Abstract

Going to make LIGO the best possible ever.

Adaptive Mode Matching in Advanced LIGO

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Preface

The era of gravitational waves astronomy was ushered in by the LIGO (Laser Interferometer Gravitational-Wave Observatory) collaboration with the detection of a binary black hole collision (Detection paper). The event that shook the foundation of space-time allowed mankind to view the cosmos in a way that had never been done previously.

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1.1 Gravitational Waves

In 1915, Albert Einstein published his theory of general relativity [13], which contains the most complete description of gravity to date. The seminal equation in this theory, called the Einstein Field Equation is

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

which is a set of 10 coupled second-order differential equations that are nonlinear and fully describes the interaction between space-time and mass-energy. Equation 1.1 is difficult to solve except in situations where specific approximations allow a user to find exact solutions such as spherical symmetry [6] [30]. In areas where the curvature is close to flat, the weak field approximation can be applied and the metric is described as

$$g_{\mu\nu} \approxeq \eta_{\mu\nu} + h_{\mu\nu} \tag{1.2}$$

where $\eta_{\mu\nu}$ is the metric of flat space time and $|h_{\mu\nu}| \ll 1$ is the perturbation due to a gravitational field.

By plugging in equation 1.2 into 1.1 and using empty space we obtain the familiar wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} = 0 \tag{1.3}$$

which has a plane-wave solution of the form $h_{\mu\nu} = A_{\mu\nu}e^{ik_\nu x^\nu}$.

Using the gauge constraint $h^{\mu\nu}_{,\nu} = 0$, it follows that $A_{\mu\nu}k^\mu = 0$ which means that the gravitational wave amplitude is orthogonal to the propagation vector.

Further imposing transverse-traceless gauge and assuming that the wave is traveling in the x^3 direction, it can be shown that the complex amplitude has physical significance expressed in the matrix

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{yx} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.4}$$

Oftentimes, the four non-zero components of equation 1.4 can be categorized into two distinct polarizations called plus and cross such that $h_+ = A_{xx} = -A_{yy}$ and $h_\times = A_{xy} = A_{yx}$.

It is natural to attempt to understand the physical interpretation of equation 1.4 as an affect on the position of a free floating particle. Consider the four-velocity, U^α, in the transverse traceless gauge where the coordinate itself is attached the particles and incorporates any small wiggles that would shake the coordinates. Of course, any free particles will follow the geodesic equation

$$\nabla U^\alpha = \frac{\mathrm{d}}{\mathrm{d}\tau}U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0 \tag{1.5}$$

where $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\gamma\alpha}(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma})$ are the famous Christoffel symbols. By evaluating the first term of the acceleration in equation 1.5,

$$\left(\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau}\right)_0 = -\Gamma^\alpha_{00} = \frac{1}{2}\eta_{\mu\nu}(h_{\beta 0,0} + h_{0\beta,0} + h_{00,\beta}) \tag{1.6}$$

However, comparing equation 1.4 and equation 1.6, it is clear that if the particle is initially at rest, then a moment later it is still at rest! The term "at rest" is actually used liberally here since the coordinate system varies along with the gravitational wave.

Alternatively, one can ask if a gravitational wave passed by a pair of particles separated by length L, what would be the effect on the distance between two points? The proper distance is defined as

$$\delta l = \int g_{\mu\nu} dx^{\mu} dx^{\nu} = \int_0^L g_{xx} dx \approx |g_{xx}(x=0)|^{1/2} \approx [1 + \frac{1}{2} h_{xx}(x=0)]L \tag{1.7}$$

which shows us two very important points about the nature of gravitational waves. Firstly, the effect is very small since the length variation, h_{xx}, is a small perturbation on flat space-time. Secondly, the effect is proportional to the initial separation between the particles. This means a detector which is large will have a better chance to measure these small effects, an important point that drove the design of the Laser Interferometer Gravitational-Wave Interferometer (LIGO).

1.1.1 Measuring Gravitational Waves with Light

Even with the theoretical formulation of gravitational waves resolved by the 1970s [26], the possibility of detecting gravitational waves by ground-based instruments was still a controversial topic among scientists in the field [11]. During the famous Chapel Hill Conference in 1957 which included some of the great minds of the era such as Wheeler, Schwinger, and Feynman, the experimental search for gravitational waves began to take hold. One thought experiment that was proposed by Feynmann [10] where he considered a bead sliding on a string with some friction as a gravitational wave passes by. As the beads slide back and forth due to the wave described by equation 1.7, there will be some heat dissipated which means the gravitational wave must carry some energy.

There is still the issue of an incredible amount of accuracy required to measure the strain from even the most dense astrophysical objects. The earliest attempts at detecting these small signals were famously done by Joseph Weber using large resonant bars and piezoelectric transducers to extract the energy from gravitational waves at the resonant frequencies of the bars. Picture of bar detectors at LHO. However, these bars are limited by thermal noise and can only detect GWs in very narrow frequency bands.

Interferometers are devices that measure small displacements by using a laser that is split by a partially transmitting mirror (or beamsplitter), which allows 50% of the light to get reflected and 50% to be transmitted. Each of the beams travel down the arms and reflect off of mirrors and return to the beamsplitter. Upon reaching the optic, the two beams recombine and by the principle of superposition the electromagnetic waves will add linearly at the output port (or antisymmetric port). Figure[]. The laser beams will gather phase as they propagate down each individual arm, and when recombining, the intensity of the light will be proportional to the phase differences between each beam. This will correspond to a differential length that is described by

$$L_{-} = l_x - l_y \tag{1.8}$$

As shown in equation 1.7, the effect of gravitational waves on the proper length between two free falling objects is proportional to the initial separation. From figure[GWparticles], it is intuitively clear that interferometry would be an ideal technique to detect signals from a gravitational wave. However, one can explicitly derive a ground-based interferometer's response to a GW from an astrophysical object.

Consider a gravitational wave source arbitrarily located in the sky with respect to an interferometer on Earth. By denoting the interferometer's Cartesian coordinates as $\{\hat{x}, \hat{y}, \hat{z}\}$ with x and y located along the arms respectively such that the z-axis points directly towards the zenith (i.e. a right-handed system.). Using the well known Euler angles, a relation from the detector frame to the source frame with coordinates $\{\hat{x}', \hat{y}', \hat{z}'\}$ can be seen in Figure[Euler].

If a gravitational wave at the source has emitted GWs with plus and cross polarizations as denoted by equation 1.4, then the detector time series can be regarded as [1] [15]

$$h(t) = F_{+}(\theta, \phi, \psi)\, h_{+}(t) + F_{\times}(\theta, \phi, \psi)\, h_{\times}(t) \tag{1.9}$$

where $F_{+}(\theta, \phi, \psi)$ and $F_{\times}(\theta, \phi, \psi$ are the antenna pattern functions that project the gravitational wave amplitudes onto the detector frame.

$$F_{+}(\theta, \phi, \psi) = -\frac{1}{2}[1 + \cos^2(\theta)]\cos(2\phi)\cos(2\psi) - \cos(\theta)\sin(2\phi)\sin(2\psi) \tag{1.10}$$

$$F_{\times}(\theta, \phi, \psi) = +\frac{1}{2}[1 + \cos^2(\theta)]\sin(2\phi)\cos(2\psi) - \cos(\theta)\sin(2\phi)\sin(2\psi) \tag{1.11}$$

If the gravitational wave is located directly above the interferometer such that $\theta = 0$ and setting $\psi = 0$, then the magnitude of the antenna pattern is equal to unity. Furthermore, by rotating about the ϕ angle such that the detector arms align with the plus polarization, the null geodesic equation (ie the path of a photon) in the interferometer frame becomes

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + [1 + h_{+}]dx^2 + [1 - h_{+}]dy^2 + dz^2 = 0 \tag{1.12}$$

Now, if the photon is traveling along the x-arm, this means that $dy^2 = dz^2 = 0$ and the metric equation transforms to

$$\frac{dt}{dx} = \sqrt{1 + h_+} \approx 1 + \frac{1}{2}h_+ \tag{1.13}$$

The amount of time required for the photon to reach the x-end mirror (starting at $t = 0$) is equal to

$$t_1 = \int_0^{L_x} [1 + \frac{1}{2}h_+(x)]\mathrm{d}x \tag{1.14}$$

where L_x is the total length of the x-arm. Upon returning to the beamsplitter, the photon's total time of flight for the x and y arms are, respectively,

$$t_2 = 2L_x + \frac{1}{2}\int_0^{L_x} \Big[h_+(x) + h_+(x + L_x)\Big]\mathrm{d}x \tag{1.15}$$

$$t_2' = 2L_y - \frac{1}{2}\int_0^{L_y} \Big[h_+(y) + h_+(y + L_y)\Big]\mathrm{d}y \tag{1.16}$$

If the gravitational wave period is much longer than the time of flight, then h_+ does not change much during the measurement, which means $h_+(\eta_i) \approx h_+(\eta_i + L_{\eta_i}) \approx constant$. By subtracting the flight times of the photons for each arm and setting $L = L_x = L_y$, the difference is proportional to the gravitational wave perturbation multiplied by the sum of arm lengths (with a factor of c to get the units right),

$$\Delta t = t_2 - t_2' = \frac{2L}{c}h_+ \tag{1.17}$$

By recasting the expression for time of flight in terms of the phase picked up laser light as it travels through space, the differential phase shift is

$$\Delta\Phi = \Phi(t_2) - \Phi(t_2') = \frac{4\pi}{\lambda}\, h_+\, L \tag{1.18}$$

The equation above is simple, however, it only works for gravitational wave signals that are not frequency dependent and it assumes that the path length can be arbitrarily long. Both points are actually not true [27] but we can alleviate these discrepancies by considering a gravitational wave signal of the form $h(t) = h_0 \mathrm{exp}(i2\pi f_{GW} t)$ and repeating the calculation between equations 1.14 and 1.18:

$$\Delta\Phi(t) = h(t)\ \tau_{RT}\ \frac{2\pi c}{\lambda}\ \mathrm{sinc}(f_{GW}\tau_{RT})e^{i\pi f_{GW}\tau_{RT}} \tag{1.19}$$

where $\tau_{RT} = 2L/c$. The response for a detector whose arm lengths are blank km have null points at the around blank Hz which means the instrument cannot be arbitrarily long, however, this point is not concerning because it is too expensive and difficult to make a terrestrial detector of this size.

How to practically measure $\Delta\Phi$ with an interferometer is explained in section 1.2.

