Mechanics of Structures
CT4145 / CT2031

MODULE : INTRODUCTION INTO CONTINUUM MECHANICS

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INTRODUCTORY REMARKS

These notes are part of the educational lecture module Mechanics of Structures CT4145 and CT2031. The theory and examples are presented in such a way that the reader should be able to study this subject as a self-tuition module. Apart from these notes some additional material can be found on the web pages. Most of this material can be downloaded via BlackBoard or via the internet at:

http://home.hetnet.nl/~t-wmn/index.html

Although this material has been prepared with great care, faults or errors may occur. I would very much appreciate the report of such faults or errors.

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1. INTRODUCTION INTO STRESSES AND STRAINS

In this module the relation between stresses and strains in a continuum will be described. With the definition of the stresses in 3D the limit state of a stress combination will be examined based on two distinctive plasticity models like von Mises and Mohr.
In order to do so the reader will be made familiar with the definition of stresses, strains and the constitutive relation between these stresses and strains. The properties of stresses and strains such as transformation rules for different coordinate systems can be formulated with help of the definition of a tensor. Both stresses and strains are tensors, which will be shown in this part of the lecture notes, and therefore have the same transformation rules. The transformations can be done analytically but also graphically with help of Mohr's circle. Although graphical methods seem to be somewhat old-fashioned they have advantageous properties which will be illustrated with examples.

1.1 Stresses in 3D

From the well known definition of a stress which is a Force per Area the stresses can be distinguished into *normal stresses* and *shear stresses*. A normal stress acts in the direction normal to the plane and a shear stress acts along a vector in the plane. Based on a three dimensional coordinate system x-y-z we can define three stresses acting on a plane. If the normal vector of the plane coincides with the x-axis, the in plane axes are y and z. The plane is therefore called a x-plane (normal in the x-direction).

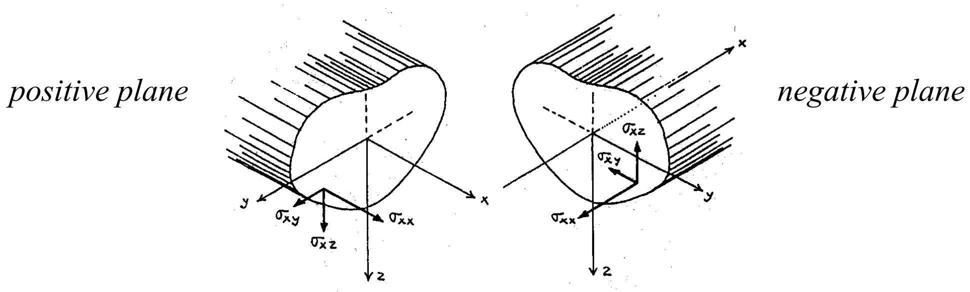


Figure 1.1 : Normal and shear stresses

Stresses will be denoted with a double index, one for the plane and one for the direction. The normal stress acting on a x-plane, acts in the x-direction and will be denoted as:

$$\underbrace{\sigma_{x}}_{x\text{-plane}}\!\underbrace{{}_{x}}_{x\text{-direction}} \qquad \sigma_{xx}$$

The two shear stresses acting on the x-plane with directions into y and z will be denoted as :

$$\sigma_{xy}$$
$$\sigma_{xz}$$

The positive directions of stresses are well defined. On a positive x-plane (direction of the outer normal of the plane coincides with the direction of the x-axis) a positive stress acts in the positive coordinate direction (left side of fig. 1). On a negative x-plane (outer normal of the plane coincides with the negative x-axis) a positive stress acts in the negative coordinate direction (right side of fig. 1).

For a cube with six faces the following stresses can be found.

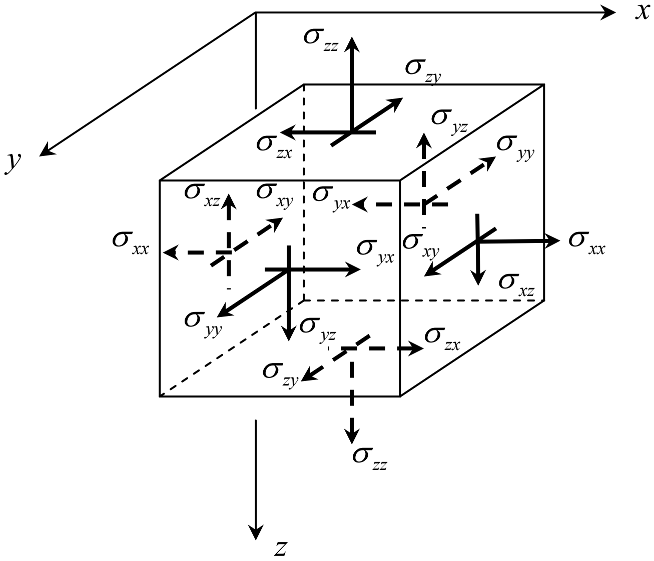


Figure 1.2 : Stresses in 3D

Out of the six faces there are only three different planes :

x-plane
y-plane
z-plane

From this it can be seen that there are only 9 different stresses as shown in figure 2.

$$\begin{array}{ll} \sigma_{xx};\sigma_{xy};\sigma_{xz} & x-plane \\ \sigma_{yy};\sigma_{yx};\sigma_{yz} & y-plane \\ \underbrace{\sigma_{zz}}_{\text{normal}};\underbrace{\sigma_{zx};\sigma_{zy}}_{\text{shear}} & z-plane \end{array}$$

Assignment:

Check for your self the equilibrium of forces due to the shown stresses in figure 1.2, acting on a cube with size dx, dy *and* dz*. How many independent stresses do we really need ?*

These nine stresses can be presented as a transposed vector :

$$\sigma_{xyz}{}^{T} = \left(\sigma_{xx},\sigma_{yy},\sigma_{zz},\sigma_{xy},\sigma_{yx},\sigma_{yz},\sigma_{zy},\sigma_{zx},\sigma_{xz}\right)$$

In this presentation the normal stresses are placed in front. Another possible presentation is a matrix presentation:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

The internal stresses are the result of an externally applied load. From the equilibrium equations a relation can be found between the stresses and the externally applied stress. In figure 3, a small specimen loaded with an external stress $\boldsymbol{p}$ on a surface A is presented with the defined stresses on the other surfaces which coincide with the x-, y- and z-planes.

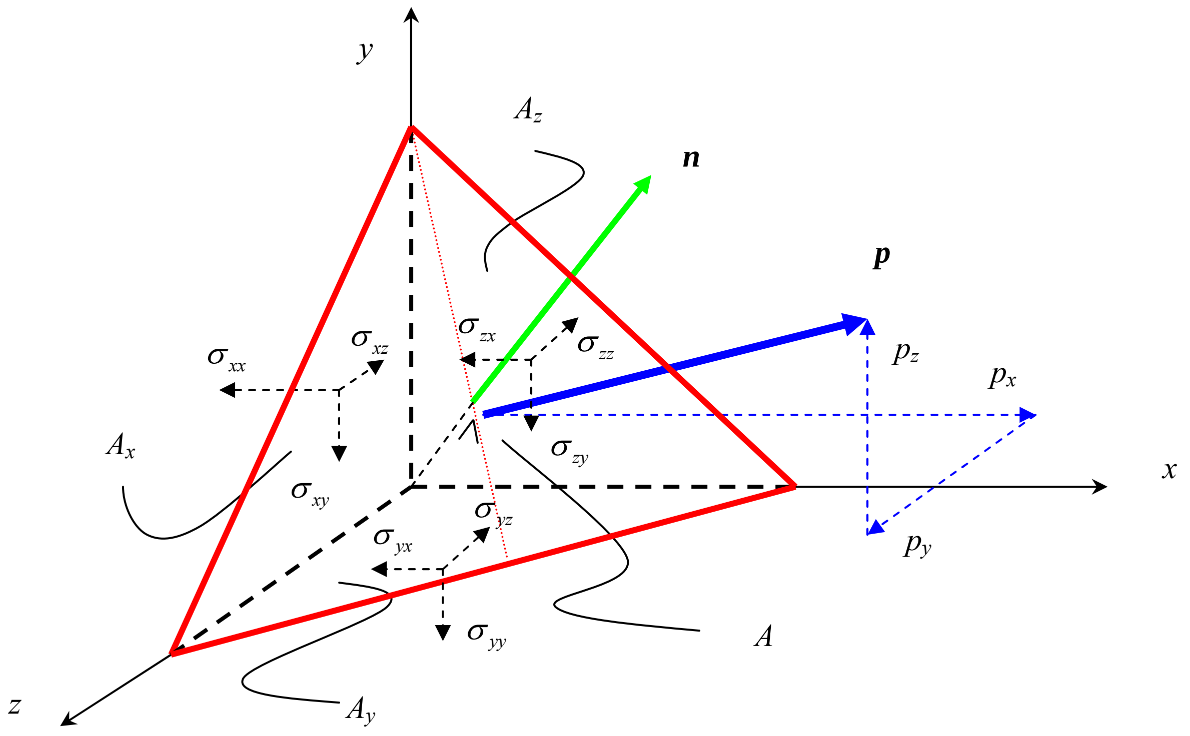


Figure 1.3 : Specimen in 3D

The stress $\boldsymbol{p}$ has components p_x, p_y and p_z. The surface A has an outer normal unit vector $\boldsymbol{n}$ with components n_x, n_y and n_z. The size of the x-, y- and z-planes can be found with:

$$
\begin{aligned}
A_x &= A \times n_x \\
A_y &= A \times n_y \\
A_z &= A \times n_z
\end{aligned}
\qquad \text{where :} \qquad
\begin{aligned}
n_x &= \cos(n,x) \\
n_y &= \cos(n,y) \\
n_z &= \cos(n,z)
\end{aligned}
\qquad (1)
$$

in which the angle is presented as the angle between the two specified vectors (e.g. ($\boldsymbol{n}$,$\boldsymbol{x}$), ($\boldsymbol{n}$,$\boldsymbol{y}$), and ($\boldsymbol{n}$,$\boldsymbol{z}$)).

Equilibrium demands equilibrium of *forces*. For each stress acting on a surface the resulting force can now be determined. The equilibrium conditions in x-, y- and z-direction yield :

$$
\begin{aligned}
A \times p_x - A_x\sigma_{xx} - A_y\sigma_{yx} - A_z\sigma_{zx} &= 0 \\
A \times p_y - A_x\sigma_{xy} - A_y\sigma_{yy} - A_z\sigma_{zy} &= 0 \\
A \times p_z - A_x\sigma_{xz} - A_y\sigma_{yz} - A_z\sigma_{zz} &= 0
\end{aligned}
$$

With use of (1) these equilibrium conditions can be rewritten as:

$$
\begin{aligned}
A \times p_x &= A \times \left(\cos(n,x)\times\sigma_{xx} + \cos(n,y)\times\sigma_{yx} + \cos(n,z)\times\sigma_{zx}\right) \\
A \times p_y &= A \times \left(\cos(n,x)\times\sigma_{xy} + \cos(n,y)\times\sigma_{yy} + \cos(n,z)\times\sigma_{zy}\right) \\
A \times p_z &= A \times \left(\cos(n,x)\times\sigma_{xz} + \cos(n,y)\times\sigma_{yz} + \cos(n,z)\times\sigma_{zz}\right)
\end{aligned}
$$

This result can be presented in matrix notation as:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos(n,x) \\ \cos(n,y) \\ \cos(n,z) \end{bmatrix} \quad or \quad \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

This result shows the relation between a stress ***p*** at a surface and the normal ***n*** to this surface. The matrix has to be a symmetrical one (prove this with the assignment of page 2).

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \qquad \text{with:} \quad \begin{aligned} \sigma_{xy} &= \sigma_{yx} \\ \sigma_{xz} &= \sigma_{zx} \\ \sigma_{yz} &= \sigma_{zy} \end{aligned}$$

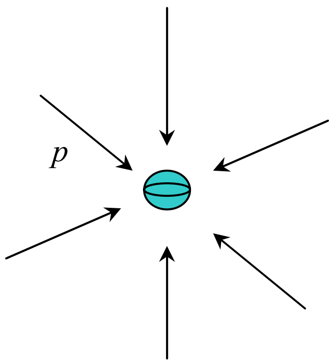
The relation between these two vectors is the matrix with the earlier defined stress components. We only have to identify 6 stress components.

1.1.1 Special stress situations

Some special cases of stress situations will be presented here. These cases will be used in examples in the next paragraphs.

Isotropic stress - If only the normal stresses have a non zero value and are alle equal to each other the stress situation is denoted as an isotropic or *hydrostatic* stress situation.

$$\sigma_{xyz}^{\;T} = (p, p, p, 0, 0, 0, 0, 0, 0)$$



Plane stress situation - If on one of the planes all stresses are equal to zero this stress situation is denoted as a plane stress situation.

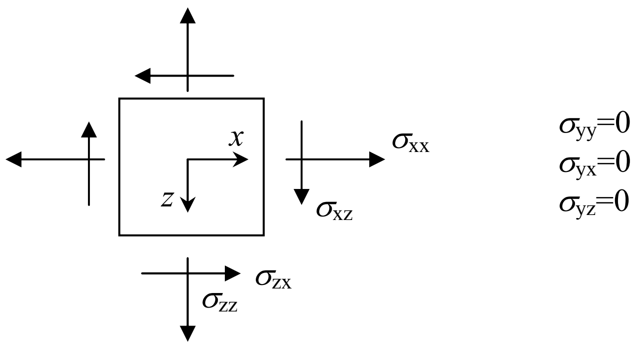


Figure 1.4 : Plane stress situation

Plane stress in beams

A special case of a plane stress situation occurs in beams. The commonly used beam theory only describes normal stresses and shear stresses within the vertical cross sections of the beam and related to these shear stresses also a shear stress on horizontal cross sections in the *x*-*y*-plane. The normal and shear stresses are distributed over the depth of the beam. For a small specimen at distance *z* from the neutral axis we can consider the stresses as uniform.

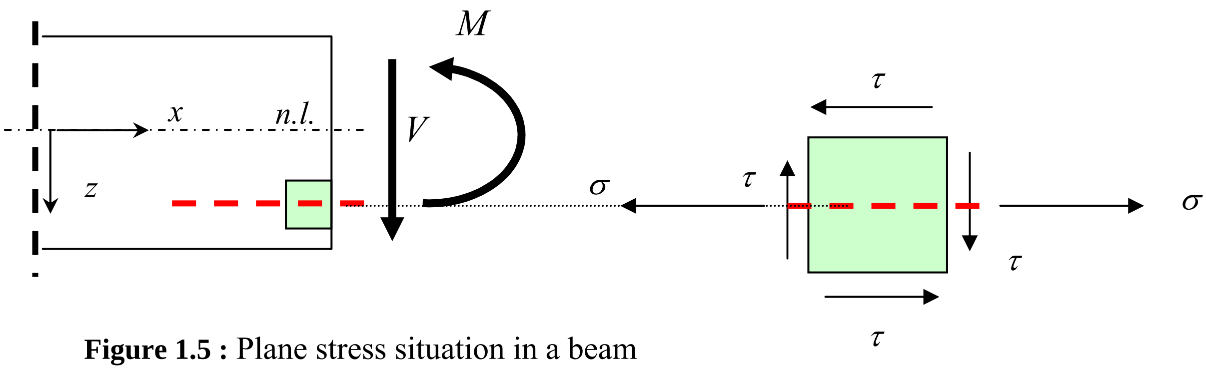


Figure 1.5 : Plane stress situation in a beam

Uniaxial stress situation - If only one of the normal stresses occur the stress situation is denoted as a uniaxial stress situation.

1.1.2 Isotropic and Deviatoric stress components

The matrix with the defined stresses in 3D can be split in a diagonal matrix with the same diagonal terms and a non diagonal matrix :

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_o & 0 & 0 \\ 0 & \sigma_o & 0 \\ 0 & 0 & \sigma_o \end{bmatrix}}_{\text{isotropic part}} + \underbrace{\begin{bmatrix} \sigma_{xx}-\sigma_o & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-\sigma_o & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\sigma_o \end{bmatrix}}_{\text{deviatoric part}}$$

The diagonal matrix is called the *isotropic stress* contribution. The second part is called the *deviatoric stress* component. The magnitude of the isotropic stress is the average of the normal stresses:

$$\sigma_o = \tfrac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

This distinction is made since the *isotropic stress* contribution will only cause a change in *volume* and the deviatoric stress component is responsible for a *distortion*. We will make use of this distinction in the chapter on Failure and Yield Criteria.

1.2 Strains

Stresses will cause deformations. The amount of deformation per unit of length is called a specific elongation or in short strain. An uniaxial example is given below.

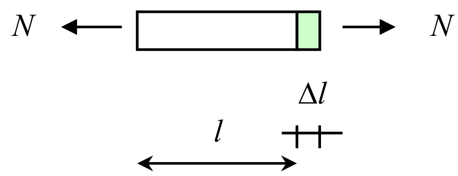


Figure 1.4 : Specific elongation

The length of a rod *l* will increase with an elongation Δ*l* due to the force *N*. The specific elongation is denoted as ε and is called the strain:

$$\varepsilon = \frac{\Delta l}{l}$$

For a three dimensional body the deformations will not only be the above mentioned elongation. Also a change of shape can be observed.

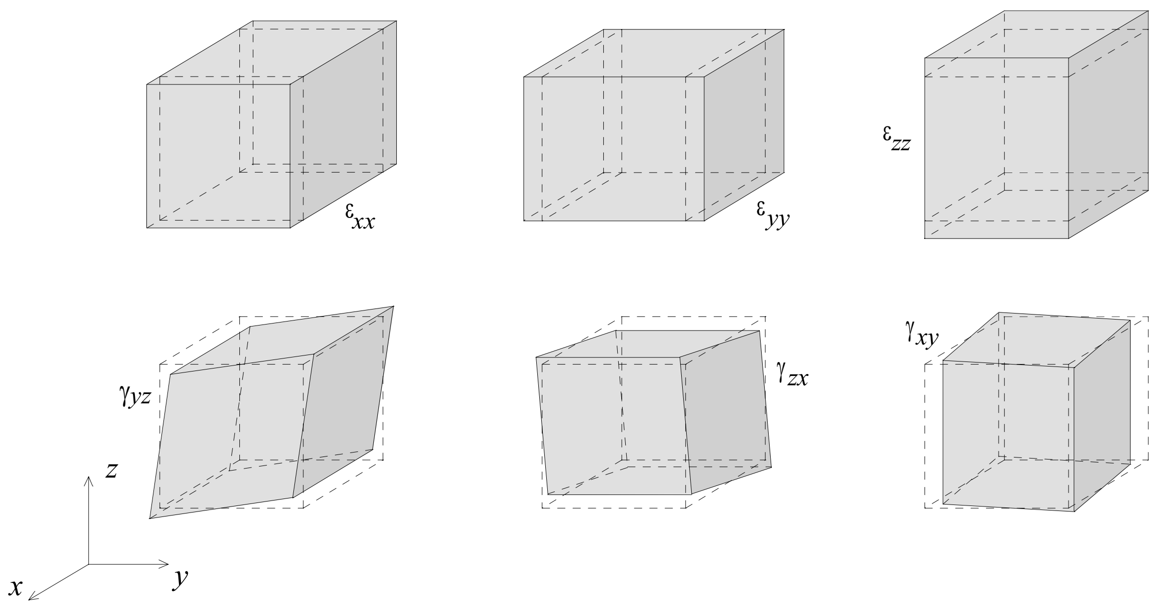


Figure 1.5 : Deformation modes

The first upper three deformed bodies describe elongations in *x*, *y* and *z*-direction. The shape of the body is unchanged. The three deformed bodies below are shape deformations. Like stresses (normal stresses and shear stresses) we observe that also deformations can be distinguished in to normal strains and shearing strains. The central question is however how these strains can be obtained from the observed displacements.

To simplify the problem we start with the uniaxial example. We assume a cross section build of fibres parallel to the x-axis. The displacement in the x-direction is denoted with u. At the left side of the specimen we assume a displacement u and at the right side an increased displacement $u+\Delta u$.

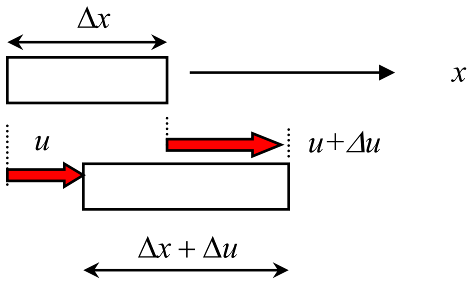


Figure 1.6 : Strain definition

The definition of the strain as the ratio of the change in length of the fibre to its original length, results in:

$$\varepsilon = \frac{\Delta l}{l} = \frac{\Delta u}{\Delta x}$$

In case the length of the specimen becomes very small this yields to the general relation between the strain and the displacement field $u(x)$:

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\mathrm{d}u}{\mathrm{d}x}$$

This result shows that the strain of a fibre is the derivative of the displacement function $u(x)$ in the direction of the fibre.

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$$

The same procedure can now be applied to a 2-dimensional problem. In figure 1.6 a specimen PRSQ is shown with dimensions Δx , Δy. The displacement field u consists of component u_x in x-direction and u_y in y-direction. Due to a deformation, point P will move to P' and Q will move to Q'.

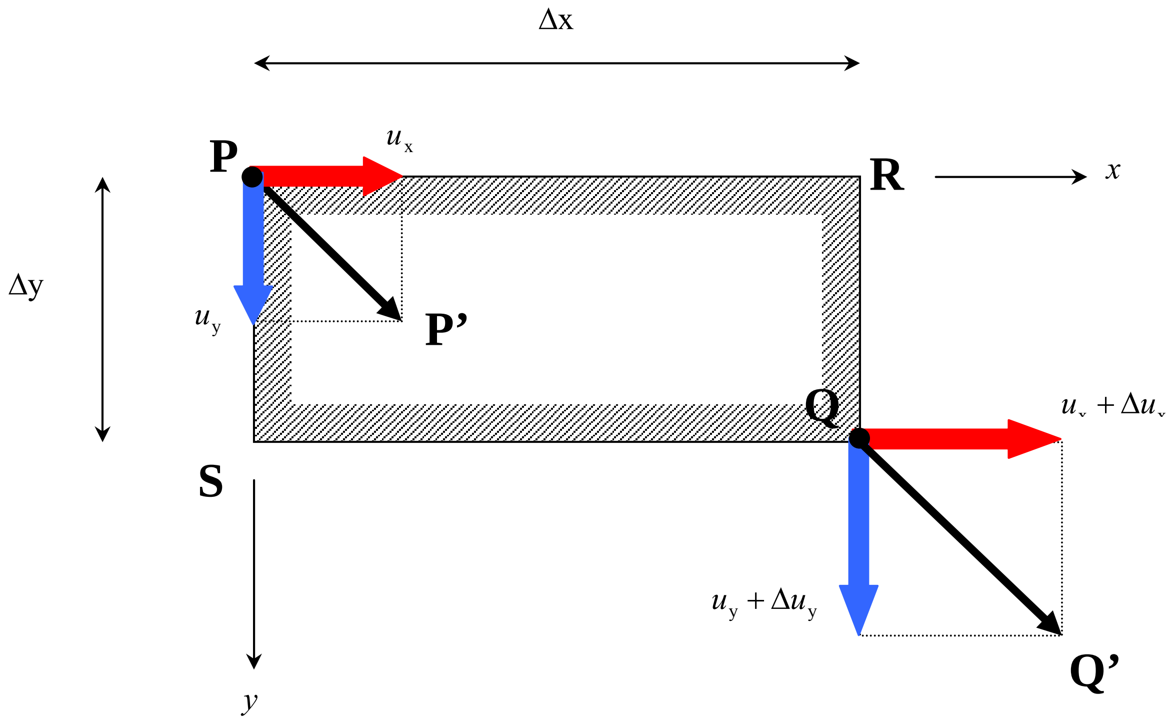


Figure 1.6 : Two-dimensional problem

For this block **PRSQ** the displacement field can be defined as:

$$\begin{cases} u_x = u_x(x,y) \\ u_y = u_y(x,y) \end{cases}$$

We assume a continuous displacement field which also has continuous derivatives :

$$\begin{cases} \Delta u_x = \dfrac{\partial u_x}{\partial x}\Delta x + \dfrac{\partial u_x}{\partial y}\Delta y \\ \Delta u_y = \dfrac{\partial u_y}{\partial x}\Delta x + \dfrac{\partial u_y}{\partial y}\Delta y \end{cases}$$

These relative displacements can be written in matrix notation as:

$$\begin{bmatrix} \Delta u_x \\ \Delta u_y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad \frac{\partial u_x}{\partial y} \neq \frac{\partial u_y}{\partial x}$$

With these relative displacements we will try to describe the strains of the fibres in x- and y-direction.

In the following figure the deformed specimen is partly shown. The fibres PR parallel to the x-axis and PS parallel to the y-axis are shown here.

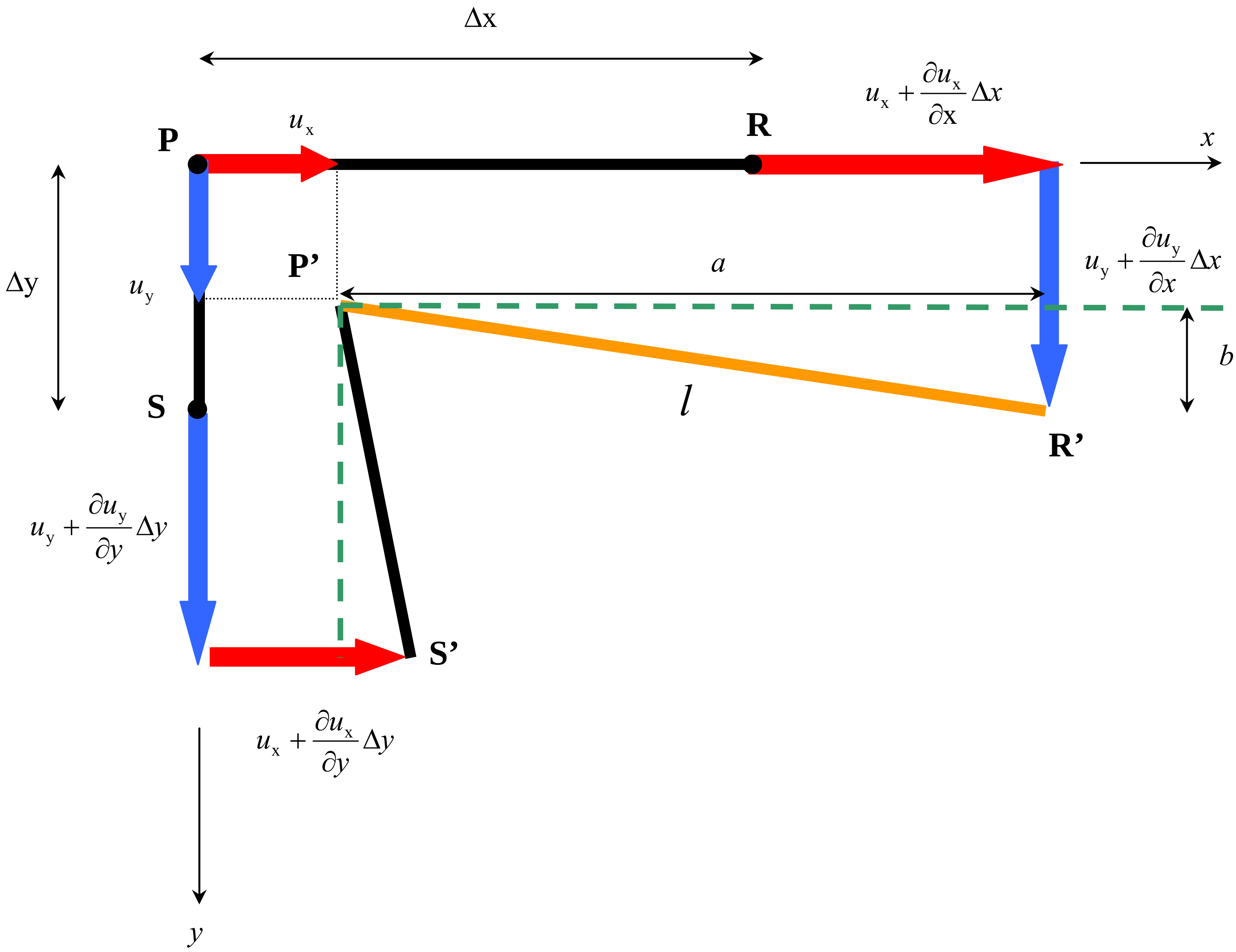


Figure 1.7 : Strains in x- and y-direction

From the graph it follows that the length of fibre P'R' has become :

$$l^2 = a^2 + b^2 = \left(\Delta x + \frac{\partial u_x}{\partial x}\Delta x\right)^2 + \left(\frac{\partial u_y}{\partial x}\Delta x\right)^2 \qquad (2)$$

With use of the classical definition of the strain, the new length of a fibre in x-direction can also be expressed as:

$$l = \Delta x + \varepsilon_{xx}\Delta x = (1+\varepsilon_{xx})\Delta x \qquad (3)$$

In which ε_{xx} is the strain in the x-direction of a fibre in x-direction. An expression for the strain in terms of the displacement field, can be found if we combine these two expressions:

$$(1+\varepsilon_{xx})^2 \times \Delta x^2 = \left(1+\frac{\partial u_x}{\partial x}\right)^2 \times \Delta x^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 \times \Delta x^2 \quad \Leftrightarrow$$

$$\varepsilon_{xx} = \sqrt{\left(1+\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2} - 1$$

This expression can be simplified by developing it into a Taylor series in both u_x and u_y :

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2 \qquad \text{(see APPENDIX 1)}$$

If the displacement gradients are small, higher order terms can be neglected and we can obtain the so called linearised expression for the strain:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

For the strain in y-direction of a fibre in y-direction (e.g. P'S') the same approach holds:

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

This result so far is very similar to the earlier obtained expression of the strain. Since the displacement field is a function of x and y we have to use the partial derivative instead of the ordinary derivative.

