# Lecture 8: Constraint Semantics & Resolution

**CS4200 Compiler Construction**

**Hendrik van Antwerpen**

**TU Delft**

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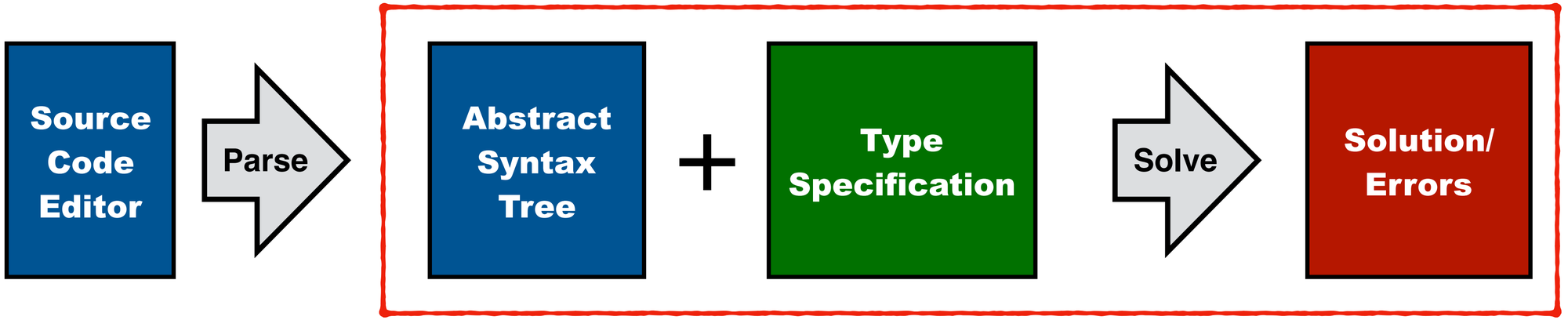
TU Delft

# This lecture




- Type checking with type specifications
- Semantics of a type specification
- Type checking algorithms
- Constraint solving for type specifications
- Term equality and unification

# Reading Material

The following papers add background, conceptual exposition, and examples to the material from the slides. Some notation and technical details have been changed; check the documentation.

This paper describes the next generation of the approach.

Addresses (previously) open issues in expressiveness of scope graphs for type systems:
- Structural types
- Generic types

Addresses open issue with staging of information in type systems.

Introduces Statix DSL for definition of type systems.

OOPSLA 2018

https://doi.org/10.1145/3276484

## Scopes as Types

HENDRIK VAN ANTWERPEN, Delft University of Technology, Netherlands
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EELCO VISSER, Delft University of Technology, Netherlands

Scope graphs are a promising generic framework to model the binding structures of programming languages, bridging formalization and implementation, supporting the definition of type checkers and the automation of type safety proofs. However, previous work on scope graphs has been limited to simple, nominal type systems. In this paper, we show that viewing *scopes as types* enables us to model the internal structure of types in a range of non-simple type systems (including structural records and generic classes) using the generic representation of scopes. Further, we show that relations between such types can be expressed in terms of generalized scope graph queries. We extend scope graphs with scoped relations and queries. We introduce Statix, a new domain-specific meta-language for the specification of static semantics, based on scope graphs and constraints. We evaluate the scopes as types approach and the Statix design in case studies of the simply-typed lambda calculus with records, System F, and Featherweight Generic Java.

CCS Concepts: • **Software and its engineering → Semantics**; **Domain specific languages**;

Additional Key Words and Phrases: static semantics, type system, type checker, name resolution, scope graphs, domain-specific language

**ACM Reference Format:**
Hendrik van Antwerpen, Casper Bach Poulsen, Arjen Rouvoet, and Eelco Visser. 2018. Scopes as Types. *Proc. ACM Program. Lang.* 2, OOPSLA, Article 114 (November 2018), 30 pages. https://doi.org/10.1145/3276484

### 1 INTRODUCTION

The goal of our work is to support high-level specification of type systems that can be used for multiple purposes, including reasoning (about type safety among other things) and the implementation of type checkers [Visser et al. 2014]. Traditional approaches to type system specification do not reflect the commonality underlying the name binding mechanisms for different languages. Furthermore, operationalizing name binding in a type checker requires carefully staging the traversals of the abstract syntax tree in order to collect information before it is needed. In this paper, we introduce an approach to the declarative specification of type systems that is close in abstraction to traditional type system specifications, but can be directly interpreted as type checking rules. The approach is based on scope graphs for name resolution, and constraints to separate traversal order from solving order.

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2475-1421/2018/11-ART114
https://doi.org/10.1145/3276484

Good introduction to unification, which is the basis of many type inference approaches, constraint languages, and logic programming languages. Read sections 1, and 2.

Baader et al. “Chapter 8 - Unification Theory.” In Handbook of Automated Reasoning, 445–533. Amsterdam: North-Holland, 2001.

https://www.cs.bu.edu/~snyder/publications/UnifChapter.pdf

CHAPTER 8

# Unification theory

Franz Baader

Wayne Snyder

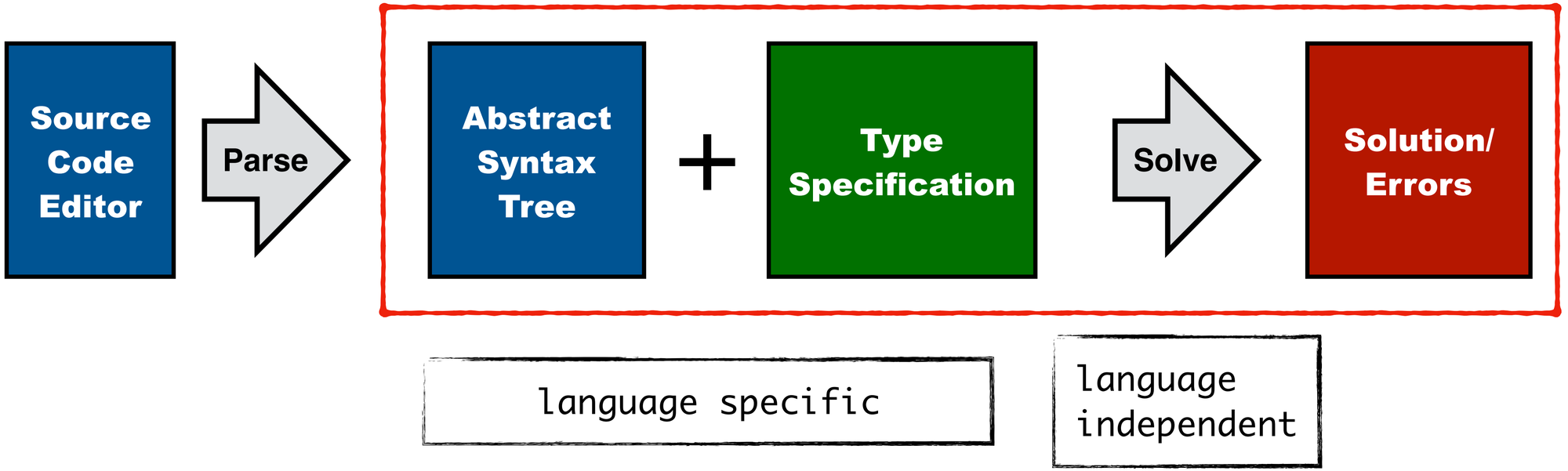
SECOND READERS: Paliath Narendran, Manfred Schmidt-Schauss, and Klaus Schulz.

