Problem 1.

(a) If A and B are not disjoint, there will be a point which $g_1(x) \le 0$, $f(x) \le s < p^*$ which is impossible as p^* is $\min_{x \in D} f(x)$. as $f(x)$ and $g(x)$ is convex function A is also convex function, and set B is like a line so it is a convex function, as well.

(b) If $\tilde{\lambda} < 0$ or $\mu < 0$, it may be unboundness for $\tilde{\lambda} u + \mu t \ge \alpha$, so $\tilde{\lambda} \ge 0$, $\mu \ge 0$

For point on B, we can get $\mu s \le \alpha$, for all $s < p^*$

and we can get $\mu p^* \le \alpha$,

and from $\tilde{\lambda} u + \mu t \ge \alpha$, we can get the result

$$\tilde{\lambda} g_1(x) + \mu f(x) \ge \alpha \ge \mu p^*$$

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(c) when $\mu > 0$, we can have $\frac{\tilde{\lambda}}{\mu} g_1(x) + f(x) \ge p^*$

for $d^* = \sup_{\lambda \ge 0} \inf_{x \in D} (\lambda g_1(x) + f(x)) \ge \frac{\tilde{\lambda}}{\mu} g_1(x) + f(x) \ge p^*$

and we have $p^* \ge d^*$, we can have $d^* = p^*$

when $\mu = 0$, we have

$\lambda g_1(x) \ge 0$ and we have $g_1(x) \le 0$ and $\lambda \ge 0$

so we have $\lambda = 0$ or $g_1(x)$ which $d^* = \sup_{\lambda \ge 0} \inf_{x \in D} f(x) = p^*$

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Problem 2.

(a) e^{-x} is a convex function which will be a convex optimization problem

$\frac{x^2}{y} \leq 0$, $y > 0$ and $x^2 \geq 0$

we can get $x = 0$

$\Rightarrow e^{-0} = 1$ #

(b)

dual problem = $\sup_{\lambda \geq 0} \inf_{x \in D} \left(e^{-x} + \frac{\lambda x^2}{y} \right)$

which is also 0 with $\lambda \geq 0$

(c)

$\Delta = p^* - d^* = 1 - 0 = 1$

but $\frac{x^2}{y} < 0$ will be null , so slater's theorem is not hold!

Problem 3.

(a)

$$f(z) - f(\bar{x}) = \left(f(z) - f(x)\right) - \left(f(\bar{x}) - f(x)\right)$$

$$\geq \left(\nabla f(x)^T(z-x)\right) - \left(\nabla f(x)^T(\bar{x}-x) + \underbrace{\frac{L}{2}\|\bar{x}-x\|^2}_{= \frac{1}{2L}\|g_c(x)\|^2}\right)$$

$$\geq \nabla f(x)^T(z-x) - \underline{\nabla f(x)^T(\bar{x}-x)} - \frac{1}{L}\|g_c(x)\|^2 + \frac{1}{2L}\|g_c(x)\|^2$$

$\downarrow$ projection theorem

$$\geq \nabla f(x)^T(z-x) + \underline{\frac{1}{L}\|g_c(x)\|^2} - \frac{1}{L}\|g_c(x)\|^2 + \frac{1}{2L}\|g_c(x)\|^2$$

$$\geq \nabla f(x)^T(z-x) + \frac{1}{2L}\|g_c(x)\|^2$$

$$\geq g_c(x)^T(z-x) + \frac{1}{2L}\|g_c(x)\|^2$$

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(b) Apply (a) with $z = x_t$ we can get

$$f(x_t) - f(x_{t+1}) \geq g_c(x_t)^T(x_t - x_t) + \frac{1}{2L}\|g_c(x_t)\|^2$$

$$\Rightarrow f(x_{t+1}) - f(x_t) \leq -\frac{1}{2L}\|g_c(x_t)\|^2$$

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(c)

$$f(x^*) - f(x_{t+1}) \geq g_c(x_t)^T(x^* - x_t) + \frac{1}{2L}\|g_c(x_t)\|^2$$

$$\geq \|g_c(x_t)\|\|x^* - x_t\|$$

$$\|g_c(x_t)\| \geq \frac{f(x_{t+1}) - f(x^*)}{\|x_t - x^*\|} \geq \frac{f(x_{t+1}) - f(x_t)}{\|x_t - x^*\|} \quad \ldots \quad \because f(x^*) \leq f(x_t)$$

(d)

$$\Delta_{t+1} - \Delta_t = f(x_{t+1}) - f(x_t) \leq -\frac{1}{2L}\|g_c(x_t)\|^2$$

$$\leq -\frac{1}{2L}\left(\frac{f(x_{t+1}) - f(x^*)}{\|x_t - x^*\|}\right)^2$$

$$\leq -\frac{1}{2L}\frac{\Delta_{t+1}^2}{\|x_0 - x^*\|^2}$$