Problem 1 (15 marks)

Let S be a set.

(a) Show that for any set T and any function $f : S \to T$, the relation $R_f \subseteq S \times S$, defined as:

$$(s, s') \in R_f \text{ if and only if } f(s) = f(s')$$

is an equivalence relation. (9 marks)

(b) Show that if $R \subseteq S \times S$ is an equivalence relation, then there exists a set T and a function $f_R : S \to T$ such that:

$$(s, s') \in R \text{ if and only if } f_R(s) = f_R(s')$$

(6 marks)

Solution

(a) To show R_f is an equivalence relation, we must show reflexivity (R), symmetry (S), and transitivity (T).

(R): For all $s \in S$ we have $f(s) = f(s)$, so $(s, s) \in R_f$. So R_f is reflexive.

(S): Suppose we have $(s, s') \in R_f$. That is $f(s) = f(s')$. But then $f(s') = f(s)$, so $(s', s) \in R_f$. Therefore R_f is symmetric.

(T): Suppose we have $(s, t), (t, u) \in R_f$. Therefore $f(s) = f(t)$ and $f(t) = f(u)$. But then, $f(s) = f(u)$, so $(s, u) \in R_f$. Therefore R_f is transitive.

(b) Let T be the set of equivalence classes of R, and define $f_R : S \to T$ as $f_R(s) = [s]$. From lectures we have $(s, s') \in R$ if and only if $[s] = [s']$. That is, if and only if $f_R(s) = f_R(s')$, as required.

Problem 2 (20 marks)

Let $\mathbb{B} = \{0, 1\}$ and consider the function $f : \mathbb{N} \to \mathbb{B}$ given by

$$f(n) = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that for all $a, b \in \mathbb{N}$:

(i) $f(a + b) = \max\{f(a), f(b)\}$

(ii) $f(ab) = \min\{f(a), f(b)\}$

(6 marks)

Solution

(a) We consider 3 cases:

- $a = b = 0$: Here we have $f(a) = f(b) = 0$, $a + b = 0$ and $ab = 0$. So $f(a + b) = 0 = \max\{f(a), f(b)\}$ and $f(ab) = 0 = \min\{f(a), f(b)\}$.
- $a > 0$, $b = 0$ (the case where $b > 0$ and $a = 0$ is symmetric): Here we have $f(a) = 1$, $f(b) = 0$, $a + b > 0$ and $ab = 0$. So $f(a + b) = 1 = \max\{f(a), f(b)\}$ and $f(ab) = 0 = \min\{f(a), f(b)\}$.
- $a, b > 0$: Here we have $f(a) = f(b) = 1$, $a + b > 0$ and $ab > 0$. So $f(a + b) = 1 = \max\{f(a), f(b)\}$ and $f(ab) = 1 = \min\{f(a), f(b)\}$.

From Problem 1, we know that $R_f \subseteq \mathbb{N} \times \mathbb{N}$, the relation given by:

$$(m, n) \in R_f \text{ if and only if } f(m) = f(n)$$

is an equivalence relation. Let $\mathbb{E} \subseteq \mathrm{Pow}(\mathbb{N})$ be the set of equivalence classes of R_f, and for $n \in \mathbb{N}$, let $[n] \in \mathbb{E}$ denote the equivalence class of n.

We would like to define binary operations, $\boxplus$ and $\boxdot$, on $\mathbb{E}$ as follows:

$$[x] \boxplus [y] := [x + y]$$
$$[x] \boxdot [y] := [xy].$$

The difficulty is that the operands $[x]$ and $[y]$ can have multiple representations (e.g. if $z \in [x]$ then $[x] = [z]$), and so it is not clear that such a definition makes sense: if we take a different representation of the operands, do we still end up with the same result? That is, if $[x] = [x']$ and $[y] = [y']$ is it the case that $[x + y] = [x' + y']$ and $[xy] = [x'y']$? Our next step is to show that such a definition makes sense.

