HW0

Q1) $X \sim \mathcal{N}(0, \sigma^2)$

$$MGF_X(t) = \mathbb{E}(e^{Xt})$$

$$= \int_{\mathbb{R}} e^{xt} p_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{xt} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-x^2/2\sigma^2}\, dx$$

Let $y = x/\sigma$

$\Rightarrow dx = \sigma\, dy$

$$MGF_X(t) = \int_{-\infty}^{\infty} e^{\sigma y t} \frac{\cancel{\sigma}}{(2\pi\cancel{\sigma^2})^{1/2}} e^{-\frac{y^2}{2}}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{\sigma y t - \frac{y^2}{2}}\, dy$$

Remark

$$zx - \frac{x^2}{2} = -\frac{1}{2}(x-z)^2 + \frac{z^2}{2}$$

$$\Rightarrow MGF_X(t) = e^{\frac{\sigma^2 t^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y-\sigma t)^2}}{(2\pi)^{1/2}} dy}_{\substack{\text{Resembles} \\ \text{Normal P.d.F}}}$$

$\Rightarrow$

$$MGF_X(t) = e^{\frac{\sigma^2 t^2}{2}}$$

Q2) **Markov's inequality**

(a) For any non-negative R.V X and any $\varepsilon > 0$,

$$P[X \geq \varepsilon] \leq \frac{E[X]}{\varepsilon}$$

Chebyshev Inequality

For any R.V X and any $\varepsilon > 0$;

$$P(|X - \mathbb{E}[X]| > \varepsilon) \leq \frac{Var[X]}{\varepsilon^2}$$

Proof

Take $Y = (X - \mathbb{E}[X])^2$

By markov's inequality

$$P((X - \mathbb{E}[X])^2 \geq \varepsilon^2) \leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^2]}{\varepsilon^2}$$

here 'ε' is 'ε^2'

$$\Rightarrow P(|X - \mathbb{E}[X]| \geq \varepsilon) \leq \frac{Var[X]}{\varepsilon^2}$$

Note:-

$$(X - \mathbb{E}[X])^2 \geq \varepsilon^2 \equiv |X - \mathbb{E}[X]| \geq \varepsilon$$

(b) **Chernoff bound**

For any R.V X and $t > \varepsilon > 0$

$$P[X \geq \varepsilon] = P\left[e^{Xt} \geq e^{\varepsilon t}\right] \leq e^{-\varepsilon t}\, MGF_X(t)$$

Proof

$f(x) = e^{x}$ is a strictly monotonically increasing function

$$\Rightarrow \quad a \geq b$$
$$\Rightarrow \quad e^{a} \geq e^{b}$$

$$\Rightarrow P[X \geq \varepsilon] = P\left[e^{Xt} \geq e^{\varepsilon t}\right]$$

since both e^{Xt} & $e^{\varepsilon t}$ are positive

by applying Markov inequality

$$P\left[e^{Xt} \geq e^{\varepsilon t}\right] \leq \frac{\mathbb{E}\left[e^{Xt}\right]}{e^{\varepsilon t}}$$

$\therefore \quad P[X \geq \varepsilon] = P[e^{Xt} \geq e^{\varepsilon t}] \leq \dfrac{\mathbb{E}[e^{Xt}]}{e^{\varepsilon t}}$

Q3)

a) Take biased coin toss,

Sample space $(S) = \{H, T\}$

Define a R.V $X : S \longrightarrow \mathbb{R}$

s.t

$$\left.\begin{aligned} X(H) &= K \\ X(T) &= 0 \end{aligned}\right\} \text{ for some } K \in (0, \infty)$$

$$P(H) = 1/K$$

$$P(T) = 1 - 1/K$$

$$\Rightarrow \mathbb{E}[X] = K \times \frac{1}{K} + 0 \times 1 - \frac{1}{K}$$

$$\Rightarrow \boxed{\mathbb{E}[X] = 1}$$

Applying markov inequality,

$$P(X \geq \varepsilon) \leq \frac{\mathbb{E}[X]}{\varepsilon}$$

$$\Rightarrow P(X \geq \varepsilon) \leq \frac{1}{\varepsilon}$$

At $\varepsilon = k$

$$\Rightarrow P(X \geq k) \leq \frac{1}{k}$$

$$\frac{1}{k} \leq \frac{1}{k}$$

$\therefore$ At $\varepsilon = k$; The inequality has become $\Rightarrow$ "tight"

b) Consider n-coins tosses

Let R.V X indicate # heads in n-tosses.

