pytrajectory Documentation

Release 0.3.3

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CHAPTER

ONE

PYTRAJECTORY USER'S GUIDE

1.1 About PyTrajectory

PyTrajectory is being developed at Dresden University of Technology at the Institute for Control Theory.

Based upon a study work of Oliver Schnabel under the supervision of Carsten Knoll in February 2013 it has been further developed by Andreas Kunze to increase its numeric performance.

If you face any problems using PyTrajectory, feel free to contact us.

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1.2 Getting Started

This section provides an overview on what PyTrajectory is and how to use it. For a more detailed view please have a look at the *PyTrajectory Modules Reference*.

Contents

1.2.1 What is PyTrajectory?

PyTrajectory is a Python library for the determination of the feed forward control to achieve a transition between desired states of a nonlinear control system.

Planning and designing of trajectories represents an important task in the control of technological processes. Here the problem is attributed on a multi-dimensional boundary value problem with free parameters. In general this problem can not be solved analytically. It is therefore resorted to the method of collocation in order to obtain a numerical approximation.

PyTrajectory allows a flexible implementation of various tasks and enables an easy implementation. It suffices to supply a function $f(x, u)$ that represents the vectorfield of a control system and to specify the desired boundary values.

1.2.2 Installation

PyTrajectory has been developed and tested on Python 2.7

Dependencies

Before you install PyTrajectory make sure you have the following dependencies installed on your system.

- numpy
- sympy
- scipy
- **optional**
  - matplotlib [visualisation]
  - ipython [debugging]

Source

To install PyTrajectory from the source files please download the latest release from here (insert link!). After the download is complete open the archive and change directory into the extracted folder. Then all you have to do is run the following command

```
$ python setup.py install
```

PyPI

The easiest way of installing PyTrajectory would be

```
$ pip install pytrajectory
```

or

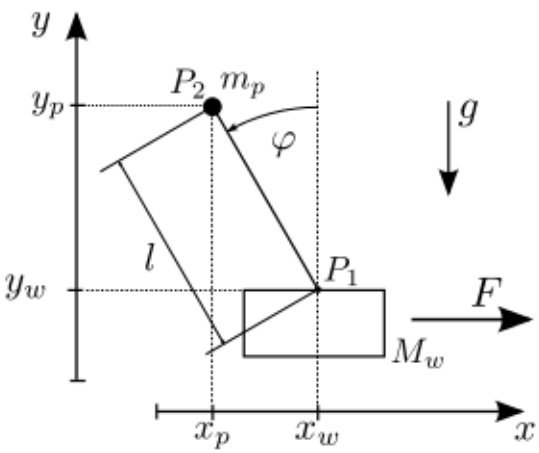
```
$ easy_install pytrajectory
```

provided that you have the Python modules *pip* or *setuptools* installed on your system.

1.2.3 Usage

In order to illustrate the usage of PyTrajectory we consider the following simple example.

A pendulum mass m_p is connected by a massless rod of length l to a cart M_w on which a force F acts to accelerate it.



A possible task would be the transfer between two angular positions of the pendulum. In this case, the pendulum should hang at first down ($\varphi = \pi$) and is to be turned upwards ($\varphi = 0$). At the end of the process, the car should be at the same position and both the pendulum and the cart should be at rest. The (partial linearised) system is represented by the following differential equations, where $[x_1, x_2, x_3, x_4] = [x_w, \dot{x_w}, \varphi, \dot{\varphi}]$ and $u = \ddot{x}_w$ is our control variable:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= u \\ \dot{x_3} &= x_4 \\ \dot{x_4} &= \frac{1}{l}(g\ sin(x_3) + u\ cos(x_3)) \end{aligned}$$

To solve this problem we first have to define a function that returns the vectorfield of the system above. Therefor it is important that you use SymPy functions if necessary, which is the case here with sin and cos.

So in Python this would be

```
>>> from sympy import sin, cos
>>>
>>> def f(x,u):
...     x1, x2, x3, x4 = x  # system variables
...     u1, = u             # input variable
...
...     l = 0.5     # length of the pendulum
...     g = 9.81    # gravitational acceleration
...
...     # this is the vectorfield
...     ff = [          x2,
...                     u1,
...                     x4,
...         (1/l)*(g*sin(x3)+u1*cos(x3))]
...
...     return ff
...
>>>
```

Wanted is now the course for $u(t)$, which transforms the system with the following start and end states within $T = 2[s]$.

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ \pi \\ 0 \end{bmatrix} \rightarrow x(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so we have to specify the boundary values at the beginning

```
>>> from numpy import pi
>>>
>>> a = 0.0
>>> xa = [0.0, 0.0, pi, 0.0]
```

and end

```
>>> b = 2.0
>>> xb = [0.0, 0.0, 0.0, 0.0]
```

The boundary values for the input variable are

```
>>> uab = [0.0, 0.0]
```

because we want $u(0) = u(T) = 0$.

Now we import all we need from PyTrajectory

```
>>> from pytrajectory import Trajectory
```

and pass our parameters.

```
>>> T = Trajectory(f, a, b, xa, xb, uab)
```

All we have to do now, to solve our problem is

```
>>> T.startIteration()
```

After the 7th iteration step with 320 spline parts, this would yield a solution which satisfies the default tolerance for the boundary values of 10^{-02}. But PyTrajectory enables you to improve its performance by altering some of its method parameters.

For example if we increase the factor for raising the spline parts (default: 2)

```
>>> T.setParam('kx', 5)
```

and don't take advantage of the system structure (integrator chains)

