Q3)

1) we need to argue that the ~~non-singular~~ non-zero singular values of a matrix A are square roots of eigen values of AA^T (or A^TA.

w.k.t A can be done singular value decomposition to USV^T

i.e $A = USV^T$

consider

AA^T

$AA^T = (USV^T).(USV^T)^T$

$= (USV^T).(VSU^T)$

$= USV^TVSU^T$ (as $S = S^T$, S is diagonal matrix)

$= US^2U^T$ (as $V^TV = I$)

consider

A^TA

$A^TA = (USV^T)^T.(USV^T)$

$= VSU^T.USV^T$

$= VS^2V^T$ (as $U^TU = I$)

let us consider eigen values of AA^T and A^TA

$AA^Tv = \lambda v$ ($v \to$ eigen vector, $\lambda \to$ eigen value)

$A^TAv' = \lambda v'$ ($v' \to$ eigen vector, $\lambda \to$ eigen values)

$US^2U^Tv = \lambda v$

multiply U^T on left side in both LHS and RHS

$U^TUS^2U^Tv = \lambda U^Tv$

$\Rightarrow S^2U^Tv = \lambda U^Tv$

$\Rightarrow \lambda$ is eigen value of S^2.

w.k.t for a diagonal matrix eigen values are entries on the diagonal.

$\therefore$ eigen values of AA^T are $\sigma_1^2, \sigma_2^2, \dots \sigma_r^2$, as they are eigen values of S^2 (Property of eigen values of diagonal matrix)

$\therefore$ non-zero singular values of matrix A are square-root of eigen values of AA^T.

similarly consider

$$A^TAv' = \lambda v'$$

$$= V S^2 V^T v' = \lambda v'$$

multiply with V^T on left side on both L.H.S and R.H.S

$$\Rightarrow V^T V S^2 V^T v' = \lambda V^T v'$$

$$\Rightarrow S^2 V^T v' = \lambda V^T v'$$

$\therefore$ eigen values of both A^TA and S^2 are same.

$\therefore$ eigen values of A^TA are $\sigma_1^2, \sigma_2^2, \dots \sigma_r^2$ (eigen values of a diagonal matrix are diagonal entries themselves)

$\therefore$ non-zero singular values of matrix A are square root of eigen values of A^TA.

b) Frobenius norm of a matrix $= \|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}$

Squared Frobenius norm of matrix $= \|A\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2$

w.k.t For a matrix A

$$\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2 = tr(AA^T)$$

from Previous Part w.k.t $AA^T = US^2U^T$

$$(AA^T)_{ij} = \sum_{m=1}^{r}\sum_{k=1}^{r} U_{ik} (S^2)_{km} (U^T)_{mj}$$

$$tr(AA^T) = \sum_{i=j} (AA^T)_{ij}$$

$$= \sum_{i=j} \sum_{m=1}^{r}\sum_{k=1}^{r} U_{ik} (S^2)_{km} (U^T)_{mi}$$

as $(S^2)_{km} = 0$ when $k \neq m$

$$tr(AA^T) = \sum_{i=j}\sum_{k=m} U_{ik} (S^2)_{kk} (U^T)_{ki}$$

$\sum_{k=1}^{r} (S^2)_{kk} \sum_{i=1}^{m} U_{ik} U^T_{ki}$

as U, U^T are orthonormal eigen matrices (orthogonal matrices)

$\sum_{i=1}^{m} U_{ik} U^T_{ki} = 1$

$\Rightarrow tr(AA^T) = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_r^2$

• Frobenius norm of a matrix is equal to sum of square of it's singular values

c) student is trying to compute eigen values and eigen vectors of AA^T, A^TA using eig function in matlab.

let $[V, D_1] = eig(A^TA)$

$[U, D_2] = eig(AA^T)$

where V, U are eigenen value matrices of A^TA and AA^T respectively and D_1, D_2 are corresponding diagonal eigen value of matrices.

when student tries to find USV^T, it is not equal to A, it may be due to the following reasons

i) Reo-ordening of eigen values and eigen vectors :

eigen vectors returned by eig function might not always be in sorted order, so eigen vectors matrices U and V might not correspond to correct singular values of A, i.e U can be in one order of eigen values, V can be corresponding to another permutation of eigen values of A, due to which when we calculate USV^T, it is not equal to A.

ii) Incorrect sign of eigen vectors :-

w.k.t If v is a eigen vector $\Rightarrow -v$ is also a eigen vector. eig function in matlab may return eigen vectors that differ in sign

$A = USV^T$

$\Rightarrow AV = US$

$\Rightarrow U^TAV = S$ (where S is SVD of A)

It is also possible that U^TAV has negative entries in diagonal entries when U, V are obtained through eig.

→ these two can possible reasons for USV^T not equal to A.

solutions:-

i) sort D_1 and D_2 in decreasing order of eigen values

ii) sort U and V corresponding to sorting in D_1 and D_2 (argsort)

iii) For every negative diagonal entry in U^TAV, change the sign of every entry the corresponding column of U and V.

d)

i) Given A is of size $m \times m$, $m \leq n$

$$P = A^TA, \quad Q = AA^T$$

w.k.t prove

$y^tPy \geq 0$ and $z^tQz \geq 0$, eigen values of P and Q are non-negative

$$y^tPy = y^tA^TAy$$
$$= (Ay)^TAy$$

let $A^Ty = x$

$\Rightarrow y^tPy = x^Tx$, i.e, sum of square of all the elements of Ay.

