CONVEX OPTIMISATION
ASSIGNMENT 1

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Question 1

The set C can also be represented as,

$$C = \{\overline{x} : \mathbf{Y}^T \overline{x} \geq \overline{0}\}$$

Where $\mathbf{Y}$ is a matrix with columns as elements of set S.

A: It is not a Subspace.

Choose $\overline{x_1}, \overline{x_2}$ from set C, if this set is subspace then $\forall \alpha_1, \alpha_2 \in \mathbb{R}; \alpha_1 \overline{x_1} + \alpha_2 \overline{x_2}$ must also be in set C. Since α_1 and α_2 can also take negative value which can the inequatlity, this relation cannot be valid. Thus it is not a Subspace

B: It is not a Affine set.

The same explanation holds, since the only difference between affine and subspace is that in case of affine $\alpha_1 + \alpha_2 = 1$, but they can take any real values thus can be negative as well. Thus in this case also the ineqality need not satisfy.

C: It is a convex set.

Choose $\overline{x_1}, \overline{x_2}$ from set C,

$$\mathbf{Y}^T (\theta \overline{x_1}) \geq \overline{0}; \mathbf{Y}^T ((1-\theta)\overline{x_2}) \geq \overline{0}$$
$$\implies \mathbf{Y}^T (\theta \overline{x_1} + (1-\theta)\overline{x_2}) \geq \overline{0}$$

This means that for any $\theta \in [0, 1]$ the above inequality satisfies which means that the set is convex.

D: It is a cone.

By definition of cone if any set is a cone then any positively scaled vector must also belong to that set.

$$\mathbf{Y}^T \overline{x} \geq \overline{0}$$
$$\implies \mathbf{Y}^T (\theta \overline{x}) \geq \overline{0}; \text{Given } \theta \geq 0$$

Observe that above conditions satisfied despite the nature of matrix $\mathbf{Y}$. Thus we can say that **none of the above queries depend on structure of S**.

Question 2

Given $f_1(\overline{x}) = ||\overline{y} - \mathbf{A}\overline{x}||_2$ and $f_2(\bar{(x)}) = ||\overline{y} - \mathbf{A}\overline{x}||_2^2$,
Useful properties

- Norm function is convex.
- Affine transformation of domain of the function does'nt change the convexity of function.

In case of $f_1(\overline{x})$, the domain is affine transformation of $\overline{x}$ and the norm is operated on it, from above properties the function is convex.

$$f_1(\overline{x}) = ||\Phi(\overline{x})||_2$$
$$\Phi : \overline{x} \to \overline{y} - \mathbf{A}\overline{x}$$

In case of $f_2(\overline{x})$,

$$f_2(\overline{x}) = \overline{x}^T\mathbf{A}^T\mathbf{A}\overline{x} - 2\overline{y}^T\mathbf{A}\overline{x} + \overline{y}^T\overline{y}$$

The above function is quadratic. The function is convex iff $\mathbf{A}^T\mathbf{A}$ is positive semi definite.

$$\mathbf{A} = \mathbf{V\Sigma V}^T$$
$$\implies \mathbf{A}^T\mathbf{A} = (\mathbf{V\Sigma V}^T)^T(\mathbf{V\Sigma V}^T) = \mathbf{V\Sigma}^T\mathbf{\Sigma V}^T$$

$\mathbf{\Sigma}^T\mathbf{\Sigma}$ has positive diagonal entries as above multiplication will yeild individual diagonal entries of $\mathbf{\Sigma}$ to square each other. Since eigen values of $\mathbf{A}^T\mathbf{A}$ are positive thus it is positive definite. Hence $f_2(\overline{x})$ is a convex function.

Question 3

(a)

Convex hull of $S = \{\overline{x_1} = (0,0), \overline{x_2} = (1,1), \overline{x_3} = (1,0)\}$ is just linear combination of all the vectors such that their coeefficients are positive and they sum up to 1. Since these are 3 non collinear points, which means that convex hull is just the triangle.

