Probability

Problem 1
A 4-letter word is selected at random from Σ^4, where $\Sigma = \{a, b, c, d, e\}$.

(a) What is the probability that the letters in the word are distinct?

(b) What is the probability that there are no vowels in the word?

(c) What is the probability that the word begins with a vowel?

(d) What is the expected number of vowels in the word?

(e) Let x be the answer to the previous question. What is the probability of the word having $\lceil x \rceil$ or more vowels?

Solution

(a) Out of $5^4 = 625$ words, there are $5 \cdot 4 \cdot 3 \cdot 2 = 120$ words consisting of distinct letters $\rightarrow$ the probability is $\frac{120}{625} = 19.2\%$.

(b) There are $3^4 = 81$ words consisting of only the letters b, c, d (not the vowels a, e); the probability is $\frac{81}{625} \approx 13\%$.

(c) Since there are two vowels, the probability that the first letter is a vowel is $\frac{2}{5}$ (the subsequent letters are irrelevant).

(d) Let X_i $(1 \leq i \leq 4)$ be a random variable that counts the number of vowels in the i-th position (so each X_i is either 0 [with probability $\frac{3}{5}$] or 1 [with probability $\frac{2}{5}$). The expected number of vowels is then $E(X_1 + X_2 + X_3 + X_4) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$ by the linearity of expectation. $E(X_i) = 0.\frac{3}{5} + 1.\frac{2}{5} = \frac{2}{5}$; so the expected number of vowels is $4.\frac{2}{5} = \frac{8}{5}$.

(e) $\lceil \frac{8}{5} \rceil = 2$. From (b), the probability that the word contains no vowels is $\frac{81}{625}$; and the probability that the word contains exactly one vowel is $\frac{4.2.3^3}{5^4}$ (there are 4 choices for the location of the vowel; 2 choices for the vowel; and 3^3 choices for the remaining 3 consonants). So the probability of having 1 or fewer vowels is $\frac{4.2.3^3+81}{625} = \frac{216+81}{625} = \frac{297}{625}$. Therefore, the probability of having $\lceil \frac{8}{5} \rceil$ or more vowels is $1 - \frac{297}{625} = \frac{328}{625} \approx 52.5\%$.

Problem 2
A black die and a red die are tossed. What is the probability that

(a) the sum of the values is even?

(b) the number on the red die is bigger than the number on the black die?

(c) the number on the red die is twice the number on the black die?

Solution

(a) Regardless of the outcome of the black die, there will be 3 outcomes of the red die (out of the possible 6) for which the sum of the values is even. Therefore the probability is $\frac{1}{2}$. (Alternatively, verify that the sum is even in 18 out of the 36 possible outcomes of both dice.)

(b) There are 6 outcomes (among the 36 possible) where the numbers are equal, and of the remaining 30, half (that is, 15 outcomes) have a higher red number than black numbers. Therefore the probability is $\frac{15}{36} = \frac{5}{12}$.

(c) The relevant outcomes are (1,2), (2,4), or (3,6); therefore the probability is $\frac{3}{36} = \frac{1}{12}$.

Problem 3
Team α faces team β in a 5-match series. Matches are either won or lost, i.e., there are no draws. It takes 3 wins to win the series. Team α has probability p $(0 < p < 1)$ of winning a match. Consider each of the following situations and calculate the probability that they will lose the whole series.

(a) They have lost the first match of the series already.

(b) They have lost one of the first two matches of the series already.

(c) They have lost the first two matches of the series already.

(d) They have lost one of the first three matches of the series already.

(e) They have lost two of the first three matches of the series already.

Solution

(a) To lose at least two more matches out of the remaining 4, the probability is $\binom{4}{2}p^2(1-p)^2 + \binom{4}{3}p(1-p)^3 + (1-p)^4$.

(b) To lose at least two more matches out of the remaining 3, the probability is $\binom{3}{2}p(1-p)^2 + (1-p)^3$.

(c) To lose at least one more match out of the remaining 3, the probability is $\binom{3}{1}p^2(1-p) + \binom{3}{2}p(1-p)^2 + (1-p)^3$ (alternatively, $1-p^3$).

(d) To lose both of the remaining two matches, the probability is $(1-p)^2$.

(e) To lose at least one of the remaining two matches, the probability is $1-p^2$.

Problem 4
Let E_1, E_2 be two events. Prove that $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$ implies $P(E_2 \setminus E_1) = 0$.

Solution

Since $E_1 \setminus E_2$ and E_2 are disjoint and $E_2 \setminus E_1$ and E_1 are disjoint, we have:

$$P(E_2 \setminus E_1) + P(E_1) = P(E_2 \cup E_1) = P(E_1 \setminus E_2) + P(E_2).$$

So $P(E_2 \setminus E_1) = P(E_1 \setminus E_2) + P(E_2) - P(E_1) = 0$ if $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$.

Problem 5† (20T2)

Suppose two players, A and B, are playing the following game:

- A starts.
- The players take turns rolling a 6-sided die.
- Whoever rolls the first 6 wins the game.

(a) What is the probability that A wins?

(b) What is the expected number of die rolls before a winner is determined?

Now suppose we consider the following addition to the rules:

- If a player rolls a number that has already been seen then they roll again until an unseen number is rolled.

(c) What is the probability that A wins this game? (4 marks)

(d) If we say a turn ends when an unseen number is rolled, what is the expected number of turns before a winner is determined?

(e) At the start of B's second turn, what is the expected number of die rolls before a winner is determined?

Problem 6

Consider the procedure given in lectures to simulate a die using a fair coin:

(A) Flip a coin 3 times.

(B) If the outcome was:

- HHH: Output 1
- HHT: Output 2
- HTH: Output 3
- HTT: Output 4
- THH: Output 5
- THT: Output 6
- TTH: Go to (A)
- TTT: Go to (A)

What is the expected number of coin flips to obtain an output?

Solution

Let E be the expected number of coin flips (starting at (A)) before we output a number. With probability $\frac{6}{8} = \frac{3}{4}$ we will output something after 3 coin flips. With probability $\frac{2}{8} = \frac{1}{4}$ we will take 3 coin flips and return to (A) where we know we will take, on average, E more coin flips. So,

$$E = \frac{3}{4}.3 + \frac{1}{4}(3 + E).$$

Solving for E yields $E = 4$.