Set theory

Problem 1

(a) How many elements in the following sets:

(i) $S_1 = \{s, y, d, n, e, y\}$
(ii) $S_2 = \{\emptyset, \{\emptyset, \emptyset\}\}$
(iii) $S_3 = \{x \in \mathbb{Z} : |x| < 20\}$
(iv) $S_4 = \{x \in \mathbb{Z} : x \text{ div } 5 = 5\}$
(v) $S_5 = \{\emptyset, 10, 20, S_3\}$
(vi) $S_6 \subseteq \mathbb{Z} \times \mathbb{Z}$ given by $S_6 = \{(n, n^2) : n \in [0, 5]\}$
(vii) $S_7 \subseteq \text{Pow}(\text{Pow}(\mathbb{Z}))$ given by $S_7 = \{(n, n^2) : n \in [0, 5]\}$
(viii) $S_1 \cup S_2$
(ix) $S_3 \cap S_4$
(x) $S_5 \setminus S_3$
(xi) $S_2 \oplus S_5$
(xii) $S_2 \times S_5$
(xiii) $S_6 \setminus (S_3 \times S_4)$
(xiv) $S_7 \setminus (S_3 \times S_4)$

(b) True or false (intervals over $\mathbb{Z}$):

(i) $[1, 10) \subseteq (1, 10]$
(ii) $(1, 10] \subseteq [1, 10)$
(iii) For all $m, n \in \mathbb{Z}$: $(m, n) = [m + 1, n - 1]$
(iv) $[1, 4) \times (0, 3] = (0, 3] \times [1, 4)$

Solution

(a) (i) $S_1 = \{s, y, d, n, e\}$ has 5 elements
(ii) S_2 has 2 elements: $\emptyset$ and $\{\emptyset\}$
(iii) $S_3 = \{-19, -18, -17, \ldots, 0, 1, \ldots, 18, 19\}$ has 39 elements
(iv) $S_4 = \{25, 26, 27, 28, 29\}$ has 5 elements
(v) S_5 has 4 elements: $\emptyset$, 10, 20, and S_3 (note: $S_3 \neq \emptyset$).
(vi) $S_6 = \{(0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$ has 6 elements: each element is an ordered pair
(vii) $S_7 = \{(0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$ has 6 elements: each element is an [open] interval

(viii) $S_1 \cup S_2 = \{s, y, d, n, e, \emptyset, \{\emptyset\}\}$ has 7 elements

(ix) $S_3 \cap S_4 = \emptyset$ has 0 elements

(x) $S_5 \setminus S_3 = \{\emptyset, 20, S_3\}$ has 3 elements

(xi) $S_2 \oplus S_5 = \{\{\emptyset\}, 10, 20, S_3\}$ has 4 elements

(xii) $S_2 \times S_5 = \{(\emptyset,\emptyset), (\{\emptyset\},\emptyset), (\emptyset, 10), (\{\emptyset\}, 10), (\emptyset, 20), (\{\emptyset\}, 20), (\emptyset, S_3), (\{\emptyset\}, S_3)\}$ has 8 elements

(xiii) $(5, 25) \in S_3 \times S_4$ and no other element of S_6 is in $S_3 \times S_4$, so $S_6 \setminus (S_3 \times S_4)$ has $6 - 1 = 5$ elements.

(xiv) S_7 is a set of intervals, and $S_3 \times S_4$ is a set of ordered pairs of integers, so no elements of S_7 is in $S_3 \times S_4$. So $S_7 \setminus (S_3 \times S_4) = S_7$ has 6 elements.

(b) (i) False: $1 \in [1, 10)$ but $1 \notin (1, 10]$

(ii) False: $10 \in (1, 10]$ but $10 \notin [1, 10)$

(iii) True (for intervals over $\mathbb{Z}$): $(m, n) = \{k \in \mathbb{Z} : m < k < n\} = \{k \in \mathbb{Z} : m + 1 \leq k \leq n - 1\} = [m + 1, n - 1]$.

(iv) True (for intervals over $\mathbb{Z}$): $[1, 4) = \{1, 2, 3\} = (0, 3]$, so $[1, 4) \times (0, 3] = [1, 4) \times [1, 4) = (0, 3] \times (0, 3] = (0, 3] \times [1, 4)$.

