GSOE9210 Engineering Decisions

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Decisions under complete uncertainty/ignorance

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Ignorance and possibility

Beware of false knowledge; it is more dangerous than ignorance.
—*George Bernard Shaw*

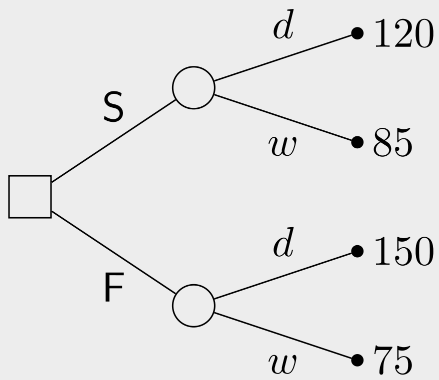
- Ignorance = possibilities without probabilities
  i.e., more than one possible state; probabilities unknown
- Actions may not determine a unique outcome; 'states' are those aspects of a situation which, when combined with possible actions, discriminate between outcomes
- Rational decision rules for choosing/evaluating actions under ignorance

Decisions under uncertainty

Example (Uncertain school fund-raising)

Proceeds of the school fund-raiser depend on the weather; on a dry day (d) the school expects to make $\$150$ for a fête (F) but only $\$120$ for a sports day (S). However, on a wet day (w) the sports day will net $\$85$ and the fête only $\$75$.

	d	w
S	120	85
F	150	75



d dry day

w wet day

Decisions under uncertainty

Problem

How to assign values to uncertain actions?

	d	w
S	120	85
F	150	75

- How should $V(\text{S})$ and $V(\text{F})$ be defined?
- How should V (over actions) depend on v (over outcomes)?

Lotteries

Definition (Lottery)

A *lottery* over a finite set of states $\mathcal{S}$, and outcomes, or *prizes*, Ω, is a function $\ell : \mathcal{S} \to \Omega$. The lottery ℓ is written:

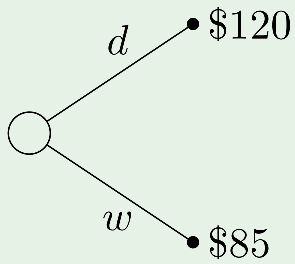
$$\ell = [s_1 : \omega_1 | s_2 : \omega_2 | \ldots | s_n : \omega_n]$$

where for each $s_i \in \mathcal{S}$, $\omega_i = \ell(s_i)$.

Example (Dry or wet?)

Choosing a sports day is represented by lottery:

$$\ell_{\mathsf{S}} = [d : \$120 | w : \$85]$$



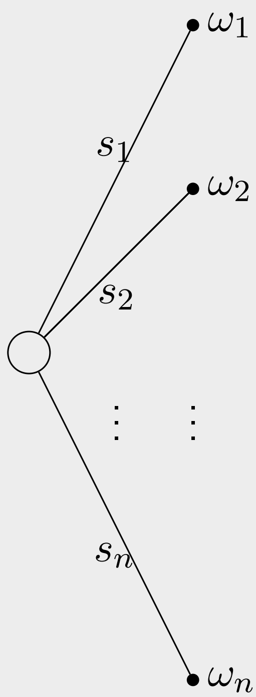
Decision problems and lotteries

$$[s_1 : \omega_1 | s_2 : \omega_2 | \ldots | s_n : \omega_n]$$

- Uncertain action = lottery
- Choosing an action = choosing a lottery

Corollary

The problem of evaluating actions amounts to the problem of determining how to compare and/or evaluate lotteries.



Decisions under uncertainty: ignorance

Example (Raffle)

There are four raffle tickets in a hat. Each ticket is either blue or red, but you don't know how many of each there are. Blue tickets win $\$3$; red ones lose ($\$0$). The cost of entering the raffle is $\$1$.

Exercises

- Draw the decision tree and table for this problem
- Should you draw a ticket in the raffle?
- What if you knew there were three blue tickets? Four? None?
- How many blue tickets would there have to be to make it worth entering?
- If there were n blue tickets $(0 \leqslant n \leqslant 4)$, what would the prize have to be to make it worthwhile entering?

Decisions under ignorance: possibilities

Definition (Decision rule)

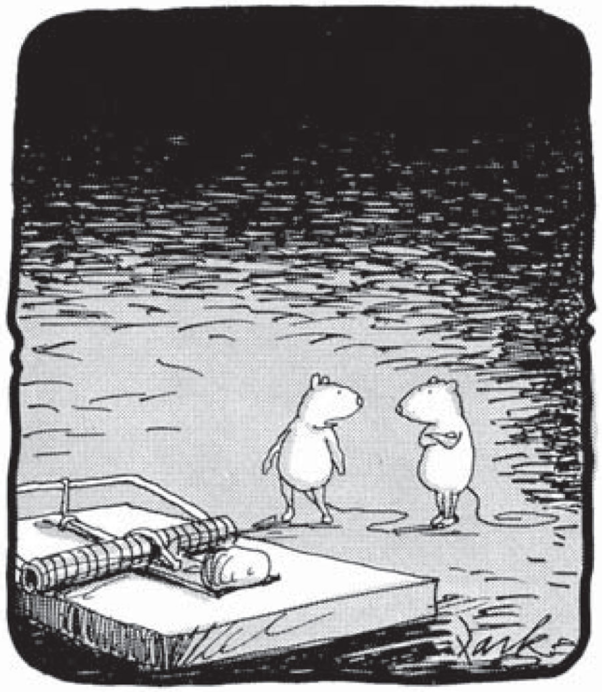
A *decision rule* is a way of choosing, for each decision problem, an action or set of actions.

