APPENDIX

We provide the synchronization conditions of Algorithm 1 and show that the synchronization will be established eventually. We need to mention that we follow similar calculation steps to the ones presented in the studies [20], [28]. However, since we use a different clock drift model in this article, i.e. Brownian motion rather than uniform distribution, the following mathematical analysis is moderately different and more realistic as compared to those in the previous studies.

Since f_w and f_r are time varying, by following [51], [52], we model the *evolution* of f_w^r as integrated white noise, i.e. *Brownian motion*, given as

$$f_w^r(t_0+\delta) \triangleq f_w^r(t_0) + \int_{t_0}^{t_0+\delta} \eta(t)dt. \tag{8}$$

Here, δ denotes the amount of time after t_0 and $\eta(t)$ denotes the instantaneous change on the relative clock frequency, namely *relative drift rate* at time t. We assume that $\eta(t) \sim \mathcal{N}(0, \sigma_\eta^2)$ and for all time instants t' and t'', $\eta(t')$ and $\eta(t'')$ are uncorrelated random variables. In order to ease the calculations in the following sections, we will consider the *average relative clock frequency* in an interval of interest $[t_0, t_0+\delta]$ rather than instantaneous relative frequency in (8). We denote this value by $\bar{f}_r^w(t_0, t_0+\delta)$ and define it as

$$\bar{f}_w^r(t_0, t_0+\delta) \triangleq \frac{1}{\delta}\int_{t_0}^{t_0+\delta} f_w^r(t)dt \triangleq f_w^r(t_0) + \frac{1}{\delta}\int_0^{\delta}\int_{t_0}^{t_0+t} \eta(u)dudt. \tag{9}$$

A. Proof of Convergence

Consider (5) and without loss of generality we assume that $C_w(t_1) - C_w(t_0) = \tau(1 + \bar{f}_w^r(t_0, t_1)) + \varepsilon$ where τ denotes the event period and ε accounts for the uncertainty of the event period (as indicated in Section VI-0a). For ease of calculations we assume that $\varepsilon \sim \mathcal{N}(0, \sigma_\tau^2)$. Define *relative frequency estimation error* as

$$\tilde{f}_w^r(t_0) \triangleq f_w^r(t_0) - \hat{f}_w^r(t_0). \tag{10}$$

Using (5), (9) and (10) we get:

$$\gamma(t_1) = \tau(\bar{f}_w^r(t_0,t_1) - \hat{f}_w^r(t_0)) + \varepsilon = \tau\tilde{f}_w^r(t_0) + \frac{\tau}{t_1 - t_0}\int_0^{t_1-t_0}\int_{t_0}^{t_0+t}\eta(u)dudt + \varepsilon. \quad (11)$$

For ease of representation, let us introduce the following notation:

$$\Phi(h) \triangleq \int_{t_h}^{t_{h+1}} \eta(t)dt, \quad (12)$$

$$\Omega(h) \triangleq \frac{\tau}{t_{h+1} - t_h}\int_0^{t_{h+1}-t_h}\int_{t_h}^{t_h+t}\eta(u)dudt, \quad (13)$$

$$\Delta(h) \triangleq \tilde{f}_w^r(t_h). \quad (14)$$

Using this notation, we denote $\gamma(t_{h+1})$ by $\Theta(h+1)$ as

$$\Theta(h+1) \triangleq \gamma(t_{h+1}) = \tau\Delta(h) + \Omega(h) + \varepsilon. \quad (15)$$

Moreover, since $\Delta(h)$, $\Omega(h)$ and ε are independent random variables, we can write $\Delta(h+1)$ as

$$\begin{aligned}\Delta(h+1) &\overset{(14)}{=} \Delta(h) + \tilde{f}_w^r(t_{h+1}) - \tilde{f}_w^r(t_h)\\ &\overset{(10)}{=} \Delta(h) + f_w^r(t_{h+1}) - f_w^r(t_h) - \hat{f}_w^r(t_{h+1}) + \hat{f}_w^r(t_h)\\ &\overset{(6),(8)}{=} \Delta(h) + \int_{t_h}^{t_{h+1}}\eta(t)dt - \beta\Theta(h+1)\\ &\overset{(15)}{=} (1-\beta\tau)\Delta(h) - \beta(\Omega(h)+\varepsilon) + \Phi(h). \quad (16)\end{aligned}$$

