1.

(a).

$\because$ the running time of the code: $C\ [i,j]\ =\ A[i,j]\ +\ B[i,j] \ \epsilon\ \mathrm{O}(1)$
the running time of the for loop: for i $\in\ [0,n)$: $\epsilon\ \mathrm{O}(\mathrm{n})$
the running time of the for loop: for j $\in\ [0,n)$: $\epsilon\ \mathrm{O}(\mathrm{n})$
$\therefore$ the upper bound of the running time of the sum is : $\mathrm{O}(\mathrm{n}) \times \mathrm{O}(\mathrm{n}) \times \mathrm{O}(1)$
$= \mathrm{O}(\mathrm{n}^2)$

(b).

$\because$ *the running time of the code*: $A[i,k] * B[k,j]\ \epsilon\ O(1 * k)\ \epsilon\ O(n)$ with $\mathrm{k}\ \epsilon\ [0,\mathrm{n})$
and the running time of the process about adding element $\epsilon\ O(1 * k)\ \epsilon\ O(n)$ with $\mathrm{k}\ \epsilon\ [0,\mathrm{n})$
the running time of the for loop: for i $\in\ [0,n)$: $\epsilon\ \mathrm{O}(\mathrm{n})$
the running time of the for loop: for j $\in\ [0,n)$: $\epsilon\ \mathrm{O}(\mathrm{n})$
$\therefore$ the upper bound of the running time of the sum is
: $\mathrm{O}(\mathrm{n}) \times \mathrm{O}(\mathrm{n}) \times (\mathrm{O}(\mathrm{n}) + \mathrm{O}(\mathrm{n})) = \mathrm{O}(\mathrm{n}^3)$

(c).

$\because$ with S,T,U,V,W,X,Y,Z *are* $\frac{\mathrm{n}}{2} \times \frac{\mathrm{n}}{2}$ *matrices*
and AB are sums of products of the smaller matrices of the S,T,U,V,W,X,Y,Z
$\because$ according the the answer from the quesion(a), the running time of the sum$\epsilon\ \mathrm{O}(\mathrm{n}^2)$
$\therefore \mathrm{T}(1) = \mathrm{O}(1)$
$T(n) = 8T\left(\frac{\mathrm{n}}{2}\right) + 4\mathrm{O}((\frac{\mathrm{n}}{2})^2)\ \ \epsilon\ 8T\left(\frac{\mathrm{n}}{2}\right) + \mathrm{O}(\mathrm{n}^2)$

(d).

Accoring to the theorem of the recurrences: $\mathrm{T}(\mathrm{n}) = \mathrm{a}^*\mathrm{T}(\frac{\mathrm{n}}{\mathrm{b}}) + \mathrm{f}(\mathrm{n})$
where $\mathrm{f}(\mathrm{n})\ \epsilon\theta(\mathrm{n}^{\mathrm{c}}(\log \mathrm{n})^{\mathrm{k}}$

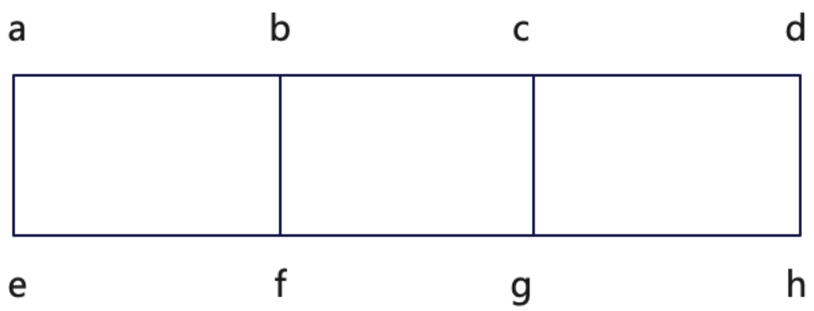
it can get: $a = 8, \mathrm{b}, 2, \mathrm{c} = 2, \mathrm{k} = 0$
$\therefore \ \mathrm{d} = \log_{\mathrm{b}}(\mathrm{a}) = 3$
$\because \ \mathrm{c} < \mathrm{d}, \text{then } \mathrm{T(n)} = \mathrm{O}(\mathrm{n}^{\mathrm{d}}) \ \epsilon \ \mathrm{O}(\mathrm{n}^3)$

2.

