Je odhad $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$ nestranný pokud $X_i \sim N(\mu,\sigma^2)$ když a) μ je známé
b) μ je neznámé?

a) μ je známé

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum_i (X_i-\mu)^2\right] = \frac{1}{n}\sum_i E[(X_i-\mu)^2] = \frac{1}{n}\sum_i \sigma^2 = \sigma^2 \Rightarrow \text{nestranný } \underline{\underline{\text{je}}}$$

b) μ neznáme, takže ho nahradíme odhadem $\hat{\mu} = \frac{1}{n}\sum_i X_i$

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum_i \left(X_i - \frac{1}{n}\sum_j X_j\right)^2\right] = \frac{1}{n}E\left[\sum_i\left(X_i^2 - 2X_i\frac{1}{n}\sum_j X_j + \frac{1}{n^2}\left(\sum_j X_j\right)\left(\sum_k X_k\right)\right)\right] =$$

$$= \frac{1}{n}\sum_i E[X_i^2] - \frac{2}{n^2}\underbrace{\sum_i\sum_j E[X_iX_j] + \frac{1}{n^2}\sum_j\sum_k E[X_jX_k]}_{\text{stejné}} =$$

$$= \frac{1}{n}\sum_i E[X_i^2] - \frac{1}{n^2}\sum_i\sum_j E[X_iX_j] = *$$

Pomocné výpočty:

$$\sigma^2 = E[(X_i-\mu)^2] = E[X_i^2] - 2\mu \underbrace{E[X_i]}_{\mu} + \mu^2 = E[X_i^2] - \mu^2 \Rightarrow E[X_i^2] = \sigma^2+\mu^2$$

$$E[X_iX_j] = \begin{cases} i=j & E[X_i^2] = \sigma^2+\mu^2 \\ i\neq j & \underbrace{E[X_i]}_{\mu}\underbrace{E[X_j]}_{\mu} = \mu^2 \end{cases}$$

Pokračujeme:

$$* = \frac{1}{n}\sum_i(\sigma^2+\mu^2) - \frac{1}{n^2}\left((n^2-n)(\mu^2) + n(\mu^2+\sigma^2)\right) = \sigma^2+\mu^2-\mu^2-\frac{1}{n}\sigma^2 = \underline{\underline{\frac{n-1}{n}\sigma^2}}$$

$\Rightarrow$ není nestranný