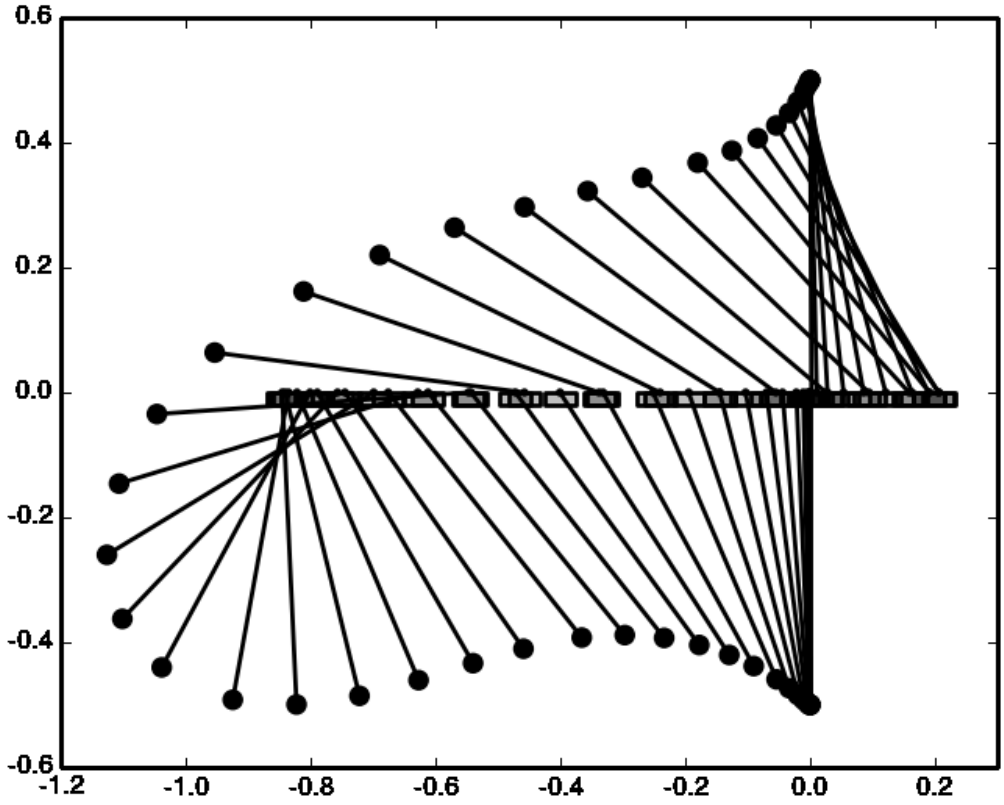
```
>>> T.setParam('use_chains', False)
```

we get a solution after 3 steps with 125 spline parts.

There are more method parameters you can change to speed things up, i.e. the type of collocation points to use or the number of spline parts for the input variables. To do so, just type:

```
>>> T.setParam('<param>', <value>)
```

Please have a look at the *PyTrajectory Modules Reference* for more information.



1.3 Background

This section is intended to give some insights into the mathematical background that is the basis of PyTrajectory.

Contents

1.3.1 Collocation Method

Given a system of autonomous differential equations

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), ..., x_n(t)) \\ &\vdots \quad \vdots \\ \dot{x}_n(t) &= f_n(x_1(t), ..., x_n(t)) \end{aligned}$$

with $t \in [a, b]$ and *Dirichlet* boundary conditions

$$x_i(a) = \alpha_i, \quad x_i(b) = \beta_i \qquad i = 1, ..., n$$

the collocation method to solve the problem basically works as follows.

We choose $N + 1$ collocation points $t_j,\ j = 0, ..., N$ from the interval $[a, b]$ where $t_0 = a,\ t_N = b$ and search for functions $P_i : [a, b] \to \mathbb{R}$ which for all $j = 0, .., N$ satisfy the following conditions:

$$P_i(t_0) = \alpha_i, \qquad P_i(t_N) = \beta_i \tag{1.1}$$

$$\frac{d}{dt}P_i(t_j) = f_i(P_1(t_j), ..., P_n(t_j)) \quad i = 1, ..., n \tag{1.2}$$

Through these demands the exact solution of the differential equation system will be approximated. The demands on the boundary values (1) can be sure already by suitable construction of the candidate functions. This results in the following system of equations.

$$\begin{aligned}
G_1^0(c) &:= \frac{d}{dt}P_1(t_0) - f(P_1(t_0)) = 0 \\
&\vdots \\
G_n^0(c) &:= \frac{d}{dt}P_n(t_0) - f(P_n(t_0)) = 0 \\
&\vdots \\
G_1^1(c) &:= \frac{d}{dt}P_1(t_1) - f(P_1(t_1)) = 0 \\
&\vdots \\
G_n^N(c) &:= \frac{d}{dt}P_n(t_N) - f(P_n(t_N)) = 0
\end{aligned}$$

Solving the boundary value problem is thus reduced to the finding of a zero point of $G = (G_1^0, ..., G_n^N)^T$, where c is the vector of all independent parameters that result from the candidate functions.

1.3.2 Candidate Functions

PyTrajectory uses cubic spline functions as candidates for the approximation of the solution. Splines are piecewise polynomials with a global differentiability. The connection points τ_k between the polynomial sections are equidistantly and are referred to as nodes.

$$t_0 = \tau_0 < \tau_1 < ... < \tau_\eta = t_N \qquad h = \frac{t_N - t_0}{\eta}$$
$$\tau_{k+1} = \tau_k + h \quad k = 0, ..., \eta - 1$$

The η polynomial sections can be created as follows.

$$S_k(t) = c_{k,0}(t - kh)^3 + c_{k,1}(t - kh)^2 + c_{k,2}(t - kh) + c_{k,3}$$
$$c_{k,l} \in \mathbb{R}, \qquad k = 1, ..., \eta,\ l = 0, ..., 3$$

Then, each spline function is defined by

$$P_i(t) = \begin{cases} S_1(t) & t_0 \le t < h \\ \vdots & \vdots \\ S_k(t) & (k-1)h \le t < kh \\ \vdots & \vdots \\ S_\eta(t) & (\eta - 1)h \le t \le \eta h \end{cases}$$

In addition to the steadiness the spline functions should be twice steadily differentiable in the nodes τ. Therefor, three smoothness conditions can be set up in all $\tau_k, k = 1, ..., \eta - 1$.

$$\begin{aligned} S_k(kh) &= S_{k+1}(kh) \\ \frac{d}{dt}S_k(kh) &= \frac{d}{dt}S_{k+1}(kh) \\ \frac{d^2}{dt^2}S_k(kh) &= \frac{d^2}{dt^2}S_{k+1}(kh) \end{aligned}$$

In the later equation system these demands result in the block diagonal part of the matrix. Furthermore, conditions can be set up at the edges arising from the boundary conditions of the differential equation system.

$$\frac{d^j}{dt^j}S_1(\tau_0) = \tilde{\alpha}_j \qquad \frac{d^j}{dt^j}S_\eta(\tau_\eta) = \tilde{\beta}_j \qquad j = 0, ..., \nu$$

The degree ν of the boundary conditions depends on the structure of the differential equation system. With these conditions and those above one obtains the following equation system ($\nu = 2$).