(a).

(i).

set the houses in the first line is A, B, C, D, and the houses in the second line is E, F, G, H.



And set the eight houses as 8 boolean variable with the variable of True and False.

the variable of True means the house use the channel 1, and the variable of False means the house use the channel 2. Take the house A as the example: A means channel 1 and ¬A means channel 2

(ii).

for the houses which are neighbouring houses, they all need to represent the same propositional formulas, take the neighbouring houses A and B as the example, the propositional formulas is: $(A \leftrightarrow \neg B) \vee (\neg A \leftrightarrow B)$
$\therefore$ the formula for the eight houses is:
$(((((((A \leftrightarrow \neg B) \leftrightarrow C) \leftrightarrow \neg D) \leftrightarrow H) \leftrightarrow \neg G) \leftrightarrow F) \leftrightarrow \neg E)$
$\vee(((((((\neg A \leftrightarrow B) \leftrightarrow \neg C) \leftrightarrow D) \leftrightarrow \neg H) \leftrightarrow G) \leftrightarrow \neg F) \leftrightarrow E)$

(iii).

as the formulas $(A \leftrightarrow \neg B) \vee (\neg A \leftrightarrow B)$ get in the question 2 (ii), it can derive that:

$(A \leftrightarrow \neg B) \vee (\neg A \leftrightarrow B)$
$= ((A \rightarrow \neg B) \wedge (\neg B \rightarrow A)) \vee ((\neg A \rightarrow B) \wedge (B \rightarrow \neg A))$
$\equiv ((\neg A \vee \neg B) \wedge (B \vee A)) \vee ((A \vee B) \wedge (\neg B \vee \neg A))$
$\equiv (\neg A \vee \neg B) \wedge (A \vee B)$
$\equiv ((\neg A \vee \neg B) \wedge A) \vee ((\neg A \vee \neg B) \wedge B)$
$\equiv ((\neg A \wedge A) \vee (A \wedge \neg B)) \vee ((\neg A \wedge B) \vee (B \wedge \neg B)) \approx$
$\equiv (A \vee \neg B) \vee (\neg A \vee B)$
$\equiv \top$

$\therefore$ for the eight houses: we can get the fomula:

$((A \leftrightarrow \neg B) \wedge (A \leftrightarrow \neg E) \wedge (\neg B \leftrightarrow F) \wedge (\neg B \leftrightarrow C) \wedge (C \leftrightarrow \neg D) \wedge (C \leftrightarrow \neg G) \wedge$
$(\neg D \leftrightarrow H)) \vee$
$((\neg A \leftrightarrow B) \wedge (\neg A \leftrightarrow E) \wedge (B \leftrightarrow \neg F) \wedge (B \leftrightarrow \neg C) \wedge (\neg C \leftrightarrow D) \wedge (\neg C \leftrightarrow G) \wedge$
$(D \leftrightarrow \neg H))$

and it can derive that:

$((A \leftrightarrow \neg B) \wedge (A \leftrightarrow \neg E) \wedge (\neg B \leftrightarrow F) \wedge (\neg B \leftrightarrow C) \wedge (C \leftrightarrow \neg D) \wedge (C \leftrightarrow \neg G) \wedge$
$(\neg D \leftrightarrow H)) \vee$
$((\neg A \leftrightarrow B) \wedge (\neg A \leftrightarrow E) \wedge (B \leftrightarrow \neg F) \wedge (B \leftrightarrow \neg C) \wedge (\neg C \leftrightarrow D) \wedge (\neg C \leftrightarrow G) \wedge$
$(D \leftrightarrow \neg H))$
$\equiv \top$

(iv).

set the eight houses as 8 boolean variable with the variable of 1,2,3

Take the house A as the example: A1 means channel 1 and A2 means channel 2 and A3 means channel 3.

