HW-0

Q1) Sauer - Shelah Lemma:-

If VC diemension of a function is 'd' then growth function

$$S^{\wedge}(n) \le \sum_{i=0}^{d} {}^{n}C_{i}$$

Corollary:-

$$S^{\wedge}(n) \le \sum_{i=0}^{d} {}^{n}C_{i} \le \left(\frac{ne}{d}\right)^{d}$$

Proof by Induction

$n = 0$

$\Rightarrow S^{\wedge}(0) \le 1$, emptyset

$d = 0$

$\Rightarrow S^{\wedge}(n) \le 1$, Can't shatter one point also

Induction Step:-

$S = [x_1, x_2 \ldots\ldots x_m] = H$

Let

$H_1 = [x_1, x_2 \ldots\ldots x_{n-1}]$

$H_2 = H \setminus H_1$

Claim ①

If set is shattered by H_1 then it is also shattered by H

Reason

$H \supseteq H_1$

$\Rightarrow VC\dim(H_1) \leq VC\dim(H) = d$

Claim 2

If a set A is shattered by H_2 then $A \cup \{x_m\}$ is shattered by H

Reason

H_2 contains collapsed dicotomies of H_1

$\Rightarrow VC\dim(H_2) + 1 \leq VC\dim(H)$

$\Rightarrow$ VC dim $(H_2) \le d-1$

By Induction.

$$\Rightarrow \quad S^{H_1}(n) \le \sum_{i=0}^{d} {}^{n-1}C_i$$

$$S^{H_2}(n) \le \sum_{i=0}^{d-1} {}^{n-1}C_i$$

$$= \sum_{j=1}^{d} {}^{n-1}C_{i-1}$$

as ${}^{n-1}C_i + {}^{n-1}C_{i-1} = {}^{n}C_i$

$$\Rightarrow S^{H}(n) \le \sum_{i=0}^{d} {}^{n}C_i$$

$\Rightarrow$ Proved by induction

Collorary Proof

$$\left(\frac{n}{d}\right)^d \sum_{i=0}^{d} \left(\frac{d}{n}\right)^d {}^{n}C_i \le \left(\frac{n}{d}\right)^d \sum_{i=0}^{d} \left(\frac{d}{n}\right)^i {}^{n}C_i$$

as $\frac{d}{n} \le 1$

$$\Rightarrow \left(\frac{d}{n}\right)^d \sum_{i=0}^{d} {}^{n}C_i \le \left(1+\frac{d}{n}\right)^d \le e^d$$

$$\therefore \sum_{i=0}^{d} {}^{n}C_i \le \left(\frac{ne}{d}\right)^d$$

Q2) Generalisation bound

$$P\left(\sup_{\alpha\in\Lambda} |R(\alpha) - R_{emp}(\alpha)| > \varepsilon\right)$$

$$\leq 4 S^{\Lambda}(2n) \exp\left(\frac{-n\varepsilon^2}{8}\right)$$

By Sauer's Lemma,

$$\leq \underbrace{4 \left(\frac{2ne}{d}\right)^{d} \exp\left(\frac{-n\varepsilon^2}{8}\right)}_{\eta}$$

$\Rightarrow$

$$\Rightarrow \ln \eta = \ln 4 + d\left[\ln 2 + \ln n + 1 - \ln d\right] - \frac{n\varepsilon^2}{8}$$

$$\varepsilon = \sqrt{\frac{8}{n}\left[\ln\left[4/\eta\right] + d\ln\left[\frac{2n}{d}\right] + d\right]}$$

$$R(\alpha) \leq R_{emp}(\alpha) + \left[\frac{8}{n}\left[\ln[4/\eta] + d\ln[2n/d] + d\right]\right]^{1/2}$$

holds with probability "$1-\eta$"

