Distributed Control of Heterogeneous Networks of Vehicles with Positive Systems Theory and Generalized H_2 Norm

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***Abstract*— For interconnections of vehicles with complex dynamics, it is commonly suggested to design controllers locally at agents so that agents track the trajectories generated by simplified dynamics. We analyze the performance of such a decoupled control architecture by considering the l_∞ norm of the local tracking error, which allows to quantify the deviation of the actual system from the simplified system and either provide a justification for the decoupled architecture or show a need to consider instead a coupled architecture. Specifically, we consider first-order protocols as the interaction mechanism among agents and general discrete-time Linear Time-Invariant (LTI) dynamics as agent models which track the trajectories generated by the first-order protocols. For the analysis and synthesis of first-order protocols, we do not assume *a priori* knowledge of the connectivity of the graph or the spectrum of the Laplacian matrix, but use results on positive systems to obtain conditions that can scale linearly with network size. We provide bounds on the loss in performance due to imperfect tracking of first order dynamics by higher order agent dynamics and due to disturbances acting locally on agents. Numerical simulations involving generic second order vehicle models with damping illustrate the applicability of the results.**

I. INTRODUCTION

A major portion of the literature on large scale networks of dynamic systems typically assumes simple agent dynamics such as single integrator dynamics. When considering interconnections of higher order agent dynamics such as quadrotors or higher order vehicle models, it is commonly proposed (for example in [1],[2],[3],[4]) to implement local tracking controllers to track the trajectories generated by these simplified dynamics. For example, the authors in [2] consider consensus based cooperative control of vehicles with different degrees of coupling between agents. A consensus loop in series with a decoupled local tracking loop is considered (see section 8.3.4 in [2]) and the stability of the overall system is implied by the series connection of two input-to-state stable systems. Scenarios where such a decoupling is not recommended are discussed briefly but a quantitative measure in making this decision is not presented. This decoupled architecture in the continuous-time setting is also considered in [4] where only a stability analysis is presented without any performance measure to decide on the applicability of the decoupled architecture. Moreover, the presented analysis would allow intersecting trajectories without preventing collisions. A similar decoupled architecture is considered in the discrete-time setting in [3],[5] along with a stability analysis and a discussion on the coupled vs decoupled loops is presented in [6]. The authors in [1] similarly propose to design cooperative control algorithms for interconnected systems assuming single integrator agent dynamics and wrap local tracking controllers around these trajectories so that complicated agents act like single integrators. This idea is further demonstrated for differentially flat systems with a simple example of a kinematic unicycle, without considering disturbances on the agents. Our goal is to rigorously analyze up-to what extent is the decoupled architecture reasonable. We use the l_∞ norm as the performance measure of the local tracking performance. Based on the inter-agent distances of the simplified dynamics, this l_∞ norm of the tracking error can be used to either argue that the trajectories remain collision free or to decide on a coupled architecture in case a collision is possible. A very closely related problem of consensus of general non-linear vehicles modeled as quasi-Linear Parameter Varing (qLPV) systems is considered in the continuous-time setting in [7]. In contrast to [7], we do not require *a priori* knowledge of the spectrum of the graph laplacian but we instead use as the first main ingredient in our analysis, the recently developed results [8], [9] on scalable analysis and control of positive systems. As a second ingredient, we use a local input-output performance result for tracking these dynamics and derive one LMI condition per agent that guarantees the stability and performance of the local tracking feedback loops in the sense of the l_2 to l_∞ norm (generalized H_2 norm [10], [11]).

Outline:

After stating the notation in Section 2, Section 3 presents the problem formulation followed by the main analysis and synthesis results in Section 4. Section 5 provides illustrative examples demonstrating the application of the theoretical results developed in Section 4. Conclusions are presented in Section 5.

II. NOTATION AND PRELIMINARIES

Let $\mathbb{R}$ denote the set of real numbers. The notation $X >(\geq)0$ denotes that all entries of the matrix X are positive (non-negative) and X^T represents the transpose of matrix X. $X \succ (\succcurlyeq)0$ means that the matrix X is symmetric positive definite (semi-definite), and $X \prec (\preccurlyeq)0$ denote that $-X$ is symmetric positive definite (semi-definite). For $X \succ 0$, let the positive number $\text{cond}(X)$ be the condition number of X. For block matrices, we use $*$ to denote required entries to make the complete matrix symmetric. Let $\mathbf{0}$ and $\mathbf{1}$ denote the vectors or matrices of all zeroes and ones of appropriate

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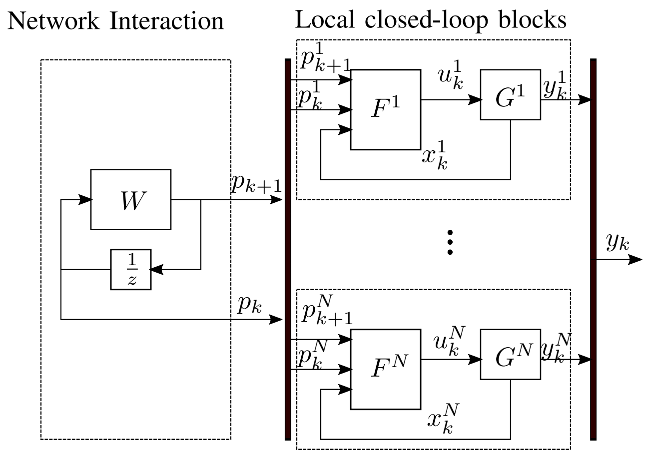