$\Rightarrow$ ~~...~~ $y^tPy \geq 0$

$z^tQz \geq 0$

$Q = AA^T$

$$\Rightarrow z^tQz = z^tAA^Tz$$
$$= (A^Tz)^T \cdot (A^Tz)$$

let $A^Tz = y$

$\Rightarrow z^tQz = y^T \cdot y$, i.e sum of squares of all elements of $y = A^Tz$

$\therefore z^tQz \geq 0$

consider eigen values of P

$$Pv = \lambda v$$

$$A^TAv = \lambda v$$

multiplying with v^T on left side on both L.H.S and R.H.S

$\Rightarrow v^TA^TAv = \lambda v^Tv$

we has proven $v^TPv \geq 0$, w.k.t $v^Tv \geq 0$

$\Rightarrow \lambda \geq 0$.

$\therefore$ eigen values of P are non-negative

consider eigen values of Q

$$Qv' = \lambda v'$$

multiply with $(v')^T$ on both sides on left on L.H.S and R.H.S

$(v')^TQv' = \lambda (v')^Tv'$

we have proven $(v')^TQv' \geq 0$ w.k.t and $(v')^Tv' \geq 0$ $\Rightarrow \lambda \geq 0$.

$\therefore$ eigen values of Q are non-negative.

ii) Given:-

u is an eigenvector of P with eigen value λ

Need to Prove:-

Au is eigenvector of Q with eigen value λ

u is eigenvector of P with eigen value λ

$\Rightarrow Pu = \lambda u$

$\Rightarrow A^TAu = \lambda u$

multiply with A on left side on L.H.S and R.H.S

$\Rightarrow AA^TAu = \lambda Au$

$AA^T = Q$

$\Rightarrow Q(Au) = \lambda (Au)$

$\therefore$ Au is eigen vector of Q with eigen value λ.

Given:-

v is an eigenvector of Q with eigen value μ.

Need to Prove:-

A^Tv is eigenvector of P with eigen value μ.

$Qv = \mu v$

$\Rightarrow AA^Tv = \mu v$

multiply with A^T on left side on L.H.S and R.H.S

$\Rightarrow A^TAA^Tv = \mu A^Tv$

as $A^TA = P$

$\Rightarrow P(A^Tv) = \mu(A^Tv)$

$\therefore A^Tv$ is eigen vector of P with eigen value μ.

u will be $n \times 1$ and v will be $m \times 1$ so u has n elements and v has m elements

iii)

Given:

v_i is eigen vector of Q

$$u_i = \frac{A^Tv_i}{\|A^Tv_i\|_2}$$

Need to Prove :-

$Au_i = \gamma_i v_i$ for some real, non-negative γ_i

consider Au_i

$$Au_i = \frac{AA^Tv_i}{\|A^Tv_i\|_2}$$

$$= \frac{\mu_i v_i}{\|A^Tv_i\|_2}$$ (as $AA^T = Q$ and v_i is eigenvector of Q)

consider

$$\|A^Tv_i\|_2 = \sqrt{(A^Tv_i)^T A^Tv_i} = \sqrt{v_i^T AA^T v_i}$$

$$= \sqrt{v_i^T \mu_i v_i}$$

$$= \sqrt{\mu_i}\, \|v_i\|_2$$

$$\Rightarrow Au_i = \frac{\mu_i v_i}{\sqrt{\mu_i}\, \|v_i\|_2}$$

$$\Rightarrow \gamma_i = \frac{\sqrt{\mu_i}}{\|v_i\|_2}$$ $\Rightarrow \gamma_i$ is real and Positive as $\mu_i \geq 0$ from Part - i.

iv)

given $u_i^T u_j = 0$ for $i \neq j$ and $v_i^T v_j = 0$ for $i \neq j$.

$$U = [v_1 | v_2 | \cdots v_m]$$

$$V = [u_1 | u_2 | \cdots | u_m]$$

consider $U^T A V$

From Part-iii w.k.t $Au_i = r_i v_i$

$$AV = [Au_1 | Au_2 | \cdots | Au_m]$$

$$= [r_1 v_1 | r_2 v_2 | \cdots | r_m v_m]$$

$U^T A V$

$$= U^T [r_1 v_1 | r_2 v_2 | \cdots | r_m v_m]$$

as $v_i^T v_j = 0$ for $i \neq j$ and $v_i^T v_i = 1$, $U = [v_1 | v_2 | \cdots | v_m]$

$$U^T [r_1 v_1 | r_2 v_2 | \cdots | r_m v_m]$$

$= [\gamma_1 U^T v_1 | \gamma_2 U^T v_2 | \cdots | \gamma_m U^T v_m]$

Consider $U^T v_i$

$= ([v_1 | v_2 | \cdots | v_m])^T v_i$ ✓

as $v_i^T v_j = 0$ when $i \neq j$, $v_i^T v_i = 1$

it will be a column vector with only i^{th} element as 1 and others 0.

$\Rightarrow$

$U^T A V = \mathrm{diag}(\gamma_1, \gamma_2, \cdots \gamma_m) = \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \gamma_m \end{bmatrix}$ ✓

let $\mathrm{diag}(\gamma_1, \gamma_2, \cdots \gamma_m) = \Gamma$

$\Rightarrow$ multiply with U on left, V^T on right on L.H.S and R.H.S

$\Rightarrow UU^T A V V^T = U \Gamma V^T$

$\Rightarrow A = U \Gamma V^T$