limitations to the detector sensitivity, and they can often be improved upon by small and gradual improvements of the system. Some examples are related to the laser source like frequency and intensity noise, or to the feedback control system, mostly sensing and control noises that couple to the main GW channels. Another important examples is scattered light: a small fraction of the laser beam can be scattered by the mirrors in directions that are different from the main beam, and part of it can recombine with the main interferometer laser, carrying a phase modulation related to the motion of the scattering object, and therefore introducing noise.

6.2 *Quantum noise*

- shot noise and radiation pressure noise recap
- quantum formalism for laser beams: two photon formalism
- squeezed states
- how squeezing is generated
- IFO response to vacuum fluctuations: beam splitter, losses
- effect of squeezed light on IFO: the IFO as a ponderomotive squeezer, frequency independent squeezing, filter cavities, frequency dependent squeezing

6.3 *Thermal noise*

Friction

Every object is in thermal contact with the environment and in thermal equilibrium at a finite temperature. This continues exchange of energy with the environment takes two complementary forms. Energy can be lost by the system due to various forms of friction: this energy is converted to thermal energy of the environment. The flip coin is clearly also true, in the sense that the energy coupling must happens in the reverse direction too: at equilibrium the system will be subject to a continuous flux of energy from the environment. This is one way to look at the equipartition theorem: every harmonic oscillator degree of freedom in the system will have an average energy $k_B T$ at equilibrium. This energy comes from the coupling with the environment, and in continuous systems it gives rise to the thermal agitation that was first discovered in Brownian motion.

There is only one mechanism that determines the coupling of energy between the system and the environment, therefore we can expect a relationship between the amount of friction and the amplitude of thermal agitation or thermal noise in the system at equilibrium, in absence of external perturbations. This result will be made formal below with the *fluctuation dissipation theorem*, but before going down that road, let's build some intuition.

Let's consider a system with only one degree of freedom x, driven by an external force F. In frequency domain we can write the system response to a monochromatic force in terms of the transfer function:

$$\tilde{x}(\omega) = T(\omega)\tilde{F}(\omega) \tag{6.1}$$

To be concrete we can focus on a simple harmonic oscillator with mass m, spring constant k and friction γ, so that the transfer function is

$$T(\omega) = \frac{1}{k - m\omega^2 + i\omega\gamma} \tag{6.2}$$

but the conclusion below will hold for a general system. The force is a real sinusoid, that we can write in the form $F(t) = \tilde{F}_0 \cos \omega t$. The system motion is also real: remember that the fact that the transfer function is complex is just a handy way to represent the fact that the system motion can be out-of-phase with respect to the force. The actual motion of the system can be obtained by taking the real part of the product of the force and the transfer function. If we write $T(\omega) = T_R(\omega) + iT_I(\omega)$, then the system motion is simply

$$x(t) = F_0 \left(T_R \cos \omega t + T_I \sin \omega t\right) \tag{6.3}$$

We can compute the work done by the external force on the system by multiplying the velocity and the force:

$$W(t) = \dot{x}(t)F(t) = \omega F_0^2 \left(T_I \cos^2 \omega t - R_R \sin \omega t \cos \omega t\right) \tag{6.4}$$

The work clearly depends on the instantaneous velocity and force values, but we can average over one oscillation cycle $T = \frac{2\pi}{\omega}$

$$\langle W \rangle = \frac{1}{T} \int_0^T \dot{x}(t)F(t)\, dt \tag{6.5}$$

It's easy to see that the only term with non zero average is the one that is proportional to the imaginary part of the transfer function:

$$\langle W \rangle = \frac{\omega T_I(\omega)}{2} F_0^2 \tag{6.6}$$

Since we are at equilibrium, the average energy per cycle is constant, so the work done by the external force must be equal to the energy loss, due to internal friction. We can then write

$$\Delta E_{cycle} = \frac{\omega T_I(\omega)}{2} F_0^2 \tag{6.7}$$

This is a first important result, which tells us that the energy loss in a system is proportional to the imaginary part of the system transfer function: if the transfer function is purely real, there is no energy loss, and so there is no friction in the system. We can go one step further and express the force amplitude in terms of the averaged system displacement and velocity

$$\left\langle x^2 \right\rangle = \frac{|T(\omega)|^2}{2} F_0^2 \tag{6.8}$$

$$\left\langle \dot{x}^2 \right\rangle = \frac{\omega^2 |T(\omega)|^2}{2} F_0^2 \tag{6.9}$$

If the we can define the system mass m, then we can write the energy dissipation in terms of the kinetic energy

$$\Delta E_{cycle} = \frac{2}{m} \frac{T_I(\omega)}{\omega |T(\omega)|^2} \langle E_K \rangle \tag{6.10}$$

which makes it even mode evident that the imaginary part of the system transfer function is intimately related to the coupling of energy from the system to the thermal environment and back.

In the case of the simple pendulum example, we can write the transfer function as

$$T(\omega) = \frac{(k - m\omega^2) + i\omega\gamma}{(k - m\omega^2)^2 + \omega^2\gamma^2} \tag{6.11}$$

and using equation 6.10 we find a result we already know, namely that the friction coefficient γ determines the energy lost per cycle

$$\Delta E_{cycle} = \frac{2\gamma}{m} \langle E_K \rangle \tag{6.12}$$

Thermal motion

The equipartition theorem states that at equilibrium, the total energy in each of the oscillatory degrees of freedom of the system is $k_B T$, but it does not tell us how this is distributed in frequency, in terms of the variable that we care about, which is the displacement x. Before proving the general fluctuation dissipation theorem, let's consider the simple case of a harmonic oscillator described by the equation of motion below

$$m\ddot{x} + kx + \gamma\dot{x} = F \tag{6.13}$$

Here the force F is not something that we control, but simply a way to describe the effect of the interaction with the thermal environment. It is therefore a force with a stochastic nature, and we can only describe it through its statistical properties. An equation like the one

above, where a determinist system is driven by a stochastic force, is called a *Langevin equation*. At this stage we can only assume that the force is completely random, meaning that the values at different times are uncorrelated:

$$\left\langle F(t)F(t')\right\rangle = F_0^2\delta(t-t') \tag{6.14}$$

where the average is meant to be a statistical averaging over the equilibrium distribution of the system. But this assumption is not enough to compute the power spectral density of the system motion due to thermal noise. What we need is a way to connect together the friction in the system, which is described by the coefficient γ with the thermal bath, which is characterized by the energy scale k_BT. We are going to do this in two steps. First of all, we shall prove again the relation between the work done by the stochastic force and the kinetic energy of the system. We simply multiply equation 6.13 by $\dot{x}$ and collect terms in a common derivative to find

$$\frac{d}{dt}\left[\frac{1}{2}kx^2+\frac{1}{2}m\dot{x}^2\right] = -\gamma\dot{x}^2 + F\dot{x} \tag{6.15}$$