Rational decision rules under ignorance:

- Optimistic *MaxiMax* rule: "nothing ventured, nothing gained"
- Wald's pessimistic *Maximin* rule: "better to be safe than sorry"
- *Hurwicz*'s mixed optimistic-pessimistic rule: degree of optimism/pessimism
- Savage's *miniMax Regret* rule: least *regret*
- Laplace's *principle of insufficient reason*

Approach: assign values (V) to actions based on preferences over outcomes (v)

MaxiMax



“Modern technology being what it is, there’s a good chance it won’t work anyway.”

MaxiMax: associate with each action states which yields the most preferred outcome (*i.e.*, preference maximal)

MaxiMax selects the actions which yield preference-maximal outcomes among these

Decisions under uncertainty

Example (Uncertain school fund-raising)

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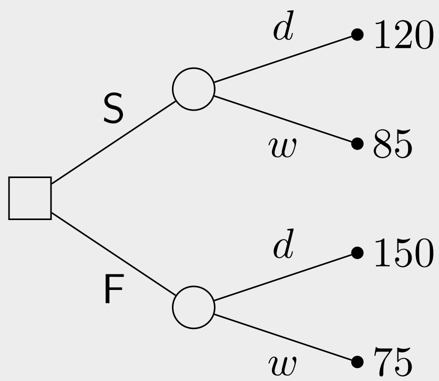

MaxiMax (MM): aim for the best

	s_1	s_2	s_3	V
A$_1$	6	0	4	6
A$_2$	2	5	1	5
A$_3$	4	3	2	4

- For each action find best possible outcome over all possible states; *i.e.*, for each row find the maxima: $V_{MM}(\mathsf{A}) = M(\mathsf{A}) = \max\{v(\omega(\mathsf{A}, s)) \mid s \in \mathcal{S}\}$
- Choose actions/rows with maximal value: A$_1$
- Equivalently: find maximum value of entire table, choose row/action with this value
- $r_{MM}(\omega) = \arg\max\{M(\mathsf{A}) \mid \mathsf{A} \in \mathcal{A}\}$

MaxiMax

	s_1	s_2
A	10	0
B	9	9

- Which action is better?
- How could ties be broken?

	s_1	s_2	s_3	V
A$_1$	6	0	4	6
A$_2$	2	6	1	6
A$_3$	4	3	2	4

Risk attitudes

- *MaxiMax* is a decision rule for extreme *risk takers*
- *MaxiMax* is 'rational' decision rule for decision-makers with risk-taking attitudes/preferences

- In some cases it may be wise to be *risk averse*: *i.e.*, avoid, reduce, or protect against risk
- What might a risk averse decision rule look like?

Decisions under uncertainty

Example (Uncertain school fund-raising)

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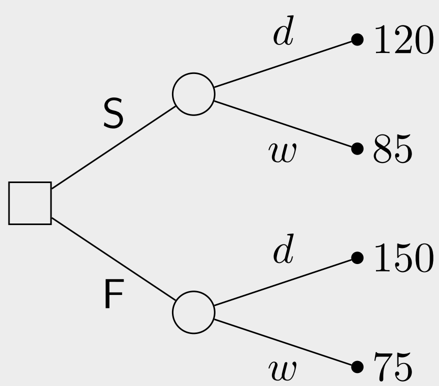

Maximin (Mm): best in the worst case

- Assume worst case/state will occur for each action

	s_1	s_2	s_3	V
A_1	6	0	4	0
A_2	2	5	1	1
A_3	4	3	2	2

- For each action find worst possible outcome under all possible cases/states; *i.e.*, for each row find minimum value: $V(\mathsf{A}) = m(\mathsf{A}) = \min\{v(\omega(\mathsf{A}, s)) \mid s \in \mathcal{S}\}$
- $m(\mathsf{A})$ sometimes called the *security level* of action A
- Choose the action/row with the maximum of these: A_3
- $r_{Mm}(\omega) = \arg\max\{m(\mathsf{A}) \mid \mathsf{A} \in \mathcal{A}\}$

Maximin

	s_1	s_2
A	10	0
B	1	1

- Which action is better?
- How could ties be broken?

	s_1	s_2	s_3	V
A_1	6	0	4	0
A_2	2	5	2	2
A_3	4	3	2	2

Hurwicz's optimism index

	s_1	s_2	s_3	M	m	$\alpha M + (1-\alpha)m$
A$_1$	6	0	4	6	0	$\frac{9}{2}$
A$_2$	2	5	1	5	1	$\frac{8}{2}$
A$_3$	4	3	2	4	2	$\frac{7}{2}$

- For each action/row, find minima (m) and maxima (M)
- Calculate weighted sum based on *optimism index* $\alpha \in [0, 1]$; *e.g.*, if $\alpha = \frac{3}{4}$, then $V(\text{A}) = \alpha M(\text{A}) + (1-\alpha)m(\text{A}) = \frac{3}{4}M + \frac{1}{4}m$.
- Choose the row/action that maximises this value: A$_1$

Exercise

What happens when $\alpha = 1$? $\alpha = 0$?