1.1.2 Detection of Gravitational Waves

The purpose of LIGO is to observe gravitational waves emanating from astrophysical objects [**?**], so it is natural to wonder how well a single detector can probe the universe. It is worthwhile to explicitly derive the inspiral horizon distance, which is how far a single detector can see a source comprised two equal mass compact objects optimally oriented in the sky relative to the detector. Signal-to-noise (SNR) is the level of interesting signals compared to the background noise, which can be expressed as

$$\rho = 2\int_{-\infty}^{\infty} \frac{\tilde{s}(f)\tilde{s}^*(f)}{S_n(f)} \tag{1.20}$$

where $S_n(f)$ is the one-sided average power spectral density of the detector noise and $\tilde{s}(f)$ is the Fourier transform of the detectors' response to a gravitational-wave signal.

Consider two dense objects with mass m_1 and m_2 rotating around each other, separated by a distance R such that quadrupole equation still holds (i.e. flat space-time with small perturbations). The detector response, in units of strain, to the gravitational waves emitted by this source is [15],

$$s(t) = \frac{\mathcal{M}}{d_L}\Theta(\theta,\phi,\psi,i)[\pi f(t)\mathcal{M}]^{2/3}\cos[\Phi(t) + constant] \tag{1.21}$$

where

$$\mathcal{M} = (1+z)\frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \tag{1.22a}$$

$$\Theta(\theta,\phi,\psi,i) = 2\sqrt{[F_+(\theta,\phi,\psi)(1+\cos^2(i))]^2 + [2F_\times\cos(i)]^2} \tag{1.22b}$$

where $\mathcal{M} = (1+z)\frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$ is the chirp mass and d_L is the luminosity distance. The orientation response, $\Theta(\theta,\phi,\psi,i)$, is a function that depends entirely on the detector orientation relative to the angular momentum vector of the binary [Figure of a binary spiraling and a detector on earth]

where F_+ and $F_\times$ are from equation 1.10 and 1.11.

As the binary loses energy to gravitational waves, the orbit will shrink as a function of time, in turn, the orbital frequency will increase

$$f(t) = \frac{1}{\pi\mathcal{M}}\left[\frac{5}{256}\frac{\mathcal{M}}{T-t}\right]^{3/8} \tag{1.23}$$

By defining the binary phase as $f(t) = \frac{1}{2\pi}\frac{\partial\mathbf{\Phi}}{\partial t}$, it is easy to recognize that the phase also evolves with time.

$$\Phi(t) = 2\pi \int_T^t f(t)dt = -2\left(\frac{T-t}{5\mathcal{M}}\right)^{5/8} \tag{1.24}$$

What this means is that the gravitational wave signal from the binary will increase in amplitude and frequency as a function of time up until the coalescence time, T.

There is actually a subtle difference between the coalescence time and the moment when adiabatic approximations fail which allows usage of the quadrupole formulation.

Plugging equation 1.21 into 1.20,

$$\rho = 8\Theta(\theta, \phi, \psi, i)\frac{r_0}{d_L}\left(\frac{\mathcal{M}}{1.2\mathrm{M}_\odot}\right)^{5/6}\zeta(f_{max}) \tag{1.25}$$

There are two important functions in the equation above that reflect the detector's performance in sensing gravitational radiation from a binary inspiral, r_0 and $\zeta(f_{max})$.

The characteristic distance, r_0 is how far the detector can see for a fixed binary's mass distribution,

$$r_0^2 = \frac{5}{192\pi^{4/3}}\left(\frac{3\mathrm{M}_\odot}{20}\right)^{5/3}\int_0^\infty \frac{1}{S_n(f)}\frac{\mathrm{d}f}{f^{7/3}} \tag{1.26}$$

Due to the integrand's dependence on $f^{-7/3}$, lower frequency improvements in the detector's noise spectrum will contribute more to the distance.

$$\zeta(f_{max}) = \frac{\int_0^{2f_{max}} \frac{1}{S_n(f)}\frac{\mathrm{d}f}{f^{7/3}}}{\int_0^\infty \frac{1}{S_n(f)}\frac{\mathrm{d}f}{f^{7/3}}} \tag{1.27}$$

The sensitivity can be different for various mass distributions; $\zeta(f_{max})$ is a normalized function that describes how well the detector's bandwidth overlaps the binary's frequency evolution. If f_{max} is higher that the minimal sensitivity frequency for the detector, then there is good overlap and $\zeta(f_{max}) \approx 1$. However, if the masses are sufficiently large, f_{max} will be lower because coalescence occurs before the objects reach a higher frequency regime and this will result in $\zeta(f_{max}) \approx 0$.

The process of two objects coalescing starts at lower frequencies and terminate at some f_{max} that depends on when the objects reach their inner-most stable circular orbit, f_{ISCO}. If $m_1 = m_2$, then the maximum frequency is[]

$$\begin{aligned} f_{max} &= \frac{f_{ISCO}}{1+z} \\ &= \frac{99\text{Hz}}{1+z}\frac{20\mathrm{M}_\odot}{M} \end{aligned} \tag{1.28}$$

where $M = m_1 + m_2$.

By plugging a few mass distributions and an estimate of the sensitivity [GWINC], it is possible to study the range of a single detector. For example, a 1.4-1.4$M_\odot$ binary that has optimal orientation with respect to the detector has horizon distance equal to...

Horizon distance and range, advanced LIGO projected

[Finn, Duncan Thesis] [27]

1.2 The LIGO Instrument

In its simplest form, the LIGO instrument is an incredibly large Michaelson interferometer. If we imagine the output as a measure of differential arm lengths, it becomes a natural way of detecting gravitational waves. However, to make a practical gravitational-wave observatory, the complexity will have to be extended beyond what Michelson and Morley used. The next sections will explore the various upgrades that were implemented to improve the sensitivity of LIGO.

1.2.1 Simple Michaelson

As shown in Figure [michelsonifo], the interferometer readout uses a photodetector that measures the total laser beam power and depends on how light in the arms constructively or destructively interferes at the beamsplitter. In Section 1.1.1, it was shown that the differential time of flight between photons traveling down the individual arms carry gravitational wave information. The difference in flight times, Δt, can be converted into how much phase,$\Delta\phi$ is accumulated by the photons as they propagate through space. But the question remains how an interferometer actually measures $\Delta\phi$.

If the input electric field of the interferometer is E_0, the beamsplitter will transmit $E_0/2$ down the x-arm and reflect $E_0/2$ down the y-arm. By setting the beamsplitter to be the origin, two beams traveling down their respective arms will have gathered a phase ϕ_i. Then, upon reflecting off the end mirrors and returning to the beamsplitter, each of the electric fields can be described by these equations

$$\begin{aligned} E_x &= \frac{iE_0}{2}e^{2i\phi_x} \\ E_y &= \frac{iE_0}{2}e^{2i\phi_y} \end{aligned} \tag{1.29}$$

Since the electromagnetic waves are linear, the resultant sum of waves at the output will be $E_{out} = E_x + E_y$. A photodiode (PD) is placed at the output (or antisymmetric) port to read out the integrated power which is related to the total electric field by

$$\begin{aligned} P_{AS} &= \int_{Area} I \, \mathrm{d}A \\ &= \int_{Area} |E_x + E_y|^2 \, \mathrm{d}A \\ &= P_{in} \cos^2(\Delta\phi) \end{aligned} \tag{1.30}$$

where $\Delta\phi = \phi_x - \phi_y = k_x L_x - k_y L_y$ and P_{in} is the input power. By using equation 1.18 and the common (or average) arm length $L_+ = \frac{L_x + L_y}{2}$, the power due to a differential phase shift is

$$P_{AS} \approx P_{in} \, (1 - 2\Delta\phi) = P_{in} \, (1 - 2kL_+ h_+) \tag{1.31}$$

There is a large DC term that is dependent on the input power and generally, it is very difficult to measure small changes in a large signal. So the next obvious method would be to shift the arms such that the output port is operating on a dark fringe, normally this is called a null-point operation. However, there are difficulties associated with this method as well. Consider shifting the phase of equation 1.30 by $\pi/2$, which would result in

$$P_{AS}|_{null} = P_{in} \sin^2(\Delta\phi) \approx P_{in} \, (kL_+ h_+)^2 \tag{1.32}$$

This results in a second-order dependence on a gravitational-wave signal that is already approximated to be very small. So a good solution to the issue of how to *read* out a gravitational wave signal can be solved using radio frequency (RF) detection methods. Consider changing the interferometer input by adding an electro-optical modulator (EOM) to sinusoidally modulate the laser frequency and expanding to first order using the Bessel functions,

$$\begin{aligned} E_{in} &= E_0 e^{i(wt + \beta \cos(\Omega t))} \\ &\approx E_0 e^{iwt} [J_0(\beta) + J_1(\beta) e^{i\Omega t} + J_1(\beta) e^{-i\Omega t}] \\ &= E_{C,in} + E_{SB+,in} + E_{SB-,in} \end{aligned} \tag{1.33}$$

where Ω and β are the modulation frequency and depth, respectively. The first term is commonly called the carrier field whereas the second and third terms are referred to as the (upper or lower) sidebands. Because there are multiple electric fields, it is useful to define an optical transfer function which transforms the interferometer's input fields to its output,

$$E_{out} = E_C + E_{SB+} + E_{SB-} = \begin{pmatrix} t_C \\ t_{SB+} \\ t_{SB-} \end{pmatrix} \begin{pmatrix} E_{C,in} & E_{SB+,in} & E_{SB-,in} \end{pmatrix} \tag{1.34}$$

The carrier transfer function, t_C has already been calculated by equations 1.30 - 1.32 and the sideband transfer functions are not much different.

$$t_{SB\pm} = r_{x,\pm} e^{i\phi_{\pm,x}} - r_{y,\pm} e^{i\phi_{\pm,y}} \tag{1.35}$$

where $\phi_{\pm,i} = (k \pm k_\Omega)\,\ell_i = (\frac{w+\Omega}{c})\ell_i$. In the current example, the sidebands and carrier fields reflect off the end mirrors identically, however, this will not be true in general when dealing with resonators that are highly frequency dependent. Plugging equation 1.35 into 1.34, the output electric field becomes

$$E_{out} = ie^{iwt}[J_0(\beta)k\ell_+ h_+ \; + \; J_1(\beta)\sin(k\Delta\ell + k_\Omega\Delta\ell)(e^{i\Omega t} + e^{-i\Omega t})] \tag{1.36}$$

