error $= 0.200$

Continuing the process in table below;

Iteration	x_l	x_u	x_m	$f(x_l)$	$f(x_u)$	$f(x_m)$	Error
1	1	2	1.5	-1.161	5.665	1.967	0.500
2	1	1.5	1.25	-1.161	1.967	0.301	0.333
3	1	1.25	1.125	-1.161	0.301	-0.459	0.200
4	1.125	1.25	1.188	-0.459	0.301	-0.086	0.100
5	1.188	1.25	1.219	-0.086	0.301	0.106	0.050
6	1.188	1.219	1.203	-0.086	0.106	0.010	0.026
7	1.188	1.203	1.195	-0.086	0.010	-0.038	0.013
8	1.195	1.203	1.199	-0.038	0.010	-0.014	0.006

Thus, root $= 1.199$

Convergence of Bisection Method

In bisection method interval is halved in every interval.
After nth iteration, size of iteration is reduced to;

$$\Delta_n = \frac{(x_u - x_l)}{2^n}$$

Now we can say that maximum error after nth iteration is $E_n = \pm \Delta_n$.

$$\Rightarrow |E_n| = \frac{(x_u - x_l)}{2^n}$$

Similarly, after $(n+1)^{th}$ iteration maximum error is given by,

$$E_{n+1} = \frac{(x_u - x_l)}{2^{n+1}}$$

$$E_{n+1} = \frac{(x_u - x_l)}{2^n \times 2}$$

$$E_{n+1} = \frac{E_n}{2}$$

This shows that error is halved after each iteration therefore we can say that bisection method converges linearly.

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2) Newton Raphson Method:

Newton-Raphson method is based on the principle that if the initial guess of the root of $f(x)=0$ is at x_i, and if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root.

