Recursion and Induction

Problem 1
Prove by induction that

$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1 \quad \text{for } n \geq 1$$

Problem 2
Let $\Sigma = \{1, 2, 3\}$.

(a) Give a recursive definition for the function $\mathsf{sum} : \Sigma^* \to \mathbb{N}$ which, when given a word over Σ returns the sum of the digits. For example $\mathsf{sum}(1232) = 8$, $\mathsf{sum}(222) = 6$, and $\mathsf{sum}(1) = 1$. You should assume $\mathsf{sum}(\lambda) = 0$.

(b) For $w \in \Sigma^*$, let $P(w)$ be the proposition that for all words $v \in \Sigma^*$, $\mathsf{sum}(wv) = \mathsf{sum}(w) + \mathsf{sum}(v)$. Prove that $P(w)$ holds for all $w \in \Sigma^*$.

(c) Consder the function $\mathsf{rev} : \Sigma^* \to \Sigma^*$ defined recursively as follows:

- $\mathsf{rev}(\lambda) = \lambda$
- For $w \in \Sigma^*$ and $a \in \Sigma$, $\mathsf{rev}(aw) = \mathsf{rev}(w)a$

Prove that for all words $w \in \Sigma^*$, $\mathsf{sum}(\mathsf{rev}(w)) = \mathsf{sum}(w)$

Problem 3
Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ recursively as follows: $f(m, 0) = 0$ for all $m \in \mathbb{N}$ and $f(m, n+1) = m + f(m, n)$.

(a) Let $P(n)$ be the proposition that $f(0, n) = f(n, 0)$. Prove that $P(n)$ holds for all $n \in \mathbb{N}$.

*(b) Let $Q(m)$ be the proposition $\forall n,\ f(m, n) = f(n, m)$. Prove that $Q(m)$ holds for all $m \in \mathbb{N}$.

Problem 4[†] (20T2)

Let $\Sigma = \{a, b\}$ and define $f : \Sigma^* \to \mathbb{R}$ recursively as follows:

- $f(\lambda) = 0$,
- $f(aw) = \frac{1}{2} + \frac{1}{2} f(w)$ for $w \in \Sigma^*$, and
- $f(bw) = -\frac{1}{2} + \frac{1}{2} f(w)$ for $w \in \Sigma^*$.

(a) What is $f(abba)$?

[†] indicates a previous exam question
[*] indicates a difficult/advanced question.

(b) Prove that $f(w) \in (-1, 1)$ for all $w \in \Sigma^*$

(c) Prove, or give a counterexample to disprove:

  (i) f is injective

  (ii) $Im(f) = (-1, 1)$

Problem 5
Let $\Sigma = \{0, 1\}$

(a) Recursively define a function $\mathsf{str2num} : \Sigma^+ \to \mathbb{N}$ that converts a non-empty word over Σ to the number that one obtains by viewing the word as a binary number. For example $\mathsf{str2num}(1100) = 12$, $\mathsf{str2num}(0111) = 7$, $\mathsf{str2num}(0000) = 0$.

(b) Recursively define a function $\mathsf{num2str} : \mathbb{N} \to \Sigma^+$ that converts a number to its (shortest) binary representation. *Hint: you may want to use* div *and* %.

(c) Writing your functions as code in the natural way,

  (i) Give an asymptotic upper bound in terms of $\text{length}(()w)$ on the running time to compute $\mathsf{str2num}(w)$.

  (ii) Give an asymptotic upper bound in terms of n on the running time to compute $\mathsf{num2str}(n)$.