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & h^3 & -h^2 & h & -1 & & & & & & & & \\ 0 & 0 & 1 & 0 & -3h^2 & 2h & -1 & 0 & & & & 0 & & & & \\ 0 & 2 & 0 & 0 & 6h & -2 & 0 & 0 & & & & & & & & \\ & & & & 0 & 0 & 0 & 1 & h^3 & -h^2 & h & -1 & & & & \\ & 0 & & & 0 & 0 & 1 & 0 & -3h^2 & 2h & -1 & 0 & & 0 & & \\ & & & & 0 & 2 & 0 & 0 & 6h & -2 & 0 & 0 & & & & \\ & & & & & & & & & & & & \ddots & & & \\ & & & & & & & & 0 & 0 & 0 & 1 & h^3 & -h^2 & h & -1 \\ & & & & & 0 & & & 0 & 0 & 1 & 0 & -3h^2 & 2h & -1 & 0 \\ & & & & & & & & 0 & 2 & 0 & 0 & 6h & -2 & 0 & 0 \\ -h^3 & h^2 & -h & 1 & & & & & & & & & & & & \\ 3h^2 & -2h & 1 & 0 & & & & 0 & & & & & & & & \\ -6h & 2 & 0 & 0 & & & & & & & & & & & & \\ & & & & & & & & & & & & 0 & 0 & 0 & 1 \\ & & & & & 0 & & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & & & & & & & 0 & 2 & 0 & 0 \end{bmatrix}}_{=:\boldsymbol{M}} \cdot \underbrace{\begin{bmatrix} c_{1,0} \\ c_{1,1} \\ c_{1,2} \\ c_{1,3} \\ c_{2,0} \\ c_{2,1} \\ c_{2,2} \\ c_{2,3} \\ \vdots \\ \\ \vdots \\ \\ \vdots \\ \\ c_{\eta,0} \\ c_{\eta,1} \\ c_{\eta,2} \\ c_{\eta,3} \end{bmatrix}}_{=:\boldsymbol{c}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \\ \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \\ \tilde{\beta}_0 \\ \tilde{\beta}_1 \\ \tilde{\beta}_2 \end{bmatrix}}_{=:\boldsymbol{r}}$$

The matrix $\boldsymbol{M}$ of dimension $N_1 \times N_2,\ N_1 < N_2$, where $N_2 = 4\eta$ and $N_1 = 3(\eta - 1) + 2(\nu + 1)$, can be decomposed into two subsystems $\boldsymbol{A} \in \mathbb{R}^{N_1 \times (N_2 - N_1)}$ and $\boldsymbol{B} \in \mathbb{R}^{N_1 \times N_1}$. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the two matrices with the respective coefficients of $\boldsymbol{c}$.

$$\begin{aligned} \boldsymbol{M}\boldsymbol{c} &= \boldsymbol{r} \\ \boldsymbol{A}\boldsymbol{a} + \boldsymbol{B}\boldsymbol{b} &= \boldsymbol{r} \\ \boldsymbol{b} &= \boldsymbol{B}^{-1}(\boldsymbol{r} - \boldsymbol{A}\boldsymbol{a}) \end{aligned}$$

With this allocation, the system of equations can be solved for $\boldsymbol{b}$ and the parameters in $\boldsymbol{a}$ remain as the free parameters of the spline function.

Use of the system structure

In physical models often occur differential equations of the form

$$\dot{x}_i = x_{i+1}$$

For these equations, it is not necessary to determine a solution through collocation. Without severe impairment of the solution method, it is sufficient to define a candidate function for x_i and to win that of x_{i+1} by differentiation.

$$P_{i+1}(t) = \frac{d}{dt}P_i(t)$$

Then in addition to the boundary conditions of $P_i(t)$ applies

$$\frac{d}{dt}P_i(t_0 = a) = \alpha_{i+1} \qquad \frac{d}{dt}P_i(t_N = b) = \beta_{i+1}$$

Similar simplifications can be made if relations of the form $\dot{x}_i = u_j$ arise.

1.3.3 Levenberg-Marquardt Method

The Levenberg-Marquardt method can be used to solve nonlinear least squares problems. It is an extension of the Gauss-Newton method and solves the following minimization problem.

$$\|F'(x_k)(x_{k+1} - x_k) + F(x_k)\|_2^2 + \mu^2\|x_{k+1} - x_k\|_2^2 \to \text{min!}$$

The real number μ is a parameter that is used for the attenuation of the step size $(x_{k+1} - x_k)$ and is free to choose. Thus, the generation of excessive correction is prevented, as is often the case with the Gauss-Newton method and leads to a possible non-achievement of the local minimum. With a vanishing attenuation, $\mu = 0$ the Gauss-Newton method represents a special case of the Levenberg-Marquardt method. The iteration can be specified in the following form.

$$x_{k+1} = x_k - (F'(x_k)^T F'(x_k) + \mu^2 I)^{-1} F'(x_k) F(x_k)$$

The convergence can now be influenced by means of the parameter μ. Disadvantage is that in order to ensure the convergence, μ must be chosen large enough, at the same time, this also leads however to a very small correction. Thus, the Levenberg-Marquardt method has a lower order of convergence than the Gauss-Newton method but approaches the desired solution at each step.

Control of the parameter μ

The feature after which the parameter is chosen, is the change of the actual residual

$$\begin{aligned} R(x_k, s_k) &:= \|F(x_k)\|_2^2 - \|F(x_k + s_k)\|_2^2 \\ s_k &:= x_{k+1} - x_k \end{aligned}$$

and the change of the residual of the linearized approximation.

$$\tilde{R}(x_k, s_k) := \|F(x_k)\|_2^2 - \|F(x_k) + F'(x_k)x_k\|_2^2$$

As a control criterion, the following quotient is introduced.

$$\rho = \frac{R(x_k, s_k)}{\tilde{R}(x_k, s_k)}$$

It follows that $R(x_k, s_k) \geq 0$ and for a meaningful correction $\tilde{R}(x_k, s_k) \geq 0$ must also hold. Thus, ρ is also positive and $\rho \to 1$ for $\mu \to \infty$. Therefor ρ should lie between 0 and 1. To control μ two new limits b_0 and b_1 are introduced with $0 < b_0 < b_1 < 1$ and for $b_0 = 0.2, b_1 = 0.8$ we use the following criteria.