The term in brackets in the left hand side is the total energy of the system, which is on average is a constant since we are at thermal equilibrium:

$$0 = \frac{d}{dt}\left\langle E\right\rangle = -\frac{2\gamma}{m}\left\langle E_K\right\rangle + \left\langle F\dot{x}\right\rangle \tag{6.16}$$

from which we can derive our first important result, that connects the work done by the stochastic thermal force with the friction coefficient:

$$\left\langle F\dot{x}\right\rangle = \frac{2\gamma}{m}\left\langle E_K\right\rangle \tag{6.17}$$

Now we need a way to relate this to the autocorrelation of the stochastic force. To achieve this, we start again from the equation of motion, and integrate it once with respect to time, between an arbitrary initial time, taken to be 0, and a time t:

$$m\dot{x}(t) - m\dot{x}(0) + k\int_0^t x(\tau)d\tau = -\gamma x(t) + \gamma x(0) + \int_0^t F(\tau)d\tau \tag{6.18}$$

To relate this with the already found relation, we multiply with the force $F(t)$ at the time t, and perform a statistical average:

$$\begin{aligned}\left\langle F(t)\dot{x}(t)\right\rangle \quad &- \quad \left\langle F(t)\dot{x}(0)\right\rangle + k\int_0^t \left\langle F(t)x(\tau)\right\rangle d\tau = \\ &- \quad \gamma\left\langle F(t)x(t)\right\rangle + \gamma\left\langle F(t)x(0)\right\rangle \\ &+ \quad \int_0^t \left\langle F(t)F(\tau)\right\rangle d\tau\end{aligned} \tag{6.19}$$

This expression might seem very complicated, but most of terms are zero. In particular, the force is completely random, and the system is

causal, so the position at a time t can depend only on the force values at previous times $t' < t$. Therefore $F(t)$ is statistically independent from the position and velocity at all times between 0 and t. Therefore we are left with

$$m \langle F(t)\dot{x}(t)\rangle = -\gamma \langle F(t)x(t)\rangle + \int_0^t \langle F(t)F(\tau)\rangle \, d\tau = -\gamma \langle F(t)x(t)\rangle + F_0^2 \tag{6.20}$$

where in the last equality we used the autocorrelation property of the stochastic force. One last step allows us to compute the first term in the right hand side. If we denote with $h(t)$ the step response of the system, which is simply the Fourier anti-transform of the transfer function, we can write

$$x(t) = x(0) + \int_0^t h(t-\tau)F(\tau)d\tau \tag{6.21}$$

We can then multiply with $F(t)$ and average. Using the force autocorrelation again we find

$$\langle F(t)x(t)\rangle = h(0)F_0^2 \tag{6.22}$$

To compute this part, we need to use the initial value theorem for the Laplace transform, which states that

$$f(0^+) = \lim_{s\to\infty} s\hat{f}(s) \tag{6.23}$$

where $f(t)$ and $\hat{f}(s)$ are Laplace transform pairs. In the case of any physical system describing a mass attached to any number of springs and contraptions, we know that for large frequencies the mass will behave like it's free, so that

$$T(\omega) \sim \frac{1}{m\omega^2} \qquad \text{for } \omega \to \infty \tag{6.24}$$

and therefore we can conclude that $h(0) = 0$. Physically, this means that any mechanical system will respond to a sudden change in force with a sudden change in velocity, but not of position: just another way to express the fact that Newton's equations of mechanics are second order. In conclusion, we found our second constitutive equation:

$$m \langle F(t)\dot{x}(t)\rangle = F_0^2 \tag{6.25}$$

We are now ready to combine equations 6.17 and 6.25 to obtain

$$F_0^2 = 2\gamma \langle E_K\rangle \tag{6.26}$$

The equipartition theorem now states that for a harmonic oscillator, the mean value of the kinetic energy is $\frac{1}{2}k_BT$ so we finally get the

desired link between the system response and the temperature of the thermal environment:

$$F_0^2 = \gamma k_B T \tag{6.27}$$

or to be more complete

$$\langle F(t_1)F(t_2)\rangle = \gamma k_B T \delta(t_1 - t_2) \tag{6.28}$$

This is a very profound results, since it links two seemingly distinct aspects of a harmonic oscillator in thermal equilibrium: the loss of energy due to internal friction and the statistical properties of the random force that acts due to the connection to the thermal environment. We can now derive an expression for the power spectral density of the thermal force acting on the mass, using the results of chapter **??**:

$$S_F(\omega) = 2\gamma k_B T \tag{6.29}$$

where the factor 2 comes from the choice of a single sided PSD instead of a double sided PSD. It is now easy to compute the system motion due to this thermal force:

$$S_x(\omega) = 2\left|T(\omega)\right|^2 \gamma k_B T = \frac{2\gamma k_B T}{(k - m\omega^2)^2 + \omega^2\gamma^2} \tag{6.30}$$

We recall that for a double sided PSD the total RMS of the signal can be recovered by integrating over frequencies ranging from $-\infty$ to $+\infty$. But for a real signal, the PSD is symmetric, so we can defined the single sided PSD as just twice the double sided PSD: the total RMS is obtained by integrating over only positive frequencies.

This is a very important result, and it shows the key feature of *thermal noise*: the power spectral density scales with the temperature of the system, and with the coefficient of friction, which as we have already showed above is related to the energy dissipation per cycle in the system. If we want to reduce the thermal noise in an oscillator like the one described here, we have two possible ways: either decreasing the temperature or decreasing the energy loss in the system.

If we compare equation 6.30 with equation 6.11, we can rewrite the thermal noise in terms of the imaginary part of the transfer function:

$$S_x(\omega) = \frac{2k_B T}{\omega}\left|T_I(\omega)\right| \tag{6.31}$$

This is the classical formulation of the *fluctuation dissipation theorem* and, although we proved it only in the case of a simple harmonic oscillator, it is valid for a general system, as we shall prove below. Let's stress again that this is a very powerful and profound result, since it links two phenomena with quite different manifestations: energy loss in the system due in this case to viscous damping, and random fluctuations of a system variable around an equilibrium position. The physical intuition behind this link is that the system responds in the same way to a transient mechanical perturbation induced by an external force, and to thermal fluctuations around the equilibrium

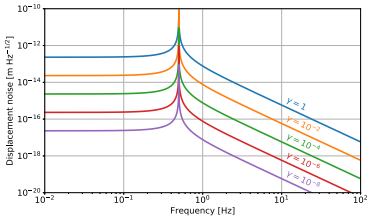

Figure 6.1: Thermal noise for a mass of 40 kg suspended to a 1 meter long pendulum, for different values of the viscous friction coefficient. The temperature is set to 300 K.

position. In both cases, the system gets kicked off an equilibrium position, and relaxes back in a way which is completely determined by the system dynamics, encoded in the transfer function.

Before moving to the actual proof of the fluctuation dissipation theorem, let's take a look at the power spectral density of thermal noise for the simple harmonic oscillator in presence of viscous damping, as shown in figure 6.1. There we shows the amplitude spectral density, and we can clearly see that above the resonant frequency of the pendulum the noise scales like f^{-2}, and the amplitude is proportional to the square root of the friction coefficient.