MaxiMax and *Maximin*

	s_1	s_2
A	100	0
B	99	99

	s_1	s_2
A	100	4
B	5	5

	s_1	s_2
A	a_1	a_2
B	b_1	b_2

Compare problems above:

- *MaxiMax* and *Maximin* choose the same action for any values of a_1, a_2, b_1, b_2, provided $a_1 > b_1 \geqslant b_2 > a_2$ is preserved; since $M(\text{A}) = a_1 > M(\text{B})$, $m(\text{B}) > a_2 = m(\text{A})$ remain unchanged; *i.e.*, the actual numbers are irrelevant for the rules
- In this case the differences $a_1 - b_1$ and $b_2 - a_2$ are irrelevant provided $M(\text{A}) > M(\text{B})$ and $m(\text{B}) > m(\text{A})$

Decisions under uncertainty

Example (Uncertain school fund-raising)

Proceeds of the school fund-raiser depend on the weather; on a dry day (d) the school expects to make $\$150$ for a fête (F) but only $\$120$ for a sports day (S). However, on a wet day (w) the sports day will net $\$85$ and the fête only $\$75$.

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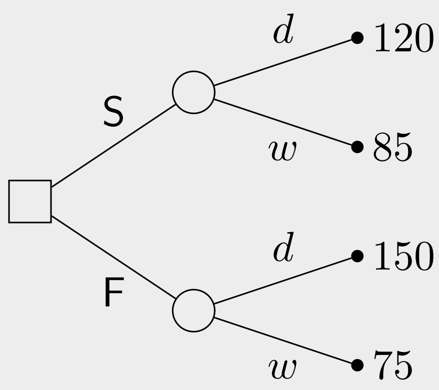


d dry day

w wet day

Best response actions

Definition (Better actions)

An action A is *better* than action B in state s iff A's outcome in s preferred to B's.

Definition (Best response)

An action A is a *best response* in state s iff no other action is better than A in state s.

- Best response actions in a state are preference maximal over all actions in that state

Best response actions

	d	w
S	120	85
F	150	75

- On dry days (d), best response is F; $v(\omega(\mathsf{F}, d)) > v(\omega(\mathsf{S}, d))$
- On wet days (w), best response is S; $v(\omega(\mathsf{S}, w)) > v(\omega(\mathsf{F}, w))$
- Choosing an action which turns out to not be a best response action causes 'regret' (*e.g.*, when discover actual state afterwards)
- A best response means 'no regret' in the given state . . .

Regret

Definition (Regret)

The *regret*, or *opportunity loss*, of an action in a state is the difference between the action's value and that of the best response for that state.

Consider fund-raising problem:

	d	w
S	120	85
F	150	75
M_s	150	85

	d	w	M_R
S	30	0	30
F	0	10	10

- The maximum regret for the sports day is 30 but only 10 for the fête
- The action which minimises the maximum regret is F

miniMax Regret

	s_1	s_2	s_3
A_1	6	0	4
A_2	2	5	1
A_3	4	3	2
	6	5	4

	s_1	s_2	s_3	V
A_1	0	5	0	5
A_2	4	0	3	4
A_3	2	2	2	2

- For each column/state s, find its maximum value (M_s)
- Construct the regret table: $R(\omega) = M_s - v(\omega)$
- For each action/row find the maximum regret: $V(\mathsf{A}) = \max\{R(\omega(\mathsf{A}, s)) \mid s \in \mathcal{S}\}$
- Choose the row/action that minimises the regret: A_3

Laplace's principle of insufficient reason

	s_1	s_2	s_3	V
A_1	6	0	4	$\frac{6}{3} + 0 + \frac{4}{3} = 3\frac{1}{3}$
A_2	2	5	1	$\frac{2}{3} + \frac{5}{3} + \frac{1}{3} = 2\frac{2}{3}$
A_3	4	3	2	$\frac{4}{3} + \frac{3}{3} + \frac{2}{3} = 3$

- *Assume* each state is equally likely
- For each row/action calculate the mean value: $V(\mathsf{A}) = \frac{1}{n}v(\omega(\mathsf{A}, s_1)) + \cdots + \frac{1}{n}v(\omega(\mathsf{A}, s_n))$
- Choose the row/action with maximum value: A_1

Exercise

How could you simplify this decision rule?

Decision rules

	s_1	s_2	s_3	s_4	V
A_1	2	2	0	1	
A_2	1	1	1	1	
A_3	0	4	0	0	
A_4	1	3	0	0	

- For a value function V on actions, a decision rule r_V is defined by:

$$r_V(p) = \arg\max\{V(\mathsf{A}) \mid \mathsf{A} \in \mathcal{A}\}$$

- Which rules agree (*i.e.*, choose the same actions)?

$$\begin{aligned} V_{MM}(\mathsf{A}) &= \max\{v(\omega(\mathsf{A}, s)) \mid s \in \mathcal{S}\} \\ V_{Mm}(\mathsf{A}) &= \min\{v(\omega(\mathsf{A}, s)) \mid s \in \mathcal{S}\} \\ V_{mMR}(\mathsf{A}) &= \max\{R(\mathsf{A}, s) \mid s \in \mathcal{S}\} \end{aligned}$$

Summary: decisions under complete uncertainty/ignorance

- Ignorance: more than one possible state; likelihoods unknown
- Lotteries: uncertain situations
- Risk attitudes
- Best response/action
- Some decision rules under complete uncertainty (ignorance):
  - Optimistic *MaxiMax* rule: “nothing ventured, nothing gained”
  - Wald’s pessimistic *Maximin* rule: “its better to be safe than sorry”
  - *Hurwicz*’s mixed optimistic-pessimistic rule: use an *optimism index* α
  - Savage’s *miniMax Regret* rule; least *opportunity loss*
  - Laplace’s *principle of insufficient reason* rule

The *Maximin* principle

Definition (The *Maximin* principle)

Assume only minimally preferred outcomes occur and choose actions that lead to most preferred among these.