With this result the earlier found expression for the relative displacements can be rewritten.

$$\begin{bmatrix} \Delta u_x \\ \Delta u_y \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad \frac{\partial u_x}{\partial y} \neq \frac{\partial u_y}{\partial x}$$

$$\begin{bmatrix} \Delta u_x \\ \Delta u_y \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \boxed{\frac{\partial u_x}{\partial y}} \\ \boxed{\frac{\partial u_y}{\partial x}} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

WHAT ARE THESE NON DIAGONAL TERMS ?

From figure 1.7 we can also observe that not only an elongation of fibres occur but also a change of the right angle between fibres. This will cause a shape deformation. We will take a closer look to this component of the deformation. Most likely we will find in this way an expression for the non- diagonal terms of the rewritten expression above.

In order to do so we redraw figure 1.7 and look at the rotations of the fibres in *x*- and *y*-direction.

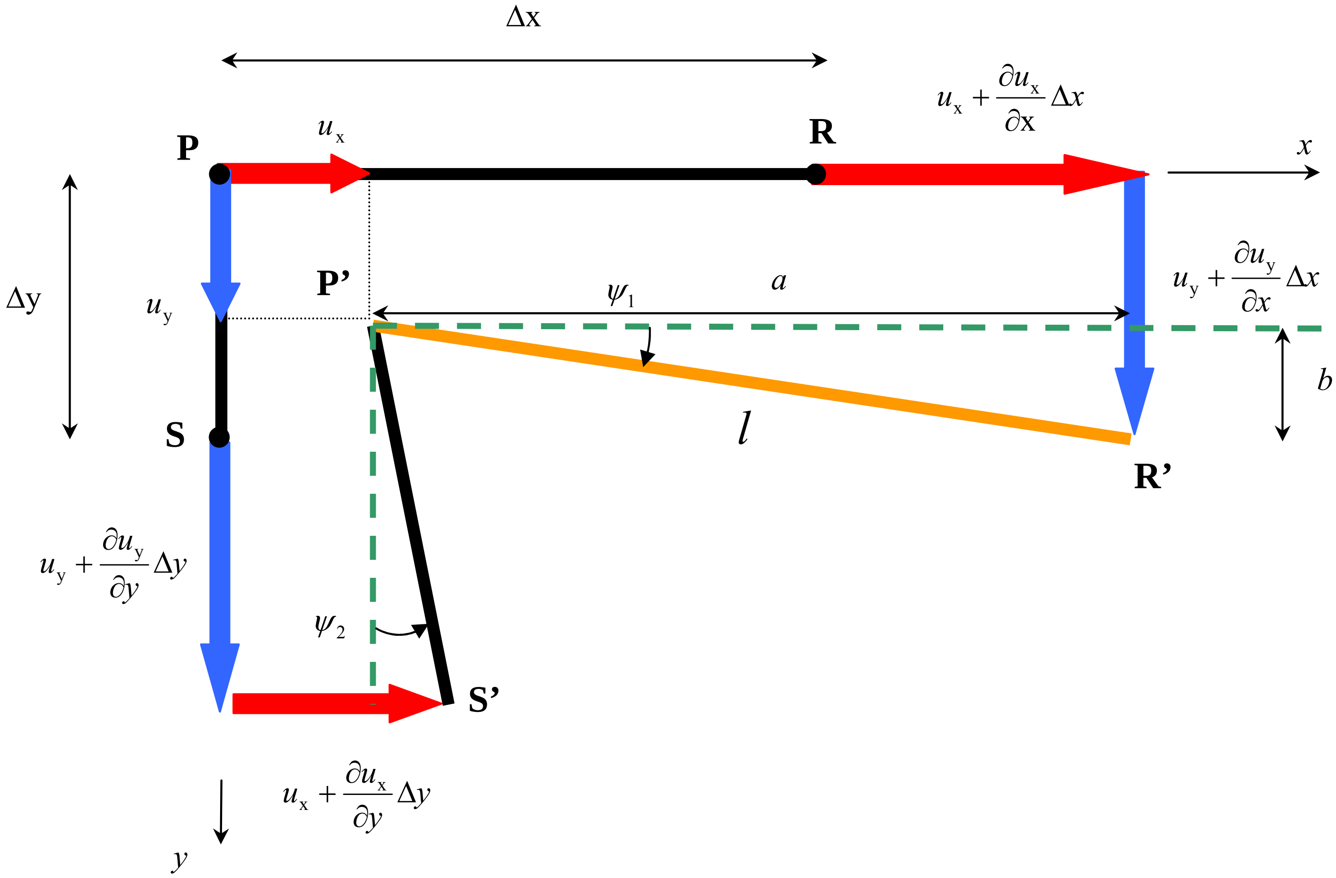


Figure 1.8 : Rotation of fibres in *x*- and *y*-direction

From figure 1.8 we can see that the shape of the specimen will be deformed due to the rotations ψ_1 and ψ_2. The total change of the right angle between the *x*- and *y*-fibres is defined as the *shear deformation* and denoted with γ:

$$\gamma = \psi_1 + \psi_2 \qquad \text{(definition)}$$

From the graph it follows that the rotations can be found with:

$$\psi_1 = \frac{b}{a} = \frac{\frac{\partial u_y}{\partial x}\Delta x}{\left(1+\frac{\partial u_x}{\partial x}\right)\Delta x} \qquad \text{and} \qquad \psi_2 = \frac{\frac{\partial u_x}{\partial y}\Delta y}{\left(1+\frac{\partial u_y}{\partial y}\right)\Delta y}$$

For small displacement gradients the expressions can be simplified to:

$$\psi_1 = \frac{\partial u_y}{\partial x}$$

$$\psi_2 = \frac{\partial u_x}{\partial y}$$

These expressions are exactly the non-diagonal terms we were looking for.

The earlier found expression for the relative displacements can be rewritten as:

$$\begin{bmatrix} \Delta u_{\mathrm{x}} \\ \Delta u_{\mathrm{y}} \end{bmatrix} = \begin{bmatrix} \varepsilon_{\mathrm{xx}} & \psi_2 \\ \psi_1 & \varepsilon_{\mathrm{yy}} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad \psi_1 \neq \psi_2$$

This matrix is not symmetric. The matrix can however be split into a symmetric part and a non-symmetric but anti-symmetric part :

$$\begin{bmatrix} \varepsilon_{\mathrm{xx}} & \psi_2 \\ \psi_1 & \varepsilon_{\mathrm{yy}} \end{bmatrix} = \begin{bmatrix} \varepsilon_{\mathrm{xx}} & \tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_2 \\ \tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_2 & \varepsilon_{\mathrm{yy}} \end{bmatrix} + \begin{bmatrix} 0 & -\tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_2 \\ \tfrac{1}{2}\psi_1 - \tfrac{1}{2}\psi_2 & 0 \end{bmatrix}$$

If we introduce a new rotational variable ω:

$$\omega = \tfrac{1}{2}\psi_1 - \tfrac{1}{2}\psi_2$$

The non-diagonal term in the symmetric matrix will be defined as:

$$\varepsilon_{xy} = \varepsilon_{yx} = \tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_2$$

This is exactly half the change of the right angle between the *x*- and *y*-fibres. This will result in:

$$\begin{bmatrix} \varepsilon_{\mathrm{xx}} & \psi_2 \\ \psi_1 & \varepsilon_{\mathrm{yy}} \end{bmatrix} = \begin{bmatrix} \varepsilon_{\mathrm{xx}} & \varepsilon_{\mathrm{xy}} \\ \varepsilon_{\mathrm{yx}} & \varepsilon_{\mathrm{yy}} \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \quad \text{with : } \varepsilon_{\mathrm{xy}} = \varepsilon_{\mathrm{yx}}$$

With this result we can also split the relative displacement in to two parts:

$$\begin{bmatrix} \Delta u_{\mathrm{x}} \\ \Delta u_{\mathrm{y}} \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{\mathrm{xx}} & \varepsilon_{\mathrm{xy}} \\ \varepsilon_{\mathrm{yx}} & \varepsilon_{\mathrm{yy}} \end{bmatrix}}_{\text{due to deformation}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}}_{\text{due to rigid body rotation}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \quad \text{with : } \varepsilon_{\mathrm{xy}} = \varepsilon_{\mathrm{yx}}$$

This is visualised in the figure below.

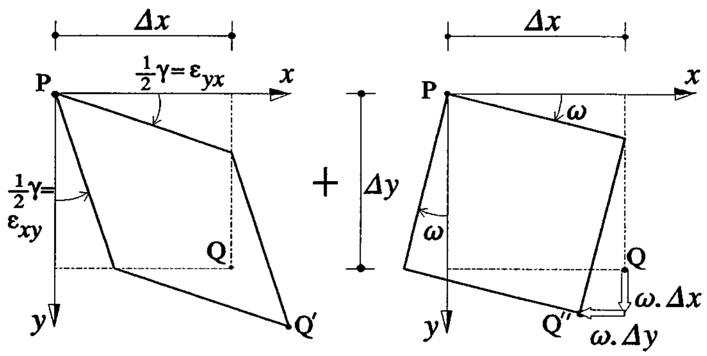


Figure 1.9 : Relative displacements due to deformation and rigid body rotation

From figure 1.8 can be seen that the total change of the right angle between the *x*- and *y*-fibre is equal to:

$$\gamma = \psi_1 + \psi_2$$

From this definition it follows that:

$$\gamma = \psi_1 + \psi_2 = 2\varepsilon_{xy} \quad \Rightarrow \quad \varepsilon_{xy} = \tfrac{1}{2}\gamma$$

The non-diagonal terms of the strain tensor are thus equal to half the total *shear deformation*.

To calculate stresses only the deformation component is important, since a rigid body displacement will not cause stresses. In most engineering textbooks therefore the expression containing the rigid body rotation is not presented.

The general formulation for the presented strains in 2D becomes:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$
$$\varepsilon_{xy} = \varepsilon_{yx} = \tfrac{1}{2}\frac{\partial u_x}{\partial y} + \tfrac{1}{2}\frac{\partial u_y}{\partial x}$$

The shear deformation γ is defined as:

$$\gamma = \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 2\varepsilon_{xy}$$

For 3D situations a similar approach leads to strains which can be presented in a matrix notation as:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad \text{with:} \quad \begin{matrix} \varepsilon_{\mathrm{xy}} = \varepsilon_{\mathrm{yx}} \\ \varepsilon_{\mathrm{xz}} = \varepsilon_{\mathrm{zx}} \\ \varepsilon_{\mathrm{yz}} = \varepsilon_{\mathrm{zy}} \end{matrix} \quad \text{and} \quad \begin{matrix} \gamma_{\mathrm{xy}} = 2\varepsilon_{\mathrm{xy}} \\ \gamma_{\mathrm{xz}} = 2\varepsilon_{\mathrm{xz}} \\ \gamma_{\mathrm{yz}} = 2\varepsilon_{\mathrm{yz}} \end{matrix}$$

The strain components of this symmetric matrix can be derived from the displacement field with:

$$\varepsilon_{ij} = \tfrac{1}{2}\frac{\partial u_i}{\partial j} + \tfrac{1}{2}\frac{\partial u_j}{\partial i} \quad \text{with :} \quad i, j = x, y, z$$

The relative displacements due to the rigid body rotations can be presented with the anti-symmetric matrix of rigid body rotations:

$$\omega = \begin{bmatrix} 0 & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & 0 & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & 0 \end{bmatrix} \quad \text{and:} \quad \omega_{ij} = \tfrac{1}{2}\frac{\partial u_i}{\partial j} - \tfrac{1}{2}\frac{\partial u_j}{\partial i} = -\omega_{ji} \quad \text{with :} \quad i, j = x, y, z$$

The compact notation with the indices *i,j* will be explained in the next chapter about transformations and tensors.

1.2.1 Special strain situation, plane strain

A special strain situation occurs when in one of the planes no strains can occur, e.g. $\varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{xz} = 0$. An example of such a condition is a cross section in elongated bodies of uniform cross section subjected to uniform loading along their longitudinal axis (in this case the *x*-axis).

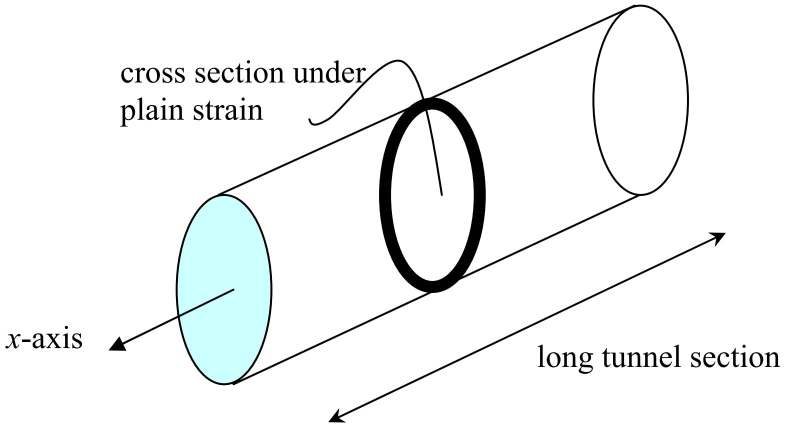


Figure 1.10 : Plane stress situation

1.2.2 Volume strain

If a cube is subjected to a volume change due to normal strains the change in volume can be described in terms of the normal strains as can be seen from figure 1.11.

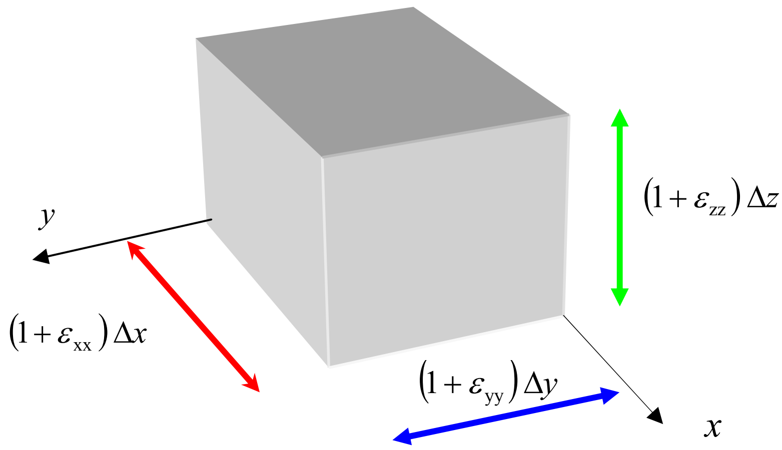


Figure 1.11 : volume change

The change in volume related to the original volume can be expressed as the volume strain *e*:

$$V + \Delta V = (1+\varepsilon_{xx})\Delta x \times (1+\varepsilon_{yy})\Delta y \times (1+\varepsilon_{zz})\Delta z$$

$$\Delta V = V\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} + \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}\right)$$

$$\Delta V \cong V\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) \quad \Rightarrow \quad \boxed{e = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}$$

Assumption:
Neglect higher order terms for small strains.

2. Transformations and tensors

With the definition of both stresses and strains we have found that both stresses and strains can be presented as symmetrical matrices:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}}_{\boxed{\text{stresses}}} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \qquad \text{with:} \quad \begin{matrix} \sigma_{xy} = \sigma_{yx} \\ \sigma_{xz} = \sigma_{zx} \\ \sigma_{yz} = \sigma_{zy} \end{matrix}$$

$$\begin{bmatrix} \Delta u_x \\ \Delta u_y \\ \Delta u_z \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}}_{\boxed{\text{strains}}} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \begin{bmatrix} 0 & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & 0 & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \qquad \text{with:} \quad \begin{matrix} \varepsilon_{xy} = \varepsilon_{yx} \\ \varepsilon_{xz} = \varepsilon_{zx} \\ \varepsilon_{yz} = \varepsilon_{zy} \end{matrix}$$

Both the stress and strain matrices relate vectors. The stress matrix relates the normal vector ***n*** to the stress vector ***p***. The strain matrix partly relates the location vector (positions) to the relative displacement vector ***u***. The specified vectors are related to some kind of coordinate system. It is interesting to know how these vectors will change if we perform a transformation of the coordinate system. Due to the similarity between stresses and strains it is most likely that both the stress and strain matrices will transform by the same rules.

2.1 Transformations

In order to investigate the behaviour of transformations of vectors and matrices we will restrict ourselves to a simple 2D situation. As an example we can use a stiffness problem in which a displacement vector ***u*** and a force vector ***F*** are involved. The stiffness matrix **K** relates these two vectors:

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \text{ or } F = \mathrm{K}.u \qquad (1)$$

With elementary calculus the stiffness matrix can be obtained:

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

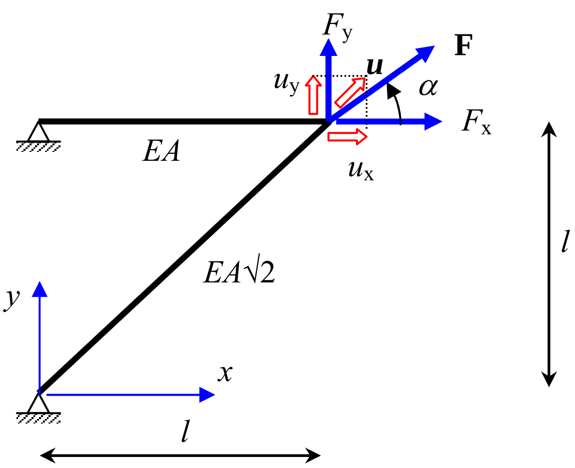


Figure 2.1 : Stiffness problem

Assignment:
Proof the given stiffness relation

Both the force and displacement vector have a physical meaning and a magnitude. This magnitude is irrespective of the coordinate system used and is therefore called an *invariant*.
If the coordinate x-y system is changed, the values of the components of both the vectors $\boldsymbol{F}$ and $\boldsymbol{u}$ will change since the coordinate system is changed from x-y in to $\bar{x}-\bar{y}$.

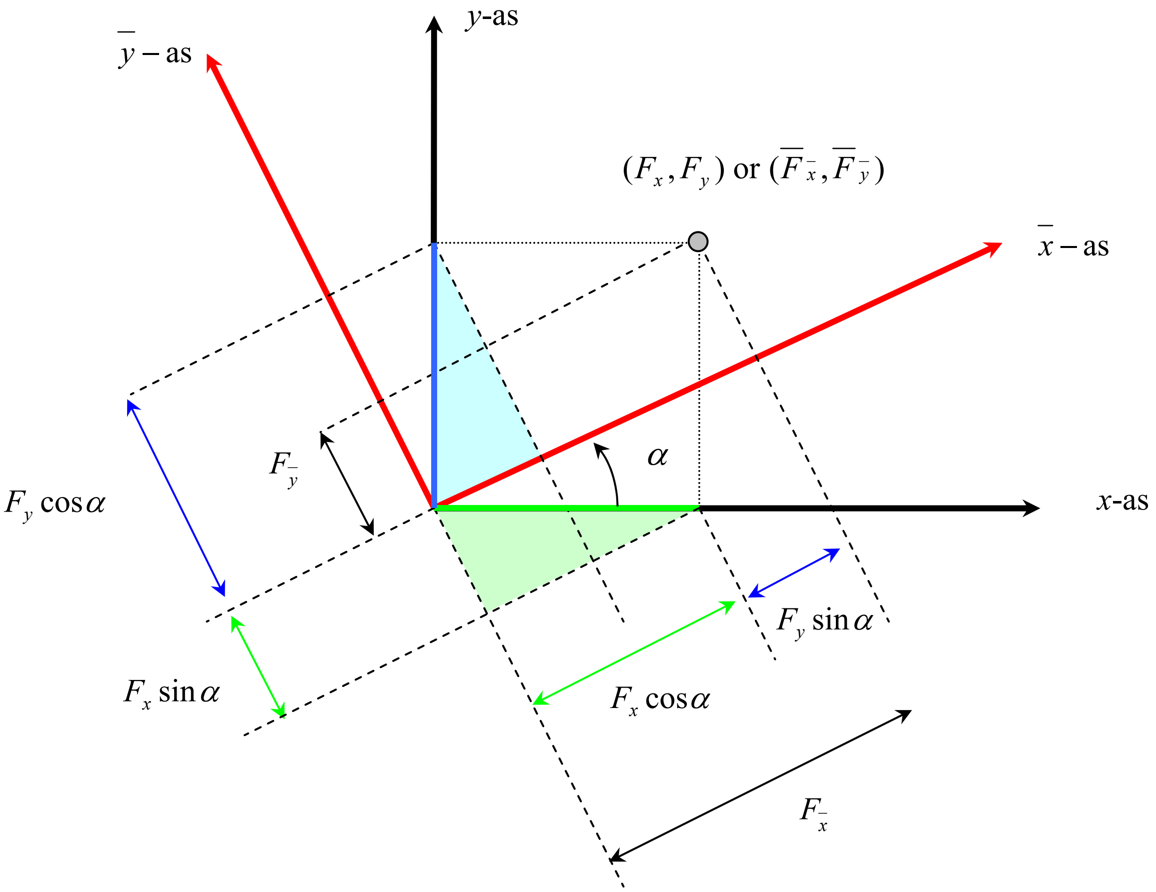


Figure 2.2 : Transformation

From the graph it can be seen that the vector $\boldsymbol{F}$ has different components in the $\bar{x}-\bar{y}$-coordinate system :

$$\bar{F}_{\bar{x}} = F_x \cos\alpha + F_y \sin\alpha$$
$$\bar{F}_{\bar{y}} = -F_x \sin\alpha + F_y \cos\alpha$$

This relation can be rewritten in matrix notation as:

$$\bar{F} = \mathrm{R} \,.\, F \qquad \text{with :} \quad \mathrm{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \qquad (2)$$

For the displacement, which is also a vector, the same transformation rule holds:

$$\bar{u} = \mathrm{R} \,.\, u \qquad (3)$$

With this result the transformation of a vector is clear. The backward transformation will be:

$$u = \mathrm{R}^{-1} \, . \, \overline{u}$$

For the transformation matrix R it holds that the inverse matrix is the same as the transposed matrix R^{T} (check this your self):

$$u = \mathrm{R}^{\mathrm{T}} \, . \, \overline{u} \qquad (4)$$

The main question now is how the stiffness matrix will transform due to a change of coordinate system. To find the transformation rule for the stiffness matrix we have to obtain the stiffness definition in the transformed coordinate system :

$$\overline{F} = \overline{\mathrm{K}} \, . \, \overline{u} \qquad (5)$$

First step is to use (2) and (1) :

$$\overline{F} = \mathrm{R} \, . \, F = \mathrm{R} \, . \, \mathrm{K} \, . \, u \qquad (6)$$

With use of (4) this expression can be rewritten as:

$$\overline{F} = \mathrm{R} \, . \, \mathrm{K}.u = \mathrm{R} \, . \, \mathrm{K} \, . \, \mathrm{R}^{\mathrm{T}} \, \overline{u} \qquad (7)$$

In this expression both the loadvector and the displacement vector are defined in the transformed coordinate system. With (5) it can be seen that the transformed stiffness matrix yields :

$$\overline{\mathrm{K}} = \mathrm{R} \, . \, \mathrm{K} \, . \, \mathrm{R}^{\mathrm{T}} \qquad (8)$$

This result can be worked out for the 2×2 example:

$$\overline{\mathrm{K}} = \begin{bmatrix} k_{\overline{xx}} & k_{\overline{xy}} \\ k_{\overline{yx}} & k_{\overline{yy}} \end{bmatrix} = \underset{\mathrm{R}}{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}} \underset{\mathrm{K}}{\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}} \underset{\mathrm{R}^{\mathrm{T}}}{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}$$

$$\begin{aligned}
k_{\overline{xx}} &= k_{xx}\cos^2\alpha + k_{xy}\sin\alpha\cos\alpha + k_{yx}\sin\alpha\cos\alpha + k_{yy}\sin^2\alpha \\
k_{\overline{xy}} &= -k_{xx}\sin\alpha\cos\alpha + k_{xy}\cos^2\alpha - k_{yx}\sin^2\alpha + k_{yy}\sin\alpha\cos\alpha \\
k_{\overline{yx}} &= -k_{xx}\sin\alpha\cos\alpha - k_{xy}\sin^2\alpha + k_{yx}\cos^2\alpha + k_{yy}\sin\alpha\cos\alpha \\
k_{\overline{yy}} &= k_{xx}\sin^2\alpha - k_{xy}\sin\alpha\cos\alpha - k_{yx}\sin\alpha\cos\alpha + k_{yy}\cos^2\alpha
\end{aligned}$$

These transformation rules can be simplified with the double-angle goniometric relations:

$$\begin{aligned}
2\cos^2\alpha &= 1+\cos 2\alpha \\
2\sin^2\alpha &= 1-\cos 2\alpha \\
2\sin\alpha\cos\alpha &= \sin 2\alpha
\end{aligned}$$

In to:

$$\begin{aligned}
k_{\overline{xx}} &= \tfrac{1}{2}(k_{xx}+k_{yy}) + \tfrac{1}{2}(k_{xx}-k_{yy})\cos 2\alpha + k_{xy}\sin 2\alpha \\
k_{\overline{yy}} &= \tfrac{1}{2}(k_{xx}+k_{yy}) - \tfrac{1}{2}(k_{xx}-k_{yy})\cos 2\alpha - k_{xy}\sin 2\alpha \\
k_{\overline{xy}} &= -\tfrac{1}{2}(k_{xx}-k_{yy})\sin 2\alpha + k_{xy}\cos 2\alpha
\end{aligned}$$

Stress example
A very similar result can also be obtained with a simple stress example. In the figure below a 2D-stress situation is given in the *x-y* plane in which we are looking for expressions for the normal and shear stresses on the inclined face with area *A* with a local coordinate system which is rotated with respect to the *x-y* system by α.

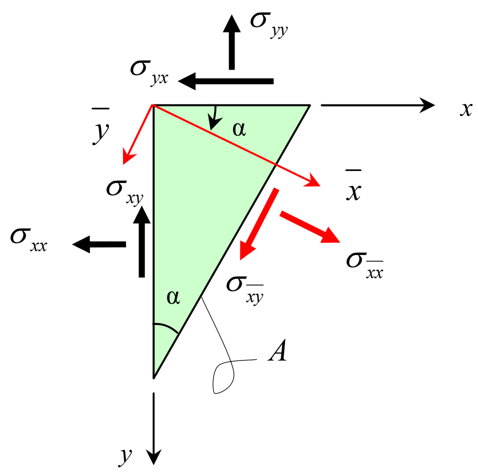


Figure 2.2b : Transformation of stresses

The resulting forces due to these stresses should satisfy the three equilibrium conditions for coplanar forces and moments on a rigid body. Moment equilibrium requires $\sigma_{xy} = \sigma_{yx}$ which reduces this system to two equilibrium conditions to be met:

horizontal equilibrium: $A\sigma_{\bar{x}\bar{x}}\cos\alpha - A\sigma_{\bar{x}\bar{y}}\sin\alpha = \sigma_{xx}A\cos\alpha + \sigma_{yx}A\sin\alpha$

vertical equilibrium: $A\sigma_{\bar{x}\bar{x}}\sin\alpha + A\sigma_{\bar{x}\bar{y}}\cos\alpha = \sigma_{xz}A\cos\alpha + \sigma_{yy}A\sin\alpha$

These equations can be rewritten as:

$$\sigma_{\bar{x}\bar{x}} = \sigma_{xx}\cos^2\alpha + \sigma_{yy}\sin^2\alpha + 2\sigma_{yx}\sin\alpha\cos\alpha$$
$$\sigma_{\bar{y}\bar{y}} = \sigma_{xx}\sin^2\alpha + \sigma_{yy}\cos^2\alpha - 2\sigma_{xy}\sin\alpha\cos\alpha$$
$$\sigma_{\bar{x}\bar{y}} = (\sigma_{yy} - \sigma_{xx})\sin\alpha\cos\alpha - \sigma_{yx}(\sin^2\alpha - \cos^2\alpha)$$

Using the double angle notation the transformation rules of the previous page are obtained.

Summary
The result is the transformation rule for a matrix which is based on the transformation rule of a vector.

$$\bar{F} = \mathrm{R}\,.\,F \quad \text{and} \quad \bar{u} = \mathrm{R}\,.\,u$$

$$F = \mathrm{R}^{\mathrm{T}}\,.\,\bar{F} \quad \text{and} \quad u = \mathrm{R}^{\mathrm{T}}\,.\,\bar{u} \qquad \text{with :} \quad \mathrm{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}; \quad \mathrm{R}^{\mathrm{T}} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\bar{\mathrm{K}} = \mathrm{R}\,.\,\mathrm{K}\,.\,\mathrm{R}^{\mathrm{T}}$$

$$k_{\bar{x}\bar{x}} = \tfrac{1}{2}(k_{xx} + k_{yy}) + \tfrac{1}{2}(k_{xx} - k_{yy})\cos 2\alpha + k_{xy}\sin 2\alpha$$
$$k_{\bar{y}\bar{y}} = \tfrac{1}{2}(k_{xx} + k_{yy}) - \tfrac{1}{2}(k_{xx} - k_{yy})\cos 2\alpha - k_{xy}\sin 2\alpha$$
$$k_{\bar{x}\bar{y}} = -\tfrac{1}{2}(k_{xx} - k_{yy})\sin 2\alpha + k_{xy}\cos 2\alpha$$

2.2 Tensors

In scientific publications the matrix notation is not used very often. The standard notation used is the tensor notation. The previous used vector notation can be written in a compact way with :

$$u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$u_i \qquad \text{with} : i = x, y, z$$

The same principle can be adopted for a matrix :

$$\mathrm{K} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}$$

$$\mathrm{K}_{ij} \qquad \text{with} : i, j = x, y, z$$

The short notation is not the only advantage. We can distinguish different rankings of tensors.