The fluctuation-dissipation theorem

The result in equation 6.31 is very general, and can be applied to any classical system. It is also possible to generalize it to a quantum system, but that is beyond the scope of this work. Here we shall prove the general form of the fluctuation dissipation theorem for a classical system. The proof is based on the idea that a system at equilibrium responds to thermal excitations in the same way it responds to an external force. We shall therefore assume that the system is initially at equilibrium with a constant force, and study how it evolves once the force is suddenly switched off at $t = 0$.

Let's therefore consider a system which was subject to a force on the form

$$f(t) = f_0 \Theta(-t) \tag{6.32}$$

where $\Theta(t)$ is the usual step function, zero for negative arguments and one for positive arguments. We assume that the force was on for an infinite time, so that the system for $t \leq 0$ is at thermal equilibrium. At any time $t > 0$ we can compute the statistical average of a system intensive observable $x(t)$ by using the corresponding probability distribution $P(x,t)$, which depends on time, since the system is no more at equilibrium after the force is switched off. We have

$$\langle x(t) \rangle = \int dx' \, P(x',t) x' = \int dx' \int dx P(x',t|x,0) P(x,0) \tag{6.33}$$

where we used the transition probability $P(x',t|x,0)$ which gives us the probability of finding the system at the position x' at the time t, knowing it was in the position x at time 0. The reason for doing this is that we know what the probability distribution at $t = 0$ was, since the system was at thermal equilibrium at that time. It is given by a Boltzmann distribution for the Hamiltonian $H(x) = H_0(x) - x f_0$ where H_0 is the Hamiltonian for the free system, without any

external force. We have therefore

$$P(x,0) = \frac{e^{-\beta H(x)}}{\int dx' e^{-\beta H(x')}} \tag{6.34}$$

where $\beta = \frac{1}{k_B T}$ as usual. Now, we can choose any force we want, so we can always assume that the force is weak enough so that we can expand the exponentials at first order in f_0:

$$\begin{aligned} e^{-\beta H(x)} &\simeq e^{-\beta H_0(x)} (1 + \beta f_0 x) && (6.35) \\ \int dx' e^{-\beta H(x')} &\simeq \int dx' e^{-\beta H_0} + \beta f_0 \int dx' e^{-\beta H_0} x' \\ &= \int dx' e^{-\beta H_0} + \beta f_0 \langle x \rangle_0 && (6.36) \end{aligned}$$

and in the last equation we have used $\langle \cdot \rangle_0$ to denote the statistical averaging for the system at equilibrium without any external force, i.e. using the Botzmann distribution that comes from the free Hamiltonian H_0. From the equation above, we can derive the first order expansion of the probability distribution for the system at equilibrium with the external force on:

$$P(x,0) \simeq P_0(x) \left[1 + \beta f_0 \left(x - \langle x \rangle_0\right)\right] \tag{6.37}$$

where $P_0(x)$ is the probability distribution for the system without external force. If we substitute this result back into the average value for x, we find:

$$\langle x(t) \rangle = \iint dx' dx x' P(x', t|x, 0) P_0(x) \left[1 + \beta f_0 \left(x - \langle x \rangle_0\right)\right] \tag{6.38}$$

For all times $t > 0$ the force is off, therefore the transition probability is just the one corresponding to the free system. But this means that for the constant term the integration over x can be carried out by noticing that we are simply propagating the equilibrium probability distribution forward in time

$$\int dx\, P(x', t|x, 0) P_0(x) = P_0(x, t) \tag{6.39}$$

Using this result we get

$$\begin{aligned} \langle x(t) \rangle &= \int dx'\, x' P_0(x', t) + \\ &+ \beta f_0 \iint dx' dx\, x' x P(x', t|x, 0) P_0(x) \\ &- \beta f_0 \iint dx' dx \left[\iint dx' dx\, x' P(x', t|x, 0) P_0(x)\right] \langle x \rangle_0 \end{aligned} \tag{6.40}$$

Let's look at each term in the equation above. The first one is simply $\langle x \rangle_0$ since the average of any variable in thermal equilibrium is constant over time. In the last term the integral is again simply $\langle x \rangle_0$. To

tackle the second term, we perform the common trick of subtracting $\langle x \rangle_0$ into the integral to both x' and x, and adding the correct terms to cancel them out of the integral. In the way we find that

$$\iint dx'dx\, x'xP(x',t|x,0)P_0(x) = \langle x \rangle_0^2 + \\ + \langle (x(t) - \langle x \rangle_0)\,(x(0) - \langle x \rangle_0) \rangle \tag{6.41}$$

We finally found the following result

$$\langle x(t) \rangle = \langle x \rangle_0 + \beta f_0 A(t) \tag{6.42}$$

where we have defined the autocorrelation of the variable x as

$$A(t) = \langle (x(t) - \langle x \rangle_0)\,(x(0) - \langle x \rangle_0) \rangle \tag{6.43}$$

Equation 6.42 gives us a relation that links the time evolution of the average value of the system observable x with thermodynamical variables. This relation is valid only for times $t > 0$, since before that time the system was at equilibrium and the considerations above are not valid, since $(x(t < 0) - \langle x \rangle_0)$ is a constant.

We know another way to compute how the average value of the system observable evolves with time, by simply using the system transfer function, or to be more precise the impulse response $\chi(\tau)$:

$$\langle x(t) \rangle = \langle x(-\infty) \rangle + \int_{-\infty}^{t} f(\tau)\chi(t-\tau)d\tau \tag{6.44}$$

Here we just used the fact that the average over the Boltzmann distribution is giving us the actual value of the system observable that we would measure, and this evolves accordingly to the system dynamics, encoded in $\chi(t)$ and the external force. We assume that at $t = -\infty$ the system was at equilibrium, so $\langle x(-\infty) \rangle = \langle x \rangle_0$, and any deviation from this value for finite times is due to the presence of the force, which is taken into account in the convolution integral above. By using the expression for the force $f(t) = f_0\Theta(-t)$ we can finally write

$$\langle x(t) \rangle - \langle x \rangle_0 = f_0 \int_0^{\infty} \Theta(\tau - t)\chi(\tau)d\tau \tag{6.45}$$

We can now collect the results from equations 6.42 and 6.45 and write the following relation, which is again valid only for $t > 0$:

$$f_0 \int_t^{\infty} \chi(\tau)d\tau = \beta f_0 A(t) \tag{6.46}$$

To get rid of the integral, we take the derivative with respect to the time t and find

$$-\chi(t) = \beta \frac{dA(t)}{dt}\Theta(t) \tag{6.47}$$

where we added an explicit $\Theta(t)$ on the right hand side to keep track that the relation is valid only for $t > 0$, while for $t < 0$ the derivative of $A(t)$ is zero, since the system is at equilibrium. We already know that the (one-sided) power spectral density of the noise in x is related to the Fourier transform of the autocorrelation $S_x(\omega) = 2\hat{A}(\omega)$, so we take the Fourier transform of the equation above and obtain

$$-\hat{\chi}(\omega) = \beta \int_{-\infty}^{\infty} e^{-i\omega t} \frac{dA}{dt} \Theta(t) dt = \beta \int_{0}^{\infty} e^{-i\omega t} \frac{dA}{dt} dt \tag{6.48}$$

