

Cost models and advanced Futhark programming

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Agenda

Parallel cost models

Prefix sums (scans)

Using scans

Auxiliary

Parallel cost models

Prefix sums (scans)

Using scans

Auxiliary

The need for cost models

Which is better?

```
import numpy as np

def inc_scalar(x):
    for i in range(len(x)):
        x[i] = x[i] + 1

def inc_par(x):
    return x + np.ones(x.shape)
```

The need for cost models

Which is better?

```
import numpy as np

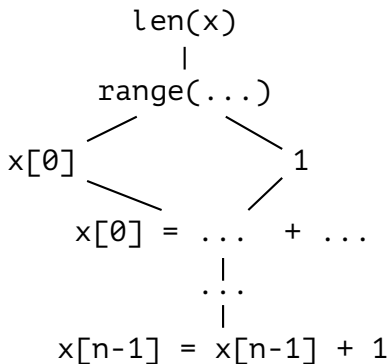
def inc_scalar(x):
    for i in range(len(x)):
        x[i] = x[i] + 1

def inc_par(x):
    return x + np.ones(x.shape)
```

Intuitively, `inc_par` is better because it is “more parallel”.

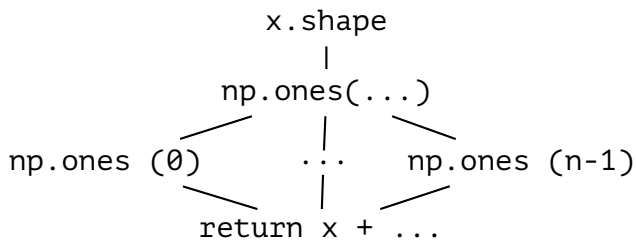
Parallel cost models make this notion precise.

Dependency DAG for `inc_scalar`



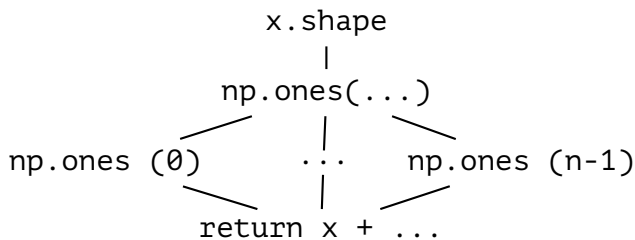
- Total count of nodes is the *work*, $W(p)$.
- Length of longest path from a leaf to the root is the *span*.
- **With an infinite number of processors, if a program p has span k , written $S(p) = k$, the program can execute in $O(k)$ time.**
- Here, $W(p) = O(n)$, $S(p) = O(n)$.

Dependency DAG for `inc_par`



What is the work and span complexity?

Dependency DAG for `inc_par`



What is the work and span complexity?

- $W(p) = O(n)$
- $S(p) = O(1)$

Parallel cost model based on work and span

Instead of giving just a simple cost-model based on the total notion of work carried out by a program, we give instead a *refined* cost model, which aims at providing both:

- a notion of how much total work (W) the program does;
- a notion of the *span*¹ (S) of the program, specifying the maximum depth required by the computation.

Notice:

- The span is the length of the longest sequence of operations that must be performed sequentially due to data dependencies.
- With an infinite number of processors, if a program p has span k , written $S(p) = k$, the program can execute in $O(k)$ time.

¹Sometimes also called *depth*.

Writing T_i for the time taken to execute an algorithm on i processors, Brent's Theorem states that

$$\frac{T_1}{p} \leq T_p \leq T_\infty + \frac{T_1}{p}$$

Proof sketch: At level j of the DAG there are M_j independent operations, which can clearly be executed by p processors in time

$$\left\lceil \frac{M_j}{p} \right\rceil$$

Sum these for each level of the DAG.



Ramification: We can simulate an “infinitely parallel” machine on a real machine at an overhead proportional to the amount of “missing” hardware parallelism.

