Hw 0.

Problem 1.

From Taylor Theorem :

$$f(x^*+t) - f(x^*) = \nabla f(x^*) \cdot t + \frac{1}{2} t^T \nabla^2 f(x^*) t + o(|t|^2)$$

According to F.O.N.C. we know that when x^* is local minimizer,

we know that $\nabla f(x^*) = 0$

Assume $\nabla^2 f(x^*) < 0$,

$$\therefore f(x^*+t) - f(x^*) = \cancelto{0}{\nabla f(x^*)} \cdot t + \frac{1}{2} t^T \nabla^2 f(x^*) t + o(|t|^2)$$

$\because \nabla^2 f(x^*) < 0 \quad \therefore \frac{1}{2} t^T \nabla^2 f(x^*) t < 0$

and when t is enough small $o(|t|^2) < \frac{1}{2} t^T \nabla f(x^*) t$

which will make $f(x^*+t) - f(x^*) < 0$, and there exist a value

where $f(x^*+t) < f(x^*)$, contradiction to $f(x^*)$ is local minimizer,

$\therefore \nabla^2 f(x^*) \geq 0$ #

Problem 2.

(a) Assume $\nabla f(x^*) \neq 0$, we can always find a direction which make $f(x^*+td) < f(x^*)$.

Contradiction to $f(x^*)$ is local minimum of every line f that passed x^*.

$\therefore \nabla f(x^*) = 0$ #

(b)

Consider $f(x_1, x_2) = (x_1 - x_2^2)(x_1 - 3x_2^2) = x_1^2 - 4x_1x_2^2 + 3x_2^4$

and $f(0,0) = 0$ but $f(2x_2^2, x_2) = -x_2^4 < 0$ when $x_2 \neq 0$

but when consider $x_1 = m x_2$

$$f(mx_2, x_2) = (mx_2 - x_2^2)(mx_2 - 3x_2^2) = m^2x_2^2 - 4mx_2^3 + 3x_2^4$$

For any x_2 close to 0, $f(mx_2, x_2)$ is still bigger than 0.

However $f(0,0) = 0$ which means it is a local minimum for every line pass through it

Problem 3.

(a)

$$\nabla^2 C = A$$

and when $A \succeq 0$, we can know that C is a convex function

when C is a convex function, we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

$$(\alpha x + (1-\alpha)y)^T A (\alpha x + (1-\alpha)y) + b^T(\alpha x + (1-\alpha)y) + c$$

$$\leq \alpha(x^TAx + b^Tx + c) + (1-\alpha)(y^TAy + b^Ty + c)$$

and we can eliminate $\alpha b^Tx + (1-\alpha)b^Ty$ and c from both side

$$\Rightarrow (\alpha x + (1-\alpha)y)^T A(\alpha x + (1-\alpha)y) \leq \alpha x^TAx + (1-\alpha)y^TAy$$

$$\alpha(1-\alpha)x^TAx - \alpha(1-\alpha)x^TAy - \alpha(1-\alpha)\,{}^TAx + \alpha(1-\alpha)y^TAy \geq 0$$

which we can get $\alpha(1-\alpha)(x-y)^T A(x-y) \ge 0$

which cannot be $x^T A x \ge 0$ and get $A \succeq 0$

so the reverse is false #

(b)

$B = A + \lambda g g^T \succeq 0$

$\Rightarrow A = B - \lambda g g^T$

C will be: $x^T(B - \lambda g g^T)x + b^T x + c \le 0$

$\Rightarrow x^T B x + \lambda x^T g g^T x + b^T x + c \le 0$

$\Rightarrow x^T B x + b^T x + (c - \lambda h^2) \le 0$

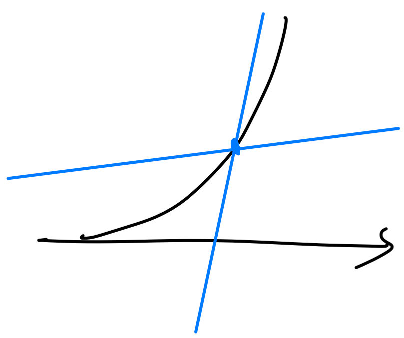
$\because B \succeq 0$ $\therefore$ we can know that C is convex function and also $g^T x + h = 0$ is a convex function as well, so their intersection is also a convex function #

Problem 4.

(a) if $\partial f(x) = \{g\}$ and $g \ne \nabla f(x)$

From the definition of subdifferential we can get

$f(z) \ge f(x) + g^T(z - x)$ but as $g \ne \nabla f(x)$ which means it will bigger than our function in the left side or right side, so its a contradiction to our definition and $g = \nabla f(x)$ #



(b)

If $\nabla f(x) \neq g_0$ and according to the definition of subdifferential, we must have some point bigger than our function value, which is a contradiction as Problem (a)

$\therefore$ when $\partial f(x) = \{g_0\}$, than $\nabla f(x) = g_0$, if f is differentiable at x #

Problem 5.

(a)

$\because$ g is a convex function and h is non-decreasing function $\begin{cases} g(\alpha x+(1-\alpha)y) \le \alpha g(x)+(1-\alpha)g(y) \\ h(y) \ge h(x), \text{ if } y \ge x \end{cases}$

$$f(\alpha x+(1-\alpha)y) = h(g(\alpha x+(1-\alpha)y)) \le h(\alpha g(x)+(1-\alpha)g(y))$$

h is a convex function $\Leftarrow$ $\le \alpha h(g(x)) + (1-\alpha) h(g(y)) = \alpha f(x) + (1-\alpha) f(y)$

$\therefore$ f is a convex function #

(b)

$\begin{cases} g(\alpha x+(1-\alpha)y) \ge \alpha g(x)+(1-\alpha)g(y) \Rightarrow g \text{ is concave} \\ h(y) \le h(x), \text{ if } y \ge x \Rightarrow h \text{ is non-increasing} \end{cases}$

$$f(\alpha x+(1-\alpha)y) = h(g(\alpha x+(1-\alpha)y)) \le h(\alpha g(x)+(1-\alpha)g(y))$$

h is a convex function $\Leftarrow$ $\le \alpha h(g(x)) + (1-\alpha) h(g(y)) = \alpha f(x) + (1-\alpha) f(y)$

$\therefore$ f is a convex function

#