How we would like to take the derivative out of the integration sign using one of the Fourier transform properties, but to do that we would need to extend the integration to negative times. Luckily, we can do that by substituting $\omega \rightarrow -\omega$ and remembering that $\hat{\chi}(-\omega) = \hat{\chi}^*(\omega)$, since the system impulse response must be real. Therefore we have

$$-2i\mathfrak{Im}\hat{\chi}(\omega) = -\hat{\chi}(\omega) + \hat{\chi}^*(\omega) = \beta \int_{-\infty}^{\infty} e^{-i\omega t} \frac{dA}{dt} dt = -\beta i\omega \hat{A}(\omega) \tag{6.49}$$

and we finally found the general form of the Fluctuation Dissipation Theorem (FDT):

$$S_x(\omega) = \frac{4k_B T}{\omega} \mathfrak{Im}\hat{\chi}(\omega) \tag{6.50}$$

which is exactly the same result we found in the simple case of a harmonic oscillator, eq. 6.31. But the proof we just described does not rely on any property of the system. It can be applied to any intensive thermodynamical observable of a general system which is in thermal contact and equilibrium with the environment, as long as we can identify the corresponding generalized force and compute the system response, which is often called the *susceptibility*.

Structural damping

In the model of the simple harmonic oscillator that we used so far we always restricted ourselves to *viscous damping*, which simply means that the dissipative force is proportional to the velocity of the mass. This is the case for some sources of friction, like for example air drag or eddy current damping. When the mirror is suspended in vacuum, there is almost no more any air that can produce viscous damping, but nevertheless we see friction in excess of what we would expect from the residual gas pressure. This is due to internal friction in the material, or in other words to deviations from a simple elastic response. Indeed, the Hooke's law assumes that the deformation of the material is instantaneous, and so perfectly in phase with the external stress. But this cannot be true in any realistic system: the response

will always be slightly delayed, meaning that the system response is out-of-phase with respect to the applied force. We already discussed above that every time the response is phase shifted, there is energy dissipation, since the system transfer function has an imaginary part. Now that we have established the fluctuation-dissipation theorem, we also know that this implies the presence of some thermal noise.

In the frequency domain, we can describe this by adding an imaginary part to the elastic constant (or to the Young's modulus if we are considering an continuous system). The simplest approximation is to just add a constant imaginary part, by specifying a *loss angle* ϕ. This is often called *structural damping*:

$$k \to k\,(1 + i\phi) \tag{6.51}$$

The elastic energy stored by the system acquires an imaginary part

$$E_E = \frac{1}{2}kx^2 + i\phi\frac{1}{2}kx^2 \tag{6.52}$$

which can be interpreted as a an energy loss over time proportional to the loss angle. Clearly a loss angle which is constant with frequency is not realistic, since we know that all physical system will eventually move in phase with the force at very low or very large frequencies. Moreover, the imaginary and real part of the system response must obey the Kramers-Kronig relations. A discussion of those relations is outside the scope of this section, so here we will simply show a simple model of a material that can give rise to an imaginary part of the elastic constant.

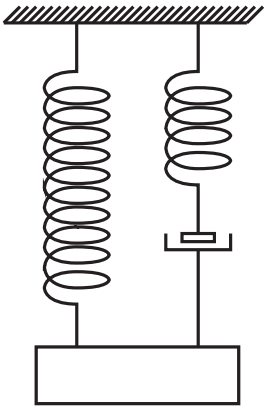

Figure 6.2: A simple model of the elastic response of a material, that can give rise to non trivial frequency dependency of the imaginary part of the elastic constant.

The standard linear elastic material model, also known as the Zener model, is shown in its simplest form in figure 6.2. The system is made of a mass m suspended with the parallel of two paths: the first composed by an ideal spring with elastic constant k, and the second by a spring with constant δk in series with a *dashpot*, i.e. a viscous damping component with friction coefficient b. Let x_0 be the length change of the first spring, with respect to equilibrium, and similarly x_1 and x_2 be the length changes of the second spring and of the dashpot. Then it's easy to show that the forces acting at the extremities of all elements are (with the same numbering convention):

$$F_0 = -kx_0 \tag{6.53}$$
$$F_1 = -\delta k x_1 \tag{6.54}$$
$$F_2 = -b\dot{x}_2 \tag{6.55}$$

Using those equations together with the constituent relations for

parallel and series connections:

$$x_0 = x_1 + x_2 \tag{6.56}$$
$$F_1 = F_2 \tag{6.57}$$

we can easily compute the total force acting on the bottom mass, going in Fourier transform to convert derivatives into multiplications with ω:

$$F = -\left[k + \frac{i\omega b}{1 + \frac{i\omega b}{\delta k}}\right] x_0 \tag{6.58}$$

With a bit of algebra we can collect the real and imaginary parts of the composite elastic constant. By defining the typical relaxation time given by $\tau = \frac{b}{\delta k}$ we can finally write

$$k_{tot} = k + \delta k \frac{\omega^2\tau^2}{1+\omega^2\tau^2} + i\delta k \frac{\omega\tau}{1+\omega^2\tau^2} \tag{6.59}$$

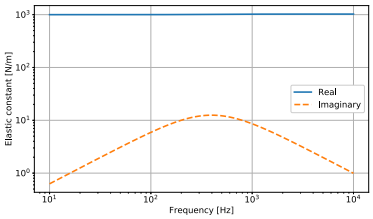

Figure 6.3: Real and imaginary part of the elastic constant for the model shown in figure 6.2. In this example $\frac{1}{2\pi\tau} \simeq 400$ Hz

Figure **??** shows the real and imaginary part of such a simple system. As one would expect, the imaginary part is tapering to zero both at low and high frequency, but it has a broad peak at a frequency determined by $(2\pi\tau)^{-1}$. In a real material one can imagine that defects and dislocations will give rise to such dissipations, with a continuous distribution of values for δk and b. This can easily give rise to almost constant loss angle as a function of frequency, over a wide range.