Language-based cost models

- Tallying up levels in an infinite DAG is impractical for real programs. Instead we prefer a *language-based cost model*
- E.g. $W(x + y)$ is defined as $W(x) + W(y)$.
- The following slides define work and span cost for a small subset of Futhark.
- Write $\llbracket e \rrbracket$ for the result of evaluating expression e (we are being intuitive about scopes and such).

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Cost model must be implementable

A provable time and space efficient implementation of NESL—Guy Blelloch and John Greiner, 1996

Simple cases

$$W(v) =$$

$$S(v) =$$

$$W(e_1 \oplus e_2) =$$

$$S(e_1 \oplus e_2) =$$

$$W(\backslash x \rightarrow e) =$$

$$S(\backslash x \rightarrow e) =$$

Simple cases

$$W(v) = 1$$

$$S(v) = 1$$

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Simple cases

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$$S(v) = 1$$

$$W(e_1 \oplus e_2) = W(e_1) + W(e_2) + 1$$

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$$W(\backslash x \rightarrow e) = 1$$

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$$W([e_1, \dots, e_n]) =$$

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$$W((e_1, \dots, e_n)) =$$

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Implications?

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$$W(\backslash x \rightarrow e) = 1$$

$$S(\backslash x \rightarrow e) = 1$$

$$W([e_1, \dots, e_n]) = W(e_1) + \dots + W(e_n) + 1$$

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Implications?

Interesting cases

$W(\text{iota } e) =$

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Work and span for map

$$W(\text{map } e_1 \ e_2) =$$

$$S(\text{map } e_1 \ e_2) =$$

Work and span for map

$$\begin{aligned} W(\text{map } e_1 \ e_2) &= \\ &W(e_1) + W(e_2) + W(e'[x \mapsto v_1]) + \dots + W(e'[x \mapsto v_n]) \\ \text{where } \llbracket e_1 \rrbracket &= \backslash x \rightarrow e' \\ \text{where } \llbracket e_2 \rrbracket &= [v_1, \dots, v_n] \end{aligned}$$

$$\begin{aligned} S(\text{map } e_1 \ e_2) &= \\ &S(e_1) + S(e_2) + \max(S(e'[x \mapsto v_1]), \dots, S(e'[x \mapsto v_n])) + 1 \\ \text{where } \llbracket e_1 \rrbracket &= \backslash x \rightarrow e' \\ \text{where } \llbracket e_2 \rrbracket &= [v_1, \dots, v_n] \end{aligned}$$

Reduction by contraction

```
let npow2 (n:i32) : i32 =  
  loop a = 2 while a < n do 2*a  
  
-- Pad a vector to make its size a power of two  
let padpow2 [n] (ne: i32) (v:[n]i32) : []i32 =  
  concat v (replicate (npow2 n - n) ne)  
  
-- Reduce by contraction  
let red (xs : []i32) : i32 =  
  let xs =  
    loop xs = padpow2 0 xs  
    while length xs > 1 do  
      let n = length xs / 2  
      in map2 (+) xs[0:n] xs[n:2*n]  
  in xs[0]
```

Work and span of loop

$$W(\text{loop } x = e_1 \text{ while } e_2 \text{ do } e_3) =$$

+

$$S(\text{loop } x = e_1 \text{ while } e_2 \text{ do } e_3) =$$

Work and span of `loop`

$W(\text{loop } x = e_1 \text{ while } e_2 \text{ do } e_3) =$
if $\llbracket e_2 \rrbracket = \text{false}$
then $1 + W(e_2)$
else $1 + W(e_2) + W(e_3[x \mapsto \llbracket e_1 \rrbracket]) +$
 $+ W(\text{loop } x = \llbracket e_3[x \mapsto \llbracket e_1 \rrbracket] \rrbracket \text{ while } e_2 \text{ do } e_3)$

$S(\text{loop } x = e_1 \text{ while } e_2 \text{ do } e_3) =$
if $\llbracket e_2 \rrbracket = \text{false}$
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else $1 + S(e_2) + S(e_3[x \mapsto \llbracket e_1 \rrbracket])$
 $+ S(\text{loop } x = \llbracket e_3[x \mapsto \llbracket e_1 \rrbracket] \rrbracket \text{ while } e_2 \text{ do } e_3)$