Considering (15) and (16), we can write the system evolution as

$$\underbrace{\begin{bmatrix}\Theta(h+1)\\ \Delta(h+1)\end{bmatrix}}_{X(h+1)} = \underbrace{\begin{bmatrix}0 & \tau\\ 0 & 1-\beta\tau\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\Theta(h)\\ \Delta(h)\end{bmatrix}}_{X(h)} + \underbrace{\begin{bmatrix}\Omega(h)+\varepsilon\\ \Phi(h) - \beta(\Omega(h)+\varepsilon)\end{bmatrix}}_{Y(h)}. \quad (17)$$

Taking the expectation of both sides of (17) yields $E\left[\!\left[X(h+1)\right]\!\right] = AE\left[\!\left[X(h)\right]\!\right]$ since the means of the random variables in the matrix $Y(h)$ are all zero. According to stability of linear systems

[53], *asymptotic convergence* will be reached if and only if the magnitudes of the eigenvalues of the matrix A are strictly smaller than one. The eigenvalues λ_1 and λ_2 of the matrix A can be obtained by solving

$$\begin{vmatrix} -\lambda & \tau \\ 0 & 1-\beta\tau-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad \begin{array}{c} \lambda_1 = 0, \\ \lambda_2 = 1-\beta\tau. \end{array} \tag{18}$$

Therefore, the synchronization will be established if and only if $|\lambda_1, \lambda_2| < 1$, and in turn

$$0 < \beta < \frac{2}{\tau} \tag{19}$$

should hold. The rate of convergence is governed by the largest eigenvalue λ_2, which means that bigger values of β will lead to faster convergence. Consequently, selecting β within the bound above will lead the system converge to the state

$$\lim_{h\to\infty} E\left[\!\left[\Theta(h+1)\right]\!\right] = E\left[\!\left[\Theta(h)\right]\!\right], \tag{20}$$

$$\lim_{h\to\infty} E\left[\!\left[\Delta(h+1)\right]\!\right] = E\left[\!\left[\Delta(h)\right]\!\right]. \tag{21}$$

Hence, due to asymptotic convergence

$$E\left[\!\left[\Delta(\infty)\right]\!\right] \overset{(16)}{=} (1-\beta\tau)\, E\left[\!\left[\Delta(\infty)\right]\!\right] \quad \Longrightarrow \quad E\left[\!\left[\Delta(\infty)\right]\!\right] = 0, \tag{22}$$

$$E\left[\!\left[\Theta(\infty)\right]\!\right] \overset{(15)}{=} \tau E\left[\!\left[\Delta(\infty)\right]\!\right] \quad \Longrightarrow \quad E\left[\!\left[\Theta(\infty)\right]\!\right] = 0. \tag{23}$$

Consequently, the prediction error $\Theta(h) = \gamma(t_h)$ eventually converges to zero as h goes to infinity. This means the WISP tag will estimate the occurrence times of the successive events with zero error in expectation.

B. Steady-State (Asymptotic) Synchronization Error

An analytical expression for the *asyptotic synchronization error* can be obtained by calculating $\lim_{h\to\infty} \mathrm{Var}(\Theta(h))$. This calculation reduces to find $\mathrm{Var}(\Theta(\infty)) = E\left[\!\left[\Theta(\infty)^2\right]\!\right]$, since it holds from (23) that $E\left[\!\left[\Theta(\infty)\right]\!\right] = 0$. Using (15), we get

$$E\left[\!\left[\Theta(\infty)^2\right]\!\right] \overset{(15)}{=} \tau^2 E\left[\!\left[\Delta(\infty)^2\right]\!\right] + E\left[\!\left[(\Omega(h)+\varepsilon)^2\right]\!\right], \tag{24}$$

since $E\left[\!\left[\Delta(\infty)\right]\!\right] = 0$ from (22) and $\Delta(h)$, $\Omega(h)$ and ε are independent. In order to calculate (24), let us derive first $E\left[\!\left[\Delta(\infty)^2\right]\!\right]$ as

$$E\left[\!\left[\Delta(\infty)^2\right]\!\right] \overset{(16),(21)}{=} (1-\beta\tau)^2 E\left[\!\left[\Delta(\infty)^2\right]\!\right] + E\left[\!\left[(\Phi(h)-\beta(\Omega(h)+\varepsilon)^2\right]\!\right]. \tag{25}$$

