CONVEX OPTIMISATION
ASSIGNMENT 1

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Question 1

The set C can also be represented as,

$$C = \{\overline{x} : \mathbf{Y}^T\overline{x} \geq \overline{0}\}$$

Where $\mathbf{Y}$ is a matrix with columns as elements of set S.

A: It is not a Subspace.

Choose $\overline{x_1}, \overline{x_2}$ from set C, if this set is subspace then $\forall \alpha_1, \alpha_2 \in \mathbb{R}; \alpha_1\overline{x_1} + \alpha_2\overline{x_2}$ must also be in set C. Since α_1 and α_2 can also take negative value which can the inequatlity, this relation cannot be valid. Thus it is not a Subspace

B: It is not a Affine set.

The same explanation holds, since the only difference between affine and subspace is that in case of affine $\alpha_1 + \alpha_2 = 1$, but they can take any real values thus can be negative as well. Thus in this case also the ineqality need not satisfy.

C: It is a convex set.

Choose $\overline{x_1}, \overline{x_2}$ from set C,

$$\mathbf{Y}^T(\theta\overline{x_1}) \geq \overline{0}; \mathbf{Y}^T((1-\theta)\overline{x_2}) \geq \overline{0}$$
$$\implies \mathbf{Y}^T(\theta\overline{x_1} + (1-\theta)\overline{x_2}) \geq \overline{0}$$

This means that for any $\theta \in [0, 1]$ the above inequality satisfies which means that the set is convex.

D: It is a cone.

By definition of cone if any set is a cone then any positively scaled vector must also belong to that set.

$$\mathbf{Y}^T\overline{x} \geq \overline{0}$$
$$\implies \mathbf{Y}^T(\theta\overline{x}) \geq \overline{0}; \text{Given } \theta \geq 0$$

Observe that above conditions satisfied despite the nature of matrix $\mathbf{Y}$. Thus we can say that **none of the above queries depend on structure of S.**

Question 2

Given $f_1(\overline{x}) = ||\overline{y} - \mathbf{A}\overline{x}||_2$ and $f_2(\bar{(}x)) = ||\overline{y} - \mathbf{A}\overline{x}||_2^2$,
Useful properties

- Norm function is convex.
- Affine transformation of domain of the function does'nt change the convexity of function.

In case of $f_1(\overline{x})$, the domain is affine transformation of $\overline{x}$ and the norm is operated on it, from above properties the function is convex.

$$f_1(\overline{x}) = ||\Phi(\overline{x})||_2$$
$$\Phi : \overline{x} \to \overline{y} - \mathbf{A}\overline{x}$$

In case of $f_2(\overline{x})$,

$$f_2(\overline{x}) = \overline{x}^T \mathbf{A}^T \mathbf{A}\overline{x} - 2\overline{y}^T \mathbf{A}\overline{x} + \overline{y}^T \overline{y}$$

The above function is quadratic. The function is convex iff $\mathbf{A}^T\mathbf{A}$ is positive semi definite.

$$\mathbf{A} = \mathbf{V\Sigma V}^T$$
$$\implies \mathbf{A}^T\mathbf{A} = (\mathbf{V\Sigma V}^T)^T(\mathbf{V\Sigma V}^T) = \mathbf{V\Sigma}^T\mathbf{\Sigma V}^T$$

$\mathbf{\Sigma}^T\mathbf{\Sigma}$ has positive diagonal entries as above multiplication will yeild individual diagonal entries of $\mathbf{\Sigma}$ to square each other. Since eigen values of $\mathbf{A}^T\mathbf{A}$ are positive thus it is positive definite. Hence $f_2(\overline{x})$ is a convex function.

Question 3

(a)

Convex hull of $S = \{\overline{x_1} = (0,0), \overline{x_2} = (1,1), \overline{x_3} = (1,0)\}$ is just linear combination of all the vectors such that their coeefficients are positive and they sum up to 1. Since these are 3 non collinear points, which means that convex hull is just the triangle.

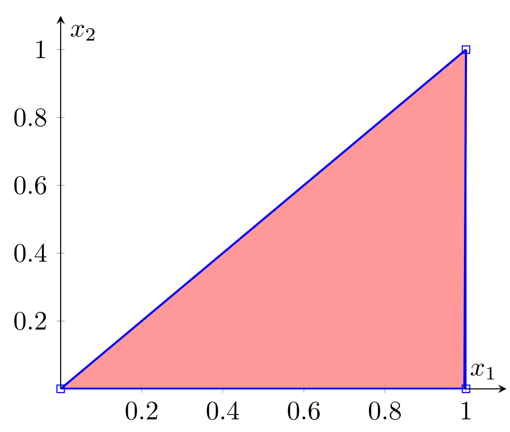
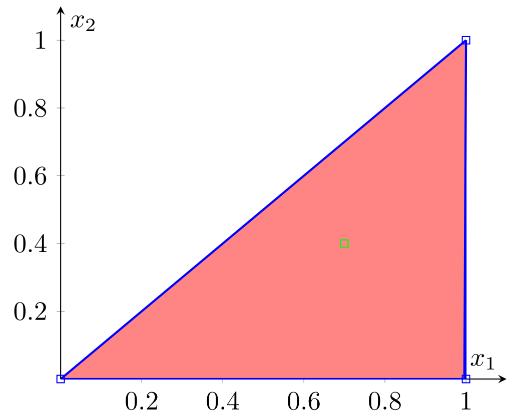


Figure 1: conv(S)

(b)

Take a point inside convex hull,



This point can be written as $\overline{v} = \sum_{i=1}^{3} \theta_i \overline{x_i}$ such that $\sum_{i=1}^{3} \theta_i = 1$. Given f is convex function which means that,

$$f(\overline{v}) = f(\sum_{i=1}^{3} \theta_i \overline{x_i})$$
$$\implies f(\overline{v}) \leq \sum_{i=1}^{3} \theta_i f(\overline{x_i}) f(\overline{v}) \leq \sum_{i=1}^{3} \theta_i max\{f(\overline{x_i})\} = max\{f(\overline{x_i})\} \sum_{i=1}^{3} \theta_i$$
$$\implies f(\overline{v}) \leq max\{f(\overline{x_i})\}$$

Clearly from above inequalities we see that value of any point in convex hull of polyhedra is always less than equal to the value of maximum value at vertices given f is convex, which means maxima occurs at vertices.

Question 4

Question 5

Question 6