Work and Span for $n^{\text{pow}2}$

Work and Span for $\text{npow2 } n$

By inspection, we have

$$W(\text{npow2 } n) = S(\text{npow2 } n) = O(\log n)$$

Work and Span for $\text{padpow2 } ne \text{ } v$

Work and Span for npow2 n

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Work and Span for padpow2 ne v

Because $\text{npow2 } n \leq 2n$, we have (where $n = \text{length } v$)

$$\begin{aligned} W(\text{padpow2 } ne \ v) &= W(\text{concat } v \ (\text{replicate } (\text{npow2 } n - n) \ ne)) \\ &= O(n) \\ S(\text{padpow2 } ne \ v) &= O(\log n) \end{aligned}$$

Work and Span for red

Work and Span for npow2 n

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&= O(n)
\end{aligned}$$

$$S(\text{padpow2 } ne \ v) = O(\log n)$$

Work and Span for red

Each loop iteration in red has span $O(1)$. Because the loop is iterated at-most $\log(2n)$ times, we have (where $n = \text{length } v$)

$$\begin{aligned}
W(\text{red } v) &= O(n) + O(n/2) + O(n/4) + \cdots + O(1) = O(n) \\
S(\text{red } v) &= O(\log n)
\end{aligned}$$

Work efficiency

A parallel algorithm is said to be *work efficient* if it has at most the same work as the best sequential algorithm.

Is red work efficient?

Work efficiency

A parallel algorithm is said to be *work efficient* if it has at most the same work as the best sequential algorithm.

Is red work efficient?

Yes, because it does $O(n)$ work, which is as good as a sequential summation.

Is it also *efficient*?

Performance Compared to the Built-in Reduction SOAC

```
-- ==  
-- entry: test_red test_reduce  
-- random input { [10000000]i32 }  
entry test_red = red  
entry test_reduce = reduce (+) 0
```

Performance Compared to the Built-in Reduction SOAC

```
-- ==  
-- entry: test_red test_reduce  
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entry test_red = red  
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$ futhark bench --backend=opencl reduce.fut  
Compiling reduce.fut...  
Results for reduce.fut:test_red:  
dataset [10000000]i32:      4675.40 $\mu$ s  
Results for reduce.fut:test_reduce:  
dataset [10000000]i32:      273.80 $\mu$ s
```

Performance Compared to the Built-in Reduction SOAC

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-- ==  
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```

If you are not using `futhark bench`, then you are probably doing it wrong.

Parallel cost models

Prefix sums (scans)

Using scans

Auxiliary

Inclusive and exclusive prefix sum

Exclusive prefix sum ("prescan")

Given

[1, 2, 3, 4]

produce

[0, 1, 3, 6]

Inclusive prefix sum

Given

[1, 2, 3, 4]

produce

[1, 3, 6, 10]

Prefix sums are scans

Generalising the addition and zero used by a prefix sum to an arbitrary associative operator \oplus and neutral element 0_{\oplus} , we get *scan*.

-- The scan in Futhark is inclusive.

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> scan (+) 0 [1,2,3,4]  
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> scan (+) 0 [1,2,3,4]  
[1, 3, 6, 10]
```

- Scans are a fundamental building block on parallelising seemingly-sequential algorithms.
- Let us see how scans can be computed in parallel.

Sequential prefix sum