$\Rightarrow$ This is Binomial distribution

$$\mathbb{E}[X] = np$$

$$Var[X] = np(1-p)$$

$$MGF_X(t) = (p(e^t - 1) + 1)^n$$

By chernoff inequality,

$$P(X \geq \varepsilon) \leq \frac{(p(e^t - 1) + 1)^n}{e^{\varepsilon t}}$$

Let's find tight bound:

$$\Rightarrow \frac{\partial}{\partial t}(RHS) = 0$$

$$\Rightarrow e^t = \frac{1-p}{p\left(\frac{n}{\varepsilon}-1\right)}$$

$$\left.\frac{\partial^2}{\partial t^2}(RHS)\right|_{e^t = \frac{1-p}{p\left(\frac{n}{\varepsilon}-1\right)}} > 0$$

$$\Rightarrow P(X \geq \varepsilon) \leq \frac{p^{\varepsilon}(1-p)^{n-\varepsilon}}{\left(\frac{n}{\varepsilon}-1\right)^{n-\varepsilon}} \quad \text{--- (i)}$$

By Chebyshev,

$$P(|X - np| \geq \varepsilon') \leq \frac{np(1-p)}{\varepsilon'^2}$$

Let $\varepsilon' > np$

$$\Rightarrow P(X \geq np + \varepsilon')$$
$$= P(|X - np| \geq \varepsilon')$$
$$(\because X \geq 0)$$

$$\Rightarrow P(X \geq \varepsilon) \leq \frac{np(1-p)}{(\varepsilon - np)^2} \quad \leftarrow \text{(ii)}$$

Take $p = 1/2$

From (i) & (ii)

(i) $\rightarrow$ $$P(X \geq \varepsilon) \leq \frac{1}{2^n \left(\frac{n}{\varepsilon} - 1\right)^{n-\varepsilon}}$$

(ii) $\rightarrow$ $$P(X \geq \varepsilon) \leq \frac{n}{4\left(\varepsilon - \frac{n}{2}\right)^2}$$

$\rightarrow$ Upper bound $= \Theta(n^{-n})$

$\rightarrow$ Upper bound $= \Theta(n^{-1})$

Clearly (i) has tighter bound than (ii)

Q4)

a) Weak law of Large Number (LLN)

Given, $x_1 \cdots x_n$ which are i.i.d.

$$\tilde{X}_n = \frac{1}{n}\sum_i X_i$$

$$\lim_{n\to\infty} P\left[\,|\tilde{X}_n - \mu| < \varepsilon\,\right] = 1$$

By chebyshev inequality,

$$P\left[\,|\tilde{X}_n - \mu| \geq \varepsilon\,\right] \leq \frac{Var[\tilde{X}_n]}{\varepsilon^2}$$

$$Var[\tilde{X}_n] = \frac{\sigma^2}{n}$$

$$P\left[\,|\tilde{X}_n - \mu| \geq \varepsilon\,\right] \leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\Rightarrow P\left[\,|\tilde{X}_n - \mu| < \varepsilon\,\right] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

$$\Rightarrow \lim_{n \to \infty} P\left[|\tilde{X}_n - \mu| < \varepsilon\right] = 1$$

Q 5)

a) Rademacher R.V

$$p_X(x) = \frac{1}{2}\left(\delta(x+1) + \delta(x-1)\right)$$

Kronecker delta

$$\mathbb{E}[e^{Xt}] = \int_{\mathbb{R}} e^{xt} p_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{xt}\left[\frac{1}{2}\left[\delta(x+1) + \delta(x-1)\right]\right] dx$$

$$= \frac{1}{2}\left[e^{-t} + e^{t}\right] \left(\int_{\mathbb{R}} \delta(x) = 1 \right)$$

$$\Rightarrow \mathbb{E}[e^{Xt}] = \frac{1}{2}\left[e^{t} + e^{-t}\right]$$

$$= \cosh(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}$$

(By taylor around 0)

$$(2n)! = (2n)(2n-1)\cdots(n+1)\,n!$$

for $n \geq 1 \Rightarrow n+1 \geq 2$

$\Rightarrow (2n)! \geq 2^n n!$.

