GSOE9210 Engineering Decisions

Victor Jauregui

v.jauregui@unsw.edu.au
www.cse.unsw.edu.au/~gs9210

Risk attitudes and Utility

1. Risk
   - Risk preference
   - Expected monetary value
2. Utility
   - Utility of money
   - Risk attitudes

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Introduction to risk preference

To gamble or not to gamble

You're offered to play the following game: a coin is tossed once. If it lands 'heads' you get $\$2000$. If it lands 'tails' you get nothing. It costs $\$1000$ to play. Would you play?

- Measured in dollars, $v_{\$}(\$x) = x$, the two have equal *Bayes* value; *i.e.*, $v_{\$}(\$1000) = 1000 = V_B([\frac{1}{2} : \$2000|\frac{1}{2} : \$0])$
- Most won't risk $\$1000$ on this bet; *i.e.*, prefer $\$1000$ to $[\frac{1}{2} : \$2000|\frac{1}{2} : \$0]$
- How can we explain this?

Is this irrational?

Do we need a new decision rule (other than *Bayes*)?

Gambles

Definition (Gamble)

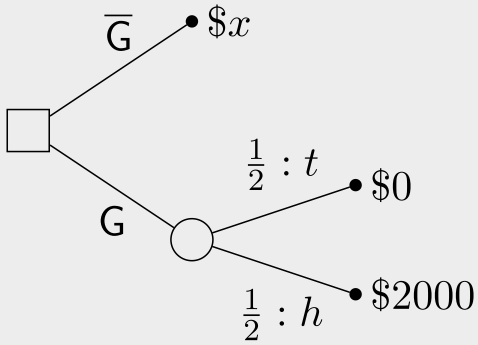
A *gamble* is a decision problem with two alternatives: one which is certain and another which is a (proper) lottery.

Examples

Whether or not to:

- bet on the toss of a coin
- bet on a horse race, a football match, *etc.*
- buy a share whose price may go up or down
- pay for insurance

Expected monetary value



$$\ell_{\overline{\mathsf{G}}} = [\$x]$$
$$\ell_{\mathsf{G}} = [\tfrac{1}{2} : \$2000 | \tfrac{1}{2} : \$0]$$

Definition (Expected monetary value)

The *expected monetary value* (EMV) of a lottery, denoted $V_\$$, is the *Bayes* value of the lottery when outcomes are valued in \$ (*i.e.*, $v = v_\$$).

$$
\begin{aligned}
V_\$(\ell_{\overline{\mathsf{G}}}) &= v_\$(\$x) = x \\
V_\$(\ell_{\mathsf{G}}) &= \tfrac{1}{2} v_\$(h) + \tfrac{1}{2} v_\$(t) \\
&= \tfrac{1}{2}(2000) + \tfrac{1}{2}(0) = 1000
\end{aligned}
$$

How much would you pay to gamble?

Risk attitude indicators

Definition (Certainty equivalent)

An agent's *certainty equivalent* for a lottery is the certain amount it would be willing to exchange for the lottery; *i.e.*, the certain amount for which an agent would be indifferent between it and the lottery.

- certainty equivalents are subjective: different decision-makers may have different certainty equivalents for the same lottery
- certainty equivalents characterise risk attitudes towards a lottery: what would a 'high' certainty equivalent mean?

Definition (Risk premium)

An agent's *risk premium* for a lottery is the difference between the lottery's fair bet value (EMV) and the agent's certainty equivalent.

Fair gambles

Definition (Fair gamble)

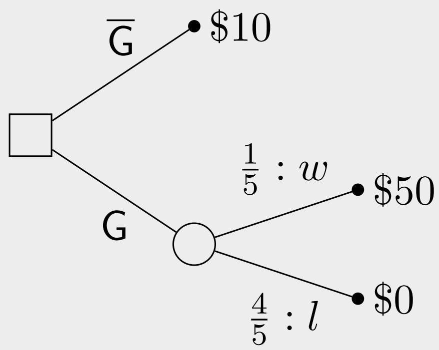
A gamble is *fair* or *unbiased* if the expected monetary value for the lottery is the same as the value of the certain outcome; *i.e.*,

$$V_\$(\ell_{\mathsf{G}}) = E(v_\$) = V_\$(\ell_{\overline{\mathsf{G}}})$$

Suppose Alice has \$10 and is offered a bet on $[\frac{1}{5} : \$50 \,|\, \frac{4}{5} : \$0]$.