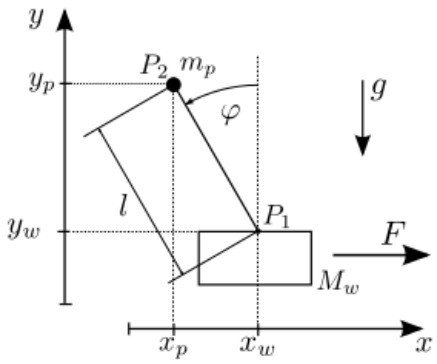
- $\rho \leq b_0$: μ is doubled and s_k is recalculated
- $b_0 < \rho < b_1$: in the next step μ is maintained and s_k is used
- $\rho \geq b_1$: s_k is accepted and μ is halved during the next iteration

1.4 Examples

The following example systems from mechanics demonstrate the application of PyTrajectory. The deriving of the model equations is omittted here.

1.4.1 Translation of the inverted pendulum

An example often used in literature is the inverted pendulum. Here a force F acts on a cart with mass M_w. In addition the cart is connected by a massless rod with a pendulum mass m_p. The mass of the pendulum is concentrated in P_2 and that of the cart in P_1. The state vector of the system can be specified using the carts position $x_w(t)$ and the pendulum deflection $\varphi(t)$ and their derivatives.



With the *Lagrangian Formalism* the model has the following state representation where $u_1 = F$ and $x = [x_1, x_2, x_3, x_4] = [x_w, \dot{x}_w, \varphi, \dot{\varphi}]$

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{m_p \sin(x_3)(-l x_4^2 + g\cos x_3)}{M_w l + m_p \sin^2(x_3)} + \frac{\cos(x_3)}{M_w l + m_p l \sin^2(x_3)} u_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{\sin(x_3)(-m_p l x_4^2 \cos(x_3) + g(M_w + m_p))}{M_w l + m_p \sin^2(x_3)} + \frac{\cos(x_3)}{M_w l + m_p l \sin^2(x_3)} u_1
\end{aligned}
$$

A possibly wanted trajectory is the translation of the cart along the x-axis (i.e. by $0.5m$). In the beginning and end of the process the cart and pendulum should remain at rest and the pendulum should be aligned vertically upwards ($\varphi = 0$). As a further condition u_1 should start and end steadily in the rest position ($u_1(0) = u_1(T) = 0$). The operating time here is $T = 1[s]$.

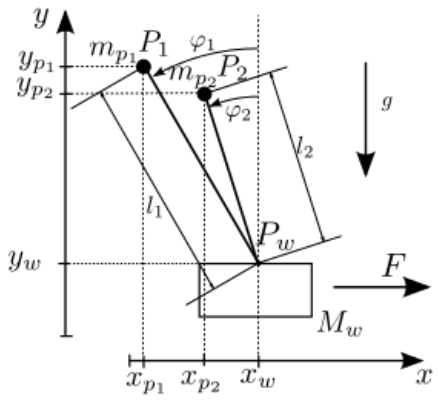
Source Code

```
# translation of the inverted pendulum

# import trajectory class and necessary dependencies
from pytrajectory import Trajectory
from sympy import sin, cos
import numpy as np

# define the function that returns the vectorfield
def f(x,u):
    x1, x2, x3, x4 = x        # system state variables
    u1, = u                   # input variable
```

```
    l = 0.5      # length of the pendulum rod
    g = 9.81     # gravitational acceleration
    M = 1.0      # mass of the cart
    m = 0.1      # mass of the pendulum

    s = sin(x3)
    c = cos(x3)

    ff = np.array([                          x2,
                    m*s*(-l*x4**2+g*c)/(M+m*s**2)+1/(M+m*s**2)*u1,
                                             x4,
              s*(-m*l*x4**2*c+g*(M+m))/(M*l+m*l*s**2)+c/(M*l+l*m*s**2)*u1
                  ])
    return ff

# boundary values at the start (a = 0.0 [s])
xa = [  0.0,
        0.0,
        0.0,
        0.0]

# boundary values at the end (b = 1.0 [s])
xb = [  0.5,
        0.0,
        0.0,
        0.0]

# create trajectory object
T = Trajectory(f, a=0.0, b=1.0, xa=xa, xb=xb)

# run iteration
T.startIteration()

# show results
T.plot()
```

1.4.2 Oscillation of the inverted double pendulum

In this example we add another pendulum to the cart in the system.



The system has the state vector $x = [x_1, \dot{x}_1, \varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2]$. A partial linearization with $y = x_1$ yields the following

system state representation where $\tilde{u} = \ddot{y}$.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \tilde{u} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{l_1}(g\sin(x_3) + \tilde{u}\cos(x_3)) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \frac{1}{l_2}(g\sin(x_5) + \tilde{u}\cos(x_5))
\end{aligned}$$

Here a trajectory should be planned that transfers the system between the following two positions of rest. At the beginning both pendulums should be directed downwards ($\varphi_1 = \varphi_2 = \pi$). After a operating time of $T = 2[s]$ the cart should be at the same position again and the pendulums should be at rest with $\varphi_1 = \varphi_2 = 0$.

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ \pi \\ 0 \\ \pi \\ 0 \end{bmatrix} \rightarrow x(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

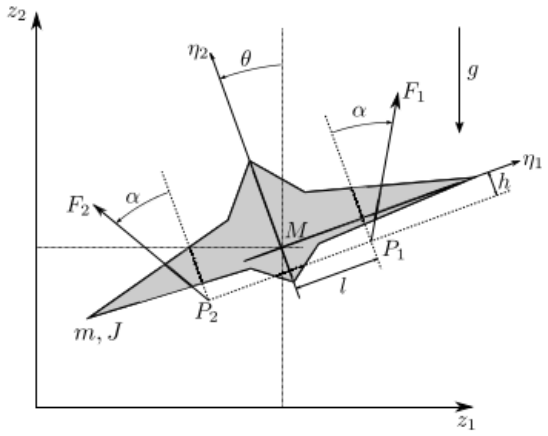
Source Code

```
# oscillation of the inverted double pendulum with partial linearization

# import trajectory class and necessary dependencies
from pytrajectory.trajectory import Trajectory
from sympy import cos, sin
import numpy as np
from numpy import pi

# define the function that returns the vectorfield
def f(x,u):
        x1, x2, x3, x4, x5, x6 = x  # system variables
        u, = u                      # input variable

    # length of the pendulums
        l1 = 0.7
        l2 = 0.5

        g = 9.81    # gravitational acceleration

        ff = np.array([         x2,
                                u,
                                x4,
                (1/l1)*(g*sin(x3)+u*cos(x3)),
                                x6,
                (1/l2)*(g*sin(x5)+u*cos(x5))
                    ])

        return ff

# system state boundary values for a = 0.0 [s] and b = 2.0 [s]
xa = [0.0, 0.0,  pi, 0.0,  pi, 0.0]
xb = [0.0, 0.0, 0.0, 0.0, 0.0, 0.0]
```

```
# boundary values for the input
uab= [0.0, 0.0]

# create trajectory object
T = Trajectory(f, a=0.0, b=2.0, xa=xa, xb=xb, g=uab)

# alter some method parameters to increase performance
T.setParam('su', 10)
T.setParam('eps', 8e-2)

# run iteration
T.startIteration()

# show results
T.plot()
```

