EE2100/EE5609 Matrix Theory

Lecture 1a: Basis and dimension

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- These notes are purposefully terse; you can generate more detailed notes, as per your writing style.
- Include a short summary of the concepts covered so far (in particular if they relate to the topic you are writing about)

RECALL: vector spaces, matrix multiplication, linear independence

Basis and dimension

A maximal linearly independent set in a vector space is called a basis. We say that a set $T \subseteq \mathcal{V}$ is maximally linearly independent if adding any vector to T makes it linearly dependent, i.e.

$$T \cup \{\underline{v}\} \text{ is linearly dependent for any } \underline{v} \in \mathcal{V}.$$

For example, the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$. The set $\{1, x, x^2, x^3\}$ is a basis for the vector space consisting of all degree 3 polynomials, etc.

Theorem 1. *If $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is a basis for $\mathcal{V}$, then every vector $\underline{v} \in \mathcal{V}$ can be written uniquely as*

$$\underline{v} = \sum_{i=1}^{k} \alpha_i \underline{v}_i.$$

The scalars α_i are called the coefficients or coordinates of the vector $\underline{v}$ in the given basis.

This result highlights the importance of a basis. There are two aspects to this theorem: First that any vector *can* be expressed as a linear combination of the vectors in the basis, and secondly, that such an expression is *unique*. This justifies the term 'basis': a set that can be used to build the vector space.

Proof. 1. *Span:* Suppose we have a $\underline{v} \in \mathcal{V}$ that cannot be written as a linear combination of $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$. Then $\underline{v} \cup \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ is a linearly independent set (why?), contradicting of maximality of $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$.

2. *Uniqueness:* Suppose there are two different ways of writing $\underline{v}$ as a linear combination

$$\underline{v} = \sum_{i=1}^{k} \alpha_i \underline{v}_i, \quad \underline{v} = \sum_{i=1}^{k} \beta_i \underline{v}_i, \text{ with } (\alpha_1, \alpha_2, \ldots) \neq (\beta_1, \beta_2, \ldots).$$

Then $\sum_{i=1}^{k} (\alpha_i - \beta_i) \underline{v}_i = 0$, giving a non-trivial linear combination that is zero. This contradicts the linear independence of $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$.

□

There could be many bases for a vector space [1]. The following result guarantees that all bases of a vector space have the same number of elements.

[1] As an excercise, try to construct three different bases for $\mathbb{R}^3$

Theorem 2. *Any two bases of a vector space have the same number of elements.*

Since all bases (maximally linearly independent sets) have the same size, one could also have defined a basis as a *maximum* size linearly independent set.

Proof. Suppose we have bases $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ and $\{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_p\}$ with $k < p$. Let $V = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \ldots & \underline{v}_k \end{pmatrix}$ and $W = \begin{pmatrix} \underline{w}_1 & \underline{w}_2 & \ldots & \underline{w}_p \end{pmatrix}$ be the corresponding matrices. Then $W = VA$ for some $k \times p$ matrix A (why ? [2]). Thus the linear independence of columns of W and columns of V implies that

$$A\underline{x} = 0 \text{ only when } \underline{x} = 0,$$

which is a contradiction since A is a fat matrix ($k < p$) [3]. □

[2] This is obtained by writing each basis vector in $\{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_p\}$ in terms of the basis vectors in $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$.

[3] This result (that $A\underline{x} = 0$ always has a non zero solution when A is fat) was used in an earlier lecture as well. We can prove it using row echelon forms, which we will defer till later in the course.

Because of Theorem **??**, it makes sense to define the following:

Definition. THE DIMENSION of a vector space is the size of any basis in the vector space. [4]

[4] What is the dimension of the set of all degree 3 polynomials f with $f(0) = 0$? You need to first check that this is a vector space.

Matrix spaces

Recall the system interpretation of a matrix. In view of that, we define the following

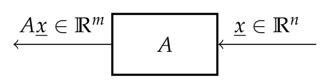

Figure 1: System interpretation of an $m \times n$ matrix A. The matrix takes an 'input' $\underline{x} \in \mathbb{R}^n$ and gives an 'output' $A\underline{x} \in \mathbb{R}^m$. The input space is $\mathbb{R}^n$ and the output space is $\mathbb{R}^m$.

1. The image space of A, written $im(A)$ [5]is the set of all possible outputs of A
$$im(A) = \{A\underline{x} : \underline{x} \in \mathbb{R}^n\}.$$
2. The null space of A, written $null(A)$ [6],is the set of all inputs that give a zero output
$$null(A) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = 0\}.$$

[5] Note that $im(A)$ lies in the output space, i.e. $im(A) \subseteq \mathbb{R}^m$. Since $A\underline{x}$ is a linear combination of columns of A, the set $im(A)$ is also called the column space of A.

[6] Note that $null(A)$ lies in the input space, i.e. $null(A) \subseteq \mathbb{R}^n$.

Characterize the column and null spaces of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

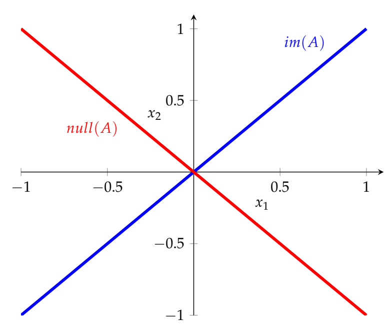

Figure 2: For the 2×2 matrix A, both $im(A)$ and $null(A)$ are in $\mathbb{R}^2$, plotted above. Verify, and similarly repeat for B.