(b) Define relations $\boxplus, \boxdot \subseteq \mathbb{E}^2 \times \mathbb{E}$ as follows:

$$((X, Y), Z) \in \boxplus \text{ if and only if there is } x \in X \text{ and } y \in Y \text{ such that } x + y \in Z$$
$$((X, Y), Z) \in \boxdot \text{ if and only if there is } x \in X \text{ and } y \in Y \text{ such that } xy \in Z$$

(i) Show that $\boxplus$ is a function.

(ii) Show that $\boxdot$ is a function.

(6 marks)

Solution

(b) For both $\boxplus$ and $\boxdot$ we need to show that the relation is functional (F) [for every $(X, Y) \in \mathbb{E}^2$ there is at most one $Z \in \mathbb{E}$ such that (X, Y) is related to Z] and total (T) [for every $(X, Y) \in \mathbb{E}^2$ there is at least one $Z \in \mathbb{E}$ such that (X, Y) is related to Z]. In both cases, let (X, Y) be an arbitrary element of $\mathbb{E}^2$ and suppose $X = [x]$ and $Y = [y]$.

(i)
- To show $\boxplus$ is total, consider $Z = [x + y]$. From the definition of $\boxplus$, we have $((X, Y), Z) = (([x], [y]), [x + y]) \in \boxplus$.
- To show $\boxplus$ is functional we need to show that if $[x] = [x']$ and $[y] = [y']$ then $[x + y] =$

$[x' + y']$. If $[x] = [x']$ and $[y] = [y']$ then $f(x) = f(x')$ and $f(y) = f(y')$, so

$$f(x + y) = \max\{f(x), f(y)\} = \max\{f(x'), f(y')\} = f(x' + y').$$

Therefore $[x + y] = [x' + y']$ as required.

(ii)
- To show $\boxdot$ is total, consider $Z = [xy]$. From the definition of $\boxdot$, we have $((X, Y), Z) = (([x], [y]), [xy]) \in \boxdot$.
- To show $\boxdot$ is functional we need to show that if $[x] = [x']$ and $[y] = [y']$ then $[xy] = [x'y']$. If $[x] = [x']$ and $[y] = [y']$ then $f(x) = f(x')$ and $f(y) = f(y')$, so

$$f(xy) = \min\{f(x), f(y)\} = \min\{f(x'), f(y')\} = f(x'y').$$

Therefore $[xy] = [x'y']$ as required.

Part (b) shows that the informal definition of $\boxplus$ and $\boxdot$ given earlier is *well-defined*, so from now we will view $\boxplus$ and $\boxdot$ as **binary operations** on $\mathbb{E}$, that is $\boxplus, \boxdot : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$.

(c) Show that for all $A, B, C \in \mathbb{E}$:

(i) $A \boxdot [1] = A$

(ii) $A \boxplus B = B \boxplus A$

(iii) $A \boxdot (B \boxplus C) = (A \boxdot B) \boxplus (A \boxdot C)$

(8 marks)

Remark

Objects that have a concept of "addition" ($\boxplus$) and "multiplication" ($\boxdot$) where:

- addition and multiplication are associative,
- both operations have identities (see (c)(i)),
- addition is commutative (see (c)(ii)), and
- multiplication distributes over addition (see (c)(iii))

are known as semirings. We have already seen a number of semirings in this course:

- The natural numbers with usual addition and multiplication,
- Integers modulo n with addition and multiplication modulo n,
- Subsets of a set X with union and intersection,
- Languages with union and concatenation,
- Binary relations with union and relational composition (see Assignment 1),
- Matrices with matrix addition and matrix multiplication.

Solution

(c) Let x, y, z be such that $A = [x]$, $B = [y]$ and $C = [z]$. We have:

(i)
$$\begin{aligned} A \boxdot [1] &= [x] \boxdot [1] \\ &= [x \cdot 1] && \text{(from (b))} \\ &= [x] = A. \end{aligned}$$

(ii)
$$\begin{aligned} A \boxplus B &= [x] \boxplus [y] \\ &= [x + y] && \text{(from (b))} \\ &= [y + x] \\ &= [y] \boxplus [x] && \text{(from (b))} \\ &= B \boxplus A. \end{aligned}$$