1.4.3 Aircraft

In this section we consider the model of a unmanned vertical take-off aircraft. The aircraft has two permanently mounted thrusters on the wings which can apply the thrust forces F_1 and F_2 independently of each other. The two engines are inclined by an angle α with respect to the aircraft-fixed axis η_2 and engage in the points $P_1 = (l, h)$ and $P_2 = (-l, -h)$. The coordinates of the center of mass M of the aircraft in the inertial system are denoted by z_1 and z_2. At the same time, the point is the origin of the plane coordinate system. The aircraft axes are rotated by the angle θ with respect to the z_2-axis.



Through the establishment of the momentum balances for the model one obtains the equations

$$\begin{aligned} m\ddot{z}_1 &= -\sin(\theta)(F_1 + F_2)\cos(\alpha) + \cos(\theta)(F_1 - F_2)\sin(\alpha) \\ m\ddot{z}_2 &= \cos(\theta)(F_1 + F_2)\sin(\alpha) + \sin(\theta)(F_1 - F_2)\cos(\alpha) - mg \\ J\ddot{\theta} &= (F_1 - F_2)(l\cos(\alpha) + h\sin(\alpha)) \end{aligned}$$

With the state vector $x = [z_1, \dot{z}_1, z_2, \dot{z}_2, \theta, \dot{\theta}]^T$ and $u = [u_1, u_2]^T = [F_1, F_2]^T$ the state space representation of the

system is as follows.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{m}(-\sin(x_5)(u_1+u_2)\cos(\alpha)+\cos(x_5)(u_1-u_2)\sin(\alpha)) \\
\dot{x}_3 &= x_4 \\
\dot{x}_2 &= \frac{1}{m}(\cos(x_5)(u_1+u_2)\cos(\alpha)+\sin(x_5)(u_1-u_2)\sin(\alpha))-g \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \frac{1}{J}(l\cos(\alpha)+h\sin(\alpha))
\end{aligned}
$$

For the aircraft, a trajectory should be planned that translates the horizontally aligned flying object from a rest position (hovering) along the z_1 and z_2 axis back into a hovering position. The hovering is to be realized on the boundary conditions of the input. Therefor the derivatives of the state variables should satisfy the following conditions. $\dot{z}_1 = \ddot{z}_1 = \dot{z}_2 = \ddot{z}_2 = \dot{\theta} = \ddot{\theta} = 0$ For the horizontal position applies $\theta = 0$. These demands yield the boundary conditions for the inputs. $F_1(0) = F_1(T) = F_2(0) = F_2(T) = \frac{mg}{2\cos(\alpha)}$

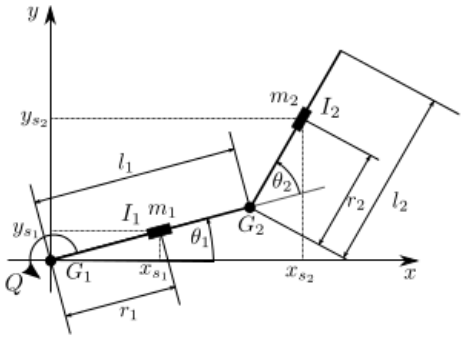
Source Code

```
# vertical take-off aircraft

# import trajectory class and necessary dependencies
from pytrajectory import Trajectory
from sympy import sin, cos
import numpy as np
from numpy import pi

# define the function that returns the vectorfield
def f(x,u):
    x1, x2, x3, x4, x5, x6 = x  # system state variables
    u1, u2 = u                  # input variables

    # coordinates for the points in which the engines engage [m]
    l = 1.0
    h = 0.1

    g = 9.81    # graviational acceleration [m/s^2]
    M = 50.0    # mass of the aircraft [kg]
    J = 25.0    # moment of inertia about M [kg*m^2]

    alpha = 5/360.0*2*pi    # deflection of the engines

    sa = sin(alpha)
    ca = cos(alpha)

    s = sin(x5)
    c = cos(x5)

    ff = np.array([             x2,
                   -s/M*(u1+u2) + c/M*(u1-u2)*sa,
                                x4,
                   -g+c/M*(u1+u2) +s/M*(u1-u2)*sa ,
                                x6,
                   1/J*(u1-u2)*(l*ca+h*sa)])

    return ff
```

```
# system state boundary values for a = 0.0 [s] and b = 3.0 [s]
xa = [ 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]
xb = [10.0, 0.0, 5.0, 0.0, 0.0, 0.0]

# boundary values for the inputs
g = [0.5*9.81*50.0/(cos(5/360.0*2*pi)),
     0.5*9.81*50.0/(cos(5/360.0*2*pi))]

# create trajectory object
T = Trajectory(f, a=0.0, b=3.0, xa=xa, xb=xb, g=g)

# don't take advantage of the system structure (integrator chains)
# (this will result in a faster solution here)
T.setParam('use_chains', False)

# also alter some other method parameters to increase performance
T.setParam('kx', 5)

# run iteration
T.startIteration()

# show results
T.plot()
```

1.4.4 Underactuated Manipulator

In this section, the model of an underactuated manipulator is treated. The system consists of two bars with the mass M_1 and M_2 which are connected to each other via the joint G_2. The angle between them is designated by θ_2. The joint G_1 connects the first rod with the inertial system, the angle to the x-axis is labeled θ_1. In the joint G_1 the actuating torque Q is applied. The bars have the moments of inertia I_1 and I_2. The distances between the centers of mass to the joints are r_1 and r_2.



