Q 1) a) Procedure to determine translation between two given images

→ Let f_2 is translated version of f_1

i.e $f_2(x,y) = f_1(x-x_0, y-y_0)$

then $F_2(u,v) = e^{-2\pi j(ux_0+vy_0)} F_1(u,v)$

$$\Rightarrow \frac{F_1(u,v)\, F_2^*(u,v)}{|F_1(u,v)\, F_2(u,v)|} = e^{j2\pi(ux_0+vy_0)}$$

$$\Rightarrow F^{-1}\left(\frac{F_1(u,v)\, F_2^*(u,v)}{|F_1(u,v)\, F_2(u,v)|}\right) = \delta(x+x_0, y+y_0)$$

→ $\delta(x+x_0, y+y_0)$ is 0 everywhere except $(-x_0, -y_0)$ so we can use this property to find translation (i.e -ve of the maximum point of fourier inverse).

⇒ The method is (based on above facts)

step 1) Calculate fourier of both images f_1 and f_2. Let those be $F_1(u,v)$, $F_2(u,v)$ respectively

step 2) Get cross power spectrum of two images

$$\frac{F_1(u,v)\, F_2^*(u,v)}{|F_1(u,v)\, F_2(u,v)|} = C(u,v)$$

step 3) Inverse fourier transform of $C = F^{-1}(C(u,v))$

→ Now the peak gives you the corresponding translation.

→ Time complexity of this procedure for $N \times N$ image

Step 1) 2 FFT's are to be taken, FFT of an $M \times N$ image has $O(MN \log MN)$ complexity

$$\Rightarrow O(N^2 \log N^2) + O(N^2 \log(N^2)) \simeq O(N^2 \log N)$$

Step 2) It has a point wise calculation. At each point it is $O(1)$

$$\Rightarrow O(N^2)$$

Step 3) Inverse FTT

$$\Rightarrow O(N^2 \log N)$$

$$\Rightarrow \text{Total} = O(N^2 \log N) + O(N^2) + O(N^2 \log N) \approx \underline{O(N^2 \log N)}$$

→ Time complexity of method if we loop over different values of translation

→ Here for each iteration we do this (i-e x_0, y_0 take diff values)

1) Image Wrapping :- Find the translated image with x_0, y_0 as translation. While wrapping we go through all point in image so $O(N^2)$

2) MSSD calculation :- We find Mean of successive square difference between transformed and reference image. This also requires us to go through all pixels. So $O(N^2)$

→ Different translation possible $= N^2$ $(x_0 = 1 \text{ to } N,\ y_0 = 1 \text{ to } N)$

$\Rightarrow$ This process total time $= N^2 O(N^2 + N^2) = O(N^4)$

$\Rightarrow$ Our previous method using FFT being $O(N^2 \log N)$ is better than this. It improves $N^2/\log N$ times.

b) Predicting rotation between two images can be done by using Fourier translational and rotational properties.

Let f_2 is obtained by translation and rotation of f_1

$$f_2(x,y) = f_1(x\cos\theta_0 + y\sin\theta - x_0,\ -x\sin\theta_0 + y\cos\theta_0 - y_0)$$

$\rightarrow$ relation btw fourier transforms

$$F_2(\mu,\nu) = e^{-j2\pi(\mu x_0 + \nu y_0)} \cdot F_1(\mu\cos\theta_0 + \nu\sin\theta_0,\ -\mu\sin\theta_0 + \nu\cos\theta_0)$$

$\rightarrow$ Let M_1, M_2 be magnitudes of F_1 and F_2

then $M_2(\mu,\nu) = M_1(\mu\cos\theta_0 + \nu\sin\theta_0,\ -\mu\sin\theta_0 + \nu\cos\theta_0)$

(Polar form $\left(\rho = \sqrt{\mu^2+\nu^2};\ \theta = \tan^{-1}\left(\frac{\nu}{\mu}\right)\right)$

$$M_1(\rho,\theta) = M_2(\rho, \theta - \theta_0)$$

Now it is similar to translation prediction in part 1, (Use cross power spectrum method)

$\Rightarrow$ We can use phase correlation between M_1 and M_2 to determine θ_0

$\Rightarrow$ This method gives rotation even when there is a translation also.

⇒ The method will be

Step 1) Calculate fourier for both images f_1 and f_2. Let them be $F_1(u,v)$, $F_2(u,v)$

Step 2) Find magnitude of fourier at each point let them be $M_1(u,v)$, $M_2(u,v)$

Step 3) Change coordinates from (u,v) to (ρ,θ) and make array correspondingly $M_1(\rho,\theta)$, $M_2(\rho,\theta)$ (Wrapping)

Step 4) Calculate translation along θ for $M_1(\rho,\theta)$ and $M_2(\rho,\theta)$ by Cross-powered spectrum method in part-1

→ θ gives you angle of rotation between two images. The reasoning for this method are the equations written before the method.

Proof of 2D-Fourier rotation theorem:-

$$F\left(I\left(x\cos\theta_0+y\sin\theta_0,\ y\cos\theta_0-x\sin\theta_0\right)\right)=\sum\sum I\left(x\cos\theta_0+y\sin\theta_0,\ y\cos\theta_0-x\sin\theta_0\right)e^{\frac{-j2\pi ux}{N}}e^{\frac{-j2\pi vy}{N}}$$

Substitute $x=\bar{x}\cos\theta_0-\bar{y}\sin\theta_0$, $y=\bar{x}\sin\theta_0+\bar{y}\cos\theta_0$

$$\Rightarrow \sum\sum I(\bar{x},\bar{y})\,e^{-j2\pi\frac{(u\cos\theta_0+v\sin\theta_0)\bar{x}}{N}}\,e^{-j2\pi\frac{(v\cos\theta_0-u\sin\theta_0)}{N}}$$

$$=F\left(u\cos\theta+v\sin\theta,\ v\cos\theta_0-u\sin\theta_0\right).$$

$$\Rightarrow F\left(I\left(x\cos\theta_0+y\sin\theta_0,\ y\cos\theta_0-x\sin\theta_0\right)\right)=F\left(u\cos\theta+v\sin\theta,\ v\cos\theta_0-u\sin\theta_0\right)$$

(We used this in 2^{nd} proof)