*Contents*



HANDBOOK OF AUTOMATED REASONING
Edited by Alan Robinson and Andrei Voronkov

# Type Checking with Specifications


Source Code Editor
Parse
Abstract Syntax Tree
+
Type Specification
Solve
Solution/ Errors
language specific
language independent

# Typing Rules

## What are typing rules?

- Predicates that specify constraints (= rule premises) on their arguments (= the program)
- Syntax-directed, match on program constructs (at least in Statix)
- *Specification* of what it means to be well-typed!

## What are the premises?

- Logical assertions that should hold for well-typed programs
- Specification language determines what assertions can be made
  - Type equality and inequality, name resolution, ...
- Determines the *expressiveness* of the specification!

## Solving

- Given an initial predicate that must hold, ...
- find an assignment for all logical variables, such that the predicate is satisfied

# Typing Checking

## Challenges for type checker implementations?

- Collecting (non-lexical) binding information before use
- Dealing with unknown (type) values

## Separation of *what* from *how*

- Typing rules says *what* is a well-typed program
- Solver says *how* to determine that a program is well-typed

## Separation of computation from program structure

- Typing rules follow the structure of the program
- Solver is flexible in order of resolution

## Approach: reusable solver for the specification language

- Support logical variables for unknowns and *infer* their values
- Automatically determine correct resolution order

# Constraint Semantics

# What gives constraints meaning?

```
ty == FUN(ty1,ty2)
Var{x} in s |-> d
ty1 == INT()
```

## What is the meaning of constraints?

- What is a valid solution?
- Or: in which models are the constraints satisfied?
- Can we describe this independent of an algorithm to find a solution?

## When are constraints satisfied?

- Formally described by the declarative semantics
- Written as $G, \phi \vDash C$
- Satisfied in a model
  - Substitution $\phi$ (read: *phi*)
  - Scope graph $G$
- Describes for every type of constraint when it is satisfied

# Semantics of (a Subset of) Statix Constraints

## Syntax

```
C = t == t              // equality
  | r in s |-> d        // name resolution
  | C /\ C              // conjunction
```

## Declarative semantics

$G,\phi \vDash t == u$ if $\phi(t) = \phi(u)$

$G,\phi \vDash r \text{ in } s \mid\!\!\to d$ if $\phi(r) = \text{Var}\{x\}$ and $\phi(d) = \text{Var}\{x\}$ and $\phi(s) = \#i$ and $\text{Var}\{x\}$ resolves to $\text{Var}\{x\}$ from $\#i$ in $G$

$G,\phi \vDash C_1 \wedge C_2$ if $G,\phi \vDash C_1$ and $G,\phi \vDash C_2$

# Using the Semantics

**Program**

```
let
  function f1(i2 : int) : int =
    i3 + 1
in
  f4(14)
end
```

**Program constraints**

```
ty1 == INT()
INT() == INT()
Var{"i"} in #s1 |-> d1
ty2 == INT()
Var{"f"} in #s0 |-> d2
ty3 == FUN(ty4,ty5)
ty4 == INT()
…
```

**Unifier ϕ (model)**

```
ϕ = { ty1 -> INT(),
      ty2 -> INT(),
      ty3 -> FUN(INT(),ty5),
      ty4 -> INT(),
      d1  -> Var{"i"},
      d2  -> Var{"f"}
    }
```

**Constraint semantics**

G,ϕ ⊨ t == u

*if* ϕ(t) = ϕ(u)

G,ϕ ⊨ r in s |-> d

*if* ϕ(r) = Var{x}

*and* ϕ(d) = Var{x}

*and* ϕ(s) = #i

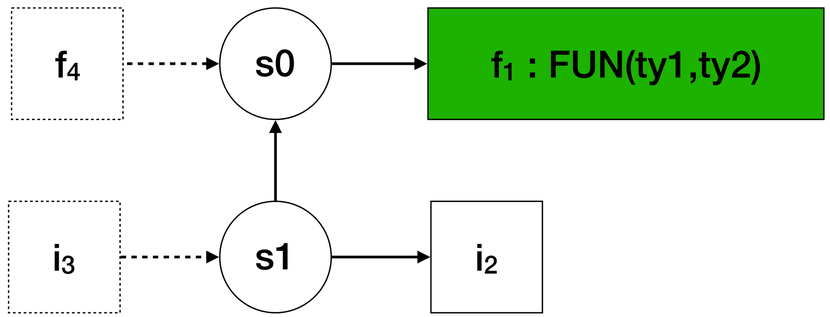
*and* Var{x} resolves to Var{x} from #i in G

G,ϕ ⊨ $C_1$ /\ $C_2$

*if* G,ϕ ⊨ $C_1$

*and* G,ϕ ⊨ $C_2$

**Scope graph G (model)**



# Different Kinds of Variables

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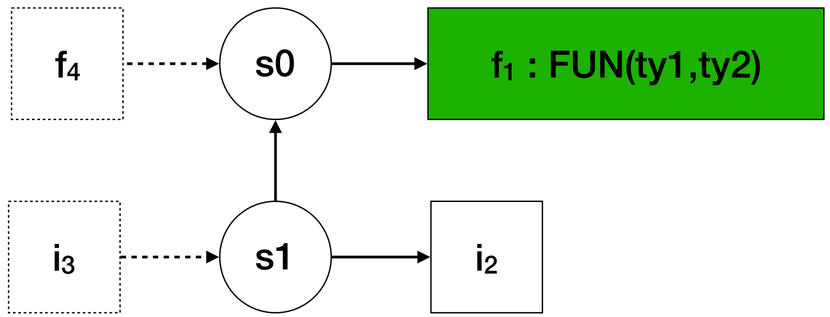