$\therefore\ A = (A1 \wedge \neg A2 \neg A3) \vee (\neg A1 \wedge A2 \neg A3) \vee (\neg A1 \wedge \neg A2 A3)$

$\therefore$ *for the houses which are neighbouring houses, they all need to represent the* same propositional formulas, take the neighbouring houses A and B as the example, the *propositional formulas is*:

$(A \leftrightarrow \neg B) \vee (\neg A \leftrightarrow B)$
$\equiv \neg(A1 \wedge B1) \wedge \neg(A2 \wedge B2) \wedge \neg(A3 \wedge B3)$

(b).

$\because$ each house chooses uniformly at random
$\therefore$ each house have two different choice. So there are 2^8 different choice for eight houses.
$\because$ there are only two network in channels will not be interference. The first viable network is A, C, F, H in channel1 and B, D, E, G in channel2. The second viable network is A, C, F, H in channel2 and B, D, E, G in channel2.
$\therefore$ the probability is $\frac{2}{2^8} = \frac{1}{2^7}$

3.

(a).

$(x \wedge 1') \vee (x' \wedge 1)$	
$= (x \wedge (x \vee x')') \vee (x' \wedge 1)$	Complement with ∨
$= (x \wedge (x \vee x')') \vee x'$	Identity of ∧
$= (x \wedge (x' \wedge x'')) \vee x'$	De Morgan's, ' over ∨
$= (x \wedge (x' \wedge x)) \vee x'$	Double complement
$= (x \wedge (x \wedge x')) \vee x'$	Commutatitivity of ∧
$= (x \wedge 0) \vee x'$	Complement with ∧
$= 0 \vee x'$	Annihilation of ∧
$= x' \vee 0$	Commutatitivity of ∨
$= x'$	Identity of ∨

(b).

$(x \wedge y) \vee x$	
$= (x \wedge y) \vee (x \wedge 1)$	Identity of ∧
$= x \wedge (y \vee 1)$	Distributivity of ∧ over ∨
$= x \wedge 1$	Annihilation of ∨
$= x$	Identity of ∧

(c).

$y' \wedge ((x \vee y) \wedge x')$	
$= y' \wedge (x' \wedge (x \vee y))$	Commutatitivity of ∧

$= y' \wedge ((x' \wedge x) \vee (x' \wedge y))$	Distributivity of ∧ over ∨
$= y' \wedge ((x \wedge x') \vee (x' \wedge y))$	Commutatitivity of ∧
$= y' \wedge (0 \vee (x' \wedge y))$	Complement with ∧
$= y' \wedge ((x' \wedge y) \vee 0)$	Commutatitivity of ∨
$= y' \wedge (x' \wedge y)$	Identity of ∨
$= (x' \wedge y) \wedge y'$	Commutatitivity of ∧
$= x' \wedge (y \wedge y')$	Associativity of ∧
$= x' \wedge 0$	Complement with ∧
$= 0$	Annihilation of ∧

4.

According to the definition of the Boolean algebra: A boolean algebra is a structure $(T, \vee, \wedge, ', 0, 1)$ where hold some laws and $0, 1 \in T$; $\vee, \wedge: T \times T \rightarrow T$; $': T \rightarrow T$.
suppose there is an other element $t \in T$ and $t \neq 0$ and $t \neq 1$ and only three element : t, 0,1 in Boolean Algebras
$\because$ only three element in Boolean Algebras
$\therefore$ t' need to equal to t
if $t' = t$, then $t \vee t' = t$
according to the law of the Complementation: $t \vee t' = 1$
$\because$ $t \neq 1$
$\therefore$ the suppose does not hold.
$\therefore$ *there are no three element Boolean Algebras*.

5.

(a).