Q3) $X_n(\omega) = \omega^n \quad ; \quad \omega \in [0,1]$

as $\lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} \omega^n$

X_n is memoryless

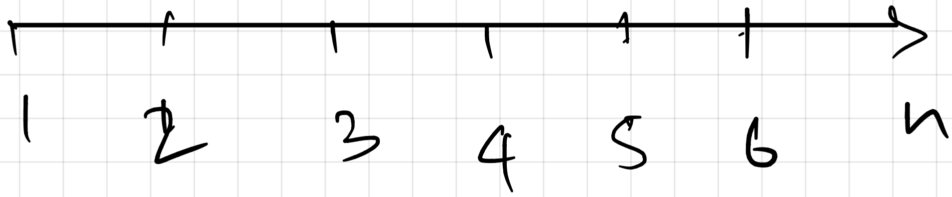
thus either $\omega = 1$

or $0 \leq \omega < 1$

$\Rightarrow \lim_{n\to\infty} \omega^n = 0$ (or) 1

$\therefore$ limit does not exist

Q4)



$j = n - 2^{k}$

$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \in (n2^{-k} - 1, (n+1)2^{-k} - 1] \\ 0 & \text{else} \end{cases}$$

width of ω when $X_n(\omega) = 1$

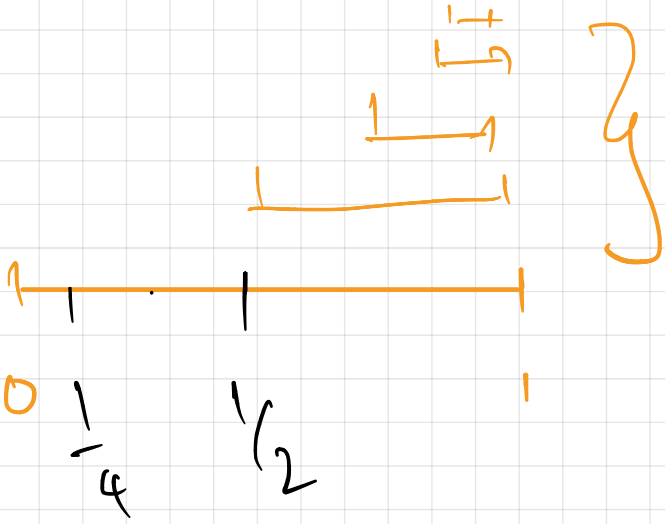
is 2^{-k}

$\Rightarrow$ as $n \to \infty$

$k = \lfloor \ln_2(n) \rfloor \to \infty$

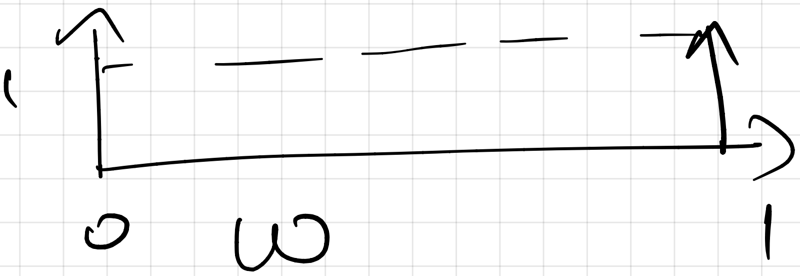
$\Rightarrow \quad 2^{-k} \to 0$

∴



decreases

$\Rightarrow$ as $n \to 0$



as $n \to \infty$ $\quad x_n(\omega) = \delta(\omega - 1)$

$$\Rightarrow \lim_{n\to\infty} x_n(\omega) = \delta(\omega-1)$$

∴ Converges a.s

Q 5)

$$X_n = \frac{(-1)^n}{n} U$$

$U \sim \text{Uniform}(0,1)$

(a) if $\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty$

then, $X_n \xrightarrow{a.s} X$

$$\sum_{n=1}^{\infty} P\left(\left|\frac{(-1)^n}{n} U - 0\right| > \varepsilon\right)$$