- *Maximin* and *miniMax Regret* follow *Maximin* principle: original values vs regrets
- *Maximin* principle is main decision principle used under complete uncertainty
- We've seen *Maximin* and *miniMax Regret* on decision tables, but what about more complex decision problems (*e.g.*, multiple decision points)?

Multi-stage decisions

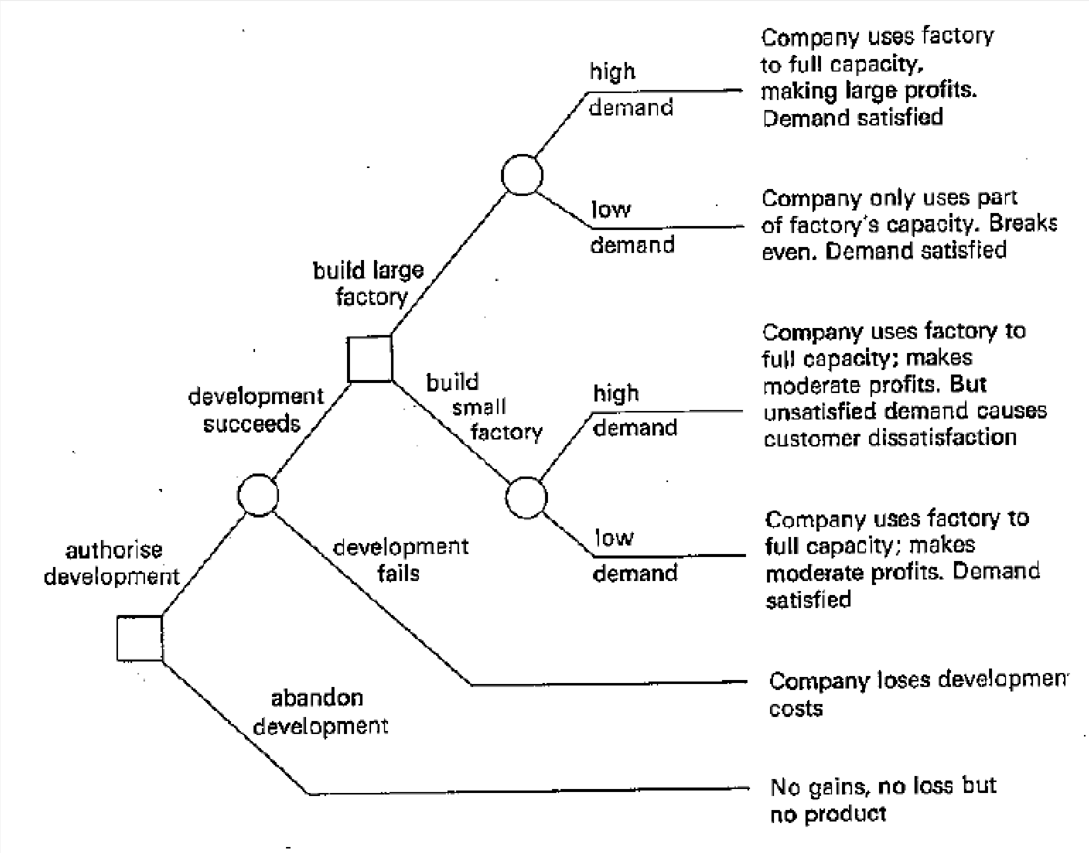
Example (Product development)

You head the R&D department of a small manufacturing company. Your company is working on a new product. The company must decide whether to develop a prototype, and, if this is successful, the scale of production (*i.e.*, the size of the factory).

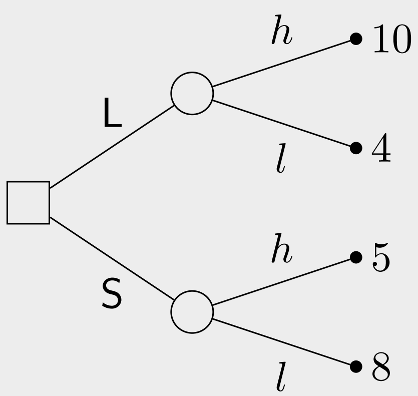
Questions

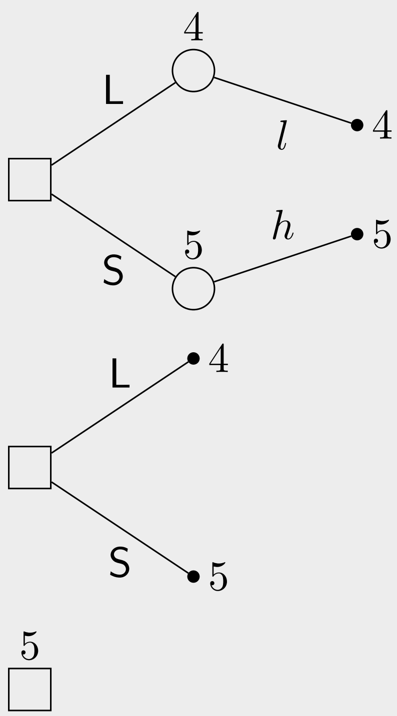
- What does *Maximin* or *miniMax Regret* mean in this problem?
- Is there a decision-table representation?