W.K.T, $e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

$$\Rightarrow e^{\frac{\sigma^2}{2}t^2} = \sum_{n=0}^{\infty} \frac{(\sigma t)^{2n}}{2^n n!}$$ take $\sigma = 1$

$$\Rightarrow e^{\frac{\sigma^2 t^2}{2}} \geq \cosh(t) \quad (\text{by } (2n)! \geq 2^n n!)$$

$\therefore$ For $\sigma = 1$; Radamacher r.v is sub-gaussian

(b) This is a trivial case.

For $X \sim \mathcal{N}(0, \sigma^2)$

$$MGF_X(t) = e^{\frac{\sigma^2 t^2}{2}} \quad [\text{Refer to Q1}]$$

$$\Rightarrow MGF_X(t) \leq e^{\frac{\sigma^2 t^2}{2}}$$

Thus this is trivial case:

For any (σ) this satisfies.

(c) Given, R.V $\mathbb{E}[X] = 0$

$$a \leq X \leq b$$

since $\mathbb{E}[X] = 0$

$$\Rightarrow a \leq 0 \leq b$$

$$\mathbb{E}[e^{Xt}] = \int_{\mathbb{R}} e^{tx} p_X(x)\,dx$$

$$= \int_a^b e^{xt} p_X(x)\,dx$$

$$\mathbb{E}[e^{Xt}] = \int_a^b e^{xt} p_X(x)\,dx \le e^{bt} \int_a^b p_X(x)\,dx$$

$$\Rightarrow \mathbb{E}[e^{Xt}] \le e^{bt}$$

for $\mathbb{E}[e^{Xt}] \le e^{\frac{\sigma^2 t^2}{2}}$

$$e^{bt} \le e^{\frac{\sigma^2 t^2}{2}}$$

$$\Rightarrow bt \le \frac{\sigma^2 t^2}{2} \Rightarrow \sigma^2 \ge \frac{2b}{t}$$

$\therefore$ For $\sigma \in \left[\sqrt{\frac{2b}{t}}, \infty\right)$

the given R.V is sub-gaussian.

Q6) Let $Y = X - \mu$; By Chernoff bound

$$P(|Y| \ge t)$$

$$= P(e^{Y\varepsilon} \le e^{-\varepsilon t}) + P(e^{-Y\varepsilon} \le e^{-\varepsilon t})$$

$$\le e^{-\varepsilon t}\left[\mathbb{E}[e^{Y\varepsilon}] + \mathbb{E}[e^{-Y\varepsilon}]\right]$$

for $\varepsilon > 0$

Lemma: If X is sub-exponential then so is $X - a$

Let $Z = X - a \Rightarrow \mathbb{E}[e^{\lambda Z}] \le e^{\frac{\sigma^2\lambda^2}{2} - a\lambda}$

$\exists\, \sigma'$ s.t. $\frac{\sigma^2\lambda^2}{2} - a\lambda \le \frac{\sigma'^2\lambda^2}{2}$ $\forall \lambda$

$$(\because \text{ from above lemma}) \leq e^{-\varepsilon t}\left[e^{\frac{\sigma^2\varepsilon^2}{2}-\mu\varepsilon} + e^{\frac{\sigma^2\varepsilon^2}{2}+\mu\varepsilon}\right]$$

$$\leq e^{\frac{\sigma^2\varepsilon^2}{2}-\varepsilon t}\left[e^{-\mu\varepsilon}+e^{\mu\varepsilon}\right]$$

By A.M $\leq$ G.M

$$\leq 2\exp\left[\frac{\sigma^2\varepsilon^2}{2}-\varepsilon t\right]$$

For a tight bound,

$$\frac{\partial}{\partial\varepsilon}\left(\frac{\sigma^2\varepsilon^2}{2}-\varepsilon t\right)=0$$

$$\Rightarrow \varepsilon = t/\sigma^2$$

$$\Rightarrow P(|X-\mu|\geq t) \leq 2\exp\left(-\frac{t^2}{2\sigma^2}\right)$$

Q7)

Given X_i's are independent are zero mean and σ_i^2-subgaussian

Let $Y = \sum_{i=1}^{n} X_i$

$$\Rightarrow \mathbb{E}[e^{\lambda Y}] = \mathbb{E}[e^{\lambda \sum_i x_i}]$$

Since x_i's independent

$$\Rightarrow \mathbb{E}[e^{\lambda Y}] = \prod_{i=1}^{n} \mathbb{E}[e^{\lambda x_i}]$$

as X_i is σ_i-subgaussian

$$\mathbb{E}[e^{\lambda x_i}] \le e^{\frac{\sigma_i^2 \lambda^2}{2}}$$

$$\Rightarrow \mathbb{E}[e^{\lambda Y}] \le \prod_{i=1}^{n} e^{\frac{\sigma_i^2 \lambda^2}{2}}$$

$$\Rightarrow \mathbb{E}[e^{\lambda \sum_{i=1}^{n} x_i}] \le e^{\sum_{i=1}^{n} \sigma_i^2 \frac{\lambda^2}{2}}$$