- A *first order tensor* is a vector which has:
  - a magnitude (length)
  - direction
  - transforms according to the earlier introduced transformation rule R

  The load and displacement vectors presented in the previous example are first order tensors.

- A *second order tensor* is a tensor that relates two first order tensors. If the components of a first order tensor ***F*** can be derived from the components of an other first order tensor ***u*** by means of a linear relation ***F***=K.***u*** then K is a second order tensor. The transformation rule for a second order tensor is the earlier presented relation $\mathrm{R.K.R}^T$.

If we can identify a linear relation as a second order tensor we know in advance that this relation transforms in the way a second order tensor transforms. From the earlier found definitions of the *stress* and *strain* it is now clear that these definitions are *second order tensors*. We can denote them as σ_{ij} and ε_{ij}.

2.3 Special mathematical properties of tensors

A first order tensor can be seen as a vector with a magnitude. Its direction is related to the coordinate system used but the magnitude or length of the vector is not. This length is called an invariant, it is constant with respect to any coordinate system used. Mathematicaly this means for the used stiffness example:

$$|F| = \sqrt{F_x^2 + F_y^2} = \sqrt{\overline{F}_x^2 + \overline{F}_y^2}$$

A second order tensor describes the relation between two first order tensors. Most likely these two vectors will not have the same direction. The question could be when will the direction of the two vectors coincide? This question can mathematically be presented as an eigenvalue problem:

$$F = \lambda u$$

Combined with (1) this results in:

$$F = \mathrm{K} \,.\, u = \lambda u$$

The rigth hand side of this expression can be written as a matrix with the use of the unity matrix I:

$$\mathrm{K} \,.\, u = \lambda \,.\, \mathrm{I} \,.u \quad \Leftrightarrow \quad (\mathrm{K} - \lambda \mathrm{I}) \,.\, u = 0$$

This latter equation is called the *eigenvalue problem*. Only for certain values of λ this system will have a non-trivial solution. The values of λ are the eigenvalues and the solutions of *u* the eigenvectors. If we apply this eigenvalue problem to the 2D stiffness example we will find:

$$\begin{bmatrix} k_{xx} - \lambda & k_{xy} \\ k_{yx} & k_{yy} - \lambda \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = 0$$

This homogeneous system of equations will only have a non zero soltution if the determinant of the matrix is zero:

$$\mathrm{Det} = (k_{xx} - \lambda)(k_{yy} - \lambda) - k_{xy} k_{yx} = 0$$

Since the non diagonal terms are equal (symmetrical matrix) we find the following characteristic polynomial which has to be zero.

$$(k_{xx} - \lambda)(k_{yy} - \lambda) - k_{xy}{}^2 = 0 \quad \Leftrightarrow$$
$$\lambda^2 - (k_{xx} + k_{yy})\lambda + (k_{xx} k_{yy} - k_{xy}^2) = 0$$

The solution of this equation yields:

$$\lambda_1, \lambda_2 = \tfrac{1}{2}(k_{xx} + k_{yy}) \pm \sqrt{\left[\tfrac{1}{2}(k_{xx} - k_{yy})\right]^2 + k_{xy}^2} \qquad (9)$$

For each solution of this eigenvalue λ_i an eigenvector u^i can be found. The eigen vectors are all independent of each other which means that they make straight angles with each other. The eigenvalues will always be the same irrespective of the choice of the coordinate system.

This means that the characteristic polynomial will always be the same regardless of the choice of the coordinate system. In order to obtain in every coordinate system the same characteristic polynomial, the constants of this polynomial are invariant and denoted with I_i:

$$\lambda^2 - I_1\lambda + I_2 = 0$$
$$I_1 = k_{xx} + k_{yy}$$
$$I_2 = k_{xx}k_{yy} - k_{xy}^2$$

The eigenvectors belonging to the eigen values form a base for the transformed matrix:

$$\overline{\mathrm{K}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

From a mathematical point of view this means that if the x-y-coordinate system is transformed into the coordinate system based on the eigenvectors u^i, the matrix K will transform into the presented matrix $\overline{\mathrm{K}}$. If one of the eigenvectors belonging to the first eigen value λ_1 is denoted as :

$$u^1 = \begin{bmatrix} u_{\bar{x}}^1 \\ u_{\bar{y}}^1 \end{bmatrix}$$

Suppose only a displacement in the $u_{\bar{x}}^1$ direction is imposed. The load will become:

$$\begin{bmatrix} F_{\bar{x}}^1 \\ F_{\bar{y}}^1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_{\bar{x}}^1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} u_{\bar{x}}^1 \\ 0 \end{bmatrix}$$

This result shows indeed a force which has the same direction as the imposed displacement.

If we translate this mathematical elaboration to the 2D stiffness example we have to try to find the direction of the force ***F*** such that this direction coincides with the observed displacment ***u***.

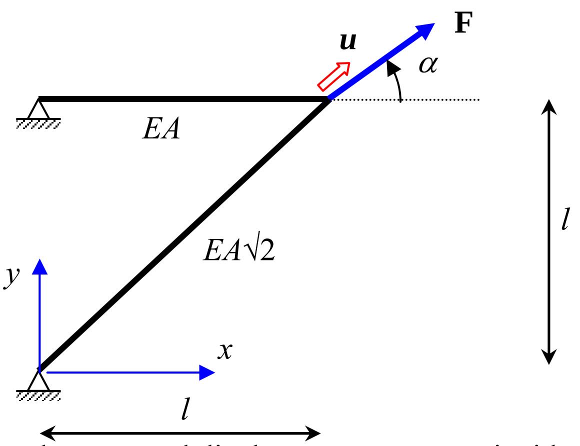


Figure 2.3 : Load vector and displacement vector coincide ?

From an engineering point of view this leaves us with two solutions :

- either we pull or push in the stiffest direction of the structure or
- we pull or push into the weakest direction of the structure

What is the stiffest direction and what will be the weakest direction ? In order to find out we can look at the components of the transformed stiffnessmatrix due to a change of coordinate system by the specified angle α. The components of the stiffness matrix will be (see also page 17):

$$\overline{\mathrm{K}} = \begin{bmatrix} k_{\overline{xx}} & k_{\overline{xy}} \\ k_{\overline{yx}} & k_{\overline{yy}} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}}_{\mathrm{R}} \underbrace{\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}}_{\mathrm{K}} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{\mathrm{R}^{\mathrm{T}}}$$

$$\begin{aligned} k_{\overline{xx}} &= \tfrac{1}{2}(k_{xx} + k_{yy}) + \tfrac{1}{2}(k_{xx} - k_{yy})\cos 2\alpha + k_{xy}\sin 2\alpha \\ k_{\overline{yy}} &= \tfrac{1}{2}(k_{xx} + k_{yy}) - \tfrac{1}{2}(k_{xx} - k_{yy})\cos 2\alpha - k_{xy}\sin 2\alpha \\ k_{\overline{xy}} &= -\tfrac{1}{2}(k_{xx} - k_{yy})\sin 2\alpha + k_{xy}\cos 2\alpha \end{aligned}$$

The diagonal terms will be extreme (maximum or minimum) if its derivative with respect to the angle α will be zero. If we elaborate this we will find:

$$\frac{\mathrm{d}k_{\overline{xx}}}{\mathrm{d}\alpha} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}k_{\overline{yy}}}{\mathrm{d}\alpha} = 0$$

This results in the following expression for the optimum angle α_o:

$$\tan 2\alpha_o = \frac{k_{xy}}{\tfrac{1}{2}\left(k_{xx} - k_{yy}\right)} \tag{10}$$

With this result the extreme or *principal values* of the diagonal terms become:

$$k_{1,2} = \tfrac{1}{2}\left(k_{xx} + k_{yy}\right) \pm \sqrt{\left[\tfrac{1}{2}\left(k_{xx} - k_{yy}\right)\right]^2 + k_{xy}^2} \tag{11}$$

This result is exactly the same as the earlier found expression (9). For the optimal direction of the angle α_o, the <u>non-diagonal terms will become zero</u>! This will result in the transformed matrix:

$$\overline{\mathrm{K}}_{\alpha} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

If the coordinate system is rotated by α_o and the load $\boldsymbol{F}$ is applied along one of the axis of this coordinate system, we find:

$$\begin{bmatrix} \overline{F_1} \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \overline{u_1} \\ 0 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 0 \\ \overline{F_2} \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} 0 \\ \overline{u_2} \end{bmatrix}$$

Which results in a displacement which has indeed the same direction as the load.

Finding the extreme or *principal values* of the stiffness matrix by rotating the coordinate system is apparently the same as solving the eigenvalue problem. For any second order tensor we now have found the tool to obtain it's extreme values by solving the eigenvalue problem.

If these results are applied to the 2D stiffness example, we can find the stiffest and weakest direction and its values for the given frame with the known stiffness relation:

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

We will find for the value of the angle α_o:

$$\tan(2\alpha_o) = \frac{2 \times \frac{1}{2}}{\frac{3}{2} - \frac{1}{2}} = 1 \quad \Rightarrow$$

two possible solutions for α_o :

$$\alpha_1 = 22{,}5^o, \alpha_2 = 112{,}5^o$$

$$k_1 = \frac{EA}{l}\left(1 + \tfrac{1}{2}\sqrt{2}\right)$$

$$k_2 = \frac{EA}{l}\left(1 - \tfrac{1}{2}\sqrt{2}\right)$$

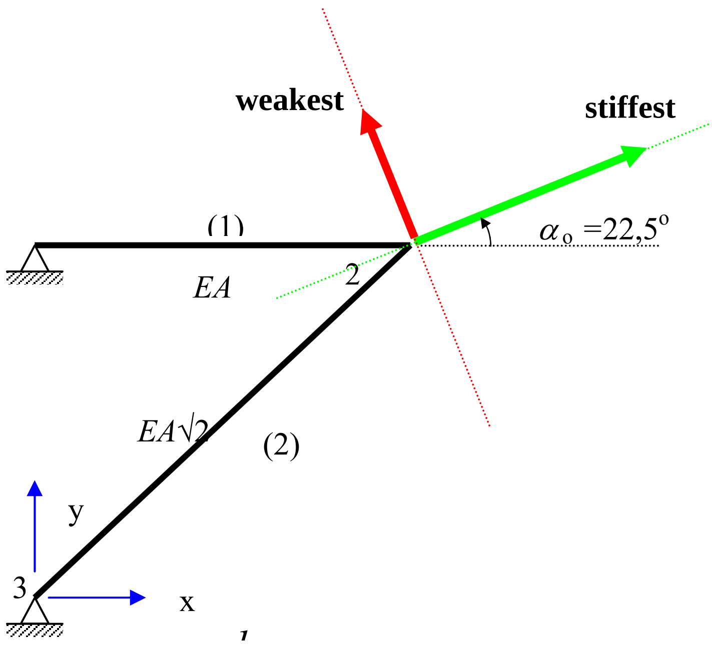


Figure 2.4 : Stiffest and weakest directions

2.3.1 Generalisation to 3D

The above presented theory can be extended to 3D. The eigenvalue problem is exactly the same however the order of the characteristic polynomial will increase. For a 3D stress tensor this will yield:

$$\begin{bmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \lambda \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = 0$$

The characteristic polynomial becomes:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

with :

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 + 2\sigma_{xy}\sigma_{yz}\sigma_{zx}$$

This second order stress tensor in 3D has three invariants. Regardless of the choice of base or coordinate system these invariants will be constant. For the special case that we choose the principal directions as a base the invariants can also be presented in terms of the principal stresses:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

2.4 Mohr's graphical Circle Method

The transformation rules we found provide a tool to calculate the components of a tensor due to a rotation α of the coordinate system :

$$\begin{aligned}
k_{\bar{x}\bar{x}} &= k_{xx}\cos^2\alpha && + k_{xy}\sin\alpha\cos\alpha && + k_{yx}\sin\alpha\cos\alpha && + k_{yy}\sin^2\alpha \\
k_{\bar{x}\bar{y}} &= -k_{xx}\sin\alpha\cos\alpha && + k_{xy}\cos^2\alpha && - k_{yx}\sin^2\alpha && + k_{yy}\sin\alpha\cos\alpha \\
k_{\bar{y}\bar{x}} &= -k_{xx}\sin\alpha\cos\alpha && - k_{xy}\sin^2\alpha && + k_{yx}\cos^2\alpha && + k_{yy}\sin\alpha\cos\alpha \\
k_{\bar{y}\bar{y}} &= k_{xx}\sin^2\alpha && - k_{xy}\sin\alpha\cos\alpha && - k_{yx}\sin\alpha\cos\alpha && + k_{yy}\cos^2\alpha
\end{aligned} \tag{1}$$

The extreme values of the tensor can be obtained with:

$$k_{1,2} = \tfrac{1}{2}\left(k_{xx} + k_{yy}\right) \pm \sqrt{\left[\tfrac{1}{2}\left(k_{xx} - k_{yy}\right)\right]^2 + k_{xy}^2} \tag{2}$$

These values occur for an optimum angle α_o of :

$$\tan 2\alpha_o = \frac{k_{xy}}{\tfrac{1}{2}\left(k_{xx} - k_{yy}\right)} \tag{3}$$

Mohr discovered from the above equations (2) and (3) a specific graphical presentation. If a coordinate system as shown below is defined, the values from equation (2) can be presented as points in this coordinate system.

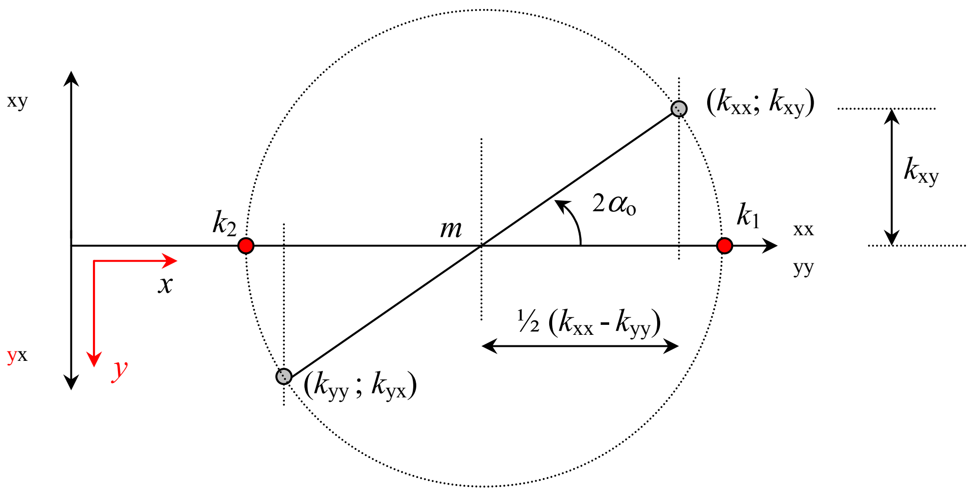


Figure 2.5 : Mohr's Circle, definition of coordinate system

This specific coordinate system needs some explanation. The diagonal terms k_{xx} and k_{yy} of the tensor k_{ij} are placed on the horizontal axis. The vertical abcis contains the non diagonal terms k_{xy} and k_{yx}. Although the values for these non diagonal terms are equal a distinction is being made here. The tensor k_{ij} is presented with two points (k_{xx}; k_{xy}) and (k_{yy}; k_{yx}) in the presented coordinate system.

Special attention is needed for the direction of a positive k_{yx} and k_{xy}. The rule here is that if the y-axis is pointing downward the positive direction of k_{yx} is also downward. This rule can be remembered with the indicated red y's.

The circle through the tensor points is called Mohr's circle. It has a centre point m and radius r which can be derived from the graph as:

$$m = \tfrac{1}{2}\left(k_{xx} + k_{yy}\right)$$
$$r = \sqrt{\left(\tfrac{1}{2}(k_{xx} - k_{yy})\right)^2 + k_{xy}^2}$$

Both the *centre point* and the *radius* can be related to the *two invariants* by:

$$m = \tfrac{1}{2} I_1 \qquad \text{with} : I_1 = k_{xx} + k_{yy}$$
$$r = \sqrt{\left(\tfrac{1}{2} I_1\right)^2 - I_2} \qquad \text{with} : I_2 = k_{xx}k_{yy} - k_{xy}^2$$

The principal values k_1 and k_2 can also be presented as points of intersection of Mohr's circle with the *xx*/*yy*-axis.

From the graph is clear that equation (2) can be obtained with :

$$k_1 = m + r$$
$$k_2 = m - r$$

By default the first principal value is the most positive one.

The graphical meaning of the principal values is clear now. But what about the principal directions (equivalent with the eigen vectors) ? In order to find the principal directions defined as the directions in which the given tensor becomes extreme, we need a special point on the circel defined as the Directional Centre **DC**. Some times this point is also referred to as the pole of the circle.

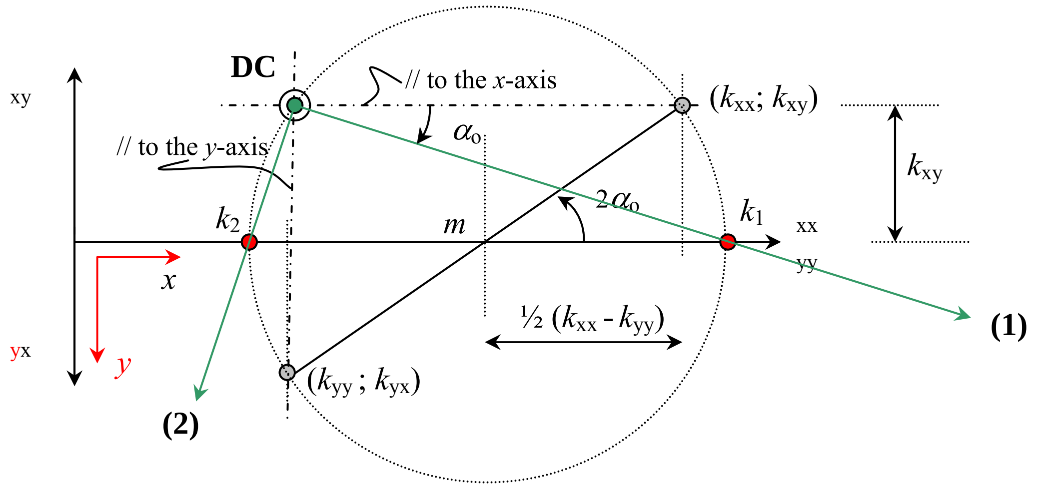


Figure 2.6 : Mohr's circle, definition of Directional Centre **DC**

The position of the DC on the circle can be found by :

- Drawing a line parallel to the x-axis through the tensor point (k_{xx}; k_{xy})
- Drawing a line parallel to the y-axis through the tensor point (k_{yy}; k_{yx})
- The intersection of the two lines and the circle is the Directional Centre **DC**

With this DC the principal directions can be found with:

- Draw a line through **DC** and k_1, this is the first principal direction, denoted as **(1)**
- Dra a line through **DC** and k_2, this is the second principal direction, denoted as **(2)**

From elementary mathematics it is known that the inner angle **DC** – k_1 – (k_{xx}; k_{xy}) is equal to half the centre point angle m – k_1 – (k_{xx}; k_{xy}). From the graph it can be seen that this inner angle is equal to α_o. The direction of (1) is either from the DC to the point k1 or from k1 to the DC. With the choosen direction the direction of (2) is fixed due to the definition of the coordinate system, thus (1) and (2) must have the same orientation as x and y. In the figure below both valid solutions are presented.

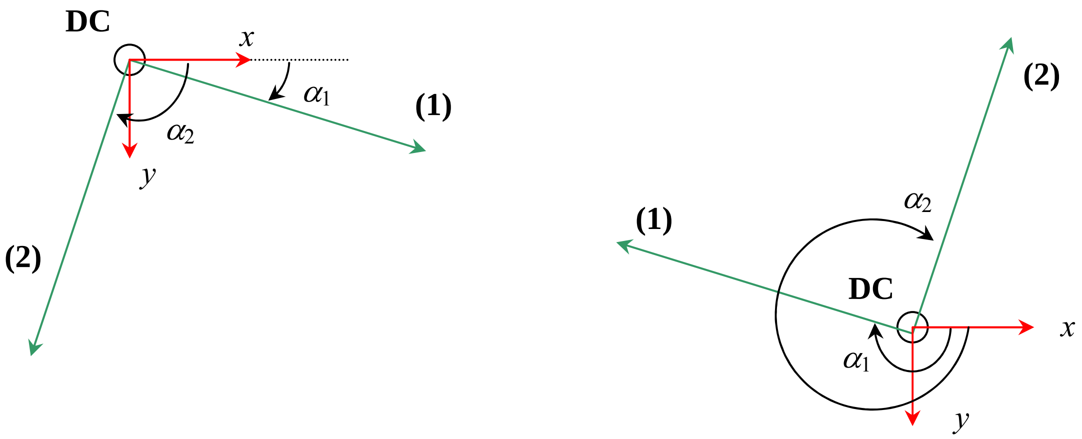


Figure 2.7 : Valid principal directions

The Directional Centre **DC** acts like a kind of hinge. If we rotate the current coordinate system over an angle α keeping its origin in the **DC** the tensor points will move along the circle to the new transformed points. In this way the transformation rules (1) of page 21 for the components of the tensor can be obtained in a graphical way.

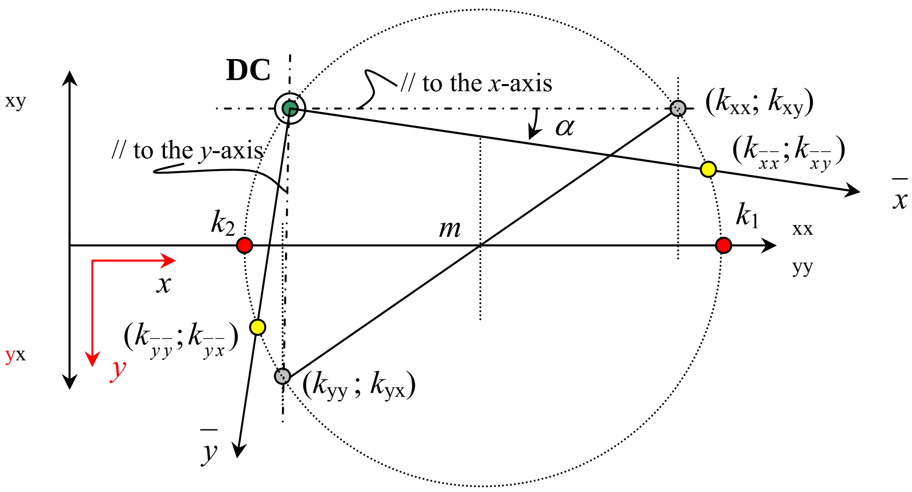


Figure 2.8 : Tensor transformation with Mohr's circle

Mohr's graphical method is best applicable to 2D tensors. Some special applications for 3D stress tensors exist and will be presented in the next chapter.

Summary

An alternative method for the tensor transformation rule is Mohr's Circle Method. The principal values and the principal directions can be obtained directly from the circle. The method has its own dedicated coordinate definition. Essential is the correct location of the Directional Centre **DC** which acts like a hinge for the rotation of the coordinate system.

2.5 Example of Mohr's Circle Method

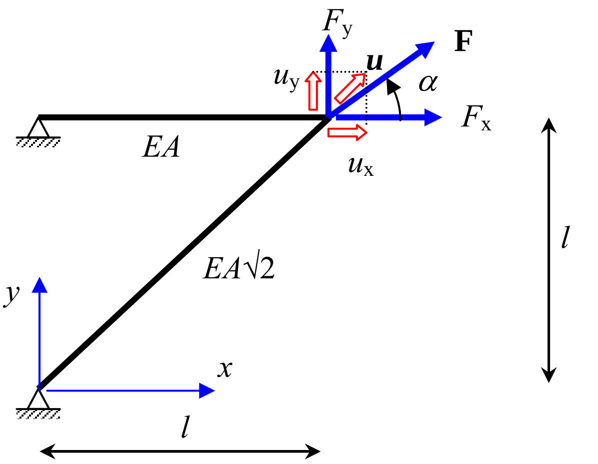
In this paragraph some examples will be shown. We start with the earlier defined stiffness problem. This problem has already been solved with the transformation formulas but here we will apply Mohr's graphical method. The second example will be a given stress situation from which we will try to find the maximum stresses (principal stresses) and the according principal directions.

2.5.1 Stiffness example

The stiffness matrix for the stiffness problem in the specified x-y-coordinate system can be described with:

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

The stiffness tensor can be represented in a graphical way by two points in the special Mohr-coordinate system. Both points will be on Mohr's Circle.



in a

Figure 2.9 : Stiffness problem

If we want to draw this circle we will have to follow the next steps :

1) Draw the combined *xx*- and *yy*-axis.
2) Put the *x*-*y* coordinate definition at the origin.
3) Draw the vertical axis and denote the *yx*-axis in the same direction as the *y*-axis.
4) Draw the veritcal *xy*-axis in the opposite direction.

These first four steps are visualised in the next graph:

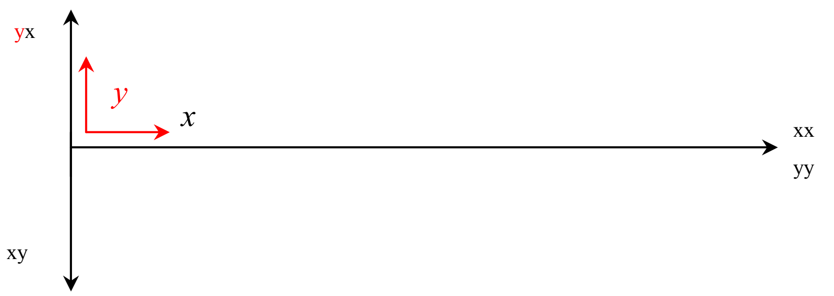


Figure 2.10 : Definition of Mohr's graph

With this setup we can continue with:

5) Put the tensor components (k_{xx}; k_{xy}) and (k_{yy}; k_{yx}) as points in the graph.
6) Draw a line through these two points.
7) Draw through the half way point a perpendicular line.
8) The point of intersection of this latter line with the x-axis is the centre m of the circle.
9) Draw a circle with centre point m and through both points (k_{xx}; k_{xy}) and (k_{yy}; k_{yx}).

These steps have been realised in the next graph.

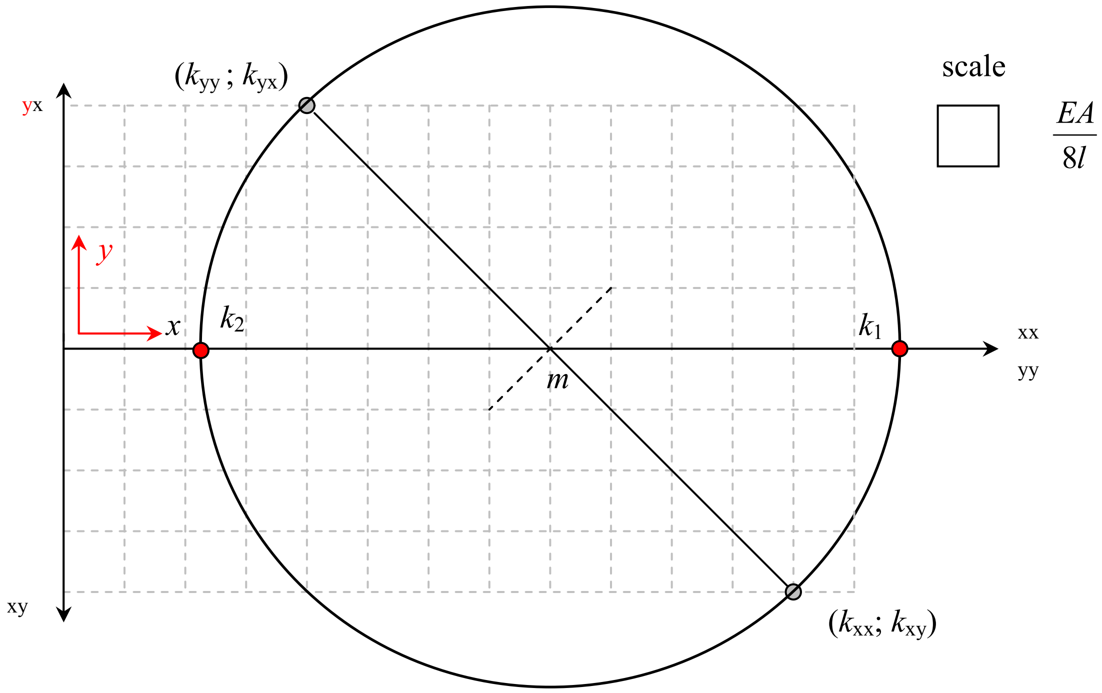


Figure 2.11 : Mohr's Circle for the stiffness tensor and principal values

In this graph the stiffness tensor k_{ij} has been presented as two points on Mohr's Circle. The graphical method for transformations is based on the fact that any transformation of the stiffness tensor will produce points which are on the circle. In the extreme situation where all non diagonal terms become zero (eigen value problem) the principal values k_1 and k_2 are found.