```
acc = 0
for i < n:
    acc = acc + input[i]
    scanned[i] = acc
```

Sequential prefix sum

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Work: $O(n)$

Span: $O(n)$

Brute force

To calculate the prefix sum of $[x_0, \dots, x_{n-1}]$, compute

$$\begin{aligned} &[sum([x_0]) \\ &sum([x_0, x_1]) \\ &\vdots \\ &sum([x_0, x_1, \dots, x_{n-1}])] \end{aligned}$$

Assume $S(sum([x_0, \dots, x_{n-1}])) = \log_2(n)$.

Brute force

To calculate the prefix sum of $[x_0, \dots, x_{n-1}]$, compute

$$\begin{aligned} & \text{sum}([x_0]) \\ & \text{sum}([x_0, x_1]) \\ & \vdots \\ & \text{sum}([x_0, x_1, \dots, x_{n-1}]) \end{aligned}$$

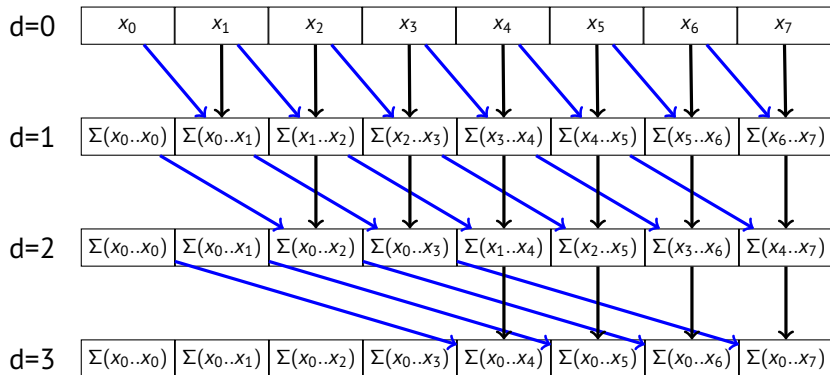
Assume $S(\text{sum}([x_0, \dots, x_{n-1}])) = \log_2(n)$.

Work: $O(\sum_{i < n} i) = O(n^2)$

Span: $O(\max(S(\text{sum}([x_0])), \dots, S(\text{sum}([x_0, \dots, x_{n-1}])))) = O(\log_2(n))$

Terrible. The sequential implementation is faster for large n !

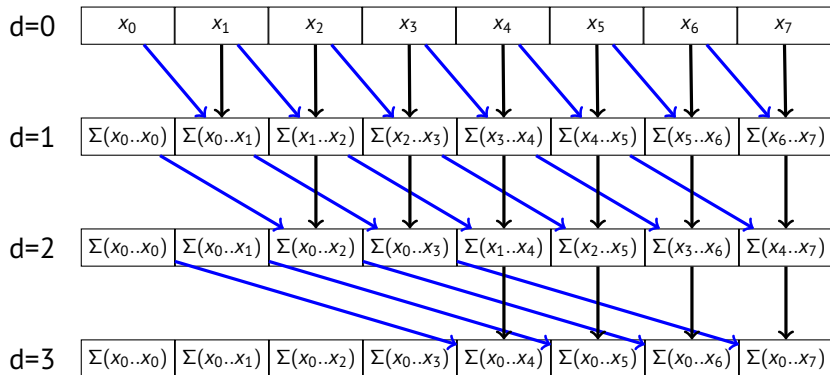
Hillis–Steele scan (1986)²



For each d , element x_i^d is updated by $x_{i-2^d}^{d-1} + x_i^{d-1}$.

²This slide took 60 minutes to make.

Hillis–Steele scan (1986)²



For each d , element x_i^d is updated by $x_{i-2^i}^{d-1} + x_i^{d-1}$.

Work: For $n = 2^m$, $O(\sum_{i < m} 2^m - 2^i) = O(n \log(n))$

Span: $\log(n)$

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Work-efficient scan

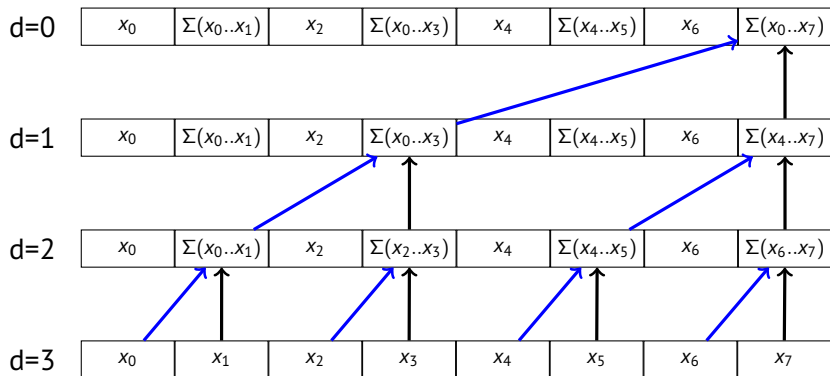
Two passes

Up-sweep Build a balanced binary tree of partial sums stored in every other cell.

Down-sweep Use the partial sums to fill out the missing parts.