Thus, $\sum_{i=1}^{n} X_i$ is $\sum_{i=1}^{n} \sigma_i^2$ - sub-gaussian

Q8)

Given, X_i is R.V $[a_i, b_i]$

Hoeffding's lemma

For R.V X which is zero mean and bounded by $[a, b]$

$$MGF_X(t) \leq e^{\frac{t^2(b-a)^2}{8}}$$

Let $S_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i))$

Let $Y_i = X_i - \mathbb{E}(X_i)$

Note: $\mathbb{E}[Y_i] = 0$

$$S_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$\Rightarrow MGF_{S_n}(t) = \prod_{i=1}^{n} MGF_{Y_i}(t/n)$$

By chernoff bond, [Note: $\mathbb{E}[S_n] = 0$]

$$P(S_n \geq t) \leq \frac{MGF_{S_n}(\varepsilon)}{e^{\varepsilon t}}$$

$$P(S_n \geq t) \leq \frac{\prod_{i=1}^{n} MGF_{Y_i}(\varepsilon/n)}{e^{\varepsilon t}}$$

$$\leq \frac{\prod_{i=1}^{n} e^{\frac{\varepsilon^2 (b_i - a_i)^2}{8n^2}}}{e^{\varepsilon t}}$$

$$\leq e^{-\varepsilon t + \sum_{i=1}^{n} \frac{\varepsilon^2}{8n^2}(b_i - a_i)^2}$$

For a tight bound:

$$\min_{\varepsilon} \{ RHS \} \equiv \min_{\varepsilon} \left\{ -\varepsilon t + \sum_{i=1}^{n} \frac{\varepsilon^2 (b_i - a_i)^2}{8n^2} \right\}$$

$$\frac{\partial}{\partial \varepsilon} = 0 \Rightarrow -t + \frac{\varepsilon}{4n^2} \sum_{i=1}^{n} (b_i - a_i)^2$$

$$\Rightarrow \varepsilon = \frac{4n^2 t}{\sum_{i=1}^{n} (b_i - a_i)^2}$$

$$\Rightarrow P(S_n \geq t) \leq \exp\left[\frac{-2n^2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right]$$

Note we did not impose any constraint on sign of

S_n & t

Here Let $Y_i' = \mathbb{E}[X_i] - X_i$

$$\Rightarrow \mathbb{E}[Y_i'] = 0$$

Let $S_n' = \frac{1}{n}\sum_{i=1}^{n} Y_i' \Rightarrow S_n = -S_n'$

$$P(-S_n' \leq -t) \equiv P(S_n' \geq t)$$

Note $MGF_{S_n'}(\varepsilon) = \prod_{i=1}^{n} MGF_{Y_i'}\left(\frac{\varepsilon}{n}\right)$

By chernoff bond

$$P(S_n \leq -t) \leq \frac{\prod_{i=1}^{n} MGF_{Y_i'}(\varepsilon/n)}{e^{t\varepsilon}}$$

$$P(S_n \leq -t) \leq e^{-\varepsilon t + \sum_{i=1}^{n} \frac{\varepsilon^2}{8}(b_i - a_i)^2}$$

This is same form as earlier

$$\Rightarrow P(S_n \leq -t) \leq \exp\left(\frac{-2n^2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Q9)(b) χ^2 - distribution

$$P_X(x,k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{\frac{k}{2}-1} e^{-x/2}$$

$$\mathbb{E}[x] = \int_0^\infty x P_X(x,k)\, dx$$

$$= \int_0^\infty \frac{x^{k/2} e^{-x/2}}{2^{k/2}\Gamma(k/2)} dx$$

$$\mathbb{E}[x] = k \qquad \left[\text{Note: } \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx\right]$$

$$E\left[e^{\lambda(X-\mu)}\right]$$

$$= \int_0^\infty \frac{e^{\lambda(x-k)} x^{\frac{k}{2}-1} e^{-x/2}}{2^{k/2}\Gamma(k/2)} dx$$

$$= \frac{e^{-\lambda k}}{2^{k/2}\Gamma(k/2)} \int_0^\infty x^{\frac{k}{2}-1} e^{-(\frac{1}{2}-\lambda)x} dx$$