By choosing the carrier signal to be on the dark fringe, $k\Delta\ell = \pi/2$ but the sidebands to be slightly off the null (Figure []) and hence leak into the anti-symmetric port, the electric field reduces to

$$E_{out} = ie^{iwt}[J_0(\beta)k\ell_+ h_+ \; + \; J_1(\beta)\sin(k_\Omega\Delta\ell)(e^{i\Omega t} + e^{-i\Omega t})] \tag{1.37}$$

Recall that the intensity is equal to the electric field squared,

$$\begin{aligned} I = |E_{out}|^2 = & |E_C|^2 + |E_{SB+}|^2 + |E_{SB-}|^2 \\ & + 2\mathbf{Re}\{\; E_{SB+}E^*_{SB-}e^{2i\Omega t} \;\} \\ & + 2\mathbf{Re}\{\; (E_C E^*_{SB-} + E_{SB+}E^*_C)e^{i\Omega t} \;\} \end{aligned} \tag{1.38}$$

The last term is referred to as the *beat note* between the carrier signal and the sidebands. It is possible to extract the term at the modulation frequency using a mixer which is an analog device that outputs the product of two inputs. Usually, the same oscillator that was used to modulate the input beam can be one of the mixer inputs, $\cos(\Omega t)$, so that the demodulated signal is

$$\begin{aligned} I_{Demod} &\propto \left[4\pi J_0(\beta)J_1(\beta)\frac{\ell}{\lambda}\sin(k_\Omega\Delta\ell)\; h_+\right]\left[\cos(\Omega t)\sin(\Omega t + \phi_{Demod})\right] \\ &= \left[4\pi J_0(\beta)J_1(\beta)\frac{\ell}{\lambda}\sin(k_\Omega\Delta\ell)\; h_+\right]\left[\sin(\phi_{Demod}) + \sin(2\Omega t + \phi_{Demod})\right] \end{aligned} \tag{1.39}$$

where ϕ_{Demod} is the phase that can be set by the user in order to account for extra phase shifts (ie. longer cables). After the mixer, there will be signals at DC, Ω, 2Ω and so on. However, the part that is linear in the gravitational wave amplitude will be at DC so a low-pass filter will allow the final signal to dominate:

$$S = 4\pi J_0(\beta) J_1(\beta) \frac{\ell}{\lambda} \sin(k_\Omega \Delta\ell) \sin(\phi_{Demod})\, h_+ \tag{1.40}$$

This shows that a RF detection technique will be linear in GW signal with no large DC offset. Setting the carrier on a null point means $\Delta\ell = \frac{k_\Omega}{k}\frac{\pi}{2}$ and allows the designer to optimize the Schnupp asymmetry length to get the best signal for some modulation frequency. This type of readout scheme was used in Enhanced LIGO and is called heterodyne detection, where the sideband fields are produced by an EOM and its efficacy depends on the local oscillator's stability [16]. In contrast, the Advanced LIGO scheme uses a homodyne detection [21] method called "DC-Readout". Here the oscillator field is produced by slightly offsetting the arms away from the dark fringe and letting a small amount of carrier light through the antisymmetric port. A gravitational wave will induce sidebands on the carrier and this will allow the same mathematics as above to achieve a linear signal in gravitational wave strain. This method benefits from naturally being co-aligned and mode matched with the signal field. All techniques follow the same logic of beating the field containing useful information with a reference field to extrapolate a linear signal but the differences come from technical noise such as laser intensity fluctuations and effective quantum noise.

[3]

1.2.2 Fabry-Perot Cavities

There are two ways to improve the LIGO detectors: one is to increase the response from gravitational waves and the other is to decrease the noise contributions. From equation 1.18, the gravitational wave signal is proportional to the optical path length that the photon travels, which means the most straightforward method of increasing the sensitivity is to make the arms as long as possible (up to the null point described by equation 1.19. Generally, there were two methods to do this: a Herriott delay line or a Fabry-Perot resonator, the differences between each method is shown in Figure[]. At the time of writing this thesis, all modern gravitational wave detectors use the latter method.

A Fabry-Perot cavity is an optical system comprised of two or more partially transmitting mirrors with one laser input. To create a resonator, the user must design a system such that once the laser has made one round trip around the optics, it is the same shape and size as when it started. Conceptually, this may seem simple but in practice, controlling and sensing any optical cavity comes with a few challenges.

To start understanding the longitudinal degree of freedom, consider a two mirror system in Figure []

which is separated by a length L with reflection and transmission coefficients: r_1, t_1, r_2, t_2. Starting with a plane wave at the input mirror with amplitude E_0, the beam will enter the cavity add on top of each other such that the reflected field [27] is

$$E_{REFL} = r_{FP}E_0 = \left(-r_1 + \frac{t_1^2 r_2 e^{-i2kL}}{1 - r_1 r_2 e^{-i2kL}} \right) E_0, \tag{1.41}$$

the transmission field is

$$E_{TRAN} = t_{FP}E_0 = \left(\frac{t_1 t_2 e^{ikL}}{1 - r_1 r_2 e^{-i2kL}} \right) E_0 \tag{1.42}$$

the circulating field is

$$E_{CIRC} = c_{FP}E_0 = \left(\frac{t_1}{1 - r_1 r_2 e^{-2ikL}} \right) E_0 \tag{1.43}$$

The fields become resonant when the cavity length is $L = n\lambda/2$ and the circulating coefficient in the cavity is maximized such that the gain is

$$\text{Gain} = c_{FP}^2|_{\text{resonating}} = \left(\frac{t_1}{1 - r_1 r_2} \right)^2 \tag{1.44}$$

Depending on the relative reflection coefficients of the input and output mirrors, the fields on resonance will be slightly different Figure[].

Frequency response of a single FP (plot):

Notice that the circulating power dependent on the cavity length and laser frequency so one might naively determine that modulating the two parameters independently cause the same effect. However, when both are changing by large amounts, they are related by a frequency dependent transfer function

$$C(s)\frac{\Delta w}{w} = -\frac{\Delta L}{L} \tag{1.45}$$

where $C(s) = \frac{1-e^{-2sL/c}}{2sL/c}$ in the Laplace domain. Only when the cavity is on or near resonance, then the frequency and length variations are related by $\frac{\Delta w}{w} = -\frac{\Delta L}{L}$.

While sweeping through either laser frequency or cavity length and measuring the reflected (or transmitted) fields, there are features of the power spectrum which relate directly to the cavity's physical properties:

Finesse, or the line width of the resonant peak, function of r_1 and r_2:

$$\mathbb{F} = \frac{\pi\sqrt{r_1 r_2}}{1 - r_1 r_2} \tag{1.46}$$

Storage Time :

$$\tau_s = \frac{L}{c\pi}\,\mathbb{F} \tag{1.47}$$

Cavity Pole:

$$f_{pole} = \frac{1}{4\pi\tau_s} \tag{1.48}$$

Free Spectral Range:

$$f_{FSR} = \frac{c}{2L} \tag{1.49}$$

Stability: In order to prove that the Fabry Perot is stable, it is useful to introduce the matrices that describe a periodic optical system which is explicitly derived in appendix A. A Fabry Perot cavity that is separated by distance L with spherical mirrors that have radii of curvature R_1 and R_2 will need to satisfy

$$0 \geq \left(1+\frac{L}{R_1}\right)\left(1+\frac{L}{R_2}\right) \geq 1 \tag{1.50}$$

in order to be geometrically stable.

Power circulating as a function of our defined parameters slightly off resonance by a length of δL:

$$P_{cav} = |c_{FP}|^2 = \frac{Gain}{1+\left(\frac{2\mathbb{F}}{\pi}\right)^2 \sin^2(k\delta L)} \tag{1.51}$$

Locking a Fabry Perot Cavity

This is described everywhere, reference [3], Drever. Just give the highlights.

Described above are the theoretical constructs of a FP cavity, but the question remains, how does one practically construct a resonant optical cavity? The answer comes from using a heterodyne sensing scheme similar to the one described in Section 1.2.1. Except the optical system is not a Michelson interferometer but rather a two mirror cavity, however, heterodyne detection can apply to a number of different geometries such as triangular or bow-tie cavities shown in Figure []. All of which are used in LIGO for various reasons.

Starting with an input laser and EOM (electro-optical modulator) that imparts upper and lower sidebands at a modulation frequency Ω, the user injects three beams into the optical system exactly as in Equation 1.33. When placing a photodetector on the reflection port, one should see the cavity's effect on each of the three electric fields.

$$E_{FP,out} = E_C + E_{SB+} + E_{SB-} = \begin{pmatrix} r_C \\ r_{SB+} \\ r_{SB-} \end{pmatrix} \begin{pmatrix} E_{C,in} & E_{SB+,in} & E_{SB-,in} \end{pmatrix} \tag{1.52}$$

where the reflection coefficients follow equation 1.41. Because sidebands are frequency shifted, they will effectively *see* a different cavity than the carrier and the total phase accumulated between the fields will be different. To make a good reference for the resonant carrier beam, the modulation frequency, k_Ω, is chosen such that sidebands are be anti-resonant in the cavity.

$$r_C = -r_1 + \frac{t_1^2 r_2 e^{-i2kL}}{1 - r_1 r_2 e^{-i2kL}} \tag{1.53}$$

and

$$r_{SB\pm} = -r_1 + \frac{t_1^2 r_2 e^{-i2(k+k_\Omega)L}}{1 - r_1 r_2 e^{-i2(k+k_\Omega)L}} \tag{1.54}$$

The formalism to read out the error signal is the shown in Equation 1.38. Using a photodetector in reflection, the error signal will be linearly proportional to the laser frequency and cavity length [3].

$$\text{Error Signal} \propto \frac{L\mathbb{F}}{\lambda}\left[\frac{\delta w}{w} + \frac{\delta L}{L}\right] \tag{1.55}$$

Application to LIGO

Frequency response of a FP Michaelson: Just write it down. References:[[4]]

1.2.3 Power-Recycled Fabry-Perot Interferometers

Power Recycling If the interferometer is operating such that the 4 km arms are exactly different in arms a pi over two times the wavelength, then the intensity of the light at antisymmetric port will be close to null. This means the power from the arms will reflect back towards the input laser. Ref[] shows the effect of adding a partially reflecting mirror to increase the optical gain of the Michaelson for the sidebands and carrier fields.

References: [Meers, Kiwamu]

1.2.4 Dual-Recycled Fabry-Perot Interferometers

One of the biggest changes made between initial and advanced LIGO was the addition of a signal recycling mirror at the antisymmetric port shown in Figure []. This extra mirror allows an extra degree of freedom in shaping the sensitivity curve such.