$\neg(p \rightarrow q)$	
$\equiv \neg(\neg p \vee q)$	Implication
$\equiv \neg\neg p \wedge \neg q$	De Morgan's, ¬ over ∨
$\equiv p \wedge \neg q$	Double negation

$(\neg p \rightarrow \neg q)$	
$\equiv \neg\neg p \vee \neg q$	Implication
$\equiv p \vee \neg q$	Double negation

$\because$ p $\wedge$ q not in (p $\wedge$ ¬q) but in (p ∨ ¬q)
$\therefore$ $\neg(p \rightarrow q) \equiv (\neg p \rightarrow \neg q)$ can not be prove

(b).

$((p \wedge q) \rightarrow r)$	
$\equiv \neg(p \wedge q) \vee r$	Implication
$\equiv (\neg p \vee \neg q) \vee r$	De Morgan's, ¬ over ∧
$\equiv r \vee (\neg p \vee \neg q)$	Commutatitivity of ∨
$\equiv r \vee (\neg q \vee \neg p)$	Commutatitivity of ∨
$\equiv (r \vee \neg q) \vee \neg p$	Associativity of ∨
$\equiv \neg p \vee (r \vee \neg q)$	Commutatitivity of ∨
$\equiv \neg p \vee (\neg q \vee r)$	Commutatitivity of ∨
$\equiv \neg p \vee (q \rightarrow r)$	Implication
$\equiv (p \rightarrow (q \rightarrow r))$	Implication

(c).

$((p \vee (q \vee r)) \wedge (r \vee p))$	
$\equiv ((p \vee q) \vee r) \wedge (r \vee p)$	Associativity of ∨
$\equiv (r \vee (p \vee q)) \wedge (r \vee p)$	Commutatitivity of ∨
$\equiv r \vee ((p \vee q) \wedge p)$	Distributivity of ∨ over ∧
$\equiv r \vee (p \wedge (p \vee q))$	Commutatitivity of ∧
$\equiv r \vee ((p \wedge p) \vee (p \wedge q))$	Distributivity of ∧ over ∨
$\equiv r \vee (p \vee (p \wedge q))$	Idempotence of ∧
$\equiv (r \vee p) \vee (p \wedge q)$	Associativity of ∨
$\equiv ((p \wedge q) \vee (r \vee p))$	Commutatitivity of ∨

6.

(a).

$\because$ binary tree data structure is either an empty tree or a node with two children that are tree

$\therefore$ it can derive that the tree with n node has x node in its left children and y node in its right children, with $x, y \epsilon [0, \text{n-1}]$ and $x + y = \text{n-1}$
$\therefore$ it can dirive the $y = \text{n-x-1}$

Sothe recurrence equation that:
Base: $T(0) = 1$

Recursive: $T(n) = \sum_{x\epsilon[0,n-1]} T(x) \times T(\text{n-x-1})$

(b).

According the Assignment 2, we can get: $count\_of\_leaf\_node$
$= 1 + count\_of\_fully\ internal\_node$
$\because$ a fully binary tree only contain the node of fully internal node the leaves node
$\therefore$ count_of_fully binary_node $= count\_of\_leaf\_for\ node$
$+count\_of\_fully\ internal\_node$
$= 1 + count\_of\_fully\ internal\_node$
$+ count\_of\_fully\ internal\_node$
$= 1 + 2 \times count\_of\_fully\ internal\_node$
$\therefore$ a full binary tree must have an odd number of nodes.

(c).

According the result from (c), we can get that $B(n) = 1 + 2T(n)$
$\because$ when $n' \leq n$: $T(n') \leq T(n)$
$\therefore B(n)\ involving\ T\ (n')\ where\ n' \leq n$

7.