Experimentally, one finds that a structural damping with a frequency independent loss angle is often a good model at the frequencies that are relevant for gravitational wave detectors, between 10 Hz and a few kHz. Let's therefore consider a harmonic oscillator with elastic constant k, mass m, and structural damping described with a frequency independent loss angle ϕ:

$$-m\omega^2\hat{x} + k(1+i\phi)\hat{x} = \hat{F} \tag{6.60}$$

from which we can see that, if $\phi \ll 1$, the eigenmode is a damped oscillation with an amplitude that follows a decaying trend described by $e^{-\omega_0\phi}$ where $\omega_0 = \frac{k}{m}$. The transfer function can be written as follows, splitting imaginary and real part:

$$\begin{aligned} T(\omega) &= \frac{1}{-m\omega^2 + k(1+i\phi)} \\ &= \frac{k - m\omega^2}{(k-m\omega^2)^2 + k^2\phi^2} - i\phi\frac{k}{(k-m\omega^2)^2 + k^2\phi^2} \end{aligned} \tag{6.61}$$

The fluctuation dissipation theorem allows us to directly write the corresponding thermal noise

$$S_X(\omega) = \frac{4k_BT}{\omega}\frac{k\phi}{(k-m\omega^2)^2 + k^2\phi^2} \tag{6.62}$$

In figure **??** we compare the level of thermal noise for a harmonic oscillator. We chose the viscous damping factor γ and the structural damping loss angle such that the quality factor of the oscillator is the same: the same amount of energy is lost per cycle at resonance. Thermal noise in the two cases have clearly different slopes: the amplitude spectral density of the displacement noise in the structural damping case has an additional $f^{-1/2}$ frequency dependency. Above the resonance frequency structural damping gives rise to a smaller noise than viscous damping, but the situation is reversed below the resonance. This result clearly shows that measuring the mechanical quality factor of a resonator is not sufficient to completely determine the thermal noise level: indeed the quality factor in the two examples is the same, since it is related to the energy loss at resonance, which depends only on the friction at that particular frequency. If we want to use measurements of quality factors to estimate thermal noise we have to make some assumptions. The most common assumption, which is well supported by experiments an observations, is that for high quality resonators in vacuum, friction is dominated by energy dissipations internal to the material, that can be described by a frequency dependent loss angle. We can then measure the quality factor of a resonator at multiple resonant frequencies, and extract the loss angle at the corresponding frequency. This provides us with a map of the frequency dependency of the loss angle. It is typically a good assumption to take the loss angle as a smooth function of frequency.

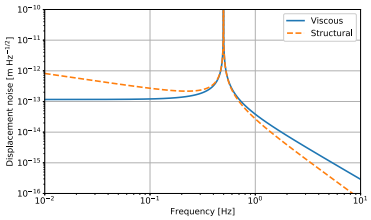

Figure 6.4: Comparison of the thermal noise in a simple harmonic oscillator, when viscous and structural damping are considered. The damping factor for the viscous term and the loss angle for the structural tarmac are adjusted to give a resonance with the same quality factor.

Suspension thermal noise

The mirror of all gravitational wave interferometers are suspended to pendulums, therefore it is important to study the contribution of the *suspension thermal noise*. The main question to be address concerns the origin of the energy loss or friction in such a system. If the pendulum is in vacuum, many source of viscous friction, such as air damping, are absent or negligible, so we are led to assume that the main conurbation to thermal noise will come from structural friction in the material. In a simple harmonic oscillator we described structural damping by assigning an imaginary part to the elastic constant, but in the case of a pendulum we have $k = \frac{mg}{l}$, since the restoring force is provided by gravity: what is the meaning of assigning a loss angle to an elastic constant which is determined by gravity? For sure there is no dissipation in the gravitational restoring force. To properly account for energy dissipations in the material, we cannot simply model the system as a pendulum with an ideal wire, but we have to

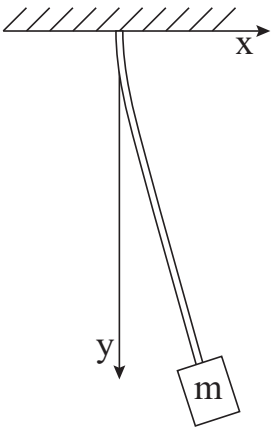

Figure 6.5: Model of a pendulum where the wire is a elastic thin rod, rigidly connected to the suspension point.

include the energy stored in the bending of the wire itself: this will be the source of structural energy dissipation.

We will model the suspension wire as a thine rod, as shown in figure 6.5. The wire is rigidly attached to both the suspension point and the mass. The suspension point cannot rotate, so the wire will always be pointing in the vertical direction near the attachment point. We instead assume that the moment of inertia of the suspended mass is small, so that the mass is free to rotate. To be more precise, the energy due to the mass rotation is rnegeglible with respect to all the other contributions.

The wire is assumed to be made of an uniform, isotropic material that we will describe with the classical equations of elasticity theory. The material is characterized by a Young's modulus Y which relates the strain and stress in each part of the wire. Here we consider that the wire is longitudinally stretched by a constant force due to gravity mg, and that the mass is allowed only small transverse motions in the x direction. This implies that the wire tension is at first order constant, and that the wire lateral displacement from the vertical is always small. If we also assume that the wire thickness is much smaller than its length, then we can use the well known thin beam approximation of elasticity theory. There are many ways to derive the differential equation that describes the response of the wire to an external force applied to the mass. We could for example write the equations that describe the force balance for each section of the wire, or use a variational principle approach based on the total elastic and kinetic energy of the system. The details are described in many textbooks on vibrations of continuous bodies, so here we will only state the result. If $y(z,t)$ denotes the transverse displacement of the wire at a distance z from the suspension point, and $F(t)$ is the external force applied to the suspended mass (assumed to be point-like), we have

$$YI\frac{\partial^4 y}{\partial x^4} - mg\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \tag{6.63}$$

where ρ is the linear mass density, m is the mass of the suspended object and I is the transverse moment of inertia

$$I = \iint_S dxdy\, y^2 = \frac{\pi}{8}r^4 \tag{6.64}$$

where the integration is over the entire section S of the wire and the result is for a cylindrical wire of radius r. At first it might seem surprising that the external force does not enter in this equation. This is however quite usual in the case of continuous bodies: indeed the equation above is not enough to determine the wire bending, and it

has to be supplemented with suitable boundary conditions for the clamp at the suspension point:

$$y(0,t) = 0 \tag{6.65}$$

$$\frac{\partial y}{\partial x}(0,t) = 0 \tag{6.66}$$

$$\tag{6.67}$$

and for the properties of the mass (no moment of inertia, external force)

$$\frac{\partial^2 y}{\partial x^2}(l,t) = 0 \tag{6.68}$$

$$YI\frac{\partial^3 y}{\partial x^3}(l,t) - mg\frac{\partial y}{\partial x}(l,t) = F \tag{6.69}$$

where l is the vertical coordinate of the wire end point. Since we are taking into account very small transverse displacements, we can safely assume that $l \simeq L$. If we assume the force to be purely sinusoidal $F(t) = Fe^{i\omega t}$ then we can look for an oscillating solution in the form $y(x,t) = y(x)e^{i\omega t}$, so that the elastic equation becomes

$$YIy'''' - mgy'' + \omega^2\rho y = 0 \tag{6.70}$$

where we used the notation y'' to indicate, for example, the second derivative of y with respect to the spatial coordinate x. This is a linear equation, so we can as usual search for a solution in the exponential form $y(x) = y_0e^{\lambda x}$, which provides us with an algebraic equation

$$YI\lambda^2 - mb\lambda^2 + \omega^2\rho = 0 \tag{6.71}$$

which is quite easy to solve

$$\lambda^2 = \frac{mg}{2YI}\left[1 \pm \sqrt{1 - \frac{4YI\rho\omega^2}{m^2g^2}}\right] \tag{6.72}$$