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```

Object language variables

**Constraint semantics**

```
G,ϕ ⊨ t == u
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G,ϕ ⊨ r in s |-> d
    if  ϕ(r) = Var{x}
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G,ϕ ⊨ C1 /\ C2
    if  G,ϕ ⊨ C1
    and G,ϕ ⊨ C2
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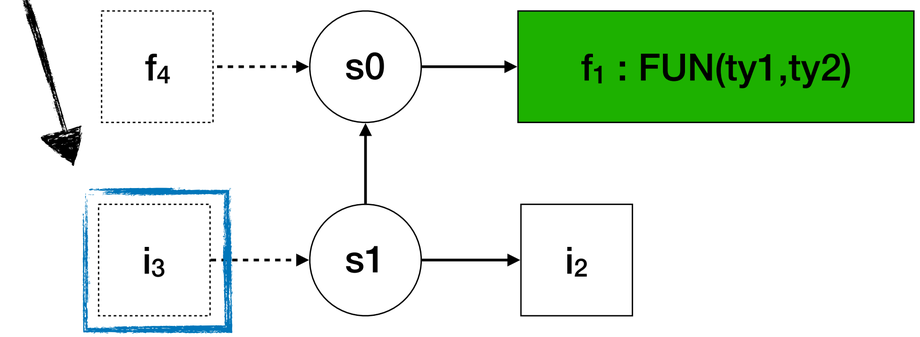

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```

Constraint / logic variables

**Constraint semantics**

```
G,ϕ ⊨ t == u
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G,ϕ ⊨ r in s |-> d
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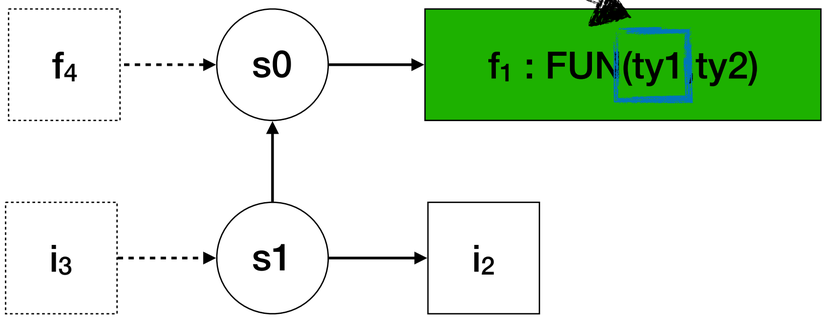

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```

Semantics meta-variables

**Constraint semantics**

$G,\phi \models t == u$
*if* $\phi(t) = \phi(u)$

$G,\phi \models r$ in $s$ |-> $d$
*if* $\phi(r) = Var\{x\}$
*and* $\phi(d) = Var\{x\}$
*and* $\phi(s) = \#i$
*and* $Var\{x\}$ resolves to $Var\{x\}$ from $\#i$ in $G$

$G,\phi \models C_1 \wedge C_2$
*if* $G,\phi \models C_1$
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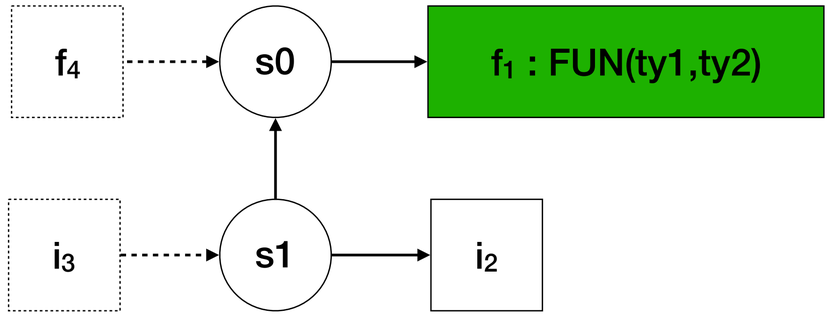

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$G,\phi \models C_1 \;/\backslash\; C_2$

*if* $G,\phi \models C_1$

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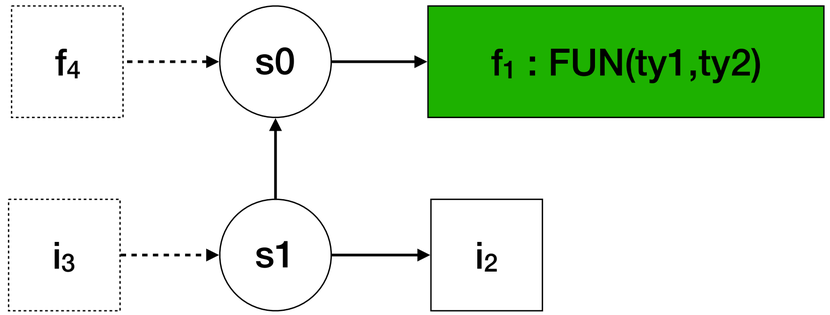
**Scope graph G (model)**



# Type Checking

# How to check types?

## What should a type checker do?

- Check that a program is well-typed!
- Resolve names, and check or compute types
- Report useful error messages
- Provide a representation of name and type information
  - Type annotated AST

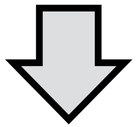
## This information is used for

- Next compiler steps (optimization, code generation, …)
- IDE (error reporting, code completion, refactoring, …)
- Other tools (API documentation, …)

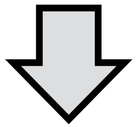
## How are type checkers implemented?

# Computing Type of Expression (recap)

```
function (a : int) = a + 1
```



```
Fun("a", INT(),
    Plus(Var("a"), Int(1)))
```



```
FUN(INT(), INT())
```

```
typeOfExp(s, Int(_)) = INT().

typeOfExp(s, Plus(e1, e2)) = INT() :-
  typeOfExp(s, e1) == INT(),
  typeOfExp(s, e2) == INT().

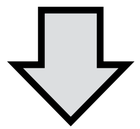
typeOfExp(s, Fun(x, te, e)) = FUN(S, T) :- {s_fun}
  typeOfTypeExp(s, te) == S,
  new s_fun, s_fun -P-> s,
  s_fun -> Var{x} with typeOfDecl S,
  typeOfExp(s_fun, e) == T.

typeOfExp(s, Var(x)) = T :-
  typeOfDecl of Var{x} in s |-> [(_, (_, T))].
```

- Can be executed top down, in premise order
- Could be written almost like this in a functional language

