variable—the angular displacement or the linear displacement—was needed to specify completely the state of the system. If we now go on to the problem of two masses attached to a stretched string, then two displacements are needed to specify the state of the system; for three masses, clearly we should need three displacements. We can refer to these as examples of systems with, respectively, two and three degrees of freedom.

A general statement is possible, as follows:

A mechanical system is said to have n degrees of freedom if n independent variables are necessary and sufficient to specify the state of the system.

4.3.1 *Two masses on a stretched string*

Let us now consider a stretched string with two particles of mass m attached to it. We impose the same conditions as in Section 4.2.4: that is, the displacements are small so that one can treat the tension T as being unchanged from its equilibrium value, and also we can make the standard small-angle approximations.

The system is shown in Figure 4.10. The length of the stretched string is $2(a+b)$ and each particle is situated distance a from an end. The displacements from equilibrium are represented by X_1 and X_2.

We can extend the arguments of Section 4.2.4 to obtain an equation of motion for each mass. However, note that as drawn, one portion of string exerts a restoring force on the left-hand mass, whereas the other string pulls it away from equilibrium. With this in mind, the two equations may be written as:

$$m\ddot{X}_1 = -T\frac{X_1}{a} + T\frac{X_2 - X_1}{2b}$$

and

$$m\ddot{X}_2 = -T\frac{X_2}{a} - T\frac{X_2 - X_1}{2b}.$$

Now assume a trial solution:

$$X_1 = A\cos(\omega t + \phi)$$
$$X_2 = B\cos(\omega t + \phi),$$

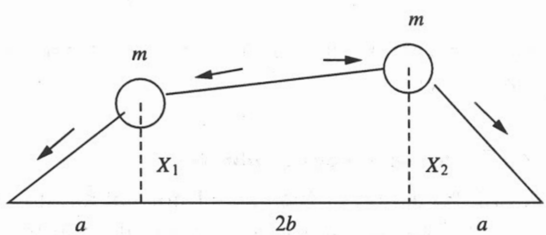

Fig. 4.10 Transverse displacements of two identical masses attached to a stretched string.

and substitute into both sides of the above equations of motion, with the result:

$$\left(\omega^2 - \frac{T}{m}\frac{a+2b}{2ab}\right)A = -\frac{T}{2mb}B \tag{4.17}$$

$$\frac{T}{2mb}A = -\left(\omega^2 - \frac{T}{m}\frac{a+2b}{2ab}\right)B. \tag{4.18}$$

If we cross-multiply these two equations and cancel the common factor of $-AB$, we end up with one equation, viz.,

$$\left(\omega^2 - \frac{T}{m}\frac{a+2b}{2ab}\right)^2 = \left(\frac{T}{2mb}\right)^2.$$

Taking square roots and rearranging gives us for ω^2:

$$\omega^2 - \frac{Ta+2Tb}{2mab} \pm \frac{Ta}{2mab} = 0.$$

The permissible values of ω are obtained from this equation. Denoting these by ω_1, ω_2 we have

$$\omega_1^2 = \frac{T}{ma}, \tag{4.19}$$

corresponding to a choice of the positive sign in the last term, and

$$\omega_2^2 = \frac{T(a+b)}{mab}, \tag{4.20}$$

corresponding to the negative sign.

The constants A and B are related by equation (4.16) or (4.17). Choosing the first of these and substituting (4.18) for ω^2, we have

$$\left[\frac{T}{ma} - \frac{T}{m}\frac{(a+2b)}{2ab}\right]A = \frac{-T}{2mb}B;$$

and, after some cancellation and rearrangements,

$$A = B.$$

Similarly, substituting (4.19) for ω^2 leads to

$$A = -B.$$

The corresponding normal modes of vibration are as shown in Figure 4.11.

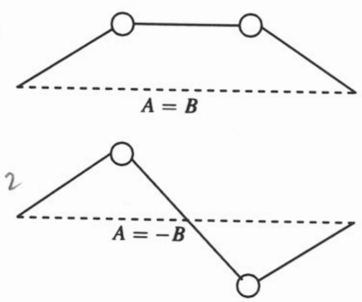

Fig. 4.11 Normal modes of transverse vibration for two masses attached to a stretched string.

4.3.2 *Normal modes of vibration*

A full discussion of the idea of normal modes of vibration, along with technical considerations like 'normal coordinates', would take us far beyond the scope and purpose of the present work. Therefore, we shall confine ourselves here to a few remarks which should help to establish the general idea.

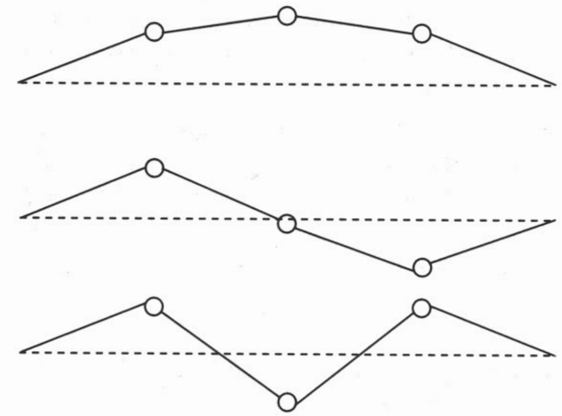

Fig. 4.12 Normal modes of transverse vibration for a system with three degrees of freedom.

We are already familiar with the idea that a system with one degree of freedom will, when started into motion, oscillate at its natural (or resonant) frequency ω_0. We simply extend this idea in an obvious way to systems with more than one degree of freedom.

For instance, to take the next step in order of complexity, a system with two degrees of freedom will have two natural frequencies. When started into motion, it will oscillate at one of these frequencies, depending on the initial conditions. However, the essential feature is that both the constituent masses oscillate at the **same** frequency, which will be one of the **two** natural frequencies for a system with two degrees of freedom.

We have already considered the normal modes of two masses on a stretched string. In the next section, we shall solve for the normal frequencies of the double pendulum. Here, the form of the normal modes is readily deduced to be as shown in Figure 4.15 below.

If we go on to systems with three degrees of freedom—and the problem of three masses attached to a stretched string will be a convenient example—then we may expect there to be three normal frequencies, and the modes of vibration will look like the illustration in Figure 4.12.

We can carry on in this way, and in the limit we can think of a string with its own mass as being a continuum with an infinite number of degrees of freedom. The normal modes of a stretched string are just the so-called standing waves and may be pictured as shown in Figure 4.13, for the first three, and so on. In Section 4.4.5 we shall see that a standing wave can be considered to be the superposition of two travelling waves.

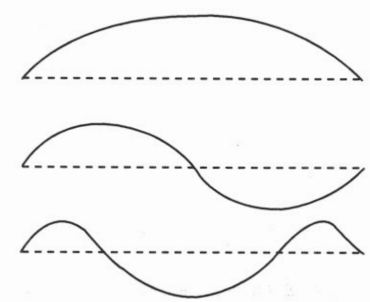

Fig. 4.13 Standing waves in a stretched string, showing the first three modes.

4.3.3 Example: *The double pendulum*

A mass m_1 hangs from a fixed point O by a string of length l_1, while a second mass m_2 hangs from m_1 by a string of length l_2. Obtain the