Multi-stage decisions



Node evaluation





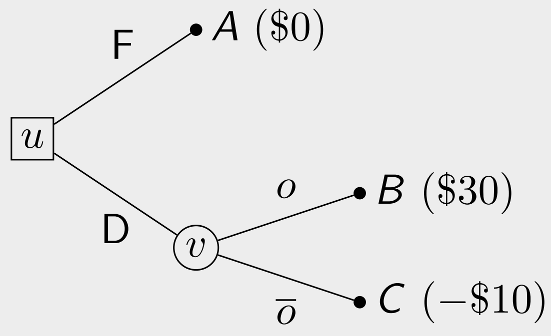
- What does *Maximin* mean in a tree?
- *Maximin* eliminates branches in chance nodes (*i.e.*, prunes the tree)
- Reduces problem to that of certainty

Node evaluation

- Each decision problem is assigned a 'value' by a decision rule
- The *Maximin* algorithm for decision trees:
  1. Begin at leaves
  2. At parent node:
     1. if a chance node, prune all children except the minimally preferred
     2. if a decision node, by *elimination principle*, prune all children except the maximally preferred
     3. propagate (unique) value up to parent node
  3. Repeat previous step until root is reached
- Value of root is value of the problem (under *Maximin*); *i.e.*, value which *Maximin* assigns to the problem

Decisions under complete uncertainty/ignorance

Problem representation: decision tables



Represented as a table:

	ω	o	$\overline{o}$
$\mathcal{A}$	F	A	A
	D	B	C

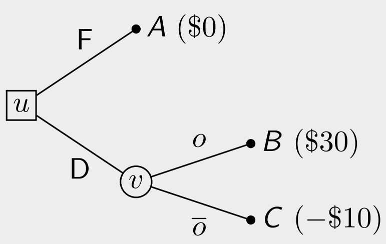
(columns: $\mathcal{S}$)

Decision tables:

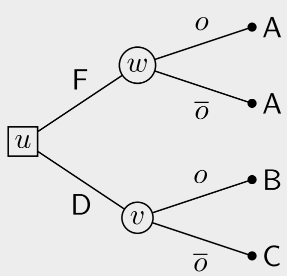
- row = action
  column = state
- Interpretation: B $= \omega(\text{D}, o)$ means "B is the outcome of action D in state o";

- Observation: each action–state pair uniquely determines an outcome
- Model as a 2-ary (dyadic) function: $\omega : \mathcal{A} \times \mathcal{S} \to \Omega$

Trees and tables

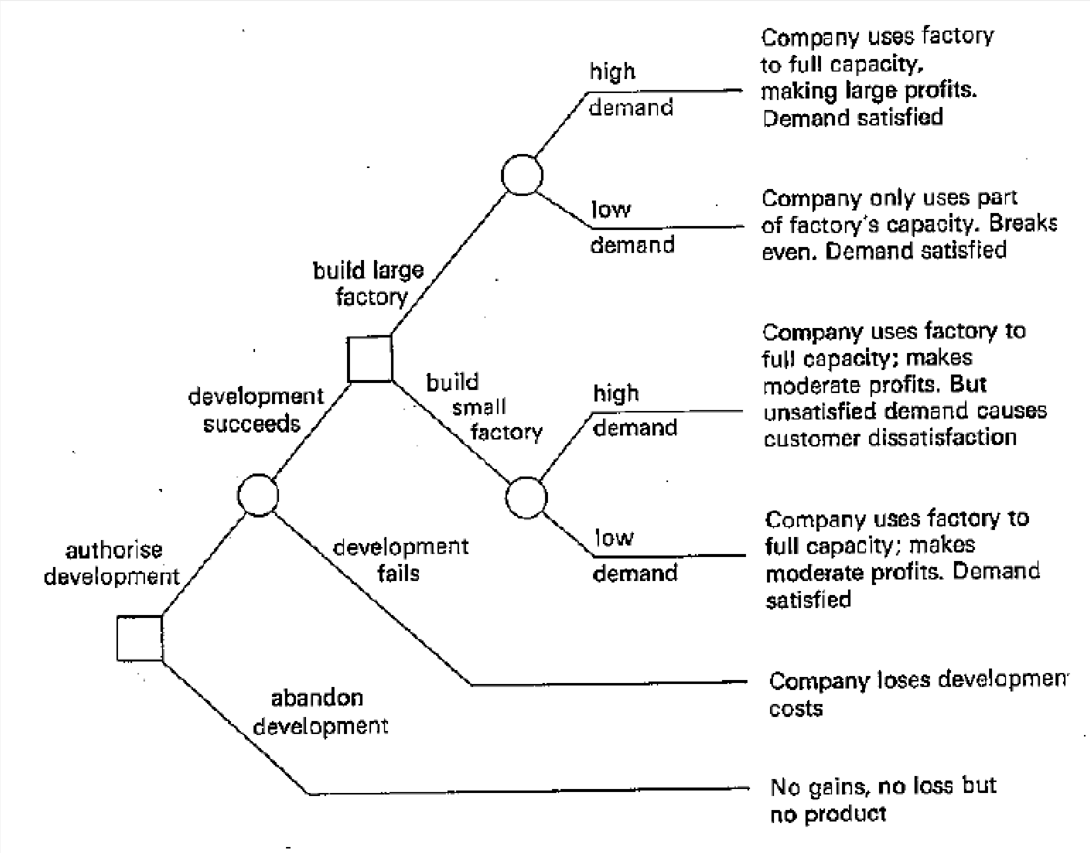


	o	$\overline{o}$
F	\$0	\$0
D	\$30	−\$10



- Multiple trees may correspond to the same table
- Going from tables (*normal form*) to trees (*extensive form*) is straight forward, but the converse can be tricky
- Which representation is better: trees or tables?
- Which representation facilitates decision analysis most?

Multi-stage decisions



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Questions

- What does *Maximin* or *miniMax Regret* mean in this problem?
- Is there a decision-table representation?