Questions:

- is the bet fair?
- should she gamble?



Risk attitude indicators

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Risk attitudes and Utility

Utility

- Should Alice bet if she *believes* the chances of winning exceed 1 in 5? Suppose she needs $\$10$ to buy dinner; should Alice gamble?
- Alice's risk preference: I'll gamble (risk going hungry) only if my chances of winning are at least even (*i.e.*, greater than 1 in 2); *i.e.*, indifferent between certain $\$10$ and $\ell = [\frac{1}{2} : \$50|\frac{1}{2} : \$0]$:

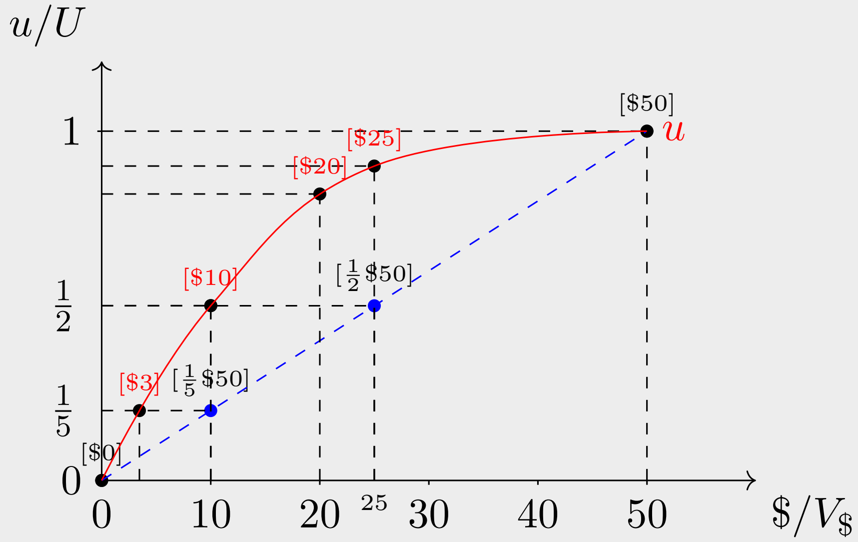
$$\begin{aligned} u(\$10) &= U([\tfrac{1}{2} : \$50|\tfrac{1}{2} : \$0]) = E_u(\ell) \\ &= V_B([\tfrac{1}{2} : \$50|\tfrac{1}{2} : \$0]) \qquad \text{using } u \text{ rather than } v_\$ \\ &= \tfrac{1}{2}u(\$50) + \tfrac{1}{2}u(\$0) \end{aligned}$$

- What does u look like?

Utility of money

Reference scale for u relative to best/worst outcomes:

$$u(\$0) = 0$$
$$u(\$50) = 1$$

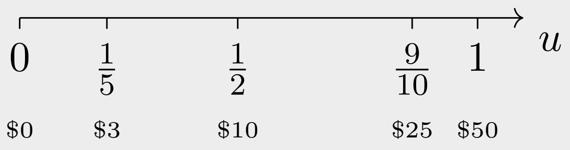


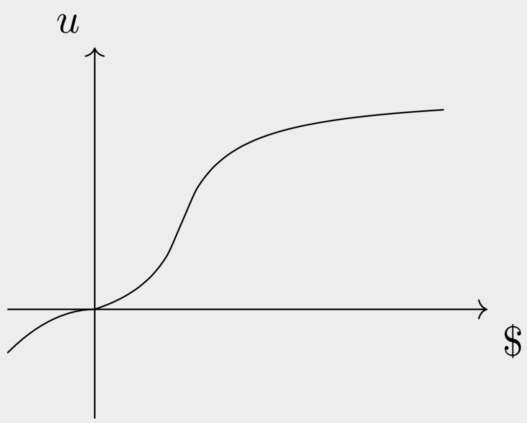
Reference lotteries lie on diagonal:

$$U([\tfrac{1}{2} : \$50|\tfrac{1}{2} : \$0]) = \tfrac{1}{2}u(\$50) + \tfrac{1}{2}u(\$0) = \tfrac{1}{2}$$
$$U([p : \$50|(1-p) : \$0]) = p$$