In order to find the principal directions which belong to these principal values we need the position of the Directional Centre **DC** on the circle. From the earlier given definition of the **DC** we have to follow the next 3 steps :

10) Draw a line parallel to the x-axis through the point (k_{xx}; k_{xy})[1].
11) Draw a line parallel to the y-axis through the point (k_{yy}; k_{yx}).
12) The intersection of these two lines on the circle is the Directional Centre **DC**.

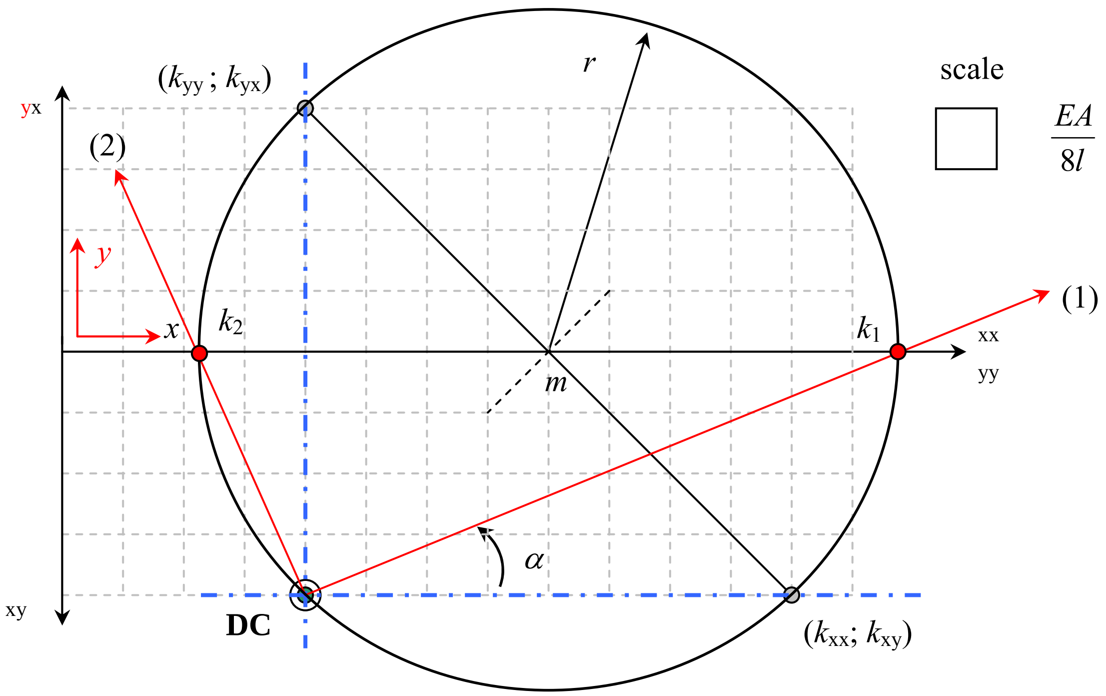


Figure 2.12 : Directional Centre **DC** and principal directions

To find the principal directions we draw lines from the **DC** through k_1 to obtain the first principal direction (1) and a line from the **DC** through k_2 to obtain the second principal direction (2).

> *Remark :*
> *The (1) – (2) directions have the same orientation as the x-y-coordinate system.*

From the graph it can be seen that the x-y-coordinate system can be transformated into the principal coordinate system (1)-(2) by rotating the x-y-system by α. This angle becomes:

$$\alpha = 22{,}5^o$$

With the circle and the position of the **DC** any transformation of the coordinate system can be investigated.

[1] In step 10 and 11 the line through the tensor point should be drawn parallel to the associated axis as indicated with the first sub index. In this example the associated axis is the x-axis respectively the y-axis but in general this is not the case. Therefore we must always use the direction of the first index of the tensor components used.

As an example we will find the transformed stiffness tensor for a rotation of the *x*-*y*-coordinate system of 45°.

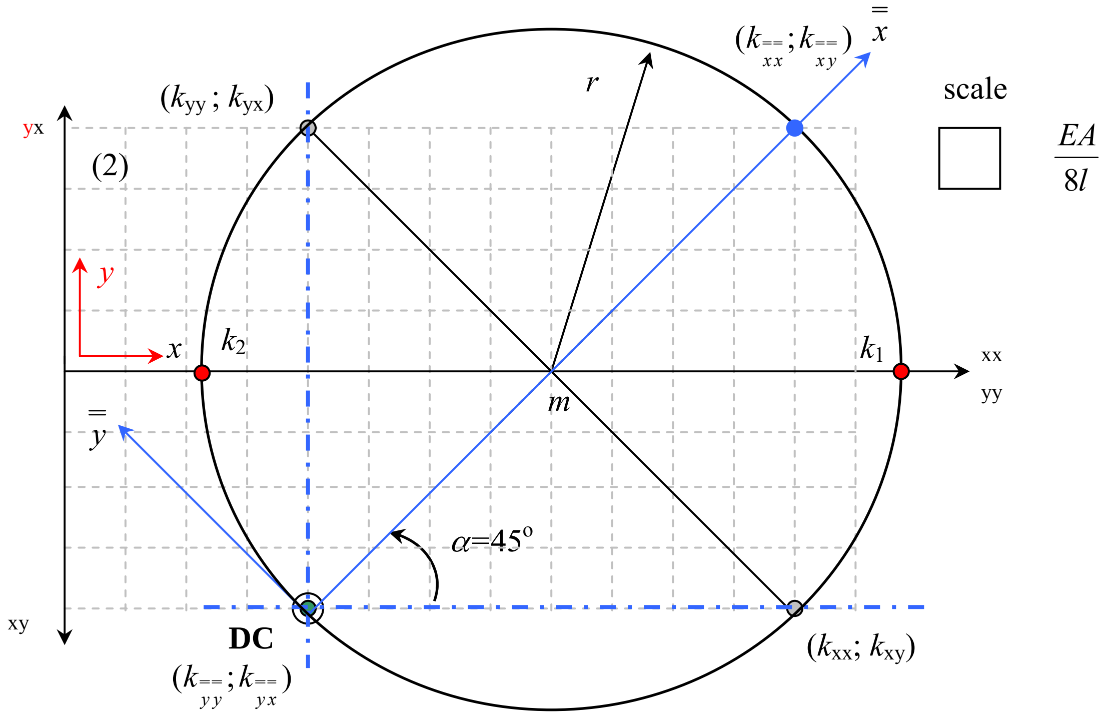


Figure 2.13 : Tensor transformation due to rotation of 45 degrees

The transformed coordinate system is denoted with $\bar{\bar{x}}-\bar{\bar{y}}$. This coordinate system is in fact the rotated *x*-*y*-coordinate system around the hinge **DC**. The intersections of the rotated axis of the coordinate system with the circle are the tranformed tensor components:

$$(k_{\bar{\bar{x}}\bar{\bar{x}}};k_{\bar{\bar{x}}\bar{\bar{y}}})$$

$$(k_{\bar{\bar{y}}\bar{\bar{y}}};k_{\bar{\bar{y}}\bar{\bar{x}}})$$

The values of these transformed components are read from the horizontal and vertical axis. This yields for these two points :

$$(k_{\bar{\bar{x}}\bar{\bar{x}}};k_{\bar{\bar{x}}\bar{\bar{y}}})=\left(\frac{3EA}{2l};-\frac{EA}{2l}\right)$$

$$(k_{\bar{\bar{y}}\bar{\bar{y}}};k_{\bar{\bar{y}}\bar{\bar{x}}})=\left(\frac{EA}{2l};-\frac{EA}{2l}\right)$$

Remark: Pay attention to the minus sign !

One of the two points coincides with the Directional Centre **DC** which is a coincidence.

2.5.2 Stress example

Mohr's method can also be used to find the extreme or principal stresses of a homogeneous loaded specimen under a plane stress situation. The principal stress occurs on planes on which the shear stresses are zero. We found that in the previous paragraphs dealing with the tensor transformation rules. In the figure below a loaded test specimen is shown from which the stresses are only known on two faces of the specimen. The material thickness *t* is constant. The specimen is loaded with a **homogenous plane stress situation,** which means that we assume the same stress situation for every point of the specimen and zero stresses on the *z*-plane of the specimen.

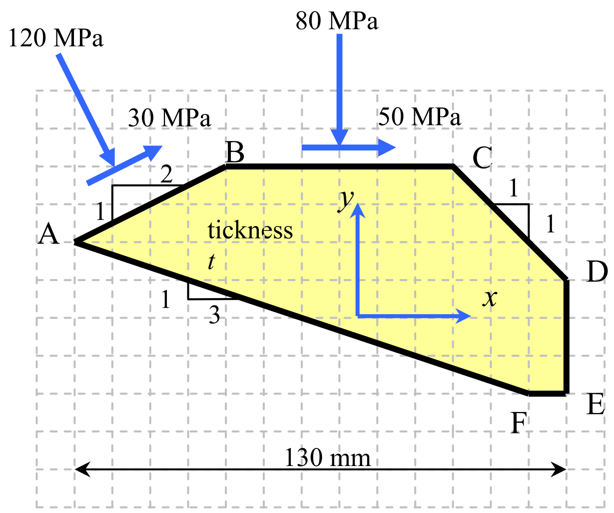


Figure 2.14 : Stresses on a specimen

Given : - on plane AB acts a compressive stress of 120 N/mm^2 and a shear stress of 30 N/mm^2. On plane BC acts a compressive stress of 80 N/mm^2 and a shear stress of 50 N/mm^2.

From this specimen we would like to know all the stresses on the other faces. The stresses on the faces AB and BC can be presented as two points in Mohr's circle. With two points a circle can be drawn and for any rotation of the coordinate system the transformed stresses can be presented as a point on this circle. In this way we can find the stresses acting on the faces AF, FE, ED and CD. In order to do so we will however need to know the Directional Centre **DC,** which acts like a hinge or pole for the rotating coordinate system.

The two faces on which the stresses are known are not perpendicular to each other. Therefore we can not use one single coordinate system to describe the four known stresses. For each surface however we can introduce a local coordinate system as is shown below.

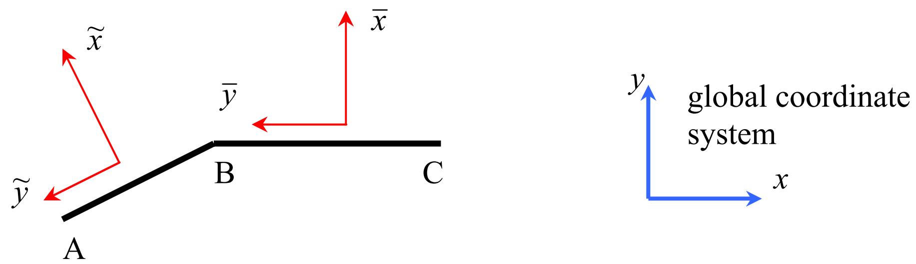


Figure 2.15 : Local coordinate system for AB and BC

From these definitions of the local coordinate systems we can denote the stresses on AB and BC as:

$$\text{AB} \quad (\sigma_{\tilde{x}\tilde{x}};\sigma_{\tilde{x}\tilde{y}}) = (-120;-30)$$
$$\text{BC} \quad (\sigma_{\bar{x}\bar{x}};\sigma_{\bar{x}\bar{y}}) = (-80;-50)$$

Since the normal stresses are negative (compression) we expect a circle mainly on the left side of the origin. The positive *xy*-axis is downward due to the choosen global coordinate system. Both shear stresses are negative and therefore the points in the graph are on the negative side of the vertical axis.

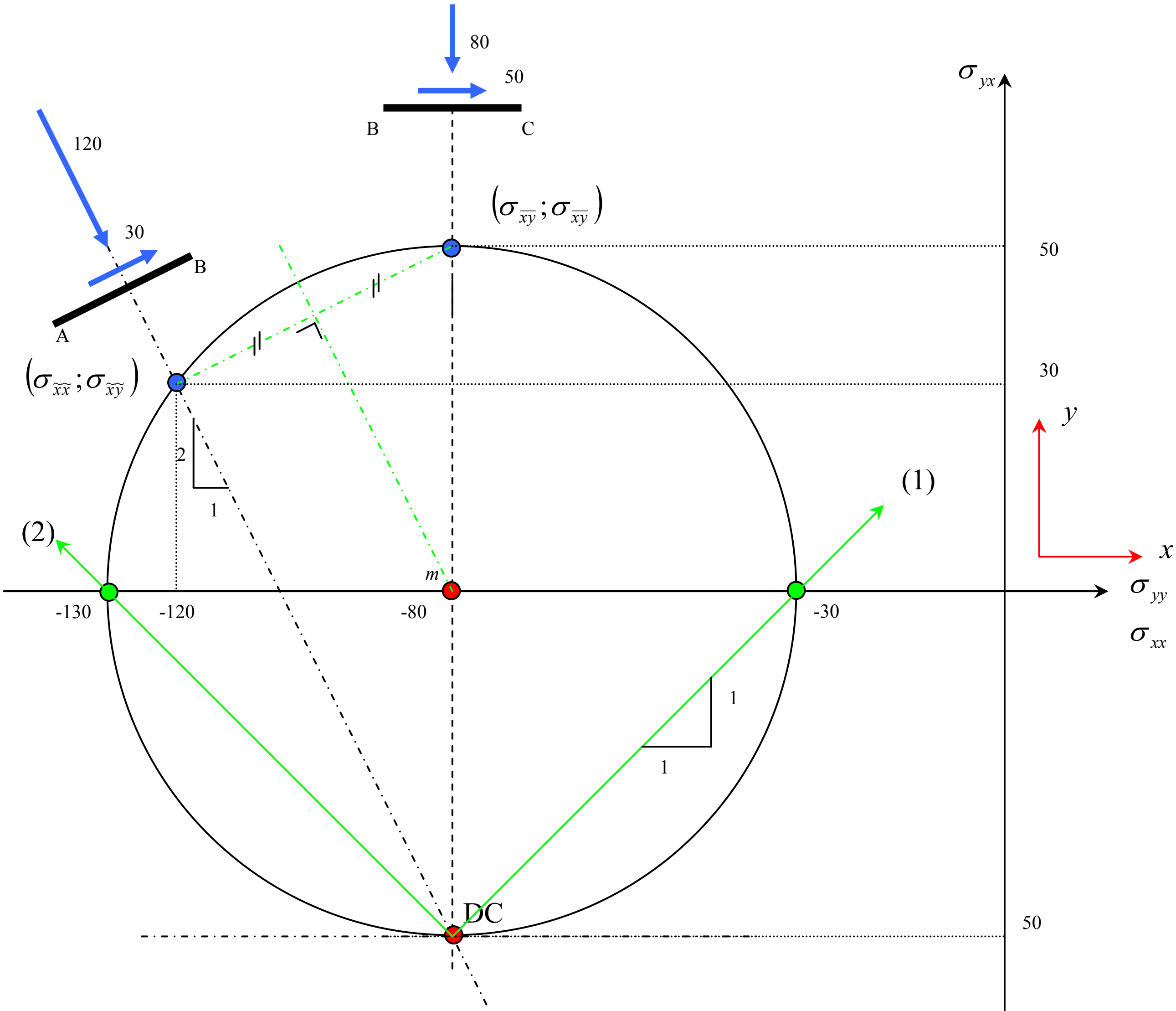


Figure 2.16 : Mohr's stress circle

The centre point of the circle can be found with the previously described steps 6-8:

6) Draw a line through the two known stress points .
7) Draw a perpendicular line throught the half way point.
8) The point of intersection of this latter line with the *x*-axis is the centre *m* of the circle.

The circle through the two known stress points and with centre point *m* can now be drawn.
The Directional Centre **DC** can be found with steps 10-12:

10) Draw a line parallel to the $\tilde{x}$ – axis through the point $\left(\sigma_{\tilde{x}\tilde{x}};\sigma_{\tilde{x}\tilde{y}}\right)$.

11) Draw a line parallel to the $\bar{x}$ – axis through the point $\left(\sigma_{\bar{x}\bar{y}};\sigma_{\bar{x}\bar{y}}\right)$.

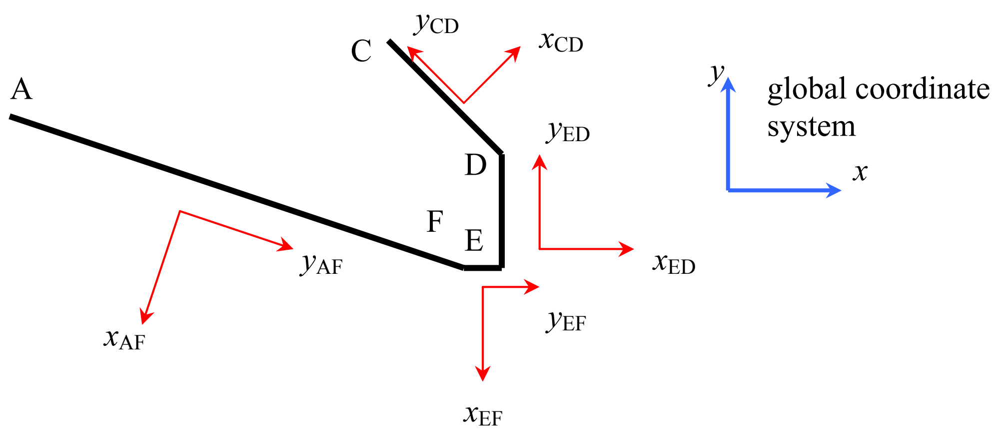
12) The intersection of these two lines on the circle is the Directional Centre **DC**.

To find the principal directions (1) and (2) we can draw lines from the DC through the extreme values σ_1 and σ_2 on the horizontal axis. This is also shown in figure 2.16. The orientation of the (1)-(2) coordinate system is the same as the global coordinate system. The result is:

$$\sigma_1 = -30 \quad \text{N/mm}^2$$

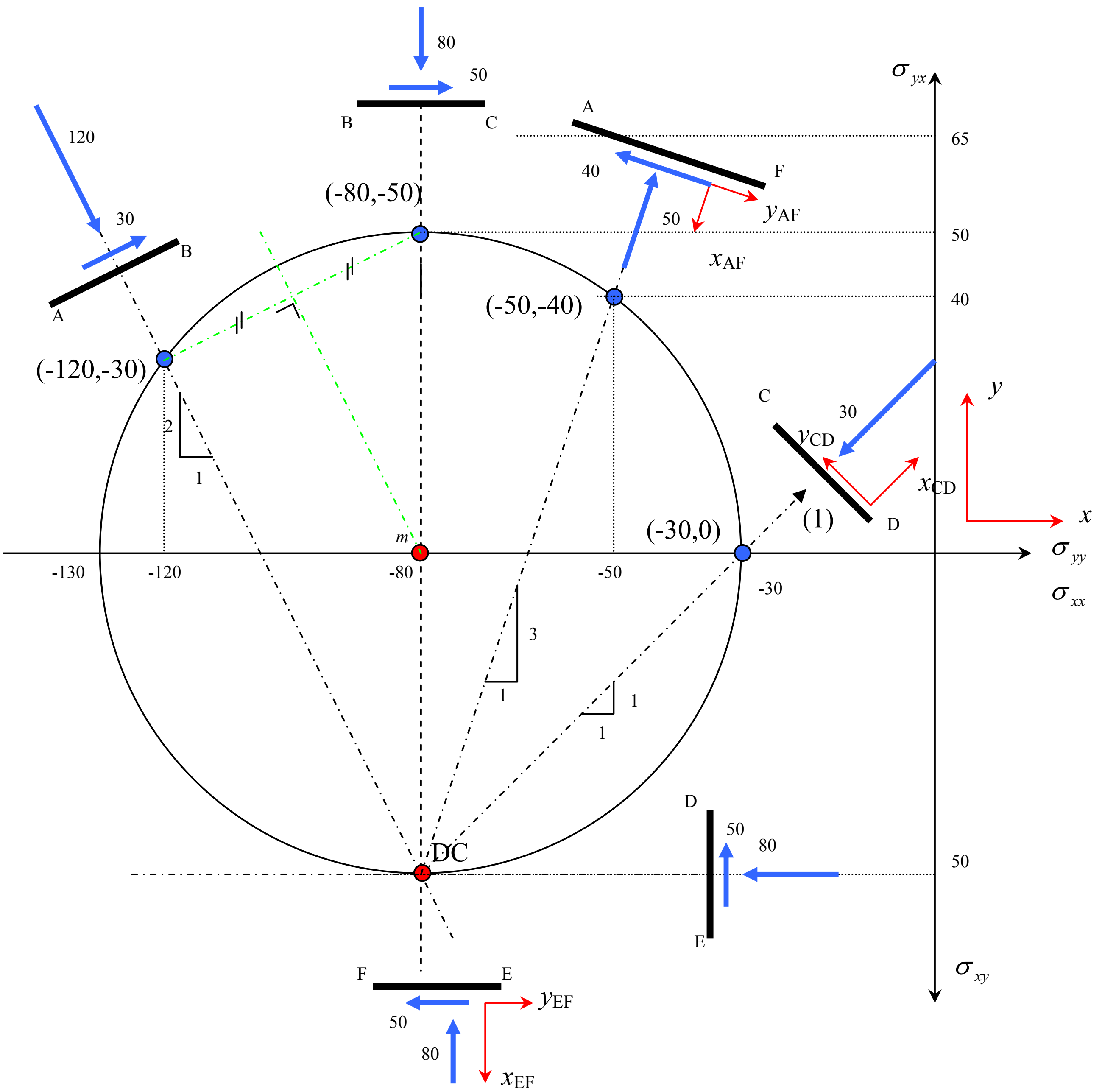
$$\sigma_2 = -130 \quad \text{N/mm}^2$$

In order to find the stresses on the other four sides we start with defining local coordinate systems on these sides and draw a line parallel to the local *x*-axis through the DC. The point of intersection with the circle will be the stress point. Normally there are two points of intersection. The point opposite the DC will be the stress point. If the DC is the only point of intersection, this will be the required point. The values can be read on the horizontal and vertical axis. The definitions of the local axis are shown in figure 2.17.



Figuur 2.17 : Local coordinate systems

With these definitions of the local global axis figure 2.18 can be obtained.

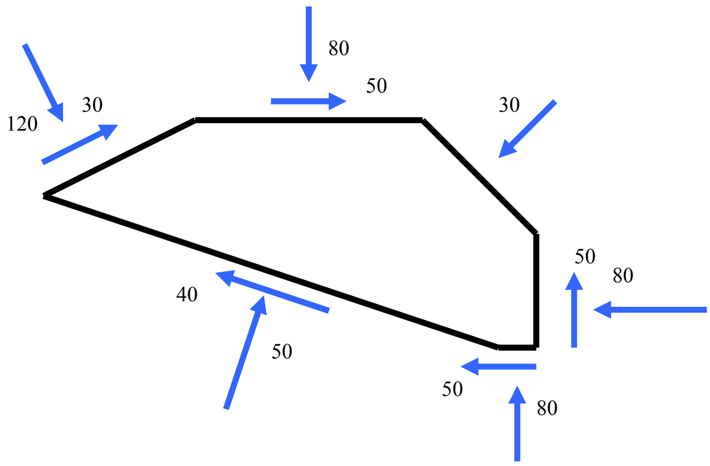


Figuur 2.18 : Stresses on all faces

The stresses found, as they act, on all faces are shown in figure 2.19.

Assignment :

Check the equilibrium of this specimen.



2.5.3 Strain example

Mohr's method can also be used to find the extreme or principal strains for a plane strain situation. From the specimen shown below the displacement field is known and given as:

$$u_x = 0{,}2\times10^{-4} + 0{,}3\times10^{-4}\,x$$
$$u_y = 2\times10^{-4}\,x + 1{,}8\times10^{-4}\,y$$

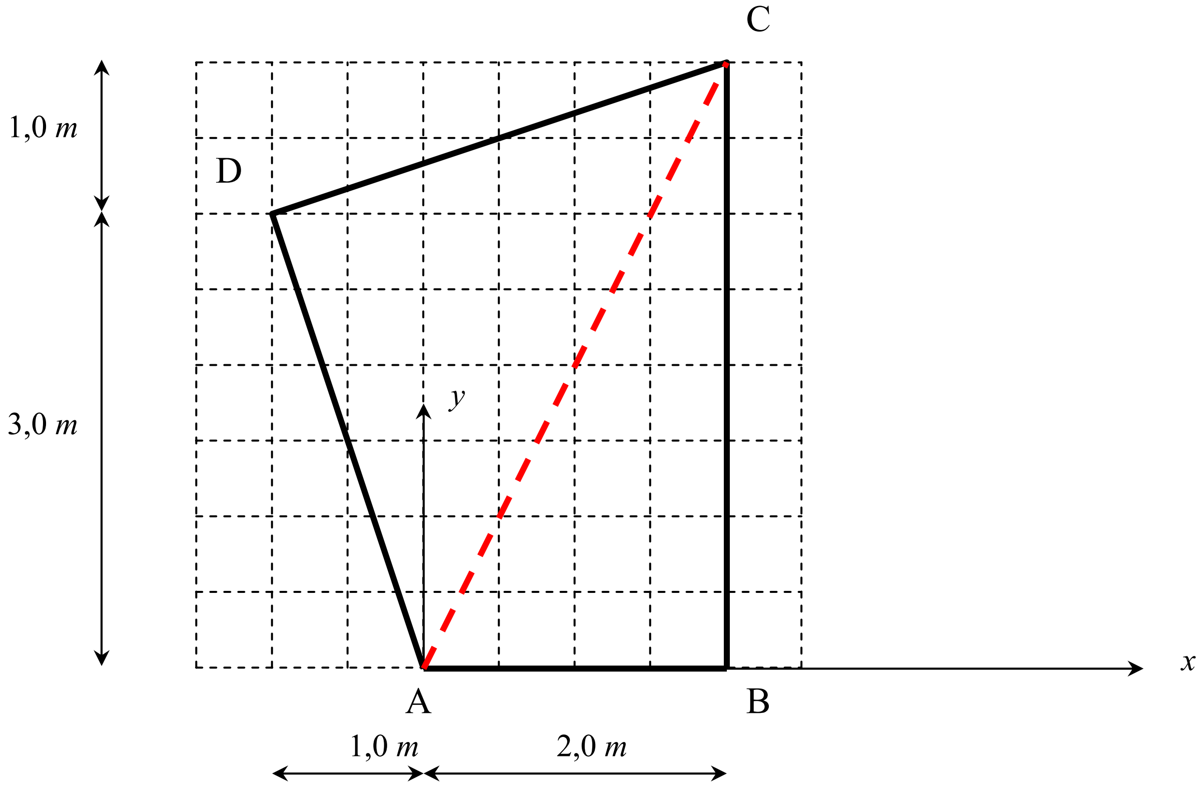


Figure 2.19 : Strain example

For any fibre the strain can be found with Mohr's circle. In this example we will compute the strain in the fibre parallel to AC.

In order to do so we have to construct Mohr's circle. From the displacement field the strain tensor can be derived with the earlier found formula :

$$\varepsilon_{ij} = \tfrac{1}{2}\frac{\partial u_i}{\partial u_j} + \tfrac{1}{2}\frac{\partial u_j}{\partial u_i} \quad \text{with :} \quad i,j = x,y$$

This results in:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0{,}3\times10^{-4}$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 1{,}8\times10^{-4}$$
$$\varepsilon_{xy} = \tfrac{1}{2}\times0 + \tfrac{1}{2}\times2{,}0\times10^{-4} = 1{,}0\times10^{-4}$$

This tensor can be represented by two points in Mohr's graph which should by on a circle. Both points can be denoted as:

$$\left(\varepsilon_{xx}, \varepsilon_{xy}\right) = \left(0{,}3 \times 10^{-4}; 1{,}0 \times 10^{-4}\right)$$
$$\left(\varepsilon_{yy}, \varepsilon_{yx}\right) = \left(1{,}8 \times 10^{-4}; 1{,}0 \times 10^{-4}\right)$$

> <u>*Important to note:*</u>
> *The directions in the strain circle are always the directions of the fibres. For the stresses we use as direction the normal to the plane on which the stresses act. Strains in the direction of the fibres are read from the horizontal axis, the vertical axis shows half the shear deformation.*

In figure 2.20 the diagram is shown.

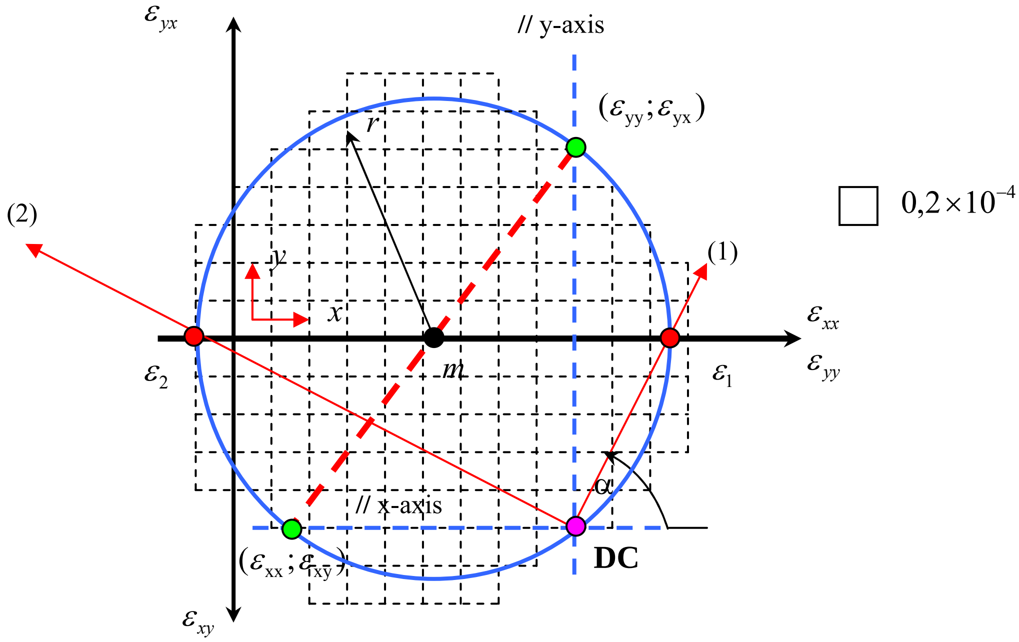


Figure 2.20 : Mohr's strain circle

From the diagram the principal values and the principal directions can be obtained :

$$\varepsilon_1 = 2{,}3 \times 10^{-4} \qquad \text{and} \qquad \tan\alpha = \frac{2}{1} = 1 \quad \Rightarrow \alpha = 63^0$$
$$\varepsilon_2 = -0{,}2 \times 10^{-4}$$

Fibres parallel to AC have directions which coincide with the principal direction (1). The strain in this direction of these fibres is therefore the principal strain ε_1.