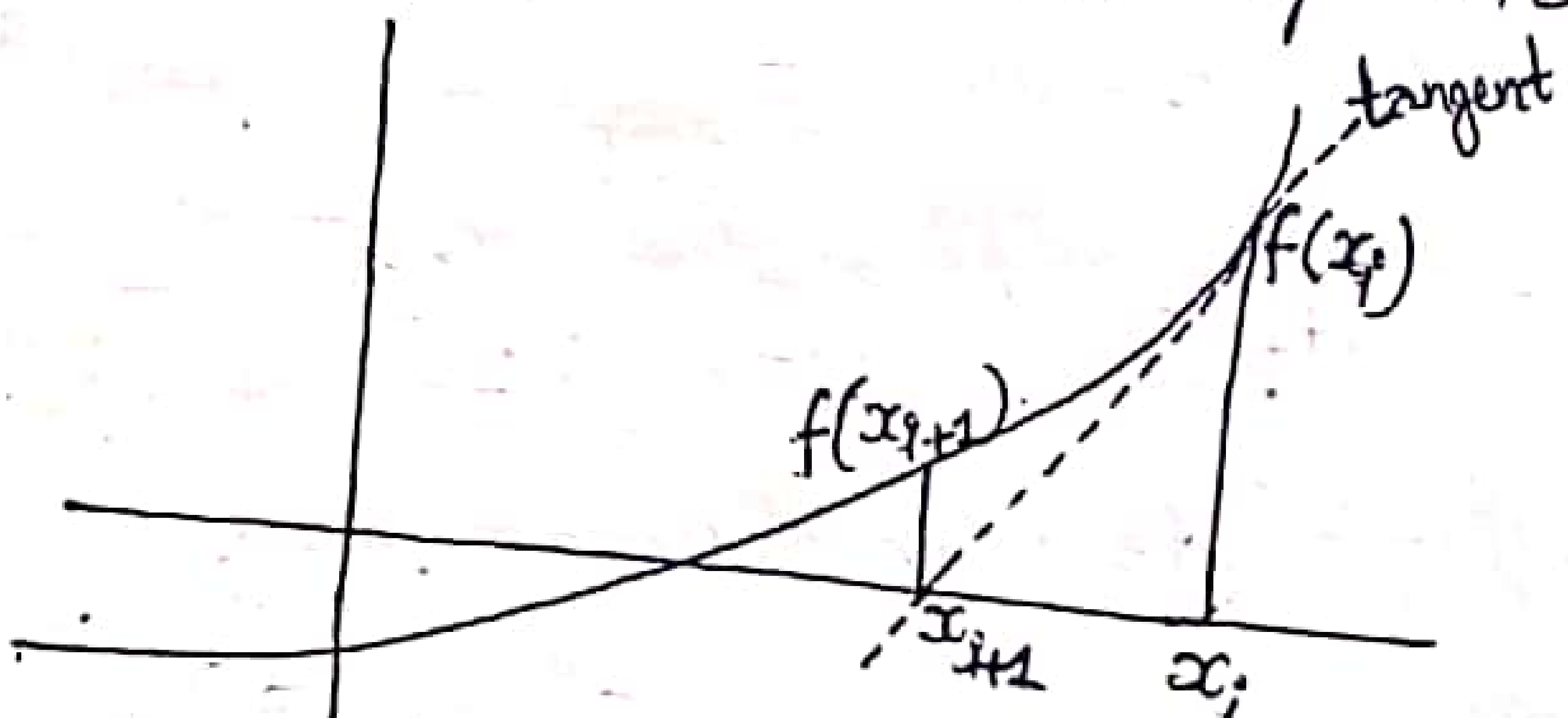

fig. Geometrical interpretation of Newton-Raphson Method.

Using the definition of the slope of a function, at $x=x_i$.

$$f'(x_i) = \frac{f(x_i)-f(x_{i+1})}{x_i - x_{i+1}}$$

If $f(x)$ have root at x_{i+1}, then $f(x_{i+1})=0$. Putting this in above equation, we get

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

This gives, $\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$

This equation is called the Newton-Raphson formula for solving non-linear equations of the form $f(x)=0$. So starting with an initial guess x_i, we find the next guess x_{i+1}, by using the above equation. We can repeat this process until the root within a desirable tolerance is found.

Deriving Newton Raphson formula from Taylors Series

Let h is an small increment in x_n to get next estimate of root

i.e, $x_{n+1} = x_n + h$ — (i)

or, $h = x_{n+1} - x_n$ — (ii)

We know that Taylor's series can be written as;

$$f(x+h) = f(x) + \frac{f'(x)h}{1!} + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \dots$$

Since h is very small, we can neglect the terms containing second and higher order terms. Then, above equation can be written as:

$$f(x+h) = f(x) + \frac{f'(x)\,h}{1!}$$

$$\text{or, } f(x_n+h) = f(x_n) + \frac{f'(x_n)\,h}{1!}$$

from (i)
$x_{n+1} = x_n + h$

value of h from (i)

$$\text{or, } f(x_{n+1}) = f(x_n) + \frac{f'(x_n)\cdot(x_{n+1}-x_n)}{1!} \quad \text{— (iii)}$$

If x_{n+1} is root of the given polynomial then,

$$f(x_{n+1}) = 0$$

from iii

$$\Rightarrow \frac{f(x_n) + f'(x_n)(x_{n+1}-x_n)}{1!} = 0$$

Solving above equation to get the value of x_{n+1} we get,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

It will be better to understand if we write algorithm before solving question of each method which are after few pages (Algorithm).

This equation is called Newton Raphson formula.

Q. Solve the equation $x^2+4x-9=0$ by using Newton–Raphson method. The solution correct upto 4 significant digit.

Soln

$f(x) = x^2+4x-9 = 0.$

$f'(x) = 2x+4$

Let us assume initial guess is 4.

Iteration 1.

$x_0 = 4$

$f(x) = 23$ & $f'(x) = 12$

$x_1 = 4 - \frac{23}{12} = 2.0833$ (Using formula)

$$\text{Error} = \left|\frac{(x_1 - x_0)}{x_1}\right|$$

$$= \left|\frac{2.08-4}{2.08}\right|$$

$$= 0.920$$

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Iteration 2

$x_0 = 2.0833$
$f(x) = 3.678$
$f'(x) = 8.166$
$x_1 = 1.633$
Error $= 0.275$

Continuing above process in table to get solution as;

Iteration	x	$f(x)$	$f'(x)$	Error
1.	4	23	12	
2	2.0833	3.678	8.166	0.920
3	1.633	0.202	7.267	0.275
4	1.606	0.001	7.211	0.017
5	1.606	0.000	7.211	0.000

Thus, root $= 1.606$.

Drawbacks of Newton-Raphson Method.

i) Division by zero → Consider a function $f(x) = 1 - x^2$. ~~has~~ It has a maximum at $x=0$ and solutions of $f(x)=0$ at $x=\pm 1$. If we start iterating from the $x_0 = 0$ (where the derivative is zero), x_1 will be undefined,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 0 - \frac{1}{0}$$
$$= \text{undefined.}$$

→ The same issue occurs if, ~~insted~~ instead of starting point, any iteration point is stationary.