# Inferring the Type of a Parameter

```
function (a      ) = a + 1
```



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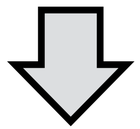

```
FUN(INT(), INT())
```

Unknown S!

```
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typeOfExp(s, Plus(e1, e2)) = INT() :-
  typeOfExp(s, e1) == INT(),
  typeOfExp(s, e2) == INT().

typeOfExp(s, Fun(x,    e)) = FUN(S, T) :- {s_fun}
                                          
  new s_fun, s_fun -P-> s,
  s_fun -> Var{x} with typeOfDecl S,
  typeOfExp(s_fun, e) == T.

typeOfExp(s, Var(x)) = T :-
  typeOfDecl of Var{x} in s |-> [(_, (_, T))].
```

- What are the consequences for our typing rules?
- Types are not known from the start, but learned gradually
- A simple top-down traversal is insufficient

# Checking classes

```
class A {
    B m() {
        return new C();
    }
}

class B {
    int i;
}

class C extends B {
   int m(A a) {
        return a.m().i;
    }
}
```

## How can we type check this program?

- Is there a possible single traversal strategy here?
- Why are the type annotations not enough?
- What strategy could be used?

## Two-pass approach

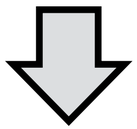
- The first pass builds a class table
- The second pass checks expressions using the class table

## Question

- Does this still work if we introduce nested classes?

# Variables and Constraints

```
function (a      ) = a + 1
```



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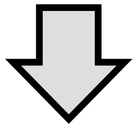
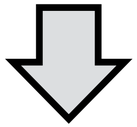


```
FUN(?S, INT()) + ?S == INT()
```



```
?S := INT()
```

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typeOfExp(s, Var(x)) = T :-
  typeOfDecl of Var{x} in s |-> [(_, (_, T))].
```

# How to check types?

## What are challenges when implementing a type checker?

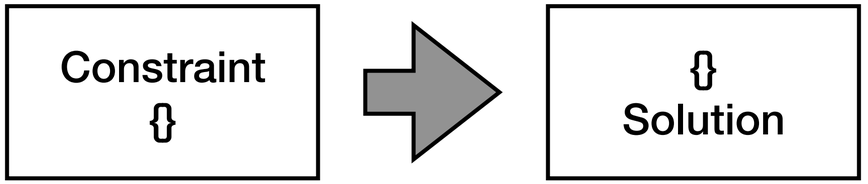
- Collecting necessary binding information before using it
- Gradually learning type information

## What are the consequences of these challenges?

- The order of computation needs to be more flexible than the AST traversal
- Support explicit logical variables during solving

# Solving Constraints

# Solving by Rewriting




# Solving by Rewriting

Non-deterministic constraint selection

$$\langle C;\ G,\ \phi \rangle \longrightarrow \langle C;\ G,\ \phi \rangle$$

$$\langle t == u,\ C;\ G,\ \phi \rangle \longrightarrow \langle C;\ G,\ \phi' \rangle \textit{ where } \text{unify}(\phi, t, u) = \phi'$$

$$\langle s1\ \text{-L->}\ s2,\ C;\ G,\ \phi \rangle \longrightarrow \langle C;\ G',\ \phi \rangle \textit{ where } \phi(s1) = \#i,\ \phi(s2) = \#j,\ G + \{\#i\ \text{-L->}\ \#j\} = G'$$

$$\langle r\ \text{in}\ s\ \text{|->}\ t,\ C;\ G,\ \phi \rangle \longrightarrow \langle t == d;\ G,\ \phi \rangle \textit{ where } \phi(r) = Ns\{x\},\ \phi(s) = \#i,\ \text{resolve}(G, \#i, Ns\{x\}) = d$$

```
def solve(C):
  if <C; {}, {}> ⟶* <{}; G, ϕ>:
    return <G, ϕ>
  else:
    fail
```

Scope graph and name resolution algorithm don't have to know about logical variables

# Solving by Rewriting

## Solver = rewrite system

- Rewrite a constraints set + solution
- Simplifying and eliminating constraints
  - Constraint selecting is non-deterministic
  - Resolution order is controlled by side conditions on rewrite rules
- Rely on (other) solvers and algorithms for base cases
  - Unification for term equality
  - Scope graph resolution
- The solution is final if all constraints are eliminated

## Does the order matter for the outcome?

- Confluence: the output is the same for any solving order
- Partly true for Statix
  - Up to variable and scope names
  - Only if all constraints are reduced

# Semantics vs Algorithm

## What is the difference?

- Algorithm computes a solution (= model)
- Semantics describes when a constraint is satisfied by a model

## How are these related?

- Soundness
  - If the solver returns $\langle G, \phi \rangle$, then $G, \phi \models C$
- Completeness:
  - If a $G$ and $\phi$ exists such that $G, \phi \models C$, then the solver returns it
  - If no such $G$ or $\phi$ exists, the solver fails
- Principality
  - The solver finds the most general $\phi$

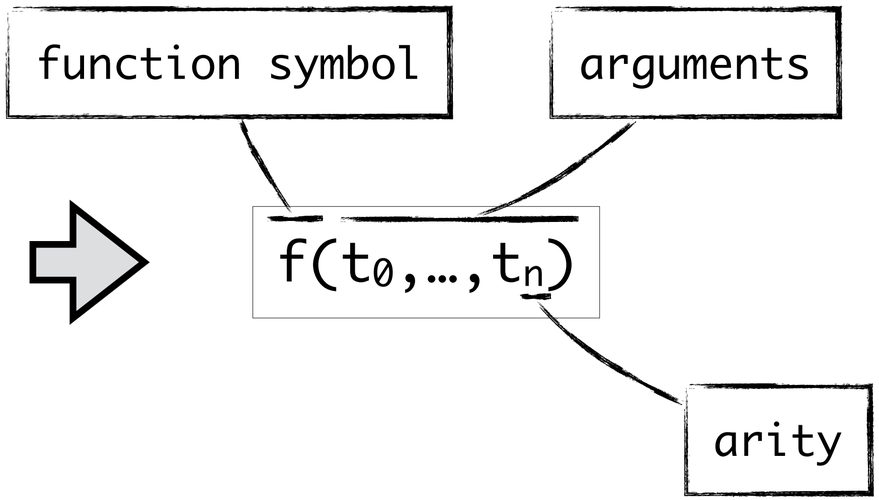
# Term Equality & Unification

# Syntactic Terms