Fig. 1. Interconnected system (7)

sizes, respectively. Let I be the identity matrix of appropriate size and let e_i be the i^{th} canonical basis vector of appropriate size. For $x \in \mathbb{R}^n$, we use the notation $\text{avg}(x) = \frac{1}{n}\mathbf{1}^T x$. Let $\otimes$ represent the Kronecker product. The norm $||x||_p$ denotes the standard Euclidean p-norm. For sequences w, define the l_2 norm by $||w||_{l_2} = \sqrt{\sum_{k=0}^{\infty} ||w_k||_2^2}$. Let the l_∞ norm be defined by $||w||_{l_\infty} = \sup_k ||w_k||_2$. For the graph theoretic notation, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected unweighted graph of order N with the set of nodes $\mathcal{V} = \{v_1, v_2, \dots v_N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and an adjacency matrix $A \in \mathbb{R}^{N \times N}$ given by $A = [a_{ij}]$ with 0 and 1 entries such that $a_{ij} = 1 \iff (v_i, v_j) \in \mathcal{E}$. The set of neighbors of node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. Let d_i be the degree of node i, i.e., the number of links at that node and let $D \in \mathbb{R}^{N \times N}$ be a diagonal matrix formed by the d_i's along the diagonal. Define the graph Laplacian matrix $L \in \mathbb{R}^{N \times N}$ as $L = D - A$. Let the maximum degree $\max_i(d_i)$ be denoted by d_{max}. Let $|\mathcal{E}|$ represent the total number of edges.

III. PROBLEM FORMULATION

We consider a large-scale interconnected system of N vehicles governed by general discrete-time LTI dynamics with state x^i, input u^i and output y^i (for agent i) neither of which are communicated with neighboring agents. At the same time, the interaction mechanism between different agents is modeled by first-order protocols with the virtual state p^i (for agent i) which is communicated with neighboring agents. The key idea is to see the first-order protocol as a reference generator, generating the reference trajectory p. We wrap local feedback controllers around the reference so that complex vehicles with higher order dynamics track these references. The resulting system dynamics and the controller structure for agent i are depicted in Fig. 1 and Fig. 2, respectively, and is discussed next in detail.

A. First order protocols

Let $p^i_k \in \mathbb{R}$ be a virtual state of the i^{th} agent at time k that is communicated with neighboring agents. Let the interconnection topology between the agents be represented by an undirected graph $\mathcal{G}$ with the corresponding Laplacian matrix L. We assume that there is one leader which we assume to be the 1st agent. The leader is externally controlled to the desired state which we assume without the loss of generality to be the origin. Define $L_c = L + K_{\text{lead}} \cdot e_1 e_1^T$ for some given $K_{\text{lead}} > 0$. Discrete-time first-order protocols [12] can be represented by

$$p^i_{k+1} = w^{ii} p^i_k + \sum_{j \in \mathcal{N}_i} w^{ij} p^j_k \tag{1}$$

where $w^{ij} \in \mathbb{R}$ represent the weights corresponding to the edge between node i and j, and w^{ii} represents the weight associated with the self-state. We can define a matrix W of weights w^{ij} with $w^{ij} = 0$ if the edge $(v_i, v_j) \notin \mathcal{E}$. By defining $p_k = [p^1_k \cdots p^N_k]^T$, the dynamics in (1) can be represented by

$$p_{k+1} = W p_k. \tag{2}$$

We restrict ourselves to the form $W = I - EL_c$, where K_{lead} is assumed fixed and E is a diagonal matrix which we consider as a design parameter (control variable) such that $W \geq 0$. Using a bound on d_{max}, one can ensure that K_{lead} and E are such that $W \geq 0$ and it does not require the knowledge of the spectrum of the Laplacian. The requirement of $W \geq 0$ is justified by scalability aspects from the theory of positive systems analysis [9], which is discussed briefly in the following section.

Remark 1: Generalizing this to the vector-valued case, i.e, when $p^i_k \in \mathbb{R}^{n_p}$ where $n_p > 1$ could be done by running independent first order protocols along each canonical direction, i. e, $p_{k+1} = (W \otimes I_{n_p}) p_k$.

B. Wrapping local tracking controllers about the first-order protocol dynamics

We now consider the physical dynamics of the N vehicles along with local tracking controllers wrapped around the simplified first-order dynamics (2). The i-th vehicle is modeled as a discrete-time LTI system

$$\begin{aligned} x^i_{k+1} &= A^i x^i_k + B^i_u u^i_k + B^i_w w^i_k \\ y^i_k &= C^i x^i_k \end{aligned} \tag{3}$$

for $i \in \{1, \cdots, N\}$, where $x^i_k \in \mathbb{R}^{n_x}$, $u^i_k \in \mathbb{R}^{n_u}$, $w^i_k \in \mathbb{R}^{n_w}$ and $y^i_k \in \mathbb{R}$ are the state, the control input, the disturbance and the output of the i^{th} system at time k, respectively. A^i, B^i_u, B^i_w and C^i are constant matrices of appropriate dimensions.