→ Even if the derivative is small but not zero, the next ~~iteration~~ will be a far worse approximation.

ii) Oscillations near local maximum and minimum → For some functions, some starting points may enter an infinite cycle, preventing convergence. ~~Let $f(x) = x^3 - 2x + 2$ and take 0 as starting point~~ In some cases the sequence of iteration will alternate between certain points without converging to a root.

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Convergence of Newton-Raphson Method:

Suppose x_r is root of $f(x)=0$ and x_n is estimated root of $f(x)=0$ such that $|x_r - x_n| = \delta << 1$. Then by Taylor's Series expansion:

$$f(x_r) = f(x_n+\delta) = f(x_n) + f'(x_n)(x_r - x_n) + R_1 \cdots (i)$$

where, R_1 is Lagrange form of the Taylor series expansion remainder and is given by the equation,

$$R_1 = \frac{1}{2!} f''(\mathcal{E})(x_r - x_n)^2$$

where, $\mathcal{E}$ is in between x_n and x_r. Since x_r is the root, eq^n (i), becomes;

$$0 = f(x_n) + f'(x_n)(x_r - x_n) + \frac{1}{2!} f''(\mathcal{E})(x_r - x_n)^2 \text{ —— } (ii)$$

Dividing equation (ii) by $f'(x_n)$ and rearranging gives,

$$\frac{f(x_n)}{f'(x_n)} + (x_r - x_n) = -\frac{f''(\mathcal{E})}{2f'(x_n)}(x_r - x_n)^2 \text{ —— } (iii)$$

We know that according to Newton Raphson formula x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Thus eq^n (iii) can be written as;

$$x_r - x_{n+1} = -\frac{f''(\mathcal{E})}{2f'(x_n)}(x_r - x_n)^2 \text{ —— } (iv).$$

Let $E_n = x_r - x_n$ and $E_{n+1} = x_r - x_{n+1}$, Eq^n (iv) can be written as;

$$E_{n+1} = -\frac{f''(\mathcal{E})}{2f'(x_n)} \cdot E_n^2$$

Taking absolute value on both sides gives;

$$|E_{n+1}| = \frac{|f''(\mathcal{E})|}{2|f'(x_n)|} E_n^2 \text{ —— } (v).$$

Eq^n (v) shows that Newton-Raphson method have quadratic rate of convergence.

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3) Secant Method:

This method uses two initial estimates but does not require to bracket the root. Geometrical representation of secant method is as shown in the figure below:-