```
terms      t, u
functions  f, g, h
```

## Generic Terms

```
INT()
FUN(INT(),INT())
```



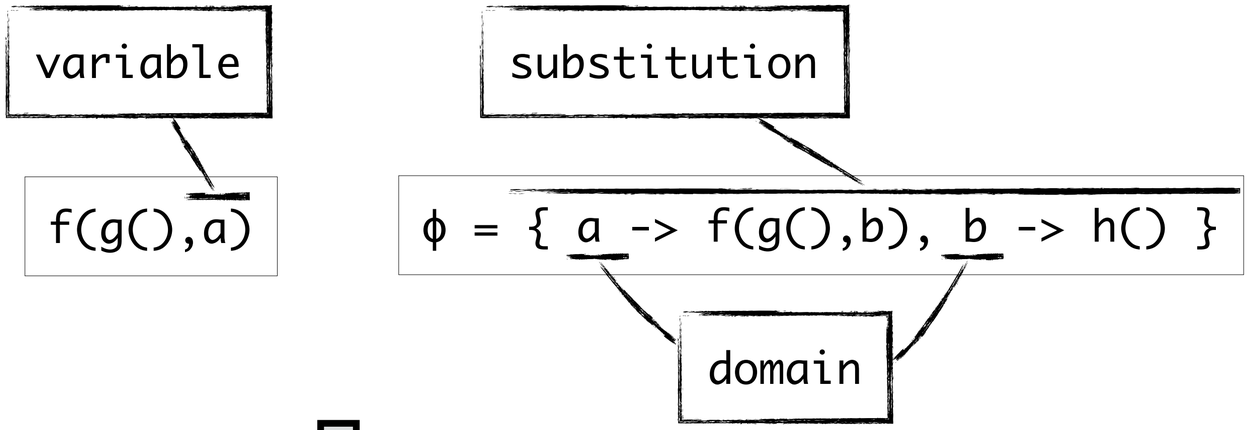

## Syntactic Equality

$f(t_0,\ldots,t_n) == g(u_0,\ldots,u_m)$ if

- $f = g$, and $n = m$
- $t_i == u_i$ for every $i$

# Variables and Substitution

| | |
|---|---|
| terms | t, u |
| functions | f, g, h |
| variables | a, b, c |
| substitution | ϕ |




$$
\begin{aligned}
\phi(a) &= t && \text{if } \{ a \to t \} \text{ in } \phi \\
\phi(a) &= a && \text{otherwise} \\
\phi(f(t_0,\ldots,t_n)) &= f(\phi(t_0),\ldots,\phi(t_n))
\end{aligned}
$$

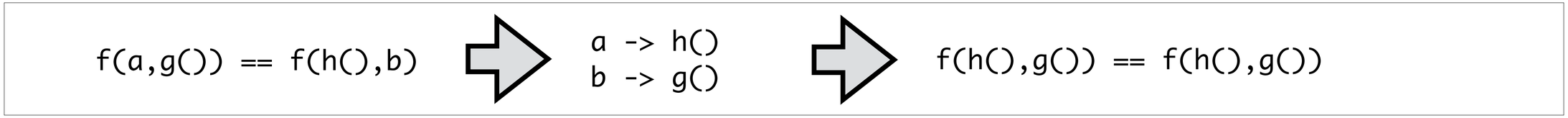
f(g(),f(g(),b))

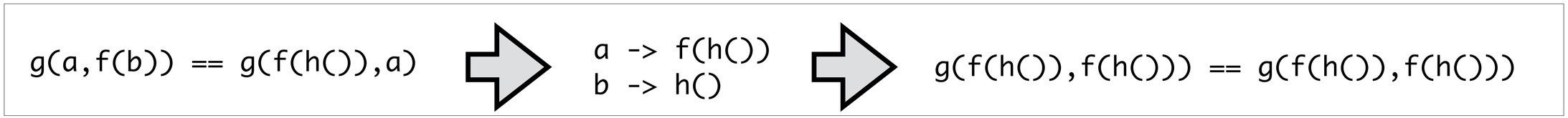
ground term: a term without variables

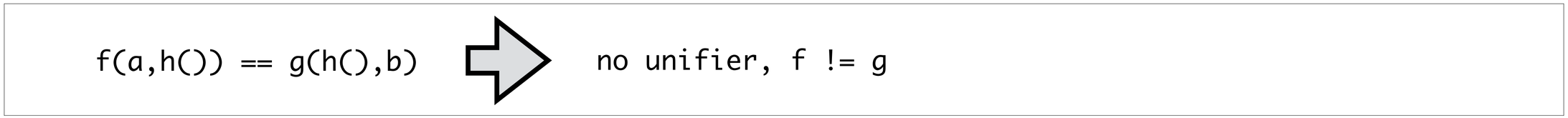
# Unifiers

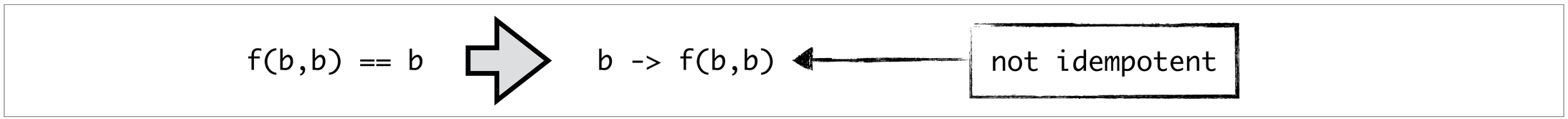
| | |
|---|---|
| terms | t, u |
| functions | f, g, h |
| variables | a, b, c |
| substitution | ϕ |

unifier: a substitution that makes terms equal



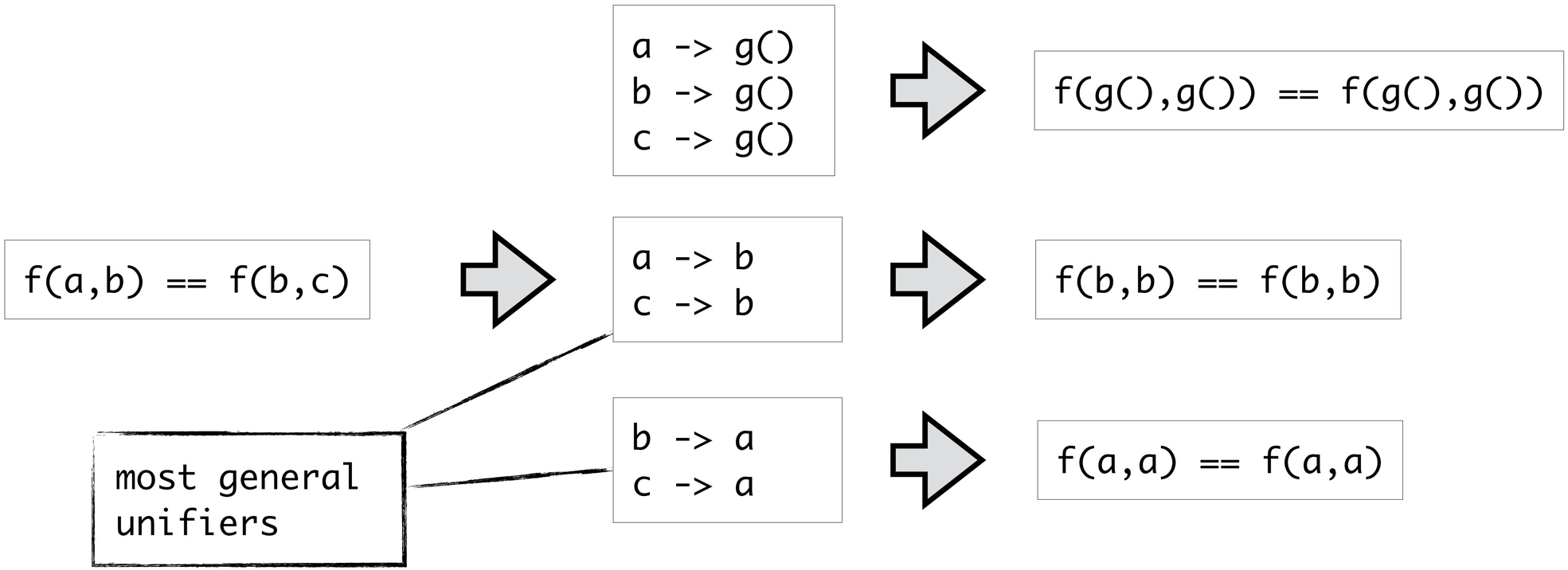



# Most General Unifiers

| | |
|---|---|
| terms | t, u |
| functions | f, g, h |
| variables | a, b, c |
| substitution | ϕ |

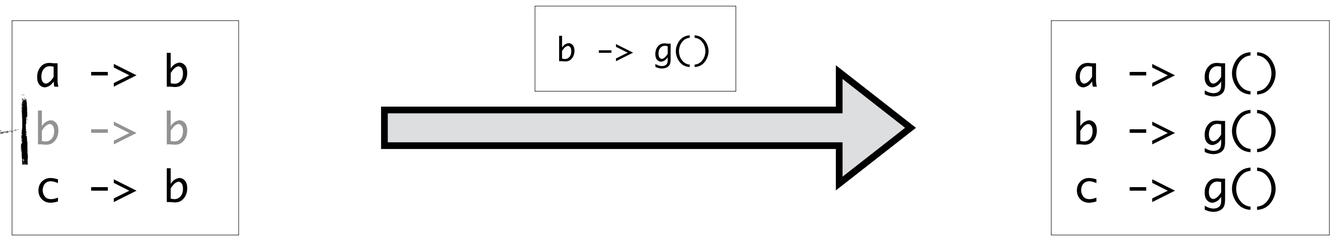


# Most General Unifiers

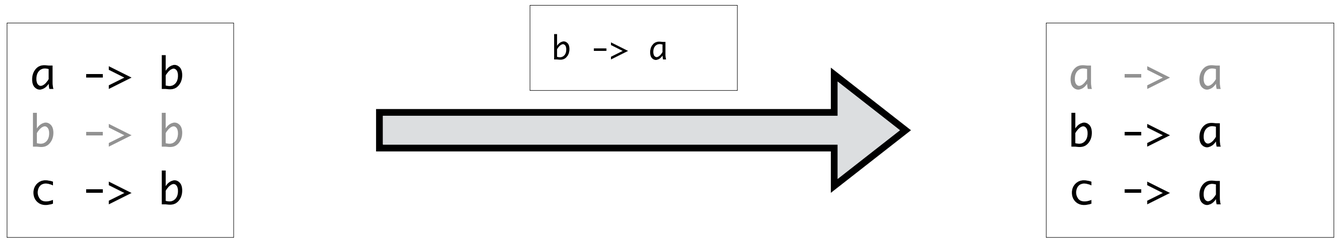
| | |
|---|---|
| terms | t, u |
| functions | f, g, h |
| variables | a, b, c |
| substitution | ϕ |

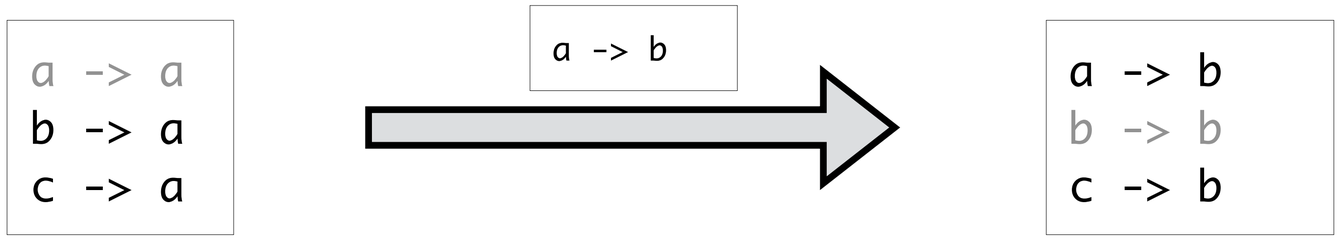
every unifier is an instance of a most general unifier

(implicit) identity case




most general unifiers are related by renaming substitutions



# Unification

| | |
|---|---|
| terms | t, u |
| functions | f, g, h |
| variables | a, b, c |
| substitution | ϕ |