(iii)
$$\begin{aligned} A \boxdot (B \boxplus C) &= [x] \boxdot ([y] \boxplus [z]) \\ &= [x] \boxdot [y + z] && \text{(from (b))} \\ &= [x \cdot (y + z)] && \text{(from (b))} \\ &= [x \cdot y + x \cdot z] \\ &= [x \cdot y] \boxplus [x \cdot z] && \text{(from (b))} \\ &= ([x] \boxdot [y]) \boxplus ([x] \boxdot [z]) && \text{(from (b))} \\ &= (A \boxdot B) \boxplus (A \boxdot C). \end{aligned}$$

Problem 3 (12 marks)

Eight houses are lined up on a street, with four on each side of the road as shown:

Each house wants to set up its own wi-fi network, but the wireless networks of neighbouring houses – that is, houses that are either next to each other (ignoring trees) or over the road from one another (directly opposite) – can interfere, and must therefore be on different channels. Houses that are sufficiently far away may use the same wi-fi channel. Your goal is to find the minimum number of different channels the neighbourhood requires.

(a) Model this as a graph problem. Remember to:

  (i) Clearly define the vertices and edges of your graph. (4 marks)

  (ii) State the associated graph problem that you need to solve. (2 marks)

(b) Give the solution to the graph problem corresponding to this scenario; and determine the minimum number of wi-fi channels required for the neighbourhood? (2 marks)

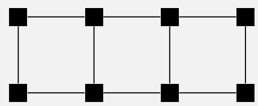
(c) How do your answers to (a) and (b) change if a house's wireless network can also interfere with those of the houses to the left and right of the house over the road? (4 marks)

Solution

(a) (i) The problem can be viewed as the problem of finding the chromatic number (i.e. minimum number of colours required to colour vertices such that no edge has two end-points the same colour) of the following graph:

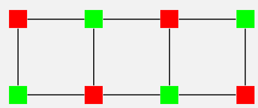
- We have one vertex for every house
- We have an edge between houses if, and only if, their wireless networks can interfere.

In this example the graph would be:



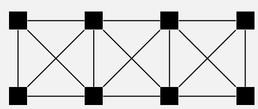
(ii) Each colour in a valid colouring represents the wi-fi channel that the house will use - so a valid colouring corresponds to a non-interfering assignment of wi-fi channels. Finding the minimum number of different channels is therefore the same as finding the minimum number of colours to validly colour the graph.

(b) Here is a 2-colouring of the graph from part (a):

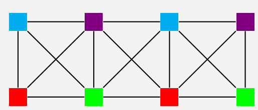


Since the graph has an edge, we cannot colour the graph with fewer than 2 colours, so the minimum number of wi-fi channels required for the neighbourhood is 2.

(c) With the additional constraint, the neighbourhood interference graph becomes:



Here is a 4-colouring of this graph:



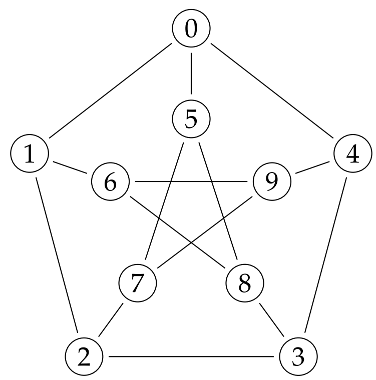
As the graph contains K_4, we cannot colour it with fewer than 4 colours, so the minimum number of wi-fi channels required increases to 4.

Discussion

- For (a) students should define the graph in a general way (not required in (c)), and justify the use of the chromatic number. Failure to do either is a major error.
- For (b) and (c) students must provide an u pper and lower bound. Omission of either is a major error.

Problem 4 (12 marks)

This is the Petersen graph:



(a) Give an argument to show that the Petersen graph does not contain a subdivision of K_5. (6 marks)

(b) Show that the Petersen graph contains a subdivision of $K_{3,3}$. (6 marks)

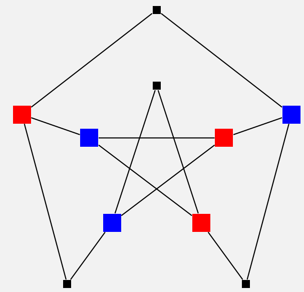
Solution

To show that a graph G contains a subdivision of H we can do the following.