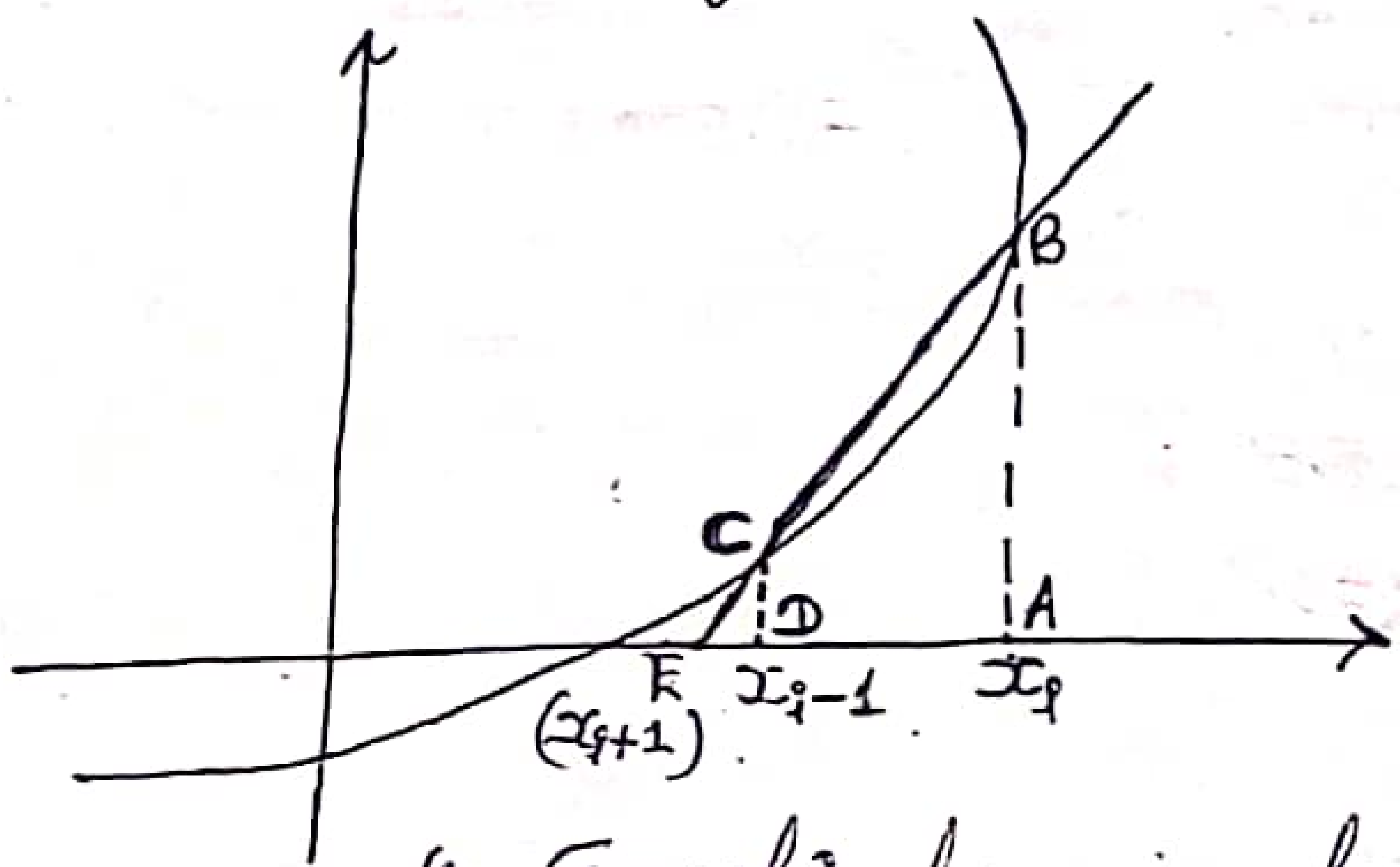

fig. Geometrical representation of the secant method

Let us consider two initial guesses x_i and x_{i-1} and draw a straight line between $f(x_i)$ and $f(x_{i-1})$ passing through the X-axis at x_{i+1}. Now, ABE and DCE are similar triangles and therefore following relation holds;

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\text{or, } \frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

Rearranging the terms we get,

$$x_{i+1} = \frac{f(x_{i-1})x_i - f(x_i)x_{i-1}}{f(x_{i-1}) - f(x_i)}$$

Adding and subtracting $f(x_i)x_i$ from numerator, we get.

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$ This equation is called secant formula.

Q. Solve the equation $2x^2+4x-10=0$ with error precision 0.04

Soln

Let us assume that initial guess $x_1 = 2$ & $x_2 = 6$.

& Specified error precision $E = 0.04$.

Iteration 1:

$x_1 = 2 \quad f(x_1) = 6$

$x_2 = 6 \quad f(x_2) = 86$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$x_3 = 1.7 \quad \text{Error} = \frac{x_3 - x_2}{x_2} = \frac{1.7 - 6}{1.7} = 2.529$

Iteration 2:

$x_1 = 6 \quad f(x_1) = 86$

$x_2 = 1.7 \quad f(1.7) = 2.58$

$x_3 = 1.568 \quad \text{Error} = 0.828$

Iteration 3:

$x_1 = 1.7 \quad f(x_1) = 2.58$

$x_2 = 1.568 \quad f(x_2) = 1.189$

$x_3 = 1.4551 \quad \text{Error} = 0.0775.$

Continuing above solution in table as;

Iteration	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	Error.
4	1.568	1.4551	1.4496	1.189	0.055	0.003

Thus the root is 1.4496.

Convergence of Secant Method:

The secant formula is given by;

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{--- (i)}$$

Let x_r be the actual root of $f(x)$ and e_i be the error estimate in ith iteration then,

$$x_{i+1} = e_{i+1} + x_r$$

$$x_i = e_i + x_r$$

$$x_{i-1} = e_{i-1} + x_r.$$

Substituting these values in (i), we get,

$$e_{i+1} = e_i - \frac{f(x_i)(e_i - e_{i-1})}{f(x_i) - f(x_{i-1})}$$

$$\text{or, } e_{i+1} = \frac{e_{i-1} f(x_i) - e_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{--- (ii)}$$

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According to Mean Value Theorem, there exists at least one point say $x = R_i$ in the interval x_i and x_r such that,

$$f'(R_i) = \frac{f(x_i) - f(x_r)}{x_i - x_r} \quad \text{—— (iii)}$$

Since $f(x_r) = 0$ and $e_i = x_i - x_r$, therefore eqn (iii) becomes,

$$f'(R_i) = \frac{f(x_i)}{e_i}$$

$$\text{or, } f(x_i) = e_i \cdot f'(R_i)$$

Similarly, $f(x_{i-1}) = e_{i-1} \cdot f'(R_{i-1})$

Substituting these in the numerator of eqn (ii) we get,

$$e_{i+1} = e_i \cdot e_{i-1} \cdot \frac{f'(R_i) - f'(R_{i-1})}{f(x_i) - f(x_{i-1})}$$

i.e, we can say, $e_{i+1} \propto e_i \cdot e_{i-1}$ —— (iv).