Take, $\left(\frac{1}{2}-\lambda\right)x = y$

$$\Rightarrow dx = \frac{dy}{\frac{1}{2}-\lambda}$$

$$\int_0^\infty x^{\frac{k}{2}-1} e^{-(\frac{1}{2}-\lambda)x} dx = \int_0^\infty \frac{y^{\frac{k}{2}-1}}{(\frac{1}{2}-\lambda)^{\frac{k}{2}}} e^{-y} dy$$

$$= \frac{\Gamma(k/2)}{(\frac{1}{2}-\lambda)^{k/2}}$$

$$\Rightarrow \mathbb{E}\left[e^{\lambda(X-\mu)}\right] = \frac{e^{-\lambda k}}{2^{k/2}(\frac{1}{2}-\lambda)^{k/2}}$$

For this to be sub-exponential

$$\frac{e^{-\lambda k}}{2^{k/2}\left(\frac{1}{2}-\lambda\right)^{k/2}} \leq e^{\frac{v^2\lambda^2}{2}} \quad ; |\lambda| \leq \frac{1}{b}$$

$$-\lambda k + \frac{k}{2}\left[\ln 2 + \ln\left(\frac{1}{2}-\lambda\right)\right] \leq \frac{v^2\lambda^2}{2}$$

$$\frac{v^2 \lambda^2}{2} + \lambda k - \frac{k}{2}\left[\ln 2 + \ln\left(\tfrac{1}{2} - \lambda\right)\right]$$

For a solution ≥ 0

$$k^2 \geq \cancel{4}\, \frac{v^2}{\cancel{2}} \left(\frac{-k}{\cancel{2}}\right) \left[\ln 2 + \ln\left(\tfrac{1}{2} - \lambda\right)\right]$$

$$k^2 + k v^2 \ln(1 - 2\lambda) \geq 0$$

$$\Rightarrow v = \begin{cases} \left[\frac{-k}{\ln(1-2\lambda)}, \infty\right) & k > 0 \\ \left(-\infty, \frac{-k}{\ln(1-2\lambda)}\right] & k < 0 \end{cases}$$

here $\lambda < \frac{1}{2}$

$\Rightarrow b = 2$

$\therefore$ Thus there exist a (v, b)

$\Rightarrow$ It is sub exponential.

(a) We have proved that

$X \sim \chi^2_k$ is a sub exponential.

with

$$\nu = \begin{cases} \left[\frac{-k}{\ln(1-2\lambda)}, \infty\right) & k>0 \\ \left(-\infty, \frac{-k}{\ln(1-2\lambda)}\right] & k<0 \end{cases}$$

$$\lambda < \frac{1}{2}$$

Note

$X \sim \chi^2_2 \equiv X \sim$ exponential distribution with $\left(\beta = \frac{1}{2}\right)$

Take,

$Y = \lambda X \Rightarrow$ it is still scaled version of χ^2_2

Lemma :- If X is a sub-exponential with (ν, b) then aX also is a sub-exponential with given $a>0$

Proof:- $Y = aX$

$$\mathbb{E}_Y[f(Y)] = \int_{\mathbb{R}} f(y)\, dF_Y(y)$$

let $y = ax$

$$= \int_{\mathbb{R}} f(ax)\, dF_X(x)$$

$$\Rightarrow \mathbb{E}_Y[f(Y)] = \mathbb{E}_X[f(aX)]$$

$$\Rightarrow \mathbb{E}_Y[e^{\lambda(Y-\mu_Y)}] = \mathbb{E}_X[e^{\lambda(aX-a\mu_X)}]$$

$$= \mathbb{E}_X[e^{\lambda a(X-\mu_X)}]$$

here λ is "$a\lambda$"

$\Rightarrow$ Y is a sub gaussian with $\left(\frac{\nu}{a}, a\lambda\right)$

By this lemma, $Y = \lambda X$ is sub-exponential and $X \sim \chi_2^2$ thus thus Y (exponential) is sub-exponential.

with

$$|\nu| \geq \frac{2}{\lambda \ln(1-2\lambda^2)}$$

$$\lambda^2 < \frac{1}{2} \Rightarrow |\lambda| < \frac{1}{\sqrt{2}}$$

Q10)