> <u>*Assignment:*</u>
> *Check with the given displacements in A and C the strain in fibre AC as found with Mohr's circle. The answer is given in the* APPENDIX.

The straining of any fibre can be found in this way by drawing a line through DC parallel to the direction of the fibre. The point of intersection with the circle is the representation of the transformed strain tensor components. On the horizontal axis the strain can be read.

Apart from straining we can also look at the shear deformation. If we want to know the change of right angle ADC we can for example take the fibre AD. In figure 2.21 the direction of fibre AD is shown in Mohr's circle.

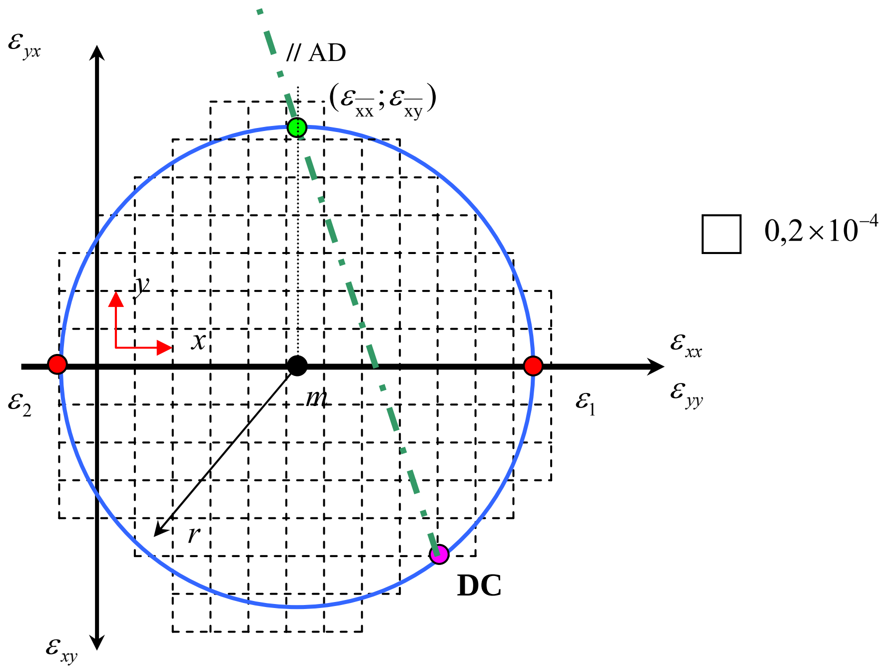


Figure 2.21 : Fibre AD

For fibres parallel to AD a local $\bar{x}-\bar{y}-$coordinate system has been introduced in which the local $\bar{x}-$axis coincides with the direction of the fibre. From the circle can be read:

$$\varepsilon_{\overline{xx}} = 1{,}05\times10^{-4}$$
$$\varepsilon_{\overline{xy}} = -1{,}25\times10^{-4}$$

The change of the right angel ADC is the shear deformation which is defined as:

$$\gamma_{\overline{xy}} = 2\varepsilon_{\overline{xy}} = -2{,}5\times10^{-4}$$

The minus sign is here not very relevant since the amount of change was asked. From the definition of figure 2.22 (left) however we can see that the angle ADC will become larger due to the negative shear deformation.

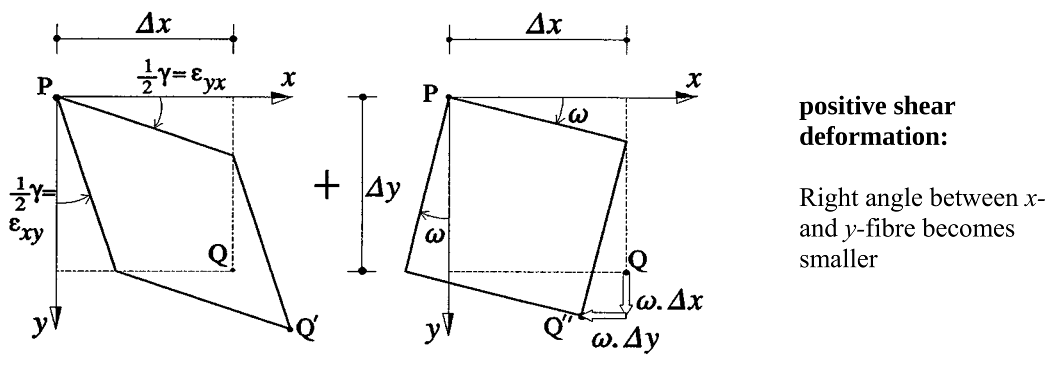


Figure 2.22 : Shear deformation

3. Assignments

With the presented theory the following questions can be answered.

Problem 1

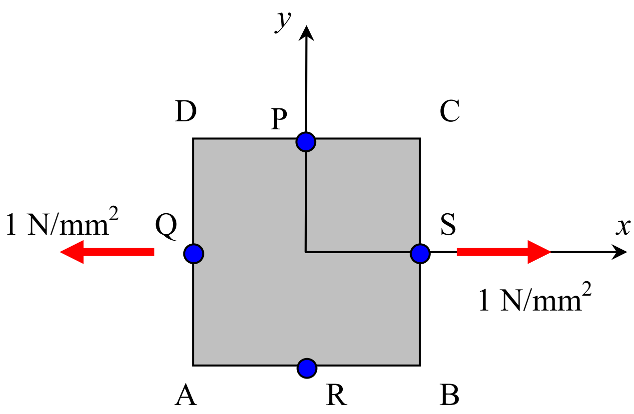


Figure 3.1 : Stress situation, problem 1

a) Draw Mohr's stress circle for the stress situation given in figure 3.1.
b) Find the stresses for a plane with an angle of 45o with the x-axis.
c) Check the equilibrium for a specimen PQRS.

Problem 2

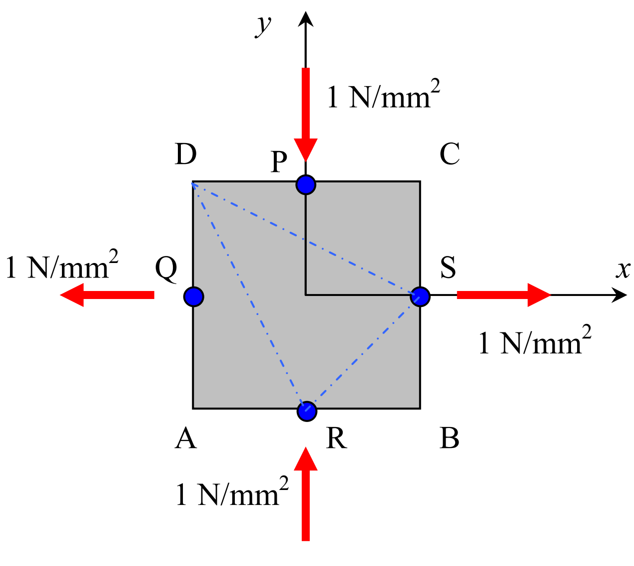


Figure 3.2 : Stress situation, problem 2

a) Draw Mohr's stress circle for the stress situation given in figure 3.2.
b) Find the principal stresses and principal direction
c) Find the stresses on plane PS
d) Find the stresses on the planes of DSR.
e) Check the equilibrium for a specimen DSR.

Problem 3

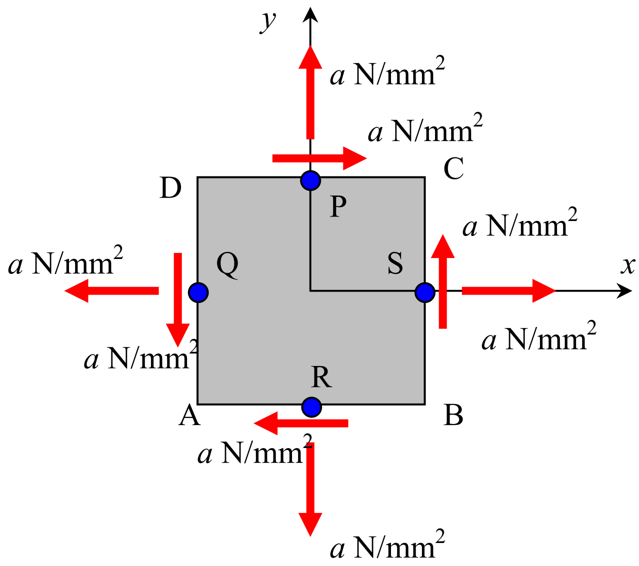


Figure 3.3 : Stress situation, problem 3

a) Draw Mohr's stress circle for the stress situation given in figure 3.3.
b) Find the principal stresses and principal direction
c) Find the stresses on the planes PA and CR.
d) Check the equilibrium for a specimen APCR.

Problem 4

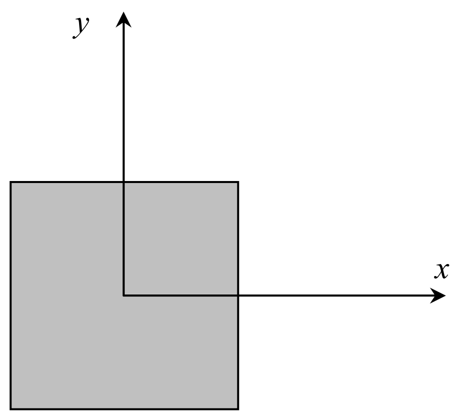


Figure 3.4 : Specimen and coordinate system of problem 4

For the presented specimen of figure 3.4 and the defined coordinate system the following stresses are given: $\sigma_{xx} = 8$ MPa; $\sigma_{yy} = -2$ MPa; $\sigma_{xy} = 0$

a) Draw Mohr's stress circle for the given stress situation.
b) Find the planes with the maximum shear stress.
c) Find the planes with zero normal stresses, show the shear stress on these planes

Problem 5

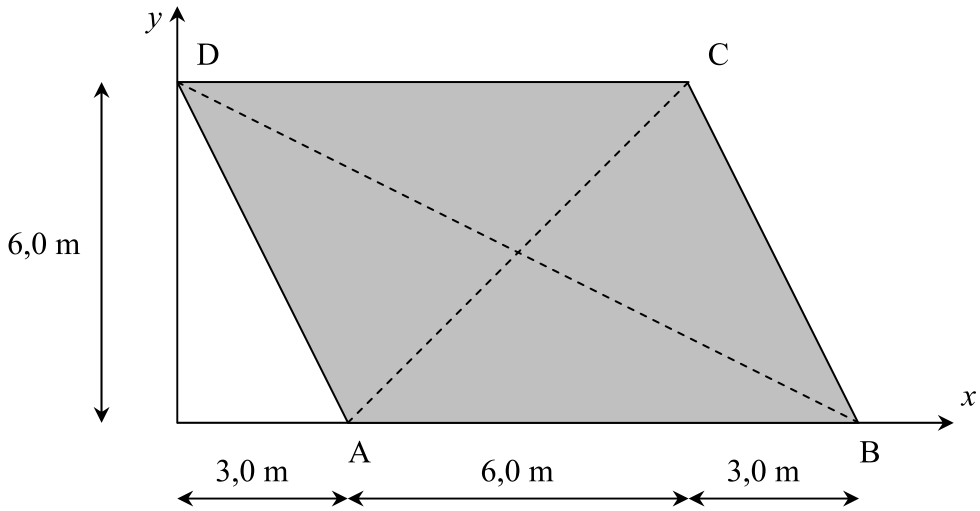


Figure 3.5 : Strain problem

In figure 3.5 a specimen is presented from which the strains are known :

$$\varepsilon_{xx} = 0; \quad \varepsilon_{xy} = \varepsilon_{yx} = 0; \quad \varepsilon_{yy} = -10^{-3}$$

a) Draw Mohr's strain circle for the given strains. Use a scale of 1 cm = $0,1\times10^{-3}$.
b) Show the position of the DC.
c) Find the principal strains and the principal directions.
d) Determine the change in length of fibres AC, BD and AD.

ANSWERS

For all planes a local x-y-coordinate system is used in which the local x-axis coincides with the outward normal to the plane.

Problem 1: stresses on PS : $\sigma_{xx} = \frac{1}{2}$ N/mm^2; $\sigma_{xy} = -\frac{1}{2}$ N/mm^2

Problem 2: stresses on PS : $\sigma_{xx} = 0$ N/mm^2; $\sigma_{xy} = -1$ N/mm^2
stresses on RS : $\sigma_{xx} = 0$ N/mm^2; $\sigma_{xy} = 1$ N/mm^2
stresses on DS : $\sigma_{xx} = -\frac{3}{5}$ N/mm^2; $\sigma_{xy} = -\frac{4}{5}$ N/mm^2
stresses on DR : $\sigma_{xx} = \frac{3}{5}$ N/mm^2; $\sigma_{xy} = -\frac{4}{5}$ N/mm^2

Problem 3 : principal direction parallel to AC and BD;
principal stresses : $\sigma_1 = 2a$; $\sigma_2 = 0$;
stresses on CR : $\sigma_{xx} = \frac{1}{5}a$ N/mm^2; $\sigma_{xy} = \frac{3}{5}a$ N/mm^2
maximum shear stress on planes parallel to x- and y-axes.

Problem 4 : plane with angle of 45 deg. : $\sigma_{xx} = 3$ N/mm^2; $\sigma_{xy} = -5$ N/mm^2
planes x-2y = C : $\sigma_{\widetilde{xx}} = 0$ N/mm^2; $\sigma_{\widetilde{xy}} = 4$ N/mm^2
planes x+2y = C : $\sigma_{\widetilde{xx}} = 0$ N/mm^2; $\sigma_{\widetilde{xy}} = -4$ N/mm^2

Problem 5 : $\Delta l^{AC} = -3\sqrt{2}$ mm; $\Delta l^{BD} = -1{,}2\sqrt{5}$ mm; $\Delta l^{AD} = -2{,}4\sqrt{5}$ mm;

4. Stress – strain relation for linear elasticity

The relation between stresses and strains or the so-called constitutive model, is different for specific materials. These models are therefore also called material-models. These models can be quite complicated since materials may behave very brittle or on the opposite very ductile. Also the composition of materials increases the complexity of these models. In the field of Civil Engineering we have quite a number of complex materials like concrete, soils and asphalt. The material models are used in modern computer applications to predict the deformations and check the safety and durability of proposed solutions with the chosen materials. In this field a lot of ongoing research is aimed to improve these predictions in order to minimize the use of rural resources.

The start of each material model however is the simplest case of a linear relation between the stresses and the strains. In this chapter we will derive this relation and investigate the number of material parameters needed to describe this relation in a unique way. In general the aim is to find the relation between the stress tensor and the strain tensor as indicated in figure 4.1

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad \boxed{\textit{linear relation}} \qquad \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

- second order strain tenson
- derived from the displacement field
- material independed

- second order stress tensor
- derived from equilibrium
- material independ

Figure 4.1: Stress strain relation

The complete relation between the 9 strains and 9 stresses consists of 9×9=81 components. However due to symmetry of both the stress and the strain tensor we only need to find 36 components of the stress strain relation. In matrix notation this looks like:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \times \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} \qquad or \quad \sigma_{ij} = A_{klij}\, \varepsilon_{ij}$$

The stress strain relation relates two second order tensors and is therefore a fourth order tensor. This fourth order tensor must[2] be a symmetric tensor which reduces the number of unknown components to 21. In this chapter we will show that for isotropic materials (same properties in all directions) only two material properties are needed to obtain the 21 components.

[2] Maxwell-Betti theorem on the reciprocal equality

4.1 Uniaxial test

From a well known pulling test the relation between the uniaxial stress and strain is known. For a *linear elastic material* this relation is also known as Hooke's law. Hooke assumed the following linear relation between stress and strain:

$$\sigma = E \times \varepsilon \quad \text{N/mm}^2$$

The constant E is denoted as the *modulus of elasticity* or Youngs's modulus. Since strains have no dimensions the dimension of the modulus of elasticity is the same as the dimension of the stress.

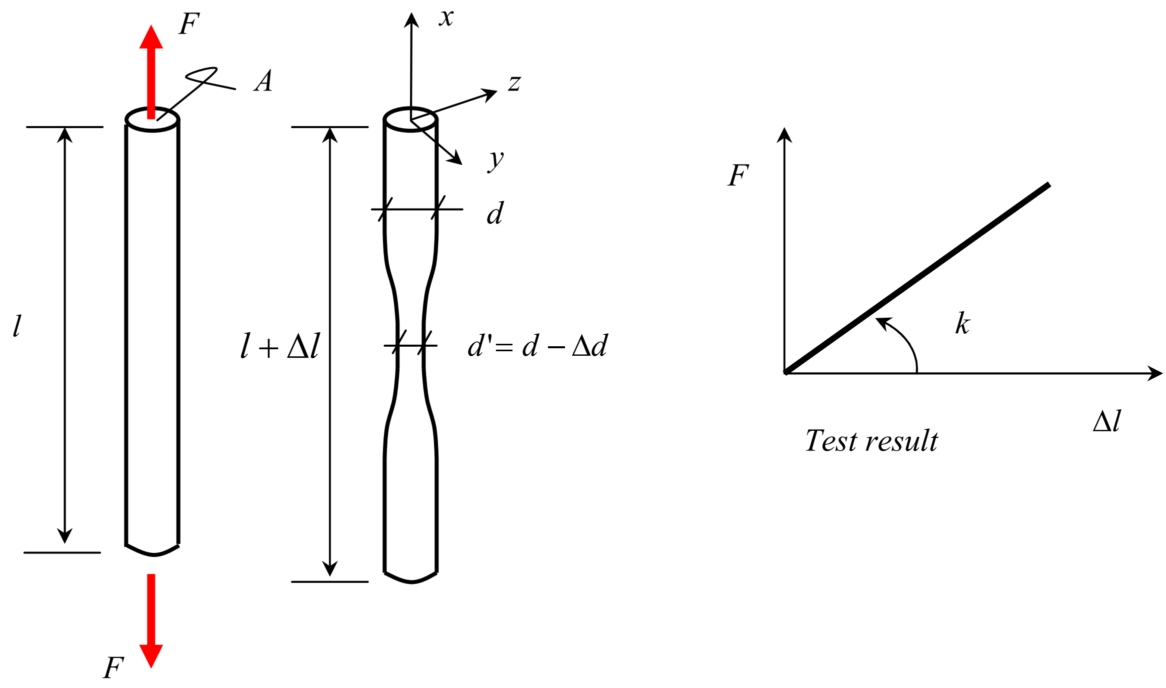


Figure 4.2 : Uniaxial test

From tests we find a linear relation between the change of length and the applied force, see figure 4.2. The slope of the relation is denoted as the stiffness k. Apparently the result yields:

$$F = k \times \Delta l$$

This test result can be related to stresses and strains with:

$$F = \sigma \times A \quad \text{and} \quad \varepsilon = \frac{\Delta l}{l}$$

We then find the well known expression:

$$F = \frac{EA}{l} \times \Delta l$$

Apart from a change in length also a change in diameter of the specimen can be observed with this pulling test.

This so called necking of the cross section is a deformation, occurring out of the plane of action (pulling direction). The amount of necking is related to the observed strain:

$$\frac{\dfrac{\Delta d}{d}}{\dfrac{\Delta l}{l}} = \text{constant} = \nu$$

This constant is called Poisson's ratio. The cross section remains a circle. The necking in both *y*- and *z*- direction is therefore the same and in this experiment only dependent on the amount of straining in the *x*-direction.

With the results of this basic experiment we will continue to find the relation between the stresses and the strains. In figure 4.3 the normal stresses and the corresponding deformations are shown as well as the shear stresses and the corresponding shear deformations.

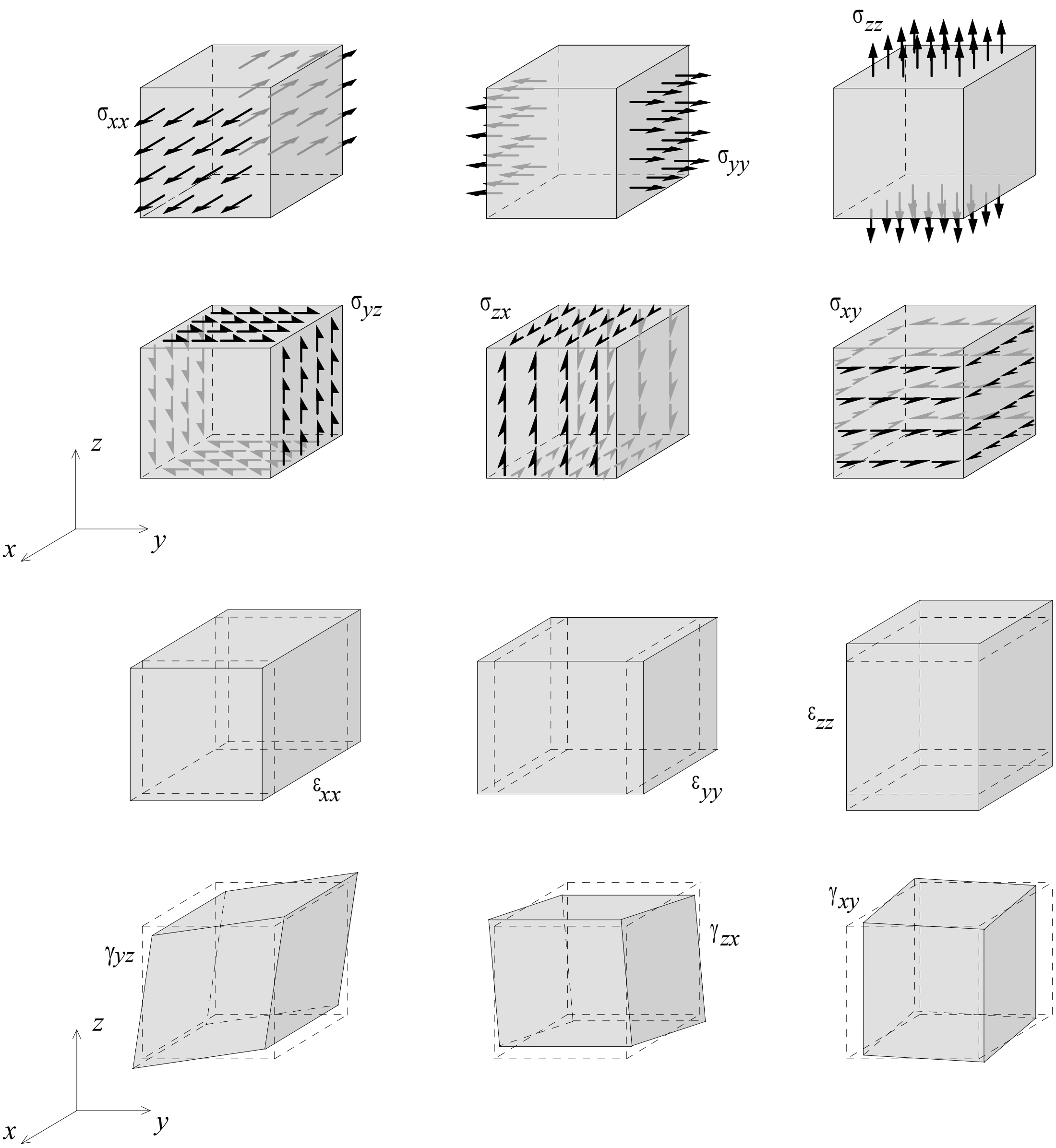


Figure 4.3 : Stresses and strains.

From these graphs can be seen that the deformations due to normal stresses are uncoupled from the shear deformations. We therefore will first try to find the relation between the normal stresses and the strains followed by the relation between the shear stresses and the shear deformation.

4.2 Normal stresses versus strains

From the experiment with only one uniaxial stress σ_{xx} we found:

Strain in the pulling direction:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

Strain in the plane perpendicular to the axis of the bar:

$$\varepsilon_{yy} = -\nu \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{zz} = -\nu \frac{\sigma_{xx}}{E}$$

In general all normal stresses can have a value which results in :

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} - \frac{\nu\sigma_{zz}}{E} \\
\varepsilon_{yy} &= -\frac{\nu\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{zz}}{E} \\
\varepsilon_{zz} &= -\frac{\nu\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E}
\end{aligned}$$

This can be presented in matrix notation as:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} . \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix}$$

The inverse relation yields:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu \\ \nu & (1-\nu) & \nu \\ \nu & \nu & (1-\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix}$$

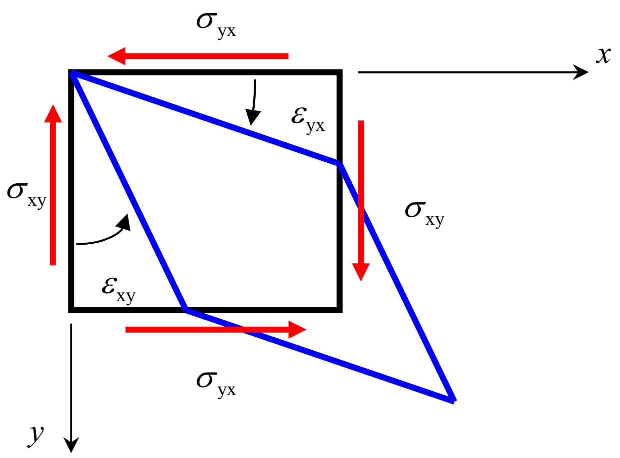
From this latter relation we can conclude that the stress strain relation is limited to Poisson’s ratio’s of $-1 < \nu < 0.5$. Metals have a Poisson ratio in the range of 0.3 while concrete has a slightly lower Poisson ratio around 0.2. Think of the meaning of a negative value for the Poisson ratio!

4.3 Shear stress versus shear deformation

The constitutive relation between the shear stress and shear deformation can be expressed as:

$$\sigma_{xy} = G \times \gamma_{xy}$$

In which G is the shear modulus and can be regarded as a material property with the dimension of a stress. In figure 4.4 the shear stress and the deformation is shown.



$$\varepsilon_{xy} = \varepsilon_{yx}$$
$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$$
thus :
$$\gamma_{xy} = 2\varepsilon_{xy}$$

Figure 4.4 : Shear stress and shear deformation in 2D

With the definition of the shear deformation γ the relation becomes:

$$\sigma_{xy} = G \times 2\varepsilon_{xy}$$
$$\sigma_{yz} = G \times 2\varepsilon_{yz}$$
$$\sigma_{zx} = G \times 2\varepsilon_{zx}$$

The inverse relation thus becomes:

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2G}$$
$$\varepsilon_{yz} = \frac{\sigma_{yz}}{2G}$$
$$\varepsilon_{zx} = \frac{\sigma_{zx}}{2G}$$

If the "material property" G is known the relation has been established. In case of a linear elastic material the shear modulus G is not an independent property but is related to the Young's modulus and the Poisson's ratio:

$$G = \frac{E}{2(1+\nu)}$$

The proof of this relation can be found in APPENDIX 2.

4.4 Complete stress strain relation in 3D

The complete relation between the stresses and the strains thus becomes:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}$$

The inverse relation can be written as:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}$$

4.5 Stress strain relation for plane stress situations

In case of a plane stress situation the general relation can be reduced to:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) & \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) & & \\ \varepsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) & \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) & & \text{with : } G = \frac{E}{2(1+\nu)} \\ \varepsilon_{xy} &= \frac{\sigma_{xy}}{2G} & \sigma_{xy} &= 2G\varepsilon_{xy} & & \end{aligned}$$

In matrix notation these relations become:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \quad or \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

The principal directions can be found either from the stress tensor or from the strain tensor.

- From the stress tensor we can find the principal stress directions with:

$$\tan 2\alpha_{\text{stress}} = \frac{\sigma_{xy}}{\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)} \qquad \text{(i)}$$

- From the strain tensor we can find the principal strain directions with:

$$\tan 2\alpha_{\text{strain}} = \frac{\varepsilon_{xy}}{\frac{1}{2}\left(\varepsilon_{xx} - \varepsilon_{yy}\right)} \qquad \text{(ii)}$$

With the relation between stresses and strains this latter result can be written in terms of stresses as:

$$\tan 2\alpha_{\text{strain}} = \frac{\dfrac{\sigma_{xy}}{2G}}{\frac{1}{2E}\left((1+\nu)\sigma_{xx} - (1+\nu)\sigma_{yy}\right)} = \frac{1}{2G} \times \frac{E}{1+\nu} \times \frac{\sigma_{xy}}{\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)} =$$

$$\frac{2(1+\nu)}{2E} \times \frac{E}{1+\nu} \times \frac{\sigma_{xy}}{\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)} = \frac{\sigma_{xy}}{\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)} = \tan 2\alpha_{\text{stress}}$$

Both expressions (i) and (ii) lead to the same result. The principal directions for the stresses and the strains for a linear elastic material are therefore the same. This property of linear elasticity can also be found in the stress and strain circle of Mohr. In both graphs the same principal directions will be found. This is only possible if in both circles the relative position of the **DC** on the circle is the same.