The modeling was taken from the thesis of Carsten Knoll (April, 2009) where in addition the inertia parameter η was introduced.

$$\eta = \frac{m_2 l_1 r_2}{I_2 + m_2 r_2^2}$$

For the example shown here, strong inertia coupling was assumed with $\eta = 0.9$. By partial linearization to the output

$y = \theta_1$ one obtains the state representation with the states $x = [\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]^T$ and the new input $\tilde{u} = \ddot{\theta}_1$.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \tilde{u} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\eta x_2^2 \sin(x_3) - (1 + \eta \cos(x_3))\tilde{u}
\end{aligned}$$

For the system, a trajectory is to be determined for the transfer between two equilibrium positions within an operating time of $T = 1.8[s]$.

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 0.4\pi \\ 0 \end{bmatrix} \rightarrow x(T) = \begin{bmatrix} 0.2\pi \\ 0 \\ 0.2\pi \\ 0 \end{bmatrix}$$

The trajectory of the inputs should be without cracks in the transition to the equilibrium positions ($\tilde{u}(0) = \tilde{u}(T) = 0$).

Source Code

```
# underactuated manipulator

# import trajectory class and necessary dependencies
from pytrajectory.trajectory import Trajectory
import numpy as np
from sympy import cos, sin
from numpy import pi

# define the function that returns the vectorfield
def f(x,u):
    x1, x2, x3, x4  = x      # state variables
    u1, = u                  # input variable

    e = 0.9     # inertia coupling

    s = sin(x3)
    c = cos(x3)

    ff = np.array([          x2,
                             u1,
                             x4,
                    -e*x2**2*s-(1+e*c)*u1
                    ])

    return ff

# system state boundary values for a = 0.0 [s] and b = 1.8 [s]
xa = [  0.0,
        0.0,
        0.4*pi,
        0.0]

xb = [  0.2*pi,
        0.0,
        0.2*pi,
        0.0]

# boundary values for the inputs
```

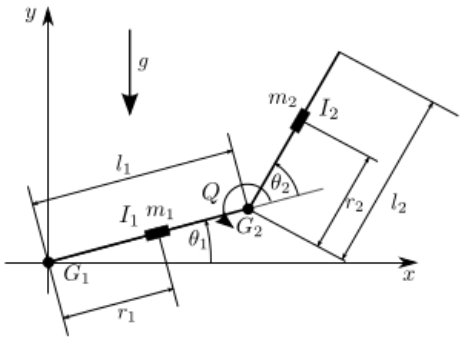
```
uab = [0.0, 0.0]

# create trajectory object
T = Trajectory(f, a=0.0, b=1.8, xa=xa, xb=xb, g=uab)

# also alter some method parameters to increase performance
T.setParam('su', 20)
T.setParam('kx', 3)

# run iteration
T.startIteration()

# show results
T.plot()
```

1.4.5 Acrobot

One further interesting example is that of the acrobot. The model can be regarded as a simplified gymnast hanging on a horizontal bar with both hands. The movements of the entire system is to be controlled only by movement of the hip. The body of the gymnast is represented by two rods which are jointed in the joint G_2. The first rod is movably connected at joint G_1 with the inertial system, which corresponds to the encompassing of the stretching rod with the hands.

For the model, two equal-length rods with a length $l_1 = l_2 = l$ are assumed with a homogeneous distribution of mass $m_1 = m_2 = m$ over the entire rod length. This does not correspond to the proportions of a man, also no restrictions were placed on the mobility of the hip joint.

The following figure shows the schematic representation of the model.



Using the previously assumed model parameters and the write abbreviations

$$
\begin{aligned}
I &= \frac{1}{3}ml^2 \\
d_{11} &= \frac{ml^2}{4} + m(l^2 + \frac{l^2}{4} + l^2\cos(\theta_2)) + 2I \\
h_1 &= -\frac{ml^2}{2}\sin(\theta_2)(\dot{\theta}_2(\dot{\theta}_2 + 2\dot{\theta}_1)) \\
d_{12} &= m(\frac{l^2}{4} + \frac{l^2}{2}\cos(\theta_1)) + I \\
\varphi_1 &= \frac{3}{2}mlg\cos(\theta_1) + \frac{1}{2}mlg\cos(\theta_1 + \theta_2)
\end{aligned}
$$

as well as the state vector $x = [\theta_2, \dot{\theta}_2, \theta_1, \dot{\theta}_1]$ one obtains the following state representation with the virtual input

$u = \ddot{\theta}_2$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -d_{11}^{-1}(h_1 + \varphi_1 + d_{12}u)
\end{aligned}$$

Now, the trajectory of the manipulated variable for an oscillation of the gymnast should be determined. The starting point of the exercise are the two downward hanging rods. These are to be transferred into another rest position in which the two bars show vertically upward within an operating time of $T = 2[s]$. At the beginning and end of the process, the input variable is to merge continuously into the rest position $u(0) = u(T) = 0$.

The initial and final states thus are

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{2}\pi \\ 0 \end{bmatrix} \rightarrow x(T) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}\pi \\ 0 \end{bmatrix}$$

Source Code

```
# acrobot

# import trajectory class and necessary dependencies
from pytrajectory.trajectory import Trajectory
import numpy as np
from sympy import cos, sin
from numpy import pi

# define the function that returns the vectorfield
def f(x,u):
    x1, x2, x3, x4 = x
    u1, = u

    m = 1.0             # masses of the rods [m1 = m2 = m]
    l = 0.5             # lengths of the rods [l1 = l2 = l]

    I = 1/3.0*m*l**2    # moments of inertia [I1 = I2 = I]
    g = 9.81            # gravitational acceleration

    lc = l/2.0

    d11 = m*lc**2+m*(l**2+lc**2+2*l*lc*cos(x1))+2*I
    h1 = -m*l*lc*sin(x1)*(x2*(x2+2*x4))
    d12 = m*(lc**2+l*lc*cos(x1))+I
    phi1 = (m*lc+m*l)*g*cos(x3)+m*lc*g*cos(x1+x3)

    ff = np.array([            x2,
                       u1,
                       x4,
                -1/d11*(h1+phi1+d12*u1)
                ])

    return ff


# system state boundary values for a = 0.0 [s] and b = 2.0 [s]
```

```
xa = [  0.0,
        0.0,
        3/2.0*pi,
        0.0]

xb = [  0.0,
        0.0,
        1/2.0*pi,
        0.0]

# boundary values for the inputs
bvin = [0.0, 0.0]

# create trajectory object
T = Trajectory(f, a=0.0, b=2.0, xa=xa, xb=xb, g=bvin)

# alter some method parameters to increase performance
T.setParam('su', 10)

# run iteration
T.startIteration()

# show results
T.plot()
```

CHAPTER

TWO

PYTRAJECTORY MODULES REFERENCE

PyTrajectory is a Python library for the determination of the feed forward control to achieve a transition between desired states of a nonlinear control system.

