Combinatorics

Problem 1

(a) In how many ways can the letters a, b, c, d, e, f be arranged so that the letters a and b are next to each other?

(b) In how many ways can the letters a, b, c, d, e, f be arranged so that the letters a and b are not next to each other?

(c) In how many ways can the letters a, b, c, d, e, f be arranged so that the letters a and b are next to each other but a and c are not?

Solution

(a) Assume a occurs before b and group a and b together as a single "letter": (ab). There are $5!$ ways to arrange this new letter with the other 4. Similarly if b occurs before a, there are $5!$ ways; giving a total of $2.5! = 240$ ways of arranging the letters so that a and b are next to each other.

(b) There are $6! = 720$ ways of arranging the letters, and from the previous question, 240 of them have a next to b. So $720 - 240 = 480$ do not have a next to b.

(c) Treat 'cab' as one long symbol: there are $4! = 24$ arrangements. Similarly 24 arrangements including 'bac'. Therefore $240 - 24 - 24 = 192$ arrangements that include either 'ab' or 'ba' but do not include 'cab' or 'bac'.

Problem 2

(a) How many well-formed formulas can be constructed from one $\vee$; one $\wedge$; two parenthesis pairs (,); and the three literals p, $\neg p$, and q?

(b) Under the equivalence relation defined by **logical equivalence**, how many equivalence classes do the formulas in part (a) form?

Solution

(a) We will count the number of well-formed formulas that use all symbols exactly once. We note that the parentheses are tied to the operations $\wedge$ and $\vee$ and there are two "shapes" of formula: $(l_1 op_1 (l_2 op_2 l_3))$ and $((l_2 op_2 l_3) op_1 l_1)$. There are $2 \times 1 = 2$ choices for op_1, op_2. There are $3 \times 2 \times 1 = 6$ choices for l_1, l_2, l_3. Therefore, there are $2.2.6 = 24$ formulas in total.

(b) We note that since $(\varphi \vee \psi)$ is logically equivalent to $(\psi \vee \varphi)$ and $(\varphi \wedge \psi)$ is logically equivalent to $(\psi \wedge \varphi)$ we can reduce the 24 formulas from above to the following six (possibly not distinct) classes:

$I.$	$(p \vee (\neg p \wedge q))$	$II.$	$(\neg p \vee (p \wedge q))$	$III.$	$(q \vee (p \wedge \neg p))$
$IV.$	$(p \wedge (\neg p \vee q))$	$V.$	$(\neg p \wedge (p \vee q))$	$VI.$	$(q \wedge (p \vee \neg p))$

Since

$$(q \vee (p \wedge \neg p)) \equiv (q \vee \bot) \equiv q \equiv (q \wedge \top) \equiv (q \wedge (p \vee \neg p))$$

we see that III and VI are the same class.

For the other cases we have:

I $(p \vee (\neg p \wedge q)) \equiv ((p \vee \neg p) \wedge (p \vee q)) \equiv (\top \wedge (p \vee q)) \equiv (p \vee q)$

II $(\neg p \vee (p \wedge q)) \equiv ((\neg p \vee p) \wedge (\neg p \vee q)) \equiv (\top \wedge (\neg p \vee q)) \equiv (\neg p \vee q)$

III $(p \wedge (\neg p \vee q)) \equiv ((p \wedge \neg p) \vee (p \wedge q)) \equiv (\bot \vee (p \wedge q)) \equiv (p \wedge q)$

IV $(\neg p \wedge (p \vee q)) \equiv ((\neg p \wedge p) \vee (\neg p \wedge q)) \equiv (\bot \vee (\neg p \wedge q)) \equiv (\neg p \wedge q)$

Each of these classes are distinct, as can be seen from the truth table:

p	q	$\neg p$	I	II	III	IV	V
T	T	F	T	T	T	T	F
T	F	F	T	F	F	F	F
F	T	T	T	T	T	F	T
F	F	T	F	T	F	F	F

So there are five equivalence classes.

Problem 3

Let A be a set with m elements and B be a set with n elements.

(a) How many functions from A to B are there?

(b) How many injective functions?

(c)* How many surjective functions?

(d) How many binary relations are there on A?

(e) How many reflexive binary relations are there on A?

(f) How many symmetric binary relations are there on A?

Problem 4† (20T2)

You are taking an exam that has 6 easy questions and 4 difficult questions. Assuming all questions are distinguishable, how many ways are there of ordering the questions so that:

(a) All the easy questions come first. (4 marks)

(b) Each pair of difficult questions is separated by at least 2 easy questions.

(c) Each pair of difficult questions is separated by at least 1 easy question.