> *Assignment:*
> *Check this yourself with a little sketch.*

4.6 Stress strain relation in principal directions

The stress strain relations found are valid for any orientation of the coordinate system. A special case of this orientation is the principal coordinate system in which the x-axis coincides with the principal axis (1) and the y-axis coincides with the principal axis (2). For a plane stress situation this results in the stress tensor:

$$\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

All shear stresses are zero (definition of principal direction) thus the previous equations can be simplified to:

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2\right) \qquad \sigma_1 = \frac{E}{1-\nu^2}\left(\varepsilon_1 + \nu\varepsilon_2\right)$$

$$\varepsilon_2 = \frac{1}{E}\left(\sigma_2 - \nu\sigma_1\right) \qquad \sigma_2 = \frac{E}{1-\nu^2}\left(\varepsilon_2 + \nu\varepsilon_1\right)$$

5. Assignments

5.1 Problem 1

In figure 5.1 a homogeneous loaded specimen is given. On two planes the stresses are known.

Additional parameters :
$E = 2$ GPa; $\nu = 0{,}5$

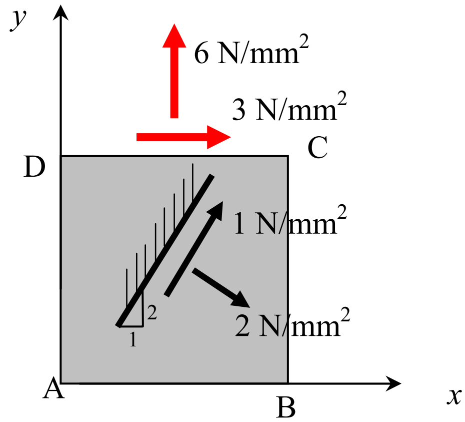


Figure 5.1 : Problem 1

Questions:
a) Draw the stress circle and find the stresses on the *x*-plane.
b) Draw the strain circle and compute the change of length of AB and the change of the right angle DAB.

5.2 Problem 2

The specimen of figure 5.2 is loaded in a homogeneous plane stress situation due to stresses on the faces of the specimen. From these stresses only the two shear stresses on AB and AC are known.

Additional information is available however; the strain of fibres parallel to CD is $-1{,}5\times10^{-3}$.

Additional parameters :
$E = 5$ GPa; $\nu = 0{,}25$

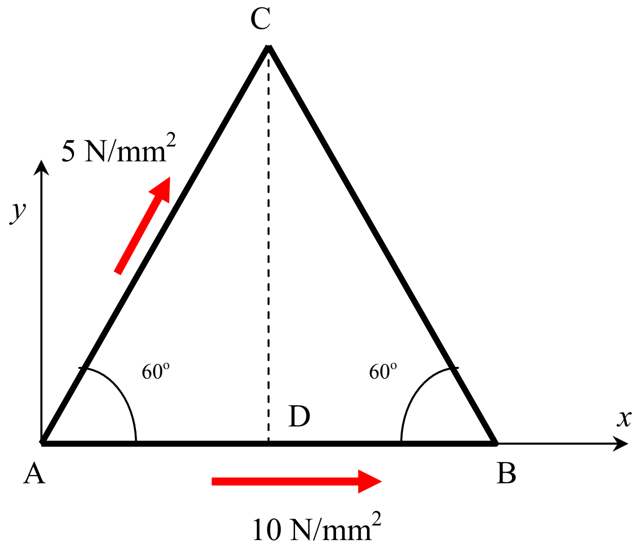


Figure 5.2 : Problem 2

Questions:
a) Find the stresses on *x*- and *y*-planes.
b) Draw the stress circle and check the given shear stresses on AB and AC
c) Draw the strain circle and check the given strain for fibres parallel to CD.

> *Hint :*
> *Use the stress – strain relation and the fact that ADC is a straight angle.*

5.3 Problem 3

A specimen is loaded with a homogeneous plane stress situation. The fibre strains in specific directions can be obtained with four mounted strain gauges G0, G45, G90 and G135 which are mounted on the specimen. The strain gauges are placed with intervals of 45 degrees between the orientation of the gauges.

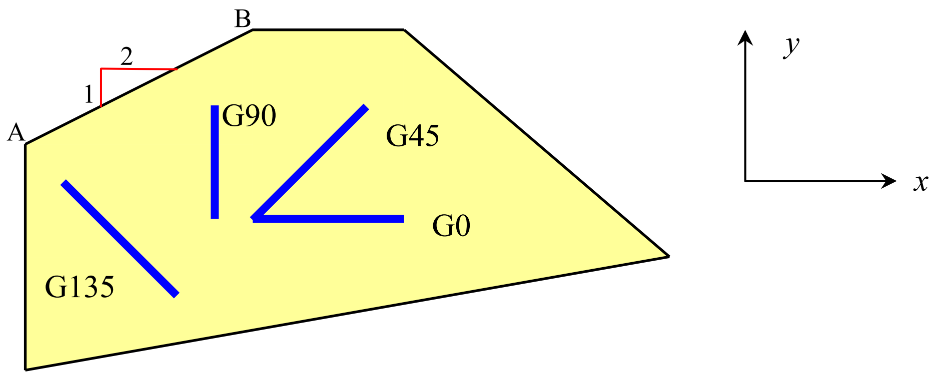


Figure 5.3 : Specimen with mounted strain gauges

During testing the reading of gauge G45 was not possible since this gauge caused a short circuit. So we only have data from three strain gauges:

$$G0 = -4.0 \times 10^{-4}$$
$$G45 = ??$$
$$G90 = 6.0 \times 10^{-4}$$
$$G135 = 5.0 \times 10^{-4}$$

From the material analysis is known:

$$E = 37500 \text{ N/mm}^2$$
$$\nu = 0.25$$
$$f_y = 35 \text{ N/mm}^2$$

Question:

a) Find the stress tensor for this stress situation in the given coordinate system and determine the principal stresses.
b) Show with Mohr circle for stresses and strains that the results are consistent with the theory.
c) Find the shear deformation of this specimen.

Additional question after reading chapter 6:

d) Find the safety factor for this stress situation using the von Mises criterion.

ANSWERS

Problem 1:	stresses :	$\sigma_{xx} = 4 \text{ N/mm}^2;$
	principal stresses :	$\sigma_1 = (5+\sqrt{10}) = 8{,}16 \text{ N/mm}^2;$ $\sigma_2 = (5-\sqrt{10}) = 1{,}84 \text{ N/mm}^2$
	principal strains :	$\varepsilon_1 = 3{,}62\times10^{-3}$ $\varepsilon_2 = -1{,}12\times10^{-3}$
	shear deformation :	$\gamma = 4{,}5\times10^{-3}$
Problem 2:	stresses :	$\sigma_{xx} = -10 \text{ N/mm}^2;\quad \sigma_{xy} = -10 \text{ N/mm}^2;$ $\sigma_{yy} = -10 \text{ N/mm}^2$
	principal stresses :	$\sigma_1 = 0 \text{ N/mm}^2;$ $\sigma_2 = -20 \text{ N/mm}^2$
	principal strains :	$\varepsilon_1 = +1\times10^{-3}$ $\varepsilon_2 = -4\times10^{-3}$
Problem 3:	stress tensor :	$\sigma_{ij} = \begin{bmatrix} -10 & -12 \\ -12 & 20 \end{bmatrix} \quad \text{N/mm}^2$
	Shear deformation:	$\gamma_{xy} = \dfrac{\sigma_{xy}}{G} = -8,0\times10^{-4}$
	principal stresses :	$\sigma_1 = 24,21 \;\; \text{N/mm}^2;$ $\sigma_2 = -14,21 \;\; \text{N/mm}^2$
	safety:	$\gamma^2\times\frac{1}{6}\left\{\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right\}-\frac{1}{3}f_y^2 \le 0$ $\gamma = 1,04$

6. Failure

With the definition of the strains, stresses and the stress-strain relation we created a model for a linear elastic material. Any strain situation can be translated with the stress-strain relation into a stress situation. This chapter will deal with the question when failure will occur. If a certain stress situation exceeds a limit we speak of failure of the material, a structure or a model. The limit state itself is in fact a model too. Depending on the behaviour of the material different models exists. Most models are based on plasticity but different models like visco-plasticity models also exists. In this chapter we will not describe all these models. Only two models will be described.

Important for any model is the description of the stress state which has to be tested to any limit or failure criteria. From the previous chapters we have seen that any stress tensor can be described in a unique way with the stress invariants, into the principal stresses. Therefore most models will be models based on principal stresses. So a good definition of a failure model could be :

> *Any combination of principal stresses that exceeds a certain limit function or value will initiate failure.*

If in any point the yield criterion is reached the material will in most cases not be able to sustain further loading in this point. However the structure as a whole does not necessarily have to fail. Due to redundancy in the structure failure may only occur at a later stage when gradually more points have reached the yield or failure criteria. It is therefore important to distinct failure at *material level* and failure at a *structural level*. In this chapter only failure at material level will be considered.

6.1 Principal stress space

Any 3D stress state in a *x*-*y*-*z*-coordinate system :

$$\sigma_{xyz} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

can be presented in terms of the principal stress tensor in the 1-2-3-coordinate system with:

$$\sigma_{123} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The standard formulae of section 2.3.1 can be used to obtain the principal values of the stresses. This principal stress tensor can also be split in to an isotropic and a deviatoric part (see section 1.1.2) with:

$$\sigma_{123} = \begin{bmatrix} \sigma_o & 0 & 0 \\ 0 & \sigma_o & 0 \\ 0 & 0 & \sigma_o \end{bmatrix} + \begin{bmatrix} \sigma_1 - \sigma_o & 0 & 0 \\ 0 & \sigma_2 - \sigma_o & 0 \\ 0 & 0 & \sigma_3 - \sigma_o \end{bmatrix} \text{ with: } \sigma_o = \tfrac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)$$

The isotropic stress is related to the first stress invariant I_1 as was shown in section 2.3.1. The deviatoric component in the 1-2-3-space can also be denoted as:

$$s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 - \sigma_o \\ \sigma_2 - \sigma_o \\ \sigma_3 - \sigma_o \end{bmatrix}$$

The stress decomposition into an isotropic and deviatoric part can be presented in the 1-2-3-space as shown in figure 6.1.

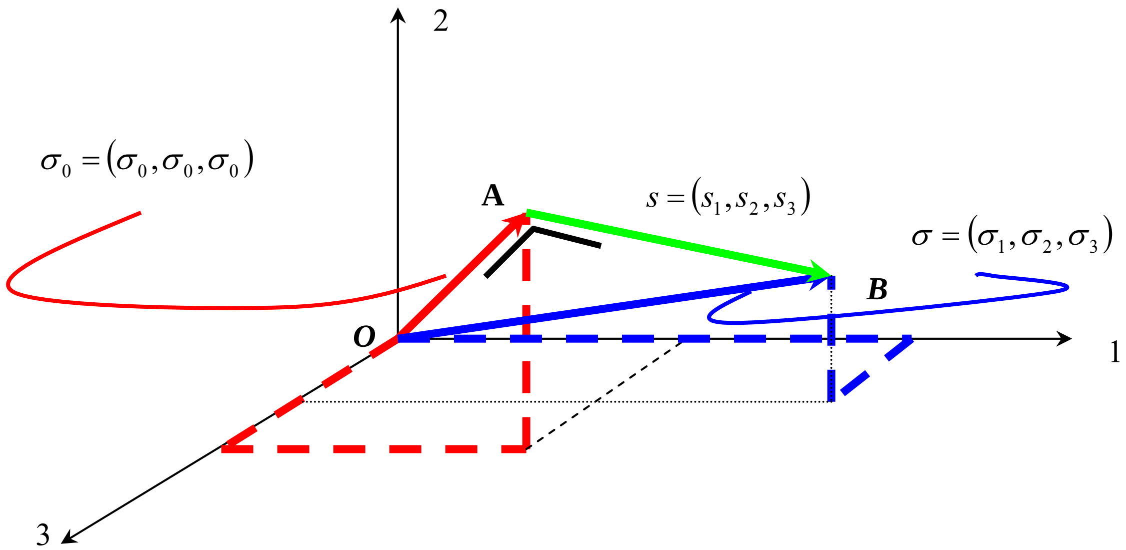


Figure 6.1 : Isotropic and deviatoric stress components in the principal stress space

The three principal stress components can be presented as a vector summation of the isotropic part and the deviatoric part of the stress. The deviatoric stress component is orthogonal to the isotropic stress component. The proof is the inproduct or so called dot product of both vectors:

$$\sigma_o \cdot s = \begin{bmatrix} \sigma_o \\ \sigma_o \\ \sigma_o \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 - \sigma_o \\ \sigma_2 - \sigma_o \\ \sigma_3 - \sigma_o \end{bmatrix} = \sigma_o \sigma_1 - \sigma_o^2 + \sigma_o \sigma_2 - \sigma_o^2 + \sigma_o \sigma_3 - \sigma_o^2 =$$

$$\sigma_o \times (\sigma_1 + \sigma_2 + \sigma_3) - 3\sigma_o^2 = \sigma_o \times 3\sigma_o - 3\sigma_o^2 = 0$$

The angle between these two vectors is therefore a straight angle. The length of the presented vectors can be found with:

$$|OB|^2 = \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)$$

$$|OA|^2 = 3\sigma_0^2$$

thus :

$$|AB|^2 = |OB|^2 - |OA|^2 = \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - 3\sigma_0^2$$

This result will be used in the next section.

6.2 Von Mises failure model

Von Mises postulated that failure occurs when the deviatoric stress exceeds a limit value. This assumption was based on the observation that many materials are not sensitive to changes in the isotropic part of the stress but very sensitive to any change in the deviatoric part of the stress. Steel e.g. will hardly fail under an isotropic stress situation. Imagine a bullet deep under water, it will not fail, even not at considerable depths. For normal engineering practice von Mises stated that a failure model for steel should be independent of the isotropic stress. Failure will thus be the result of shape deformation due to the deviatoric stress component only. A graphical representation of this idea is shown in figure 6.2.

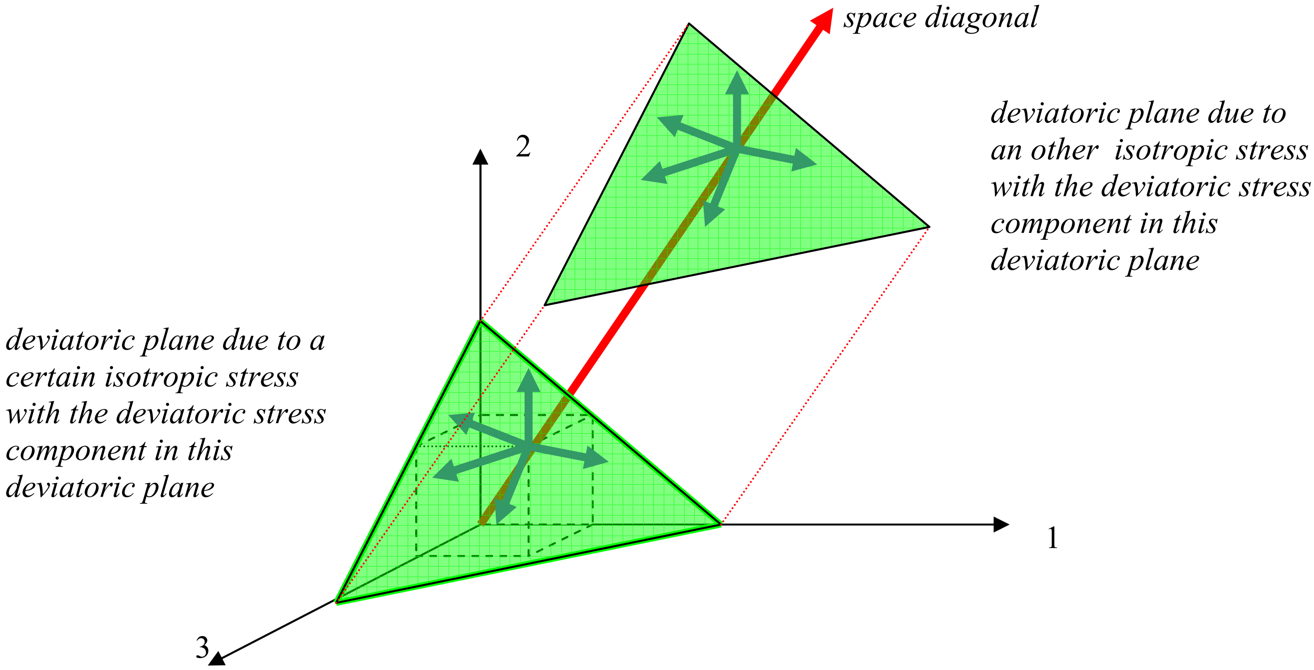


Figure 6.2 : Failure when the deviatoric stress component exceeds a limit value

For different values of the isotropic stress a deviatoric plane can be drawn which is orthogonal to the space diagonal of the principal stress space, 1-2-3-space. The deviatoric component of the principal stress lies within this plane.

The limitation of the deviatoric stress component results in the deviatoric or π-plane in a circle. If the length of the deviatoric stress component is smaller than the radius of the limit circle failure will not occur. In figure 6.3 this failure criterion is presented and yields :

$$\left|\bar{s}\right| \le s_{\max}$$

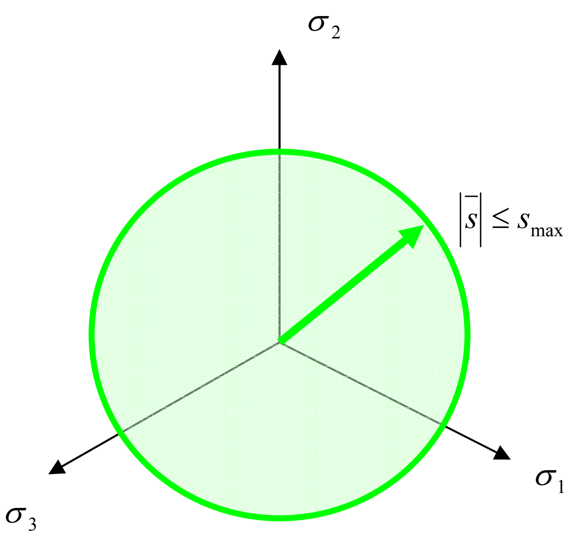


Figure 6.3 : Von Mises criterion

This failure criterion is known as von Mises yield or failure criterion.

The length of the deviatoric component was found in section 6.1 as:

$$|\bar{s}| = \sqrt{\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - 3\sigma_0^2}$$

This length is, according to von Mises, limited:

$$\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - 3\sigma_0^2 \le s_{\max}^2$$

With the definition of the isotropic stress this criterion can be elaborated to:

$$\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - 3 \times \left[\tfrac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)\right]^2 \le s_{\max}^2 \quad \Leftrightarrow$$
$$\tfrac{1}{3}\left[2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - 2\sigma_1\sigma_2 - 2\sigma_2\sigma_3 - 2\sigma_3\sigma_1\right] \le s_{\max}^2 \quad \Leftrightarrow$$
$$\tfrac{1}{3}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right] \le s_{\max}^2$$

This last expression is the von Mises formula in terms of principal stresses:

$$\tfrac{1}{3}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right] \le s_{\max}^2 \qquad \text{(von Mises Formula)}$$

The constant $s_{\max}$ has to be determined with experiments.

6.2.1 Von Mises yield criterion based on a uniaxial test

To find the parameter $s_{\max}$ in the von Mises formula a simple uniaxial test can be performed.

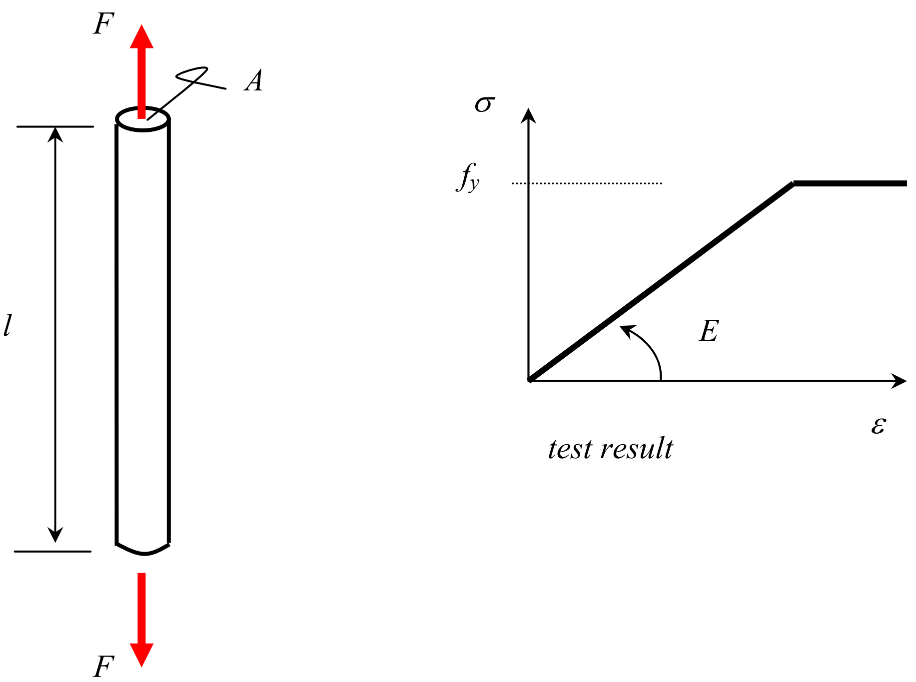


Figure 6.4 : Uniaxial test

The applied force causes a normal stress σ at a cross section. The stress will increase due to an increasing load up to the elastic stress limit, the yield stress f_y. All stresses on the outside are zero which results in a uniaxial limit stress situation which can be described in terms of principal stresses as:

$$\sigma_1 = f_y; \quad \sigma_2 = 0; \quad \sigma_3 = 0;$$

With this "test result" the parameter $s_{\max}$ in the von Mises criterion can be found as:

$$\tfrac{1}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\leq s_{\max}^2$$
$$\tfrac{1}{3}\left[(f_y)^2+(f_y)^2\right]=s_{\max}^2 \quad \Leftrightarrow$$
$$s_{\max}^2=\tfrac{2}{3}f_y^{\,2}$$

The von Mises criterion thus becomes:

$$\tfrac{1}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\leq \tfrac{2}{3}f_y^2$$

In most literature this result is presented as:

$$\tfrac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\leq \tfrac{1}{3}f_y^2$$

The criterion found can be presented in the 1-2-3-space as a tube with the space diagonal as it centre-line as can be seen from figure 6.5.

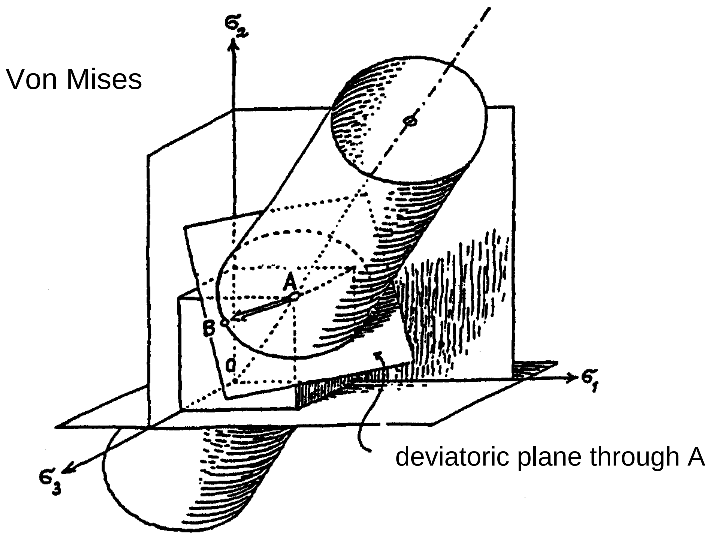


Figure 6.5 : Von Mises criterion in the principal stress space

From this graph can be observed that as long as any principal stress combination is within the tube, failure will not occur. Also can be seen that the value of the isotropic stress is of no interest with respect to failure. The tube remains the same for large isotropic stresses in both the positive and negative domain.

This model is for engineering practice applicable for a variety of ductile materials like alloys such as steel and aluminum.

An alternative way of finding the von Mises criterion can be found in the APPENDIX 3. This method shows that the deformation energy which is responsible for a change in shape of a material will lead to the von Mises criterion.

6.2.2 Von Mises criterion for plane stress situations

The von Mises criterion can also be used in plane stress situations. From figure 6.5 can be seen that the plane of intersection with one of the 1-2 , 2-3 or 3-1 planes results in an ellipse as is shown in figure 6.6 for the intersection with the 1-2-plane.

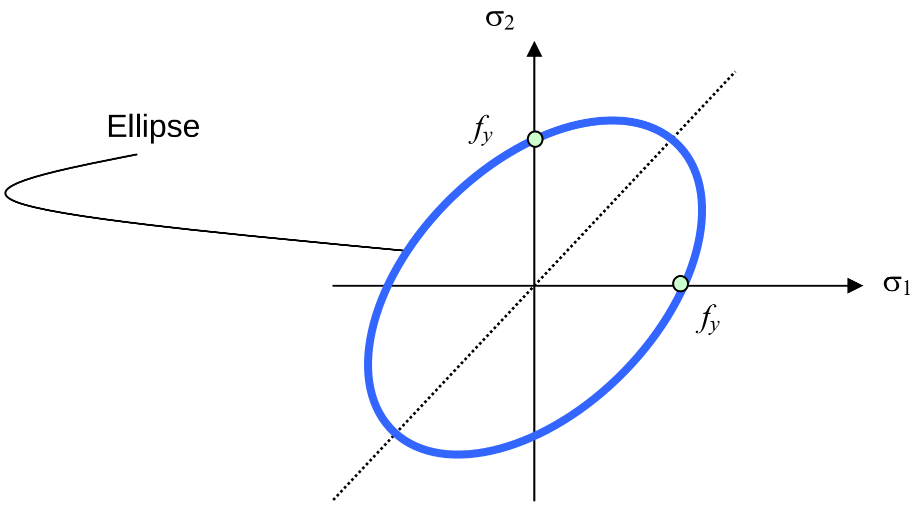


Figure 6.6 : Von Mises criterion in plane stress situation

6.2.3 Von Mises criterion for beams

In section 1.1.1 the stress situation in beams was presented as a special case of a plane stress situation.

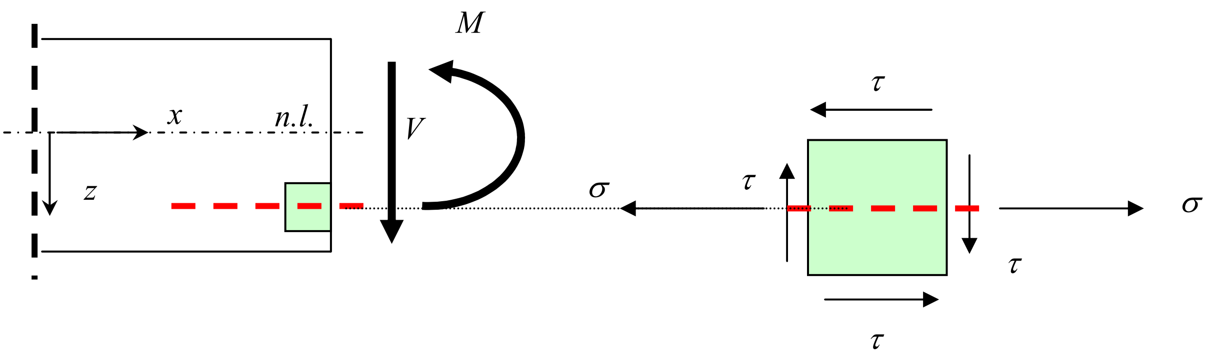


Figure 6.7 : Plane stress situation in beams

At a certain distance z from the neutral axis the stresses on a small specimen can be regarded as a homogeneous plane stress situation. With the transformation formula the principal stresses due to a specific normal stress σ and shear stress τ can be found as:

$$\sigma_1 = \tfrac{1}{2}\sigma + \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$
$$\sigma_2 = \tfrac{1}{2}\sigma - \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$
$$\sigma_3 = 0$$

This result can be used in combination with the von Mises criterion:

$$\tfrac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\le \tfrac{1}{3}f_y^2$$

Which results in the von Mises yield criterion for beams in terms of the normal stress σ and shear stress τ:

$$\sqrt{\sigma^2+3\tau^2}\le f_y$$

This formula is also known as the Huber-Hencky yield criterion. In fact this formula is a special presentation of the von Mises criterion for beams.