- Starting with G.
- Repeat any of the following operations as many times as necessary:
  - Delete edges
  - Delete vertices
  - Contract a vertex of degree 2 (delete the vertex and connect its two neighbours with an edge)
- Obtain H.

(a) We observe that in each of the above operations we will never increase the degree of a vertex. Each of the vertices in the Petersen graph has degree 3, but each vertex in K_5 has degree 4. Therefore the Petersen graph does not contain a subdivision of K_5.

(b) Here is a subdivision of $K_{3,3}$ which is clearly a subgraph of the Petersen graph. The two partitions are indicated by red and blue vertices. The black vertices subdivide edges.



Discussion

- Full marks for clear logical arguments
- Minor errors (4 marks) for small omissions such as not indicating the approach for finding a subdivision (two different approaches were given in the lecture, the students should indicate which one they are using).
- Major errors (3 marks) for a clear, but incorrect proof
- Shows progress (1 mark) for demonstrating some understanding

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Problem 5 (20 marks)

Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

$$\begin{aligned} R^0 &:= I = \{(x,x) : x \in S\}, \text{ and} \\ R^{n+1} &:= R^n \cup (R; R^n) \text{ for } n \geq 0 \end{aligned}$$

(a) Prove that for all $i, j \in \mathbb{N}$, if $i \leq j$ then $R^i \subseteq R^j$. *Hint: Let $P_i(j)$ be the proposition that $R^i \subseteq R^j$ and prove that $P_i(j)$ holds for all $j \geq i$.* (4 marks)

(b) Let $P(n)$ be the proposition that for all $m \in \mathbb{N}$: $R^n; R^m = R^{n+m}$. Prove that $P(n)$ holds for all $n \in \mathbb{N}$. *Hint: Use results from Assignment 1* (4 marks)

(c) Prove that if there exists $i \in \mathbb{N}$ such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$. (4 marks)

(d) If $|S| = k$, explain why $R^{k^2} = R^{k^2+1}$. (2 marks)

(e) If $|S| = k$, show that R^{k^2} is transitive. (2 marks)

(f)* If $|S| = k$ show that R^{k^2} is the minimum (with respect to $\subseteq$) of all reflexive and transitive relations that contain R. (4 marks)

Remark

The relation at the limit[a] as n tends to infinity, $R^* = \lim_{n\to\infty} R^i$, is known as the **reflexive, transitive closure of** R, and is closely connected to the Kleene star operator.

[a] Because $R^j \subseteq R^i \subseteq S \times S$ for all $j \leq i$, the Knaster-Tarski theorem ensures this limit always exists, even for infinite S.

Solution

(a) For any i, let $P_i(j)$ be the proposition that $R^i \subseteq R^j$. We will prove that $P_i(j)$ holds for all $j \geq i$ by induction on j.

Base case $j = i$: Clearly $R^i \subseteq R^i$, so $P_i(i)$ holds.

Inductive case. Suppose $P_i(j)$ holds, that is, $R^i \subseteq R^j$. We have:

$$R^{j+1} = R^j \cup (R; R^j) \quad \supseteq \quad R^j \quad \supseteq \quad R^i,$$

so $P_i(j+1)$ also holds. Therefore, by the principle of mathematical induction, $P_i(j)$ holds for all $j \geq i$.

(b) We will prove $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base case $n = 0$: For all m,

$$\begin{aligned} R^{0+m} &= R^m \\ &= I; R^m && \text{(From Q1(b))} \\ &= R^0; R^m && \text{(Def. of } R^0\text{)} \end{aligned}$$

Inductive case. Suppose $P(n)$ holds, that is, for all m, $R^n; R^m = R^{n+m}$. We have, for all m:

$$\begin{aligned} R^{n+1}; R^m &= ((R^n \cup (R; R^n)); R^m && \text{(Definition)} \\ &= (R^n; R^m) \cup ((R; R^n); R^m) && \text{(From Assignment 1)} \\ &= (R^n; R^m) \cup (R; (R^n; R^m)) && \text{(From Assignment 1)} \\ &= R^{n+m} \cup (R; R^{n+m}) && \text{(IH)} \\ &= R^{n+m+1} && \text{(Definition)} \\ &= R^{(n+1)+m} \end{aligned}$$

So $P(n+1)$ holds. Therefore, by the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbb{N}$.

