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Q7) Given $\frac{\partial I}{\partial t} = c\left(\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}\right)$

→ Since our image is a function of x, y, t I am denoting it as $I(x, y, t)$

→ Let us assume its fourier transform in x, y is $G(u, v, t)$
i.e $F(I(x,y,t)) = G(u,v,t)$

→ Let us try to apply fourier to given equation. So Let us find their individual fourier

$$F\left(\frac{\partial I}{\partial t}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d I(x,y,t)}{dt} e^{-j2\pi(ux+vy)} \, dx \, dy$$

$$= \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(x,y,t) \, e^{-j2\pi(ux+vy)} \, dx \, dy$$

(As integral independent of t)

$$\Rightarrow F\left(\frac{\partial I}{\partial t}\right) = \frac{d}{dt}\left(G(u,v,t)\right)$$

$$F\left(\frac{d^2 I}{dx^2}\right) = -4\pi^2 u^2 F(I) = -4\pi^2 u^2 G(u,v,t)$$

$$F\left(\frac{\partial^2 I}{\partial y^2}\right) = -4\pi^2 v^2 F(I) = -4\pi^2 v^2 G(u,v,t)$$

→ Now apply fourier to given PDE

$$\Rightarrow F\left(\frac{\partial I}{\partial t}\right) = c\,F\left(\frac{\partial^2 I}{\partial x^2}\right) + c\,F\left(\frac{\partial^2 I}{\partial y^2}\right)$$

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$$\Rightarrow \frac{d}{dt} G(u,v,t) = -4\pi^2 c(u^2+v^2)\, G(u,v,t)$$

→ This is a first order P.D.E

$$\Rightarrow \frac{d\, G(u,v,t)}{G(u,v,t)} = \underbrace{-4\pi^2 c(u^2+v^2)}_{\text{independent of } t}\, dt$$

$$\Rightarrow \left[\ln G(u,v,t)\right]_0^t = -4\pi^2 c(u^2+v^2)\, t$$

$$\Rightarrow \boxed{G(u,v,t) = G(u,v,0)\, e^{-4\pi^2 c(u^2+v^2)t}}$$

$$I(x,y,t) = F^{-1}(G(u,v,t))$$

$$\Rightarrow I(x,y,t) = F^{-1}\left(G(u,v,0)\, e^{-4\pi^2 c(u^2+v^2)t}\right)$$

$$\Rightarrow I(x,y,t) = F^{-1}(G(u,v,0)) \oplus F^{-1}\left(e^{-4\pi^2 c(u^2+v^2)t}\right)$$

(Convolution theorem of fourier transform)

→ We know $F^{-1}(G(u,v,0)) = I(x,y,0)$

→ Let us find $F^{-1}\left(e^{-4\pi^2 c(u^2+v^2)t}\right)$

We know $\boxed{F\left(\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}\right) = e^{-2\pi^2\sigma^2(u^2+v^2)}}$

Fourier of one gaussian is another gaussian.

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By comparing $2\pi^2\sigma^2 = 4\pi^2 ct$
$\Rightarrow \sigma^2 = 2ct$

$$\Rightarrow F^{-1}\left(e^{-4\pi^2 c(u^2+v^2)t}\right) = \frac{1}{2\pi(2ct)} e^{-\frac{x^2+y^2}{2(2ct)}}$$

$$= (G(0, 2ct))(x,y)$$

↳ Gaussian with mean $= 0$, variance $= 2ct$

$$\Rightarrow \boxed{I(x,y,t) = I(x,y,0) \oplus G(0,2ct)(x,y)}$$

→ The value of standard deviation is $\sqrt{2ct}$