We know that, the order of convergence of an iteration process is p iff

$$e_i \propto c e_{i-1}^p$$

$$\text{or, } e_{i+1} \propto e_i^p.$$

Substituting these values in eqn (iv).

$$e_i^p \propto e_{i-1}^p \cdot e_{i-1}$$

$$\text{or, } e_i^p \propto e_{i-1}^{p+1}$$

$$\text{or, } e_i \propto c e_{i-1}^{(p+1)/p}$$

i.e, $p^2 - p - 1 = 0$ which has the solution,

$$p = \frac{1 \pm \sqrt{5}}{2}$$

Since p is always positive we have $p = 1.618$

It follows that the order of convergence of the secant method is 1.618 & convergence is reflected to as superlinear convergence.

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4) Fixed-Point Iteration Method:

In fixed-point iteration method we rearrange the function $f(x)=0$ such that x is on the left-hand side of the equation. This can be expressed as below:

$$f(x)=0 \quad \text{---} \ (i).$$

is written as,

$$x=g(x) \quad \text{---} \ (ii).$$

Eqn (ii) is called fixed point equation. This can be achieved by algebraic manipulation or by simply adding x to both sides of the original equation, example;

$$x^2+3x-1=0 \Rightarrow x=\frac{1-x^3}{3}$$

$$\sin(x)=0 \Rightarrow x=\sin(x)+x.$$

Since eqn (i) and eqn (ii) are similar, root of equation (ii) is also root of equation (i). Root of eqn (ii) is given by the point of intersection of straight line $y=x$ and the curve $x=g(x)$. This point of intersection is called fixed point of $g(x)$. Point p is a fixed point of function $f(x)$ if and only if $f(p)=p$. For example, if function f is defined on the real numbers by $f(x)=x^2-3x+4$ then 2 is a fixed point of f, because $f(2)=2$.

Eqn (ii) provides a convinent way of predicting the next value of x on the basis of current value of x. If x_0 is the initial guess to the root, next approximated value of root can be calculated as $x_1=g(x_0)$.

Similarly, further approximation can be given as;

$$x_2=g(x_1).$$

After generalizing this, we get $\boxed{x_{n+1}=g(x_n)}$ — (iii)

This point is called fixed point formula and gives rise to the sequence $x_0, x_1, x_2, \dots$ which is hoped to converge to a point x (fixed point).