Figure comparing the three cases of increased sensitivity.

References: [Kiwamu]

1.2.5 Fundamental Noise Sources

Ref: Evan Hall, GWINC The proceeding sections describe ways to increase the response of LIGO to gravitational waves; equally as important is the science of characterizing and reducing the noise contributions from everything else.

Noise budget:

Quantum Noise

One fundamental noise source that is limiting LIGO's sensitivity comes from the fluctuations of quantum vacuum entering the anti-symmetric port and coupling to the input laser. A quantum mechanical description of an interferometer was constructed by Caves [7][8] [9], where he used electric field operators to show that vacuum fluctuations are the cause of radiation pressure and shot noise in an interferometer.

Quantum states: It is well known in physics [31] [19] that a solution to the quantum harmonic oscillator in the energy eigenbasis employs the annihilation and creation operators, $\hat{a}^\dagger$ and $\hat{a}$, to factorize the Hamiltonian

$$\hat{H} = \hbar w(\hat{N} + 1/2) \tag{1.56}$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$ is the number operator. When using this formalism to create a coherent electromagnetic field, it is useful to define a unitary operator that displaces the vacuum state [17]:

$$\hat{D} \equiv \hat{D}(\alpha) \equiv \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{\alpha^*\hat{a}} \tag{1.57}$$

$\hat{D}(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha)$

$$\begin{aligned} \hat{D}^\dagger \, \hat{a} \, \hat{D} &= \hat{a} + \alpha \\ \hat{D}^\dagger \, \hat{a}^\dagger \, \hat{D} &= \hat{a}^\dagger + \alpha^* \end{aligned} \tag{1.58}$$

$$|\alpha\rangle = \hat{D} \, |0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} \, |0\rangle \tag{1.59}$$

Radiation Pressure

One might naively think that power fluctuations in the laser cause radiation pressure effects on the test masses which will result in noise. However, if the 50/50 beamsplitter is perfect, then the momentum transfer to each test mass will be a common length change and will not vary the intensity at the antisymmetric port (or the symmetric port for that matter).

The concept of quantum radiation pressure arises from considering plane wave waves entering the interferometer from both the symmetric and anti-symmetric ports. This method is similar to the input-output methods of section 1.2.1, however, the difference being that the beamsplitter will couple the electric fields from the input laser and quantum vacuum.

To start, consider the electric fields combining at the beamsplitter from both ports to strike the mirrors, respectively,

$$E_x = \frac{1}{\sqrt{2}}\left[iE_0 + E_{AS,in}\right] \tag{1.60a}$$

$$E_y = \frac{1}{\sqrt{2}}\left[E_0 + iE_{AS,in}\right] \tag{1.60b}$$

The intensities hitting each test mass will be

$$|E_x|^2 = \frac{1}{2}\left[|E_0|^2 + |E_{AS,in}|^2 + i(E_0 E^*_{AS,in} - E^*_0 E_{AS,in})\right] \tag{1.61a}$$

$$|E_y|^2 = \frac{1}{2}\left[|E_0|^2 + |E_{AS,in}|^2 - i(E_0 E^*_{AS,in} - E^*_0 E_{AS,in})\right]` \tag{1.61b}$$

The differential momentum transfer between the two masses will be equal to the differences in intensities:

$$\begin{aligned} \mathbf{P} &= \frac{2\hbar\omega}{c}\left(|E_x|^2 - |E_y|^2\right) \\ &= \frac{2\hbar\omega}{c}\left(E_0 E^*_{AS,in} - E^*_0 E_{AS,in}\right) \\ \Rightarrow \hat{\mathbf{P}} &= \frac{2\hbar\omega}{c}\left(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1\right) \end{aligned} \tag{1.62}$$

The last part of equation 1.62 replaces the classical electric fields with creation and annihilation operators for the symmetric input mode ($\hat{a}_1^\dagger$, $\hat{a}_1$) and antisymmetric input mode ($\hat{a}_2^\dagger$, $\hat{a}_2$) modes. Recall that there is a well established convention denoting the wave function of the two dimensional modes of the quantum harmonic oscillator [17], $|\alpha, \beta\rangle$, where α denotes the input symmetric mode and β refers to the input antisymmetric mode.

$$|\alpha, 0\rangle = \hat{D}_1(\alpha)\,|0, 0\rangle \tag{1.63}$$

The interesting results arising from this formulation is the expectation value

$$\langle\hat{\mathbf{P}}\rangle = \langle\alpha, 0|\,\hat{\mathbf{P}}\,|\alpha, 0\rangle = 0 \tag{1.64}$$

And the variance

$$\begin{aligned}
\Delta\mathbf{P}^2 &= \langle\hat{\mathbf{P}}^2\rangle - \langle\hat{\mathbf{P}}\rangle^2 \\
&= \left(\frac{2\hbar\omega}{c}\right)^2 \langle\alpha, 0|\, \hat{a}_1^\dagger\hat{a}_2\hat{a}_1\hat{a}_2^\dagger + \hat{a}_2^\dagger\hat{a}_1\hat{a}_2\hat{a}_1^\dagger - \hat{a}_2^\dagger\hat{a}_1\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2\hat{a}_2\hat{a}_1^\dagger\, |\alpha, 0\rangle \\
&= \left(\frac{2\hbar\omega}{c}\right)^2 |\alpha|^2
\end{aligned} \tag{1.65}$$

For some time interval, input laser power consisting of $\langle N\rangle = |\alpha|^2$ photons for a time interval ΔT is,

$$P_{in} = \frac{\hbar\omega}{\Delta T}|\alpha|^2 \tag{1.66}$$

To solve for the amplitude spectral density of the displacement, Newton's second law can be applied in the frequency domain

$$M(2\pi f)^2\tilde{x}(f) = \tilde{F}(f) = \frac{\Delta\mathbf{P}}{\Delta T} \tag{1.67}$$

The force spectrum is white because the impacting photons arrive randomly.

Solving for the displacement,

$$\tilde{x}_{RP}(f) = \sqrt{\frac{\hbar\omega}{\Delta T}P_{in}}\frac{1}{2Mc(\pi f)^2} \tag{1.68}$$

The noise spectral density for $M = 40kg$, $P_{in} = 125\mathrm{W}$, and $\lambda = 1064\mathrm{nm}$

$$\tilde{h}_{RP}(f) = 2.04 \times 10^{-20}\frac{1}{f^2}\left[\frac{\mathrm{Strain}}{\sqrt{\mathrm{Hz}}}\right] \tag{1.69}$$

Shot Noise Another way that quantum fluctuations can vary the antisymmetric port output is by adding phase noise. Imagine holding the test masses rigidly such that the only effects on the output light is due to a phase change in the laser light in the arms. By using the same formulation for radiation pressure but propagating the fields back to the beam splitter, equations 1.60 will have extra phase,

$$E_{x,out} = \frac{1}{\sqrt{2}}\Big[iE_0 + E_{AS,in}\Big]e^{-i2kL_x} \tag{1.70a}$$

$$E_{y,out} = \frac{1}{\sqrt{2}}\Big[E_0 + iE_{AS,in}\Big]e^{-i2kL_y} \tag{1.70b}$$

Antisymmetric port output electric field is

$$\begin{aligned}
E_{AS,out} &= \frac{1}{\sqrt{2}}(E_{x,out} + iE_{y,out}) \\
&= ie^{-ikL_x - ikL_y}\big[\cos(\Delta\phi)E_0 - \sin(\Delta\phi)E_{AS,in}\big]
\end{aligned} \tag{1.71}$$

where $\Delta\phi = k(L_x - Ly)$ Output intensity,

$$\begin{aligned}
\mathbf{I}_{AS,out} &= |E_{AS,out}|^2 \\
&= \Big[\cos^2(\Delta\phi)|E_0|^2 + \sin^2(\Delta\phi)|E_{AS,in}|^2 - \sin(\Delta\phi)\cos(\Delta\phi)\big[E_0 E^*_{AS,in} + E^*_0 E_{AS,in}\big]\Big] \qquad (1.72)\\
\Rightarrow \hat{\mathbf{I}}_{AS,out} &= \Big[\cos^2(\Delta\phi)a_1^\dagger a_1 + \sin^2(\Delta\phi)a_2^\dagger a_2 - \sin(\Delta\phi)\cos(\Delta\phi)\big[a_1^\dagger a_2 + a_2^\dagger a_1\big]\Big]
\end{aligned}$$

The expectation value for the intensity using an input coherent laser with α and quantum vacuum input at the antisymmetric port is

$$\begin{aligned}
\langle\hat{\mathbf{I}}\rangle &= \langle\alpha, 0|\,\hat{\mathbf{I}}_{AS,out}\,|\alpha, 0\rangle \\
&= \cos^2(\Delta\phi)|\alpha|^2
\end{aligned} \tag{1.73}$$

which matches the classical description of a Michelson output.

Following the same methods to calculate the radiation pressure, photon number variance is

$$\begin{aligned}
\Delta\mathbf{I} &= \sqrt{\langle\hat{\mathbf{I}}^2\rangle - \langle\hat{\mathbf{I}}\rangle^2)} \\
&= |\alpha|\big[\cos^2(\Delta\phi)\;\big]
\end{aligned} \tag{1.74}$$

$$I = N\cos^2(\Delta\phi) \tag{1.75}$$

$$\frac{\partial I}{\partial\phi} = -2|\alpha|^2\cos(\Delta\phi)\sin(\Delta\phi) \tag{1.76}$$

$$\delta\phi = \sqrt{\frac{\hbar\omega}{P_{in}}}\cot(\Delta\phi) \tag{1.77}$$

Here $\delta\phi$ is the microscopic change in phase due to shot noise, whereas $\Delta\phi$ is the DC offset in the arm lengths to begin with. So in general, the shot noise contribution is dependent on the amount of light present at the anti-symmetric port and this will vary depending on what type of read out that is implemented.

$$\tilde{h}_{SN}(f) = 8.2\times10^{-22}\left[\frac{\text{Strain}}{\sqrt{\text{Hz}}}\right] \tag{1.78}$$

Unsurprisingly, the variance is proportional to the amount of photons present and the phase difference between the interferometer arms. Interestingly, there are a few subtleties associated with measuring the noise, it is assumed here that the interferometer readout is a simple photodetector located at the antisymmetric port. However, is it possible to measure the light at two different phases (90 degrees apart) and subtract the results. Also, in section 1.2.1, there is a freedom to choose the readout scheme of the interferometer (heterodyne or homodyne) which will also affect the overall quantum noise.