Given, x is sub-exponential

$$\mathbb{E}\left[e^{\lambda(x-\mu)}\right] \leq e^{\frac{v^2\lambda^2}{2}} \quad ; |\lambda| \leq \frac{1}{b}$$

By chernoff bount

$$P[X \geq \mu+t] \leq \frac{\mathbb{E}[e^{\lambda x}]}{e^{(\mu+t)\lambda}}$$

$$\leq \frac{\mathbb{E}\left[e^{\lambda(x-\mu)}\right]}{e^{t\lambda}}$$

$$\Rightarrow P[x \geq \mu+t] \leq e^{\frac{v^2\lambda^2}{2} - t\lambda} \quad ; |\lambda| \leq \frac{1}{b}$$

$$\forall t > 0$$

Q 11)

Given, X_i's are independent

$$\Rightarrow MGF_{\sum X_i}(t) = \prod_i MGF_{X_i}(t)$$

$$\Rightarrow \mathbb{E}\left[e^{t\sum_i X_i}\right] = \prod_i \mathbb{E}\left[e^{tX_i}\right]$$

$$\Rightarrow \mathbb{E}\left[e^{t\sum X_i}\right] e^{-t\sum_i \mathbb{E}[X_i]} = \prod_i \mathbb{E}\left[e^{t(X_i - \mathbb{E}[X_i])}\right]$$

$$\Rightarrow \mathbb{E}\left[e^{t\left(\sum_i X_i - \mathbb{E}\left[\sum_i X_i\right]\right)}\right] \leq \prod_i e^{\frac{\nu_i^2 t^2}{2}} \quad ; \ |t| < \frac{1}{b_i} \ \forall i$$

take $b^* = \max\{b_i\}$

$$\Rightarrow \quad |t| < \frac{1}{b^*}$$

$$\Rightarrow \mathbb{E}\left[e^{t\left[\sum x_i - \mathbb{E}\left[\sum_i x_i\right]\right]}\right] \le e^{\sum_i \nu_i^2 \frac{t^2}{2}}$$

$$; \quad |t| < \frac{1}{b^*}$$

$\Rightarrow$ Sub-exponential

with $\left(\left[\sum_i \nu_i^2\right]^{1/2}, b^*\right)$

Q(12)

Bernstein condition

$$\left|\mathbb{E}\left[(X-\mu)^k\right]\right| \le \frac{1}{2} k!\, \sigma^2 b^{k-2}$$

By taylor expansion around μ

$$e^{\lambda(x-\mu)} = 1 + \lambda(x-\mu) + \frac{\lambda^2}{2}(x-\mu)^2 + \sum_{k=3}^{\infty} \frac{(\lambda(x-\mu))^k}{k!}$$

$$\Rightarrow \mathbb{E}\left[e^{\lambda(x-\mu)}\right] \le \left|\mathbb{E}\left[e^{\lambda(x-\mu)}\right]\right|$$

$(\because a \le |a| \ \forall a \in \mathbb{R})$

$$\Rightarrow \mathbb{E}\left[e^{\lambda(X-\mu)}\right] \le 1 + \frac{\lambda^2\sigma^2}{2} + \sum_{k=3}^{\infty} \frac{|\lambda|^k}{k!} \left|\mathbb{E}\left[(X-\mu)^k\right]\right|$$

$$\left(\because \left|\sum_i x_i\right| \le \sum_i |x_i|\right)$$

$$\Rightarrow \quad \le 1 + \frac{\lambda^2\sigma^2}{2} + \frac{1}{2}\sum_{k=3}^{\infty} |\lambda|^k \sigma^2 b^{k-2}$$

$$\le 1 + \frac{\lambda^2\sigma^2}{2} + \frac{\lambda^2\sigma^2}{2}\sum_{k=3}^{\infty} (|\lambda| b)^{k-2}$$

$$\le 1 + \frac{\lambda^2\sigma^2}{2}\sum_{k=0}^{\infty} (|\lambda| b)^{k}$$

$$\le 1 + \frac{\lambda^2\sigma^2}{2(1-|\lambda| b)}$$

$$\left(\because 1 + x \le e^x\right)$$

$$\Rightarrow \mathbb{E}\left[e^{\lambda(X-\mu)}\right] \le e^{\frac{\lambda^2\sigma^2/2}{1-b|\lambda|}}$$