Utility of money

On Alice's utility scale the monetary outcomes are arranged as follows:





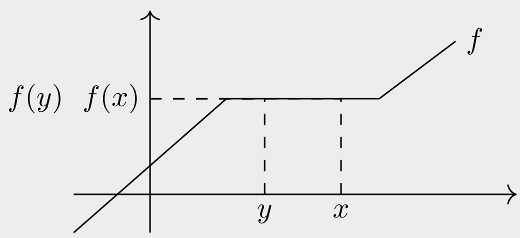
Question

What properties do typical utility functions for money have?

Utility values should increase with increasing money

Utility Utility of money

Functions on ordered sets



Definition (Monotonic increasing function)

A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is *monotonically increasing*, or *non-decreasing*, iff for any $x, y \in \mathbb{R}$, if $x \geqslant y$, then $f(x) \geqslant f(y)$.

Examples: the following are non-decreasing functions on $\mathbb{R}$: $f(x) = \frac{1}{10}x$, $f(x) = x$, $f(x) = c$, for any fixed $c \in \mathbb{R}$

Exercise

Does this imply the converse; *i.e.*, if $f(x) \geqslant f(y)$, then $x \geqslant y$?

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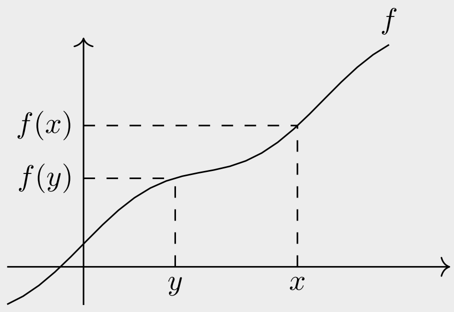
Strictly increasing functions

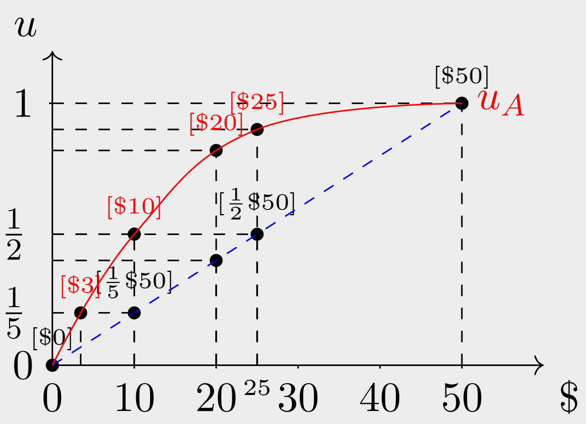


Definition (Strictly increasing function)

A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* iff for any $x, y \in \mathbb{R}$, if $x > y$, then $f(x) > f(y)$.

Examples: $f(x) = \frac{1}{10}x$, $f(x) = x$, $f(x) = 3x + 2$, $f(x) = x^2$ for $x \geqslant 0$, $f(x) = \log_2 x$

Utility for money



How much money is $[\frac{1}{2}\$50]$ worth to Alice? $10

The EMV of $[\frac{1}{2}\$50]$ is $25. How much of that amount is Alice willing to give up for a certain $10? Up to $\$25 - \$10 = \$15$

Definition (Certainty equivalent)

An agent's *certainty equivalent* for a lottery is the value x_c for which the agent would be indifferent between it and the lottery; *i.e.*, $u(x_c) = U(\ell)$.

