Set theory

Problem 1

(a) How many elements in the following sets:

(i) $S_1 = \{s, y, d, n, e, y\}$
(ii) $S_2 = \{\emptyset, \{\emptyset, \emptyset\}\}$
(iii) $S_3 = \{x \in \mathbb{Z} : |x| < 20\}$
(iv) $S_4 = \{x \in \mathbb{Z} : x \text{ div } 5 = 5\}$
(v) $S_5 = \{\emptyset, 10, 20, S_3\}$
(vi) $S_6 \subseteq \mathbb{Z} \times \mathbb{Z}$ given by $S_6 = \{(n, n^2) : n \in [0, 5]\}$
(vii) $S_7 \subseteq \text{Pow}(\text{Pow}(\mathbb{Z}))$ given by $S_7 = \{(n, n^2) : n \in [0, 5]\}$
(viii) $S_1 \cup S_2$
(ix) $S_3 \cap S_4$
(x) $S_5 \setminus S_3$
(xi) $S_2 \oplus S_5$
(xii) $S_2 \times S_5$
(xiii) $S_6 \setminus (S_3 \times S_4)$
(xiv) $S_7 \setminus (S_3 \times S_4)$

(b) True or false (intervals over $\mathbb{Z}$):

(i) $[1, 10) \subseteq (1, 10]$
(ii) $(1, 10] \subseteq [1, 10)$
(iii) For all $m, n \in \mathbb{Z}$: $(m, n) = [m + 1, n - 1]$
(iv) $[1, 4) \times (0, 3] = (0, 3] \times [1, 4)$

Problem 2
Prove, or give a counterexample to disprove for all sets A, B, C:

(a) $A \cup B = A \cap B$ if and only if $A = B$

(b) $\text{Pow}(A) \times \text{Pow}(B) = \text{Pow}(A \times B)$

(c) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

(d) $A \oplus (B \setminus C) = (A \oplus B) \setminus (A \oplus C)$

† indicates a previous exam question
* indicates a difficult/advanced question.

Problem 3

Proof assistant

https://www.cse.unsw.edu.au/~cs9020/cgi-bin/logic/22T2/set_theory/set01

Use the laws of set operations and any derived rules given in lectures to prove the following:

(a) $B \cup (A \cap \emptyset) = B$

(b) $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$

(c) $(A \cap B) \cup (A \cup B^c)^c = B$

(d) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(e) $(A \cup B) \cap A = A$

Problem 4
Let $\Sigma = \{a, b, c\}$ and $\Phi = \{a, c, e\}$.

(a) How many words are in the set Σ^2?

(b) What are the elements of $\Sigma^2 \setminus \Phi^*$?

(c) Is it true that $\Sigma^* \setminus \Phi^* = (\Sigma \setminus \Phi)^*$? Why?

Problem 5
Let $\Sigma = \{a, b, c\}$. Prove, or give a counterexample to disprove, for all languages $X, Y, Z \subseteq \Sigma^*$:

(a) $(XY)Z = X(YZ)$

(b) $X \subseteq X^*$

(c) $(XY)^* = (X^*)(Y^*)$

(d) $X(Y \cup Z) = XY \cup XZ$

(e) $X \cup YZ = (X \cup Y)(X \cup Z)$

Problem 6† (2020 T2)

(a) Prove, or give a counterexample to disprove for all sets A, B, C, D:

  (i) $(A \oplus B) = (B \oplus A)$

  (ii) $A \cup (B \oplus C) = (A \oplus B) \cup (A \oplus C)$

(iii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(b) (i) The Laws of Set Operations only define equality between sets. How can they be used to show, say, $A \subseteq B$?

(ii) Use the Laws of Set Operations to show

$$A \oplus B \subseteq A \cup B.$$

Partial marks are available for a proof that does not use the Laws of Set Operations.

Problem $7^\dagger$ (2020 T3)

(a) Prove, or provide a counterexample to disprove, the following for all sets A, B:

(i) If there is a set X such that $A \cap X = B \cap X$ then $A = B$.

(ii) If there is a set X such that $A \oplus X = B \oplus X$ then $A = B$.

(iii) If there is a set X such that $A \cap X = B \cap X$ and $A \cup X = B \cup X$ then $A = B$.

(b) Let Σ be a finite set. Prove, or provide a counterexample to disprove, the following for all languages $X, Y, Z \subseteq \Sigma^*$:

(i) If $XY = YZ$ then $X^*Y = YZ^*$.

(ii) If $X^*Y = YZ^*$ then $XY = YZ$.

Problem $8^\dagger$ (2021 T3)
Prove, or provide a counterexample to disprove, the following for all sets A, B:

(a) $A \oplus B^c = A^c \oplus B$

(b) $A^c \oplus B^c = (A \oplus B)^c$

Problem 9^*
Use the laws of set operations to show the following hold for all sets A, B, C:

(a) $A \oplus B = B \oplus A$

(b) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(c) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

(d) $A \oplus \emptyset = A$

(e) $A \oplus A = \emptyset$

(f) $A \cap (\mathcal{U} \oplus A) = \emptyset$

NB

These observations, together with the commutativity, associativity, and identity laws (for $\cap$) show that $(\mathrm{Pow}(\mathcal{U}), \oplus, \cap, \emptyset, \mathcal{U})$ forms what is known as a Boolean ring.

Problem 10*

(a) Prove the associativity laws follow from the eight other laws of set operations. That is, show

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (A \cap B) \cap C = A \cap (B \cap C)$$

using only the commutativity, distribution, identity and complement laws.

(b) Prove $(A^c)^c = A$ without using uniqueness of complement

(c) Prove de Morgan's laws with only the laws of set operations.

(d) Prove, using the laws of set operations:

$$((A \cup B) \cap (B \cup C)) \cap (C \cup A) = ((A \cap B) \cup (B \cap C)) \cup (C \cap A).$$