Relations

Problem 1

True or false:

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* must also be totally ordered.

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* must also be totally ordered.

(c) Every finite poset has a Hasse diagram.

(d) Every finite poset has a topological sorting.

(e) Every finite poset has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite poset cannot have a maximum element.

Solution

(a) True. Because we can compare every symbol, we can compare any pair of words.

(b) True. Because we can compare every symbol, we can compare any pair of words.

(c) True, by the definition of a Hasse diagram.

(d) True. We can generate an order as follows:

- Choose a minimal element of the poset. This is the next element in the topological sort.
- Remove the chosen element from the poset.
- Repeat until there are no elements left.

A proof that this is a topological sort is left to the reader.

(e) False. Consider the poset $(\{0,1\}, R)$ where R is the relation $R = \{(0,0),(1,1)\}$ (i.e. 0 and 1 are incomparable). This poset has no minimum element.

(f) True. Total order means that every element is comparable to every other. Because the set is finite, it must contain a maximal element. Therefore, this maximal element will be a maximum (as it is comparable to every other element of the set).

(g) False. Consider the poset given by the set of real numbers: $[0,1]$, together with the usual ordering $\leq$. This set has a maximum element, 1.

Problem 2

Give an example of a relation which is:

(a) Symmetric, transitive, not reflexive, and not antireflexive

(b) Antisymmetric and antireflexive

(c) Reflexive, Antisymmetric, not transitive.

Solution

(a) Consider $R \subseteq \mathbb{N} \times \mathbb{N}$ given by $(x, y) \in R$ if $xy > 0$. We have that $(x, y) \in R$ if and only if x and y are both positive. Therefore:

- $(x, y) \in R$ if and only if $(y, x) \in R$, so R is symmetric;
- If $(x, y), (y, z) \in R$ then x, y, and z are all positive, so $(x, z) \in R$. Therefore R is transitive;
- $(0, 0) \notin R$ so R is not reflexive; and
- $(1, 1) \in R$ so R is not antireflexive.

(b) Consider $R \subseteq \mathbb{N} \times \mathbb{N}$ given by $(x, y) \in R$ if $x < y$. We have:

- There are no x, y such that $x < y$ and $y < x$, so R is (vacuously) antisymmetric.
- There is no x such that $x < x$, so R is antireflexive.

(c) Consider $R \subseteq \mathbb{N} \times \mathbb{N}$ given by $(x, y) \in R$ if $y - x \in \{0, 1\}$. We have:

- For all $x \in \mathbb{N}$: $x - x = 0$, so $(x, x) \in R$ and R is reflexive.
- If $(x, y) \in R$ and $(y, x) \in R$ then $y - x \in \{0, 1\}$ and $x - y \in \{0, 1\}$. Since $x - y = -(y - x)$, it follows that $y - x = 0$, and so $x = y$. Therefore R is antisymmetric.
- We have $(0, 1) \in R$ and $(1, 2) \in R$ but $(0, 2) \notin R$. Therefore R is not transitive.

Problem 3
Let S be a set, and let $I = \{(x, x) : x \in S\}$ be the equality relation on S. Show that a binary relation $R \subseteq S \times S$ is both symmetric and antisymmetric if, and only if $\emptyset \subseteq R \subseteq I$.

Solution

"Only if": Suppose R is symmetric and antisymmetric. Clearly $\emptyset \subseteq R$, so it remains to show that $R \subseteq I$.
Take any $(a, b) \in R$. As R is symmetric, we have $(b, a) \in R$. As R is antisymmetric, we have that $a = b$. Therefore $(a, b) = (a, a) \in I$. Therefore $R \subseteq I$.

"If": Now suppose that $\emptyset \subseteq R \subseteq I$. We have:

- If $(a, b) \in R$ then $(a, b) \in I$, so $a = b$, so $(a, b) = (b, a)$ and therefore $(b, a) \in R$. So R is symmetric.

- If $(a, b), (b, a) \in R$ then, from the previous case, we have $a = b$, so R is antisymmetric.

Problem 4
Prove, or provide a counterexample to disprove, that for all sets S and all binary relations $R_1, R_2 \subseteq S \times S$:

(a) If R_1 and R_2 are transitive, then $R_1 \cup R_2$ is transitive.

(b)† If R_1 and R_2 are partial orders, then $R_1 \cap R_2$ is a partial order. (18S2)

(c)† If R_1 and R_2 are symmetric, then $R_1 \setminus R_2$ is symmetric. (19T3)

Solution

(a) This is false. Consider $R_1, R_2 \subseteq \{1,2\}$ given by $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$. Both R_1 and R_2 are transitive (vacuously, as there is no x, y, z such that $(x,y) \in R_i$ and $(y,z) \in R_i$). But $(1,2), (2,1) \in R_1 \cup R_2$ and $(1,1) \notin R_1 \cup R_2$, so $R_1 \cup R_2$ is not transitive.

(b) This is true.

(c) This is true. If $(a,b) \in R_1 \setminus R_2$ then:

- $(a,b) \in R_1$. As R_1 is symmetric, $(b,a) \in R_1$.
- $(a,b) \notin R_2$. As R_2 is symmteric, it follows that $(b,a) \notin R_2$ (as otherwise $(a,b) \in R_2$).

Therefore $(b,a) \in R_1 \setminus R_2$.

Problem 5† (20T2)

Let $(B, \preceq)$ be a partially ordered set, A be a set, and $f : A \to B$ a function from A to B. Define $R \subseteq A \times A$ as follows:

$$(a,b) \in R \quad \text{if and only if} \quad f(a) \preceq f(b)$$

(a) Give a counterexample to show that in general R is not a partial order.