Remark 2: The system defined above is a multi-input single output system. In the light of Remark 1, it is possible to generalize the results to multiple outputs, whenever p^i_k is a vector-valued signal.

Assumption 1: The disturbance signal w^i for all vehicles is bounded in the sense of the l_2 norm and this bound is known a priori, i.e, $||w^i||_{l_2} \leq \beta$.

The controller running on board first calculates p^i_{k+1} according to (1) which is the desired output at time $k+1$ and uses this to compute the control input u^i_k as

$$u^i_k = F^i_1 x^i_k + F^i_2 p^i_k + F^i_3 p^i_{k+1} \tag{4}$$

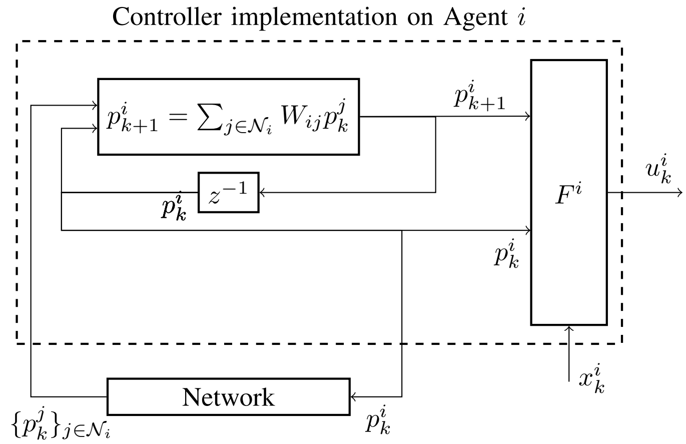

Fig. 2. Controller implemented on agent i

where $F_1^i \in \mathbb{R}^{n_u\times n_x}$, $F_2^i \in \mathbb{R}^{n_u\times 1}$ and $F_3^i \in \mathbb{R}^{n_u\times 1}$ represent the control variables. This is shown in Fig. 2. The closed-loop agent dynamics with $\eta_k^i = \begin{bmatrix} x_k^i \\ p_k^i \end{bmatrix}$ as state and the tracking error $e_k^i = p_k^i - y_k^i$ as output can then be represented along with (1) by

$$\forall i: \quad \begin{cases} \eta_{k+1}^i &= \mathcal{A}^i\eta_k^i + \mathcal{B}_q^i q_k^i + \mathcal{B}_w^i w_k^i \\ e_k^i &= \mathcal{C}^i \eta_k^i \end{cases} \tag{5}$$

where $q_k^i = p_{k+1}^i$ and

$$\mathcal{A}^i = \begin{bmatrix} A^i + B_u^i F_1^i & B_u^i F_2^i \\ \mathbf{0} & 0 \end{bmatrix}, \mathcal{B}_q^i = \begin{bmatrix} B_u^i F_3^i \\ I \end{bmatrix}, \mathcal{B}_w^i = \begin{bmatrix} B_w^i \\ 0 \end{bmatrix},$$
$$\mathcal{C}^i = \begin{bmatrix} -C^i & I \end{bmatrix}. \tag{6}$$

The overall interconnected system is shown in Fig. 1 and is governed by

$$\begin{aligned} p_{k+1} &= Wp_k \\ \forall i: \quad &\begin{cases} \eta_{k+1}^i &= \mathcal{A}^i\eta_k^i + \mathcal{B}_q^i q_k^i + \mathcal{B}_w^i w_k^i \\ e_k^i &= \mathcal{C}^i \eta_k^i \end{cases} \end{aligned} \tag{7}$$

where $q_k^i = p_{k+1}^i$, $\eta_k^i = \begin{bmatrix} x_k^i \\ p_k^i \end{bmatrix}$ and initial conditions $p_0^i = y_0^i = Cx_0^i \ \ \forall i$.

C. Objectives

For the setup described above, we wish to design controllers (E and $\{F_1^i, F_2^i, F_3^i\}_{i=1}^N$) and analyze stability and performance of the interconnected system (7). More precisely, we want to find an upper bound on the peak norm of the tracking error, i.e, $||e^i||_{l_\infty}$ under bounded disturbances (see Assumption 1). This bound can be seen as a measure of how far the actual trajectories of the vehicles are as compared to their virtual trajectories generated by the first-order protocols. This can then be used to conclude that the actual trajectories are collision-free by simply looking at the virtual trajectories generated by the first-order protocols. It can also serve to validate or invalidate the applicability of the decoupled architecture. For example, if the tracking errors are large, it would make sense to communicate rather the actual outputs of the vehicles and change the architecture. For a detailed discussion on a comparison between these architectures, see [13]. We further define a coarse-grained performance measure to quantify the deviation of the actual system in comparison to the simplified system as

$$\zeta_k = |\text{avg}(y_k) - \text{avg}(p_k)|. \tag{8}$$

If the desired relative displacements between agents i and j is denoted by r^{ij} (which for formation stabilization can be set to 0), we define a measure for cooperation for the actual system and the simplified system as

$$\begin{aligned} J_k &= \frac{1}{|\mathcal{E}|} \sum_{(v_i,v_j)\in\mathcal{E}} |y_k^j - y_k^i - r^{ij}|, \\ M_k &= \frac{1}{|\mathcal{E}|} \sum_{(v_i,v_j)\in\mathcal{E}} |p_k^j - p_k^i - r^{ij}|, \end{aligned} \tag{9}$$

respectively. Note that $\mathcal{E}$ may not be known exactly. $\mathcal{E}$ can either be estimated or (9) can be interpreted as a measure up to a scaling.

