

CONVEX OPTIMISATION

ASSIGNMENT 1

TADIPATRI UDAY KIRAN REDDY
EE19BTECH11038

March 25, 2022

Question 1

The set C can also be represented as,

$$C = \{\bar{x} : \mathbf{Y}^T \bar{x} \geq \bar{0}\}$$

Where \mathbf{Y} is a matrix with columns as elements of set S .

A: It is not a Subspace.

Choose \bar{x}_1, \bar{x}_2 from set C , if this set is subspace then $\forall \alpha_1, \alpha_2 \in \mathbb{R}; \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2$ must also be in set C . Since α_1 and α_2 can also take negative value which can the inequatlity, this relation cannot be valid. Thus it is not a Subspace

B: It is not a Affine set.

The same explanation holds, since the only difference between affine and subspace is that in case of affine $\alpha_1 + \alpha_2 = 1$, but they can take any real values thus can be negative as well. Thus in this case also the inequality need not satisfy.

C: It is a convex set.

Choose \bar{x}_1, \bar{x}_2 from set C ,

$$\begin{aligned} \mathbf{Y}^T(\theta \bar{x}_1) &\geq \bar{0}; \mathbf{Y}^T((1 - \theta) \bar{x}_2) \geq \bar{0} \\ \implies \mathbf{Y}^T(\theta \bar{x}_1 + (1 - \theta) \bar{x}_2) &\geq \bar{0} \end{aligned}$$

This means that for any $\theta \in [0, 1]$ the above inequality satisfies which means that the set is convex.

D: It is a cone.

By definition of cone if any set is a cone then any positively scaled vector must also belong to that set.

$$\begin{aligned} \mathbf{Y}^T \bar{x} &\geq \bar{0} \\ \implies \mathbf{Y}^T(\theta \bar{x}) &\geq \bar{0}; \text{ Given } \theta \geq 0 \end{aligned}$$

Observe that above conditions satisfied despite the nature of matrix \mathbf{Y} . Thus we can say that **none of the above queries depend on structure of S .**

Question 2

Given $f_1(\bar{x}) = \|\bar{y} - \mathbf{A}\bar{x}\|_2$ and $f_2(\bar{x}) = \|\bar{y} - \mathbf{A}\bar{x}\|_2^2$,

Useful properties

- Norm function is convex.
- Affine transformation of domain of the function doesn't change the convexity of function.

In case of $f_1(\bar{x})$, the domain is affine transformation of \bar{x} and the norm is operated on it, from above properties the function is convex.

$$f_1(\bar{x}) = \|\Phi(\bar{x})\|_2$$

$$\Phi : \bar{x} \rightarrow \bar{y} - \mathbf{A}\bar{x}$$

In case of $f_2(\bar{x})$,

$$f_2(\bar{x}) = \bar{x}^T \mathbf{A}^T \mathbf{A} \bar{x} - 2\bar{y}^T \mathbf{A} \bar{x} + \bar{y}^T \bar{y}$$

The above function is quadratic. The function is convex iff $\mathbf{A}^T \mathbf{A}$ is positive semi definite.

$$\mathbf{A} = \mathbf{V} \mathbf{\Sigma} \mathbf{V}^T$$

$$\implies \mathbf{A}^T \mathbf{A} = (\mathbf{V} \mathbf{\Sigma} \mathbf{V}^T)^T (\mathbf{V} \mathbf{\Sigma} \mathbf{V}^T) = \mathbf{V} \mathbf{\Sigma}^T \mathbf{\Sigma} \mathbf{V}^T$$

$\mathbf{\Sigma}^T \mathbf{\Sigma}$ has positive diagonal entries as above multiplication will yield individual diagonal entries of $\mathbf{\Sigma}$ to square each other. Since eigen values of $\mathbf{A}^T \mathbf{A}$ are positive thus it is positive definite. Hence $f_2(\bar{x})$ is a convex function.

Question 3

(a)

Convex hull of $S = \{\bar{x}_1 = (0,0), \bar{x}_2 = (1,1), \bar{x}_3 = (1,0)\}$ is just linear combination of all the vectors such that their coefficients are positive and they sum up to 1. Since these are 3 non collinear points, which means that convex hull is just the triangle.

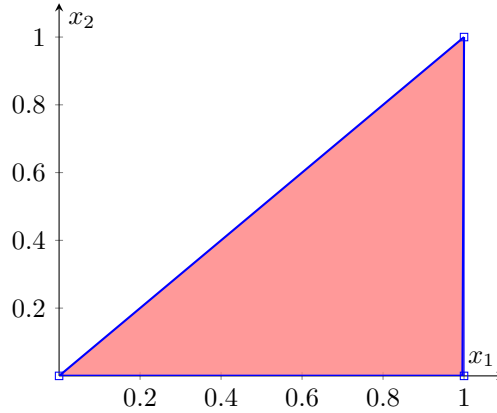
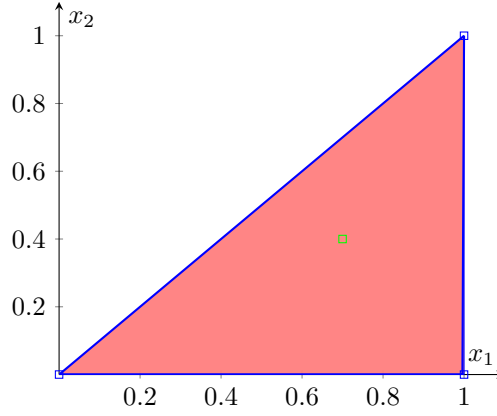


