Nechť x_i, $i = 1 \dots N$ má exponenciální rozložení s parametrem θ

$p(\theta) = \lambda e^{-\lambda\theta}$ $\quad \theta \geq 0$ $\quad$ je apriorní rozložení θ (též exponenciální, s parametrem λ)

Odvoďte $\hat{\theta}_{MSE}$ a $\hat{\theta}_{MAP}$.

$$p(x_i \mid \theta) = \theta e^{-x_i \theta} \qquad x_i \geq 0$$

$$p(x \mid \theta) = \prod_{i=1}^{N} \theta e^{-x_i \theta} = \theta^N e^{-\theta \sum_{i=1}^{N} x_i}$$

$$p(\theta \mid x) = \frac{p(x\mid\theta)\, p(\theta)}{p(x)} = \frac{p(x\mid\theta)\, p(\theta)}{\int_0^{+\infty} p(x\mid\theta)\, p(\theta)\, d\theta} = \frac{\theta^N e^{-\theta\sum x_i} \cdot \lambda e^{-\lambda\theta}}{\int_0^{+\infty} \theta^N e^{-\theta\sum x_i} \lambda e^{-\lambda\theta}\, d\theta}$$

<u>MSE odhad:</u>

$$\hat{\theta}_{MSE} = \int \theta\, p(\theta\mid x)\, d\theta = \frac{\lambda \int \theta^{N+1} e^{-\theta(\sum x_i + \lambda)}\, d\theta}{\lambda \int \theta^N e^{-\theta(\sum x_i + \lambda)}\, d\theta} \overset{\text{per partes}}{=} \frac{\left[\theta^{N+1} \frac{1}{-(\sum x_i + \lambda)} e^{-\theta(\sum x_i + \lambda)}\right]_0^{+\infty} - \int_0^{+\infty} \frac{N+1}{-(\sum x_i + \lambda)} \theta^N e^{-\theta(\sum x_i + \lambda)}\, d\theta}{\int_0^{+\infty} \theta^N e^{-\theta(\sum x_i + \lambda)}\, d\theta}$$

(hranatá závorka $= 0$; vykrátí se)

$$= \frac{N+1}{\sum_{i=1}^{N} x_i + \lambda}$$

<u>MAP odhad:</u>

$$\hat{\theta}_{MAP} = \arg\max_{\theta} p(\theta\mid x) = \arg\max_{\theta} \log p(\theta \mid x)$$

$$\frac{\partial}{\partial\theta} \log p(\theta\mid x) = \frac{\partial}{\partial\theta}\left(N \log\theta + \log\lambda - \theta\left(\sum_{i=1}^{N} x_i + \lambda\right)\right) = \frac{N}{\theta} - \left(\sum_{i=1}^{N} x_i + \lambda\right) = 0$$

$$\Rightarrow \quad \hat{\theta}_{MAP} = \frac{N}{\sum_{i=1}^{N} x_i + \lambda}$$

per partes

$$\int f g' = fg - \int f' g$$

$f(\theta) = \theta^{N+1}$ $\qquad f'(\theta) = (N+1)\theta^N$

$g'(\theta) = e^{-\theta(\sum x_i + \lambda)}$ $\qquad g(\theta) = \frac{1}{-(\sum x_i + \lambda)} e^{-\theta(\sum x_i + \lambda)}$