Problem 2

Prove, or give a counterexample to disprove for all sets A, B, C:

(a) $A \cup B = A \cap B$ if and only if $A = B$

(b) $\mathrm{Pow}(A) \times \mathrm{Pow}(B) = \mathrm{Pow}(A \times B)$

(c) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

(d) $A \oplus (B \setminus C) = (A \oplus B) \setminus (A \oplus C)$

Solution

(a) This is true. If $A = B$ then

$$\begin{aligned} A \cup B &= A \cup A \\ &= A && \text{(Idempotence)} \\ &= A \cap A && \text{(Idempotence)} \\ &= A \cap B. \end{aligned}$$

Conversely, if $A \cap B = A \cup B$, then

$$A \subseteq A \cup B = A \cap B \subseteq A,$$

so $A = A \cap B = A \cup B$. Similarly,

$$B \subseteq A \cup B = A \cap B \subseteq B,$$

so $B = A \cap B = A$.

(b) This is false. Consider $A = \{0\}$, $B = \{1\}$.

Then $\text{Pow}(A) = \{\emptyset, \{0\}\}$, $\text{Pow}(B) = \{\emptyset, \{1\}\}$ and $A \times B = \{(0,1)\}$.

Therefore $(\emptyset, \emptyset) \in \text{Pow}(A) \times \text{Pow}(B)$, but $\text{Pow}(A \times B) = \{\emptyset, \{(0,1)\}\}$, so $(\emptyset, \emptyset) \notin \text{Pow}(A \times B)$.

So $\text{Pow}(A) \times \text{Pow}(B) \neq \text{Pow}(A \times B)$.

(c) This is true.

We have $(x,y) \in A \times (B \setminus C)$;
if and only if $x \in A$ and $y \in B \setminus C$;
if and only if $x \in A$, $y \in B$ and $y \notin C$;
if and only if $(x,y) \in A \times B$ and $(x,y) \notin A \times C$;
if and only if $(x,y) \in (A \times B) \setminus (A \times C)$.

Therefore $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

(d) This is false. Consider $A = \{0\}$ and $B = C = \emptyset$. Then

$$B \setminus C = \emptyset \setminus \emptyset = \emptyset;$$

and

$$A \oplus B = A \oplus C = \{0\} \oplus \emptyset = \{0\}.$$

Therefore

$$A \oplus (B \setminus C) = A \oplus \emptyset = A = \{0\},$$

but

$$(A \oplus B) \setminus (A \oplus C) = A \setminus A = \emptyset.$$

Problem 3

Proof assistant

https://www.cse.unsw.edu.au/~cs9020/cgi-bin/logic/21T3/set_theory/set01

Use the laws of set operations and any derived rules given in lectures to prove the following:

(a) $B \cup (A \cap \emptyset) = B$

(b) $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$

(c) $(A \cap B) \cup (A \cup B^c)^c = B$

(d) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(e) $(A \cup B) \cap A = A$

Solution

(a)

$$\begin{aligned} B \cup (A \cap \emptyset) &= B \cup (A \cap (A \cap A^c)) && \text{(Complement with } \cap) \\ &= B \cup ((A \cap A) \cap A^c) && \text{(Associativity of } \cap) \\ &= B \cup (A \cap A^c) && \text{(Idempotence of } \cap) \\ &= B \cup \emptyset && \text{(Complement with } \cap) \\ &= B && \text{(Identity of } \cup) \end{aligned}$$

(b)

$$\begin{aligned} (C \cup A) \cap (B \cup A) &= (A \cup C) \cap (B \cup A) && \text{(Commutatitivity of } \cup) \\ &= (A \cup C) \cap (A \cup B) && \text{(Commutatitivity of } \cup) \\ &= A \cup (C \cap B) && \text{(Distributivity of } \cup \text{ over } \cap) \\ &= A \cup (B \cap C) && \text{(Commutatitivity of } \cap) \end{aligned}$$

(c)

(d)

(e)

Problem 4
Let $\Sigma = \{a, b, c\}$ and $\Phi = \{a, c, e\}$.

(a) How many words are in the set Σ^2?

(b) What are the elements of $\Sigma^2 \setminus \Phi^*$?

(c) Is it true that $\Sigma^* \setminus \Phi^* = (\Sigma \setminus \Phi)^*$? Why?

Solution

(a) $\Sigma^2 = \{aa, ab, ac, ba, \ldots, cc\}$, hence $|\Sigma^2| = 3 \cdot 3 = 9$.

(b) $\Sigma^2 \setminus \Phi^* = \{ab, ba, bb, bc, cb\}$, that is, all words in Σ^2 with the letter b.