Losses!?!?!?!?!?!?

Seismic Noise

Seismic noise will be the low frequency barrier for all terrestrial gravitational-wave detectors. By using seismic isolation platforms and quadruple suspensions, the noise contribution can be attenuated for frequencies larger than 1Hz.

Compare low and high noise microseisms [5]

Thermal Noise

Thermal Noise [[29]]

Suspension Thermal Noise [18]

Substrate Thermal Noise [[27]]

Coating Thermal Noise [[20]]

Thermal-Optical Noise[[14]]

Newtonian Noise

Although there are some noise sources that can be reduced using increasingly complicated techniques such as various levels of seismic isolation or quantum nondemolition devices as shown above. [[28] and [22]]

Newtonian noise caused by fluctuating gravitational fields from the wave motion of the ground is one that cannot easily be reduced but possibly be measured and fed-forward or subtracted.[[12]]

1.3 Squeezed States of Light

Virtually all undergraduate quantum mechanics textbooks include a section on the harmonic oscillator[Shankar, Sakura]. An interesting result when solving the Schrodinger equation using creation ($\hat{a^\dagger}$) and annihilation $\hat{a}$ operators is the existence of a non-zero energy ground state.

This effect leads to Quantum Noise, which is a fundamental source that can be improved by modifying the quantum vacuum using correlated photons and injecting the states of light into the antisymmetric port of the interferometer. Within the LIGO community, this procedure of modifying quantum vacuum is called squeezing. Caves analytically derived the effects of quantum noise on the interferometer as well as the improvement due to

Losses!?!?!?!?!?!? References: [Caves, Dwyer, Kwee, Miao]

Chapter 2

The Fundamentals of Mode-Matching

Theory section of modematching. Gaussian beams, define relevant quantities, Gouy Phase!

In order to properly explain the effects of electromagnetic waves in the LIGO interferometers, it is useful to review the

References: [Kognelik and Li]

2.1 Gaussian Beam Optics

When dealing with length sensing degrees of freedoms such as section 1.2.2, the simple plane wave approximation is sufficient in describing the dynamics, however, when trying to understand the misalignment and mode-mismatch signals, it is necessary to incorporate Gaussian beams and associated their higher order modes.

Consider the famous Maxwell's equations in vacuum:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu \, \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon} \end{aligned} , \tag{2.1}$$

Concentrating on the electric field in vacuum, we arrive at the Helmholtz Equation

$$(\nabla^2 + k^2)\mathbf{U}(\mathbf{r}, t) = 0 \tag{2.2}$$

where $k = \frac{2\pi\nu}{c}$ is the wave number and $\mathbf{U}(\mathbf{r}, t)$ is the complex amplitude which can describe either the electric or magnetic fields.

It is possible to express the solution to equation 2.2 as a plane wave with a modulated complex envelope

$$\mathbf{U}(\mathbf{r}) = \mathbf{A}(\mathbf{r})e^{-ikz)} \tag{2.3}$$

By imposing the constraints which force the envelope to vary slowly with respect to the z-axis within the distance of one wavelength $\lambda = 2\pi/k$,

$$\left|\frac{\partial^2 \mathbf{A}}{\partial z^2}\right| \ll \left|k\frac{}{\partial}\mathbf{A}\partial z\right| \tag{2.4a}$$

$$\left|\frac{\partial^2 \mathbf{A}}{\partial z^2}\right| \ll \left|k\frac{\partial^2 \mathbf{A}}{\partial x^2}\right| \tag{2.4b}$$

$$\left|\frac{\partial^2 \mathbf{A}}{\partial z^2}\right| \ll \left|k\frac{\partial^2 \mathbf{A}}{\partial y^2}\right| \tag{2.4c}$$

the partial differential equation which arises is called the Paraxial Helmholtz Equation:

$$\nabla_T^2 A(r) - i2k\frac{\partial A(r)}{\partial z} = 0 \tag{2.5}$$

where $\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian. A simple solution for equation 2.5 is the complex paraboloidal wave

$$A(\mathbf{r}) = \frac{A_0}{q(z)}e^{\frac{-ikr^2}{2q(z)}}, \quad q(z) = z + iz_0 \tag{2.6}$$

where z_0 is the Rayleigh range and is directly proportional to the square of the waist size. In order to separate the amplitude and phase portions of the wave, it is useful to rewrite $q(z)$ as

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)} \tag{2.7}$$

Plugging equation 2.7 into 2.6 leads directly to the complex amplitude for a Gaussian Beam

$$U(r) = A_0\frac{W_0}{W(z)}e^{-\frac{r^2}{W^2(z)}}e^{-ikz-ik\frac{r^2}{2R(z)}+i\phi(z)} \tag{2.8}$$

where

$$W(z) = W_0\sqrt{1+\left(\frac{z}{z_0}\right)^2} \tag{2.9a}$$

$$R(z) = z\left[1+\left(\frac{z}{z_0}\right)^2\right] \tag{2.9b}$$

$$\phi(z) = \tan^{-1}\left(\frac{z}{z_0}\right) \tag{2.9c}$$

$$W_0 = \sqrt{\frac{\lambda z_0}{\pi}} \tag{2.9d}$$

Gouy Phase

The Gouy phase is - Gouy Phase + graph

Intensity

- Intensity and Power

Hermite-Gauss Modes

The fundamental Gaussian beam is not the only solution which can be used to solve equation 2.5. In fact, there exists a complete set of solutions that can solve the paraxial Helmholtz Equation in rectangular coordinates, which are referred to as the Hermite Gauss modes

$$\begin{aligned} U_{mn}(x,y,z) = A_{mn}\left[\frac{W_0}{W(z)}\right]\mathbb{G}_m\left(\frac{\sqrt{2}x}{W(z)}\right)\mathbb{G}_n\left(\frac{\sqrt{2}y}{W(z)}\right) \\ \times \exp\left\{-ikz - \frac{ik(x^2+y^2)}{2R(z)} + i(m+n+1)\phi(z)\right\} \end{aligned} \tag{2.10}$$

where,

$$\mathbb{G}_(u) = \mathbb{H}(u)\exp(-u^2/2) \tag{2.11}$$

and $\mathbb{H}(u)$ are the well known Hermite polynomials. It is important to mention that the Gouy phase of the complex amplitude is different than the fundamental Gaussian beam by a factor of $(m+n+1)$. Also, the intensity distribution of these higher order modes are much different. Both of these facts will become extremely important in the following wavefront sensing discussion.

It is useful to normalize the Hermite-Gauss modes with respect to the overall power, which are derived in Appendix[].

How higher order modes show up in resonators, Gouy Phase, HOM spacing.

Laguerre Modes

Another complete set of alternative solutions to equation 2.5 exists which are called the Laguerre-Gauss modes

$$\begin{aligned} V_{\mu\nu}(\rho,\theta,z) = A_{\mu\nu}\left[\frac{W_0}{W(z)}\right]\mathbb{L}_\nu^\mu\left(\frac{\sqrt{2}x}{W(z)}\right) \\ \times \exp\left\{-ikz - \frac{ik\rho^2}{2R(z)} + i(\mu+2\nu+1)\phi(z)\right\} \end{aligned} \tag{2.12}$$

where $\mathbb{L}_\nu^\mu\left(\frac{\sqrt{2}x}{W(z)}\right)$ is the Laguerre polynomial function. Both equations 2.10 and 2.12 are able to fully describe any complex electromagnetic amplitude; and because they both form complete sets, there is a rotation which can map from one basis to the other [ref Bond and Biejergasern]

$$U_{\mu\nu}^{LG}(x,y,z) = \sum_k^N i^k b(n,m,k) U_{N-k,k}^{HG}(x,y,z) \tag{2.13}$$

where

$$b(n,m,k) = \sqrt{\left(\frac{(N-k)!k!}{2^N n!m!}\right)}\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}[(1-t)^m(1+t)^m]|_{t=0} \tag{2.14}$$

2.1.1 Misalignment and Higher Order Modes

Morrison and Anderson derived a simplistic way of how small misalignments and mismodematched cavities can couple the fundamental Gaussian beam into various higher order modes. This is done by taking a linear cavity and using its perfectly matched Gaussian beam as a reference, and then varying the input electric field with small perturbations and expanding in terms of the cavity modes. As long as the mismatches are small, it is possible to consider only the first few terms of the expansion which have gained power from the fundamental mode.

Consider the first three modes of equation 2.10 in one dimension and normalized to set the total optical power to unity (derived in Appendix[]):

$$\begin{aligned} U_0(r) &= \left(\frac{2}{\pi w^2(z)}\right)^{1/4} e^{-r^2/w^2(z)} \\ U_1(r) &= \left(\frac{2}{\pi w^2(z)}\right)^{1/4} \frac{2r}{w(z)} \quad e^{-r^2/w^2(z)} \qquad , \\ U_2(r) &= \left(\frac{2}{\pi w^2(z)}\right)^{1/4} \frac{1}{\sqrt{2}}\left(\frac{4r^2}{w^2(z)} - 1\right)e^{-r^2/w^2(z)} \end{aligned} \tag{2.15}$$

Beam Axis Tilted

If the input beam into an optical cavity is tilted by an angle α with respect to the nominal cavity axis as shown in Figure [], the wave front of the input beam will have an extra phase propagation relative to the cavity that is approximately proportional to $e^{ik\alpha r}$. By implementing the small angle approximation, which is valid if the misalignment is much smaller than the divergence angle of the fundamental mode $k\alpha r << 1$, the resultant input beam is

$$\Psi \approx U_0(r)e^{ik\alpha r} \approx U_0(r)(1 + ik\alpha r) = U_0(r) + \frac{ik\alpha w(z)}{\sqrt{2\pi}}U_1(r) \tag{2.16}$$

Here the factor associated with the first higher order mode is complex, indicating there is a 90 degree phase difference between the fundamental and off axis mode.

