Due: Monday, 18th July, 12:00 (AEST)

Submission is through WebCMS/give and should be a single pdf file, maximum size 2Mb. Prose should be typed, not handwritten. Use of LaTeX is encouraged, but not required.

Discussion of assignment material with others is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

Problem 1 (15 marks)

Let S be a set.

(a) Show that for any set T and any function $f : S \to T$, the relation $R_f \subseteq S \times S$, defined as:

$$(s, s') \in R_f \text{ if and only if } f(s) = f(s')$$

is an equivalence relation. (9 marks)

(b) Show that if $R \subseteq S \times S$ is an equivalence relation, then there exists a set T and a function $f_R : S \to T$ such that:

$$(s, s') \in R \text{ if and only if } f_R(s) = f_R(s')$$

(6 marks)

Problem 2 (20 marks)

Let $\mathbb{B} = \{0, 1\}$ and consider the function $f : \mathbb{N} \to \mathbb{B}$ given by

$$f(n) = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that for all $a, b \in \mathbb{N}$:

(i) $f(a + b) = \max\{f(a), f(b)\}$

(ii) $f(ab) = \min\{f(a), f(b)\}$

(6 marks)

From Problem 1, we know that $R_f \subseteq \mathbb{N} \times \mathbb{N}$, the relation given by:

$$(m, n) \in R_f \text{ if and only if } f(m) = f(n)$$

is an equivalence relation. Let $\mathbb{E} \subseteq \mathrm{Pow}(\mathbb{N})$ be the set of equivalence classes of R_f, and for $n \in \mathbb{N}$, let $[n] \in \mathbb{E}$ denote the equivalence class of n.

We would like to define binary operations, $\boxplus$ and $\boxdot$, on $\mathbb{E}$ as follows:

$$[x] \boxplus [y] := [x + y]$$
$$[x] \boxdot [y] := [xy].$$

The difficulty is that the operands $[x]$ and $[y]$ can have multiple representations (e.g. if $z \in [x]$ then $[x] = [z]$), and so it is not clear that such a definition makes sense: if we take a different representation of the operands, do we still end up with the same result? That is, if $[x] = [x']$ and $[y] = [y']$ is it the case that $[x + y] = [x' + y']$ and $[xy] = [x'y']$? Our next step is to show that such a definition makes sense.

(b) Define relations $\boxplus, \boxdot \subseteq \mathbb{E}^2 \times \mathbb{E}$ as follows:

$$((X, Y), Z) \in \boxplus \text{ if and only if there is } x \in X \text{ and } y \in Y \text{ such that } x + y \in Z$$
$$((X, Y), Z) \in \boxdot \text{ if and only if there is } x \in X \text{ and } y \in Y \text{ such that } xy \in Z$$

(i) Show that $\boxplus$ is a function.

(ii) Show that $\boxdot$ is a function.

(6 marks)

Part (b) shows that the informal definition of $\boxplus$ and $\boxdot$ given earlier is *well-defined*, so from now we will view $\boxplus$ and $\boxdot$ as **binary operations** on $\mathbb{E}$, that is $\boxplus, \boxdot : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$.

(c) Show that for all $A, B, C \in \mathbb{E}$:

(i) $A \boxdot [1] = A$

(ii) $A \boxplus B = B \boxplus A$

(iii) $A \boxdot (B \boxplus C) = (A \boxdot B) \boxplus (A \boxdot C)$

(8 marks)

Remark

Objects that have a concept of "addition" ($\boxplus$) and "multiplication" ($\boxdot$) where:

- addition and multiplication are associative,
- both operations have identities (see (c)(i)),
- addition is commutative (see (c)(ii)), and
- multiplication distributes over addition (see (c)(iii))

are known as semirings. We have already seen a number of semirings in this course:

- The natural numbers with usual addition and multiplication,
- Integers modulo n with addition and multiplication modulo n,
- Subsets of a set X with union and intersection,
- Languages with union and concatenation,
- Binary relations with union and relational composition (see Assignment 1),
- Matrices with matrix addition and matrix multiplication.

Problem 3

(12 marks)

Eight houses are lined up on a street, with four on each side of the road as shown:

Each house wants to set up its own wi-fi network, but the wireless networks of neighbouring houses – that is, houses that are either next to each other (ignoring trees) or over the road from one another (directly opposite) – can interfere, and must therefore be on different channels. Houses that are sufficiently far away may use the same wi-fi channel. Your goal is to find the minimum number of different channels the neighbourhood requires.

(a) Model this as a graph problem. Remember to:

 (i) Clearly define the vertices and edges of your graph. (4 marks)

 (ii) State the associated graph problem that you need to solve. (2 marks)

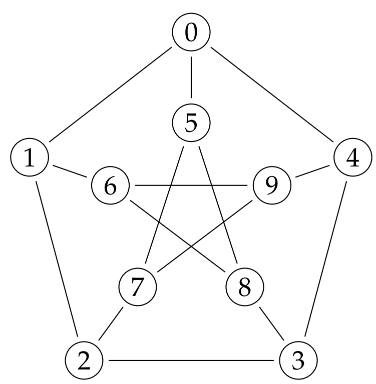
(b) Give the solution to the graph problem corresponding to this scenario; and determine the minimum number of wi-fi channels required for the neighbourhood? (2 marks)

(c) How do your answers to (a) and (b) change if a house's wireless network can also interfere with those of the houses to the left and right of the house over the road? (4 marks)