(c) No; for example, $ab \in \Sigma^*$ and $ab \notin \Phi^*$, hence $ab \in \Sigma^* \setminus \Phi^*$; but $\Sigma \setminus \Phi = \{b\}$, hence $ab \notin (\Sigma \setminus \Phi)^*$.

Problem 5
Let $\Sigma = \{a, b, c\}$. Prove, or give a counterexample to disprove, for all languages $X, Y, Z \subseteq \Sigma^*$:

(a) $(XY)Z = X(YZ)$

(b) $X \subseteq X^*$

(c) $(XY)^* = (X^*)(Y^*)$

(d) $X(Y \cup Z) = XY \cup XZ$

(e) $X \cup YZ = (X \cup Y)(X \cup Z)$

Solution

(a) This is true.

(b) This is true.

(c) This is false. Consider $X = \{a\}$, and $Y = \{b\}$. Then $abab \in (XY)^*$ but $abab \notin (X^*)(Y^*)$

(d) This is true.

(e) This is false. Consider $X = \{a\}$, and $Y = Z = \emptyset$. Then $aa \in (X \cup Y)(X \cup Z)$, but $aa \notin X \cup YZ$.

Problem 6[†] (2020 T2)

(a) Prove, or give a counterexample to disprove for all sets A, B, C, D:

(i) $(A \oplus B) = (B \oplus A)$

(ii) $A \cup (B \oplus C) = (A \oplus B) \cup (A \oplus C)$

(iii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(b) (i) The Laws of Set Operations only define equality between sets. How can they be used to show, say, $A \subseteq B$?

(ii) Use the Laws of Set Operations to show

$$A \oplus B \subseteq A \cup B.$$

Partial marks are available for a proof that does not use the Laws of Set Operations.

Solution

(a) (i) This is true.

(ii) This is false. Consider $A = \{0\}$ and $B = C = \{1\}$. Then

$$A \cup (B \oplus C) = \{0\} \cup (\{1\} \oplus \{1\}) = \{0\} \cup \emptyset = \{0\},$$

but

$$(A \oplus B) \cup (A \oplus C) = (\{0\} \oplus \{1\}) \cup (\{0\} \oplus \{1\}) = \{0,1\} \cup \{0,1\} = \{0,1\}.$$

(iii) This is true.

(b) (i) We know that $A \subseteq B$ if and only if $A = A \cap B$. So we could show $A = A \cap B$ using the laws of set operations. Similarly we could show $B = A \cup B$.

(ii)

Problem 7[†] (2020 T3)

(a) Prove, or provide a counterexample to disprove, the following for all sets A, B:

  (i) If there is a set X such that $A \cap X = B \cap X$ then $A = B$.
  (ii) If there is a set X such that $A \oplus X = B \oplus X$ then $A = B$.
  (iii) If there is a set X such that $A \cap X = B \cap X$ and $A \cup X = B \cup X$ then $A = B$.

(b) Let Σ be a finite set. Prove, or provide a counterexample to disprove, the following for all languages $X, Y, Z \subseteq \Sigma^*$:

  (i) If $XY = YZ$ then $X^*Y = YZ^*$.
  (ii) If $X^*Y = YZ^*$ then $XY = YZ$.

Problem 8*
Use the laws of set operations to show the following hold for all sets A, B, C:

(a) $A \oplus B = B \oplus A$

(b) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(c) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

(d) $A \oplus \emptyset = A$

(e) $A \oplus A = \emptyset$

(f) $A \cap (\mathcal{U} \oplus A) = \emptyset$

NB

These observations, together with the commutativity, associativity, and identity laws (for $\cap$) show that $(\mathrm{Pow}(\mathcal{U}), \oplus, \cap, \emptyset, \mathcal{U})$ forms what is known as a Boolean ring.

Problem 9*

Proof assistant

https://cgi.cse.unsw.edu.au/~cs9020/cgi-bin/logic/21T2/adv_set_theory/set02

(a) Prove the associativity laws follow from the eight other laws of set operations. That is, show

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (A \cap B) \cap C = A \cap (B \cap C)$$

using only the commutativity, distribution, identity and complement laws.

(b) Prove $(A^c)^c = A$ without using uniqueness of complement

(c) Prove de Morgan's laws with only the laws of set operations.