Definition (Risk premium)

The *risk premium* of an agent for lottery ℓ is the difference between the EMV of the lottery and the certainty equivalent: $V_\$(\ell) - x_c$.

Repeated trials

Example (Alice and Bob)

Alice and her twin, Bob, have $\$10$ each and they are offered, separately, 4 to 1 odds on a team in two different football matches (*e.g.*, home and away). They believe the team has a 2 in 5 chance of winning each match.

- Should Alice bet?

  Outcomes if both gamble:

$$\ell_{AB} = \left[\tfrac{9}{25} : (\$0, \$0) \middle| \tfrac{6}{25} : (\$0, \$50) \middle| \tfrac{6}{25} : (\$50, \$0) \middle| \tfrac{4}{25} : (\$50, \$50)\right]$$

  If Alice and Bob share the risk/gain then:

$$(\$x, \$y) \sim \$\left(\tfrac{x+y}{2}\right) \qquad \textit{i.e. } u_A(x, y) = u_A\left(\tfrac{x+y}{2}\right)$$

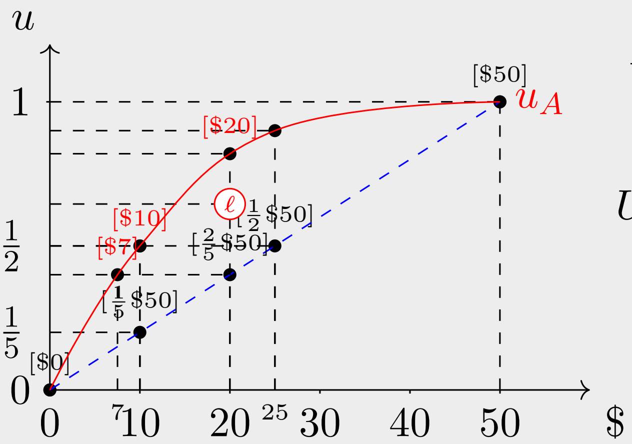
  So for Alice:

$$\begin{aligned} \ell_A &= \left[\tfrac{9}{25} : \$0 \middle| \tfrac{6}{25} : \$25 \middle| \tfrac{6}{25} : \$25 \middle| \tfrac{4}{25} : \$50\right] \\ &= \left[\tfrac{9}{25} : \$0 \middle| \tfrac{12}{25} : \$25 \middle| \tfrac{4}{25} : \$50\right] \end{aligned}$$

Repeated trials

Where does ℓ_A fit in in the scheme of things?

$$\ell_A = \left[\tfrac{9}{25} : \$0 \middle| \tfrac{12}{25} : \$25 \middle| \tfrac{4}{25} : \$50\right]$$

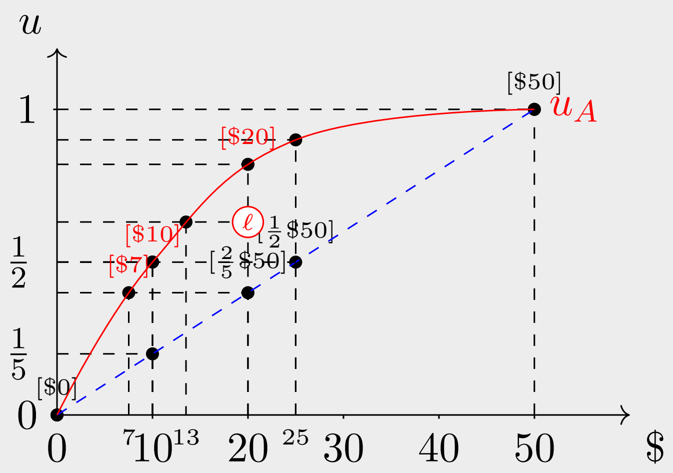


$$V_\$(\ell_A) = \tfrac{12}{25}(25) + \tfrac{4}{25}(50) = 20$$

$$\begin{aligned} U_A(\ell_A) &= \tfrac{9}{25}(0) + \tfrac{12}{25}u_A(\$25) + \tfrac{4}{25}(1) \\ &\simeq 0 + \tfrac{12}{25}\left(\tfrac{9}{10}\right) + \tfrac{4}{25} = \tfrac{4}{25}\left(\tfrac{37}{10}\right) \\ &> \tfrac{4}{25}\left(\tfrac{35}{10}\right) = \tfrac{14}{25} > \tfrac{1}{2} = u_A(\$10) \end{aligned}$$

Alice should bet, sharing the risk and the winnings!