Actions to strategies

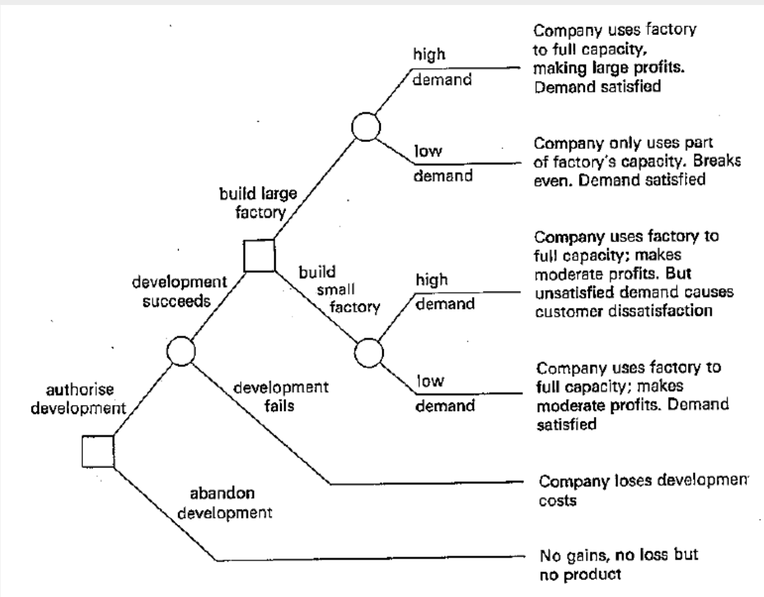
In a decision tree:

- Recall that a decision table is a representation of the outcome mapping $\omega : \mathcal{A} \times \mathcal{S} \to \Omega$
- Observation: following a path from the root to a leaf leads to a unique outcome
- Generalising:
  - A 'state' must include branches at chance nodes
  - An 'action' must include branches at decision nodes

Definition (Strategy)

A *strategy* (or *policy* or *plan*) is a procedure that specifies the selection of an action at every *reachable* decision point.

Normalisation



- States:

s_1	s_2	s_3
s, h	s, l	f

- A strategy must specify an action at each *reachable* decision point; *e.g.*, "Authorise development (Au), if development succeeds (s), then build large factory (L)"

Normalisation

Encoding:

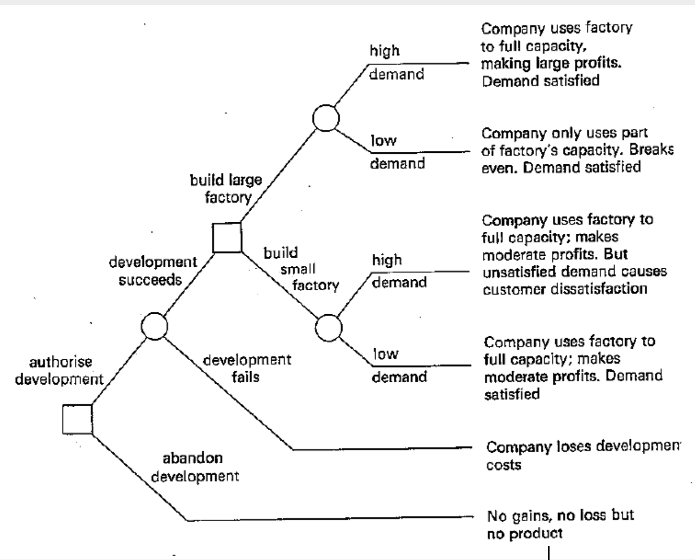
- α/A says:
  At the decision node reached via path α choose action A.
- Example: Au;s/S:
  If development has been authorised (Au) and has succeeded (s), choose to build a small factory (S).
- Strategies for this problem:

  A_1 Au;s/L

  A_2 Au;s/S

  A_3 Ab

Normalisation



Code	*Description*
fc	full capacity
pc	partial capacity
lp	large profits
mp	moderate profits
be	break even
ldc	lose dev. costs
sat	demand satisfied
dis	dissatisfaction
sq	status quo

	s, h	s, l	f
Au;s/L	fc,lp,sat	pc,be,sat	ldc
Au;s/S	fc,mp,dis	fc,mp,sat	ldc
Ab	sq	sq	sq

Normalisation

Outcome values:

ω	v
fc,lp,sat	10
pc,be,sat	4
ldc	-1
fc,mp,dis	5
fc,mp,sat	8
sq	0

Decision table:

	s,h	s,l	f
Au;s/L	10	4	-1
Au;s/S	5	8	-1
Ab	0	0	0

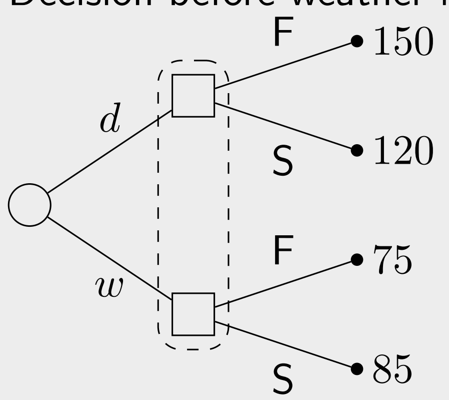
Exercises

- Find the *Maximin* and *miniMax Regret* strategies for this problem.
- Evaluate this problem under *MaxiMax*, *Maximin*, *miniMax Regret* using both normal and extensive forms.

