Probability

Problem 1
A 4-letter word is selected at random from Σ^4, where $\Sigma = \{a, b, c, d, e\}$.

(a) What is the probability that the letters in the word are distinct?

(b) What is the probability that there are no vowels in the word?

(c) What is the probability that the word begins with a vowel?

(d) What is the expected number of vowels in the word?

(e) Let x be the answer to the previous question. What is the probability of the word having $\lceil x \rceil$ or more vowels?

Problem 2
A black die and a red die are tossed. What is the probability that

(a) the sum of the values is even?

(b) the number on the red die is bigger than the number on the black die?

(c) the number on the red die is twice the number on the black die?

Problem 3
Team α faces team β in a 5-match series. Matches are either won or lost, i.e., there are no draws. It takes 3 wins to win the series. Team α has probability p $(0 < p < 1)$ of winning a match. Consider each of the following situations and calculate the probability that they will lose the whole series.

(a) They have lost the first match of the series already.

(b) They have lost one of the first two matches of the series already.

(c) They have lost the first two matches of the series already.

(d) They have lost one of the first three matches of the series already.

(e) They have lost two of the first three matches of the series already.

Problem 4
Let E_1, E_2 be two events. Prove that $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$ implies $P(E_2 \setminus E_1) = 0$.

Problem 5† (20T2)
Suppose two players, A and B, are playing the following game:

† indicates a previous exam question
* indicates a difficult/advanced question.

- A starts.
- The players take turns rolling a 6-sided die.
- Whoever rolls the first 6 wins the game.

(a) What is the probability that A wins?

(b) What is the expected number of die rolls before a winner is determined?

Now suppose we consider the following addition to the rules:

- If a player rolls a number that has already been seen then they roll again until an unseen number is rolled.

(c) What is the probability that A wins this game? (4 marks)

(d) If we say a turn ends when an unseen number is rolled, what is the expected number of turns before a winner is determined?

(e) At the start of B's second turn, what is the expected number of die rolls before a winner is determined?

Problem 6
Consider the procedure given in lectures to simulate a die using a fair coin:

(A) Flip a coin 3 times.

(B) If the outcome was:

- HHH: Output 1
- HHT: Output 2
- HTH: Output 3
- HTT: Output 4
- THH: Output 5
- THT: Output 6
- TTH: Go to (A)
- TTT: Go to (A)

What is the expected number of coin flips to obtain an output?