CHAPTER 7

Making Decisions

7.1 Expected Utility

This chapter concerns situations in which decisions need to be taken under uncertainty. Consider the following scenario : you are asked if you wish to take a bet on the outcome of tossing a fair coin. If you bet and win, you gain £100. If you bet and lose, you lose £200. If you don't bet, the cost to you is zero. We can set this up using a two state variable x, with $\mathrm{dom}(x) = \{\mathsf{win}, \mathsf{lose}\}$, a decision variable d with $\mathrm{dom}(d) = \{\mathsf{bet}, \mathsf{no\ bet}\}$ and utilities as follows:

$$U(\mathsf{win}, \mathsf{bet}) = 100, \quad U(\mathsf{lose}, \mathsf{bet}) = -200, \quad U(\mathsf{win}, \mathsf{no\ bet}) = 0, \quad U(\mathsf{lose}, \mathsf{no\ bet}) = 0 \tag{7.1.1}$$

Since we don't know the state of x, in order to make a decision about whether or not to bet, arguably the best we can do is work out our expected winnings/losses under the situations of betting and not betting[240]. If we bet, we would expect to gain

$$U(\mathsf{bet}) = p(\mathsf{win}) \times U(\mathsf{win}, \mathsf{bet}) + p(\mathsf{lose}) \times U(\mathsf{lose}, \mathsf{bet}) = 0.5 \times 100 - 0.5 \times 200 = -50$$

If we don't bet, the expected gain is zero, $U(\mathsf{no\ bet}) = 0$. Based on taking the decision which maximises expected utility, we would therefore be advised not to bet.

Definition 52 (Subjective Expected Utility)**.** The utility of a decision is

$$U(d) = \langle U(d, x) \rangle_{p(x)} \tag{7.1.2}$$

where $p(x)$ is the distribution of the outcome x and d represents the decision.

7.1.1 Utility of money

You are a wealthy individual, with £1,000,000 in your bank account. You are asked if you would like to participate in a fair coin tossing bet in which, if you win, your bank account will become £1,000,000,000. However, if you lose, your bank account will contain only £1000. Assuming the coin is fair, should you take the bet?

If we take the bet our expected bank balance would be

$$U(\mathsf{bet}) = 0.5 \times 1,000,000,000 + 0.5 \times 1000 = 500,000,500.00 \tag{7.1.3}$$

If we don't bet, our bank balance will remain at £1,000,000. Based on expected utility, we are therefore advised to take the bet. (Note that if one considers instead the amount one will win or lose, one may show that the difference in expected utility between betting and not betting is the same, exercise(73)).

Whilst the above is a correct mathematical derivation, few people who are millionaires are likely to be willing to risk losing almost everything in order to become a billionaire. This means that the subjective utility of money is not simply the quantity of money. In order to better reflect the situation, the utility of money would need to be a non-linear function of money, growing slowly for large quantities of money and decreasing rapidly for small quantities of money, exercise(68).

7.2 Decision Trees

Decision trees are a way to graphically organise a sequential decision process. A *decision tree* contains decision nodes, each with branches for each of the alternative decisions. Chance nodes (random variables) also appear in the tree, with the utility of each branch computed at the leaf of each branch. The expected utility of any decision can then be computed on the basis of the weighted summation of all branches from the decision to all leaves from that branch.

Example 29 (Party)**.** Consider the decision problem as to whether or not to go ahead with a fund-raising garden party. If we go ahead with the party and it subsequently rains, then we will lose money (since very few people will show up); on the other hand, if we don't go ahead with the party and it doesn't rain we're free to go and do something else fun. To characterise this numerically, we use:

$$p(Rain = \mathsf{rain}) = 0.6,\ p(Rain = \mathsf{no\ rain}) = 0.4 \tag{7.2.1}$$

The utility is defined as

$$U\left(\mathsf{party}, \mathsf{rain}\right) = -100, \quad U\left(\mathsf{party}, \mathsf{no\ rain}\right) = 500, \quad U\left(\mathsf{no\ party}, \mathsf{rain}\right) = 0, \quad U\left(\mathsf{no\ party}, \mathsf{no\ rain}\right) = 50 \tag{7.2.2}$$

