Appendix

Decoupling Approach for Solving Linear Quadratic Discrete-Time Games with Application to Consensus Problem

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Algorithm for Problem 1

Algorithm 1 Nash Equilibrium via coupled Riccati difference equations

Input: each agent position and velocity or the state x^i at current time k
Output: each control inputs u_k^i and its agent position and velocity x_{k+1}^i

Initialization $P_T^i = Q_T^i$,
for $k = 1 : T - 1$ **do**
  for $j = T - 1 : -1 : 1$ **do**
    for $i = 1 : N$ **do**
      $S^i = G^i {R^{ii}}^{-1} {G^i}^T$
      $\Lambda_k = I + \sum_{i=1}^{N} S^i P_{j+1}^i$
      $P_j^i = Q^i + F^T P_{j+1}^i \Lambda_k^{-1} F$
    end for
  end for
  for $i = 1 : N$ **do**
    $u_k^i = -{R^{ii}}^{-1} {G^i}^T P_{k+1}^i \Lambda_k^{-1} F x_k$
  end for
  $x_{k+1} = F x_k + \sum_{i=1}^{N} G^i u_k^i$
end for

Algorithm for Problem 2

To evaluate this method, the same step as in the previous problem, we initialize $P_T = Q_T$. Therefore, we have the algorithm as below:

Algorithm 2 LQDTG via single Riccati difference equations

Input: agent position and velocity or the state x at current time k
Output: control input u_k and its agent position and velocity x_{k+1}

Initialization $P_T = Q_T$,
for $k = 1 : T - 1$ **do**
  for $j = T - 1 : -1 : 1$ **do**
    $P_j = Q + F^T P_{j+1} F + F^T P_{j+1} G K_j$
  end for
  $K_k = -(R + G^T P_{k+1} G)^{-1} G^T P_{k+1} F$
  $u_k = K_k x_k$
  $x_{k+1} = F x_k + G u_k$
end for

Algorithm for Problem 3

The same step as in the previous problem, we initialize $\tilde{P}_T = \tilde{Q}_T$. To implement the decoupling framework in the open-loop solution, we provide the systematic way as in the Algorithm 3.

Algorithm 3 Optimal control via decoupling framework

Input: relative state z at current time k

Output: relative control input a_k, relative states z_{k+1}, transformed control input $\hat{u}_k$, and transformed states $\hat{x}_{k+1}$

Initialization $\tilde{P}_T = \tilde{Q}_T$, $\Phi^\dagger = \text{pinv}(\Phi)$
for $k = 1 : T-1$ **do**
  for $j = T-1 : -1 : 1$ **do**
    $\tilde{P}_j = \tilde{Q} + \tilde{F}^T\tilde{P}_{j+1}\tilde{F} + \tilde{F}^T\tilde{P}_{j+1}\tilde{G}\tilde{K}_j$
  end for
  $\tilde{K}_k = -(\tilde{R} + \tilde{G}\tilde{P}_{k+1}\tilde{G})^{-1}\tilde{G}^T\tilde{P}_{k+1}\tilde{F}$
  $a_k = \tilde{K}_k z_k$
  $z_{k+1} = \tilde{F}z_k + \tilde{G}a_k$
  $\hat{u}_k = \Phi^\dagger a_k$
  $\hat{x}_{k+1} = F\hat{x}_k + G\hat{u}_k$
end for

Proof of Theorem 2

Recall the dynamics

$$x_{k+1} = Fx_k + Gu_k, \tag{1}$$

and the cost function

$$J_k = \frac{1}{2}x_T^T Q_T x_T + \frac{1}{2}\sum_{l=0}^{T-1}\Big(x_{k+l+1}^T Q x_{k+l+1} + u_{k+l}^T R u_{k+l}\Big), \tag{2}$$

To begin the proof, we firstly define the Hamiltonian function of (1) and (2) as:

$$H_k = \frac{1}{2}\sum_{l=0}^{T-1}\left(x_{k+l+1}^T Q x_{k+l+1} + u_{k+l}^T R u_{k+l}\right) + \lambda_{k+1}^T(Fx_k + Gu_k) \tag{3}$$