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t₀,...,tₙ) and u == g(u₀,...,uₘ):
    if f == g and n == m:
      for i := 1 to n:
        unify(tᵢ, uᵢ)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

*t == a*
*instantiate variable*

*u == b*
*instantiate variable*

*b == b*
*equal variables*

*t == f(t₀,...,t₅), u == f(u₀,...,u₅)*
*matching terms*

*t == f(t₀,...,t₅), u == g(u₀,...,u₃)*
*mismatching terms*

*t == f(t₀,...,t₅), u == b*
*swap terms*

*t == a, u == k(g(a,f()))*
*recursive terms*

*t == a, u == k(u₀,...,u₅)*
*extend unifier*

# Properties of Unification

## Soundness

- If the algorithm returns a unifier, it makes the terms equal

## Completeness

- If a unifier exists, the algorithm will return it

## Principality

- If the algorithm returns a unifier, it is a most general unifier

## Termination

- The algorithm always returns a unifier or fails

# Conclusion

# Summary

## What is the meaning of constraints?

- Formally described by constraint semantics
- Semantics classify solutions, but do not compute them
- Semantics are expressed in terms of other theories
  - Syntactic equality
  - Scope graph resolution

## What techniques can we use to implement solvers?

- Constraint simplification
  - Simplification rules
  - Depends on built-in procedures to unify or resolve names
- Unification
  - Unifiers make terms with variables equal
  - Unification computes most general unifiers

## What is the relation between solver and semantics?

- Soundness: any solution satisfies the semantics
- Completeness: if a solution exists, the solver finds it
- Principality: the solver computes most general solutions

# Efficient Unification with Union-Find

TU Delft

# Complexity of Unification