We represent this situation in fig(7.1). The question is, should we go ahead with the party? Since we don't know what will actually happen to the weather, we compute the expected utility of each decision:

$$U\left(\mathsf{party}\right) = \sum_{Rain} U(\mathsf{party}, Rain)p(Rain) = -100 \times 0.6 + 500 \times 0.4 = 140 \tag{7.2.3}$$

$$U\left(\mathsf{no\ party}\right) = \sum_{Rain} U(\mathsf{no\ party}, Rain)p(Rain) = 0 \times 0.6 + 50 \times 0.4 = 20 \tag{7.2.4}$$

Based on expected utility, we are therefore advised to go ahead with the party. The maximal expected utility is given by (see `demoDecParty.m`)

$$\max_{Party} \sum_{Rain} p(Rain)U(Party, Rain) = 140 \tag{7.2.5}$$

Example 30 (Party-Friend)**.** An extension of the Party problem is that if we decide not to go ahead with the party, we have the opportunity to visit a friend. However, we're not sure if this friend will be in. The question is should we still go ahead with the party?

We need to quantify all the uncertainties and utilities. If we go ahead with the party, the utilities are as before:

$$U_{party}\left(\mathsf{party}, \mathsf{rain}\right) = -100, \quad U_{party}\left(\mathsf{party}, \mathsf{no\ rain}\right) = 500 \tag{7.2.6}$$

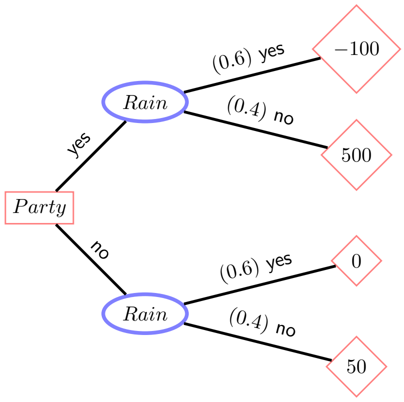


Figure 7.1: A decision tree containing chance nodes (denoted with ovals), decision nodes (denoted with squares) and utility nodes (denoted with diamonds). Note that a decision tree is *not* a graphical representation of a Belief Network with additional nodes. Rather, a decision tree is an explicit enumeration of the possible choices that can be made, beginning with the leftmost decision node, with probabilities on the links out of 'chance' nodes.

with

$$p(Rain = \mathsf{rain}) = 0.6, \quad p(Rain = \mathsf{no\ rain}) = 0.4 \tag{7.2.7}$$

If we decide not to go ahead with the party, we will consider going to visit a friend. In making the decision not to go ahead with the party we have utilities

$$U_{party}\left(\mathsf{no\ party}, \mathsf{rain}\right) = 0, \quad U_{party}\left(\mathsf{no\ party}, \mathsf{no\ rain}\right) = 50 \tag{7.2.8}$$

The probability that the friend is in depends on the weather according to

$$p(Friend = \mathsf{in}|\mathsf{rain}) = 0.8, \quad p(Friend = \mathsf{in}|\mathsf{no\ rain}) = 0.1, \tag{7.2.9}$$

The other probabilities are determined by normalisation. We additionally have

$$U_{visit}\left(\mathsf{friend\ in}, \mathsf{visit}\right) = 200, \quad U_{visit}\left(\mathsf{friend\ out}, \mathsf{visit}\right) = -100 \tag{7.2.10}$$

