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ASSIGNMENT ON:
METHODS OF PROOF:
DIRECT PROOF
INDIRECT PROOF
MATHAMATICAL INDUCTION
SUBMITTED TO:
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* METHODS OF PROOF

→ Direct Proof.
→ Indirect Proof.
→ Mathematical Induction.

Direct Proof:- A direct proof is one of the most familiar forms of proof. We use it to prove statements of the form "if P the q" or "p implies q" which we can write as $p \Rightarrow q$. The method of the proof is to take an original statement p, which we assume to be true, and use it to show directly that another statement q is true. So a direct proof has the following steps.

- Assume the statement p is true.
- Use what we know about p and other facts as necessary to deduce that another statement q is true, that is $p \Rightarrow q$ is true.

Example.
Directly prove that if n is an odd integer then n^2 is also an odd integer.

Solution.

that n is an odd integer, then by definition $n = 2k+1$ for some integer k. We will now use this to show that n^2 is also an odd integer.

$$n^2 = (2k+1)^2 \qquad \text{Since } n = 2k+1$$
$$= (2k+1)(2k+1)$$
$$= 4k^2 + 2k + 2k + 1$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

Hence we have shown that n^2 has the form of an odd integer since $2k^2 + 2k$ is an integer. Therefore we have shown that $p \Rightarrow q$ and so we have completed our proof.

Example 2.

Let a, b and c be integers, directly prove that if a divides b and a divides c then a also divides b+c.

Solution.

Let a, b and c be integers and assume that a divides b and a divides c, by definition, there is some integer k such that $b = ak$. Also as a divides c, by definition, there is some integer l such that $c = al$. Note that we use different letters k and l to stand for the integers because we don't know if b and c are equal or not. We will now use these two facts to get our conclusion. So,

$$b+c = (ak)+(al) \quad \text{by our definition of } b \text{ and } c$$

$$= a(k+l)$$

Hence a divides $b+c$ since $k+l$ is an integer.

Indirect proof:- Proof by Contrapositive:

An important proof technique called the indirect method follows from the tautology $P(p \to q) \leftrightarrow ((\sim q) \to (\sim p))$. This states that an implication is equivalent to its contrapositive. Thus to prove $p \to q$ indirectly, we assume that q is false (the statement $\sim q$) and show that p is then false (the statement $\sim p$).

Example.

Let n be an integer. Prove that if n^2 is odd, then n is odd.

Solution:

Let $p: n^2$ is odd and $q: n$ is odd. We have to prove that $p \to q$ is true. Instead we prove the contrapositive, $\sim q \to \sim p$. Thus suppose that n is not odd so that n is even. Then $n = 2k$, where k is an integer. We have $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, so n^2 is even. We thus show that if n is even, then n^2 is even, Which is the contrapositive of the given statement. Hence the given statement has been proved.

Proof by contradiction:-

This is another important proof technique. This method is based on the tautology $((p \to q) \wedge (\sim q)) \to (\sim p)$. Thus the rule of inference

$$p \to q$$
$$\sim q$$
$$\therefore \sim p$$

is valid. Informally, this ~~stat~~ states that, if a statement p implies a false statement q, then p must be false. This is often applied to the case where q is an absurdity or contradiction, that is, a statement that is always false. An example is given by taking q as the contradiction $r \wedge (\sim r)$. Thus any statement that implies a contradiction must be false. In order to use proof by contradiction, suppose we wish to show that a statement q logically follows from statements $p_1, p_2, \ldots p_n$. Assume that $\sim q$ is true (that is, q is false). as an extra hypothesis and that $p_1, p_2, \ldots, p_n$ are also true. If this enlarged hypothesis $p_1 \wedge p_2 \wedge \ldots \wedge p_n, \sim q$ implies a contradiction, then at least one of the statements $p_1, p_2, \ldots, p_n, \sim q$ must be false. This means that if all the p_i's are true, then $\sim q$ must be false, so q must be true. Thus q follows from $p_1, p_2, \ldots, p_n$. This is proof by contradiction.

Example:-

Prove that there is no rational number p/q whose square is 2. In other words, show that $\sqrt{2}$ is irrational.