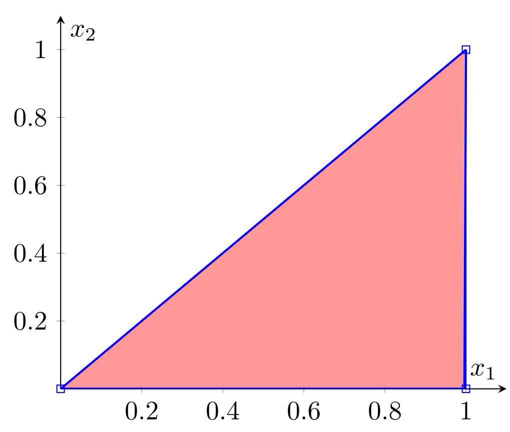
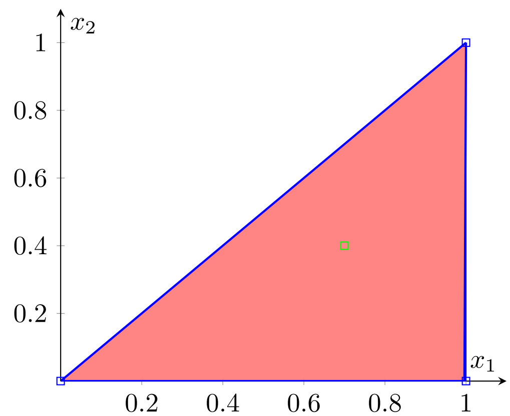


Figure 1: conv(S)

(b)

Take a point inside convex hull,



This point can be written as $\overline{v} = \sum_{i=1}^{3} \theta_i \overline{x_i}$ such that $\sum_{i=1}^{3} \theta_i = 1$. Given f is convex function which means that,

$$f(\overline{v}) = f(\sum_{i=1}^{3} \theta_i \overline{x_i})$$

$$\implies f(\overline{v}) \leq \sum_{i=1}^{3} \theta_i f(\overline{x_i}) f(\overline{v}) \leq \sum_{i=1}^{3} \theta_i max\{f(\overline{x_i})\} = max\{f(\overline{x_i})\} \sum_{i=1}^{3} \theta_i$$

$$\implies f(\overline{v}) \leq max\{f(\overline{x_i})\}$$

Clearly from above inequalities we see that value of any point in convex hull of polyhedra is always less than equal to the value of maximum value at vertices given f is convex, which means maxima occurs at vertices.

Question 4

Useful result

Block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0$ iff $A \geq 0$ and $D \geq CA^{-1}B$.

Using the above result the set C_1 implies that $A \geq 0$ and $z \geq \overline{b}^T A^{-1} \overline{b}$ and before $A \in \mathbb{S}^n$ now $S \in \mathbb{S}^n_{++}$. The set formed by the above deduced equations is same as set C_2. And as set C_1 represents set of positive semi definite matrices which means it is convex, subsequently so is set C_2. Thus both set C_1 and C_2 are convex.

Question 5

(a)

Let $\overline{x}^*$ be one of the optimum solution to the the function f on set convex set C. Further it is given that bot function f and set C are symmentric convex.

If $\overline{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a solution then the following vector is also a solution $\tilde{\overline{x}} = \begin{pmatrix} x_2^* \\ x_1^* \end{pmatrix}$ since the function is symmentric $f(\overline{x}) = f(\tilde{\overline{x}})$.