IV. THEORETICAL RESULTS

In this section, we begin by presenting two preliminary theorems (Theorem 1 and Theorem 2) which are needed as building blocks in our analysis. Analogous to the result reported in [10] and [11], for continuous-time dynamical systems, we first report a result which gives an LMI condition to find a bound on $||e||_{l_\infty}$ for inputs with a finite $||w||_{l_2}$ for discrete-time dynamical systems which corresponds to the generalized H_2 norm of the system.

Theorem 1: For a discrete-time dynamical system governed by

$$\begin{aligned} \eta_{k+1} &= \mathcal{A}\eta_k + \mathcal{B}_q q_k + \mathcal{B}_w w_k, \qquad \eta_0 = 0, \\ e_k &= \mathcal{C}\eta_k, \end{aligned} \tag{10}$$

where $\eta_k \in \mathbb{R}^{n_\eta}$, $q_k \in \mathbb{R}^{n_q}$, $w_k \in \mathbb{R}^{n_w}$ and $e_k \in \mathbb{R}^{n_e}$, if $\exists K \succ 0$, $\gamma_q > 0$, $\gamma_w > 0$ such that

$$\begin{bmatrix} \mathcal{A}^T \\ \mathcal{B}_q^T \\ \mathcal{B}_w^T \end{bmatrix} K \begin{bmatrix} \mathcal{A} & \mathcal{B}_q & \mathcal{B}_w \end{bmatrix} \prec \begin{bmatrix} K & 0 & 0 \\ 0 & \gamma_q^2 I & 0 \\ 0 & 0 & \gamma_w^2 I \end{bmatrix} \tag{11}$$

and

$$\begin{bmatrix} K & \mathcal{C}^T \\ * & I \end{bmatrix} \succ 0 \tag{12}$$

hold, then, $\forall w, q \in l_2, \ \forall k$,

$$||\eta_k|| \leq \frac{(\gamma_q||q||_{l_2} + \gamma_w||w||_{l_2})}{\sqrt{\lambda_{\min}(K)}}$$

and

$$||e_k||_2 \leq \gamma_q||q||_{l_2} + \gamma_w||w||_{l_2}.$$

Proof: See Appendix. ■

Theorem 2 ([9]): Consider a discrete-time autonomous system governed by

$$p_{k+1} = Wp_k \tag{13}$$

where $p_k \in \mathbb{R}^N$ and $W \geq 0$.

The following statements are equivalent

(1) W is Schur

(2) $\lim_{k\to\infty} p_k = \mathbf{0} \;\; \forall p_0 \in \mathbb{R}^N$

(3) $\exists$ diagonal $P \succ 0: \;\; P - W^T P W \succ 0$

(4) $\exists$ diagonal $P: \begin{bmatrix} P & W^T P \\ PW & P \end{bmatrix} \succ 0$

Proof: See [9]. ∎

Remark 3: As P is a diagonal matrix, the matrix in condition (4) is sparse based on the sparsity structure of W. Sparse techniques for Semi-Definite Programs (SDP) can be exploited to decompose this problem into smaller SDPs [14] and the verification of condition (4) leads to an analysis condition that scales linearly with N under some conditions on the sparsity structure of W. Details can be found in [14].

Building on Theorem 1 and Theorem 2, we now present one of the main results of this section.

Theorem 3: For the interconnected system (7) with given control variables F_1^i, F_2^i, F_3^i and E and matrices defined in (6), if for $0 < \alpha < 1$, $\exists P$ diagonal such that

$$\left[\begin{array}{c|c} \alpha^2 P & W^T P \\ \hline PW & P \end{array}\right] \succ 0, \tag{14}$$

and $\forall i$, $\exists K^i \succ 0$ and $\gamma_q^i > 0$, $\gamma_w^i > 0$ such that

$$\begin{bmatrix} {\mathcal{A}^i}^T \\ {\mathcal{B}_q^i}^T \\ {\mathcal{B}_w^i}^T \end{bmatrix} K^i \begin{bmatrix} \mathcal{A}^i & \mathcal{B}_q^i & \mathcal{B}_w^i \end{bmatrix} \prec \begin{bmatrix} K^i & 0 & 0 \\ 0 & {\gamma_q^i}^2 I & 0 \\ 0 & 0 & {\gamma_w^i}^2 I \end{bmatrix} \tag{15}$$

and

$$\begin{bmatrix} K^i & {\mathcal{C}^i}^T \\ * & I \end{bmatrix} \succ 0 \tag{16}$$

then the interconnected system, defined in (7), is stable in the sense that the trajectories remain bounded for bounded disturbances $||w^i||_{l_2} \leq \beta$ such that $\forall k$

$$|e_k^i|_2 \leq \gamma_q^i \sqrt{\frac{\text{cond}(P)}{1-\alpha^2}}||p_0||_2 + \gamma_w^i \beta \tag{17}$$

and

$$||\eta_k^i||_2 \leq \frac{\gamma_q^i \sqrt{\frac{\text{cond}(P)}{(1-\alpha^2)}}||p_0||_2 + \gamma_w^i \beta}{\sqrt{\lambda_{\min}(K^i)}}. \tag{18}$$

