Let's transform Hull-White Model for better!

$$x(t) = r(t) - f(0, t)$$

Kshitij Anand

02-02-2025

Introduction to the Hull-White Model

- The Hull-White model is a widely used interest rate model that allows the simulation of future short rates, enabling the modeling of future interest rate scenarios.
- Once interest rates are modeled, we can price interest rate derivatives efficiently.
- The Hull-White model belongs to the class of no-arbitrage models. The current term structure of interest rates is used as input.
- In this booklet, I present a transformation in the Hull-White Model for easy calibration.

The Hull-White Model Equation

The short rate $r(t)$ under the Hull-White model is governed by the following stochastic differential equation:

$$dr(t) = k(t)[\theta(t) - r(t)]dt + \sigma(t)dW(t),$$

- $r(t)$: The short-term interest rate at time t. It represents the instantaneous risk-free rate at which borrowing or lending occurs over an infinitesimally short period.
- $k(t)$: The mean reversion speed of the short rate process. It determines how quickly $r(t)$ reverts to its long-term mean $\theta(t)$.
- $\theta(t)$: The long-term mean level of the short rate. It represents the level towards which $r(t)$ is expected to revert over time.
- $\sigma(t)$: The volatility of the short-rate process. Quantifies the uncertainty or randomness in the evolution of $r(t)$ over time.

What's the issue?

The current term structure of interest rates is incorporated into the above model via $\theta(t)$ using the following relation:

$$\theta(t) = \frac{1}{k(t)}\frac{\partial f(0,t)}{\partial t} + f(0,t) + \frac{1}{k(t)}\int_0^t e^{-2\int_u^t k(s)\,ds}\sigma(u)^2\,du.$$

where:

$$f(0,T) = -\frac{\partial \ln P(0,T)}{\partial T},$$

The term $\frac{\partial f(0,t)}{\partial t}$ is quite problematic, especially when the initial forward curve is not smooth. This motivates us to present a transformed dynamics of the short rate.

Transformed Equation

Define

$$x(t) \triangleq r(t) - f(0,t).$$

Then, the model dynamic is as follows:-

$$\boxed{dx(t) = (y(t) - k(t)x(t))\,dt + \sigma(t)dW(t)}$$

where

$$x(0) = 0 \quad \text{and} \quad y(t) = \int_0^t e^{-2\int_u^t k(s)ds}\sigma(u)^2\,du.$$

Dynamics of Zero Coupon Bonds

The price of a zero-coupon bond under the new dynamics is given as:

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp\left(-k(t)G(t, T) - \frac{1}{2}y(t)G(t, T)^2\right)$$

where,

$$G(t, T) = \int_t^T e^{-\int_t^u k(s)ds}\, du.$$

Final Comments

- Transformation $x(t) = r(t) - f(0, t)$ helps to eliminate the explicit dependence on the initial forward curve $f(0, t)$.
- It allows us to work with the deviations of the short rate $r(t)$ from the initial term structure rather than the absolute values.
- The term $\frac{\partial f(0,t)}{\partial t}$ in the drift of $r(t)$ can be problematic, especially when the initial yield curve is not smooth. This transformation helps to handle such cases.
- Calibrating the interest rate through the transformed dynamics leads to ease and greater accuracy.

Hints on Derivation

- The above dynamics can be derived using the forward rate dynamics from the HJM model, followed by basic calculus to obtain the transformed dynamics.
- The proof serves as an excellent exercise for revising engineering mathematics. Interested individuals can contact me for the full derivation.
- References: Volume 2, Interest Rate Modeling by Andersen and Piterbarg.