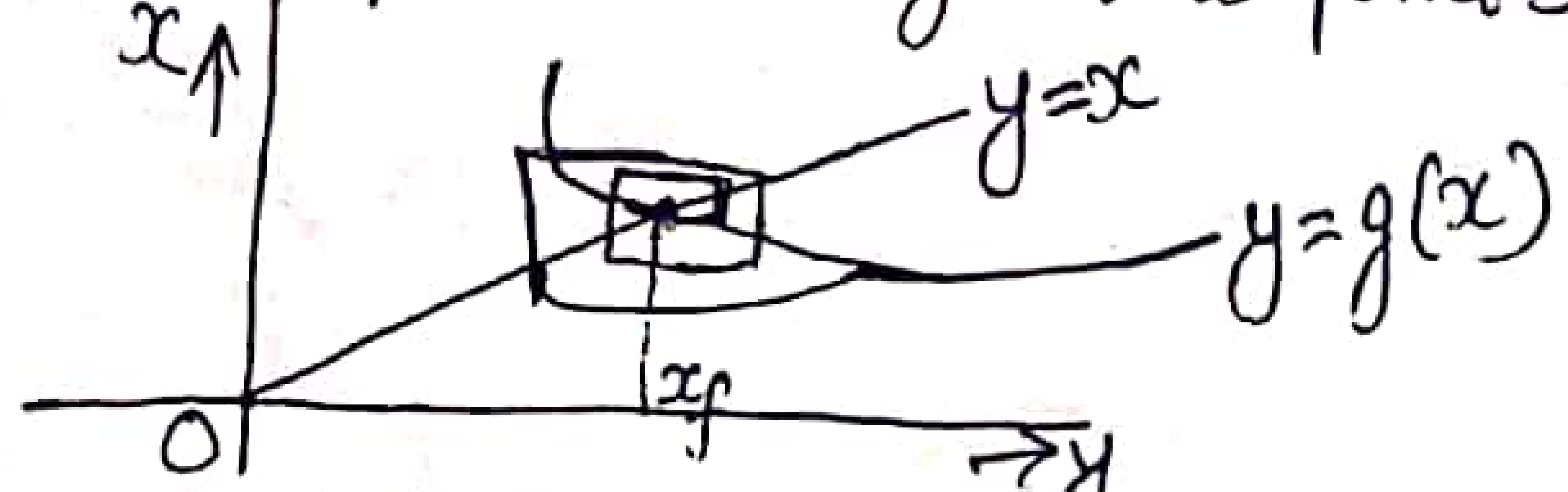

fig. Working of fixed point method

Q. Solve the equation $x^2-6x+8=0$ by assuming error precision 0.01.

Solution:

Above equation can be written as $x = x^2-5x+8$.

Assume that initial value of guess is 1.

Iteration 1

$x_0 = 1$

$x_1 = 4$ $x = x^2-5x+8 = 1^2-5\times1+8$

$$\text{Error} = \frac{4-1}{4} = 0.75$$

Iteration 2

$x_0 = 4$

$x_1 = 4$

Error = 0.0000

Since, Absolute value of error is less than specified limit.

$\therefore$ Root = 4.

Q. Solve the equation $1+1/2\sin(x)-x=0$ by assuming error precision 0.001.

Solution:

Above equation can be written as $x = 1+1/2\sin(x)$.

Assume that initial ~~gu~~ value of guess is 0.

Iteration 1:

$x_0 = 0$

$x_1 = 1$

Error = 1

Iteration 2:

$x_0 = 1$

$x_1 = 1.42$

Error = 0.296

Iteration 3:

$x_0 = 1.42$

$x_1 = 1.494$

Error = 0.0492

Iteration 4:

$x_0 = 1.494$

$x_1 = 1.4985$

Error = 0.0027

Iteration 5:

$x_0 = 1.4985$

$x_1 = 1.4986$

Error = 0.000103

Since, Absolute value of error is less than specified limit. Root = 1.4986.

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Convergence of Fixed-Point Method:

Convergence of this iteration process depend on the nature of $g(x)$. We can theoritically prove this as follows:-

→ The iteration formula is, $x_{n+1} = g(x_n)$ —— (i)

Let x_f be the root of given equation then,

$$x_f = g(x_f). \quad \text{—— (ii)}$$

Subtracting above two equations we get,

$$x_f - x_{n+1} = g(x_f) - g(x_n) \quad \text{—— (iii)}$$

According to MVT, there is at least one point say $x = R$ in the interval between x_f & x_n such that,

$$g'(R) = \frac{g(x_f) - g(x_n)}{x_f - x_n}$$

or, $g(x_f) - g(x_n) = g'(R)(x_f - x_n)$

Substituting this equation in (iii).

$$x_f - x_{n+1} = g'(R)(x_f - x_n). \quad \text{—— (iv)}$$

→ If e_i represents the error in the ith iteration, then eqn (iv) becomes,

$$e_{n+1} = g'(R)\, e_n \quad \text{—— (v)}$$

→ This gives that the error decreases with each iteration only if $g'(R) < 1$.

→ Hence we say that fixed point method converges only under the condition $g'(R) < 1$. i.e, Linearly convergent.

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⊛. Algorithms for each Iterative Methods:

1) Bisection Method: (Algorithm)

Error

→ Decide initial values for x_1 and x_2 and stopping criteria E.

→ Compute $f_1 = f(x_1)$ & $f_2 = f(x_2)$

→ If $f_1 * f_2 > 0$, then x_1 & x_2 do not bracket the root and goto stop.

→ Compute $x_0 = (x_1 + x_2)/2$ and compute $f_0 = f(x_0)$

→ If $f_1 * f_2 < 0$ then,
 set $x_2 = x_0$
 Set $f(x_1) = f(x_0)$
else.
 Set $x_1 = x_0$
 set $f_1 = f_0$

→ If absolute value of $(x_2 - x_1)/x_2$ is less than error E, then
 $root = (x_1 + x_2)/2$
 write the value of root
 goto Stop.
else.
 goto step 4.

→ Stop.

2) Newton-Raphson Method: (Algorithm)

→ Assign an initial guess to x, say x_0 & precision E.