Proof: See Appendix. ∎

Corollary 3.1: Condition (17) implies the following bounds on the loss in performance due to imperfect tracking.

$$\zeta_k \leq \gamma_q^{\text{avg}} \sqrt{\frac{\text{cond}(P)}{1-\alpha^2}}||p_0||_2 + \gamma_w^{\text{avg}} \beta \tag{19}$$

and

$$J_k \leq M_k + \frac{\sum_i \gamma_q^i \cdot d^i}{|\mathcal{E}|} \sqrt{\frac{\text{cond}(P)}{(1-\alpha^2)}}||p_0||_2 + \frac{\sum_i \gamma_w^i \cdot d^i}{|\mathcal{E}|}\beta \tag{20}$$

where γ_q^{avg} and γ_w^{avg} denote the arithmetic mean of $\{\gamma_q^1, \cdots, \gamma_q^N\}$ and $\{\gamma_w^1, \cdots, \gamma_w^N\}$ respectively.

Proof: See Appendix ∎

The analysis conditions (14), (15) and (16) can be converted into synthesis conditions for finding optimal control variables F_1^i, F_2^i, F_3^i and E. For this purpose, define matrices,

$$\mathcal{A}_o^i = \begin{bmatrix} A^i & 0 \\ \mathbf{0} & 0 \end{bmatrix}, \mathcal{B}_o^i = \begin{bmatrix} B_u^i \\ 0 \end{bmatrix}, \mathcal{F}^i = \begin{bmatrix} F_1^i & F_2^i \end{bmatrix}. \tag{21}$$

The following theorem gives LMI conditions for synthesis.

Theorem 4: For the interconnected system (7), if there exist diagonal matrices $P \succ 0$ and X such that for an $0 < \alpha < 1$,

$$\begin{bmatrix} \alpha^2 P & P^T - L_c^T X^T \\ P - X L_c & P \end{bmatrix} \succ 0, \tag{22}$$

and $\forall i$, $\exists Q^i \succ 0$, Y^i, F_3^i, $\gamma_p^i > 0$,$\gamma_w^i > 0$ such that

$$\left[\begin{array}{c|cc|c} Q^i & * & & * \\ \hline \mathbf{0} & {\gamma_q^i}^2 I & \mathbf{0} & * \\ & \mathbf{0} & {\gamma_w^i}^2 I & \\ \hline \mathcal{A}_o^i Q^i + \mathcal{B}_o^i Y^i & \begin{matrix} B_u^i F_3^i \\ 1 \end{matrix} & \begin{matrix} B_w \\ \mathbf{0} \end{matrix} & Q^i \end{array}\right] \succ 0, \tag{23}$$

$$\left[\begin{array}{c|c} Q^i & * \\ \hline \begin{bmatrix} -C^i & I \end{bmatrix} Q^i & I \end{array}\right] \succ 0, \tag{24}$$

then $\mathcal{F}^i = Y^i {Q^i}^{-1}$, F_3 and $E = P^{-1} X$ stabilize the networked system in the sense of (17) and the performance bounds (19) and (20) hold.

Proof: See Appendix. ∎

Remark 4: We note that (14) and (22) are not LMIs in α and P. We run a standard bisection algorithm and find the minimum α that renders the LMIs feasible. See the discussion on solving quasi-convex optimization problems with a bisection algorithm [15].

Extension of the analysis and synthesis results to more general non-linear agent dynamics modeled as quasi-Linear Parameter Varying (qLPV) systems is straightforward and is demonstrated in the continuous-time setting in [7].

V. PRACTICAL ASPECTS AND SCALABILITY

A. Remarks on tuning

In order to consider feedback control input in the performance channel, one could replace the LMI condition (24) by

$$\left[\begin{array}{c|c} Q^i & * \\ \hline \begin{bmatrix} -C^i & 1 \\ \mathbf{0} & 0 \end{bmatrix} Q^i + \rho \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} Y^i & I \end{array}\right] \succ 0, \tag{25}$$

where $\rho \in \mathbb{R}$ acts as a tuning knob punishing the feedback control effort in the performance channel.

B. Synthesis with H_∞techniques

Alternatively, one could use standard H_∞ loop-shaping techniques to obtain a controller and then use the analysis LMIs to obtain l_∞ bounds on the error. We note that a controller that guarantees a finite H_∞ norm (induced l_2 to l_2 norm) of the closed loop necessarily leads to a finite induced l_2 to l_∞ norm (See [7], [13]).

