CS 663 Home Work Assignment 2

Trasula Umesh Karthikeya | Gnana Mahesh Vetcha | Kajjayam Varun Gupta
22b0913 | 22b0949 | 22b1030

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1 Question 5

Given a 1D ramp image $I(x) = cx + d$, where c and d are scalar coefficients.

1.1 Filtering with a Zero-Mean Gaussian

We need to find the result of the image when it is filtered by a Gaussian with mean 0 and standard deviation σ. Let $h(x)$ be the Gaussian PDF

$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

For filtering the image $I(x)$ with the above Gaussian filter, we need to compute the convolution of $I(x)$ with $h(x)$:

$$J(x) = (I * h)(x) = \int_{-\infty}^{\infty} I(x')h(x - x')\,dx'$$

Let's substitute $I(x') = cx' + d$:

$$J(x) = \int_{-\infty}^{\infty} (cx' + d)h(x - x')\,dx'$$

$$\implies \int_{-\infty}^{\infty} (cx')h(x - x')\,dx' + \int_{-\infty}^{\infty} d * h(x - x')\,dx'$$

$$\implies \int_{-\infty}^{\infty} (cx')h(x - x')\,dx' + d * \int_{-\infty}^{\infty} h(x - x')\,dx'$$

Since $\int_{-\infty}^{\infty} h(x - x')\,dx' = 1$ as Gaussian integrates to 1 because total probability 1.

$$\implies d + \int_{-\infty}^{\infty} (cx')h(x - x')\,dx'$$

Let's re-write x' as $x - (x - x')$ then

$$\implies d + \int_{-\infty}^{\infty} (c(x - (x - x')))h(x - x')\,dx'$$

As x is constant,

$$\implies d + cx \int_{-\infty}^{\infty} h(x - x')\,dx' - \int_{-\infty}^{\infty} c(x - x')h(x - x')\,dx'$$

Since $\int_{-\infty}^{\infty} h(x - x')\,dx' = 1$ as Gaussian integrates to 1 because total probability 1.

$$\implies d + cx - c\int_{-\infty}^{\infty} (x - x')h(x - x')\,dx'$$

Since $\int_{-\infty}^{\infty} (x - x')h(x - x')\,dx' = 0$ as expectation of x in Gaussian is 0.

$$\implies cx + d$$

Therefore, after applying the Gaussian filter, the image remains unchanged:

$$J(x) = cx + d$$

1.2 Filtering with a Bilateral Filter

A bilateral filter will consider both spatial distance and intensity difference. Let $B(x)$ be a bilateral filter then it's form is

$$B(x) = \frac{1}{W(x)} \int_{-\infty}^{\infty} I(x') e^{-\frac{(x-x')^2}{2\sigma_s^2}} e^{-\frac{(I(x)-I(x'))^2}{2\sigma_r^2}} \, dx'$$

where $W(x)$ is the normalization factor:

$$W(x) = \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{2\sigma_s^2}} e^{-\frac{(I(x)-I(x'))^2}{2\sigma_r^2}} \, dx'$$

For our ramp image $I(x) = cx + d$, the intensity difference $I(x) - I(x') = c(x - x')$. Therefore:

$$e^{-\frac{(I(x)-I(x'))^2}{2\sigma_r^2}} = e^{-\frac{c^2(x-x')^2}{2\sigma_r^2}}$$

Now consider,

$$e^{-\frac{(x-x')^2}{2\sigma_s^2}} e^{-\frac{c^2(x-x')^2}{2\sigma_r^2}}$$
$$\implies e^{-(\frac{(x-x')^2}{2\sigma_s^2} + \frac{c^2(x-x')^2}{2\sigma_r^2})}$$
$$\implies e^{-\frac{(x-x')^2}{2}(\frac{1}{\sigma_s^2} + \frac{c^2}{\sigma_r^2})}$$
$$\implies e^{-\frac{(x-x')^2}{2}(\frac{\sigma_r^2+\sigma_s^2 c^2}{\sigma_r^2\sigma_s^2})}$$
$$\implies e^{-\frac{(x-x')^2}{2\frac{\sigma_r^2\sigma_s^2}{\sigma_r^2+\sigma_s^2 c^2}}}$$

This can be written

$$e^{-\frac{(x-x')^2}{2\sigma_{\text{combined}}^2}}$$

Where

$$\sigma_{\text{combined}}^2 = \frac{\sigma_r^2\sigma_s^2}{\sigma_r^2 + \sigma_s^2 c^2}$$

Thus, the bilateral filter becomes a convolution with a Gaussian, and as in the case of the Gaussian filter, the output is the same as the input:

$$J(x) = cx + d$$