$$= \sum_{n=1}^{\infty} P\left(\frac{U}{n} > \varepsilon\right)$$

There is no 'n' which holds always true as $0 \leq U \leq 1$

$\Rightarrow \quad X_n \not\xrightarrow{a.s} X$

(b) If $X_n \xrightarrow{P} X$

then,

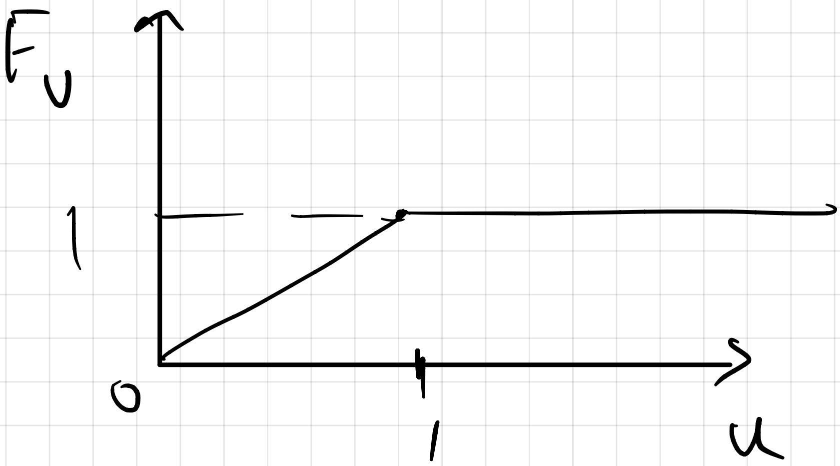
$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$$

$$\lim_{n\to\infty} P\left(\left|\frac{(-1)^n}{n} U - 0\right| \geq \varepsilon\right)$$

$$= P\left(\frac{U}{n} \geq \varepsilon\right)$$

$$= P(U \geq n\varepsilon)$$

$$= 1 - F_U(n\varepsilon)$$



if $n\varepsilon \geq 1$

$= 0$

$\Rightarrow$

$\Rightarrow \quad n \geq 1/\varepsilon$

we can always find n for an ε

$\Rightarrow \quad X_n \xrightarrow{P} X$

(c) $\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \forall x$ which $F_X(x)$ is cont