Before diving into substituting those values of λ into the expression for y and enforcing the Boundary equations, let's look at the frequencies involved in the problem. The values of ω are those corresponding to frequencies up to a few hundred hertzs. Inside the square root, the other frequency involved is determined by the factor

$$\frac{4YI\rho}{m^2g^2} \tag{6.73}$$

Let's plug in some numbers, by considering a fused silica fiber, with a diameter of $0.4mm$, suspending a mass of 10 kg. Fused silica has a Young's modulus of about 72 GPa and a density of 2200 kg/m^3. This yields:

$$\left[\frac{4YI\rho}{m^2g^2}\right]^{-1/2} \simeq 4.4 \times 10^5 \text{rad/s} \tag{6.74}$$

which correspond to frequencies of about 70 kHz. Therefore, for all the frequency range we are interested in, we can approximate the square root in equation 6.72 to one. At first glance this might seem like a bad idea, since then one of the possible values for λ is 0. But this simply means that we have to consider as possible solutions also a constant and a linear term in x. Indeed, the approximation above corresponds to ignoring the kinetic term in equation 6.71:

$$YIy'''' - mgy'' = 0 \tag{6.75}$$

which clearly admits as solutions $y = A + Bx + Ce^{\lambda x} + De^{-\lambda x}$ where $\lambda = \sqrt{\frac{mg}{YI}}$. With the same numbers used above for fused silica, we get $\lambda^{-1} \simeq 0.7mm$. This is the typical length scale of the region of the fiber which is bent. In enforcing the boundary conditions, we take for example $L \simeq 1$ m and therefore we can assume that $\lambda L \gg 1$. This simplifies the equations and allows us to finally find the fiber motion due to external force applied to the suspended mass:

$$y(x) = \frac{F}{mg\lambda}\left(e^{-\lambda x} + \lambda x - 1\right) \tag{6.76}$$

In particular, we can focus on the displacement of the end mass. The wire has a length L, but since it is bent, the end point is a coordinate l which might be slightly different. We can compute that by integrating the wire displacement along the entire length:

$$L = \int_0^l \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \simeq \int_0^l \left[1 + \frac{1}{2}\cdot\left(\frac{dy}{dx}\right)^2\right] dx \tag{6.77}$$

Using the approximation $\lambda L \gg 1$ and $\lambda l \gg 1$, we can write the transverse displacement of the end point as

$$\delta = \frac{F}{mg\lambda}\left(l\lambda - 1\right) \tag{6.78}$$

invert it to get F and substitute the result into the equation for L. At the lowest order in $\frac{1}{\lambda l}$ we get

$$L \simeq l\left[1 + \frac{1}{2}\left(\frac{\delta}{l}\right)^2\left(1 + \frac{1}{2l\lambda}\right)\right] \tag{6.79}$$

which we can invert, at first order in $\frac{1}{\lambda l}$ to obtain

$$l \simeq L\left[1 - \frac{1}{2}\left(\frac{\delta}{L}\right)^2\left(1 + \frac{1}{2l\lambda}\right)\right] \tag{6.80}$$

This confirms our earlier assumption that any change in the wire length is quadratic in the transverse displacement and only a small

correction is due to the wire bending. So, at the lowest order in $\frac{1}{\lambda l}$ we can simply write the suspended mass displacement as

$$\delta = \frac{mg}{L}\left[1 + \frac{1}{\lambda L}\right] \cdot F \tag{6.81}$$

We can interpreter this as a modification of the spring constant coming from the pendulum, depending on the material properties. To further understand this, we can write the potential energies involved in the wire bending and displacement. First of all, we have a gravitational potential energy contribution, which is

$$V_g = mg(L - l) = \frac{1}{2}\frac{mg}{L}\left(1 + \frac{1}{2\lambda L}\right)\delta^2 \tag{6.82}$$

Then we have the energy stored in the elastic deformation of the wire, which can be compute with the usual expression

$$V_e = \frac{1}{2}YI\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{1}{2}\frac{mg}{L}\left(\frac{1}{2\lambda L}\right)\delta^2 \tag{6.83}$$

Those expression show that we can interpreter the total spring constant as the sum of a gravitational term and an elastic term

$$k_t = k_g + k_e \tag{6.84}$$

$$k_g = \frac{mg}{L}\left(1 + \frac{1}{2\lambda L}\right) \tag{6.85}$$

$$k_e = \frac{mg}{L}\left(\frac{1}{2\lambda L}\right) \tag{6.86}$$

At this point we can go back to the original problem, which was to calculate the thermal noise in a suspended mass, due to structural damping in the wire material. We start by describing the mechanical loss in terms of a microscopic loss angle which gives an imaginary part to the Young's modulus:

$$Y \to Y\,(1 + i\phi) \tag{6.87}$$

Since the loss angle is typically very small, so we can always expand to first order in ϕ, so that

$$\lambda \to \lambda\left(1 - i\frac{\phi}{2}\right) \tag{6.88}$$

Substitution of this into the expression for the spring constant finally shows us how to properly account for the material internal losses, when there are conservative forces like gravity in action:

$$k_t = \frac{mg}{L}\left[1 + \frac{1}{\lambda L}\left(1 + i\frac{\phi}{2}\right)\right] \tag{6.89}$$

At first order in the usual parameter $\frac{1}{\lambda L}$ we can write this result in a very interesting form, denoting the lossless total spring constant with $k_t^{(0)}$:

$$k_t = k_t^{(0)} \left(1 + i\phi \frac{k_e}{k_t^{(0)}}\right) \tag{6.90}$$

We see that the effect of the mechanical loss of the material is reduced by a factor which takes into account the ratio of the elastic stiffness over the total stiffness. Using equations 6.82 and 6.83 we can also rewrite this *dilution factor* $\mathcal{D}$ in terms of the ratio of stored energies:

$$\phi_{eff} = \frac{V_e}{V_g + V_e}\phi = \mathcal{D}\phi \tag{6.91}$$

This dilution effect has a simple and intuitive explanation. We already discussed above that one way to interpreter the loss angle is as the fraction of energy which is dissipated in the system, averaged over an oscillation period. Clearly the kind of structural damping we are considering is related to some form of internal friction of the material, and this local friction can only access the energy which is stored locally in the form of elastic potential energy. The amount of energy which is lost at each cycle is then $\delta E = \phi V_e$. But if we look at the entire suspension, the total energy is mostly stored in the gravitational potential, and the loss angle that matters is an effective loss angle ϕ_{eff} which tells us what fraction of the total energy is lost: $\delta E = \phi_{eff} E_{tot}$. Comparison of the last two equations allow us to write the effective loss angle as the structural damping times the dilution factor.