```
terms         t, u
functions     f, g, h
variables     a, b, c
substitution  ϕ
```

## Space complexity

- Exponential
- Representation of  unifier

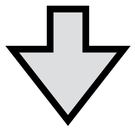
## Time complexity

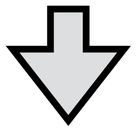
- Exponential
- Recursive calls on terms

## Solution

- Union-Find algorithm
- Complexity growth can be considered constant

$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) == h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$





$a_1 \rightarrow f(a_0,a_0)$
$a_2 \rightarrow f(f(a_0,a_0), f(a_0,a_0))$
$a_i \rightarrow \ldots\ 2^{i+1}-1$ subterms …
$b_1 \rightarrow f(a_0,a_0)$
$b_2 \rightarrow f(f(a_0,a_0), f(a_0,a_0))$
$b_i \rightarrow \ldots\ 2^{i+1}-1$ subterms …

fully applied

$a_1 \rightarrow f(a_0,a_0)$
$a_2 \rightarrow f(a_1,a_1)$
$a_i \rightarrow \ldots$ 3 subterms …
$b_1 \rightarrow f(a_0,a_0)$
$b_2 \rightarrow f(a_1,a_1)$
$b_i \rightarrow \ldots$ 3 subterms …

triangular

# Set Representatives

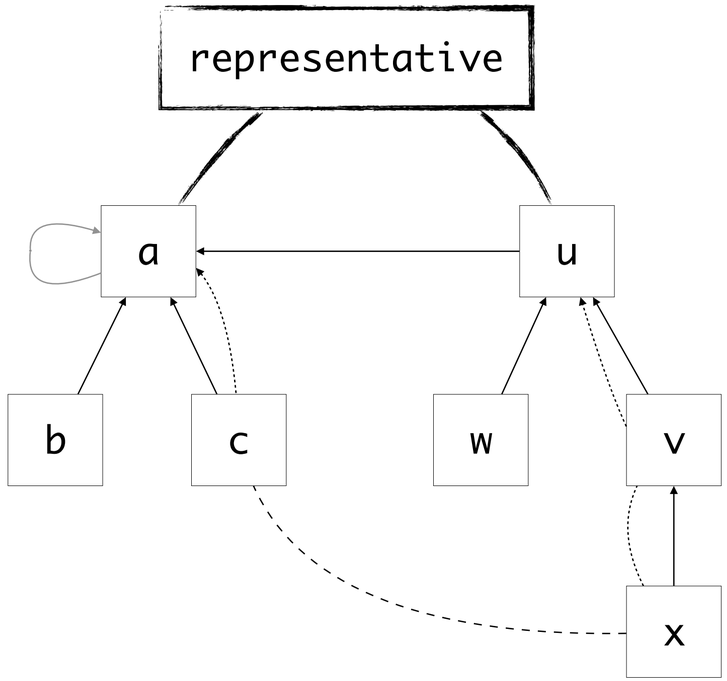
```
FIND(a):
  b := rep(a)
  if b == a:
     return a
  else
     return FIND(b)



UNION(a1,a2):
  b1 := FIND(a1)
  b2 := FIND(a2)
  LINK(b1,b2)

LINK(a1,a2):
  rep(a1) := a2
```

```
a == b
c == a
u == w
v == u
x == v
x == c
```



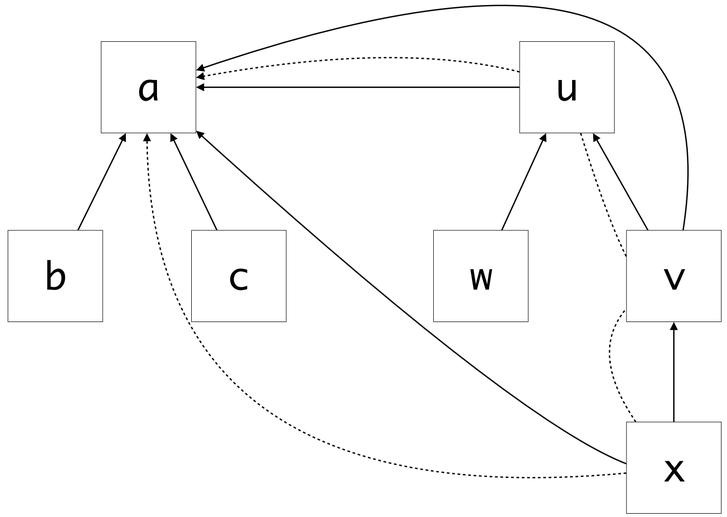
# Path Compression

```
FIND(a):
  b := rep(a)
  if b == a:
     return a
  else
     b := FIND(b)
     rep(a) := b
     return b

UNION(a1,a2):
  b1 := FIND(a1)
  b2 := FIND(a2)
  LINK(b1,b2)