Contents

2.1 `trajectory` Module

class `pytrajectory.trajectory.`**`Trajectory`**(*ff*, *a=0.0*, *b=1.0*, *xa=None*, *xb=None*, *g=None*, ***kwargs*)

Base class of the PyTrajectory project.

Trajectory manages everything from analysing the given system over initialising the spline functions, setting up and solving the collocation equation system up to the simulation of the resulting initial value problem.

After the iteration has finished, it provides access to callable functions for the system and input variables as well as some capabilities for visualising the systems dynamic.

Parameters

- **ff** (*callable*) – Vectorfield (rhs) of the control system
- **a** (*float*) – Left border
- **b** (*float*) – Right border
- **xa** (*list*) – Boundary values at the left border
- **xb** (*list*) – Boundary values at the right border
- **g** (*list*) – Boundary values of the input variables

`mparam` dict

Dictionary with method parameters

key	default value	meaning
sx	5	Initial number of spline parts for the system variables
su	5	Initial number of spline parts for the input variables
kx	2	Factor for raising the number of spline parts
delta	2	Constant for calculation of collocation points
maxIt	7	Maximum number of iteration steps
eps	1e-2	Tolerance for the solution of the initial value problem
tol	1e-5	Tolerance for the solver of the equation system
algo	'leven'	The solver algorithm to use
use_chains	True	Whether or not to use integrator chains
colltype	'equidistant'	The type of the collocation points
use_sparse	True	Whether or not to use sparse matrices

startIteration()
This is the main loop.

At first the equations that have to be solved by collocation will be determined according to the integrator chains.

Next, one step of the iteration is done by calling `iterate()`.

After that, the accuracy of the found solution is checked. If it is within the tolerance range the iteration will stop. Else, the number of spline parts is raised and another step starts.

analyseSystem()
Analyses the systems structure and sets values for some of the method parameters.

By now, this method determines the number of state and input variables and searches for integrator chains.

setParam(*param=''*, *val=None*)
Method to assign value `val` to method parameter `param`.

Parameters

- **param** (*str*) – Parameter of which to alter the value
- **val** – New value for the passed parameter

iterate()
This method is used to run one iteration step.

First, new splines are initialised for the variables that are the upper end of an integrator chain.

Then, a start value for the solver is determined and the equation system is build.

Next, the equation system is solved and the resulting numerical values for the free parameters are written back.

As a last, the initial value problem is simulated.

getGuess()
This method is used to determine a starting value (guess) for the solver of the collocation equation system.

If it is the first iteration step, then a vector with the same length as the vector of the free parameters with arbitrarily values is returned.

Else, for every variable a spline has been created for, the old spline of the iteration before and the new spline are evaluated at specific points and a equation system is solved which ensures that they are equal in these points.

The solution of this system is the new start value for the solver.

initSplines()
This method is used to initialise the provisionally splines.

buildEQS()
Builds the collocation equation system.

solveEQS()
This method is used to solve the collocation equation system.

G(*c*)
Returns the collocation system evaluated with numeric values for the independent parameters.

DG(*c*)
Returns the jacobian matrix of the collocation system w.r.t. the independent parameters evaluated at `c`.

setCoeff()
Set found numerical values for the independent parameters of each spline.

This method is used to get the actual splines by using the numerical solutions to set up the coefficients of the polynomial spline parts of every created spline.

simulateIVP()
This method is used to solve the initial value problem.

checkAccuracy()
Checks whether the desired accuracy for the boundary values was reached.

It calculates the difference between the solution of the simulation and the given boundary values at the right border and compares its maximum against the tolerance set by self.eps

x(*t*)
Returns the system state at a given (time-) point `t`.

u(*t*)
Returns the state of the inputs at a given (time-) point `t`.

dx(*t*)
Returns the state of the 1st derivatives of the system variables at `t`.

plot()
Plot the calculated trajectories and error functions.

This method calculates the error functions and then calls the `utilities.plot()` function.

save(*fname=None*)
Save system data, callable solution functions and simulation results.

clear()
This method is intended to delete some attributes of the object that are no longer neccessary after the iteration has finished.

TODO: implement this ;-P

2.2 `spline` Module

`pytrajectory.spline.`**fdiff**(*func*)
This function is used to get the derivative of of a callable splinefunction.

class `pytrajectory.spline.`**CubicSpline**(*a=0.0*, *b=1.0*, *n=10*, *tag=''*, *bc=None*, *bcd=None*, *bcdd=None*, *steady=True*)
This class represents a cubic spline.

It simultaneously provides access to the spline function itself as well as to its derivatives up to the 3rd order. Furthermore it has its own method to ensure the steadiness and smoothness conditions of its polynomial parts in the joining points.

For more information see: *Candidate Functions*

Parameters

- **a** (*float*) – Left border of the spline interval.
- **b** (*float*) – Right border of the spline interval.
- **n** (*int*) – Number of polynomial parts the spline will be devided into.
- **tag** (*str*) – The 'name' of the spline object.
- **bc** (*tuple*) – Boundary values for the spline function itself.
- **bcd** (*tuple*) – Boundary values for the splines 1st derivative
- **bcdd** (*tuple*) – Boundary values for the splines 2nd derivative
- **steady** (*bool*) – Whether or not to call `makesteady()` when instanciated.

prov_evalf (*x*, *d*)

This function yields a provisionally evaluation of the spline while there are no numerical values for its free parameters. It returns a two vectors which reflect the dependence of the splines coefficients on its free parameters (independent coefficients).

Parameters

- **x** (*real*) – The point to evaluate the spline at
- **d** (*int*) – The derivation order

evalf (*x*, *d*)

Returns the value of the splines `d`-th derivative at `x`.

Parameters

- **x** (*float*) – The point to evaluate the spline at
- **d** (*int*) – The derivation order

f (*x*)

This is just a wrapper to evaluate the spline itself.

df (*x*)

This is just a wrapper to evaluate the splines 1st derivative.

ddf (*x*)

This is just a wrapper to evaluate the splines 2nd derivative.

dddf (*x*)

This is just a wrapper to evaluate the splines 3rd derivative.

makesteady ()

This method sets up and solves equations that satisfy boundary conditions and ensure steadiness and smoothness conditions of the spline in every joining point.