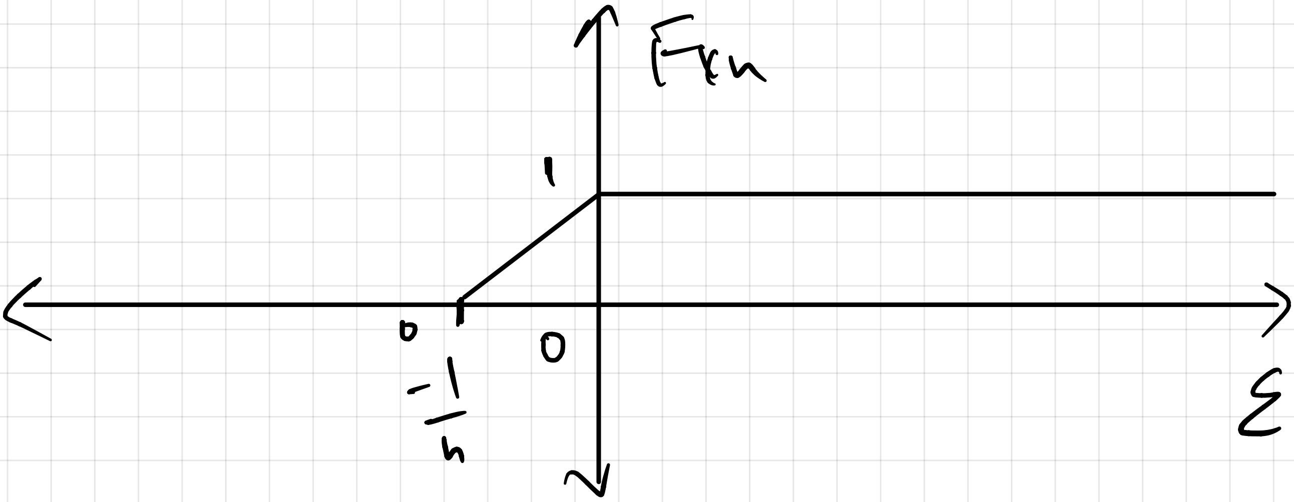
$F_{X_n}(x)$

Case $n =$ odd

$$F_{X_n}(\varepsilon) = P\left(\frac{-U}{n} \leq \varepsilon\right)$$

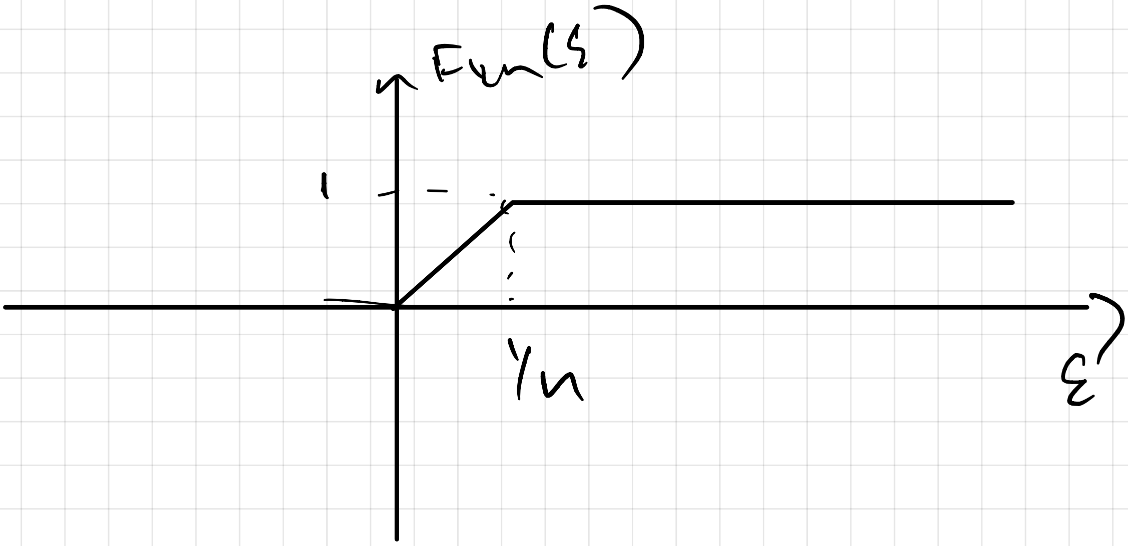
$$= P(U \geq -n\varepsilon)$$

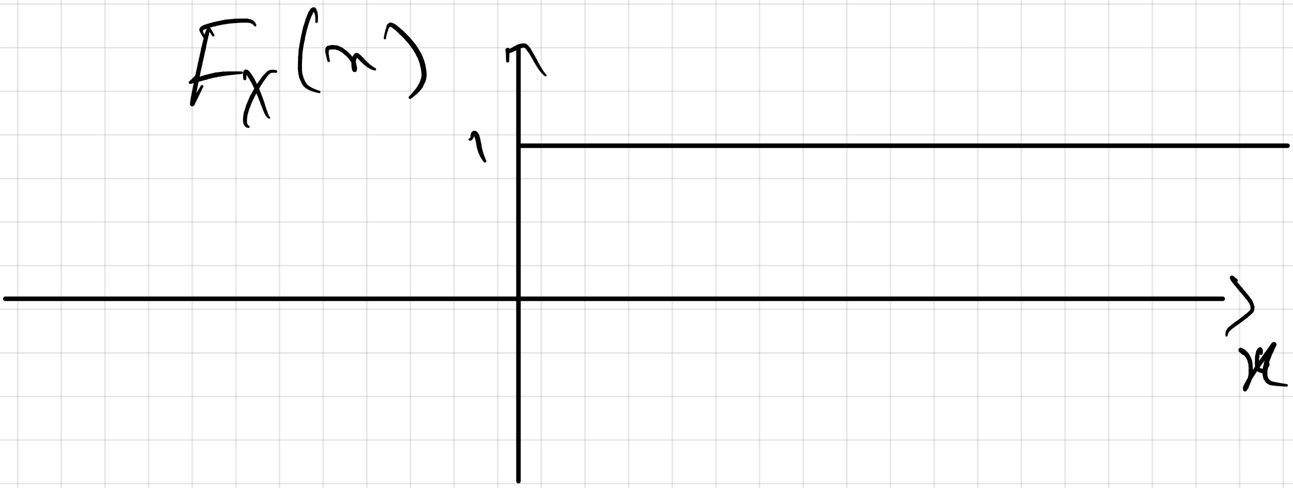
$$= 1 - F_U(-n\varepsilon)$$



Case n = even

$$F_{X_n}(\varepsilon) = F_U(n\varepsilon)$$





$$\lim_{n \to \infty} F_{x_n}(x) = \begin{cases} 1 - F_u(-\infty) & \text{if } n \text{ is odd} \\ F_u(\infty) & \text{else} \end{cases}$$

$$1 - F_u(-\infty) = F_u(\infty) = 1$$

$$\therefore \quad X_n \xrightarrow{d} X$$