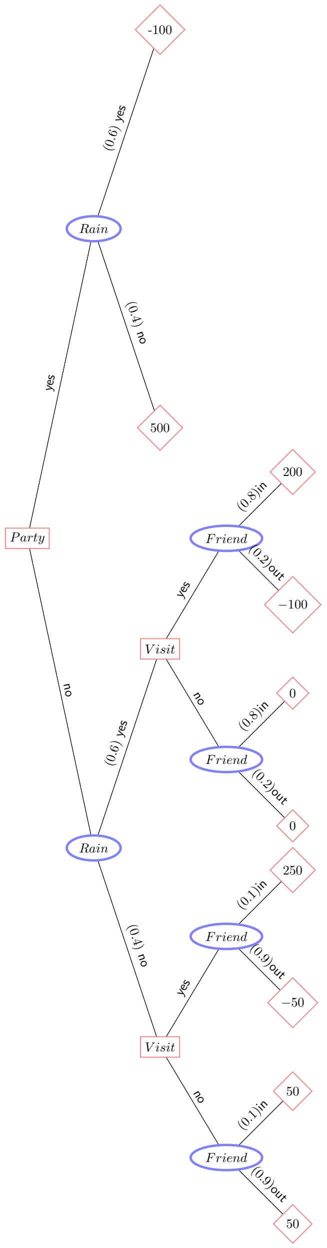
with the remaining utilities zero. The two sets of utilities add up so that the overall utility of any decision sequence is $U_{party} + U_{visit}$. The decision tree for the Party-Friend problem is shown is fig(7.2). For each decision sequence the utility of that sequence is given at the corresponding leaf of the DT. Note that the leaves contain the total utility $U_{party} + U_{visit}$. Solving the DT corresponds to finding for each decision node the maximal expected utility possible (by optimising over future decisions). At any point in the tree choosing that action which leads to the child with highest expected utility will lead to the optimal strategy. Using this, we find that the optimal expected utility has value 140 and is given by going ahead with the party, see `demoDecPartyFriend.m`.

- In DTs the same nodes are often repeated throughout the tree. For a longer sequence of decisions, the number of branches in the tree can grow exponentially with the number of decisions, making this representation impractical.
- In this example the DT is asymmetric since if we decide to go ahead with the party we will not visit the friend, curtailing the further decisions present in the lower half of the tree.

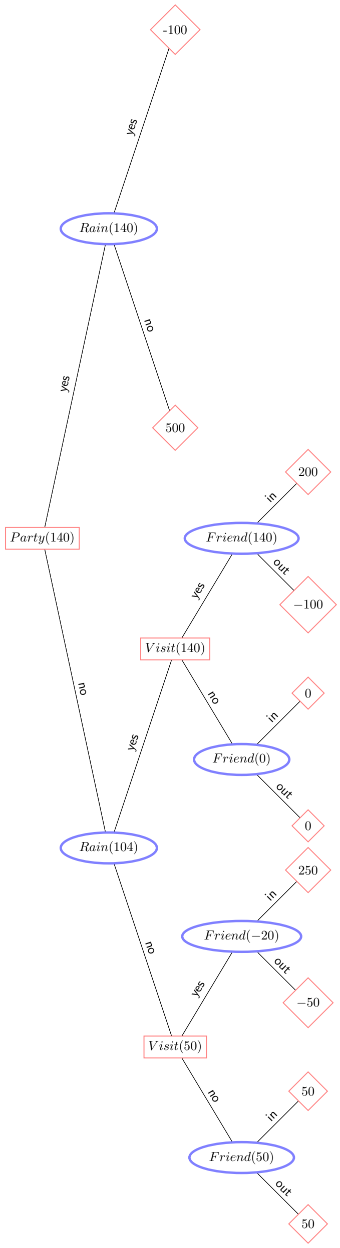
Mathematically, we can express the optimal expected utility U for the Party-Visit example by summing over un-revealed variables and optimising over future decisions:

$$\max_{Party} \sum_{Rain} p(Rain) \max_{Visit} \sum_{Friend} p(Friend|Rain) \left[U_{party}(Party, Rain) + U_{visit}(Visit, Friend)\mathbb{I}\left[Party = \mathsf{no}\right]\right] \tag{7.2.11}$$

where the term $\mathbb{I}\left[Party = \mathsf{no}\right]$ has the effect of curtailing the DT if the party goes ahead. To answer the question as to whether or not to go ahead with the party, we take that state of *Party* that corresponds to



(a)



(b)