C. Scalability

The feasibility LMI conditions (14) and (22) for analysis and synthesis of first-order protocols are *large* SDPs with a sparsity structure that depends on the sparsity structure of W (which depends on the underlying graph $\mathcal{G}$). Referring to Remark 3, these LMIs can be decomposed into N *smaller* LMIs using ideas from [14] thereby allowing a scalable analysis. The analysis LMIs $\{(15)$ and $(16)\}$ and synthesis LMIs $\{(23)$ and $(24)\}$ are N separate conditions with the size of each LMI corresponding to the respective order of the agent dynamics. The complexity thus scales linearly with the network size N. This also means that if we have a homogeneous network with identical agent dynamics, these LMIs are identical for all i which reduces to a single LMI.

VI. ILLUSTRATIVE EXAMPLES

In this section we illustrate the application of the theoretical results on illustrative examples. The code used for producing the results in this section is made available at [16].

A. Perfect tracking controller

Together with a general first order protocol (2), consider the following dynamics for all first-order agents $i \in \{1, 2, \cdots, N\}$

$$\begin{aligned} x_{k+1}^i &= a^i x_k^i + b_u^i u_k^i + b_w^i w_k^i \\ y_k^i &= x_k^i \end{aligned} \tag{26}$$

with the controller

$$u_k^i = f_1^i x_k^i + f_2^i p_{k+1}^i. \tag{27}$$

It can be seen that for $f_1^i = -a^i/b_u^i$ and $f_2^i = 1/b_u^i$, we have the closed loop agent dynamics as

$$\begin{aligned} x_{k+1}^i &= p_{k+1}^i + b_w^i w_k^i \\ y_k^i &= x_k^i \end{aligned} \tag{28}$$

along with the first order protocol (2). It can be shown that minimizing γ_q^i subject to constraint (15) gives ${\gamma_q^i}^* = 0$. This can be easily seen intuitively by observing that we have perfect tracking for arbitrary p_k^i whenever $w_k^i \equiv 0$.

Thus applying Theorem 3, we get the following $\forall k$

$$|e_k^i|_2 \leq \gamma_w^i \beta \tag{29}$$

$$\zeta_k \leq \gamma_w^{\text{avg}} \beta \tag{30}$$

$$J_k \leq M_k + \frac{\sum_i \gamma_w^i \cdot d^i}{|\mathcal{E}|} \beta. \tag{31}$$

We can interpret this as follows. For scenarios where we have very good tracking controllers acting locally on the agents, i.e, when $\gamma_q \approx 0$, the performance loss due to input disturbances acting on the agents can be estimated a priori by (30) and (31). Moreover, the boundedness of trajectories is implied just by analyzing the system of simple agents as seen in (29). It also has an interesting implication for non-linear differentially flat systems [17] where it is possible to achieve perfect tracking in the absence of disturbances. Additionally, bound (31) could be used to further prove collision avoidance by making an energy based argument as in [18].

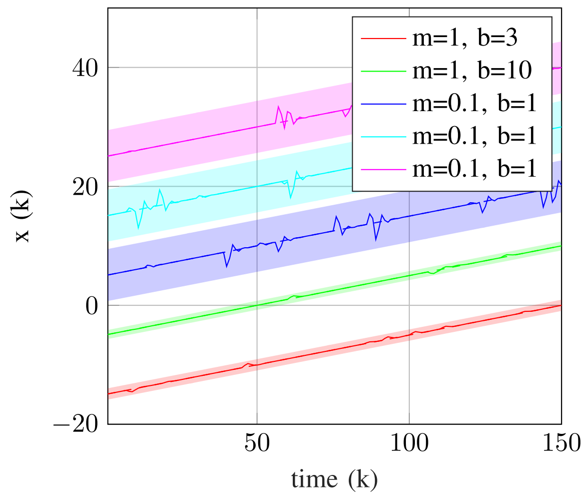

Fig. 3. Disturbance rejection during platooning

B. Generic vehicle model

We now consider a more realistic example of a generic second order vehicle model given by

$$m\ddot{q} + b\dot{q} = u + w \tag{32}$$

where m, b, q, u and w represent the mass, damping coefficient, position, forcing input and disturbance input, respectively. We write this model in a state space form and discretize using Zero-Order-Hold (ZOH) with a time-step of 0.1 to obtain a discrete-time state space model.

1) Disturbance rejection in Platooning: We first consider a platooning scenario with a heterogeneous network of agents perturbed by an l_2 input disturbance. We consider a group of five agents modeled by (32) with

- Agent 1: $m^1 = 1$, $b^1 = 3$
- Agent 2: $m^2 = 1$, $b^2 = 10$
- Agents 3, 4 and 5: $m^{3,4,5} = 0.1$, $b^{3,4,5} = 1$

A desired platoon velocity of 0.1 units is considered. In the absence of disturbances, a constant feed-forward control input will lead to the desired trajectory. The error dynamics are again governed by (32) where the control input is the deviation from the feedforward control input. A controller for the error dynamics of each agent is designed by solving the following convex optimization problem for each i:

$$\min \ \gamma_q^i + \gamma_w^i \quad \text{s.t. (23), (25).}$$

The closed loop dynamics under a disturbance signal with $\beta = 100$ along with theoretical bounds (depicted by shaded regions) are shown in Figure 3. Note that because agent 3, 4 and 5 have the least mass and are identical, the deviation bound are the greatest whereas agents 1 and 2 are heavy, leading to a smaller bound. We can conclude that as long as the bounds on the input disturbances are valid, the trajectories will remain collision free. Note that if $\beta \cdot (\gamma_w^i + \gamma_w^j)$ is higher than the distance between the nominal trajectories of agents i and j, generated by the first-order protocol,the bounding boxes would intersect allowing the possibility of a collision