For typical fused silica fibers, the material structural damping is characterized by a loss angle of the order of $\phi = 10^{-4}$. Assuming four 1-m-long fibers suspending a 40 kg mirror, and with a diameter of 0.4 mm each, we get that the dilution factor is

$$\mathcal{D} \simeq \frac{1}{2\lambda L} \simeq 3 \times 10^{-4} \tag{6.92}$$

Therefore the effective loss angle of the full suspension is much smaller than the material, and we have $\phi_{eff} \sim 3 \times 10^{-8}$. We can then compute the corresponding thermal noise by approximating the full suspension to a harmonic oscillator with $k \simeq \frac{mg}{L}$ and loss angle ϕ_{effect}:

$$S_X(\omega) = \frac{4k_B T}{\omega} \frac{k\phi_{eff}}{(k - m\omega^2)^2 + k^2\phi_{eff}^2} \tag{6.93}$$

The result is shown in figure 6.6 for the parameters above.

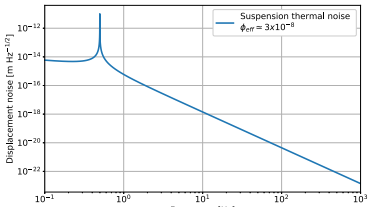

Figure 6.6: Suspension thermal noise for a 40 kg mirror suspended with four fused silica fibers, assuming a length of 1 m, diameter of 0.4 mm for each fiber and a material loss angle of 1×10^{-4}.

This dilution effect explains how we can obtain very low effective loss angle (and therefore thermal noise) in suspended mirrors, even if the material loss angle is not very high. Moreover, the interpretation of the loss angle as an energy loss ratio allow us to generalize the idea of dilution: each source of mechanical loss contributes to the effective total loss angle weighted by the ratio of the energy stored in that source over the total potential energy in the system.

Direct approach to thermal noise computation

The fluctuation dissipation theorem is a very powerful tool to compute the thermal noise in any system, but in some cases it might be difficult to compute the system transfer function, especially when continuous systems are involved and the variable we are measuring is not a simple coordinate. The example we have in mind is clearly the phase noise induced on a gaussian beam reflected from a mirror.

Moreover, we can make the connection between thermal noise and energy lost due to dissipations even more compelling. In this section we will develop a *direct approach* to the computation of thermal noise, deriving a direct relationship between the energy dissipated on a system excited by an external force and its thermal noise. The discussion presented here follows directly the original approach by Levin [1].

[1] Yu. Levin. Internal thermal noise in the ligo test masses: A direct approach. *Phys. Rev. D*, 57:659–663, Jan 1998. DOI: 10.1103/PhysRevD.57.659. URL https://link.aps.org/doi/10.1103/PhysRevD.57.659

The key ingredient in this derivation is a result we already presented at the beginning of this chapter, in section 6.3: the imaginary part of the transfer function of a linear system is directly related to the averaged energy lost in a cycle, when the system is driven by a sinusoidal force. In eq. 6.7 we showed that the energy dissipated in a cycle is given by

$$W_{diss} = \frac{\omega}{2} \Im\mathfrak{m}\left[T(\omega)\right] F_0^2 \tag{6.94}$$

This result suggests that to compute the imaginary part of the system transfer function we can perform the following mental experiment: apply a sinusoidal force to the system, with a frequency ω, and compute the energy dissipated in one cycle. Then use the equation above to extract the imaginary part of the transfer function.

To be more precise, let's consider a system described by a generalized coordinate $y(\mathbf{r}, t)$ which we assume can be a continuous function of some position $\mathbf{r}$ and time. To be concrete we can think of y as the displacement of the reflective surface of a mirror. What we measure is not y directly, but rather some scalar linear function, defined as

$$x(t) = \int f(\mathbf{r}) y(\mathbf{r}, t) d\mathbf{r} \tag{6.95}$$

where f is a weighting function that describes the measurement. Pushing forward our example, if we probe the mirror surface with a gaussian beam, and measure the phase of the reflected beam with respect to the incoming beam, the function f is proportional to the intensity profile of the beam.

We can describe the incoming beam with a field $\psi(\mathbf{r})$. Upon reflection off the mirror, the field will acquire an additional phase $e^{-2ikx(\mathbf{r},t)}$. To measure this phase we beat the reflected field with a reference, which in the optimal case has the same spatial profile as the incoming beam. The result is an overlap integral $I(t) = \int d^2\mathbf{r}\psi^*(\mathbf{r})\psi(\mathbf{r})e^{-2ikx(\mathbf{r},t)}$. For small motions $x(\mathbf{r},t)$ the phase can be approximated at first order in the displacement, to get $\phi(t) = -2k\int d^2\mathbf{r}\psi^*(\mathbf{r})\psi(\mathbf{r})x(\mathbf{r},t)$. Therefore the weighting function in this case is proportional to the beam intensity $f(\mathbf{r}) = 2k\,|\psi(\mathbf{r})|$.

Taking the fluctuation dissipation theorem as a staring point, we have to compute the system transfer function from an external force $F(t)$ to the variable $x(t)$ we are measuring. In general $x(t)$ is not one of the usual generalized coordinates used to describe a linear system, i.e. one that correspond to an eigenmode. Nevertheless, in a Hamiltonia formulation, we can always change coordinates so that $x(t)$ is one of the Hamiltonian coordinates, to which corresponds a generalized force $F(t)$. The fluctuation dissipation theorem needs the transfer function from this generalized force to the measured coordinate x. We can describe the action of this external force with an interaction Hamiltonian with the simple form

$$H_{int} = -F(t)x \tag{6.96}$$

and if we substitute into this expression the definition of the observable x we get

$$H_{int} = -\int F(t)f(\mathbf{r})y(\mathbf{r},t)d\mathbf{r} \tag{6.97}$$

One way to interpreter this is that the force consists of a distributed pressure applied to the system original coordinate, with a time varying profile defined by our measurement weight

$$P(\mathbf{r},t) = F(t)f(\mathbf{r}) \tag{6.98}$$

This defines the force we need to apply to the system. Therefore, to compute the thermal noise for the observable $x(t)$ we need to perform the following computation

1. apply a sinusoidal pressure on the system with profile $P(\mathbf{r},t) = F_0 f(\mathbf{f})\cos\omega t$;
2. compute the energy W_{diss} that is dissipated in the system, at equilibrium, when driven by the force described above, averaged over a cycle;
3. use equation 6.7 to compute the imaginary part of the transfer function and substitute into the fluctuation-dissipation theorem to obtain

$$S_x(\omega) = \frac{k_b T}{2\omega^2}\frac{W_{diss}}{F_0^2} \tag{6.99}$$

The power of this direct approach to the computation of thermal noise is twofold.

First of all, it is often simpler to compute the response of the system to the pressure profile above, rather than have to compute all the eigen-modes and eigen-functions, that would be necessary to compute the transfer function. This is particularly true for complex continuous system for which analytical expressions for the response are not available or practical, forcing us to resort to numerical methods, such as finite element analysis.