Please see the documentation for more details: *Candidate Functions*

set_coeffs (*c_sol*)

This function is used to set up numerical values for the spline coefficients.

It replaces the symbolic coefficients of the polynomial parts with the calculated values.

Parameters **c_sol** (*numpy.ndarray*) – Array with numerical values for the free spline coefficients

2.3 `solver` Module

class `pytrajectory.solver.`**`Solver`**(*F*, *DF*, *x0*, *tol=0.01*, *maxx=10*, *algo='leven'*)

This class provides solver for the collocation equation system.

Parameters

- **F** (*callable*) – The callable function that represents the equation system
- **DF** (*callable*) – The function for the jacobian matrix of the eqs
- **x0** (*numpy.ndarray*) – The start value for the sover
- **tol** (*float*) – The (absolute) tolerance of the solver
- **maxx** (*int*) – The maximum number of iterations of the solver
- **algo** (*str*) – The solver to use

`solve`()

This is just a wrapper to call the chosen algorithm for solving the collocation equation system.

`leven`()

This method is an implementation of the Levenberg-Marquardt-Method to solve nonlinear least squares problems.

For more information see: *Levenberg-Marquardt Method*

`gauss`()

`newton`()

2.4 `simulation` Module

class `pytrajectory.simulation.`**`Simulation`**(*ff*, *T*, *start*, *u*, *dt=0.01*)

This class simulates the initial value problem that results from solving the boundary value problem of the control system.

Parameters

- **ff** (*callable*) – Vectorfield of the control system.
- **T** (*float*) – Simulation time.
- **u** (*callable*) – Function of the input variables.
- **dt** (*float*) – Time step.

`rhs`(*t*, *x*)

Retruns the right hand side (vectorfield) of the ode system.

`calcStep`()

Calculates one step of the simulation.

`simulate`()

Starts the simulation

Returns

- *List of numpy arrays with time steps and simulation data of system and*

- *input variables.*

2.5 `utilities` Module

class `pytrajectory.utilities.`**`IntegChain`**(*lst*)
This class provides a representation of a integrator chain consisting of sympy symbols.

For the elements $(x_i)_{i=1,\dots,n}$ the relation $\dot{x}_i = x_{i+1}$ applies:

Parameters **lst** (*list*) – Ordered list of elements for the integrator chain

`elements` **tuple**
Ordered list of all elements that are part of the integrator chain

`upper` **sympy.Symbol**
Upper end of the integrator chain

`lower` **sympy.Symbol**
Lower end of the integrator chain

class `pytrajectory.utilities.`**`Animation`**(*drawfnc*, *simdata*, *plotsys*=[ ], *plotinputs*=[ ])
Provides animation capabilities.

Given a callable function that draws an image of the system state and smiulation data this class provides a method to created an animated representation of the system.

Parameters

- **drawfnc** (*callable*) – Function that returns an image of the current system state according to `simdata`
- **simdata** (*numpy.ndarray*) – Array that contains simulation data (time, system states, input states)
- **plotsys** (*list*) – List of tuples with indices and labels of system variables that will be plotted along the picture
- **plotinputs** (*list*) – List of tuples with indices and labels of input variables that will be plotted along the picture

class **`Image`**
This is just a container for the drawn system.

`reset`()

`Animation.`**`get_axes`**()

`Animation.`**`set_limits`**(*ax='ax_img'*, *xlim=(0, 1)*, *ylim=(0, 1)*)

`Animation.`**`set_label`**(*ax='ax_img'*, *label=''*)

`Animation.`**`animate`**()
Starts the animation of the system.

`Animation.`**`save`**(*fname*, *fps=None*, *dpi=200*)
Saves the animation as a video file or animated gif.

`pytrajectory.utilities.`**`blockdiag`**(*M*, *bshape=None*, *sparse=False*)
Takes blocks of shape `bshape` from matrix `M` and creates block diagonal matrix out of them.

Parameters

- **M** (*numpy.ndarray*) – Matrix to take blocks from

- **bshape** (*tuple*) – Shape of one block
- **sparse** (*bool*) – Whether or not to return created matrix as sparse matrix

Examples

```
>>> A = np.ones((4, 2))
>>> print A
[[ 1.  1.]
 [ 1.  1.]
 [ 1.  1.]
 [ 1.  1.]]
>>> B = blockdiag(A, (2, 2))
>>> print B
[[ 1.  1.  0.  0.]
 [ 1.  1.  0.  0.]
 [ 0.  0.  1.  1.]
 [ 0.  0.  1.  1.]]
```

pytrajectory.utilities.**plot**(*sim*, *H*, *fname=None*)

This method provides graphics for each system variable, manipulated variable and error function and plots the solution of the simulation.

Parameters

- **sim** (*tuple*) – Contains collocation points, and simulation results of system and input variables
- **H** (*dict*) – Dictionary of the callable error functions
- **fname** (*str*) – If not None, plot will be saved as <fname>.png

2.6 log Module

class pytrajectory.log.**Logger**(*fname=None*, *mode='w'*, *suppress=False*, *verbosity=0*)

This class the output of the log data.

It can simultaneously write to a specified file and the standard output as well as just the file. In addition a loglevel can be set to suppress some information.

Parameters

- **fname** (*str*) – The name of the log file to which all information will be written
- **mode** (*str*) – Either 'w' if an existing file should be overwritten or 'a' if it should be appended
- **suppress** (*bool*) – Whether or not to suppress output to the screen
- **verbosity** (*int*) – The level of verbosity that restricts the output

write(*text*, *verblvl=0*)

Writes log information if verblvl is less or equal to the level of verbosity.

Parameters

- **text** (*str*) – The information to log.
- **verblvl** (*int*) – The 'inportance' of the information.

class pytrajectory.log.**Timer**(*label='~'*, *verbose=True*)
Provides a context manager that takes the time of a code block.

Parameters

- **label** (*str*) – The 'name' of the code block which is timed
- **verbose** (*bool*) – Whether or not to output the elapsed time at exit

pytrajectory.log.**set_file**(*fname=None*, *suppress=True*)
Sets a file to which all log information are written.

pytrajectory.log.**msg**(*label*, *text*, *lvl=0*)

pytrajectory.log.**info**(*text*, *lvl=0*)

pytrajectory.log.**logtime**(*text*, *lvl=0*)

pytrajectory.log.**warn**(*text*, *lvl=0*)

pytrajectory.log.**error**(*text*, *lvl=0*)

CHAPTER

THREE

INDICES AND TABLES

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