Beam Axis Displaced

If the input beam is displaced in the transverse direction by a quantity Δr, the resultant waveform will be

$$\begin{aligned}
\Psi &= U_0(r + \Delta r) \\
&= \left(\frac{2}{\pi w^2(z)}\right)^{1/4} e^{-(r+\Delta r)^2/w^2(z)} \\
&= \left(\frac{2}{\pi w^2(z)}\right)^{1/4} e^{-(r^2+2r\Delta r+\Delta r^2)/w^2(z)} \\
&\approx \left(\frac{2}{\pi w^2(z)}\right)^{1/4} e^{-r^2/w^2(z)} e^{-2r\Delta r/w^2(z)} \\
&\approx \left(\frac{2}{\pi w^2(z)}\right)^{1/4} e^{-r^2/w^2(z)} \left(1 - \frac{2r\Delta r}{w^2(z)}\right) \\
&= \left(U_0(r) - \sqrt{\frac{2}{\pi}}\frac{\Delta r}{w(z)}U_1(r)\right)
\end{aligned} \tag{2.17}$$

Similarly to a tilted input beam axis, the displaced beam axis couples power to the first higher order mode, however, the latter does not have a 90 degree phase difference previously seen in the former. This point is of extreme importance when trying to discern between the two effects as shown in Section []. Although comparing the two cases in Figure, one can already seen the difference between the wavefronts in the near field, $z << z_R$, and the far field, $z >> z_R$.

In the near field, there is no phase difference due to a displaced beam, but there is one for a tilted beam. Conversely, in the far field, there is no phase difference due to a tilted beam, but there is one from a displaced beam. In order to implement a closed loop feedback system, the wavefront sensors discussed in Section 2.2 will use this precise logic to extract an error signal.

2.1.2 Mode Mismatch and Higher Order Modes

Waist Size Shifted

By considering the effect of evaluating the fundamental mode at the waist position, $z = 0$, but changing the waist size by a small amount ϵ, it is possible to see coupling into higher order modes by expanding to first order.

$$\begin{aligned}
\Psi &= U_0\big(r, w(z) = w_0/(1+\epsilon)\big) \\
&= \left(\frac{2}{\pi w_0^2}\right)^{1/4} \sqrt{1+\epsilon} \quad e^{-r^2(1+\epsilon)^2/w_0^2} \\
&\approx \left(\frac{2}{\pi w_0^2}\right)^{1/4} (1+\epsilon/2) \quad e^{-r^2/w_0^2} \quad e^{-2r^2\epsilon/w_0^2} \\
&\approx \left(\frac{2}{\pi w_0^2}\right)^{1/4} (1+\epsilon/2) \quad e^{-r^2/w_0^2} \quad (1 - 2r^2\epsilon/w_0^2) \\
&\approx \left(\frac{2}{\pi w_0^2}\right)^{1/4} \left(1 + 2\epsilon\left(\frac{1}{4} - \frac{r^2}{w_0^2}\right)\right) \quad e^{-r^2/w_0^2} \\
&= U_0(r) \; - \; \frac{\epsilon}{\sqrt{2}} U_2(r)
\end{aligned} \tag{2.18}$$

Changing the waist size by a small amount will couple the fundamental mode to the in-phase second order Hermite Gauss mode.

Waist Position Shifted

To repeat the process from above with a waist position shift, it is useful to start with a more general equation that includes the phase that is gained from including the radius of curvature,

$$\Psi = \left(\frac{2}{\pi w^2(z)}\right)^{1/4} \quad e^{-r^2/w^2(z)} \quad e^{-ikr^2/2R(z)} \tag{2.19}$$

where $R(z)$ is from equation 2.9b. It is also useful to approximate the shift in waist position along the longitudinal axis is small compared to the Rayleigh range of the beam, $\Delta z << z_0$, which leaves the waist size approximately the same and the radius of curvature inversely proportional to the shift.

$$w^2(\Delta z) = w_0^2\left[1 + \left(\frac{\Delta z}{z_0}\right)^2\right] \approx w_0^2 \tag{2.20a}$$

$$R(\Delta z) = \Delta z\left(1 + \left(\frac{z_0}{\Delta z}\right)^2\right) \approx \frac{z_0^2}{\Delta z} \tag{2.20b}$$

Plugging the equations above into 2.19,

$$
\begin{aligned}
\Psi &\approx \left(\frac{2}{\pi w_0^2}\right)^{1/4} \quad e^{-r^2/w_0^2} \quad e^{-ikr^2\Delta z/2z_0^2} \\
&\approx \left(\frac{2}{\pi w_0^2}\right)^{1/4} \quad e^{-r^2/w_0^2} \quad \left(1 - \frac{ikr^2\Delta z}{2z_0^2}\right) \\
&= U_0(r) - \left(\frac{2}{\pi w_0^2}\right)^{1/4} e^{-r^2/w_0^2} \quad \frac{ikr^2\Delta z}{2z_0^2} \\
&= U_0(r) - i\frac{\Delta z}{2kw_0^2}\left(4U_2(r) + U_0(r)\right)
\end{aligned}
\tag{2.21}
$$

The equations above show that a fundamental Gaussian mode that is shifted in waist position will couple power to the second order Hermite Gauss mode. Although changes in the waist size or position couple power to the same mode, they differ by a 90 degrees in phase as denoted by the extra factor of i in the coupling coefficient. By recognizing the two effects are in different quadrature phases will allow a user to design a system to distinguish between the different types of physical couplings, this is shown in Section 2.2.

In order to be physically valid one would need to consider the full two dimensional space so that the equation would encapsulate the full transverse mode, however, the x and y components would follow the exact same derivation. On that point, it is important to note that only the mode mismatch couplings from either a varying waist position or size has higher order modes that are circularly symmetric.

2.2 Wavefront Sensing

Heterodyne detection via modal decomposition of the full electric field allows the use of wavefront sensors to extract an error signal from the optical system. Hefetz et.al Ref[Sigg and Nergis] created a formalism to describe the use of wavefront sensors by creating frequency sidebands which accumulate a different Gouy phase than the electric field at the carrier frequency when passed through the optical system. By observing the demodulated signal of the intensity, it is possible to obtain a linear signal that corresponds to a physical misalignment or mode mismatch.

Fundamentally, the purpose of wavefront sensing is to detect the content of higher order modes due to physical disturbances of the optical cavity (ie. mode mismatch or misalignment). In other words, it is examining the difference of basis sets between the incoming eigenmodes and the cavity eigenmodes.

Consider a general equation for an electric field which is a linear combination of all higher order modes of the complex amplitude

$$
E(x, y, z) = \sum_{m,n}^{\infty} a_{mn} U_{mn}(x, y, z) \tag{2.22}
$$

where $U_{mn}(x, y, z)$ are the eigenmodes described in equation 2.10 (or 2.12) and a_{mn} is the complex amplitude. It is also convenient in the following analysis to use vectors when describing the composition of the electric fields.

$$|E(x, y, z)\rangle = \begin{pmatrix} E_{00} \\ E_{01} \\ E_{10} \\ E_{20} \\ E_{02} \end{pmatrix} \tag{2.23}$$

When creating a theory that involves laser beams, it is useful to define operators that are important in describing physical situations. For example, laser beams propagate through space and pick up phase according to equation 2.10 which can be represented by the spatial propagation operator,

$$\hat{P}_{mn,kl} = \delta_{mn}\delta_{kl} \quad \exp[-ik(z_2 - z_1)]\exp[i(m + n + 1)\phi(z)] \tag{2.24}$$

However, it is useful to compare how the fundamental Gaussian mode propagates compared to the higher order modes,

$$\hat{\eta}_{\mu\nu} = \begin{pmatrix} e^{i\phi} & 0 & 0 & 0 & 0 \\ 0 & e^{2i\phi} & 0 & 0 & 0 \\ 0 & 0 & e^{2i\phi} & 0 & 0 \\ 0 & 0 & 0 & e^{3i\phi} & 0 \\ 0 & 0 & 0 & 0 & e^{3i\phi} \end{pmatrix} \tag{2.25}$$

From the above diagonal elements, it is clear that the higher order modes have an extra phase compared to the fundamental 00 mode, this effect will be extremely important on how an error signal can be derived from the optical system.

$$|E(x, y, z_2)\rangle = \hat{M}(x, y, z_1, z_2)\,|E(x, y, z_1)\rangle \tag{2.26}$$

where $\hat{M}(x, y, z_1, z_2)$ is the misalignment operator. Since we are using the paraxial approximation, the z-components of the misalignment operator are small so we can approximate $\hat{M}_(x, y, z_1, z_2) \approx \hat{M}_(x, y)$ and the expectation value is

$$M_{mn,kl} = \langle U_{mn}(x, y, z_1)|\, M(x, y)\, |U_{kl}(x, y, z_2)\rangle \tag{2.27}$$

where the product is an integral over the transverse space $\iint_{D(x,y)} \mathrm{d}x\mathrm{d}y$

$$\hat{\Theta}_{\mu\nu} = \begin{pmatrix} 1 & 2i\theta_x & 2i\theta_y & 0 & 0 \\ 2i\theta_x & 1 & 0 & 0 & 0 \\ 2i\theta_y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.28}$$

$$\hat{\mathbb{D}}_{\mu\nu} = \begin{pmatrix} 1 & \alpha_x/\omega_0 & \alpha_y/\omega_0 & 0 & 0 \\ \alpha_x/\omega_0 & 1 & 0 & 0 & 0 \\ \alpha_y/\omega_0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.29}$$

$$\hat{\mathbb{Z}}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \Delta z_x & \Delta z_y \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \Delta z_x & 0 & 0 & 1 & 0 \\ \Delta z_y & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.30}$$

where $\Delta z_{(x,y)} = \frac{i}{\sqrt{2}} \frac{\lambda b}{2\pi\omega_0}$

$$\hat{\mathbb{Z}}_{0,\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \Delta z_{0,x} & \Delta z_{0,y} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \Delta z_{0,x} & 0 & 0 & 1 & 0 \\ \Delta z_{0,y} & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.31}$$

where $\Delta z_{0,(x,y)} = \frac{1}{\sqrt{2}} \frac{\omega' - \omega_0}{\omega_0}$

Figure: Beat frequency between two modes.

Example: Fabry Perot Cavity

Example: Simple Michaelson

2.3 Effects of Mode-matching on Squeezing

Still not clear to me. In the most simple sense, a loss is akin to coupling quantum vacuum into a squeezed state. References: [Miao, Sheon, Kimble, Dwyer]

Chapter 3

Simulating Mode-Matching with Finesse

3.1 How it works

[24] Summary of how Finesse works (input output matrix), how it handles HOMs

3.2 Finesse Simulations

3.2.1 ALIGO Design with FC and Squeezer

3.2.2 Looking at just Modal Change

3.2.3 QM Limited Sensitivity

3.3 Results

* Signal recycling cavity mismatches

    * Mismatches before the OMC

    * Mismatch contour graph: Comparing all of ALIGO cavities

    * Optical Spring pops up at 7.4 Hz in the Signal-to-Darm TF, re-run with varying SRM Trans which should.