Figure 1: $\text{conv}(S)$

(b)

Take a point inside convex hull,



This point can be written as $\bar{v} = \sum_{i=1}^3 \theta_i \bar{x}_i$ such that $\sum_{i=1}^3 \theta_i = 1$. Given f is convex function which means that,

$$\begin{aligned}
 f(\bar{v}) &= f\left(\sum_{i=1}^3 \theta_i \bar{x}_i\right) \\
 \Rightarrow f(\bar{v}) &\leq \sum_{i=1}^3 \theta_i f(\bar{x}_i) \leq \sum_{i=1}^3 \theta_i \max\{f(\bar{x}_i)\} = \max\{f(\bar{x}_i)\} \sum_{i=1}^3 \theta_i \\
 &\Rightarrow f(\bar{v}) \leq \max\{f(\bar{x}_i)\}
 \end{aligned}$$

Clearly from above inequalities we see that value of any point in convex hull of polyhedra is always less than equal to the value of maximum value at vertices given f is convex, which means maxima occurs at vertices.

Question 4

Question 5

(a)

Let \bar{x}^* be one of the optimum solution to the the function f on set convex set C . Further it is given that bot function f and set C are symmentric convex.

If $\bar{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a solution then the following vector is also a solution $\tilde{\bar{x}} = \begin{pmatrix} x_2^* \\ x_1^* \end{pmatrix}$ since the function is symmentric $f(\bar{x}) = f(\tilde{\bar{x}})$.

By convex nature of the function,

$$\begin{aligned}
 f(\alpha \bar{x}^* + (1 - \alpha) \tilde{\bar{x}}) &\leq \alpha f(\bar{x}^*) + (1 - \alpha) f(\tilde{\bar{x}}) \\
 \Rightarrow f(\alpha \bar{x}^* + (1 - \alpha) \tilde{\bar{x}}) &\leq f(\bar{x}) = f(\tilde{\bar{x}}) = \min_{\bar{x} \in C} \{f(\bar{x})\} \\
 \Rightarrow f(\alpha \bar{x}^* + (1 - \alpha) \tilde{\bar{x}}) &= \min_{\bar{x} \in C} \{f(\bar{x})\}
 \end{aligned}$$

This means that any point on line joining the points \bar{x}^* and $\tilde{\bar{x}}$ is a solution the optimisation problem.

Setting $\alpha = 0.5$, $\bar{a} = \frac{\bar{x}^* + \tilde{\bar{x}}}{2} = \begin{pmatrix} \frac{x_1^* + x_2^*}{2} \\ \frac{x_2^* + x_1^*}{2} \end{pmatrix}$. Now observe that obtained \bar{a} has equal coordinates. Thus there exists an optimum \bar{x} with both coordinates equal.

(b)

AM-GM Inequality

Given $x_i \geq 0$,

$$\frac{1}{N} \sum_{i=1}^N x_i \geq \sqrt[N]{\prod_{i=1}^N x_i}$$

We can solve the given optimisation problem using above inequality.

$$\begin{aligned} \sqrt[n]{x_1 x_2 x_3 \dots x_n} &\leq \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{n} \\ \implies x_1 x_2 x_3 \dots x_n &\leq \frac{1}{n^n} \end{aligned}$$

Therefore, optimal value is $\frac{1}{n^n}$.

Question 6

(a)

Let $g(\bar{x}) = \bar{x}^T \bar{x}$ then $f(\bar{x}, t)$ can be written as $tg(\frac{\bar{x}}{t})$. Now we look at the epigraph of f .

$$\begin{aligned} \text{epi}(f) &= \left\{ \begin{pmatrix} \bar{x} \\ t \\ z \end{pmatrix} : t > 0; z \geq f(\bar{x}, t) \right\} \\ \text{epi}(f) &= \left\{ \begin{pmatrix} \bar{x} \\ t \\ z \end{pmatrix} : t > 0; z \geq tg\left(\frac{\bar{x}}{t}\right) \right\} = \left\{ \begin{pmatrix} \bar{x} \\ z \\ t \end{pmatrix} : t > 0; \frac{z}{t} \geq g\left(\frac{\bar{x}}{t}\right) \right\} \end{aligned}$$

Now let us look at perspective of $\text{epi}(f)$,

$$\text{perspective}(\text{epi}(f)) = \left\{ \begin{pmatrix} \bar{x}/t \\ z/t \end{pmatrix} : t > 0; \frac{z}{t} \geq g\left(\frac{\bar{x}}{t}\right) \right\}$$

We know that g is convex as it is a quadratic function with positive definite hessian matrix. And above set is nothing but epigraph of g thus it is also convex. Now since epigraph of g is convex and it is same as perspective of epigraph of f . This implies that epigraph of f is also convex. Thus we can conclude that f is also convex as it's epigraph is convex.

(b)

(c)

Let $\tilde{x} = \frac{\bar{x}}{\|\bar{x}\|_2}$ then,

$$f(\bar{x}) = \begin{cases} \|\bar{x}\|_2 - 1 & \text{if } \|\bar{x}\|_2 \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

We can rewrite the equation as,

$$f(\bar{x}) = \max\{\|\bar{x}\|_2 - 1, 0\}$$

We know that max of convex functions is convex, $\|\bar{x}\|_2 - 1$ is convex since it is just norm function but with a bias and 0 is trivially convex. Thus f is convex.