(a).

$$p_1(n+1) = p_1(n) \times \frac{1}{2}$$

$$p_2(n+1) = p_2(n) \times \frac{1}{4} + p_1(n) \times \frac{1}{2}$$

$$p_3(n+1) = p_3(n) \times \frac{1}{2} + p_2(n) \times \frac{1}{4}$$

$$p_4(n+1) = p_4(n) \times \frac{1}{2} + p_2(n) \times \frac{1}{4}$$

$$p_5(n+1) = p_5(n) \times \tfrac{1}{1} + p_3(n) \times \tfrac{1}{2} + p_4(n) \times \tfrac{1}{2} + p_2(n) \times \tfrac{1}{4}$$

$$= p_5(n) + p_3(n) \times \tfrac{1}{2} + p_4(n) \times \tfrac{1}{2} + p_2(n) \times \tfrac{1}{4}$$

(b).

(i).

Base: $p_1(0) = 1$

Inductive: Assume $p_1(n) = \frac{1}{2^n}$

$$\text{the } p_1(n+1) = p_1(n) \times \frac{1}{2} = \frac{1}{2^n} \times \frac{1}{2} = \frac{1}{2^{n+1}}$$

$\therefore$ $p_1(n)$ implies $p_1(n+1)$

Therefore, by induction, $p_1(n) = \frac{1}{2^n}$ holds for all n.

(ii).

Base: $p_2(0) = 0$

Inductive: Assume $p_2(n) = 2(\frac{1}{2^n} - \frac{1}{4^n})$

$$\text{the } p_2(n+1) = p_2(n) \times \tfrac{1}{4} + p_1(n) \times \tfrac{1}{2}$$

$$= 2 \times (\tfrac{1}{2^n} - \tfrac{1}{4^n}) \times \tfrac{1}{4} + \tfrac{1}{2^n} \times \tfrac{1}{2}$$

$$= \frac{1}{2^{n+1}} - \frac{2}{4^n + 1} + \frac{1}{2^{n+1}}$$

$$= 2(\frac{1}{2^{n+1}} - \frac{1}{4^{n+1}})$$

$\therefore$ $p_2(n)$ implies $p_2(n+1)$

Therefore, by induction, $p_2(n) = 2(\frac{1}{2^n} - \frac{1}{4^n})$ holds for all n.

(iii).

Base: $p_3(0) = 0$

Inductive: Assume $p_3(n) = (n\text{-}2)\dfrac{1}{2^n} + \dfrac{2}{4^n}$

$$\text{the } p_3(n+1) = p_3(n) \times \tfrac{1}{2} + p_2(n) \times \tfrac{1}{4}$$

$$= ((n\text{-}2) \times \tfrac{1}{2^n} + \tfrac{2}{4^n}) \times \tfrac{1}{2} + 2(\tfrac{1}{2^n} \text{-} \tfrac{1}{4^n}) \times \tfrac{1}{4}$$

$$= (n\text{-}2) \times \frac{1}{2^{n+1}} + \frac{4}{4^{n+1}} + \frac{1}{2^{n+1}} \text{-} \frac{2}{4^{n+1}}$$

$$= (n\text{-}1) \times \frac{1}{2^{n+1}} + \frac{2}{4^{n+1}}$$

$\therefore$ $p_3(n)$ implies $p_3(n+1)$

Therefore, by induction, $p_3(n) = (n\text{-}2)\dfrac{1}{2^n} + \dfrac{2}{4^n}$ holds for all n.

Base: $p_4(0) = 0$

Inductive: Assume $p_4(n) = (n\text{-}2)\dfrac{1}{2^n} + \dfrac{2}{4^n}$

$$\text{the } p_4(n+1) = p_4(n) \times \tfrac{1}{2} + p_2(n) \times \tfrac{1}{4}$$

$$= ((n\text{-}2) \times \tfrac{1}{2^n} + \tfrac{2}{4^n}) \times \tfrac{1}{2} + 2(\tfrac{1}{2^n} \text{-} \tfrac{1}{4^n}) \times \tfrac{1}{4}$$

$$= (n\text{-}2) \times \frac{1}{2^{n+1}} + \frac{4}{4^{n+1}} + \frac{1}{2^{n+1}} \text{-} \frac{2}{4^{n+1}}$$

$$= (n\text{-}1) \times \frac{1}{2^{n+1}} + \frac{2}{4^{n+1}}$$

$\therefore$ $p_4(n)$ implies $p_4(n+1)$

Therefore, by induction, $p_4(n) = (n\text{-}2)\dfrac{1}{2^n} + \dfrac{2}{4^n}$ holds for all n.

