CONVEX OPTIMISATION ASSIGNMENT

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Question 1

(a)

Problem (2), (3) and (4) are always convex but Problem (3) are not always convex because hessian of objective for problem (1) is,

$$\mathbf{H} = \mathbf{A}^T\mathbf{A} + \alpha\mathbf{I}$$

Positive definitness of hessian is dependent on the value α.

(b)

$$\Delta Objective = \left(\partial \frac{\overline{x}^T(\mathbf{A}^T\mathbf{A} + \alpha\mathbf{I})\overline{x} - \overline{y}^T\mathbf{A}\overline{x} + \overline{y}^T\overline{y}}{\partial \overline{x}}\right)^T$$
$$\implies \Delta Objective = 2(\mathbf{A}^T\mathbf{A} + \alpha\mathbf{I})\overline{x} - \overline{y}^T\mathbf{A}$$

(c)

$$\Delta Objective = 0$$
$$\implies \overline{x}^* = 0.5(\mathbf{A}^T\mathbf{A} + \alpha\mathbf{I})^{-1}\mathbf{A}^T\overline{y}$$

(d)

(e)

Question 2

$f_2(\overline{x}) = \sum_i ||\mathbf{A_i}\overline{x} - \overline{b}||_2$

Apply epigraph trick on the objective,

$$||\mathbf{A_i}\overline{x} - \overline{b}||_2 \le t_i$$
$$\begin{aligned} \min \quad & \overline{1}^T\overline{t} \\ \text{s.t} \quad & ||\mathbf{A_i}\overline{x} - \overline{b}||_2 \le t_i \end{aligned}$$

The above is in form of *SOCP*.

$f_1(\overline{x}) = \sum_i ||\mathbf{A_i}\overline{x} - \overline{b}||_1$

$$
\begin{aligned}
&||\mathbf{A_i}\overline{x} - \overline{b}||_1 \leq \overline{1}^T\overline{t_i} \\
\implies &\mathbf{A_i}\overline{x} - \overline{b} \leq \overline{t_i}; \mathbf{A_i}\overline{x} - \overline{b} \geq -\overline{t_i} \\
\min \quad & \sum_i \overline{1}^T\overline{t_i} \equiv T\overline{1} \\
\text{s.t} \quad & \mathbf{A_i}\overline{x} - \overline{b} \leq \overline{t_i} \\
& \mathbf{A_i}\overline{x} - \overline{b} \geq -\overline{t_i}
\end{aligned}
$$

The above formulation is a *LP*.

$f_\infty(\overline{x}) = \sum_i ||\mathbf{A_i}\overline{x} - \overline{b}||_\infty$

$$
\begin{aligned}
&||\mathbf{A_i}\overline{x} - \overline{b}||_\infty \leq t_i \\
\implies &\mathbf{A_i}\overline{x} - \overline{b} \leq \overline{1}t_i; \mathbf{A_i}\overline{x} - \overline{b} \geq -\overline{1}t_i \\
\min \quad & \overline{1}^T\overline{t} \\
\text{s.t} \quad & \mathbf{A_i}\overline{x} - \overline{b} \leq \overline{t_i} \\
& \mathbf{A_i}\overline{x} - \overline{b} \geq -\overline{t_i}
\end{aligned}
$$

The above formulation is a *LP*.

Question 3

(a)

To prove that the objective function is quasi-convex we ned to show that all α level subsets are convex.

$$
\begin{aligned}
\frac{\overline{\mu}^T\overline{x}}{||\mathbf{V}\overline{x}||_2} \leq \alpha &\implies \frac{\overline{x}^T\overline{\mu}\overline{\mu}^T\overline{x}}{\overline{x}^T\mathbf{V}^T\mathbf{V}\overline{x}} \leq \alpha^2 \\
&\implies \overline{x}^T\left(\alpha^2\mathbf{V}^T\mathbf{V} + \overline{\mu}\overline{\mu}^T\right)\overline{x} \geq 0
\end{aligned}
$$

Given that $\mathbf{V}$ is symmentric which means $\mathbf{V}^T\mathbf{V}$ and outer product of matrices is positive defnite matrix. Therefore sum of positive semidefnite matrices is positive semidefnite. Thus the given objective function is quasi-convex.