Chapter 4

Experimental Mode Matching Cavities at Syracuse

In conjunction with Sandoval et al, the adaptive modematching table top experiment was able to show the feasibility of a fully dynamical system.

4.1 Adaptive Mode Matching

Reque Real time digital system and model.

Signal chain.

4.2 Actuators

4.2.1 Thermal Lenses

Fabian's work and UFL paper.

4.2.2 Translation Stages

4.3 Sensors

Recall in Section 2.2 that the RF modal decomposition technique relies on comparing the Gouy phase of the higher order modes to the fundamental Gaussian mode using an array of RF photodiodes to extract an error signal. The complication arises when using this method to extract the beat note between the fundamental mode and symmetric donut mode because the photodetector arrays must match the higher order mode geometry. This is easier to deal with when trying to sense angular distortions from a misaligned cavity because the 01 and 10 modes can be sensed with a split photodetector, however, directly using that device will not work to sense mode mismatch due to the symmetry of the 02 and 20 modes. To date, Advanced LIGO uses no RF sensors to detect mode matching so the next sections will provide methods of sensing mode mismatch which can be integrated into advanced LIGO and used for dynamic closed loop feedback control.

4.3.1 Bullseye Photodiodes (BPD)

It is clear that using a quadrant photodetector to measure the symmetric $U_{20} + U_{02}$ mode which arises from mismatch is futile because the integrated power on each side will be exactly the same due to donut mode symmetry. So it is simplest to try subtracting the inner beam power from the outer ring of the electric field in order to measure a phase difference using a specialized RF photodiode called a Bullseye Photodetector shown in Figure [].

BPD Calibration

To use the BPDs, there are a few steps required in designing the optical setup which is different than the standard PDH or WFS method. Most notably, the beam size incident on the BPD must be tuned such that the zero crossing of the $(U_{20} + U_{02})$ matches the boundary between the inner and outer segments. This condition is met if $\omega_0 = \sqrt{2}r_0$ and Appendix C shows the power ratio of the outer to inner segments when this is true,

$$\text{Power Ratio} = \frac{\mathrm{P}_2 + \mathrm{P}_3 + \mathrm{P}_4}{\mathrm{P}_1} = \frac{e^{-2r_0^2/\omega_0^2}}{1 - e^{-2r_0^2/\omega_0^2}} \approx 0.582 \tag{4.1}$$

Picture of BPD

Pitch and Yaw sensing matrix

Demodulation phase

Another constraint on using this method is that the Gouy phase separation between successive BPDs must be close to 45 or 135 degrees so that the error signals from each BPD can be orthogonalized, this is in contrast to angular wavefront sensors which require 90 degree Gouy phase separation.

4.3.2 Mode Converters

The bullseye photodiodes can be difficult to calibrate and manufacture so a particularly interesting method of sensing mode mismatch is to convert the fields with a cylindrical telescope. Using lenses which contain separate curvatures for each direction (x or y), one being flat and the other is curved with focal length, f. The idea is to break the cylindrical symmetry of the donut mode (Figure []) and convert the beam into a pringle mode, then, an error signal can extracted using the radio frequency quadrant photodiode that the angular wavefront sensors employ

Vector formalism for laser beams and optical cavities

Consider an optical cavity that is longitudinally resonant on the TEM_{00} mode using the Pound-Drever-Hall technique shown in Section 1.2.2 and also locked with angular wavefront sensors shown in Section 2.2. When there is a small mismatch between the waist size and position of the input beam relative to the cavity, this is equivalent to coupling the TEM-00 mode into higher order modes. Since mode matching is only concerned with coupling to the second order modes, the resultant field in reflection (r_{FP}) of a Fabry-Perot resonator will have the form

$$|U_{refl}\rangle = r_{FP}\begin{pmatrix} U_{00} \\ 0 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ U_{20} \\ U_{02} \end{pmatrix} \tag{4.2}$$

$$\epsilon = \frac{1}{\sqrt{2}}\left(\frac{\delta w}{w_0} + i\frac{\delta z}{z_R}\right) \tag{4.3}$$

where δw and δz are the mismatches in waist size and position, respectively.

O'Neil et al [25] wrote down a formalism that explicitly showed the effects of cylindrical telescopes on the full range of Hermite Gauss modes. An arbitrary mode converter is comprised of two cylindrical lenses that can be rotated about the axis of propagation shown in Figure []:

$$\hat{M} = \hat{R}\hat{C} \tag{4.4}$$

where $\hat{R}$ is a rotation operator and $\hat{C}$ is the mode converting operator. Since mode matching primarily couples power into the 02 and 20 modes of the cavity, we can use a 3x3 matrix. For rotations about the axis of propagation, the operator is

$$\hat{R}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\Delta\theta) & \sin(\Delta\theta) \\ 0 & -\sin(\Delta\theta) & \cos(\Delta\theta) \end{pmatrix} \tag{4.5}$$

For the mode converter, the operator is

$$\hat{C}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\Omega_{20}} & 0 \\ 0 & 0 & e^{i\Omega_{02}} \end{pmatrix} \tag{4.6}$$

where $\Omega_{mn} = (m+\frac{1}{2})\tan^{-1}\left(\frac{d}{z_{R,x}}\right)+(n+\frac{1}{2})\tan^{-1}\left(\frac{d}{z_{R,y}}\right)$ and d is the distance from one cylindrical lens to the waist. Ω_{mn} is the amount of extra phase that a higher order mode will experience due to the mode converter. It is important to note, the matrices above show that the Gaussian beam is only astigmatic within the region of between the cylindrical lenses and unchanged outside of the telescope. If the beam reflected from the cavity is well mode matched to the cylindrical telescope, then a mode converter will introduce an astigmatism and vary the separate Rayleigh ranges [2],

$$\frac{z_{R,x}}{z_{R,y}} = \frac{1+d/f}{1-d/f} \tag{4.7}$$

Of course, the tuning of θ, d and f is left up to the optical designer's choice. The obvious selection for θ should rotate the output beam such that the pringle mode intensities are centered on the individual quadrant photodiodes. The choice of separation distance d and focal length f are a bit more subtle. In Figure [] from O'Neil [25], the only conversion that transforms the diagonal HG mode into a symmetric donut mode is with $\Delta\Omega = \pi/2$. This implies the converse is true if one desires to transform the donut mode into an HG mode of the same order. Making this choice of phase propagation will automatically constrain the optical setup,

$$\frac{\pi}{2} = 2\left[\tan^{-1}\left(\frac{d}{z_{R,x}}\right) - \tan^{-1}\left(\frac{d}{z_{R,y}}\right)\right] \tag{4.8}$$

$$\Rightarrow \sqrt{2}-1 = \frac{d}{z_{R,y}} = \frac{z_{R,x}}{d} \tag{4.9}$$

The equation above only shows a relation between the Rayleigh ranges and the separation. However, by imposing mode matching conditions it is possible also constrain the focal length of the cylindrical lenses as well.

$$f = \left[\frac{1}{R_x(d)} - \frac{1}{R_y(d)}\right]^{-1} = \sqrt{2}\,d \tag{4.10}$$

Extracting information from mode

Mueller et al [23] showed that the error signal from mode mismatch could be extracted by using an RF detection scheme. Using this formalism, the error signal on a quadrant photodetector from a mismatched cavity after a mode converter is

$$
\begin{aligned}
S &\propto \sin(\Omega t)\mathrm{Im}\left\{\epsilon^*\left[\int_{A1,A3}\hat{M}^\dagger_{00}\left\langle U_{00}\right|\left[\hat{M}^\dagger_{20}\left|20\right\rangle+\hat{M}^\dagger_{02}\left|02\right\rangle\right]-\int_{A2,A4}\hat{M}^\dagger_{00}\left\langle U_{00}\right|\left(\hat{M}^\dagger_{20}\left|20\right\rangle+\hat{M}^\dagger_{02}\left|02\right\rangle\right)\right]\right\} \\
&\propto \sin(\Omega t)\mathrm{Im}\left\{\epsilon^*\left[\int_{A1,A3}\left\langle U_{00}\right|\left(\left|20\right\rangle-\left|02\right\rangle\right)-\int_{A2,A4}\left\langle U_{00}\right|\left(\left|20\right\rangle-\left|02\right\rangle\right)\right]\right\}
\end{aligned}
\tag{4.11}
$$

In the above equation, $\hat{C}_{ij}$ was chosen with $\Delta\Omega = \pi/2$ and $\hat{R}_{ij}$ should rotate the beams such that the intensities in Figure ModeConverter are aligned with the quadrant photodiodes.

Chapter 5

Mode Matching Cavities at LIGO Hanford

5.1 Active Wavefront Control System

5.2 SRC

The importance of mode-matching actually goes beyond reducing the amount of losses in coupled cavities. It also is important for cavity stability. If we look at the g-factor of a cavity, it is required through ABCD transformations that the values lay between 0 and 1. For the signal recycling cavity, if the round trip gouy phase is off by a few millimeters, the stability of the cavity can be compromised.

5.2.1 SR3 Heater

Beckhoff code

5.2.2 SRM Heater

Test set up in OSB optics lab

5.3 Frequency Scanning SRC with Squeezer

5.4 Mode Matching Squeezer to OMC

5.5 Contrast Defect

Chapter 6

Solutions for Detector Upgrades

* A full modal picture, sensors and actuators
    * SR3 Heater
    * SRM Heater
    * Bullseye photodetectors
    * Operation: range (in terms of watts and
    * Translation stages
    * Mechanical description (Solidworks designs)
    * Constraints (range, vacuum, alignment, integration)
    * Electronics
    * Software

Appendices

Appendix A

Resonator Formulas

Equation 1.50 describes the stability condition for a two mirror Fabry-Perot cavity. It is worthwhile to derive the criterion for geometric stability from the ray matrix tools commonly used in optics.