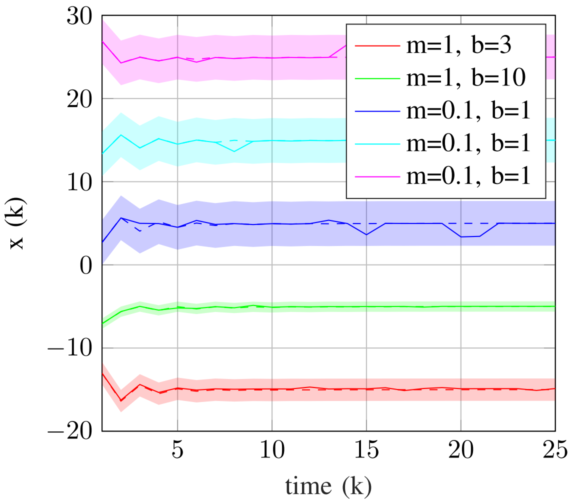

Fig. 4. Formation attainment trajectories.

and thereby demonstrating a need for a coupled architecture or a redesign.

2) Formation forming at arbitrary locations: We now illustrate the validity of the approach for the problem of formation attainment at a desired point. We consider a network of these five agents with a first order protocol running on board to agree upon the desired locations. We consider a graph randomly generated by setting the probability of having an edge between any two nodes to 0.6. We assume that the agent 1 is externally controlled as in (2) with $K_{\text{lead}} = 5$. We solve the convex optimization problem

$$\min \gamma_q + \gamma_w \quad \text{s.t. (23), (24), (22) hold.}$$

Figure 4 shows the trajectories for the formation attainments problem along with the theoretical bounds depicted by shaded regions. The dashed trajectories are the trajectories of the simplified dynamics (first order protocol) which are reasonably well tracked. Since the shaded regions do not intersect, we can conclude that the actual trajectories are collision free by simply studying the simplified dynamics. On the contrary, if the shaded regions do intersect, there would be a possibility of a collision which calls for a redesign or a coupled architecture.

VII. CONCLUSIONS

We have analyzed the performance of a decoupled network architecture of LTI vehicles by considering first-order protocols as the interaction mechanism and wrapping local controllers around these dynamics. LMIs that scale linearly with the number of agents can be obtained for bounding the l_∞ norm of local tracking error. Initial steps towards extending analysis LMIs to synthesis for local controller design and protocol design have been presented. The applicability of the synthesized controllers on generic second order vehicle models was demonstrated in simulations along with scenarios where a coupled architecture would be recommended.

APPENDIX

Proof: [Theorem 1] Condition (11) is equivalent to the existence of $\epsilon > 0$ such that $\forall \eta_k \in \mathbb{R}^{n_\eta}, \forall w_k \in \mathbb{R}^{n_w}$, $\forall q_k \in \mathbb{R}^{n_q}$,

$$\eta_{k+1}^T K \eta_{k+1} \leq \eta_k^T (K - \epsilon I)\eta_k + (\gamma_q^2 - \epsilon)||q_k||_2^2 + (\gamma_w^2 - \epsilon)||w_k||_2^2.$$

Considering the storage function $V_k = \eta_k^T K \eta_k$, we have,

$$\begin{aligned} V_{k+1} &\leq V_k + \gamma_q^2 ||q_k||_2^2 + \gamma_w^2 ||w_k||_2^2 \\ &\leq V_0 + \gamma_q^2 \sum_{i=0}^{k} ||q_i||_2^2 + \gamma_w^2 \sum_{i=0}^{k} ||w_i||_2^2 \end{aligned}$$

Then, using $\eta_0 = 0$, we obtain the following bound $\forall w, q \in l_2$

$$V_k \leq \gamma_q^2 \sum_{i=0}^{k-1} ||q_i||_2^2 + \gamma_w^2 \sum_{i=0}^{k-1} ||w_i||_2^2 \leq \gamma_q^2 ||q||_{l_2}^2 + \gamma_w^2 ||w||_{l_2}^2. \tag{33}$$

Using $\lambda_{\min}(K)||\eta_k||_2^2 \leq V_k$,

$$||\eta_k||^2 \leq \frac{(\gamma_q^2 ||q||_{l_2}^2 + \gamma_w^2 ||w||_{l_2}^2)}{\lambda_{\min}(K)},$$

which implies

$$||\eta_k|| \leq \frac{(\gamma_q ||q||_{l_2} + \gamma_w ||w||_{l_2})}{\sqrt{\lambda_{\min}(K)}}.$$