→ Evaluate $f(x_0)$ & $f'(x_0)$.

→ Find the improved estimate of x_0 as;

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

→ Check for the accuracy of latest estimate.
 "Compare the relative approximate error with predefined value E."
 " if $|(x_1 - x_0)/x_1| < E$ stop, otherwise continue

→ Replace x_0 by x_1 & repeat step 3 & 4.

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3) Secant Method: (Algorithm):

→ Input two initial guesses (say x_0 & x_1) & precision E.

→ Evaluate $f(x_0)$ & $f(x_1)$.

→ Estimate the new value of the root as;

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

→ Find the absolute relative approximate error $[E_a]$ as;

$$E_a = \left| \frac{x_2 - x_1}{x_2} \right|$$

→ Compare the approximate error (E_a) with prespecified relative error (E).

If $E_a > E$,

$$x_0 = x_1$$
$$f(x_0) = f(x_1)$$
$$x_1 = x_2$$

goto step3.

else, goto Stop.

→ Stop

4) Fixed-Point Iteration Method: (Algorithm)

→ Input initial guess (say x_0) & error estimate (say E).

→ Convert $f(x) = 0$ to the form $x = g(x)$.

→ Now estimate the value of the root x_1 as; $x_1 = g(x_0)$

→ Find the absolute relative approximate error (E_a) as;

$$|E_a| = \left| \frac{x_1 - x_0}{x_1} \right|.$$

→ Compare absolute relative approximate error, E_a with the pre-specified relation error E.

If $|E_a| > E$,

$$x_0 = x_1$$

goto step3.

else, goto stop

→ Stop.

Note: According to these algorithms practice C program for each method.

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(Lesser imp. topics onwards this)
don't waste much more time on this

Synthetic Division:

→ It is a method of performing long divisions of polynomial with less writing and fewer calculations.

constant term

→ Let $P(x)$ be a polynomial of degree n. If we divide $P(x)$ by $(x-c)$ we get another polynomial $q(x)$, which is quotient of degree $n-1$.

→ Assume,
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + x_0$$
And,
$$q(x) = b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_0$$
So, $P(x) = q(x) \cdot (x-a)$.

or, $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = (b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_0)(x-c)$.

→ Comparing the coefficients on both sides of the equation,
$$b_{n-1} = a_n$$
$$b_{n-2} = a_{n-1} + a b_{n-1}$$
$$\vdots$$
$$b_0 = a_1 + a b_1$$
Thus, $b_n = 0$ & $b_{i-1} = a_i + a b_i$, where $i = 1 \dots n$.

Q. Solve the following problem using synthetic division method.
$(x^4 - 3x + 5) \div (x-4)$

Sol^n

Given, $(x^4 - 3x + 5) \div (x-4)$

Set the synthetic division,
$$x-4 \overline{)\, x^4 + 0x^3 + 0x^2 - 3x + 5}$$
Then it becomes,

$$\begin{array}{r|rrrrr} 4 & 1 & 0 & 0 & -3 & 5 \\ & \downarrow & 4 & 16 & 64 & 244 \\ \hline & 1 & 4 & 16 & 61 & \boxed{249} \end{array}$$

coefficients

multiplied by constant term 4 at each time to bottom term.

Remainder = 249

Remainder Theorem:

→ Remainder theorem states that if the polynomial $f(x)$ is divided by $(x-c)$, then the remainder is $f(c)$.

→ This means that we can apply synthetic division & the last two number is the remainder. This remainder is the functional value of $f(x)$ at $x=c$.

→ Thus the remainder theorem is useful for evaluating polynomial at a given value of x.

Q. Solve using remainder theorem $f(x)=3x^3+x^2+x-5$ to find $f(-2)$.

Soln

We have to use synthetic division as described above. What the difference is what the final answer is going to be. This time, we are looking for functional value, so our answer will not be a quotient, but only the remainder,

Given, $f(x)=3x^3+x^2+x-5$.

$$\begin{array}{r|rrrr} -2 & 3 & 1 & 1 & -5 \\ & \downarrow & -6 & 10 & -22 \\ \hline & 3 & -5 & 11 & -27 \end{array}$$

Remainder $=-27$

So, $f(-2)=-27$.