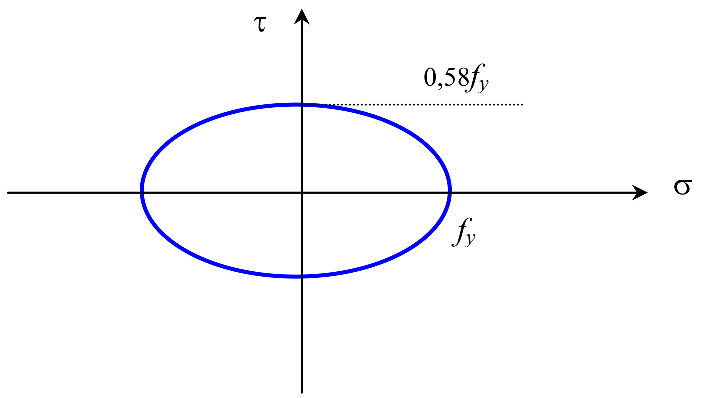


Figure 6.8 : Huber-Hencky yield criterion

6.3 Tresca's failure model

Tresca assumed failure if the maximum shear stress in the material exceeds a certain limit denoted with *c*. In case of a plane stress situation the maximum shear stress can be found with Mohr's stress circle as can be seen from figure 6.9. With only one non zero principal stress this is an example of a uniaxial stress situation.

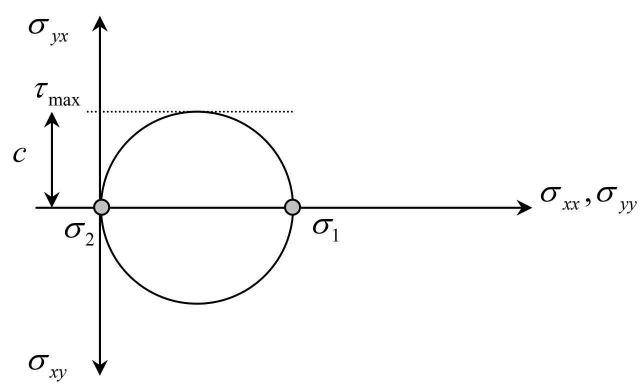


Figure 6.9 : Mohr's stress circle for a uniaxial stres situation

If the yield stress is denoted with f_y , the maximum shear stress and thus Tresca's limit value c becomes:

$$c = \tfrac{1}{2} f_y$$

Mohr's circle in the presented 1-2-plane is therefore bounded by:

$$|\sigma_1 - \sigma_2| \le 2c$$

We can extent this to 3D from which will be found that any principal stress combination is bounded according to Tresca by:

$$|\sigma_1 - \sigma_2| \le 2c$$
$$|\sigma_2 - \sigma_3| \le 2c \qquad \text{(six faces in the 1-2-3-space)}$$
$$|\sigma_3 - \sigma_1| \le 2c$$

Each of the circle 1-2, 2-3, and 1-3 is bounded by c. In figure 6.10 this is shown.

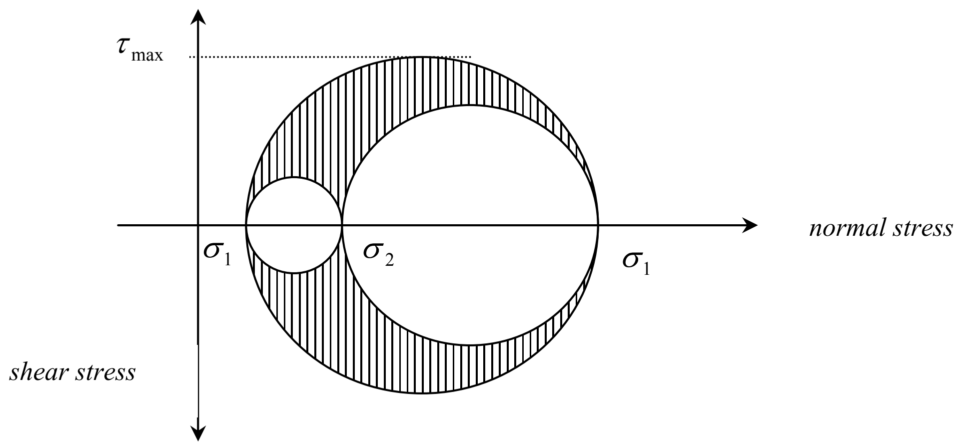


Figure 6.10 : Tresca's circles

In the presented example all principal stresses are non zero. Tresca's criterion states that the largest circle is decisive. In the 1-2-3 principal stress space Tresca can be seen as a six faced tube, a hexagon.

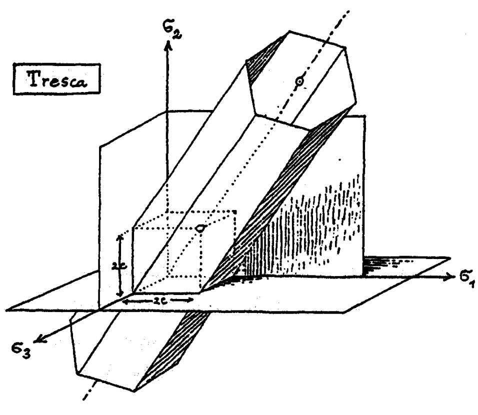


Figure 6.11 : Tresca's hexagon in the 1-2-3 principal stress space

6.3.1 Tresca in plane stress situations

From figure 6.11 it can be seen that like von Mises also Tresca's criterion is independent of any isotropic stress. All combinations of the three principal stresses which are inside the hexagon will not cause yielding. In a plane stress situation Treca's criterion can be presented as shown in figure 6.12.

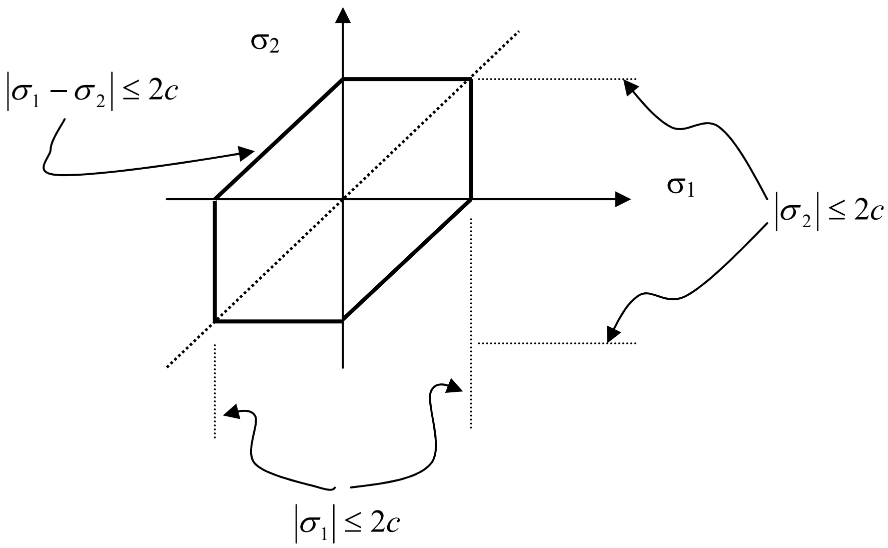


Figure 6.12 : Tresca's criterion in the 1-2-principal stress plane

The six bounding lines can be found from the general formulation on the previous page since the third principal stress is zero.

6.3.2 Tresca in beams

Like the von Mises criterion also the Tresca criterion can be used on beams. With the principal stresses :

$$\sigma_1 = \tfrac{1}{2}\sigma + \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$
$$\sigma_2 = \tfrac{1}{2}\sigma - \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$
$$\sigma_3 = 0$$

and the Tresca criterion :

$$|\sigma_1 - \sigma_2| \le 2c$$
$$|\sigma_2 - \sigma_3| \le 2c$$
$$|\sigma_3 - \sigma_1| \le 2c$$

we can find :

$$\sqrt{\sigma^2 + 4\tau^2} \le f_y$$

This formula limits the combination of normal stresses and shear stresses in beams and is shown in figure 6.13.

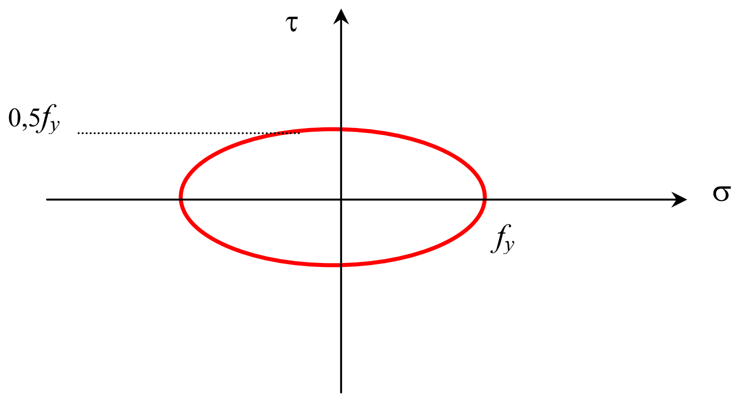


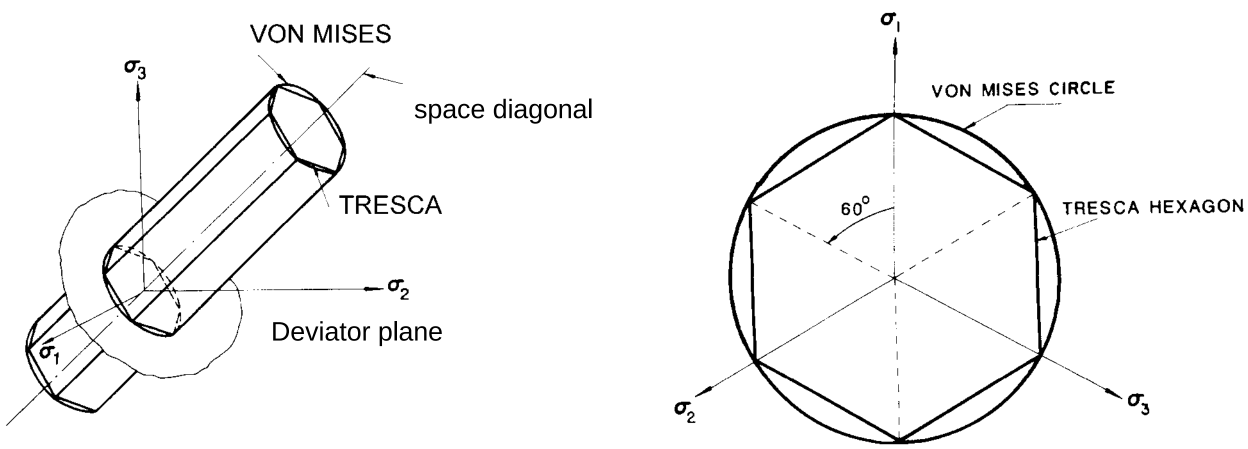
Figure 6.13 : Tresca's criterion for beams in terms of σ and τ.

The maximum shear stress is limited to :

$$\tau_{\max} = \sqrt{\tfrac{1}{4}} f_y = 0.5 f_y$$

6.4 Von Mises versus Tresca

The two presented models started with different assumptions. Von Mises based on deformation due to the deviatoric stress component and Tresca based on a maximum shear stress criterion. Von Mises model shows a continuous function which can be presented as a tube, Tresca's model is a discontinuous model build out of six faces in the three dimensional principal stress space. If we compare the two models we find for the three dimensional principal stress space:

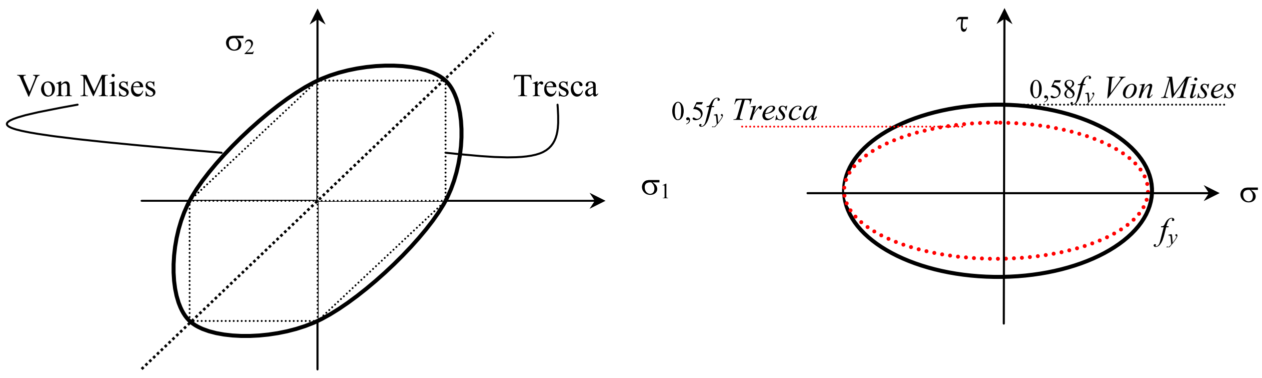


a) 1-2-3 space

b) Deviator plane

Figure 6.14 : Von Mises versus Tresca

Also for the presented plane stress situations we can compare the two models.



a) 1-2 plane stress situation *b) beams*

Figure 6.14 : Von Mises versus Tresca (2)

Tresca's model fits precisely inside the von Mises criterion. This is not always the case as can be seen from APPENDIX 4. However the differences between both models are small and most alloys follow von Mises which is also the most suitable model to implement into computer code.

6.4.1 Example

The stress tensor for a stress situation is given as:

$$\sigma_{xyz} = \begin{bmatrix} 25 & 50 & 0 \\ 50 & 100 & 0 \\ 0 & 0 & -50 \end{bmatrix} \quad \text{N/mm}^2$$

From the material used the yield stress is specified as : $f_y = 250 \text{ N/mm}^2$

Question : Find the safety for the stress situation based on Tresca and von Mises.
You may assume that all stress components are proportional to each other.

Answer : Both models use principal stresses. For this stress tensor we find:

$$\sigma_3 = -50 \text{ N/mm}^2 \quad \sigma_2 = 0 \text{ N/mm}^2 \quad \sigma_1 = 125 \text{ N/mm}^2$$

According to Tresca the principal stresses are bounded by :

$$\gamma \times |\sigma_1 - \sigma_2| \leq 250 \quad \gamma \times 125 \leq 250 \quad \Rightarrow \quad \gamma = 2{,}0$$
$$\gamma \times |\sigma_2 - \sigma_3| \leq 250 \quad \gamma \times 50 \leq 250 \quad \Rightarrow \quad \gamma = 5{,}0$$
$$\gamma \times |\sigma_3 - \sigma_1| \leq 250 \quad \gamma \times 175 \leq 250 \quad \Rightarrow \quad \gamma = 1{,}43 \quad \text{smallest safety}$$

With von Mises we find :

$$\gamma^2 \times \frac{1}{6}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\} - \tfrac{1}{3} f_y^2 \leq 0$$

$$\frac{\gamma^2}{6}\left\{125^2 + 50^2 + 175^2\right\} \leq \frac{250^2}{3} \quad \Rightarrow \quad \gamma = 1{,}60$$

Remark:

The von Mises criterion is a quadratic stress criterion:

$$\tfrac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\leq\tfrac{1}{3}f_y^2$$

If all principal stresses are proportional enlarged to:

$$\gamma\sigma_1$$
$$\gamma\sigma_2$$
$$\gamma\sigma_3$$

The final check will become:

$$\gamma^2\times\frac{1}{6}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}\leq\tfrac{1}{3}f_y^2$$

This result can also be presented with graph's in the (1)-(3)- principal plane.

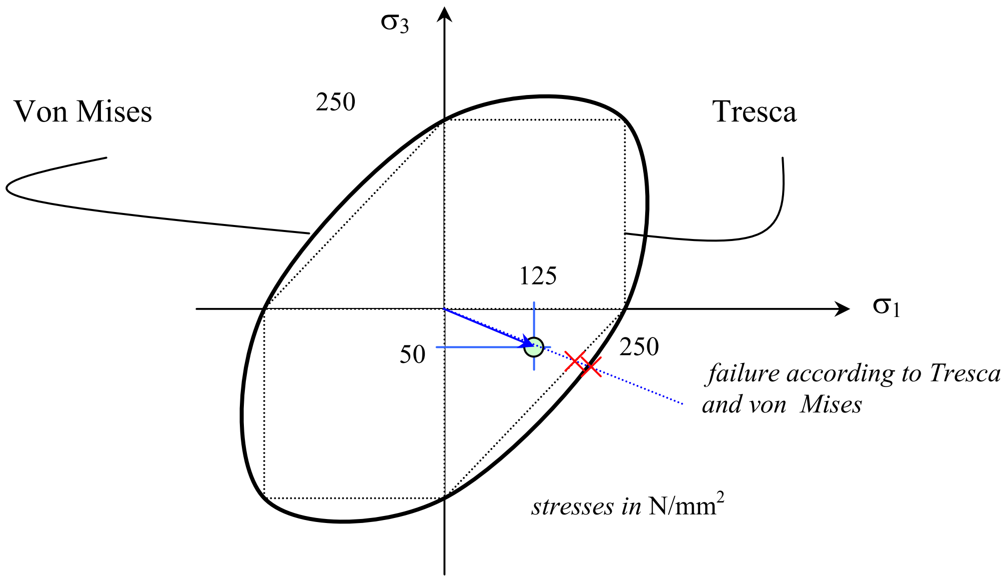


Figure 6.15 : Interpretation of the results

The found safety factors can be interpreted as the ratio of the stress vector from the origin to the marked intersection with the yield function and the blue stress vector (125, -50) in the (1)-(3)-principal plane. From this graph it becomes clear why the safety according to von Mises is larger than according to Tresca.

Assignment:

Answer the additional question posed in problem 5.3.

7. APPENDIX

7.1 Strain formulation

Find the Taylor series for the strain definition: $\varepsilon_{\mathrm{xx}} = \sqrt{\left(1+\dfrac{\partial u_{\mathrm{x}}}{\partial x}\right)^2 + \left(\dfrac{\partial u_{\mathrm{y}}}{\partial x}\right)^2} - 1$.

Start with finding a Taylor series for the general expression:

$$f(x,y) = \sqrt{(1+x)^2 + y^2} = \sqrt{1+2x+x^2+y^2}$$

In order to find the Taylor series we need the partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{1\times(2+2x)}{2\sqrt{1+2x+x^2+y^2}} = \frac{1+x}{\sqrt{1+2x+x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1\times(2y)}{2\sqrt{1+2x+x^2+y^2}} = \frac{y}{\sqrt{1+2x+x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1\times\sqrt{1+2x+x^2+y^2} - \dfrac{(1+x)\times(2+2x)}{2\sqrt{1+2x+x^2+y^2}}}{\left(\sqrt{1+2x+x^2+y^2}\right)^2} = \frac{y^2}{\left(1+2x+x^2+y^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1\times\sqrt{1+2x+x^2+y^2} - \dfrac{y\times(2y)}{2\sqrt{1+2x+x^2+y^2}}}{\left(\sqrt{1+2x+x^2+y^2}\right)^2} = \frac{(1+x)^2}{\left(1+2x+x^2+y^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{0 - \dfrac{(1+x)\times(2y)}{2\sqrt{1+2x+x^2+y^2}}}{\left(\sqrt{1+2x+x^2+y^2}\right)^2} = -\frac{y+xy}{\left(1+2x+x^2+y^2\right)^{\frac{3}{2}}}$$

The values of these derivatives at the origin (0,0) become:

$$\frac{\partial f(0,0)}{\partial x} = 1; \quad \frac{\partial f(0,0)}{\partial y} = 0; \quad \frac{\partial^2 f(0,0)}{\partial x^2} = 0; \quad \frac{\partial^2 f(0,0)}{\partial y^2} = 1; \quad \frac{\partial^2 f(0,0)}{\partial x \partial y} = 0;$$

The Taylor series approximation at a small distance *x,y* from the origin thus becomes:

$$T(x,y) = f(0,0) + \frac{\partial f(0,0)}{\partial x}x + \frac{\partial f(0,0)}{\partial y}y + \tfrac{1}{2}\left(\frac{\partial^2 f(0,0)}{\partial x^2}x^2 + 2\frac{\partial^2 f(0,0)}{\partial x \partial y}xy + \frac{\partial^2 f(0,0)}{\partial y^2}y^2\right) \Rightarrow$$

$$T(x,y) = 1 + x + \tfrac{1}{2}y^2$$

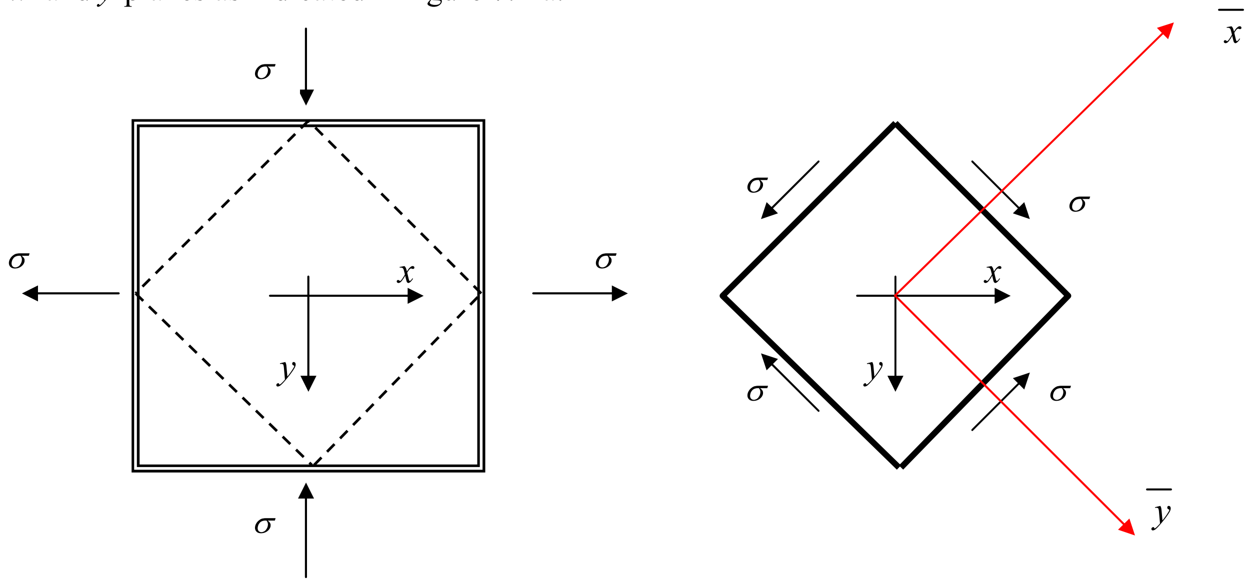
The Taylor series approximation for the strain definition thus becomes:

$$\varepsilon_{\mathrm{xx}} = \frac{\partial u_{\mathrm{x}}}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u_{\mathrm{y}}}{\partial x}\right)^2$$

7.2 Shear modulus G

The shear modulus G for a linear elastic material is related to the elasticity modulus E and the Poisson's ratio ν.

To investigate this relation we consider a plane stress situation were only normal stresses acts on the x- and y-planes as indicated in figure 7.1-a.



a) Tension and compression on specimen *b)Planes with only shear stresses*

Figure 7.1 : Plane stress with only normal stresses

For planes under 45° we can find with Mohr's circle a pure shear situation as indicated in figure 7.2-b. This circle is presented below in figure 7.2.

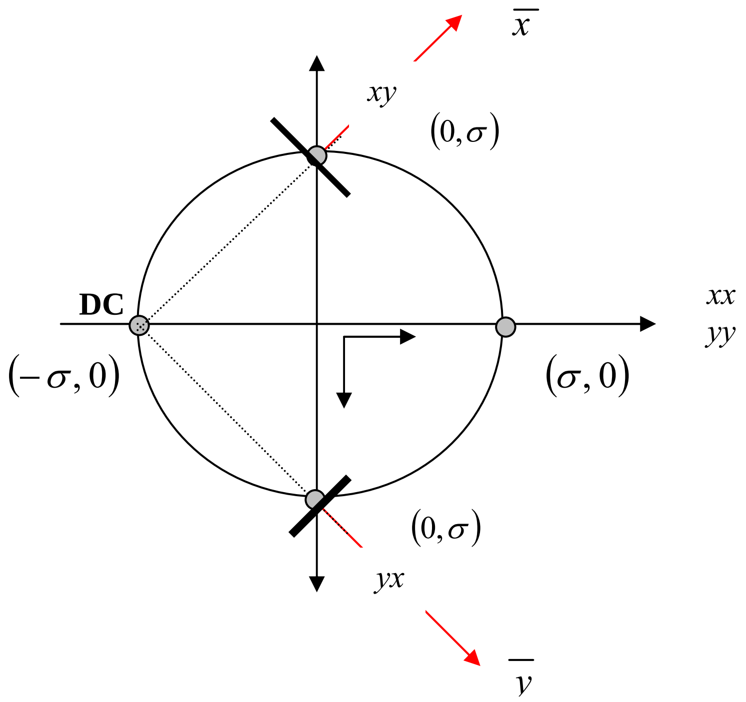


Figure 7.2. : Mohr's circle, pure shear situation

The strains in the original *x*-*y*-coordinate system can be found with the stress-strain relations:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) = \frac{1+\nu}{E}\sigma$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) = -\frac{1+\nu}{E}\sigma$$

$$\varepsilon_{xy} = 0$$

These strains can be presented with two points on the horizontal axis in Mohr's strain circle (see figure 7.3).

The shear deformation or shear strain of fibres with an angle of 45° with the *x*-axis can be found with the constitutive relation for pure shear:

$$\gamma_{\bar{x}\bar{y}} = 2\varepsilon_{\bar{x}\bar{y}} = \frac{\sigma_{\bar{x}\bar{y}}}{G} = \frac{\sigma}{G} \qquad \text{thus:} \quad \varepsilon_{\bar{x}\bar{y}} = \varepsilon_{\bar{y}\bar{x}} = \frac{\sigma}{2G}$$

The strain of these fibres are zero since the normal stresses are zero, see figure 7.2:

$$\varepsilon_{\bar{x}\bar{x}} = \frac{1}{E}\left(\sigma_{\bar{x}\bar{x}} - \nu\sigma_{\bar{y}\bar{y}}\right) = 0; \qquad \varepsilon_{\bar{y}\bar{y}} = \frac{1}{E}\left(\sigma_{\bar{y}\bar{y}} - \nu\sigma_{\bar{x}\bar{x}}\right) = 0$$

These strains can be presented with two points on the vertical axis in Mohr's strain circle (figure 7.3).

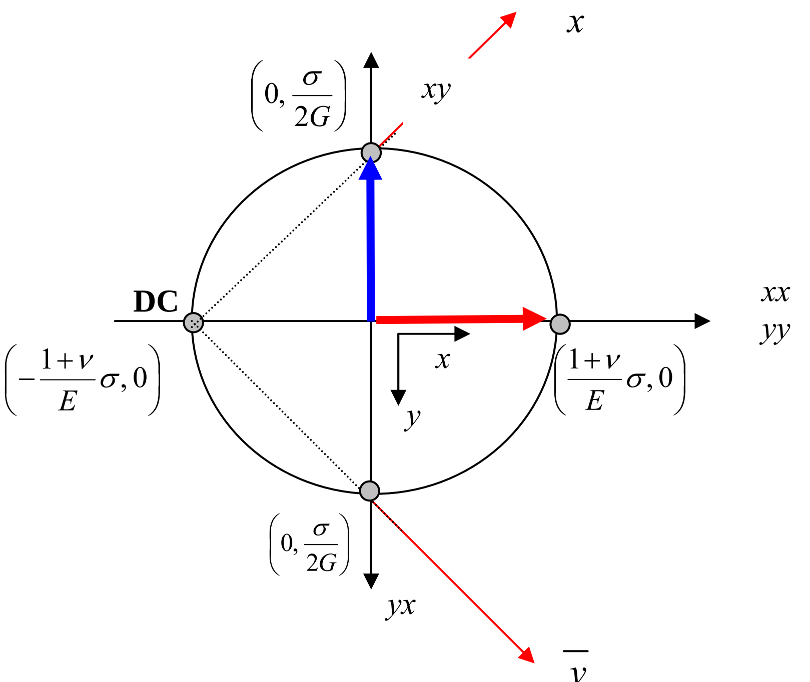


Figure 7.3 : Mohr's strain circle

The radius of a circle is constant which requires : $\dfrac{1+\nu}{E}\sigma = \dfrac{\sigma}{2G}$

From this follows the relation between *E*, *G* and *ν*.

$$G = \frac{E}{2(1+\nu)}$$

7.3 Von Mises criterion based on deformation energy

The von Mises criterion can also be found in an alternative way. Von Mises stated that the amount of shape deformation energy for an alloy material is limited. He therefore claimed that not the volume change but the change of shape causes failure in a material. In this appendix the energy needed for the shape deformation will be investigated based on the earlier found stress strain relations.

7.3.1 Stains and stresses due to shape deformation

Any deformation can be split into two parts, a volume change and a change of shape. Most alloys can withstand a considerable change of volume before failure occurs. For normal engineering purposes we can therefore conclude that failure is independent of a change of volume. For a 2D situation an example of change in volume and change in shape is given in the graph below.