Representing information

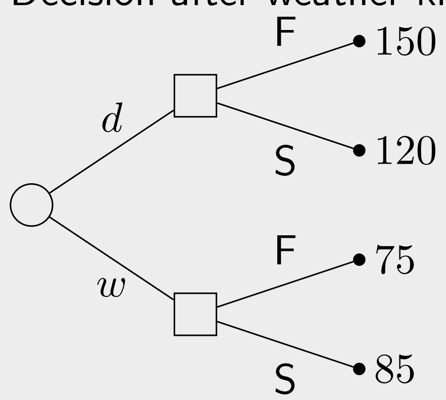
Consider the fund-raiser example.

- Decision before weather known:



- Decision nodes part of the same *information set*
- Possible choices: F, S only

- Decision after weather known:



- Decision nodes distinguishable
- Possible choices: d/F, d/S, w/F, w/S

Decisions under complete uncertainty/ignorance

Indifference: equal preference

- Which action below is preferred above under *Maximin*?

	s_1	s_2
A	1	0
B	0	1

Definition (Indifference)

If two actions A and B are *equally preferred* then the agent is said to be *indifferent* between A and B.

- Indifference means an agent prefers two alternatives equally, not that it doesn't *know* which it prefers

Indifference classes

Definition (Indifference class)

An *indifference class* is a non-empty set of all actions/outcomes between which an agent is indifferent.

- For a given action $\mathsf{A} \in \mathcal{A}$, the indifference class of A is given by

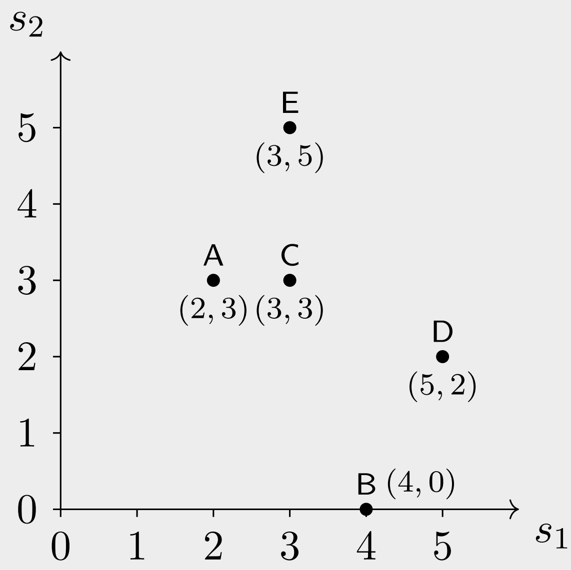
$$I(\mathsf{A}) = \{a \in \mathcal{A} \mid V(a) = V(\mathsf{A})\}$$

- Indifference classes partition set of all actions
- Different agents have different preferences over outcomes/actions, hence different indifference classes
- Different decision rules evaluate actions differently; *i.e.*, produce different indifference classes

Graphical representation

	s_1	s_2
A	2	3
B	4	0
C	3	3
D	5	2
E	3	5

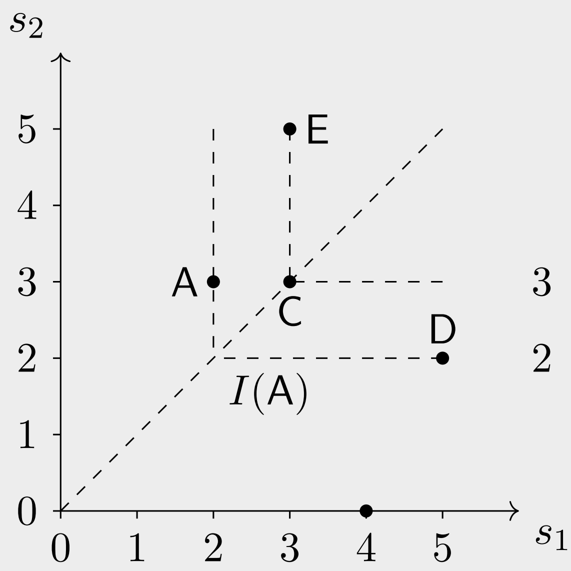
Let $v_i(a) = v(a, s_i)$ be the value of action a in state s_i. Each action a corresponds to a point (v_1, v_2), where $v_i = v(a, s_i)$.



Indifference curves: *Maximin*

For the pure actions below:

	s_1	s_2
A	2	3
B	4	0
C	3	3
D	5	2
E	3	5



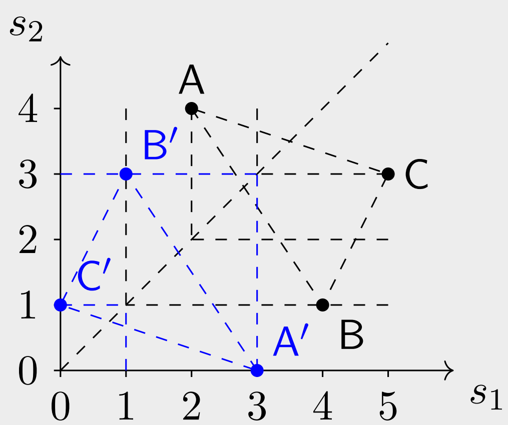
Consider curves of all points representing strategies with same *Maximin* value; *i.e.*, *Maximin indifference curves*.