(b) (i) State a restriction on f that will ensure R is a partial order, and

(ii) Prove that under that restriction R is a partial order.

(c) Prove that $R \cap R^{\leftarrow}$ is an equivalence relation.

Solution

(a) Consider the set $A = B = \{0,1\}$ with the partial order $\preceq$ being less than or equal to. Let $f : A \to B$ be defined as $f(0) = f(1) = 0$. Then we have $(0,1) \in R$ as $f(0) \preceq f(1)$ and $(1,0) \in R$ as $f(1) \preceq f(0)$, but $0 \neq 1$, so R is not antisymmetric. Therefore R is not a partial order.

(b) (i) f is an injection.

(ii) We need to show that if f is an injection, then R is a partial order. That is, we need to show that R is reflexive, antisymmetric, and transitive.

- (R): We have, for all $x \in A$, $f(x) \preceq f(x)$ because $\preceq$ is reflexive. Therefore $(x,x) \in R$, so R is reflexive.
- (AS): Suppose $(x,y) \in R$ and $(y,x) \in R$. Then we have $f(x) \preceq f(y)$ and $f(y) \preceq f(x)$. Since $\preceq$ is antisymmetric, it follows that $f(x) = f(y)$. Because f is an injection, it follows that $x = y$. Therefore R is antisymmetric.

- (T): Suppose $(x, y), (y, z) \in R$. Then $f(x) \preceq f(y)$ and $f(y) \preceq f(z)$. Therefore $f(x) \preceq f(z)$ because $\preceq$ is transitive. Therefore $(x, z) \in R$, and so R is transitive.

(c) To show that $R \cap R^{\leftarrow}$ is an equivalence relation, we need to show that it is reflexive, symmetric, and transitive.

- (R): We have, for all $x \in A$, $f(x) \preceq f(x)$ because $\preceq$ is reflexive. Therefore $(x, x) \in R$ and $(x, x) \in R^{\leftarrow}$. Therefore $(x, x) \in R \cap R^{\leftarrow}$, so $R \cap R^{\leftarrow}$ is reflexive.
- (S): Suppose $(x, y) \in R \cap R^{\leftarrow}$. Because $(x, y) \in R$, we have $(y, x) \in R^{\leftarrow}$. Also, because $(x, y) \in R^{\leftarrow}$ we have $(y, x) \in R$. Therefore $(y, x) \in R \cap R^{\leftarrow}$, so $R \cap R^{\leftarrow}$ is symmetric.

  Note

  We do not need to use the definition of R here!
- (T): Suppose $(x, y), (y, z) \in R \cap R^{\leftarrow}$. Because $(x, y), (y, z) \in R$ we have $f(x) \preceq f(y) \preceq f(z)$, so $(x, z) \in R$ because $\preceq$ is transitive. Because $(x, y), (y, z) \in R^{\leftarrow}$ we have $f(z) \preceq f(y) \preceq f(x)$, so $(x, z) \in R^{\leftarrow}$ because, again, $\preceq$ is transitive. Therefore $(x, z) \in R \cap R^{\leftarrow}$, so $R \cap R^{\leftarrow}$ is transitive.

Problem 6† (17S2)

Let $\Sigma = \{0, 1\}$. We define the *prefix relation*, $\preceq$, on Σ^* as follows: $w \preceq w'$ if, and only if, $w' = wv$ for some $v \in \Sigma^*$.

(a) Show $\preceq$ is a partial order on Σ^*.

(b) Draw the Hasse diagram for the poset $(\Sigma^{\leq 2}, \preceq)$.

(c) (i) What is $\mathsf{lub}(010101, 011011)$ (if it exists)?

(ii) What is $\mathsf{glb}(010101, 011011)$ (if it exists)?

(d) Is a sorting of Σ^* by the

(i) lexicographic order

(ii) lenlex order

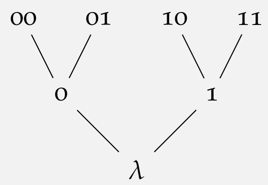
a topological sort of $(\Sigma^*, \preceq)$?

Solution

(a) $w = w.\lambda$ so $w \preceq w$ for all $w \in \Sigma^*$, so $\preceq$ is reflexive.

If $w \preceq w'$ and $w' \preceq w$ then $w' = wv$ and $w = w'v'$ for some $v, v' \in \Sigma^*$. But then $w = wvv'$, so $vv' = \lambda$, so $v = \lambda$ and $w = w'$. So $\preceq$ is antisymmetric.

Finally if $w \preceq w'$ and $w' \preceq w''$ then $w' = wv$ and $w'' = w'v'$ for some $v, v' \in \Sigma^*$. But then $w'' = (wv)v' = w(vv')$, so $w \preceq w''$. Thus $\preceq$ is transitive and therefore a partial order.

(b) 

(c) (i) $\mathsf{lub}(010101, 011011)$ does not exist as 010101 and 011011 cannot both be prefixes of the same word.

(ii) $\mathsf{glb}(010101, 011011) = 01$, the longest common prefix of 010101 and 011011.

(d) (i) Yes, if $w \preceq w'$ then w occurs before w' in the lexicographic ordering, so a lexicographic sort is a valid topological sort.

(ii) Yes, if $w \preceq w'$ then w occurs before w' in the lenlex ordering, so a lenlex sort is a valid topological sort.

Problem 7*
Consider the poset $(\mathbb{N}, |)$. Show that for all $x, y \in \mathbb{N}$:

(a) $\gcd(x, y) = \text{glb}(x, y)$

(b) $\text{lcm}(x, y) = \text{lub}(x, y)$