Conditions to be met for Euler-Lagrange equations are:

$$\lambda_k = \frac{\partial H_k}{\partial x_k} = Qx_k + F^T\lambda_{k+1} \tag{4}$$

$$\lambda_T = \frac{\partial (x_T^T Q_T x_T)}{\partial x_T} = P_T x_T \tag{5}$$

and for all $k \in [0, T-1]$

$$\frac{\partial H_k}{\partial u_k} = 0 \rightarrow Ru_k + G\lambda_{k+1} = 0$$

$$u_k^* = -R^{-1}G^T\lambda_{k+1} \tag{6}$$

Substituting (6) to (1), we have a closed-loop state as:

$$x_{k+1} = Fx_k - GR^{-1}G^T\lambda_{k+1} \tag{7}$$

From (5), supposed that the co-state is linearly related with the state vector then we have

$$\lambda_k = P_k x_k \quad \text{and} \quad \lambda_{k+1} = P_{k+1}x_{k+1} \tag{8}$$

Then, equation (7) can be rewritten as

$$x_{k+1}^\star = Fx_k - GR^{-1}G^T P_{k+1} x_{k+1^\star}$$

$$0 = Fx_k - \left[I + GR^{-1}G^T P_{k+1}\right] x_{k+1^\star}$$

If $\left[I + GR^{-1}G^T P_{k+1}\right]$ is invertible, we have

$$x^*_{k+1} = \left[I + GR^{-1}G^T P_{k+1}\right]^{-1} F x_k \tag{9}$$

From (4) and (8), we have

$$P_k x_k = Q x_k + F^T P_{k+1} x_{k+1} \tag{10}$$

and replacing x_{k+1} by (7), we obtain

$$P_k x_k = Q x_k + F^T P_{k+1} \left[I + GR^{-1}G^T P_{k+1}\right]^{-1} F x_k \tag{11}$$

Neglecting x_k from the left and right hand-side from (11), we have

$$P_k = Q + F^T P_{k+1} \left[I + GR^{-1}G^T P_{k+1}\right]^{-1} F \tag{12}$$

Using matrix inversion lemma, form of (12) can be rewritten as:

$$P_k = Q^i + F^T P_{k+1} F - F^T P_{k+1} G \Big(R + G^T P_{k+1} G\Big)^{-1} G^T P_{k+1} F \tag{13}$$

Proof of Theorem 3

Recall the relative dynamics:

$$z_{k+1} = \tilde{F} z_k + \tilde{G} a_k, \tag{14}$$

and the new relative cost

$$\tilde{J}_k = \frac{1}{2} z_T^T \tilde{Q}_T z_T + \frac{1}{2} \sum_{l=0}^{T-1} \Big(z^T_{k+l+1} \tilde{Q} z_{k+l+1} + a^T_{k+l} \tilde{R} a_{k+l}\Big), \tag{15}$$

Firstly, define the Hamiltonian that is composed from (14) and (15) as:

$$\tilde{H}_k = \frac{1}{2} \sum_{l=0}^{T-1} \Big(z^T_{k+l+1} \tilde{Q} z_{k+l+1} + a^T_{k+l} \tilde{R} a_{k+l}\Big) + \tilde{\lambda}^T_{k+1} \Big(\tilde{F} z_k + \tilde{G} a_k\Big)$$

Conditions to be met for Euler-Lagrange equation are:

$$\tilde{\lambda}_k = \frac{\partial \tilde{H}_k}{\partial z_k} = \tilde{Q} z_k + \tilde{F}^T \tilde{\lambda}_{k+1}, \tag{16}$$

$$\tilde{\lambda}_T = \frac{1}{2} \Big(\frac{\partial z_T^T \tilde{Q}_T z_T}{\partial z_T}\Big) = \tilde{P}_T z_T, \tag{17}$$

and for all $k \in [0, T-1]$

$$\begin{aligned} \frac{\partial \tilde{H}_k}{\partial a_k} = 0 \rightarrow \tilde{R} a_k + \tilde{G} \lambda_{k+1} &= 0 \\ a^\star_k &= -\tilde{R}^{-1} \tilde{G}^T \tilde{\lambda}_{k+1}. \end{aligned} \tag{18}$$

