Page No.
Date

2) Consider two 2D signals $f(x,y), g(x,y)$

I) Consider both $f(x,y)$ and $g(x,y)$ are continuous 2D signals in a continuous domain.

Since correlation of f and g is given by

$$f \oplus g\,(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, g(u+x, v+y)\, dx\, dy$$

∴ 2D fourier transform of $f(x,y)$ is:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi j u x} \cdot e^{-2\pi j v y}\, dx\, dy \quad - (2)$$

Consider FT of Convolution of f and g

$$F(f \oplus g)(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(f(x,y) \oplus g(x,y) \right) e^{-2\pi j u x}\, e^{-2\pi j v y}\, dx\, dy$$

$$\cancel{= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, g(x+i, y+j)\, di\, dj \right] e^{-2\pi j u x}\, e^{-2\pi j v y}\, dx\, dy}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j_1)\, g(x+i, y+j_1)\, di\, dj_1 \right) e^{-2\pi j u x}\, e^{-2\pi j v y}\, dx\, dy$$

Interchanging integrals

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j_1) \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x+i, y+j_1)\, e^{-2\pi j (ux + vy)}\, dx\, dy \right) di\, dj_1$$

Replacing $x+i$ as x', $y+j_1$ as y'

$dx = dx'$, $dy = dy'$ as $x \to \infty \Rightarrow x' \to \infty$ | $y \to \infty \Rightarrow y' \to \infty$

$x \to -\infty \Rightarrow x' \to -\infty$ | $y \to -\infty \Rightarrow y' \to -\infty$

i, j_1 Independent of x, y

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j_1) \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x', y')\, e^{-2\pi j (u(x'-i) + v(y'-j_1))}\, dx'\, dy' \right) di\, dj_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j) \underbrace{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x', y')\, e^{-2\pi j (u x' + v y')}\, dx'\, dy' \right)}_{\text{From (2) this term is } G(u,v)} e^{2\pi j (ui + vj_1)}\, di\, dj_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j)\, G(u,v) \cdot e^{2\pi j (ui + v j_1)}\, di\, dj_1$$

Page No.
Date

Since (u,v) are Independent of i, j_1

$$= G(u,v) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(i,j_1)\, e^{-2\pi j((-u)i + (-v)j_1)}\, di\, dj_1$$

from (2)

$$= G(u,v)\, F(-u,-v)$$

$$\therefore \boxed{F(f \oplus g\,(u,v)) = G(u,v)\, F(-u,-v)}$$ $\therefore$ Hence proved

where F and G are DFT of f and g respectively.

II) Consider both $f(x,y)$ and $g(x,y)$ are 2D descrete signals of size $M \times N$

Fourier transform for a descrete signal $z(x,y)$ of size $M \times N$ is

$$Z(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} z(x,y)\, e^{-2\pi j\left(\frac{ux}{M} + \frac{vy}{N}\right)}. \quad -(3)$$

Correlation of f and g is

$$f \oplus g\,(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, g(x+u, y+v) \quad -(4)$$

Consider Fourier of Correlation f and g

$$F(f \oplus g\,(u,v)) = \sum_{p=0}^{M-1}\sum_{q=0}^{N-1} (f \oplus g)(p,q)\, e^{-2\pi j\left(\frac{up}{M} + \frac{vq}{N}\right)}$$

from (4)

$$= \sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\left(\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, g(x+u, y+v)\right) e^{-2\pi j\left(\frac{up}{M} + \frac{vq}{N}\right)}$$

Interchanging Summations

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} \underset{\substack{\downarrow \\ \text{Independent of } p,q}}{f(x,y)} \cdot g(p+x, q+y)\, e^{-j2\pi\left(\frac{up}{M}+\frac{vq}{N}\right)}$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \sum_{p=0}^{M-1}\sum_{q=0}^{N-1} g(p+x, q+y)\, e^{-j2\pi\left(\frac{up}{M}+\frac{vq}{N}\right)}$$

Replace, $x' = p+x$, $y' = q+y$

$p=0,\ x'=x$; $p=M-1,\ x' = M-1+x$ | $q=0,\ y'=y$; $q=N-1,\ y'=N-1+y$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \sum_{x'=x}^{x'=M-1+x}\sum_{y'=y}^{y'=N-1+y} g(x',y')\, e^{-j2\pi\left(\frac{ux'}{M}+\frac{vy'}{N}\right)} \cdot e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \underbrace{\sum_{x'=x}^{x'=M-1+x}\sum_{y'=y}^{y'=N-1+y}}_{\text{one time period in } x\text{-direction} \to M \text{ \& } y\text{-direction} \to N} g(x',y')\, e^{-j2\pi\left(\frac{ux'}{M}+\frac{vy'}{N}\right)}$$

from (3)

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, G(u,v) \cdot e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

u, v independent of x, y)

$$= G(u,v) \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{(-u)x}{M}+\frac{(-v)y}{N}\right)}$$

from (3)

$$= G(u,v)\, F(-u,-v)$$

$$\therefore \boxed{\mathcal{F}(f \oplus g\,(u,v)) = G(u,v)\, F(-u,-v)}$$

where F and G are DFT of f and g respectively.

$\therefore$ Hence proved