Using the Schur complement, condition (12) can be converted to the following equivalent condition

$$K - C^T C \succ 0 \tag{34}$$

which implies

$$V_k = \eta_k^T K \eta_k \geq \eta_k^T C^T C \eta_k = e_k^T e_k \quad \forall \eta_k \in \mathbb{R}^{n_\eta}$$

which together with equation (33) implies $\forall w, q \in l_2$, $\forall k$

$$||e_k||_2^2 \leq V_k \leq \gamma_q^2 ||q||_{l_2}^2 + \gamma_w^2 ||w||_{l_2}^2$$

and

$$||e_k||_2 \leq \gamma_q ||q||_{l_2} + \gamma_w ||w||_{l_2}.$$

■

Proof: [Theorem 3] Condition (14) is equivalent to the existence of a diagonal $P \succ 0$ such that

$$\alpha^2 P - W^T P W \succ 0$$

which implies

$$p_{k+1}^T P p_{k+1} \leq \alpha^2 p_k^T P p_k \quad \forall p_k \in \mathbb{R}^{n_p}.$$

Therefore,

$$||p_k||_2^2 \leq \alpha^{2k} \text{cond}(P) ||p_0||_2^2 \quad \forall p_0 \in \mathbb{R}^{n_p}.$$

The infinite sum for a converging geometric series gives

$$||p||_{l_2}^2 \leq \frac{1}{1-\alpha^2} \text{cond}(P) ||p_0||_2^2$$

and

$$||p^i||_{l_2} \leq ||p||_{l_2} \leq \sqrt{\frac{\text{cond}(P)}{1-\alpha^2}} ||p_0||_2$$

The bounds (17) and (18) are then obtained by simply applying theorem 1 and using the bound derived above on $||p^i||_{l_2}$. ■

Proof: [Corollary 3.1] Using definitions (8) and bound (17), we can derive the bound on ν_k as

$$\begin{aligned}\zeta_k &= |\text{avg}(y_k) - \text{avg}(p_k)| \\ &= |\text{avg}(e_k)| \\ &\leq \frac{1}{N}\sum_{i=1}^{N}(\gamma_q^i\sqrt{\frac{\text{cond}(P)}{1-\alpha^2}}||p_0||_2 + \gamma_w^i\beta) \\ &\leq \gamma_q^{\text{avg}}\sqrt{\frac{\text{cond}(P)}{1-\alpha^2}}||p_0||_2 + \gamma_w^{\text{avg}}\beta\end{aligned}$$

Now using definition (9) and bound (17), we can derive the bound on J_k as

$$\begin{aligned}J_k &= \frac{1}{|\mathcal{E}|}\sum_{(v_i,v_j)\in\mathcal{E}}|y_k^j - y_k^i| \\ &= \frac{1}{|\mathcal{E}|}\sum_{(v_i,v_j)\in\mathcal{E}}|(p_k^j - p_k^i) - (e_k^j - e_k^i)| \\ &\leq M_k + \frac{1}{|\mathcal{E}|}\sum_{(v_i,v_j)\in\mathcal{E}}|e_k^j| + |e_k^i| \\ &= M_k + \frac{1}{|\mathcal{E}|}\sum_{i=1}^{N}d^i|e_k^i| \\ &\leq M_k + \frac{\sum_i\gamma_q^i\cdot d^i}{|\mathcal{E}|}\sqrt{\frac{\text{cond}(P)}{(1-\alpha^2)}}||p_0||_2 + \frac{\sum_i\gamma_w^i\cdot d^i}{|\mathcal{E}|}\beta\end{aligned} \tag{35}$$

■

Proof: [Theorem 4] Using the Schur complement, condition (15) can be written in a compact form as

$$\left[\begin{array}{ccc|c} K^i & 0 & 0 & {\mathcal{A}^i}^T K^i \\ 0 & {\gamma_q^i}^2 I & 0 & {\mathcal{B}_q^i}^T K^i \\ 0 & 0 & {\gamma_w^i}^2 I & {\mathcal{B}_w^i}^T K^i \\ \hline K^i\mathcal{A}^i & K^i\mathcal{B}_q^i & K\mathcal{B}_w^i & K^i \end{array}\right] \succ 0.$$

A congruence transformation with a non-singular matrix

$$\begin{bmatrix} {K^i}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & {K^i}^{-1} \end{bmatrix}$$

gives us the equivalent condition

$$\left[\begin{array}{ccc|c} Q^i & 0 & 0 & Q^i{\mathcal{A}^i}^T \\ 0 & {\gamma_q^i}^2 I & 0 & {\mathcal{B}_q^i}^T \\ 0 & 0 & {\gamma_w^i}^2 I & {\mathcal{B}_w^i}^T \\ \hline \mathcal{A}^i Q^i & \mathcal{B}_q^i & \mathcal{B}_w^i & Q^i \end{array}\right] \succ 0 \tag{36}$$

where $Q^i = {K^i}^{-1}$.

Similarly, condition (16), i.e,

$$\begin{bmatrix} K^i & {\mathcal{C}^i}^T \\ * & I \end{bmatrix} > 0$$

is equivalent to

$$\begin{bmatrix} Q^i & Q^i{\mathcal{C}^i}^T \\ \mathcal{C}^i Q^i & I \end{bmatrix} > 0. \tag{37}$$

With definitions (6), (21) and $\mathcal{F}^i Q^i = Y^i$, we get the equivalent conditions (23) and (24).

By substituting $W = I - EL_c$ in condition (14) and defining $X = PE$, we obtain condition (22). The proof then follows simply as a consequence of Theorem 3.

■

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