Number Theory

Problem 1

How many numbers are there between 100 and 1000 that are

(a) divisible by 3?

(b) divisible by 5?

(c) divisible by 15?

Problem 2

(a) What is:

- (i) $\gcd(420, 720)$?
- (ii) $\mathrm{lcm}(420, 720)$?
- (iii) $720 \text{ div } 42$?
- (iv) $5^{20} \ \% \ 7$?

(b) True or false:

- (i) $42|7$?
- (ii) $7|42$?
- (iii) $3 + 5|9 + 23$?
- (iv) $27 =_{(6)} 33$?
- (v) $-1 =_{(7)} 22$?

Problem 3[†] (2020 T2)

Prove, or give a counterexample to disprove:

(a) For all $x \in \mathbb{R}$:

$$|\lfloor x \rfloor| = \lfloor |x| \rfloor$$

(b) For all $x \in \mathbb{Z}$:

$$42 | x^7 - x$$

(c) For all $x, y, z \in \mathbb{Z}$, with $z > 1$ and $z \nmid y$:

$$(x \text{ div } y) =_{(z)} ((x \ \% \ z) \text{ div } (y \ \% \ z))$$

[†] indicates a previous exam question

[*] indicates a difficult/advanced question.

Problem 4
Prove that for all $m, n, p \in \mathbb{Z}$ with $n \geq 1$:

(a) $0 \leq (m \,\%\, n) < n$

(b) $m =_{(n)} p$ if, and only if $(m \,\%\, n) = (p \,\%\, n)$

Problem 5
Suppose $m =_{(n)} m'$ and $p =_{(n)} p'$. Prove that:

(a) $m + p =_{(n)} m' + p'$

(b) $m \cdot p =_{(n)} m' \cdot p'$

Problem 6

(a) Prove that the 4 digit number $n = abcd$ is:

  (i) divisible by 5 if and only if the last digit d is divisible by 5.

  (ii) divisible by 9 if and only if the digit sum $a + b + c + d$ is divisible by 9.

  (iii) divisible by 11 if and only if $a - b + c - d$ is divisible by 11.

(b) Find a similar rule to determine if a 4 digit number is divisible by 7.

Problem 7† (2020 T3)
The following process leads to a rule for determining if a large number n is divisible by 17:

- Remove the last digit, b, of n leaving a smaller number a.
- Let $n' = a - 5b$.
- Repeat with n' in place of n.

So, for example, if $n = 12345$, then $n' = 1234 - 5 \cdot 5 = 1209$. Repeating would create $120 - 5 \cdot 9 = 75$; $7 - 5 \cdot 5 = -18$; and so on.

Prove that $17 | n$ if and only if $17 | n'$.

Problem 8† (2021 T3)
Prove or disprove the following:

(a) For all $x, y, z \in \mathbb{N}$:
$$x + \gcd(y, z) = \gcd(x + y, x + z)$$

(b) For all $x, y, z \in \mathbb{N}$:
$$x \cdot \gcd(y, z) = \gcd(xy, xz)$$

Problem 9*
Prove that for $m, n \in \mathbb{Z}$:
$$\gcd(m, n) \cdot \operatorname{lcm}(m, n) = |m| \cdot |n|$$

Problem 10*
Prove that for all $n \in \mathbb{Z}$:
$$\gcd(n, n+1) = 1.$$

Problem 11*
Prove that for all $x, y, z \in \mathbb{Z}$:
$$\gcd(\gcd(x, y), z) = \gcd(x, \gcd(y, z)).$$