Figure 7.2: Solving a Decision Tree. **(a)**: Decision Tree for the Party-Visit, example(30). **(b)**: Solving the DT corresponds to making the decision with the highest expected future utility. This can be achieved by starting at the leaves (utilities). For a chance parent node x, the utility of the parent is the expected utility of that variable. For example, at the top of the DT we have the *Rain* variable with the children -100 (probability 0.6) and 500 (probability 0.4). Hence the expected utility of the *Rain* node is $-100 \times 0.6 + 500 \times 0.4 = 140$. For a decision node, the value of the node is the optimum of its child values. One recurses thus backwards from the leaves to the root. For example, the value of the *Rain* chance node in the lower branch is given by $140 \times 0.6 + 50 \times 0.4 = 104$. The optimal decision sequence is then given at each decision node by finding which child node has the maximal value. Hence the overall best decision is to decide to go ahead with the party. If we decided not to do so, and it does not rain, then the best decision we could take would be to not visit the friend (which has an expected utility of 50). A more compact description of this problem is given by the influence diagram, fig(7.4). See also `demoDecPartyFriend.m`.

the maximal expected utility above. The way to read equation (7.2.11) is to start from the last decision that needs to be taken, in this case $Visit$. When we are at the $Visit$ stage we assume that we will have previously made a decision about $Party$ and also will have observed whether or not is is raining. However, we don't know whether or not our friend will be in, so we compute the expected utility by averaging over this unknown. We then take the optimal decision by maximising over $Visit$. Subsequently we move to the next-to-last decision, assuming that what we will do in the future is optimal. Since in the future we will have taken a decision under the uncertain $Friend$ variable, the current decision can then be taken under uncertainty about $Rain$ and maximising this expected optimal utility over $Party$. Note that the sequence of maximisations and summations matters – changing the order will in general result in a different problem with a different expected utility[1].

[1]If one only had a sequence of summations, the order of the summations is irrelevant – likewise for the case of all maximisations. However, summation and maximisation operators do not in general commute.

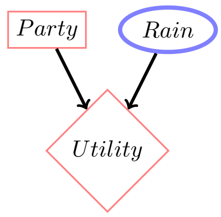


Figure 7.3: An influence diagram which contains random variables (denoted with ovals/circles) Decision nodes (denoted with squares) and Utility nodes (denoted with diamonds). Contrasted with fig(7.1) this is a more compact representation of the structure of the problem. The diagram represents the expression $p(rain)u(party, rain)$. In addition the diagram denotes an ordering of the variables with $party \prec rain$ (according to the convention given by equation (7.3.1)).

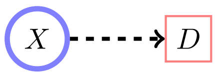
7.3 Extending Bayesian Networks for Decisions

An *influence diagram* is a Bayesian Network with additional Decision nodes and Utility nodes [137, 148, 162]. The decision nodes have no associated distribution; the utility nodes are deterministic functions of their parents. The utility and decision nodes can be either continuous or discrete; for simplicity, in the examples here the decisions will be discrete.

A benefit of decision trees is that they are general and explicitly encode the utilities and probabilities associated with each decision and event. In addition, we can readily solve small decision problems using decision trees. However, when the sequence of decisions increases, the number of leaves in the decision tree grows and representing the tree can become an exponentially complex problem. In such cases it can be useful to use an Influence Diagram (ID). An ID states which information is required in order to make each decision, and the order in which these decisions are to be made. The details of the probabilities and rewards are not specified in the ID, and this can enable a more compact description of the decision problem.

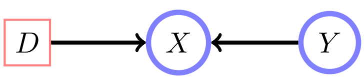
7.3.1 Syntax of influence diagrams

Information Links An *information link* from a random variable into a decision node:



indicates that the state of the variable X will be known before decision D is taken. Information links from another decision node d in to D similarly indicate that decision d is known before decision D is taken. We use a dashed link to denote that decision D is not functionally related to its parents.

Random Variables Random Variables may depend on the states of parental random variables (as in Belief Networks), but also Decision Node states:



As decisions are taken, the states of some random variables will be revealed. To emphasise this we typically shade a node to denote that its state will be revealed during the sequential decision process.

Utilities A utility node is a deterministic function of its parents. The parents can be either random variables or decision nodes.

In the party example, the BN trivially consists of a single node, and the Influence Diagram is given in fig(7.3). The more complex Party-Friend problem is depicted in fig(7.4). The ID generally provides a more compact representation of the structure of problem than a DT, although details about the specific probabilities and utilities are not present in the ID.