The binary tree does not actually exist as a recursive pointer structure, but is just a communications concept.

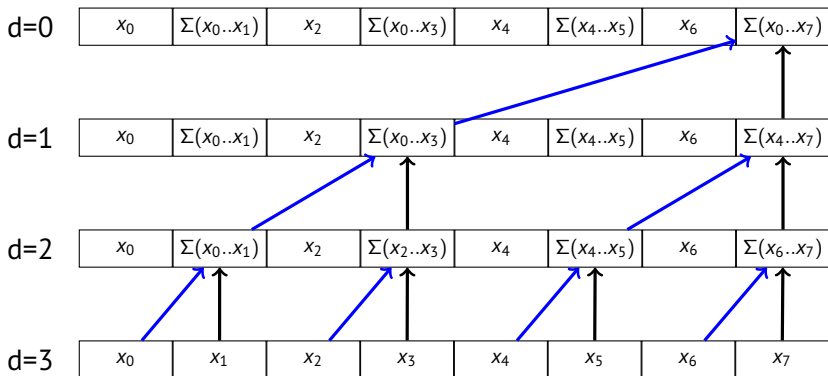
Up-sweep (“reduction phase”)³



$$x_i^d = x_{i-2^{m-d-1}}^{d+1} + x_i^{d+1}$$

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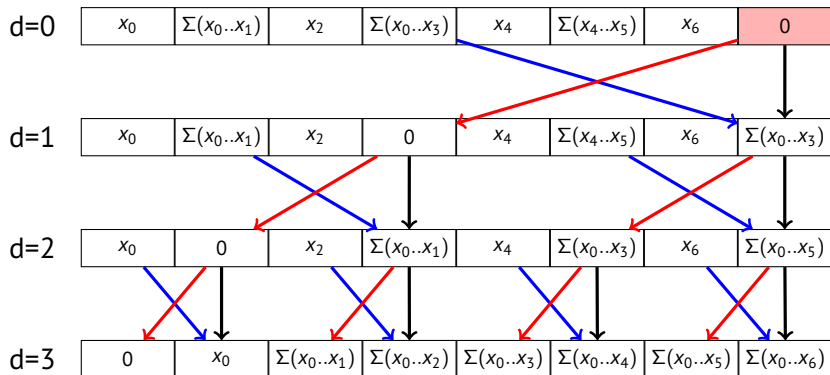
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Work: For $n = 2^m$, $O(\sum_{i < m} 2^i) = O(n)$

Span: $\log(n)$

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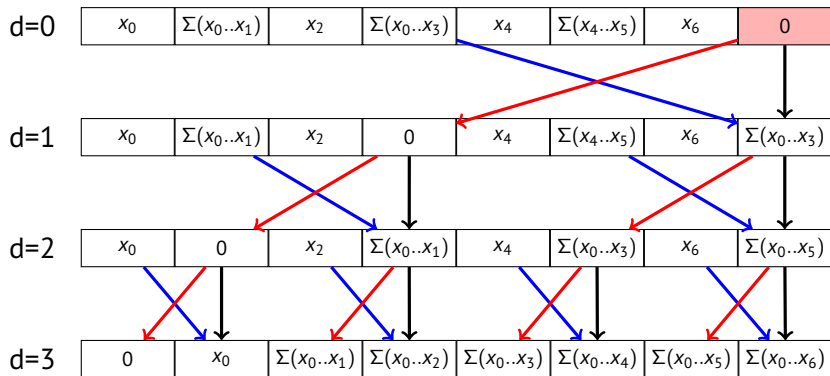
Down-sweep⁴



Inverse indexing of the up-sweep phase.

⁴This slide took 15 minutes to make.

Down-sweep⁴



Inverse indexing of the up-sweep phase.

Work: For $n = 2^m$, $O(\sum_{i \leq m} 2^i) = O(n)$

Span: $\log(n)$

⁴This slide took 15 minutes to make.

Work efficient scan

Complexity of *scan* on size- n input

Work: $O(n)$

Span: $\log(n)$

- Optimal, as *reduce* is the same.
- Can now depend on scan as a relatively cheap building block.

Real-world scan implementations are often very different for technical reasons, but we can depend on these asymptotics when analysing and designing parallel algorithms.

Parallel cost models

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Filtering

Suppose we wish to remove negative elements from the list

let as = [-1, 2, -3, 4, 5, -6]

Filtering

Suppose we wish to remove negative elements from the list

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let as = [-1, 2, -3, 4, 5, -6]
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For each element, see if we want to keep it:

```
let keep = map (\a -> if a >= 0 then 1 else 0) as  
-- [ 0, 1, 0, 1, 1, 0]
```

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```

```
let offsets1 = scan (+) 0 keep  
-- [ 0, 1, 1, 2, 3, 3]
```

Filtering

Suppose we wish to remove negative elements from the list

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```

```
let offsets1 = scan (+) 0 keep  
-- [ 0, 1, 1, 2, 3, 3]
```

```
let offsets = map (\x -> x - 1) offsets1  
-- [-1, 0, 0, 1, 2, 2]
```

Filtering

Suppose we wish to remove negative elements from the list

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let as = [-1, 2, -3, 4, 5, -6]
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-- [ 0, 1, 1, 2, 3, 3]
```

```
let offsets = map (\x -> x - 1) offsets1  
-- [-1, 0, 0, 1, 2, 2]
```

`offsets[i]` now indicates position in filtered list iff `keep[i] == 1`.

scatter

scatter xs is vs computes equivalent of the imperative pseudocode

```
for j < n:  
    xs[is[j]] = vs[j]
```

- Out-of-bound writes are ignored
- Writing different values to same index is *undefined*⁵
- Work $O(n)$, span $O(1)$

Just what we need for filtering!

⁵reduce_by_index handles conflicts with provided operator.

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Just what we need for filtering!

```
scatter (replicate (last offsets1) 0)  
  (map2 (\i k -> if k == 1 then i else -1)  
    offsets keep)  
  as
```

⁵reduce_by_index handles conflicts with provided operator.

Implementing filter

```
let filter 'a (p: a -> bool) (as: []a): []a =  
  let keep = map (\a -> if p a then 1 else 0) as  
  let offsets1 = scan (+) 0 keep  
  let num_to_keep = reduce (+) 0 keep  
  in if num_to_keep == 0  
    then []  
    else scatter (replicate num_to_keep as[0])  
                  (map2 (\i k ->  
                        if k == 1  
                        then i-1  
                        else -1)  
                      offsets1 keep)  
    as
```

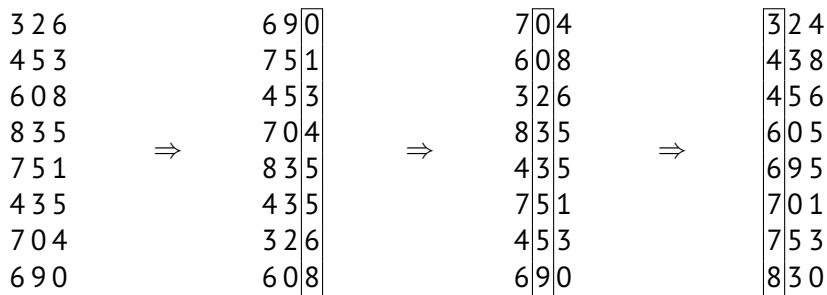
Radix sort

- Many classical sorting algorithms are a poor fit for data parallelism, but *radix sort* works well.
- Radix-2 sort works by repeatedly partitioning elements according to one bit at a time, while preserving the ordering of the previous steps.