Repeated trials



- The individual bets are favourable for both Alice and Bob
- Despite this neither Alice nor Bob would take their respective individual bets
- However, they should bet together over multiple bets/trials

Risk attitudes

Definition (Risk attitudes)

An agent is:

- *risk averse* iff its certainty equivalent is less than the lottery's expected value; *i.e.*, it values the lottery to be worth less than the expected value.
- *risk seeking* (*risk prone*) iff its certainty equivalent is greater than the lottery's expected value.
- *risk-neutral* otherwise.

Exercises

- What is Alice's certainty equivalent for the lottery with Bob?
- The risk premium in what range if the agent is: risk averse? risk seeking? risk neutral?

Risk attitudes

For a lottery:

Definition (Risk averse)

An agent is *risk averse* if its utility function is concave down over the range of possible outcomes.

Definition (Risk seeking)

An agent is *risk seeking* if its utility function is concave up (convex) over the range of possible outcomes.

Definition (Risk neutral)

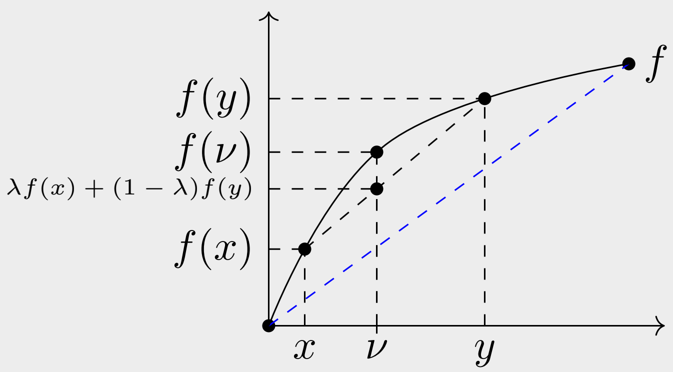
An agent is *risk neutral* if its utility function both concave down and up; *i.e.*, linear.

Concave and convex functions

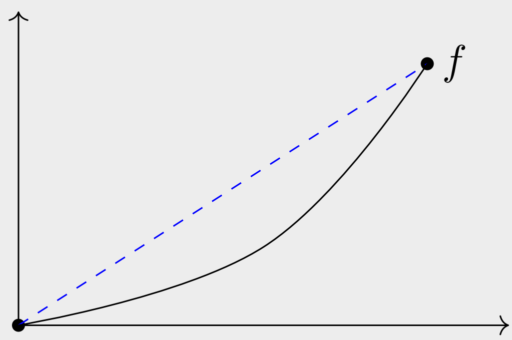
Definition (Concave and convex)

A function $f : \mathbb{R} \to \mathbb{R}$ is *concave down* in the interval $[a, b]$ if for all $x, y \in [a, b]$, and all $\lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \geqslant \lambda f(x) + (1 - \lambda)f(y)$, and *concave up* (or *convex*) if $f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$.

concave



convex



where $\nu = \lambda x + (1 - \lambda)y$

Summary: risk attitudes and utility

- Not all quantities (*e.g.*, \$) accurately represent ‘true’ preference
- Measure preference in terms of utility; agent must calibrate utilities against uncertain outcomes (lotteries)
- An agent’s utility is personal/subjective; *i.e.*, particular to him. Different agents may have different utilities for the same ‘outcome’
- Utility functions are non-decreasing; this means that over many trials *Bayes* utilities approach expected values
- The shape of an agent’s utility curve/function determines its risk attitude