(c) Suppose $R^i = R^{i+1}$. Let $P(j)$ be the proposition that $R^j = R^i$. We will prove that $P(j)$ holds for all $j \geq i$.

Base case $j = i$: Clearly $P(i)$ holds as $R^i = R^i$.

Inductive case. Suppose $P(j)$ holds, that is, $R^j = R^i$ for some $j \geq i$. We will show that $P(j+1)$ holds. We have:

$$\begin{aligned} R^{j+1} &= R^j \cup (R; R^j) && \text{(Definition)} \\ &= R^i \cup (R; R^i) && \text{(IH)} \\ &= R^{i+1} && \text{(Definition)} \\ &= R^i && \text{(Given)} \end{aligned}$$

So $P(j)$ implies $P(j+1)$. So by the principle of mathematical induction, $P(j)$ holds for all $j \geq i$.

(d) For all $j \in \mathbb{N}$ we have $R^j \subseteq S \times S$, so we have that $|R^j| \leq k^2$. From (a) we have that

$$R^0 \subseteq R^1 \subseteq R^2 \subseteq \cdots \subseteq R^{k^2} \subseteq R^{k^2+1}.$$

Since the cardinalities of each of these k^2+2 relations is bounded below by 0 and bounded above by k^2, it follows that there exists $i, j \in [0, k^2+1]$, with $i < j$, such that $|R^i| = |R^j|$. Since $R^i \subseteq R^j$ it follows that $R^i = R^j$, and indeed $R^i = R^h$ for all $h \in [i, j]$. In particular $R^i = R^{i+1}$. It then follows from (c) that $R^h = R^i$ for all $h \geq i$, in particular $R^{k^2} = R^i = R^{k^2+1}$ as required.

(e) From (d) we have that if $|S| = k$ then $R^{k^2} = R^{k^2+1}$. Now suppose $(a, b) \in R^{k^2}$ and $(b, c) \in R^{k^2}$. Then

$$\begin{aligned} (a,c) &\in R^{k^2}; R^{k^2} && \text{(Definition of ;)} \\ &= R^{2k^2} && \text{(From (b))} \\ &= R^{k^2} && \text{(From (c))} \end{aligned}$$

So R^{k^2} is transitive.

(f) From (a) we have $I = R^0 \subseteq R^{k^2}$ so R^{k^2} is reflexive; and from (e) we have that R^{k^2} is transitive. It remains to show that $R^{k^2} \subseteq T$ for any reflexive, transitive relation T with $R \subseteq T$.

We will first show that

Claim 1

If $R_1 \subseteq T_1$ and $R_2 \subseteq T_2$ then $(R_1; R_2) \subseteq (T_1; T_2)$.

If $(a, c) \in (R_1; R_2)$ then there exists a b such that

$$(a, b) \in R_1 \subseteq T_1 \quad \text{and} \quad (b, c) \in R_2 \subseteq T_2.$$

But then $(a, c) \in (T_1; T_2)$. So $(R_1; R_2) \subseteq (T_1; T_2)$.

We next show:

Claim 2

If $R \subseteq T$ then $R^i \subseteq T^i$ for all i.

Suppose $R \subseteq T \subseteq S \times S$. Let $P(i)$ be the proposition that $R^i \subseteq T^i$. We will prove, by induction, that $P(i)$ holds for all $i \in \mathbb{N}$.

Base case: $i = 0$. We have $R^0 = T^0 = I$ so the proposition $P(0)$ holds.