(d) Prove, using the laws of set operations:

$$((A \cup B) \cap (B \cup C)) \cap (C \cup A) = ((A \cap B) \cup (B \cap C)) \cup (C \cap A).$$

Solution

(a) Answer witheld for Challenge 2

(b)

$$\begin{aligned}
(A^c)^c &= (A^c)^c \cup \emptyset && \text{(Identity of } \cup\text{)}\\
&= (A^c)^c \cup (A \cap A^c) && \text{(Complement with } \cap\text{)}\\
&= (A^c)^c \cup (A^c \cap A) && \text{(Commutatitivity of } \cap\text{)}\\
&= ((A^c)^c \cup A^c) \cap ((A^c)^c \cup A) && \text{(Distributivity of } \cup \text{ over } \cap\text{)}\\
&= ((A^c)^c \cup A) \cap ((A^c)^c \cup A^c) && \text{(Commutatitivity of } \cap\text{)}\\
&= ((A^c)^c \cup A) \cap (A^c \cup (A^c)^c) && \text{(Commutatitivity of } \cup\text{)}\\
&= ((A^c)^c \cup A) \cap \mathcal{U} && \text{(Complement with } \cup\text{)}\\
&= ((A^c)^c \cup A) \cap (A \cup A^c) && \text{(Complement with } \cup\text{)}\\
&= (A \cup A^c)^c) \cap (A \cup A^c) && \text{(Commutatitivity of } \cup\text{)}\\
&= A \cup ((A^c)^c \cap A^c) && \text{(Distributivity of } \cup \text{ over } \cap\text{)}\\
&= A \cup (A^c \cap (A^c)^c) && \text{(Commutatitivity of } \cap\text{)}\\
&= A \cup \emptyset && \text{(Complement with } \cap\text{)}\\
&= A && \text{(Identity of } \cup\text{)}
\end{aligned}$$

(c) Answer witheld until after Assignment 1

(d)

$$\begin{aligned}
&((A \cup B) \cap (B \cup C)) \cap (C \cup A)\\
&\quad= ((B \cup A) \cap (B \cup C)) \cap (C \cup A) && \text{(Commutatitivity of } \cup\text{)}\\
&\quad= (B \cup (A \cap C)) \cap (C \cup A) && \text{(Distributivity of } \cup \text{ over } \cap\text{)}\\
&\quad= (C \cup A) \cap (B \cup (A \cap C)) && \text{(Commutatitivity of } \cap\text{)}\\
&\quad= ((C \cup A) \cap B) \cup ((C \cup A) \cap (A \cap C)) && \text{(Distributivity of } \cap \text{ over } \cup\text{)}\\
&\quad= (B \cap (C \cup A)) \cup ((C \cup A) \cap (A \cap C)) && \text{(Commutatitivity of } \cap\text{)}\\
&\quad= ((B \cap C) \cup (B \cap A)) \cup ((C \cup A) \cap (A \cap C)) && \text{(Distributivity of } \cap \text{ over } \cup\text{)}\\
&\quad= ((B \cap A) \cup (B \cap C)) \cup ((C \cup A) \cap (A \cap C)) && \text{(Commutatitivity of } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((C \cup A) \cap (A \cap C)) && \text{(Commutatitivity of } \cap\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup (((C \cup A) \cap A) \cap C) && \text{(Associativity of } \cap\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup (((A \cup C) \cap A) \cap C) && \text{(Commutatitivity of } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup (((A \cup C) \cap (A \cup \emptyset)) \cap C) && \text{(Identity of } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((A \cup (C \cap \emptyset)) \cap C) && \text{(Distributivity of } \cup \text{ over } \cap\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((A \cup ((C \cap \emptyset) \cup \emptyset)) \cap C) && \text{(Identity of } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((A \cup ((C \cap \emptyset) \cup (C \cap C^c))) \cap C) && \text{(Complement with } \cap\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((A \cup (C \cap (\emptyset \cup C^c))) \cap C) && \text{(Distributivity of } \cap \text{ over } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((A \cup (C \cap (C^c \cup \emptyset))) \cap C) && \text{(Commutatitivity of } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((A \cup (C \cap C^c)) \cap C) && \text{(Identity of } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup ((A \cup \emptyset) \cap C) && \text{(Complement with } \cap\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup (A \cap C) && \text{(Identity of } \cup\text{)}\\
&\quad= ((A \cap B) \cup (B \cap C)) \cup (C \cap A) && \text{(Commutatitivity of } \cap\text{)}
\end{aligned}$$