By convex nature of the function,

$$f(\alpha\overline{x}^* + (1-\alpha)\tilde{\overline{x}}) \leq \alpha f(\overline{x}^*) + (1-\alpha) f(\tilde{\overline{x}})$$
$$\implies f(\alpha\overline{x}^* + (1-\alpha)\tilde{\overline{x}}) \leq f(\overline{x}) = f(\tilde{\overline{x}}) = min_{\overline{x}\in C}\{f(\overline{x})\}$$
$$\implies f(\alpha\overline{x}^* + (1-\alpha)\tilde{\overline{x}}) = min_{\overline{x}\in C}\{f(\overline{x})\}$$

This means that any point on line joining the points $\overline{x}^*$ and $\tilde{\overline{x}}$ is a solution the optimisation problem. Setting $\alpha = 0.5$, $\overline{a} = \frac{\overline{x}^* + \tilde{\overline{x}}}{2} = \begin{pmatrix} \frac{x_1^* + x_2^*}{2} \\ \frac{x_2^* + x_1^*}{2} \end{pmatrix}$. Now observe that obtained $\overline{a}$ has equal coordinates. Thus there exists an optimum $\overline{x}$ with both coordinates equal.

(b)

AM-GM Inequality
Given $x_i \geq 0$,

$$\frac{1}{N}\sum_{i=1}^{N} x_i \geq \sqrt[N]{\Pi_{i=1}^{N} x_i}$$

We can solve the given optimisation problem using above inequality.

$$\sqrt[n]{x_1 x_2 x_3 ... x_n} \leq \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}$$
$$\implies x_1 x_2 x_3 ... x_n \leq \frac{1}{n^n}$$

Therefore, optimal value is $\frac{1}{n^n}$.

Question 6

(a)

Let $g(\overline{x}) = \overline{x}^T\overline{x}$ then $f(\overline{x}, t)$ can be written as $tg(\frac{\overline{x}}{t})$. Now we look at the epigraph of f.

$$epi(f) = \left\{ \begin{pmatrix} \overline{x} \\ t \\ z \end{pmatrix} : t > 0; z \geq f(\overline{x}, t) \right\}$$
$$epi(f) = \left\{ \begin{pmatrix} \overline{x} \\ t \\ z \end{pmatrix} : t > 0; z \geq tg\left(\frac{\overline{x}}{t}\right) \right\} = \left\{ \begin{pmatrix} \overline{x} \\ z \\ t \end{pmatrix} : t > 0; \frac{z}{t} \geq g\left(\frac{\overline{x}}{t}\right) \right\}$$

Now let us look at perspective of $epi(f)$,

$$perspctive(epi(f)) = \left\{ \begin{pmatrix} \overline{x}/t \\ z/t \end{pmatrix} : t > 0; \frac{z}{t} \geq g\left(\frac{\overline{x}}{t}\right) \right\}$$

We know that g is convex as it is a quadratic function with positive definite hessian matrix. And above set is nothing but epigraph of g thus it is also convex. Now since epigraph of g is convex and it is same as perspective of epigraph of f. This implies that epigraph of f is also convex. Thus we can conclude that f is also convex as it's epigraph is convex.

(b)

If function f is quasi convex then it's domain and all its α-sublevel sets are convex are called as "Quasi-convex".
Consider the below α-sublevel set,

$$S_\alpha = \left\{ \begin{pmatrix} \overline{x} \\ t \end{pmatrix} : \frac{\overline{x}^T \overline{x}}{t^2} \leq \alpha \right\} \equiv \left\{ \begin{pmatrix} \overline{x} \\ t \end{pmatrix} : \begin{pmatrix} \frac{\overline{x}}{t} \end{pmatrix}^T \begin{pmatrix} \frac{\overline{x}}{t} \end{pmatrix} \leq \alpha \right\}$$

Perspective of the above set is nothing but a norm ball which is convex. And this satisfies for any α which means that any α level subset is convex and already domain is convex thus this function is Quasi-convex.

(c)

Let $\tilde{\overline{x}} = \frac{\overline{x}}{||\overline{x}||_2}$ then,

$$f(\overline{x}) = \begin{cases} ||\overline{x}||_2 - 1 & if ||x||_2 \geq 1 \\ 0 & otherwise \end{cases}$$

We can rewrite the equation as,

$$f(\overline{x}) = max\{||\overline{x}||_2 - 1, 0\}$$

We know that max of convex functions is convex, $||x||_2 - 1$ is convex since it is just norm function but with a bias and 0 is trivially convex. Thus f is convex.