Second, this formulation makes the link between thermal noise and dissipated energy very clear and direct. The loss angle of the system described directly the amount of energy lost per cycle, so that $W_{diss} = \phi E_{cycle}$. It is natural to divide the energy stored in the system in its components: for example as we did above for the suspension system, by considering the gravitational energy and the elastic energy. In this way it is straightforward to assign different loss angles to different energies or different parts of the system. Each loss angle will contribute to the total effective loss angle with a dilution factor which is naturally described as the energy stored in the corresponding part of the system, over the total energy.

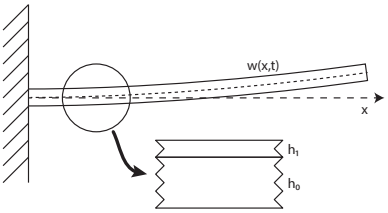

Figure 6.7: A simple cantilever with a thin film deposited on one face.

As an example, let's consider a simple cantilever, i.e. a thin beam fixed at one extreme and free to move at the other extreme, as shown in figure 6.7. The cantilever is uniform along its length, but not along its section: we assume that the cantilever is made of a substrate with thickness h_0, width b, Young's modulus Y_0 and density ρ_0, plus a thin film deposited on one entire face, with thickness h_1, Young's modulus Y_1 and density ρ_1. We assume that the film is very thin, i.e. $h_1 \ll h_0$. We can also work in the thin beam approximation, assuming that the thickness of the whole cantilever is small with respect to its length $h_0 \ll L$.

In the thin beam approximation the entire thickness of the cantilever bends following the central neutral line, which is described by a displacement $w(x,t)$ from the horizontal equilibrium position. Therefore $w(x,t)$ describes the vertical component of the displacement field, along the z coordinate, while $-z\frac{\partial w(x,t)}{\partial x}$ describes the horizontal component, along the x coordinate. In this approximation there is only one component to the strain tensor

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w(x,t)}{\partial x^2} \tag{6.100}$$

The total elastic energy can then be written as usual as an integral

over the entire cantilever:

$$V_e = \frac{1}{2}\iiint Y(x,y,z)\epsilon_{xx}\epsilon_{xx}\,dxdydz \tag{6.101}$$

$$= \frac{1}{2}\iiint Y(x,y,z)z^2\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dxdydz \tag{6.102}$$

The integration along the transverse cantilever direction y is trivial and gives just the cantilever width b. The integration along the thickness z must take into account the varying Young's modulus:

$$\int_{-\frac{h_0}{2}}^{+\frac{h_0}{2}+h_1} z^2 Y(z)dz \simeq \frac{h_0^3}{12}Y_0 + \frac{h_0^2 h_1}{4}Y_1 \tag{6.103}$$

in the approximation $h_1 \ll h_0$. We can therefore define a transverse rigidity as

$$R = \frac{bh^3}{12}\left(Y_0 + Y_1\frac{3h_1}{h_0}\right) \tag{6.104}$$

and write the total elastic energy as

$$V_e = \frac{1}{2}R\int_0^L \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dz \tag{6.105}$$

This equation, together with the definition of R, shows us how the energy is divided between the substrate and the thin film, and thus provides us a way to compute the dilution factor:

$$V_e = V_{sub.} + V_{film} \tag{6.106}$$

$$V_{sub.} = \frac{1}{2}\frac{bh^3}{12}Y_0\int_0^L \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dz \tag{6.107}$$

$$V_{film} = \frac{1}{2}\frac{bh^3}{12}Y_1\frac{3h_1}{h_0}\int_0^L \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dz \tag{6.108}$$

$$\mathcal{D} = \frac{V_{film}}{V_e} = \frac{3h_1Y_1}{h_0Y_0 + 3h_1Y_1} \simeq \frac{Y_1}{Y_0}\frac{3h_1}{h_0} \tag{6.109}$$

In many cases the loss angle of the thin film is different from the loss angle of the bulk material the cantilever is made of, and often it is much worse. In any case, we can write an effective loss angle for the entire system using the dilution factor

$$\phi_{eff} = (1-\mathcal{D})\,\phi_{bulk} + \mathcal{D}\phi_{film} \tag{6.110}$$

One important note here is that the dilution factor, and hence the effective loss angle, can depend on which resonant mode of the cantilever we are considering. Indeed, the ratio of the energies in the film and in the substrate can be different for different modes, since the deformation depends on the mode. Therefore, even if the loss angle is frequency independent, modes at different frequencies can have different loss angles and therefore different quality factors.

How to measure the loss angle

Multi-layer dielectric coatings

The thin film example we discussed in the previous section was not chosen arbitrarily, but it was an introduction to the topic of coating thermal noise. Indeed we already stated that interferometric gravitational wave detectors use high reflectivity mirrors, but we haven't described how those mirrors are actually built. Currently, the best reflective coatings are made using thin films, made of alternating layers with different materials. It turns out that the thermal noise due to energy dissipation in the thin film is an important source of noise for GW detectors.

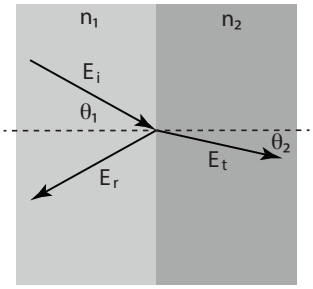

Figure 6.8: Reflection and transmission of electromagnetic waves at an interface.

Every time an electromagnetic wave propagate through then interface between two different materials, there is a partial reflection, determined by the refractive indexes of the two materials n_1 and n_2 (the wave propagating from 1 to 2), the wave polarization (either s or p) and the angle of incidence θ_1. The complex amplitude reflection r and transmission t coefficients are given by the Fresnel's laws:

$$r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{6.111}$$

$$t_s = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{6.112}$$

$$r_p = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \tag{6.113}$$

$$t_p = \frac{2n_2 \cos\theta_1}{n_2 \cos\theta_1 + n_2 \cos\theta_2} \tag{6.114}$$

where θ_2 is the angle of transmitted wave, determined by the Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{6.115}$$

Those equations can be derived from the continuity requirements for the electric and magnetic fields at the interface. Their derivation is discussed in many classical electromagnetism textbooks, and it is therefore not repeated here. For simplicity, we can assume normal incidence, meaning that $\theta_1 = \theta_2 = 0$, and moreover we don't need to distinguish between the two light polarizations. In this case the reflection and transmission coefficients are

$$r = \frac{n_1 - n_2}{n_1 + n_2} \tag{6.116}$$

$$t = \frac{2n_1}{n_1 + n_2} \tag{6.117}$$

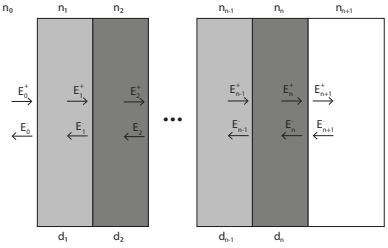
Figure 6.9: Definition of the refractive index, thickness and propagating fields in a multi-layer coating.

Thermal noise in coatings

6.4 Sesmic noise

Newtonian noise

6.5 Scattered light

6.6 Other noise sources

Electronic noise

Control noise

Residual gas noise

6.7 Noise budget