(b)

$$
\begin{aligned}
\overline{z} &= \frac{\overline{x}}{\overline{\mu}^T\overline{x}} \\
\implies \frac{\overline{z}}{\overline{1}^T\overline{z}} &= \frac{\frac{\overline{x}}{\overline{\mu}^T\overline{x}}}{\frac{\overline{1}^T\overline{x}}{\overline{\mu}^T\overline{x}}} \\
\implies & \boxed{\overline{x} = \frac{\overline{z}}{\overline{1}^T\overline{z}}}
\end{aligned}
$$

$$
\frac{\overline{\mu}^T\overline{x}}{||\mathbf{V}\overline{x}||_2} = \frac{\overline{\mu}^T\frac{\overline{z}}{\overline{1}^T\overline{z}}}{||\mathbf{V}\frac{\overline{z}}{\overline{1}^T\overline{z}}||_2} = \frac{sgn\left(\overline{1}^T\overline{z}\right)}{||\mathbf{V}\overline{z}||_2}
$$

Given $\overline{\mu}^T\overline{x} \geq 0$ and $\overline{1}^T\overline{x} = 1$ which means $\overline{1}^T\overline{z} = \frac{1}{\overline{\mu}^T\overline{x}} > 0$.

$$\implies \boxed{\frac{\overline{\mu}^T\overline{x}}{||\mathbf{V}\overline{x}||_2} = \frac{1}{||\mathbf{V}\overline{z}||_2}}$$

$$||\overline{x}||_1 \leq L \implies ||\frac{\overline{z}}{\overline{1}^T\overline{z}}||_1 \leq L$$
$$||\overline{z}||_1 \leq L\overline{1}^T\overline{z}$$

Now transformed problem is,

$$\begin{aligned} \min \quad & ||\mathbf{V}\overline{z}||_2 \\ \text{s.t} \quad & ||\overline{z}||_1 \leq L\overline{1}^T\overline{z} \end{aligned}$$

The above transformed problem has both convex objective and constraints thus it is convex optimisation problem.

Question 4

(a)

Lemma: $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} - (I + \mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$.
If $g(x)$ is convex then so is $\overline{a}^T g(x)\overline{a}$ because the map linear with respective to $g(x)$.
Now it is sufficient to prove the convexity of $\mathbf{X}^{-1}$. We do this by contradiction assume that the function is not convex which means,

$$(\alpha\mathbf{A})^{-1} + ((1-\alpha)\mathbf{B})^{-1} < (\alpha\mathbf{A} + (1-\alpha)\mathbf{B})^{-1}$$
$$\frac{1}{\alpha}\mathbf{A}^{-1} + \frac{1}{1-\alpha}\mathbf{B}^{-1} < \alpha\mathbf{A}^{-1} - \frac{1-\alpha}{\alpha^2}(I + \frac{1-\alpha}{\alpha}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$$

Since the matrices are positve semi-definite multiplication on inequality will not change the sign.

$$\frac{1}{1-\alpha}\mathbf{B}^{-1} < -\frac{1-\alpha}{\alpha^2}(I + \frac{1-\alpha}{\alpha}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$$
$$\implies \left(\frac{1-\alpha}{\alpha}\mathbf{A}^{-1}\mathbf{B}\right)^2 + \left(\frac{1-\alpha}{\alpha}\mathbf{A}^{-1}\mathbf{B}\right) + I < 0$$

Since $\mathbf{A} \geq 0$ and $\mathbf{B} \geq 0$ so is $\mathbf{A}^{-1}\mathbf{B} \geq 0 \implies \frac{1-\alpha}{\alpha}\mathbf{A}^{-1}\mathbf{B} \geq 0$.Therfore the above obtained sum is just sum of positive semi definite matrices which is positive semi definite but we got negative definite which is a contradiction. Thus our assumption is wrong. Therfore $\mathbf{X}^{-1}$ is convex and so is $\overline{a}^T\mathbf{X}^{-1}\overline{a}$.

(b)

(c)

(d)

(e)

Question 5

Question 6

Primal is

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} \min \quad & \overline{x}^T \begin{bmatrix} 1 & -0.5 \\ -0.5 & 2 \end{bmatrix} \overline{x} + \begin{bmatrix} -1 & 0 \end{bmatrix} \overline{x} \\ \text{s.t} \quad & \begin{bmatrix} 1 & -2 \\ 1 & 4 \\ 5 & -76 \end{bmatrix} \overline{x} \leq \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix} \end{aligned}$$

Dual is

(a)

The above objective is in quadratic form and eigen decomposition of the hessian is

$$\begin{bmatrix} 1 & -0.5 \\ -0.5 & 2 \end{bmatrix} = \begin{bmatrix} -0.92387953 & 0.38268343 \\ -0.38268343 & -0.92387953 \end{bmatrix} \begin{bmatrix} 0.79289322 & 0 \\ 0 & 2.20710678 \end{bmatrix} \begin{bmatrix} -0.92387953 & -0.38268343 \\ 0.38268343 & -0.92387953 \end{bmatrix}$$

Here both eigen values are positive, which implies that hessian is positive semidefnite. With linear constraints, The problem is convex and is a QP.