Example with radix-10

3 2 6		6 9 0		7 0 4		3 2 4
4 5 3		7 5 1		6 0 8		4 3 8
6 0 8		4 5 3		3 2 6		4 5 6
8 3 5		7 0 4		8 3 5		6 0 5
7 5 1	⇒	8 3 5		4 3 5		6 9 5
4 3 5		4 3 5	⇒	7 5 1		7 0 1
7 0 4		3 2 6		4 5 3		7 5 3
6 9 0		6 0 8		6 9 0		8 3 0

Example with radix-10



- Radix sort is not as general as a comparison-based sort.
- Assumes sorting key can be decomposed into “digits”.

Sorting `xs:[n]u32` by bit `b`

```
-- 1 if bit b set.  
let check_bit b x =  
    (i32.u32 (x >> u32.i32 b)) & 1
```

Sorting xs:[n]u32 by bit b

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let check_bit b x =  
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let bits = map (check_bit b) xs  
let bits_neg = map (1-) bits  
let offs = reduce (+) 0 bits_neg
```

Sorting `xs : [n]u32` by bit `b`

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let check_bit b x =  
    (i32.u32 (x >> u32.i32 b)) & 1  
  
let bits = map (check_bit b) xs  
let bits_neg = map (1-) bits  
let offs = reduce (+) 0 bits_neg
```

Example

```
b = 0  
xs = [0, 1, 2, 3, 4]  
bits = [0, 1, 0, 1, 0]  
bits_neg = [1, 0, 1, 0, 1]  
offs = 3
```

```
let idxs0 = map2 (*)  
              bits_neg  
              (scan (+) 0 bits_neg)  
let idxs1 = map2 (*)  
              bits  
              (map (+offs) (scan (+) 0 bits))
```

```
let idxs0 = map2 (*)  
              bits_neg  
              (scan (+) 0 bits_neg)  
let idxs1 = map2 (*)  
              bits  
              (map (+offs) (scan (+) 0 bits))
```

Example

```
bits      = [0, 1, 0, 1, 0]  
bits_neg  = [1, 0, 1, 0, 1]  
offs      = 3  
idxs0     = [1, 0, 2, 0, 3]  
idxs1     = [0, 4, 0, 5, 0]  
map2 (+) idxs0 idxs1  
          = [1, 4, 2, 5, 3]
```

Then **scatter** as when **filtering**.

The whole step

```
let check_bit b x =  
    (i32.u32 (x >> u32.i32 b)) & 1  
  
let radix_sort_step [n]  
    (xs: [n]u32) (b: i32): [n]u32 =  
    let bits = map (check_bit b) xs  
    let bits_neg = map (1-) bits  
    let offs = reduce (+) 0 bits_neg  
    let idxs0 = map2 (*) bits_neg  
                    (scan (+) 0 bits_neg)  
    let idxs1 = map2 (*) bits  
                    (map (+offs) (scan (+) 0 bits))  
    let idxs2 = map2 (+) idxs0 idxs1  
    let idxs  = map (\x->x-1) idxs2  
    let xs' = scatter (copy xs) idxs xs  
    in xs'
```


Radix sort in Futhark

```
let radix_sort [n] (xs: [n]u32): [n]u32 =  
    loop xs for i < 32 do radix_sort_step xs i
```

See worked example at <https://futhark-lang.org/examples/radix-sort.html>

Segmented scan

```
val segmented_scan [n] 't
  : (op: t -> t -> t) -> (ne: t)
  -> (flags: [n]bool) -> (as: [n]t)
  -> [n]t
```

true starts a segment and false continues a segment.