Problem 4

(12 marks)

This is the Petersen graph:



(a) Give an argument to show that the Petersen graph does not contain a subdivision of K_5. (6 marks)

(b) Show that the Petersen graph contains a subdivision of $K_{3,3}$. (6 marks)

Problem 5 (20 marks)
Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

$$\begin{aligned} R^0 &:= I = \{(x,x) : x \in S\}, \text{ and} \\ R^{n+1} &:= R^n \cup (R; R^n) \text{ for } n \geq 0 \end{aligned}$$

(a) Prove that for all $i, j \in \mathbb{N}$, if $i \leq j$ then $R^i \subseteq R^j$. *Hint: Let $P_i(j)$ be the proposition that $R^i \subseteq R^j$ and prove that $P_i(j)$ holds for all $j \geq i$.* (4 marks)

(b) Let $P(n)$ be the proposition that for all $m \in \mathbb{N}$: $R^n; R^m = R^{n+m}$. Prove that $P(n)$ holds for all $n \in \mathbb{N}$. *Hint: Use results from Assignment 1* (4 marks)

(c) Prove that if there exists $i \in \mathbb{N}$ such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$. (4 marks)

(d) If $|S| = k$, explain why $R^{k^2} = R^{k^2+1}$. (2 marks)

(e) If $|S| = k$, show that R^{k^2} is transitive. (2 marks)

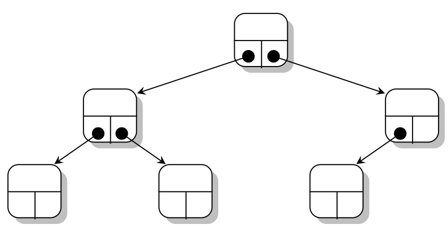
(f)* If $|S| = k$ show that R^{k^2} is the minimum (with respect to $\subseteq$) of all reflexive and transitive relations that contain R. (4 marks)

Remark

The relation at the limit[a] as n tends to infinity, $R^* = \lim_{n \to \infty} R^i$, is known as the **reflexive, transitive closure of** R, and is closely connected to the Kleene star operator.

[a]Because $R^j \subseteq R^i \subseteq S \times S$ for all $j \leq i$, the Knaster-Tarski theorem ensures this limit always exists, even for infinite S.

Problem 6 (20 marks)
A *binary tree* is a data structure where each node is linked to at most two successor nodes:



If we include empty binary trees (trees with no nodes) as part of the definition, then we can simplify the description of the data structure. Rather than saying a node has 0, 1, or 2 successor nodes, we can instead say that a node has exactly two *children*, where a child is a binary tree. That is, we can abstractly define the structure of a binary tree as follows:

- (B): An empty tree, τ
- (R): An ordered pair $(T_{\text{left}}, T_{\text{right}})$ where T_{left} and T_{right} are trees.

So, for example, the above tree would be defined as the tree T where:

$$\begin{aligned}
&T = (T_1, T_2), \text{ where}\\
&\quad T_1 = (T_3, T_4) \text{ and } T_2 = (T_5, \tau), \text{ where}\\
&\qquad T_3 = T_4 = T_5 = (\tau, \tau)
\end{aligned}$$

That is,

$$T = \Big(\big((\tau,\tau), (\tau,\tau) \big), \big((\tau,\tau), \tau \big) \Big)$$

A *leaf* in a binary tree is a node that has no successors (i.e. it is of the form (τ, τ)). A *fully-internal* node in a binary tree is a node that has exactly two successors (i.e. it is of the form (T_1, T_2) where $T_1, T_2 \neq \tau$). The example above has 3 leaves (T_3, T_4, and T_5) and 2 fully-internal nodes (T and T_1). For technical reasons (that will become apparent) we assume that an empty tree has 0 leaves and -1 fully-internal nodes.

(a) Based on the recursive definition above, recursively define a function $\mathsf{count}(\mathsf{T})$ that counts the number of nodes in a binary tree T. (4 marks)

(b) Based on the recursive definition above, recursively define a function $\mathsf{leaves}(\mathsf{T})$ that counts the number of leaves in a binary tree T. (4 marks)

(c) Based on the recursive definition above, recursively define a function $\mathsf{internal}(\mathsf{T})$ that counts the number of fully-internal nodes in a binary tree T. (4 marks)

(d) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves}(\mathsf{T}) = \mathsf{internal}(\mathsf{T}) + 1$. Prove that $P(\mathsf{T})$ holds for all binary trees T. Your proof should be based on your answers given in (b) and (c). (8 marks)

Problem 7* (5 marks)

Let Σ be a finite set, totally ordered by $<$. Give a formal, recursive definition of the lexicographic ordering $\leq_{\text{lex}} \subseteq \Sigma^* \times \Sigma^*$.

Advice on how to do the assignment

Collaboration is encouraged, but all submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted via WebCMS (or give) as a single pdf file.
- When giving answers to questions, we always would like you to prove/explain/motivate your answers. You are being assessed on your understanding and ability.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Some of the questions are very easy (with the help of external resources). You may make use of external material provided it is properly referenced[1] – however, answers that depend too heavily on external resources may not receive full marks if you have not adequately demonstrated ability/understanding.

[1] Proper referencing means sufficient information for a marker to access the material. Results from the lectures or textbook can be used without proof, but should still be referenced.