(b)

After solving the problem with $CVXPY$ we get,

$$x_1^* = -3; x_2^* = 0$$
$$\lambda_1^* = 5.167; \lambda_2^* = 1.834; \lambda_3^* = 0$$

(c)

KKT Conditions

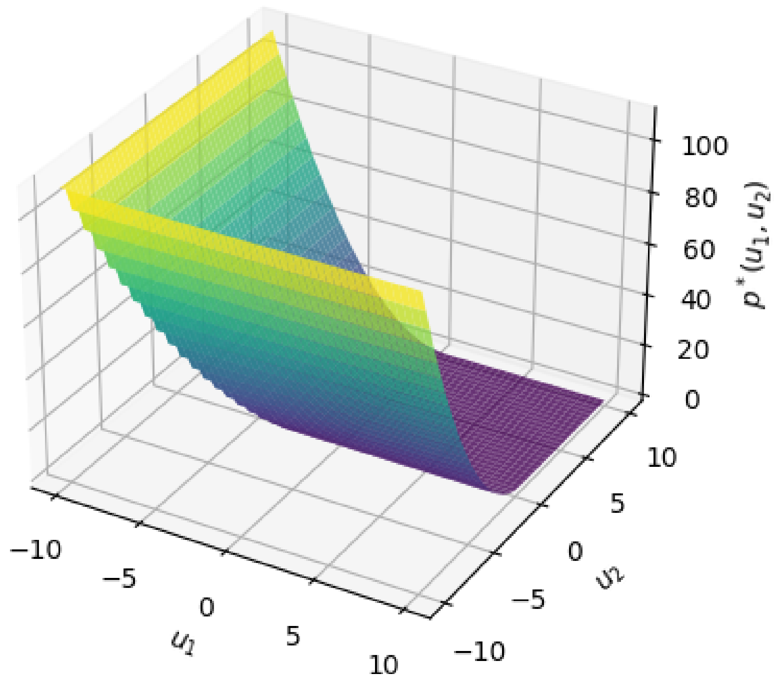
1. $f_i(x^*) \leq 0$, Satisfied.

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \\ 5 & -76 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} \leq \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \implies \begin{bmatrix} -3 \\ -3 \\ -15 \end{bmatrix} \leq \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

2. $\lambda_i^* \geq 0$, Satisfied.
3. $\lambda_i^* f_i(x^*) = 0$, Satisfied.

$$\begin{bmatrix} 5.167 & 1.834 & 0 \end{bmatrix} \left(\begin{bmatrix} -3 \\ -3 \\ -15 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right) = 0$$

(d)

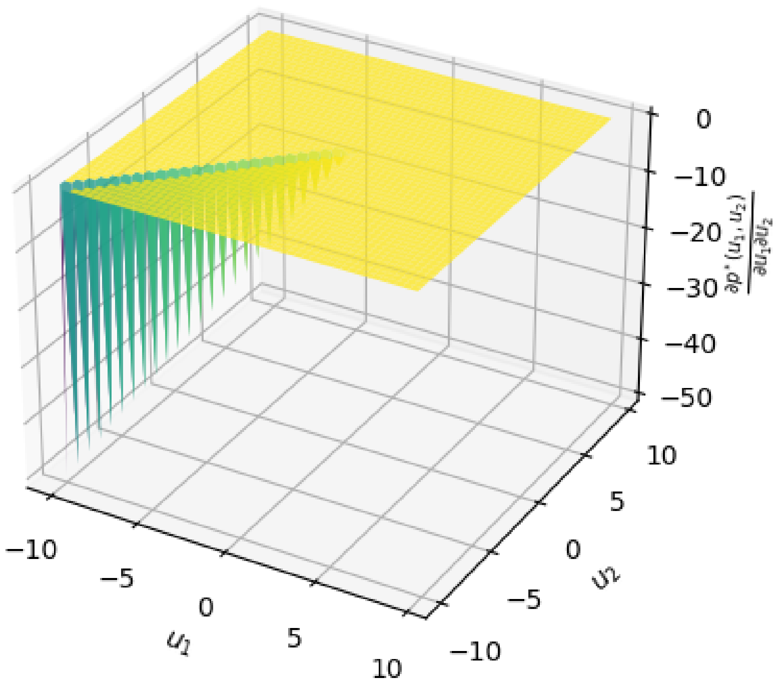


(e)

From above graph it seems like $p^*(u_1, u_2)$ is a convex function.

(f)

Numerically derivated at the given point is 0.


0
−10
−20
−30
−40
−50
$\frac{\partial \phi^*(u_1,u_2)}{\partial u_1 \partial u_2}$
10
5
0
−5
−10
u_2
−10
−5
0
5
10
u_1