Graph Theory

Problem 1
True or false?

(a) The complete bipartite graph $K_{5,5}$ has no cycle of length five.

(b) If T is a tree with at least four edges, then $\chi(T) = 3$.

(c) Let C_n denote a cycle on n vertices. For all $n \geq 5$ it holds $\chi(C_n) \neq \chi(C_{n-1})$.

(d) It is possible to remove two edges from K_6 so that the resulting graph has a clique number of 4.

Solution

(a) True: $K_{5,5}$ is 2-colourable but a cycle of length five requires 3 colours.

(b) False: Trees are 2-colourable, so $\chi(T) \leq 2$ for all trees T.

(c) True: Odd-length cycles have chromatic number 3, whereas even-length cycles have chromatic number 2.

(d) True: Consider two edges that do not have a vertex in common. Any collection of 5 vertices will necessarily contain both endpoints of at least one of these edges. If we remove these edges, then the resulting graph cannot have a 5-clique. It will have a 4-clique, which can be found by taking any four vertices that do not include both endpoints of one of the removed edges.

Problem 2
What is the minimum number of edges that need to be removed from K_5 so that the resulting graph has a chromatic number of

(a) 3?

(b) 2?

(c) 1?

Solution

(a) 2 edges. To achieve $\chi = 3$, one needs (at least) to avoid having any 4-cliques. Removing one edge leaves a 4-clique (actually two such cliques — including any one but not both of that edge's endpoints, plus the remaining three vertices). Removing two edges suffices — remove any pair of edges which do not share a common vertex; the remaining graph can then be coloured with 3 colours.

(b) 4 edges. A chromatic number of 2 means that the graph is bipartite, with two groups of nodes where each group can be painted with one colour. To minimise the number of *removed* edges, we want to have as many edges as possible in the remaining bipartite graph. We therefore look at *complete bipartite* graphs with a total of 5 vertices. As $K_{1,4}$ has four edges and $K_{2,3}$ has six edges, the latter is the better choice. To reach it we need to remove 4 of the edges in K_5.

(c) 10 edges. A chromatic number of 1 means a fully disconnected graph, with no edges at all. Therefore all 10 edges of the original graph must be removed.

Problem 3
Consider the complete 3-partite graphs $K_{4,1,1}$, $K_{3,2,1}$, $K_{2,2,2}$.

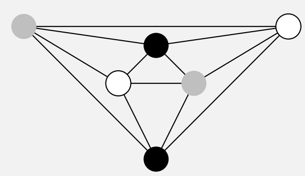
(a) What is the chromatic number of each of these graph?

(b) Which of these graphs are planar?

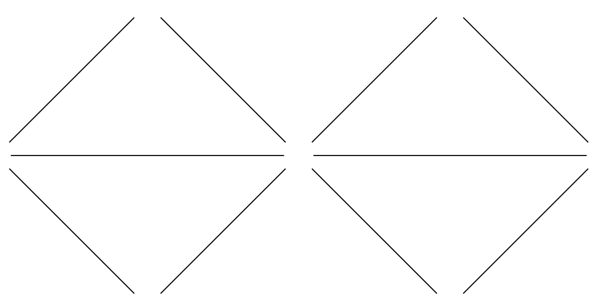
Solution

(a) For every 3-partite graph three colours suffice: use a different colour for each of the groups of vertices. Three colours are also necessary: any three vertices selected from three different groups will form a clique.

(b) $K_{4,1,1}$ can be easily drawn without intersections. $K_{3,2,1}$ contains $K_{3,3}$ hence is not planar. A planar drawing of $K_{2,2,2}$ is:



Problem 4
Consider the following graph, G:



(a) What is the chromatic number of G?

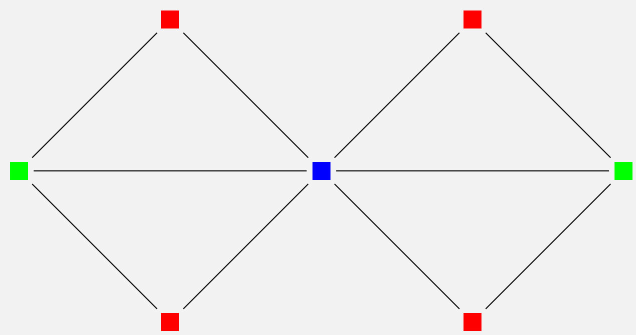
(b) What is the clique number of G?

(c) Does G have a Hamiltonian path and/or a Hamiltonian cycle?

(d) Does G have an Eulerian path and/or an Eulerian cycle?

Solution

(a) Here is a 3-colouring of the graph.



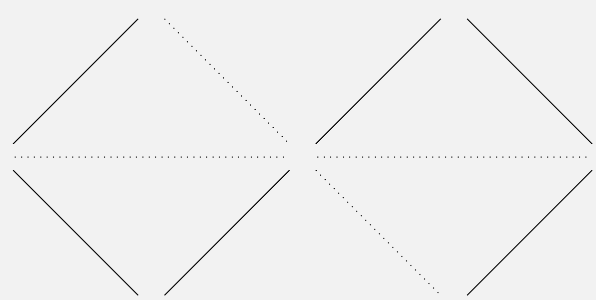
We cannot do any better because the graph contains a 3-clique, so $\chi(G) = 3$.

(b) As observed the graph has a 3-clique, so $3 \leq \kappa(G)$. Since

$$3 \leq \kappa(G) \leq \chi(G) \leq 3,$$

it follows that the clique-number of G is 3.

(c) Here is a Hamiltonian path:



It does not contain a Hamiltonian circuit because there is only one vertex disjoint path from the left-side to the right-side.

(d) The graph has two vertices of odd degree, so it contains an Eulerian path, but not an Eulerian circuit.

Problem 5
Draw a single graph with 6 vertices and 10 edges that satisfies each of the following:

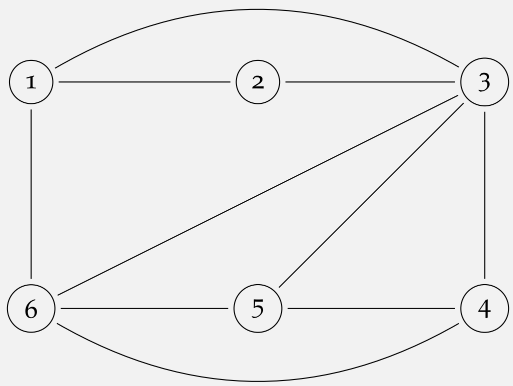
(a) is planar,

(b) contains a Hamiltonian circuit, and

(c) does not contain an Eulerian path.

Solution

Here is a graph with 6 vertices and 10 edges:



(a) It is planar because it has been drawn in the plane with no crossing edges.

(b) 1-2-3-4-5-6-1 is a Hamiltonian circuit.

(c) It has four vertices of odd degree (1,3,4,5), therefore it cannot have an Eulerian path.

Problem 6† (20T2)

(a) Prove that a graph with 6 vertices and at least 13 edges must have at least two vertices of degree 5.

(b) Prove that a graph with 6 vertices and at least 13 edges has clique number at least 4.

(c) Prove that a graph with 6 vertices and at least 13 edges is non-planar.

(d) Draw a planar graph with 6 vertices and 12 edges that has clique number 3. Justify each property.