Inductive case. Suppose $P(i)$ holds, that is $R^i \subseteq T^i$. Then, from Claim 1 we have $(R; R^i) \subseteq (T; T^i)$. Therefore,

$$R^{i+1} = R^i \cup (R; R^i) \subseteq T^i \cup (T; T^i) = T^{i+1}.$$

So $P(i+1)$ holds. Therefore, by the principle of induction $P(i)$ holds for all $i \in \mathbb{N}$.

Finally we prove:

Claim 3

If T is reflexive and transitive, then $T^i = T$ for all $i \in \mathbb{N}$, with $i > 0$.

Let $Q(i)$ be the proposition that $T^i = T$. We will show that $Q(i)$ holds for all $i > 0$ by induction.

Base case: $i = 1$. We have $T^1 = I \cup T = T$ because T is reflexive. So $Q(1)$ holds.

Inductive case. Suppose $Q(i)$ holds, that is $T^i = T$. As T is transitive, we have that $T = (T; T)$, so:

$$T^{i+1} = T \cup (T; T^i) = T \cup (T; T) = T \cup T = T.$$

Therefore $Q(i+1)$ holds. Therefore, by induction, $Q(i)$ holds for all $i > 0$.

Putting Claims 2 and 3 together we have for any reflexive, transitive relation T with $R \subseteq T$:

$$R \subseteq T \quad \text{so} \quad R^{k^2} \subseteq T^{k^2} = T$$

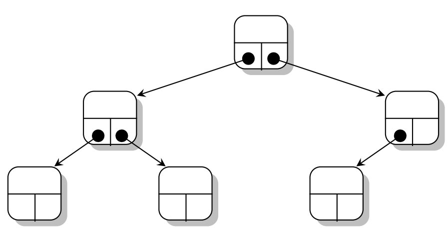
as required.

Discussion

- Generally looking for induction (or some other sound method).
- Minor errors include: not properly referencing earlier results (applied once per part); missing a base case in induction proofs
- Major errors include: missing inductive case; reasoning with "..." in (a) or (d)

Problem 6 (20 marks)

A *binary tree* is a data structure where each node is linked to at most two successor nodes:



If we include empty binary trees (trees with no nodes) as part of the definition, then we can simplify the description of the data structure. Rather than saying a node has 0, 1, or 2 successor nodes, we can instead say that a node has exactly two *children*, where a child is a binary tree. That is, we can abstractly define the structure of a binary tree as follows:

- (B): An empty tree, τ
- (R): An ordered pair $(T_{\text{left}}, T_{\text{right}})$ where T_{left} and T_{right} are trees.

So, for example, the above tree would be defined as the tree T where:

$$\begin{aligned} &T = (T_1, T_2), \text{ where} \\ &\quad T_1 = (T_3, T_4) \text{ and } T_2 = (T_5, \tau), \text{ where} \\ &\quad\quad T_3 = T_4 = T_5 = (\tau, \tau) \end{aligned}$$

That is,

$$T = \Big(\big((\tau,\tau), (\tau,\tau) \big), \big((\tau,\tau), \tau \big) \Big)$$

A *leaf* in a binary tree is a node that has no successors (i.e. it is of the form (τ, τ)). A *fully-internal* node in a binary tree is a node that has exactly two successors (i.e. it is of the form (T_1, T_2) where $T_1, T_2 \neq \tau$). The example above has 3 leaves (T_3, T_4, and T_5) and 2 fully-internal nodes (T and T_1). For technical reasons (that will become apparent) we assume that an empty tree has 0 leaves and -1 fully-internal nodes.

(a) Based on the recursive definition above, recursively define a function $\mathsf{count}(\mathsf{T})$ that counts the number of nodes in a binary tree T. (4 marks)

(b) Based on the recursive definition above, recursively define a function $\mathsf{leaves}(\mathsf{T})$ that counts the number of leaves in a binary tree T. (4 marks)

(c) Based on the recursive definition above, recursively define a function $\mathsf{internal}(\mathsf{T})$ that counts the number of fully-internal nodes in a binary tree T. (4 marks)

(d) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves}(\mathsf{T}) = \mathsf{internal}(\mathsf{T}) + 1$. Prove that $P(\mathsf{T})$ holds for all binary trees T. Your proof should be based on your answers given in (b) and (c). (8 marks)