In order to calculate (25), the first step is to obtain the following expectation:

$$E\left[\!\left[(\Phi(h)-\beta(\Omega(h)+\varepsilon)^2\right]\!\right] = E\left[\!\left[(\Phi(h)^2\right]\!\right] - 2\beta E\left[\!\left[\Phi(h)\Omega(h)\right]\!\right] + \beta^2 E\left[\!\left[\Omega(h)^2\right]\!\right] + \beta^2 E\left[\!\left[\varepsilon^2\right]\!\right]. \tag{26}$$

The equality (26) holds due to the fact that $E\left[\!\left[\Phi(h)\varepsilon\right]\!\right] = E\left[\!\left[\Phi(h)\right]\!\right] E\left[\!\left[\varepsilon\right]\!\right] = 0$ and $E\left[\!\left[\varepsilon\Omega(h)\right]\!\right] = E\left[\!\left[\varepsilon\right]\!\right] E\left[\!\left[\Omega(h)\right]\!\right] = 0$ (since $E\left[\!\left[\varepsilon\right]\!\right] = E\left[\!\left[\Phi(h)\right]\!\right] = E\left[\!\left[\Omega(h)\right]\!\right] = 0$ and ε is independent from $\Phi(h)$ and $\Omega(h)$). The following *stochastic integrals* are required for (26):

$$\begin{aligned}
E\left[\!\left[\Phi(h)^2\right]\!\right] &= \int_0^{t_{h+1}-t_h} \sigma_\eta^2 dt = \sigma_\eta^2 (t_{h+1}-t_h), && (27)\\
E\left[\!\left[\Omega(h)^2\right]\!\right] &= \frac{\tau^2}{(t_{h+1}-t_h)^2}\sigma_\eta^2 \int_0^{t_{h+1}-t_h} t^2 dt = \sigma_\eta^2 \frac{\tau^2(t_{h+1}-t_h)}{3}, && (28)\\
E\left[\!\left[\Phi(h)\Omega(h)\right]\!\right] &= \frac{\tau}{t_{h+1}-t_h}\sigma_\eta^2 \int_0^{t_{h+1}-t_h} t dt = \sigma_\eta^2 \frac{\tau(t_{h+1}-t_h)}{2}, && (29)\\
E\left[\!\left[\varepsilon^2\right]\!\right] &= \sigma_\tau^2. && (30)
\end{aligned}$$

The derivation of the expectations and stochastic integrals above relies on the assumptions we mentioned at the beginning of this Appendix: (i) $\eta(t')$ and $\eta(t'')$ are uncorrelated for all $t' \neq t''$, therefore $E\left[\!\left[\eta(t')\eta(t'')\right]\!\right] = 0$; (ii) since $E\left[\!\left[\eta(t')\right]\!\right] = 0$, therefore $E\left[\!\left[(\eta(t'))^2\right]\!\right] = \mathrm{Var}(\eta(t')) = \sigma_\eta^2$.

Putting these expected values into (26), then in turn into (25) lead:

$$E\left[\!\left[\Delta(\infty)^2\right]\!\right] \;=\; \frac{\sigma_\eta^2(t_{h+1}-t_h)\left(1-\beta\tau+\frac{\beta^2\tau^2}{3}\right)+\beta^2\sigma_\tau^2}{(2\beta\tau-\beta^2\tau^2)}. \tag{31}$$

Finally, putting (31) into (24) leads to

$$E\left[\!\left[\Theta(\infty)^2\right]\!\right] \;=\; \frac{\sigma_\eta^2(t_{h+1}-t_h)\left(\tau^2-\beta\tau^3-\frac{\beta^2\tau^4}{3}\right)+\beta^2\sigma_\tau^2\tau^2}{(2\beta\tau-\beta^2\tau^2)}+\sigma_\eta^2\frac{\tau^2(t_{h+1}-t_h)}{3}+\sigma_\tau^2. \tag{32}$$

Without loss of generality, assume that $t_{h+1}-t_h \approxeq \tau$. Therefore, the asymptotic estimation error of Algorithm 1 is given by

$$Var\left(\underset{h\to\infty}{\gamma(t_h)}\right) \;=\; E\left[\!\left[\Theta(\infty)^2\right]\!\right]=\frac{\sigma_\eta^2\left(\tau^2-\beta\tau^3+\frac{\beta^2\tau^4}{3}\right)+\beta^2\sigma_\tau^2\tau}{(2\beta-\beta^2\tau)}+\frac{\sigma_\eta^2\tau^3}{3}+\sigma_\tau^2 \tag{33}$$

which results in (7).