Example

```
segmented_scan (+) 0
  [true, false, true, false, false, true]
  [0, 1, 2, 3, 4, 5]
== scan (+) 0 [0,1] ++
   scan (+) 0 [2,3,4] ++
   scan (+) 0 [5]
== [0, 1, 2, 5, 9, 5]
```

Segmented reduction

```
val segmented_reduce [n] 't
  : (op: t -> t -> t) -> (ne: t)
  -> (flags: [n]bool) -> (as: [n]t)
  -> []t
```

Example

```
segmented_reduce (+) 0
  [true, false, true, false, false, true]
  [0, 1, 2, 3, 4, 5]
== reduce (+) 0 [0,1] ++
   reduce (+) 0 [2,3,4] ++
   reduce (+) 0 [5]
== [1, 9, 5]
```

Generalised histograms

Like `scatter`, but uses a provided reduce-like operator to handle multiple writes to same index.

Type

```
val reduce_by_index [k] [n] 'a :  
    (dest: *[k]a)  
  -> (f: a -> a -> a) -> (ne: a)  
  -> (is: [n]i32) -> (vs: [n]a) -> *[k]a
```

Semantics

```
for index in 0..k-1:  
  i = is[index]  
  v = vs[index]  
  dest[i] = f(as[i], v)
```

Futhark uses parallel implementation with GPU *atomics*.

Proving associativity and neutral elements

```
let op (x, i) (y, j) : (i32, i32) =  
    if x < y then (y, j) else (x, i)
```

```
let argmax [n] (xs: [n]i32) =  
    reduce op  
        (i32.smallest, -1)  
        (zip xs (iota n))
```

- Is op associative?
- Is (i32.smallest, -1) a neutral element?

argmax: associativity

```
(a 'op' b) 'op' c
== ((ax, ai) 'op' (bx, bi)) 'op' (cx, ci)
== let (x, i) = if ax < bx then (bx, bi)
                                else (ax, ai)
   in if x < cx then (cx, ci)
       else (x, i)
```

```
a 'op' (b 'op' c)
== (ax, ai) 'op' ((bx, bi) 'op' (cx, ci))
== let (x, i) = if bx < cx then (cx, ci)
                                else (bx, bi)
   in if ax < x then (x, i)
       else (ax, ai)
```

Enumerate all possible comparisons between ax, bx, and cx and show associativity for each.

E.g. for **!(ax < bx) && bx < cx && cx < ax**

```
let (x, i) = if ax < bx then (bx, bi)
              else (ax, bx)
in if x < cx then (cx, ci)
    else (x, i)
== if ax < cx then (cx, ci)
    else (ax, ai)
== (ax, ai)
```

```
let (x, i) = if bx < cx then (cx, ci)
              else (bx, bi)
in if ax < x then (x, i)
    else (ax, ai)
== if ax < cx then (cx, ci)
    else (ax, ai)
== (ax, ai)
```



argmax: neutral element

```
(a 'op' (i32.smallest, -1))  
== ((x, i) 'op' (i32.smallest, -1))  
== if x < i32.smallest then (i32.smallest, -1)  
    else (x, i)  
== (x, i)
```

```
((i32.smallest, -1) 'op' a)  
== ((i32.smallest, -1) 'op' (x, i))  
== if i32.smallest < x then (x, i)  
    else (i32.smallest, -1)  
== (x, i)
```


Commutativity?

Exercise for home: The `argmax` operator is not commutative. Try to come up with a counterexample, and see if you can change its definition such that it becomes commutative.

Commutativity?

Exercise for home: The `argmax` operator is not commutative. Try to come up with a counterexample, and see if you can change its definition such that it becomes commutative.

Commutative reductions

Futhark has a `reduce_comm` function that can be used for commutative operators. This runs faster than normal `reduce`. Not necessary for built-in operators.

Summary

- *Work* measures the total number of operations, *span* measures the longest chain of dependencies.
- Language-based cost models let us reason about program performance in a hardware-agnostic and composable way.
- Scans are a useful building block in advanced data parallel algorithms, but an efficient implementation is not straightforward.