Finding Multiple roots by using Newton-Raphson method:

All real roots of polynomials can be found by repeatedly applying Newton's Method and synthetic division. Actually synthetic division is used to obtain polynomial of degree $n-1$ from polynomial of degree n. Root of polynomial can be found by using Newton's formula given below and initial guess;

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

Suppose x_r is one of the root of polynomial of degree n. Now deflate the polynomial by dividing it by $x-x_r$ to get another polynomial of degree $n-1$ and use x_r as initial guess for finding next root. Continue this process until polynomial of degree one is achieved as quotient. This polynomial will be of the form $a_1x+a_0=0$. Now final root can be calculated as; $x_r=-\frac{a_0}{a_1}$.

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Algorithm:

→ Input the degree & coefficient of polynomial.

→ Input initial guess x_0 & limit E.

while $n>1$,

→ Find the root using Newton Raphson method as;

$$x_r = x_0 - \frac{f(x_0)}{f'(x_0)}$$

→ Divide the polynomial by $x-x_r$ to get the polynomial of degree $n-1$.

→ set $x_0 = x_1$

→ $n = n-1$

end while.

$$1^{st} \text{ root} = -\frac{a_0}{a_1}$$

→ Stop.

Example: Find roots of equation $(x^4-2x^3-13x^2+38x-24)=0$ using newton raphson method. fourth synthetic division.

Solution: Assume, Initial guess $(x_0)=0.5$, Error Limit $(E)=0.01$

Step 1: Use Newton Raphson method to get root, Thus, 4^{th} root of the equation $=1$.

Step 2: Now use synthetic division deflate the polynomial by $(x-1)$, which gives the polynomial:

$$x^3-x^2-14x+24=0$$

Again use Newton Raphson method, ~~wh~~ with,

Initial guess $(x_0)=1$, Error limit $(E)=0.01$

Third root of the equation $=2$

Step 3: Again, Use synthetic division to deflate the polynomial by $(x-2)$, which gives the polynomial: $x^2+x-12=0$.

From Newton Raphson method,

Initial guess $(x_0)=2$, Error Limit $(E)=0.01$

Second root of the equation $=3$.

Step 4: Again, Use synthetic division to deflate the polynomial by $(x-2)$, which gives the polynomial $x+4=0$.

Now, first root of the polynomial is calculated using the formula given below:

$$\text{root} = -\frac{a_0}{a_1}$$

Thus, First Root $=-4$

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Horner's Method for Polynomial Evaluation:

→ It is a way to optmize the task of evaluating a polynomial evaluation. The method splits the polynomial into its individual terms & solve them individually. This method seperates the lowest degree term in the polynomial. Thus it represent any polynomial in the form: $K * x + 3$.

where, K is the group of all terms with degree higher than one & C is the term with degree zero.

→ Consider a general polynomial equation,

$$P(x) = \sum_{i=0}^{n} a_i x_i$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

where, $a_0, a_1, a_2, \dots, a_n$ are the real numbers & we wish to evaluate the polynomial at point say x_0.

→ To accomplish this, we need to define the given polynomial in terms of noted multiplication.

$$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n.$$

$$\text{or, } P(x) = a_0 + x(a_1 + a_2 x + \dots + a_n x^{n-1})$$

$$\text{or, } P(x) = a_0 + x(a_1 + x(a_2 + \dots + a_n x^{n-2}))$$

$$\vdots$$

$$P(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + x(a_4 + \dots (a_{n-1} + x(a_n) \dots)$$

→ Now we can define a new sequence of constants as follows:-

$$b_n = a_n$$

$$b_{n-1} = a_{n-1} + b_n x_0$$

Then b_0 is the value of $p(x_0)$.

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Algorithm:

→ Enter the degree of the polynomial.
→ Enter the value at which the polynomial to be evaluated, x.
→ Initially set $b_n = a_n$.
→ While $n > 0$

$$b_{n-1} = a_{n-1} + b_n * x$$

→ End While.
→ Display the value of b_0, which is the value of polynomial at x.
→ Stop.

Q. Evaluate the polynomial $p(x) = 3x^3 - 4x^2 + 5x - 6$ at $x = 2$.

Solution:

We know that,

$$a_3 = 3,\ a_2 = -4,\ a_1 = 5,\ a_0 = -6$$

Now new sequence of constants can be determined by using recursive formula as below;

$$b_3 = a_3 = 3$$

$$\begin{aligned} b_2 &= a_2 + b_3 * x \\ &= -4 + 3*2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} b_1 &= a_1 + b_2 * x \\ &= 5 + 2*2 \\ &= 9 \end{aligned}$$

$$\begin{aligned} b_0 &= a_0 + b_1 * x \\ &= -6 + 9*2 \\ &= 12 \end{aligned}$$

Thus $p(2) = 12$.

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