Logic

Problem 1
Let F be the set of well-formed formulas with propositional variables from $\textsc{Prop}$. Define a relation, $R \subseteq F \times F$ by $(\varphi, \psi) \in R$ if $\varphi \models \psi$. Prove or give a counter-example to disprove:

(a) R is a partial order.

(b) $R \cup R^{\leftarrow}$ is an equivalence relation.

(c) $R \cap R^{\leftarrow}$ is an equivalence relation.

Problem 2
Prove that $\neg N$ follows logically from $H \wedge \neg R$ and $(H \wedge N) \rightarrow R$.

Problem 3
Consider the formulae $\phi_1 = (r \rightarrow p)$ and $\phi_2 = (p \rightarrow (q \vee \neg r))$. Transform the formula $\phi = (\neg q \rightarrow (\phi_1 \wedge \phi_2))$ into

(a) **DNF**, and

(b) **CNF**.

Simplify the result as much as possible.

Problem 4
Let $(T, \wedge, \vee, ', 0, 1)$ be a Boolean Algebra. Define $\oplus : T \times T \rightarrow T$ as follows:

$$x \oplus y = (x \wedge y') \vee (x' \wedge y)$$

(a) Prove using the laws of Boolean Algebra that for all $x \in T$, $x \oplus 1 = x'$.

(b) Prove using the laws of Boolean Algebra that $x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$.

(c) Find a Boolean Algebra (and x, y, z) which demonstrates that $x \oplus (y \wedge z) \neq (x \oplus y) \wedge (x \oplus z)$

Problem 5† (20T2)
Prove, or give a counterexample to disprove, for all propositional formulas φ, ψ, θ:

(a) $((\varphi \wedge \psi) \rightarrow \theta) \equiv (\varphi \rightarrow (\psi \rightarrow \theta))$

(b) $((\varphi \leftrightarrow \psi) \wedge (\psi \leftrightarrow \theta)) \equiv (\varphi \leftrightarrow \theta)$

† indicates a previous exam question
* indicates a difficult/advanced question.

(c) $(\varphi \wedge \psi) \models (\varphi \leftrightarrow \psi)$

(d) $(\varphi \to \psi), (\neg\varphi \to \theta) \models (\psi \vee \theta)$

(e) $(\varphi \to (\psi \to \theta)) \models ((\varphi \to \psi) \to \theta)$