Change of "volume" *Change of shape*

Figure 7.4 : Change in volume and shape

Change of volume is caused by the *isotropic* stress component or *hydrostatic* stress :

$$\sigma_0 = \tfrac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

The change of shape or *distorsion* is caused by the *deviatoric* part of the stress tensor:

$$\begin{Bmatrix} s_{xx} \\ s_{yy} \\ s_{zz} \\ s_{xy} \\ s_{yz} \\ s_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} - \sigma_0 \\ \sigma_{yy} - \sigma_0 \\ \sigma_{zz} - \sigma_0 \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

The strains of fibres in x-, y- or z-directions can also be written as a part due to the deviatoric stress and a part due to the isotropic stress. For a fibre in x-direction holds:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right) = \frac{1}{E}\left(\sigma_{xx} + \nu\sigma_{xx} - \nu\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right) =$$

$$\varepsilon_{xx} = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)}{E} = \underbrace{\boxed{\frac{1+\nu}{E}\sigma_{xx}}}_{\textit{deviatoric strain}} - \frac{3\sigma_0}{E}$$

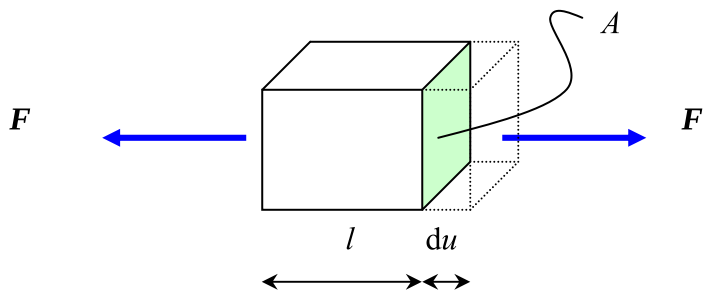
The deviatoric strain tensor e_{ij} can thus be presented as:

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1+\nu}{E}\sigma_{xx} \\ \dfrac{1+\nu}{E}\sigma_{yy} \\ \dfrac{1+\nu}{E}\sigma_{zz} \\ \dfrac{\sigma_{xy}}{2G} \\ \dfrac{\sigma_{yz}}{2G} \\ \dfrac{\sigma_{zx}}{2G} \end{Bmatrix}$$

(check this your self !)

7.3.2 Deformation energy

The amount of deformation energy can be calculated based on the deviatoric stress and deviatoric strain. An expression for the deformation energy can be found based on the simple model given in figure 7.5.



Work:

$$\mathrm{d}W = F \times (\mathrm{d}u) = \sigma \times A \times \mathrm{d}\left(\frac{\sigma \times l}{E}\right)$$

$$\mathrm{d}W = \frac{A \times l}{E}\sigma\,\mathrm{d}\sigma = \frac{V}{E} \times \sigma\,\mathrm{d}\sigma$$

$$\frac{W}{V} = \frac{\sigma^2}{2E} = \tfrac{1}{2}\varepsilon\sigma$$

Figure 7.5 : Deformation energy and Work

Per unit of volume V the force $\boldsymbol{F}$ produces an amount of work W. During the deformation of the material this amount of work will be stored in the material as *deformation energy*:

$$W = \tfrac{1}{2}\varepsilon\sigma$$

The amount of deformation energy is therefore equal to the specified area in the stress-strain diagram.

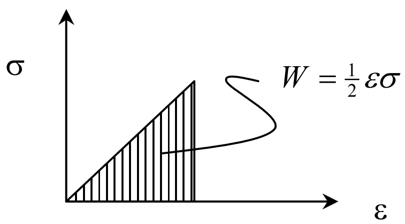


Figure 7.6 : Deformation energy

With the expressions for the deviatoric stress and deviatoric strain the *part* of the deformation energy which causes *distorsion* (change of shape) can be found with:

$$W = \tfrac{1}{2} e_{ij} s_{ij} \qquad \Leftrightarrow$$

$$W = \frac{1}{2}\begin{Bmatrix} \dfrac{1+\nu}{E}\sigma_{xx} \\ \dfrac{1+\nu}{E}\sigma_{yy} \\ \dfrac{1+\nu}{E}\sigma_{zz} \\ \dfrac{\sigma_{xy}}{2G} \\ \dfrac{\sigma_{yz}}{2G} \\ \dfrac{\sigma_{zx}}{2G} \end{Bmatrix} \cdot \begin{Bmatrix} \sigma_{xx}-p \\ \sigma_{yy}-p \\ \sigma_{zz}-p \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

$$W = \frac{1+\nu}{6E}\left\{\left(\sigma_{xx}-\sigma_{yy}\right)^2 + \left(\sigma_{yy}-\sigma_{zz}\right)^2 + \left(\sigma_{zz}-\sigma_{xy}\right)^2\right\} + \frac{1+\nu}{E}\left\{\sigma_{xy}{}^2 + \sigma_{yz}{}^2 + \sigma_{zx}{}^2\right\}$$

According to von Mises this amount of energy is bounded and can be regarded as a material limit. This limit is of course independent of the coordinate system and must be *invariant*. We can therefore choose to express this part of the deformation energy in terms of the *principal stresses*. The von Mises limit function then becomes:

$$W = \frac{1+\nu}{6E}\left\{\left(\sigma_1-\sigma_2\right)^2 + \left(\sigma_2-\sigma_3\right)^2 + \left(\sigma_3-\sigma_1\right)^2\right\}$$

In order to find the limit value of this energy a simple uniaxial test can be used. If the material yields at a yield stress f_y the stored deformation energy which leads to distorsion can be found with :

$$\sigma_1 = f_y \quad \sigma_2 = \sigma_3 = 0$$

$$\overline{W} = \frac{1+\nu}{E}\left(2 f_y^2\right) = \frac{1+\nu}{3E} f_y{}^2$$

This is the limit value for the distorsion energy. The von Mises criterion thus becomes:

$$\frac{1+\nu}{6E}\left\{\left(\sigma_1-\sigma_2\right)^2 + \left(\sigma_2-\sigma_3\right)^2 + \left(\sigma_3-\sigma_1\right)^2\right\} \le \overline{W} \quad \Leftrightarrow$$

$$\frac{1+\nu}{6E}\left\{\left(\sigma_1-\sigma_2\right)^2 + \left(\sigma_2-\sigma_3\right)^2 + \left(\sigma_3-\sigma_1\right)^2\right\} \le \frac{1+\nu}{3E} f_y{}^2 \quad \Leftrightarrow$$

$$\frac{1}{6}\left\{\left(\sigma_1-\sigma_2\right)^2 + \left(\sigma_2-\sigma_3\right)^2 + \left(\sigma_3-\sigma_1\right)^2\right\} - \tfrac{1}{3} f_y^2 \le 0$$

This is exactly the same expression as found in paragraph 6.2.1.

7.4 Von Mises based on a shear test

The von Mises criterion

$$\tfrac{1}{3}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right)\le s_{\max}^2$$

was tuned in section 6.2.1 with a uniaxial test. It is also possible to find the von Mises parameter $s_{\max}$ with a different test, e.g. a shear test as is seen in figure 7.7.

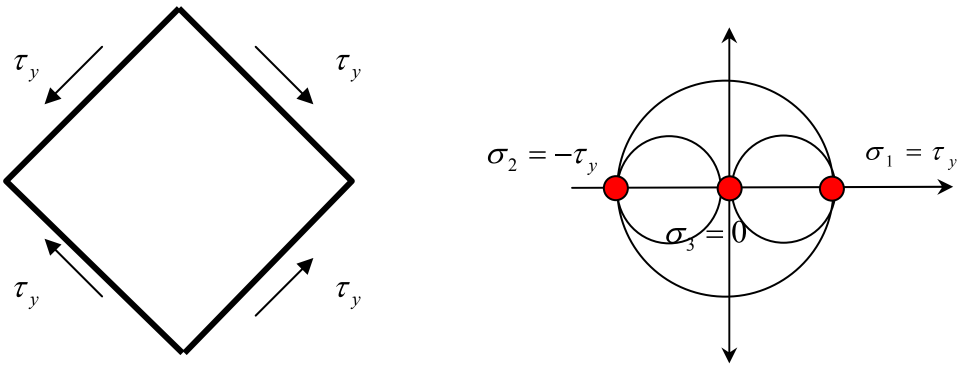


Figure 7.7 : Pure shear

From Mohr's stress circle the principal stresses can be found. The von Mises criterion thus becomes:

$$\tfrac{1}{3}\left((\tau_y+\tau_y)^2+(-\tau_y)^2+(-\tau_y)^2\right)=s_{\max}^2 \qquad \Rightarrow \qquad s_{\max}^2=2\tau_y^2$$

If the maximum shear stress is assumed as:

$$\tau_y=0.5f_y$$

The von Mises criterion then becomes:

$$\tfrac{1}{6}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right)\le \tfrac{1}{4}f_y^2$$

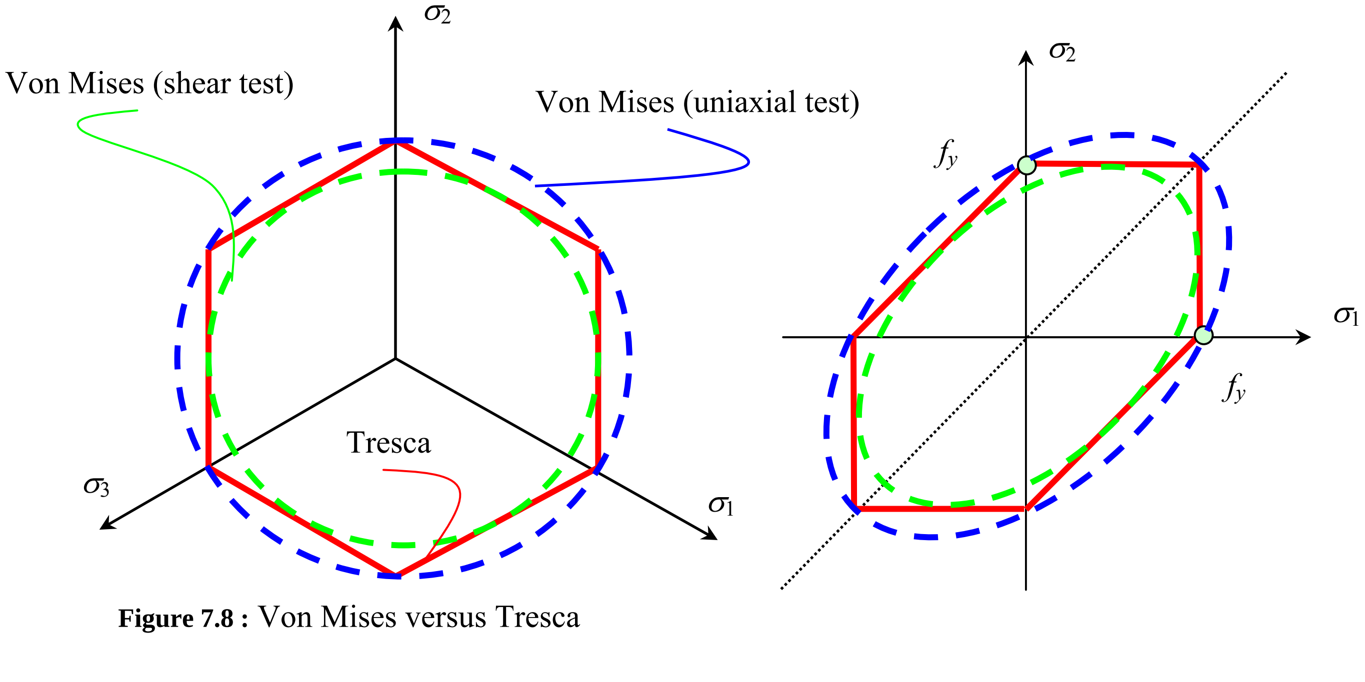


Figure 7.8 : Von Mises versus Tresca

7.5 Continuation of strain example from paragraph 2.5.3

The strain in fibres AC in the strain example of paragraph 2.5.3. can also be found in an alternative way.

The definition of the strain according to paragraph 1.2 is :

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial \mathrm{x}}$$

In other words, the strain is the relative displacement projected to the original *x*-direction of the fibre. In the example the displacement field is given and the displacements in A en C can be found as:

$$u_x^A = 0{,}2\times 10^{-4}\ \text{m}; \quad u_y^A = 0\ \text{m}$$
$$u_x^B = 0{,}8\times 10^{-4}\ \text{m}; \quad u_y^B = 11{,}2\times 10^{-4}\ \text{m}$$

The strain in direction AC can be found by projecting all the displacements to the direction AC and calculate the relative displacement of C with respect to A in the direction AC. In the graph below this procedure is visualised.

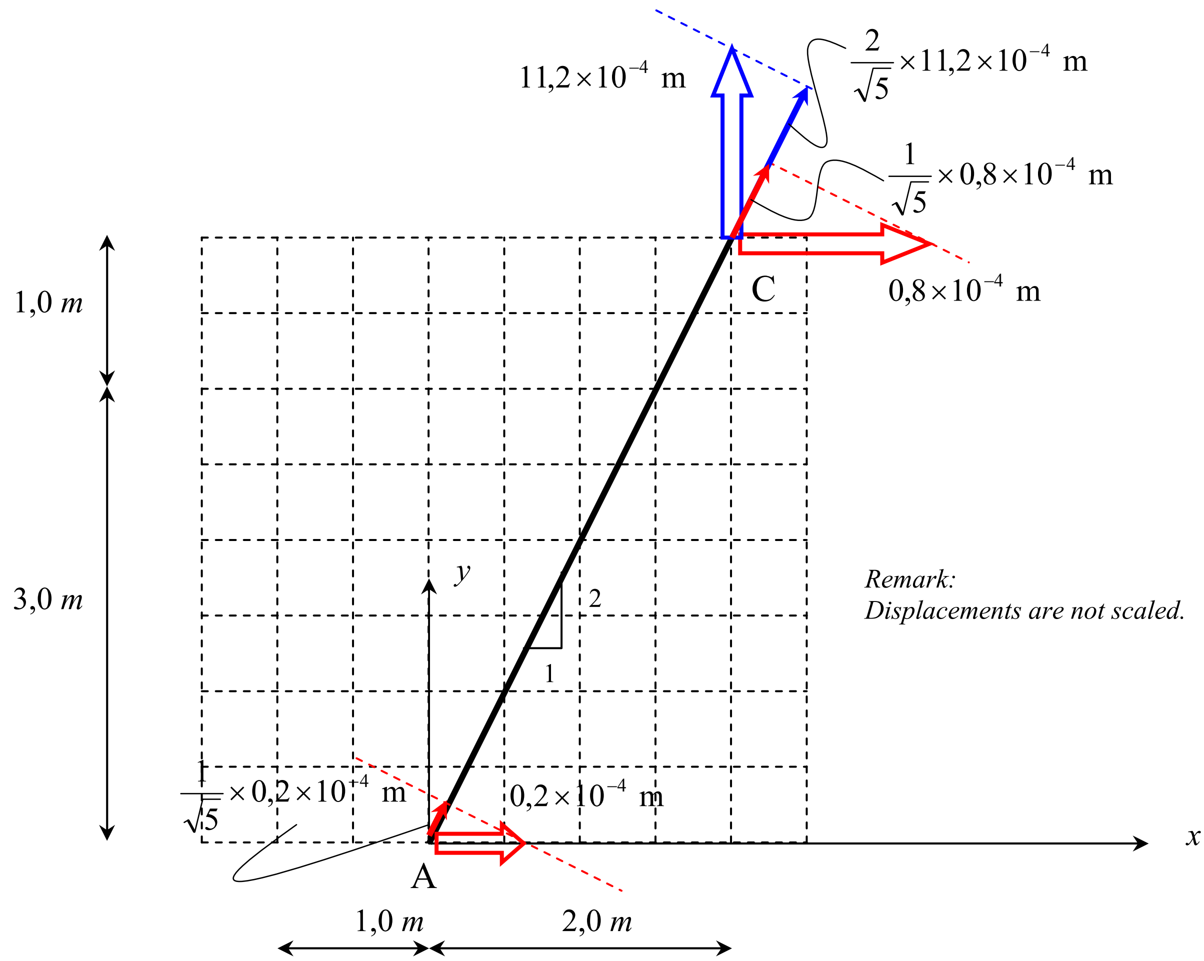


Figure 7.9 : Relative displacement of C in the direction AC

The relative displacement of C towards A in the original direction AC can be found as:

$$\Delta l^{AC} = \frac{2}{\sqrt{5}} \times 11{,}2 \times 10^{-4} + \frac{1}{\sqrt{5}} \times 0{,}8 \times 10^{-4} - \frac{1}{\sqrt{5}} \times 0{,}2 \times 10^{-4} = \frac{23}{\sqrt{5}} \times 10^{-4} \text{ m}$$

The strain of fibre AC thus becomes:

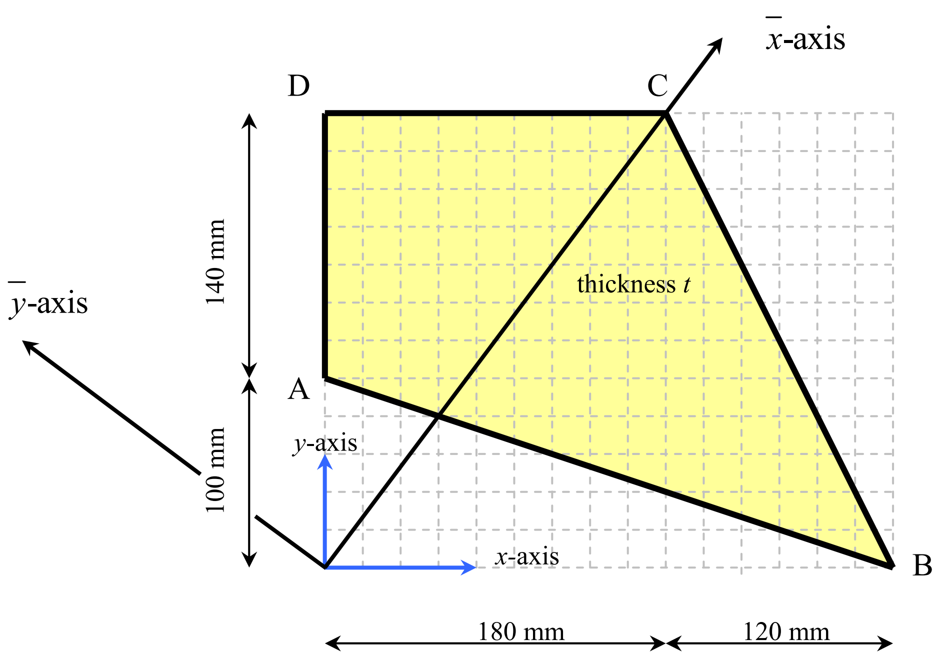
$$\varepsilon^{AC} = \frac{\Delta l^{AC}}{\sqrt{20}} = 2{,}3 \times 10^{-4}$$

This result is in full agreement with the earlier found result in paragraph 2.5.3.

7.6 Example of an examination

A fictive homogeneous isotropic specimen is shown in the figure below. The thickness of the specimen is *t*. The specimen is loaded in a homogeneous plane stress situation. The displacement field is denoted in the ($\bar{x}, \bar{y}$)-coordinate system as:

$$u_{\bar{x}} = 4\times10^{-4} + 18\times10^{-4}\bar{x} - 20\times10^{-4}\bar{y}$$
$$u_{\bar{y}} = 4\times10^{-4}\bar{x} + 6\times10^{-4}\bar{y}$$



Given specifications : E = 62500 N/mm^2, ν = 0,25 en f_y = 240 N/mm^2.

Remark : Make use of the examinations formulas leaf.

Questions:

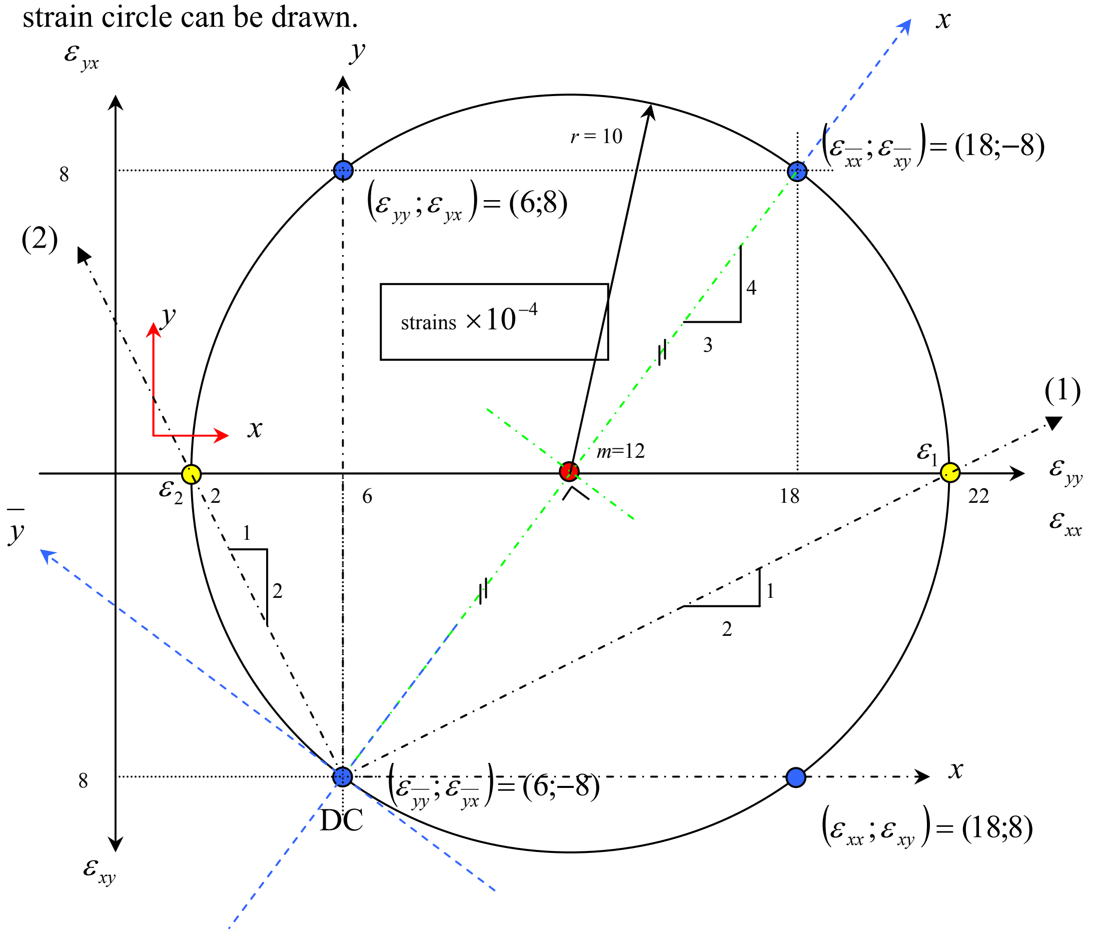
a) Draw Mohr's strain circle using a scale of : 1 cm $\hat{=}$ 2×10^{-4} . Clearly show the Directional Centre DC and the principal directions. Specify all relevant values in the drawing.
b) Derive from the circle the strains of fibres parallel to AD and DC.
c) Compute the principal stresses and draw Mohr's stress circle. Use as scale: 1 cm $\hat{=}$ 10 N/mm^2. Clearly show the position of the Directional Centre DC and the principal directions!
d) Derive from the stress circle the stresses on the faces AB, BC, CD and DA. Show in a separate graph of the specimen all stresses with the directions in which these stresses acts and the magnitude of these stresses.
e) The material follows the von Mises yield criterion:
    - Describe the starting point of the governing equation of this yield criterion.
    - Calculate the safety of the plane stress situation according to von Mises.

Answer

a) With the given displacement field the strain tensor can be found with:

$$\varepsilon_{\bar{x}\bar{x}} = \frac{\partial u_{\bar{x}}}{\partial \bar{x}} = 18 \times 10^{-4}; \quad \varepsilon_{\bar{y}\bar{y}} = \frac{\partial u_{\bar{y}}}{\partial \bar{y}} = 6 \times 10^{-4}; \quad \varepsilon_{\bar{x}\bar{y}} = \frac{1}{2}\left(\frac{\partial u_{\bar{x}}}{\partial \bar{y}} + \frac{\partial u_{\bar{y}}}{\partial \bar{x}}\right) = -8 \times 10^{-4}$$

The tensor can be shown as two points in Mohr's graph for strains. Subsequently Mohr's strain circle can be drawn.



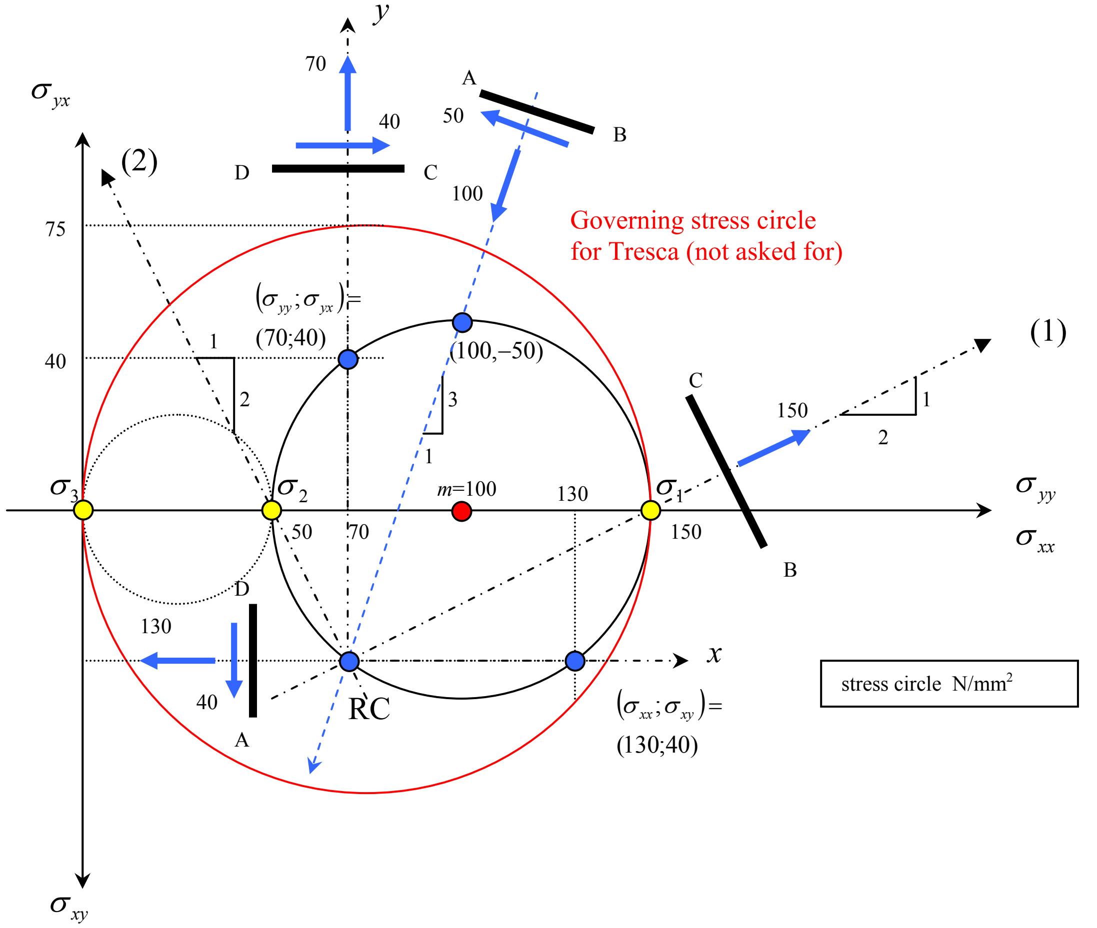
b) Fibres parallel to AD and DC are fibres of the tensor in the *x*-*y*-coordinate system. From Mohr's strain circle the components of this tensor can be obtained by drawing a line through the DC parallel to the direction of the fibre. From this follows:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} \times 10^{-4}$$

c) From the principal strains we can compute the principal stresses:

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) = \frac{62{,}5 \times 10^3}{1-0{,}25^2}(22 \times 10^{-4} + 0{,}25 \times 2 \times 10^{-4}) = 150 \text{ N/mm}^2$$

$$\sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) = \frac{62{,}5 \times 10^3}{1-0{,}25^2}(2 \times 10^{-4} + 0{,}25 \times 22 \times 10^{-4}) = 50 \text{ N/mm}^2$$



d) Mohr's stress circle is shown above. For each face of the specimen a local coordinate system is shown. By default the local x-axis is chosen as the out of plane normal to the face. This is not necessary but convenient to avoid errors. Pay attention to the position of the DC. It's relative position in the stress circle is the same as in the strain circle.

e) The principal stresses are : (0; 50; 150). The safety according to von Mises is:

$$\frac{\gamma^2}{6}\left[(150-50)^2+(50-0)^2+(0-150)^2\right]\le \tfrac{1}{3}\times 240^2 \quad\Rightarrow\quad \gamma=1{,}81$$

The safety according to Tresca is : $\gamma = 120/75 \quad\Rightarrow\quad \gamma = 1{,}60$ (maximum stress circle)

FORMULAS

Principal values for a second order tensor:

$$k_{1,2} = \tfrac{1}{2}\left(k_{xx} + k_{yy}\right) \pm \sqrt{\left[\tfrac{1}{2}\left(k_{xx} - k_{yy}\right)\right]^2 + k_{xy}^2}$$

Stress-strain relations:

$$\begin{cases} \varepsilon_{xx} = \dfrac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \\ \varepsilon_{yy} = \dfrac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) \\ \varepsilon_{xy} = \dfrac{\sigma_{xy}}{2G} \end{cases} \text{ of } \begin{cases} \sigma_{xx} = \dfrac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \\ \sigma_{yy} = \dfrac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \\ \sigma_{xy} = 2G\varepsilon_{xy} \end{cases} \text{ where } G = \frac{E}{2(1+\nu)}$$

Strains:

$$\varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) \quad i, j = x, y$$

Tresca:

$$\begin{aligned} \left|\sigma_1 - \sigma_2\right| &\le 2c \\ \left|\sigma_2 - \sigma_3\right| &\le 2c \\ \left|\sigma_3 - \sigma_1\right| &\le 2c \end{aligned}$$

Von Mises (based on tension or shear):

$$\tfrac{1}{6}\left(\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right) \le \tfrac{1}{3} f_y^2$$

$$\tfrac{1}{6}\left(\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right) \le \tau_y^2$$