(d) Each pair of difficult questions is separated by at most 1 easy question.

(e) Each pair of difficult questions is separated by exactly 1 easy question.

Problem 5

We want to tile a $2 \times n$ rectangle with 2×1 tiles so that the rectangle is completely covered and no tiles are overlapping. For example, here are two different ways to tile a 2×3 rectangle:

How many different ways (ignoring symmetry) are there of tiling a $2 \times n$ rectangle with 2×1 tiles in this way?

Solution

Let $T(n)$ be the number of ways of tiling a $2 \times n$ rectangle. We will find a formula for $T(n)$.
First we observe that $T(1) = 1$ and $T(2) = 2$.
For $n > 2$, let us consider how we fill the left-most positions in the tiling. We can either fill it with a single vertically oriented tile, or with 2 horizontally oriented tiles, and there are no other ways. Let us count how many ways there are of tiling in each of these cases. In the first case we have a $2 \times (n-1)$ rectangle remaining to tile, and we know how many ways there are of doing this: $T(n-1)$. In the second case we have a $2 \times (n-2)$ rectangle remaining to tile, and we can do this in $T(n-2)$ ways. So, in total, there are $T(n-1) + T(n-2)$ ways to tile a $2 \times n$ rectangle. That is $T(n) = T(n-1) + T(n-2)$. We can solve this: $T(n) = \textsc{Fib}_{n+1}$, the $(n+1)$-th Fibonacci number.

Problem 6

A tennis doubles match consists of two teams of two players per team. Ordering between teams, and within teams is not considered.

(a) How many different tennis doubles matches can be made with 4 players?

(b) How many different tennis doubles matches can be made with 5 players?

(c) How many different tennis doubles matches can be made from n players?

Solution

(a) Three possible approaches:

- Identify one player, A, say. We observe that the doubles match is completely determined once we choose A's partner, and there are 3 ways to do this.
- Take an arrangement of the four players. The first two in the arrangement make up one team and the second two make up the other team. There are $4! = 24$ arrangements of the four players, but we have duplication. Since the order in each team doesn't matter, we need to divide this by 2 for each team; and since the order of the teams does not matter we further divide by 2. So the total number of matches is $\frac{24}{2.2.2} = 3$.

- Let us first assume the teams are ordered/identified. To choose the players in the first team, we need to pick a subset of size 2. There are $\binom{4}{2} = 6$ ways of doing this. Once we have chosen the first team, the second team is determined, but we can also view it as choosing a subset of size 2 from the remaining 2 players: $\binom{2}{2} = 1$ possibility. This gives $6 \times 1 = 6$ ways of having ordered teams, so to account for the order of teams being irrelevant, we divide this by the number of ways of ordering the teams (2); giving a total of $\frac{6}{2} = 3$ matches.

(b) Three possible approaches:

- Choose one player to sit out. There are 5 ways of doing this. Once a player is sitting out, we know from the previous question that there are 3 ways to organise the match. Giving a total of $5 \times 3 = 15$ matches.
- Take an arrangement of the five players. The first two make up one team, the next two make up the second team, and the remaining person sits out. There are $5! = 120$ arrangements of the five players, but we need to account for the order within teams (divide by 2 for each team), and the order of the teams (divide by 2). This gives $\frac{120}{2.2.2} = 15$ matches.
- As with the previous question, there are $\binom{5}{2} = 10$ ways to choose the players in the first team; and $\binom{3}{2} = 3$ ways to choose the players in the second team. There are 2 ways of ordering the teams, so if the order does not matter there are $\frac{10 \times 3}{2} = 15$ possible matches.

(c) Three possible approaches (note: all answers are the same, just presented differently):

- Choose $(n-4)$ players to sit out (equivalently 4 players to play). This can be done in $\binom{n}{n-4} = \binom{n}{4}$ ways. Once the players have been chosen, there are 3 ways of arranging them as shown in (a). Giving the total number of matches as $3 \times \binom{n}{4}$.
- Take an arrangement of the n players. The first two make up one team, the next two make up the second team, and the remaining $n-4$ players sit out. There are $n!$ arrangements of the n players, but we need to account for the order within teams (divide by 2 for each team); the order of the teams (divide by 2); and the order of the people sitting out (divide by $(n-4)!$). This gives the total number of matches as $\frac{n!}{2.2.2.(n-4)!} = \frac{\Pi(n,4)}{8}$.
- Assume an ordering on the teams. There are $\binom{n}{2}$ ways to choose the first team, and $\binom{n-2}{2}$ ways to choose the second team. Dividing by 2 to account for the ordering of the teams gives the total number of matches as $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$.