Graphing regret

- Consider three actions:

	s_1	s_2
A	2	4
B	4	1
C	5	3

	s_1	s_2
A	3	0
B	1	3
C	0	1

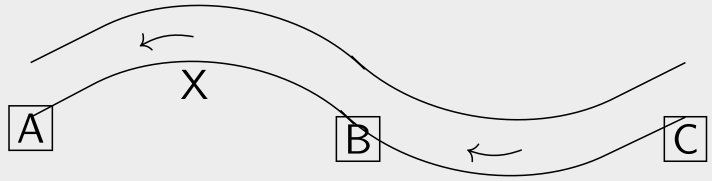


- Regrets and indifference curves for *miniMax Regret* in blue

Exercises

In regard to preference over actions, what is the relation between *Maximin* and *miniMax Regret*?

River example



Example (River logistics)

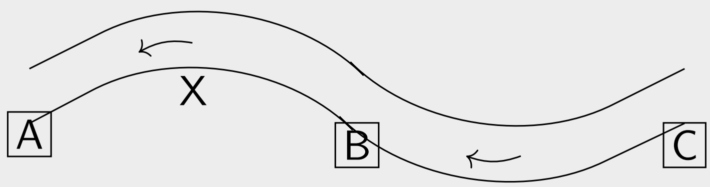
Alice's warehouse is located at X on a river that flows down-stream from C to A. She delivers goods to a client at C via motor boats. On some days a (free) goods ferry (f) travels up the river, stopping at A then B and C, but not at X.

The fuel required (litres) to reach C from each starting point:

	A	X	B	C
To C from:	4	3	2	0

Alice wants to minimise fuel consumption (in litres).

River example



	f	$\overline{f}$
A	4	0
B	3	1
C	1	1

Alice considers three possible ways to get to C (from starting point X):

- A via A, by floating down the river
- B via B, by travelling up-stream to B
- C by travelling all the way to C

Outcomes are measured in *litres left* in a four-litre tank.

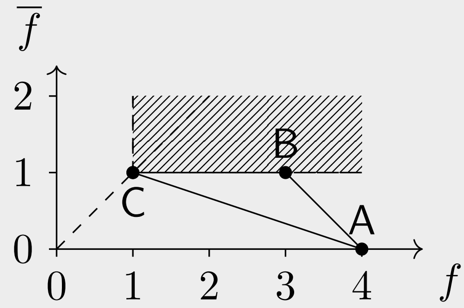
Exercise

Let $w : \Omega \to \mathbb{R}$ denote fuel consumption in litres. What transformation $f : \mathbb{R} \to \mathbb{R}$ is responsible for the values $v : \Omega \to \mathbb{R}$ in the decision table?

River example

- Axes correspond to payoffs in each of the two states; *i.e.*, payoff v_1 in state $s_1 = f$ and v_2 in $s_2 = \overline{f}$
- Actions graphed below:

	f	$\overline{f}$
A	4	0
B	3	1
C	1	1



- Option C not a better response than B under any circumstances (*i.e.*, in any state)
- C worse than B in some cases and no better in all others; C can be *discarded*

Generalised dominance

Definition (Strict dominance)

Strategy A *strictly dominates* B iff every outcome of A is more preferred than the corresponding outcome of B.

Definition (Weak dominance)

Strategy A *weakly dominates* B iff every outcome of A is no less preferred than the corresponding outcome of B, and some outcome is more preferred.

	s_1	s_2	s_3
A	3	4	2
B	4	4	3
C	5	6	3

Exercise

Which strategies in the decision table shown are dominated?

Dominance and best response

Corollary

Strategy A strictly dominates B iff A is a better response than B in each possible state.

Corollary

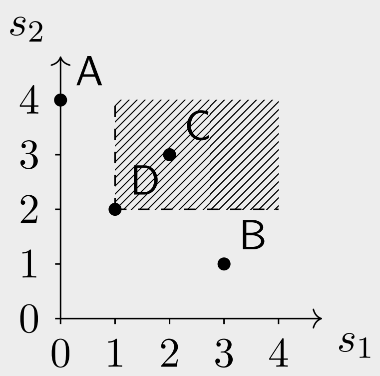
Strategy A weakly dominates B iff A is a better response than B in some possible state and B is not a better response than A in any state.

Dominance principle

A rational agent should never choose a dominated strategy.

Admissible actions

	s_1	s_2
A	0	4
B	3	1
C	2	3
D	1	2



Definition (Admissible)

An action is *admissible* iff it is not dominated by any other action. An action which is not admissible is said to be *inadmissible*. The set of all admissible actions is called the *admissible frontier*.

Exercises

Which actions above are admissible?

Dominance: *MaxiMax* and *Maximin*

	s_1	s_2	M	m
A	2	2	2	2
B	2	1	2	1
C	1	1	1	1

Definition (Dominance elimination)

A decision rule is said to satisfy (strict/weak) *dominance elimination* if it never chooses actions that are (strictly/weakly) dominated.

- Dominated actions can be discarded under any rule that satisfies dominance elimination

Dominance summary

Rules that satisfy strict/weak dominance elimination.

Rule	Strict	Weak
MaxiMax	$\surd$	$\times$
Maximin	$\surd$	$\times$
Hurwicz's	$\surd$	$\times$
miniMax Regret	$\surd$	$\times$
Laplace's	$\surd$	$\surd$

Exercise

Verify the properties above.

Rule axioms

The following criteria can be used to assess the suitability of decision rules:

Axiom of dominance

A decision rule should never choose a dominated action.

Axiom of invariance

A decision rule's choices should be independent of representation.

Axiom of solubility

A decision rule should always select at least one action.

Axiom of independence

Adding a duplicate state should not affect a rule's decision.

Summary: decisions under complete uncertainty

- *Maximin* in extensive form
- Multi-stage decisions
- Extensive to normal form translation
- Information in extensive form
- Graphical visualisation
- Indifference
- Dominance and admissibility