Teorie odhadů

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Je odhad $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$ nestranný pokud $X_i \sim N(\mu,\sigma^2)$?

a) μ je známé

$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum_i^n (X_i-\mu)^2\right] = \frac{1}{n}\sum_i^n E[(X_i-\mu)^2] = \frac{1}{n}\sum_i^n \sigma^2 = \sigma^2 \Rightarrow$ nestranný

b) μ neznáme $\Rightarrow \hat{\mu} = \frac{1}{n}\sum_i X_i$

$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum_i^n \left(X_i - \frac{1}{n}\sum_j^n X_j\right)^2\right] = \frac{1}{n}E\left[\sum_i^n\left(X_i^2 - 2X_i\frac{1}{n}\sum_j X_j + \frac{1}{n^2}\sum_j X_j \sum_k X_k\right)\right] =$

$= \frac{1}{n}\sum_i E[X_i^2] - \frac{2}{n^2}\sum_i\sum_j E[X_iX_j] + \frac{n}{n^3}\sum_j\sum_k E[X_jX_k] = \frac{1}{n}\sum_i E[X_i^2] - \frac{1}{n^2}\sum_i\sum_j E[X_iX_j] =$

$= \frac{1}{n}\sum_i(\sigma^2+\mu^2) - \frac{1}{n^2}\left(\underbrace{(n^2-n)}_{i\neq j}\cdot\mu^2 + \underbrace{n}_{i=j}(\sigma^2+\mu^2)\right) = \sigma^2 + \cancel{\mu^2} - \frac{1}{n^2}\left(\cancel{n^2\mu^2} - \cancel{n\mu^2} + n\sigma^2 + \cancel{n\mu^2}\right) =$

$= \sigma^2 - \frac{1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2 \Rightarrow$ není nestranný

z definice rozptylu $\sigma^2 = E[(X_i-\mu)^2] = E[X_i^2] - 2\mu\underbrace{E[X_i]}_{\mu} + \mu^2 = E[X_i^2] - \mu^2$

$E[X_iX_j] = \begin{cases} i=j & E[X_i^2] \overset{*}{=} \sigma^2+\mu^2 \\ i\neq j & E[X_i]\cdot E[X_j] = \mu^2 \end{cases}$

$* \quad E[X_i^2] = E[\underbrace{X_i-\mu}_{a} + \underbrace{\mu}_{b}]^2 = E[(X_i-\mu)^2] + 2E[(X_i-\mu)\cdot\mu] + E[\mu^2] =$

$= \sigma^2 + 2\mu E[X_i-\mu] + \mu^2 = \sigma^2+\mu^2$

$E[X_i] = \mu$

$2\mu E[\mu-\mu] = 0$