- Vectors in underline, matrices in capital. Assume all matrices have real entries unless stated otherwise.
- For a vector $\underline{x}$, we denote by x_i the i^{th} entry of $\underline{x}$.
- The inequality $A \geq 0$ means that the matrix A is positive semi definite. The inequality $\underline{x} \geq 0$ means that all the entries of the vector $\underline{x}$ are non negative.

1. You are given a vector $\underline{y}$ and a matrix A. Consider the following four optimization problems:

$$\min_{\underline{x}} \quad \left\|A\underline{x} - \underline{y}\right\|_2^2 + \alpha \left\|\underline{x}\right\|_2^2 \qquad (1)$$

$$\begin{aligned} \min_{\underline{x}} \quad & \|\underline{x}\|_2 \\ \text{s.t.} \quad & \left\|A^\top \left(\underline{y} - A\underline{x}\right)\right\|_2 \leq \alpha. \end{aligned} \qquad (2)$$

$$\min_{\underline{x}} \quad \left\|A\underline{x} - \underline{y}\right\|_2^2 + \alpha \left\|\underline{x}\right\|_1 \qquad (3)$$

$$\begin{aligned} \min_{\underline{x}} \quad & \|\underline{x}\|_1 \\ \text{s.t.} \quad & \left\|A^\top \left(\underline{y} - A\underline{x}\right)\right\|_\infty \leq \alpha. \end{aligned} \qquad (4)$$

   Here $\alpha > 0$ is fixed.

   (a) *(2 pts)* Which of the above optimization problems are convex ?

   (b) *(1 pts)* Find the gradient of the objective in (1).

   (c) *(1 pts)* Find an expression for $\underline{x}_1^\star$, the optimum point of (1).

   (d) *(2 pts)* Assume that α is *large enough*: in particular assume that

   $$\left\|(A^\top A + \alpha I)^{-1} A\underline{y}\right\|_2 \leq 1.$$

   If $\underline{x}_2^\star$ is an optimum point of (2), show that $\|\underline{x}_1^\star\|_2 \geq \|\underline{x}_2^\star\|_2$.

   (e) *(3 pts)* Reformulate (4) as an LP and (3) as a QP.

2. *(4 pts)* Given 2×2 matrices $A_1, A_2, \ldots, A_k$ and 2×1 vectors $\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_k$, you wish to find $\underline{x}$ such that $A_i\underline{x} \approx \underline{b}_i$. You formulate three possible objectives

$$f_2(\underline{x}) = \sum_i \|A_i\underline{x} - \underline{b}_i\|_2, \quad f_1(\underline{x}) = \sum_i \|A_i\underline{x} - \underline{b}_i\|_1, \text{ and } f_\infty(\underline{x}) = \sum_i \|A_i\underline{x} - \underline{b}_i\|_\infty.$$

   Formulate the minimization of each of these functions as an LP, QP, QCQP or SOCP.

3. *(4 pts)* Given vector $\underline{\mu}$ and symmetric matrix V, consider the problem

$$\begin{aligned} \max_{\underline{x}} \quad & \frac{\underline{\mu}^\top \underline{x}}{\|V\underline{x}\|_2} \\ \text{s.t.} \quad & \|x\|_1 \leq L, \quad \underline{1}^\top \underline{x} = 1, \quad \underline{\mu}^\top \underline{x} \geq 0. \end{aligned} \qquad (5)$$

   (a) Prove that the objective is quasi-concave.

   (b) Make the change of variable $\underline{z} = \underline{x}/\underline{\mu}^\top \underline{x}$ (what is the value of $\underline{z}/\underline{1}^\top \underline{z}$?) to reduce the above to a convex optimization problem.

4. *(10 pts)* For the parts (a)-(c) below, assume the functions have domain $\mathbb{S}_{++}^n$.

   (a) For a fixed vector $\underline{a}$, show that the function $f(X) = \underline{a}^\top X^{-1} \underline{a}$ (with $\text{dom} f = \mathbb{S}_{++}^n$) is convex.

   (b) Show that the diagonal entries of X^{-1} are convex functions of X.

   (c) Show that $\text{trace}(X^{-1})$ is a convex function of X.

(d) Consider the following convex optimization problem (A, B are given matrices)

$$\begin{aligned} \min_{X} \quad & \underline{a}^\top X^{-1} \underline{a} \\ \text{s.t.} \quad & AX = B, \\ & X \geq 0. \end{aligned}$$

Reformulate this problem as an SDP.

