Question - 5

I) Given $f(x,y)$ is real

RTP $F^*(u,v) = F(-u,-v)$ where $F(u,v)$ is DFT of $f(x,y)$

fourier transform for a descrete signal $f(x,y)$ of size $M \times N$ is

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi j\left(\frac{ux}{M}+\frac{vy}{N}\right)} \quad \text{—} (1)$$

Complex conjugate

$$F^*(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f^*(x,y)\, e^{-2\pi(-j)\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f^*(x,y)\, e^{2\pi j\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$\because$ Given $f(x,y)$ is real $f(x,y) = f^*(x,y)$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{2\pi j\left(\frac{ux}{M}+\frac{yv}{N}\right)}$$

This can be written as

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi j\left(\frac{(-u)x}{M}+\frac{(-v)y}{N}\right)}$$

From (1)

$$= F(-u,-v)$$

$$\therefore \boxed{F^*(u,v) = F(-u,-v)}$$

II) ~~f(x,y)~~ Given $f(x,y)$ is real and even where a function $f(x,y)$ is an even function if $f(x,y) = f(-x,-y)$

RTP: $F(u,v)$ is real and even.

Consider, $f(x,y)$ be of size $M \times N$ then

~~F(u,v)=~~

fourier transform of $f(x,y)$ is

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \quad -② $$

Inverse Discrete fourier Transform (IDFT)

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

Consider,

$$F(-u,-v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp\left(-2\pi j\left(\frac{(-u)x}{M} + \frac{(-v)y}{N}\right)\right)$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp\left(2\pi j\left(\frac{ux}{M} + \frac{vy}{N}\right)\right)$$

Replace ~~x by~~ $x' = -x$, $y' = -y$

$x=0, x'=0$	$y=0, y'=0$
$x=M-1, x'=1-M$	$y=N-1, y'=1-N$

$$= \sum_{x'=1-M}^{0} \sum_{y'=1-N}^{0} f(-x',-y') \exp\left(2\pi j\left(\frac{u(-x')}{M} + \frac{v(-y')}{N}\right)\right)$$

$$= \sum_{x'=1-M}^{0} \sum_{y'=1-N}^{0} f(-x',-y') \exp\left(-2\pi j\left(\frac{ux'}{M} + \frac{vy'}{N}\right)\right)$$

$\because f(x,y)$ is even

$$= \sum_{x'=1-M}^{0} \sum_{y'=1-N}^{0} f(x',y') \exp\left(-2\pi j\left(\frac{ux'}{M} + \frac{vy'}{N}\right)\right)$$

one time

If we carefully observe the above sum is a period sum along both x-axis (Time period M) and y-axis (Time period N)

$\because$ It is a Discrete Fourier Transform it is same as $F(u,v)$ [from (2)]

$\therefore F(-u,-v) = F(u,v)$

$\boxed{\therefore F(u,v) \text{ is even}}$ — (3)

Now from previous part result

$$F^*(u,v) = F(-u,-v)$$

$\because F(u,v)$ is even from (3)

$$F^*(u,v) = F(u,v)$$

$\because$ Complex conjugate of $F(u,v)$ is itself

$\boxed{F(u,v) \text{ is real}}$

$\therefore F(u,v)$ is also real and even

$\therefore$ Hence proved