Consider two plane waves traveling in space shown by figure [], they differ by two quantities: the axial and angular separations, y and θ, respectively. These two quantities can be transformed via these optical matrices:

Lens

$$\hat{F}_i = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_i} & 1 \end{pmatrix} \tag{A.1}$$

Curved Mirror

$$\hat{M}_i = \begin{pmatrix} 1 & 0 \\ \frac{2}{R_i} & 1 \end{pmatrix} \tag{A.2}$$

Space

$$\hat{D}_i = \begin{pmatrix} 1 & d_i \\ 0 & 1 \end{pmatrix} \tag{A.3}$$

Periodic Fabry-Perot with two mirrors is

$$\begin{pmatrix} y_{m+1} \\ \theta m + 1 \end{pmatrix} = \hat{M}_{FP} \begin{pmatrix} y_m \\ \theta m \end{pmatrix} \tag{A.4}$$

where

$$\hat{M}_{FP} = \hat{M}_i \hat{D}_i \hat{M}_i \hat{D}_i \tag{A.5}$$

is the optical transfer matrix. The goal is to find a geometric condition that is dependent on the optical transfer matrix which keeps the axial displacement from diverging.

$$\begin{pmatrix} y_{m+1} \\ \theta m + 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_m \\ \theta m \end{pmatrix} \tag{A.6}$$

which means

$$\theta_m = \frac{y_{m+1} - A\, y_m}{B} \tag{A.7a}$$

$$\theta_{m+1} = \frac{y_{m+2} - A\, y_{m+1}}{B} \tag{A.7b}$$

Solving for y_{m+2}

$$y_{m+2} = (A + D)\, y_{m+1} - \det(\hat{M}_{FP}) y_m \tag{A.8}$$

Assuming a geometrical solution where $y_m = y_o h^m$ and plugging into the equation above,

$$h^2 = (A + D)\, h - \det(\hat{M}_{FP}) \tag{A.9}$$

which is a quadratic equation that has two solutions and can be further simplified if the index of refraction for the entire system remains constant such that $\det(\hat{M}_{FP}) = 1$. Plugging back into y_m and doing some algebra

$$y_m \propto \sin(m\phi) \tag{A.10}$$

where $\phi = \cos^{-1}(\frac{A+D}{2})$, which is also referred to as the round trip Gouy phase of the cavity. In order for y_m to be harmonic instead of hyperbolic and hence confined, this condition must be met

$$\frac{|A + D|}{2} \leq 1 \tag{A.11}$$

By actually calculating the terms of $\hat{M}_{FP}$ and doing even more algebra, it is clear that

$$0 \geq \left(1 + \frac{L}{R_1}\right)\left(1 + \frac{L}{R_2}\right) \geq 1 \tag{A.12}$$

which is what was stated in equation 1.50.

There is a simpler and less algebraic way to reach the same conclusion by looking at the Rayleigh range of a finite Gaussian beam for a simple cavity. In Table II of Kogelnik and Li ref[Kognelik], there is an expression for the Rayleigh range

$$
\begin{aligned}
z_R^2 &= \frac{L(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{(R_1 + R_2 - 2L)^2} \\
&= \frac{g_1 g_2 (1 - g_1 g_2}{(g_1 - g_2 - 2g_1 g_2)^2}
\end{aligned}
\tag{A.13}
$$

If the Rayleigh range is a real number, then once again, equation 1.50 must be true.

Effect of higher order modes into the cavity, mode scanning. All comes from round trip Gouy phase. - RT Gouy Phase - HOM Coupling

Appendix B

Hermite Gauss Normalization

According to equation 2.10, the higher order modes in the Hermite Gauss basis has the intensity profile,

$$I_{mn}(x,y,z) = |A_{mn}|^2 \left[\frac{W_0}{W(z)}\right]^2 \mathbb{G}_n^2\left(\frac{\sqrt{2}x}{W(z)}\right)\mathbb{G}_n^2\left(\frac{\sqrt{2}y}{W(z)}\right) \tag{B.1}$$

It is useful to normalize the first few lowest order modes with respect to the total optical power since the Gaussian beam will couple to them the most due either misalignment or mode mismatch as seen in section [].

In one dimension, the total optical power for the first 3 modes are

$$\begin{aligned}
P_0(x,y,z) &= \int_{-\infty}^{\infty} |A_0|^2 \left[\frac{W_0}{W(z)}\right] e^{-2x^2/w^2(z)} dx \\
P_1(x,y,z) &= \int_{-\infty}^{\infty} |A_1|^2 \left[\frac{W_0}{W(z)}\right] \frac{8x^2}{w^2(z)} e^{-2x^2/w^2(z)} dx \\
P_2(x,y,z) &= \int_{-\infty}^{\infty} |A_2|^2 \left[\frac{W_0}{W(z)}\right] \left(\frac{8x^2}{w^2(z)} - 2\right)^2 e^{-2x^2/w^2(z)} dx
\end{aligned} \tag{B.2}$$

In two dimensions, the total optical power for the first 3 modes are

$$\begin{aligned}
P_{00}(x,y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A_{00}|^2 \left[\frac{W_0}{W(z)}\right]^2 e^{-2x^2/w^2(z)} e^{-2y^2/w^2(z)} dxdy \\
P_{10}(x,y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A_{10}|^2 \left[\frac{W_0}{W(z)}\right]^2 \frac{8x}{w^2(z)} e^{-2x^2/w^2(z)} e^{-2y^2/w^2(z)} dxdy \\
P_{20}(x,y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A_{20}|^2 \left[\frac{W_0}{W(z)}\right]^2 \left(\frac{8x^2}{w^2(z)} - 2\right)^2 e^{-2x^2/w^2(z)} e^{-2y^2/w^2(z)} dxdy
\end{aligned} \tag{B.3}$$

By setting the equations above to unity, the normalization factors become

$$
\begin{aligned}
A_0 &= \left(\frac{2}{\pi w_0^2}\right)^{1/4} \\
A_1 &= \left(\frac{2}{\pi w_0^2}\right)^{1/4} \frac{1}{\sqrt{2}} \\
A_2 &= \left(\frac{2}{\pi w_0^2}\right)^{1/4} \frac{1}{\sqrt{8}}
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
A_{00} &= \sqrt{\frac{2}{\pi w_0^2}} \\
A_{10} &= \sqrt{\frac{1}{\pi w_0^2}} \\
A_{20} &= \sqrt{\frac{1}{4\pi w_0^2}}
\end{aligned}
\tag{B.5}
$$

Therefore the normalized modes are

Appendix C

Bullseye Photodiode Characterization

C.1 DC

$$\begin{aligned}\text{Power} &= \int_A^B |A_{00}|^2 e^{\frac{-2r^2}{\omega_0^2}} 2\pi r dr \\ &= -|A_{00}|^2 \frac{\pi\omega_0^2}{2} e^{\frac{-2r^2}{\omega_0^2}} \Big|_A^B\end{aligned} \tag{C.1}$$

$$\mathrm{P}_{in} = \text{Power}\Big|_0^{r_0} = |A_{00}|^2 \frac{\pi\omega_0^2}{2}[1 - e^{\frac{-2r_0^2}{\omega_0^2}}] \tag{C.2}$$

$$\mathrm{P}_{out} = \text{Power}\Big|_{r_0}^{\infty} = |A_{00}|^2 \frac{\pi\omega_0^2}{2}[e^{\frac{-2r_0^2}{\omega_0^2}}] \tag{C.3}$$

$$\mathrm{P}_{total} = \mathrm{P}_{in} + \mathrm{P}_{out} \tag{C.4}$$

$$\omega = \sqrt{\frac{\mathrm{P}_{total}}{|A_{00}|^2 \pi/2}} \tag{C.5}$$

$$\text{DC Power Ratio} = \frac{P_{out}}{P_{in}} = \frac{e^{-2r_0^2/\omega_0^2}}{1 - e^{-2r_0^2/\omega_0^2}} \approx 0.582 \tag{C.6}$$

C.2 RF

$$\begin{aligned}\mathrm{P}_{RF} &= \int_A^B |A_{01}|^2 (1 - \frac{2r^2}{\omega_0^2}) e^{\frac{-2r^2}{\omega_0^2}} 2\pi r dr \\ &= -|A_{00}|^2 \frac{\pi}{2}\omega_0^2 e^{\frac{-2r^2}{\omega_0^2}} \left(1 + \frac{4r_4}{\omega_0^4}\right)\Big|_A^B\end{aligned} \tag{C.7}$$

$$\mathrm{P}_{in} = \mathrm{P}_{RF}\Big|_{0}^{r_0} = -|A_{01}|^2 \frac{\pi}{2}\omega_0^2 \left(e^{\frac{-2r^2}{\omega_0^2}} \big(1 + \frac{4r_4}{\omega_0^4}\big) - 1 \right) \tag{C.8}$$

$$\mathrm{P}_{out} = \mathrm{P}_{RF}\Big|_{r_0}^{\infty} = -|A_{01}|^2 \frac{\pi}{2}\omega_0^2 e^{\frac{-2r^2}{\omega_0^2}} \big(1 + \frac{4r_4}{\omega_0^4}\big) \tag{C.9}$$

$$\text{RF Power Ratio} = \frac{P_{out}}{P_{in}} = \frac{e^{-2r_0^2/\omega_0^2}}{1 - e^{-2r_0^2/\omega_0^2}} \approx 2.7844 \tag{C.10}$$

Appendix D

Overlap of Gaussian Beams

Referenced in section

The full Gaussian beam overlap is important in quantitatively defining the amount of power loss obtained when a cavity is mismatched a incoming laser field.

First we define an arbitrary Gaussian beam in cylindrical coordinates:

$$\begin{aligned} |A(r)\rangle &= \frac{A_0}{q(z)} e^{\frac{-ikr^2}{2q(z)}} \\ &= \frac{A_0}{q(z)} e^{\frac{-ikr^2(z-iz_0)}{2|q(z)|^2}} \end{aligned} \tag{D.1}$$

where A_0 is a real amplitude, $q(z) = z + iz_0$ is the complex beam parameter, k is the wave number, and r is the radial variable in the transverse direction.

First we normalize the overlap integral to unity:

$$\langle A(r)|A(r)\rangle = \frac{|A_0|^2}{z^2 + z_0^2} \int_0^\infty e^{\frac{-kr^2 z_0}{|q(z)|^2}} 2\pi r dr = 1 \tag{D.2}$$

Normalization factor is this:

$$A_0 = \sqrt{\frac{kz_0}{\pi}} \tag{D.3}$$

For two Gaussian beams with arbitrary q-parameters

$$|A_i\rangle = \frac{A_{0,i}}{q_i} e^{\frac{-ikr^2(z-iz_0)}{2|q_i|^2}} \tag{D.4}$$

where $z_{0,i}$ is the waist size of one particular beam.

The amplitude overlap is:

$$\langle A_1 | A_2 \rangle = 2i \frac{z_{0,1} z_{0,2}}{q_1 - q_2^*} \tag{D.5}$$

So the power overlap is:

$$\text{Power Overlap} = | \langle A_1 | A_2 \rangle |^2 = 4 \frac{z_{0,1} z_{0,2}}{|q_1 - q_2^*|^2} \tag{D.6}$$

List of Figures

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