Substituting (18) to (14), we have a closed-loop state as:

$$z^\star_{k+1} = \tilde{F} z_k - \tilde{G} \tilde{R}^{-1} \tilde{G}^T \tilde{\lambda}_{k+1} \tag{19}$$

From (17), supposed that the co-state is linearly related with the state vector then we have

$$\tilde{\lambda}_k = \tilde{P}_k z_k \quad \text{and} \quad \tilde{\lambda}_{k+1} = \tilde{P}_{k+1} z_{k+1}.$$

Then equation (19) can be rewritten as

$$\begin{aligned} z^\star_{k+1} &= \tilde{F} z_k - \tilde{G} \tilde{R}^{-1} \tilde{G}^T \tilde{P}_{k+1} z_{k+1^\star} \\ 0 &= \tilde{F} z_k - \left[I + \tilde{G} \tilde{R}^{-1} \tilde{G}^T \tilde{P}_{k+1}\right] z_{k+1^\star} \end{aligned}$$

If $\left[I+\tilde{G}\tilde{R}^{-1}\tilde{G}^T\tilde{P}_{k+1}\right]$ is invertible **Jank**, we have

$$z^\star_{k+1} = \left[I+\tilde{G}\tilde{R}^{-1}\tilde{G}^T\tilde{P}_{k+1}\right]^{-1}\tilde{F}z_k. \tag{20}$$

From (16) and the previous statement, we have

$$\tilde{P}_k z_k = \tilde{Q}z_k + \tilde{F}^T\tilde{P}_{k+1}z_{k+1}$$

and replacing z_{k+1} by (20), we obtain

$$\tilde{P}_k z_k = \tilde{Q}z_k + \tilde{F}^T\tilde{P}_{k+1}\left[I+\tilde{G}\tilde{R}^{-1}\tilde{G}^T\tilde{P}_{k+1}\right]^{-1}\tilde{F}z_k \tag{21}$$

Neglecting z_k from the left and right-hand side in (21) we have

$$\tilde{P}_k = \tilde{Q} + \tilde{F}^T\tilde{P}_{k+1}\left[I+\tilde{G}\tilde{R}^{-1}\tilde{G}^T\tilde{P}_{k+1}\right]^{-1}\tilde{F} \tag{22}$$

Using matrix inversion lemma, we have

$$\begin{aligned}
\tilde{P}_k &= \tilde{Q} + \tilde{F}^T\tilde{P}_{k+1}\tilde{F} - \tilde{F}^T\tilde{P}_{k+1}\tilde{G}(\tilde{R}+\tilde{G}^T\tilde{P}_{k+1}\tilde{G})^{-1}\tilde{G}^T\tilde{P}_{k+1}\tilde{F} \\
&= \tilde{Q} + \tilde{F}^T\tilde{P}_{k+1}\Big[I - \tilde{G}(\tilde{R}+\tilde{G}^T\tilde{P}_{k+1}\tilde{G})^{-1}\tilde{G}^T\tilde{P}_{k+1}\Big]\tilde{F} \\
&= \tilde{Q} + \tilde{F}^T\tilde{P}_{k+1}\Big[I - \tilde{G}\Big((I+\tilde{G}^T\tilde{P}_{k+1}\tilde{G}\tilde{R}^{-1})\tilde{R}\Big)^{-1}\tilde{G}^T\tilde{P}_{k+1}\Big]\tilde{F} \\
&= \tilde{Q} + \tilde{F}^T\tilde{P}_{k+1}\tilde{R}^{-1}\Big[I - \tilde{G}(I+\tilde{G}^T\tilde{P}_{k+1}\tilde{G}\tilde{R}^{-1})^{-1}\tilde{G}^T\tilde{P}_{k+1}\Big]\tilde{F} \\
&= \tilde{Q} + \tilde{F}^T\tilde{P}_{k+1}\Big[I + \tilde{G}\tilde{R}^{-1}\tilde{G}^T\tilde{P}_{k+1}\Big]^{-1}\tilde{F}
\end{aligned}$$

This concludes the proof.