Solution

(a) We define count recursively on trees as follows:

- $\mathsf{count}(\tau) = 0$
- $\mathsf{count}([T_1, T_2]) = 1 + \mathsf{count}(T_1) + \mathsf{count}(T_2)$

(b) We define leaves recursively on trees as follows:

- $\mathsf{leaves}(\tau) = 0$
- $\mathsf{leaves}([\tau, \tau]) = 1$ (without this condition $\mathsf{leaves}(T) = 0$ for all trees)
- $\mathsf{leaves}([T_1, T_2]) = \mathsf{leaves}(T_1) + \mathsf{leaves}(T_2)$ if T_1 and T_2 are not both empty.

(c) We define internal recursively on trees as follows:

- $\mathsf{internal}(\tau) = -1$
- $\mathsf{internal}([\tau, \tau]) = 0$ (without this condition $\mathsf{internal}(T) = -1$ for all trees)
- $\mathsf{internal}([T_1, T_2]) = 1 + \mathsf{internal}(T_1) + \mathsf{internal}(T_2)$ if T_1 and T_2 are not both empty. (As $\mathsf{internal}(\tau) = -1$, we don't have to consider separately the case where one child is empty)

(d) We prove $P(\mathsf{T})$ by structural induction on T.

Base case: ($\mathsf{T} = \tau$) From the definitions of leaves and internal we have:

$$\mathsf{leaves}(\tau) = 0 = 1 + (-1) = 1 + \mathsf{internal}(\tau).$$

So $P(\tau)$ holds.

Base case: (T $= [\tau, \tau]$) (NB: this can also be considered a subcase of the inductive case) From the definitions of leaves and internal we have:

$$\mathsf{leaves}([\tau,\tau]) = 1 = 1 + 0 = 1 + \mathsf{internal}([\tau,\tau]).$$

So $P([\tau,\tau])$ holds.

Inductive case: (T $= [T_1, T_2]$) Assume $P(T_1)$ and $P(T_2)$ hold and T_1 and T_2 are not both empty. That is,

$$\mathsf{leaves}(T_1) = 1 + \mathsf{internal}(T_1) \quad \text{and} \quad \mathsf{leaves}(T_2) = 1 + \mathsf{internal}(T_2).$$

We will show that $P([T_1, T_2])$ holds. We have:

$$\begin{aligned}
\mathsf{leaves}([T_1,T_2]) &= \mathsf{leaves}(T_1) + \mathsf{leaves}(T_2) && \text{(Def. of leaves)}\\
&= (1 + \mathsf{internal}(T_1)) + (1 + \mathsf{internal}(T_2)) && \text{(Induction Hypothesis)}\\
&= 1 + (1 + \mathsf{internal}(T_1) + \mathsf{internal}(T_2)) && \\
&= 1 + \mathsf{internal}([T_1,T_2]) && \text{(Def. of internal)}
\end{aligned}$$

So $P[T_1, T_2]$ holds when T_1 and T_2 are not both empty.

By the Principle of Structural Induction, we have that $P(T)$ holds for all trees T.

Discussion

For each part:

- Omitting a base case in definitions or proofs is a minor error.
- Any results that are not based on the recursive structure (or previous answers) can only obtain a maximum of 50% of the marks.

Problem 7* (5 marks)

Let Σ be a finite set, totally ordered by $<$. Give a formal, recursive definition of the lexicographic ordering $\leq_{\text{lex}} \subseteq \Sigma^* \times \Sigma^*$.

Solution

We can define $\leq_{\text{lex}}$ recursively as follows:

- $\lambda \leq_{\text{lex}} v$ for all $v \in \Sigma^*$
- $v \not\leq_{\text{lex}} \lambda$ for any $v \neq \lambda$
- For $w = aw' \neq \lambda$ and $v = bv' \neq \lambda$ we have

$$w \leq_{\text{lex}} v \quad \text{if and only if} \quad (a < b) \text{ or } (a = b \text{ and } w' \leq_{\text{lex}} v').$$