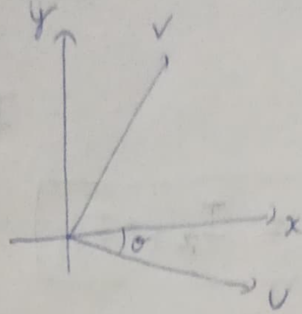
Q 6) $U = x\cos\theta - y\sin\theta \quad , \quad V = x\sin\theta + y\cos\theta$

A)

$\Rightarrow$



$x = U\cos\theta + V\sin\theta$

$y = V\cos\theta - U\sin\theta$

→ By using chain rule

$$\frac{\partial I}{\partial U} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial U} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial U}$$

$$I_U = \frac{\partial I}{\partial U} = I_x\cos\theta + I_y(-\sin\theta) \rightarrow (I_x, I_y)\cdot(\cos\theta, -\sin\theta)$$

$$\nabla I(x,y)\cdot U$$

$$\frac{\partial I}{\partial V} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial V} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial V}$$

$$I_V = \frac{\partial I}{\partial V} = I_x\sin\theta + I_y\cos\theta$$

$$\rightarrow \frac{\partial^2 I}{\partial U^2} = \frac{\partial}{\partial U}\left(\frac{\partial I}{\partial U}\right) = \frac{\partial I_U}{\partial U} = \frac{\partial I_U}{\partial x}\frac{\partial x}{\partial U} + \frac{\partial I_U}{\partial y}\frac{\partial y}{\partial U}$$

$$= (I_{xx}\cos\theta - I_{xy}\sin\theta)\cos\theta + (I_{xy}\cos\theta - I_{yy}\sin\theta)(-\sin\theta)$$

$$= I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\sin\theta\cos\theta - I_{xy}\sin\theta\cos\theta$$

$$\rightarrow \frac{\partial^2 I}{\partial V^2} = \frac{\partial}{\partial V}\left(\frac{\partial I}{\partial V}\right) = \frac{\partial I_V}{\partial V} = \frac{\partial I_V}{\partial x}\frac{\partial x}{\partial V} + \frac{\partial I_V}{\partial y}\frac{\partial y}{\partial V}$$

$$= (I_{xx}\sin\theta + I_{xy}\cos\theta)\sin\theta + (I_{xy}\sin\theta + I_{yy}\cos\theta)\cos\theta$$

$$= I_{xx}\sin^2\theta + I_{yy}\cos^2\theta + I_{xy}\sin\theta\cos\theta + I_{xy}\sin\theta\cos\theta$$

$$\Rightarrow I_{uu} + I_{vv} = \frac{\partial^2 I}{\partial u^2} + \frac{\partial^2 I}{\partial v^2} = I_{xx}(\sin^2\theta + \cos^2\theta) + I_{yy}(\sin^2\theta + \cos^2\theta)$$
$$+ I_{xy}(2\sin\theta\cos\theta - 2\sin\theta\cos\theta)$$
$$= I_{xx} + I_{yy}$$

$$\Rightarrow \boxed{I_{uu} + I_{vv} = I_{xx} + I_{yy}}$$

B) Find second derivative in the direction of $\left(\frac{I_x}{\sqrt{I_x^2+I_y^2}}, \frac{I_y}{\sqrt{I_x^2+I_y^2}}\right) \rightarrow$ Assume g is direction

Given $I_g = \nabla I(x,y) \cdot g$

$$\Rightarrow I_g = \frac{I_x \times I_x}{\sqrt{I_x^2+I_y^2}} + \frac{I_y \times I_y}{\sqrt{I_x^2+I_y^2}}$$

$$\Rightarrow I_g = \sqrt{I_x^2+I_y^2}$$

$I_{gg} = \nabla I_g(x,y) \cdot g$ (similarily)

$$\Rightarrow I_{gg} = \frac{\partial I_g}{\partial x} \times \frac{I_x}{\sqrt{I_x^2+I_y^2}} + \frac{\partial I_g}{\partial y} \times \frac{I_y}{\sqrt{I_x^2+I_y^2}}$$

$$\Rightarrow I_{gg} = \frac{I_x I_{xx} + I_y I_{xy}}{\sqrt{I_x^2+I_y^2}} \times \frac{I_x}{\sqrt{I_x^2+I_y^2}} + \frac{I_x I_{xy} + I_y I_{yy}}{\sqrt{I_x^2+I_y^2}} \times \frac{I_y}{\sqrt{I_x^2+I_y^2}}$$

$$\Rightarrow I_{gg} = \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2+I_y^2}$$

c) Second directional derivative $\perp^{lar}$ to derivative gradient. Assume p is the direction

$$I_{gg} + I_{pp} = I_{xx} + I_{yy} \quad \text{(from part A)}$$

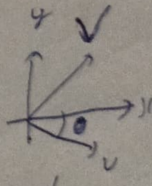
$$\Rightarrow I_{pp} = I_{xx} + I_{yy} - I_{gg}$$

$$\Rightarrow I_{pp} = I_{xx} + I_{yy} - \left(\frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2} \right)$$

$$\Rightarrow I_{pp} = \frac{I_x^2 I_{yy} + I_y^2 I_{xx} - 2 I_x I_y I_{xy}}{I_x^2 + I_y^2}$$

I_{pp} we are calculating in p direction i.e $\left(\frac{I_y}{\sqrt{I_x^2 + I_y^2}}, \frac{-I_x}{\sqrt{I_x^2 + I_y^2}} \right) \to \perp^{lar}$ to g

→ This is direction corresponding to $(\cos\theta, -\sin\theta)$ in I_{vv} calculated in part A



v vector is $(\cos\theta, -\sin\theta)$

→ Justification is substituting corresponding values also gives same answer