LINK(a1,a2):
  rep(a1) := a2
```

```
  …
x == b
x == c
x == w
x == v
```



# Tree Balancing

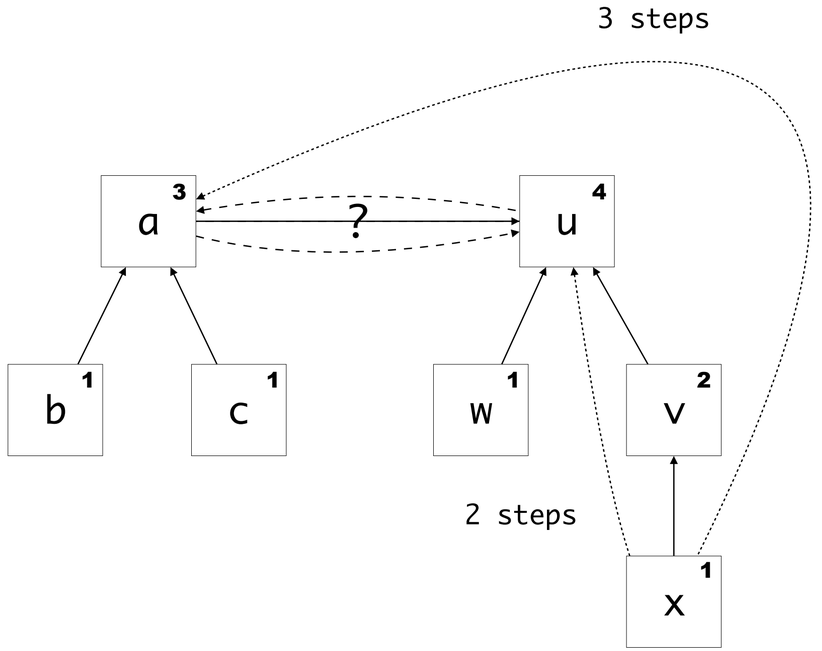
```
FIND(a):
  b := rep(a)
  if b == a:
     return a
  else
     b := FIND(b)
     rep(a) := b
     return b

UNION(a1,a2):
  b1 := FIND(a1)
  b2 := FIND(a2)
  LINK(b1,b2)

LINK(a1,a2):
  if size(a2) > size(a1):
     rep(a1) := a2
     size(a2) += size(a1)
  else:
     rep(a2) := a1
     size(a1) += size(a2)
```

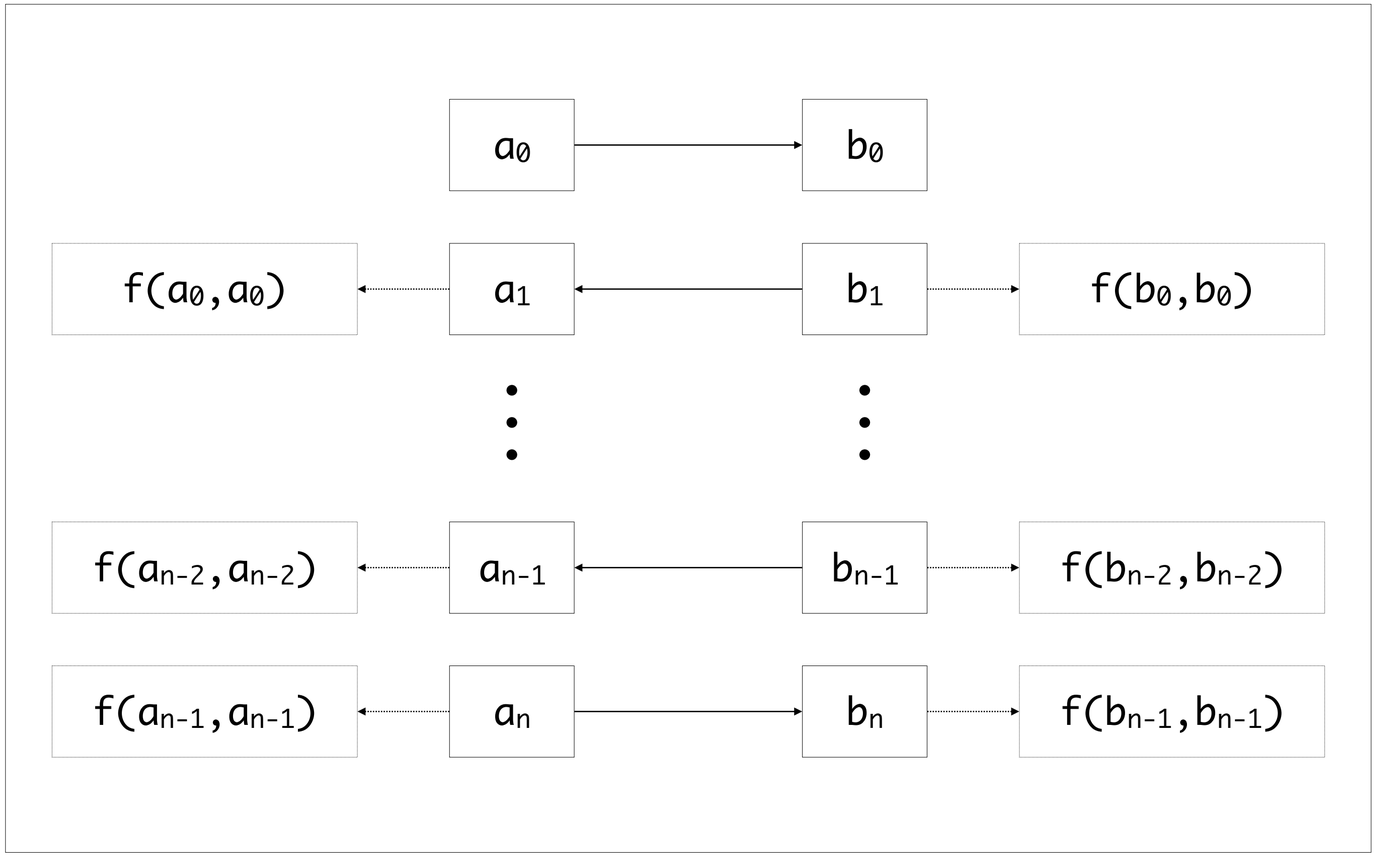
…

x == c



# The Complex Case

$$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) == h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$$




$$a_n == b_n$$

$$f(a_{n-1},a_{n-1}) == f(b_{n-1},b_{n-1})$$

$$a_{n-1} == b_{n-1} \qquad a_{n-1} == b_{n-1}$$

$$f(a_{n-2},a_{n-2}) == f(b_{n-2},b_{n-2})$$

$$\vdots$$

$$a_1 == b_1 \qquad a_1 == b_1$$

$$f(a_0,a_0) == f(b_0,b_0)$$

$$a_0 == b_0 \qquad a_0 == b_0$$

How about occurrence checks? Postpone!

# Union-Find

## Main idea

- Represent unifier as graph
- One variable represent equivalence class
- Replace substitution by union & find operations
- Testing equality becomes testing node identity

## Optimizations

- Path compression make recurring lookups fast
- Tree balancing keeps paths short

## Complexity

- Linear in space and almost linear (inverse Ackermann) in time
- Easy to extract triangular unifier from graph
- Postpone occurrence checks to prevent traversing (potentially) large terms

```
Martelli, Montanari. An Efficient Unification Algorithm. TOPLAS, 1982
```