(iv).

Base: $p_5(0) = 0$

Inductive: Assume $p_5(n) = 1\text{-}(2n\text{-}1)\dfrac{1}{2^n} \text{-} \dfrac{2}{4^n}$

$$\text{the } p_5(n+1) = p_5(n) + p_3(n) \times \tfrac{1}{2} + p_4(n) \times \tfrac{1}{2} + p_2(n) \times \tfrac{1}{4}$$

$$= 1\text{-}(2n\text{-}1) \times \tfrac{1}{2^n} \text{-} \tfrac{2}{4^n} + ((n\text{-}2) \times \tfrac{1}{2^n} + \tfrac{2}{4^n}) \times \tfrac{1}{2} +$$

$$((\text{n-2}) \times \frac{1}{2^n} + \frac{2}{4^n}) \times \frac{1}{2} + 2 \times (\frac{1}{2^n} - \frac{1}{4^n}) \times \frac{1}{4}$$

$$= 1\text{-}(2\text{n-}1) \times \frac{1}{2^n} - \frac{2}{4^n} + (\text{n-2}) \times \frac{1}{2^n} + \frac{2}{4^n} + \frac{1}{2} \times (\frac{1}{2^n} - \frac{1}{4^n})$$

$$= 1 + (1\text{-}2\text{n} + \text{n-}2 + \frac{1}{2}) \times \frac{1}{2^n} - \frac{1}{2} \times \frac{1}{4^n}$$

$$= 1\text{-}(2\text{n} + 1) \times \frac{1}{2^{n+1}} - \frac{2}{4^{n+1}}$$

$\therefore$ $p_5(n)$ implies $p_5(n+1)$

Therefore, by induction, $p_5(n) = 1\text{-}(2n\text{-}1)\frac{1}{2^n} - \frac{2}{4^n}$ holds for all n.

(c).

The expected value of X3 $= \sum_{i=\{0,1,2,3)} p_{length}(3) \times i$

$$= p_1(3) \times 0 + p_2(3) \times 1 + p_3(3) \times 2 + p_4(3) \times 2 + p_5(3) \times 3$$

$$= (\frac{1}{2^3} \times 0) + (2 \times (\frac{1}{2^3} - \frac{1}{4^3}) \times 1) + ((\frac{1}{2^3} + \frac{2}{4^3}) \times 2)$$

$$+((\frac{1}{2^3} + \frac{2}{4^3}) \times 2) + ((1\text{-}5 \times \frac{1}{2^3} - \frac{2}{4^3}) \times 3)$$

$$= 0 + \frac{14}{64} + \frac{20}{64} + \frac{20}{64} + \frac{66}{64}$$

$$= \frac{120}{64}$$

$$= \frac{15}{8}$$

8.

(a).

In n time, D may have been infected. And Dcan be infected by B in $(n + 1)$ time

$\therefore$ $P_D(n+1) = P_D(n) + \frac{1}{2}(P_B(n)\text{-}P_D(n))$

(b).

consider the time when D has not been infected in n time, it has two situation:
situation 1: only A is infected
situation 2: A and B are both infected

$\therefore$ in situation1 B is not infected, so $P_B(n) = \frac{1}{2^n}$

$\because$ B can be infected from A one time, so there are n different changes for A to infect B

$\therefore$ in situation2 B is not infected, so $P_B(n) = \frac{1}{2^n} \times n = \frac{n}{2^n}$

$\therefore P_D(n) = 1\text{-}\frac{1}{2^n}\text{-}\frac{n}{2^n} = 1\text{-}\frac{n+1}{2^n}$