(e) Reformulate the following problem as an SDP

$$\begin{aligned} \min_{X} \quad & \operatorname{trace}(X^{-1}) \\ \text{s.t.} \quad & AX = B, \\ & X \geq 0. \end{aligned}$$

For all the parts, you may find the following result useful: For any matrix $A \in \mathbb{S}^m_{++}$, vector $\underline{a} \in \mathbb{R}^m$, and scalar b,

$$\begin{pmatrix} A & \underline{a} \\ \underline{a}^\top & b \end{pmatrix} \geq 0 \quad \text{if and only if } b - \underline{a}^\top A \underline{a} \geq 0.$$

5. You are given the *samples* of a function f at certain points, for e.g. assume you are given the values $f(0), f(1), f(2), \ldots, f(N-1)$. Our goal is to interpolate to find the values of f on all points in the interval $[0, N-1]$, *in such a way that the interpolation is convex.* We know the data points $(i, f(i))$, and we wish to somehow fit a convex curve through the points. This may not be possible for all configurations of these points, so we are allowed to *ignore* some of these data points.

We will assume that we are given an $x \in [0, N-1]$, and we try to construct the interpolated value. Consider the following three potential techniques, resulting in three interpolated functions $f_1(x), f_2(x)$ and $f_3(x)$, written as linear programs:

$$\begin{aligned} f_1(x) = \min_{m,c} \quad & mx + c \\ \text{s.t.} \quad & f(i) \leq mi + c \quad \text{for } i = 0, 1, \ldots, N-1. \end{aligned} \tag{6}$$

$$\begin{aligned} f_2(x) = \max_{\underline{\alpha}} \quad & \sum_{i=0}^{N-1} \alpha_i f(i) \\ \text{s.t.} \quad & \sum_{i=0}^{N-1} i\alpha_i = x, \\ & \underline{\alpha} \geq 0, \quad \underline{1}^\top \underline{\alpha} = 1. \end{aligned} \tag{7}$$

$$\begin{aligned} f_3(x) = \min_{\underline{\alpha}} \quad & \sum_{i=0}^{N-1} \alpha_i f(i) \\ \text{s.t.} \quad & \sum_{i=0}^{N-1} i\alpha_i = x, \\ & \underline{\alpha} \geq 0, \quad \underline{1}^\top \underline{\alpha} = 1. \end{aligned} \tag{8}$$

(a) *(2 pts)* Give an interpretation for the functions $f_1(x), f_2(x)$ and $f_3(x)$.

(b) *(4 pts)* Comment on the convexity/concavity of f_1, f_2 and f_3 (in general).

(c) *(3 pts)* Prove that (in general) $f_1(x) = f_2(x)$.

(d) *(3 pts)* Given an x, suppose m^*, c^* are optima of (6), and $\underline{\alpha}^*$ is the optima of (7). If the value of $f(i)$ contributes to the value of $f_2(x)$ (i.e. the value $f(i)$ is not *ignored*) , then $\alpha_i^* \neq 0$. Show that if $f(i)$ contributes to the value of $f_2(x)$, then $f(i) = m^* i + c^*$. Using this, argue that if no three points among $(i, f(i))$ are collinear, then $\underline{\alpha}^*$ can have at most two non zero entries.

(e) *(2 pts)* For $N = 4, f(0) = 1, f(1) = 0, f(2) = 2, f(3) = 1$, sketch a plot of $f_1(x), f_2(x)$ and $f_3(x)$.

6. Consider the following program

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1^2 + 2x_2^2 - x_1 x_2 - x_1 \\ \text{s.t.} \quad & x_1 - 2x_2 \leq u_1, \\ & x_1 + 4x_2 \leq u_2, \\ & 5x_1 - 76x_2 \leq 1 \end{aligned} \tag{9}$$

with variables x_1, x_2 and parameters u_1, u_2.

(a) *(2 pts)* Is this problem convex ? If so, categorize it as an LP, QP, QCQP, SDP (or none of these).

(b) *(3 pts)* Using `CVX` or `CVXPY`, solve this problem with parameter values $u_1 = -2$, $u_2 = -3$ to find the optimal primal variables $x_1^\star$, $x_2^\star$ *and* optimal dual variables $\lambda_1^\star, \lambda_2^\star$ and $\lambda_3^\star$.

(c) *(2 pts)* Verify that the KKT conditions hold for the primal and the dual optimal variables.

(d) *(2 pts)* Let $p^\star(u_1, u_2)$ be the optimal value of the problem with parameters u_1, u_2. Sketch some level curves for $p^\star(u_1, u_2)$.

(e) *(2 pts)* From the sketch above, do you think $p^\star$ convex ?

(f) *(2 pts)* (Numerically) Compute the partial derivatives of $p^\star$ at $u_1 = -2, u_2 = -3$, and verify their relationship to the optimal dual variables $\lambda_1^\star$ and $\lambda_2^\star$.