CRLB: $X \sim N(\mu, \sigma^2) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\mathcal{L}(\mu|x_i) = \sum_{i=1}^{N} -\frac{(x_i-\mu)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)$$

CRLB PRO NORMÁLNÍ ROZDĚLENÍ

① ODHAD μ, POKUD σ^2 ZNÁME

$$\frac{\partial \mathcal{L}}{\partial \mu} = \sum_i \frac{x_i-\mu}{\sigma^2} \qquad \left(\text{regularita: } E\left[\frac{\partial \mathcal{L}}{\partial \mu}\right] = E\left[\frac{x-\mu}{\sigma^2}\right] = 0\right)$$

$$\frac{\partial^2 \mathcal{L}}{\partial \mu^2} = -\frac{N}{\sigma^2} \Rightarrow \mathrm{var}(\hat{\mu}) \geq \frac{\sigma^2}{N}$$

② ODHAD σ^2, POKUD μ ZNÁME

$\boxed{\text{CVIČENÍ 5: PŘ 1}}$

$$\frac{\partial \mathcal{L}}{\partial \sigma^2} = \sum_i \frac{(x_i-\mu)^2}{2(\sigma^2)^2} - \frac{1}{2}\frac{1}{\sigma^2} \qquad \left(\text{regularita: } E\left[\frac{(x_i-\mu)^2}{2(\sigma^2)^2} - \frac{1}{2\sigma^2}\right] = 0\right)$$

$$\frac{\partial^2 \mathcal{L}}{\partial (\sigma^2)^2} = \sum_i -\frac{(x_i-\mu)^2}{(\sigma^2)^3} + \frac{1}{2}\frac{1}{(\sigma^2)^2} \Rightarrow -E\left[\frac{\partial^2 \mathcal{L}}{\partial (\sigma^2)^2}\right] = N\left(\frac{\sigma^2}{(\sigma^2)^3} - \frac{1}{2}\frac{1}{(\sigma^2)^2}\right) = \frac{N}{2(\sigma^2)^2}$$

$\boxed{\text{CVIČENÍ 4: PŘ 1}}$

$$\Rightarrow \mathrm{var}(\hat{\sigma}^2) \geq \frac{2(\sigma^2)^2}{N}$$

③ DŮKAZ PRO FI $\mathcal{I}(\theta)$

$$\frac{\partial^2 \log p(\theta|x)}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\frac{\partial p(\theta|x)}{\partial \theta}}{p(\theta|x)}\right) = \frac{\frac{\partial^2 p(\theta|x)}{\partial \theta^2} p(\theta|x) - \left(\frac{\partial p(\theta|x)}{\partial \theta}\right)^2}{p^2(\theta|x)} =$$

$$= \frac{1}{p(\theta|x)} \frac{\partial^2 p(\theta|x)}{\partial \theta^2} - \left(\frac{\frac{\partial p(\theta|x)}{\partial \theta}}{p(\theta|x)}\right)^2 = \frac{1}{p(\theta|x)}\frac{\partial^2 p(\theta|x)}{\partial \theta^2} - \left(\frac{\partial \ln p}{\partial \theta}\right)^2$$

$$E\left[\frac{1}{p}\frac{\partial^2 p}{\partial \theta^2}\right] = \int_{\mathbb{R}} \frac{\partial^2 p}{\partial \theta^2} d\theta = \frac{\partial^2}{\partial \theta^2}\int_{\mathbb{R}} p(\theta|x)\, d\theta = 0 \qquad \square$$