Solution: This statement is a good candidate for proof by contradiction, because we could not check all possible rational numbers to demonstrate that none had a square of 2. Assume $(p/q)^2 = 2$ for some integers p and q, which have no common factors. If the original choice of p/q is not in lowest terms, we can replace it with its equivalent lowest-term form. Then $p^2 = 2q^2$, so p^2 is even. This implies that p is even, since the square of an odd no. is odd. Thus ~~$p=2q$~~ $p = 2n$ for some integer n. We see that $2q^2 = p^2 = (2n)^2 = 4n^2$, so $q^2 = 2n^2$. Thus q^2 is even, and so q is even. We now have that both p and q are even and therefore have a common factor of 2. This is contradiction to the assumption. Thus the assumption must be false.

Mathematical Induction :-

Mathematical Induction is a proof technique. Suppose that the statement to be proved can be put in the form $\forall n \geq n_0 \; P(n)$, where n_0 is some fixed integer. That is, suppose that we wish to show that $P(n)$ is true for all $n \geq n_0$. The following result shows how this can be done. Suppose that (a) $P(n_0)$ is true and (b) if $P(k)$ is true for some $k \geq n_0$, then $P(k+1)$ must also be true. Then $P(n)$ is true for all $n \geq n_0$. This result is called the principle of mathematical induction. Thus to prove the truth of a statement $\forall n \geq n_0 \; P(n)$ using the principle of mathematical induction, we must begin by proving directly that the first proposition $P(n_0)$ is true. This is called the basis step of the induction and is generally very easy.

Then we must prove $P(k) \rightarrow P(k+1)$ is a tautology for any choice of $k \geq n_0$. Since the only case where an implication is false is if the antecedent is true and the consequent is false, this step is usually done by showing that, if $P(k)$ were true, then $P(k+1)$ would also have to be true. Note that this is not the same as assuming that $P(k)$ is true for some value of k. This step is called the induction step, and some work will usually be required to show that the implication is always true.

Example :-

Show by mathematical induction that, for all $n \geq 1$, $1+2+3+\dots+n = \frac{n(n+1)}{2}$

Solution:- Let $P(n)$ be the predicate $1+2+3+\dots+n = \frac{n(n+1)}{2}$

In this example, $n_0 = 1$.

Basis step:

We must first show that $P(1)$ is true.

$P(1)$ is the statement

$$1 = \frac{1(1+1)}{2}$$

$$= \frac{1(2)}{2}$$

$$= 1$$, which is clearly true.

INDUCTION STEP:

We must now show that for $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ must also be true.

We assume that for some fixed $k \geq 1$.

$$1+2+3+\dots+k = \frac{k(k+1)}{2}. \quad (i)$$

We now wish to show the truth of $P(k+1)$:

$$1+2+3+\dots+(k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

The left hand side of $P(K+1)$ can be written as $1+2+3+---+K+(K+1)$, and we have

$$(1+2+3+---+K)+(K+1) = \frac{K(K+1)}{2} + (K+1)$$

using (i) to replace $1+2+---+K$.

$$= \frac{K(K+1)+2(K+1)}{2}$$

$$= \frac{(K+1)(K+2)}{2}$$

$$= \frac{(K+1)((K+1)+1)}{2}$$

= the right hand side of $P(K+1)$.

Thus we have shown that the left hand side of $P(K+1)$ equals the right-hand side of $P(K+1)$.
By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \geq 1$.

Example: Use mathematical induction to prove the inequality.

$$n < 2^n$$

for all positive integers n.

Solution:- Let $P(n)$ be the proposition "$n < 2^n$."

BASIS STEP:

$P(1)$ is true, since $1 < 2^1 = 2$.

INDUCTIVE STEP:

Assume that $P(k)$ is true for the positive integer k. That is, assume that $k < 2^k$.
We need to show that $P(k+1)$ is true. That is, we need to show that $k+1 < 2^{k+1}$. Adding 1 to both sides of $k < 2^k$, and then noting that $1 \leq 2^k$, gives

$$k+1 < 2^k + 1 \leq 2^k + 2^k = 2^{k+1}$$

We have shown that $P(k+1)$ is true, namely, that $k+1 < 2^{k+1}$, based on the assumption that $P(k)$ is true. The induction step is complete.

Therefore, by the principle of mathematical induction, it has been shown that $n < 2^n$ is true for all positive integers n.