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Mode transformations in terms of the constituent Hermite–Gaussian or Laguerre–Gaussian modes and the variable-phase mode converter

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Abstract

Recently the passage of light beams through mode-order-preserving optical elements has been described in terms of the constituent Hermite–Gaussian modes [L. Allen et al., Phys. Rev. E 60 (1999) 7497]. When described instead in terms of the constituent Laguerre–Gaussian modes, interesting parallels between the two formulations emerge. These parallels are employed to construct a variable-phase mode converter. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently a matrix formulation describing transformations of the phase and intensity structure of monochromatic light beams was developed [1]. This formulation is analogous to the Jones-matrix formulation of polarisation transformations, an important tool in many branches of optics [2].

Just like the Jones-matrix formulation is based on one particular basis set of orthogonal polarisation states, namely linear polarisations, the matrix formulation of laser mode transformations is based on one particular basis set of orthogonal laser modes, namely the Hermite–Gaussian modes. As any light beam can be described as a superposition of linearly polarised Hermite–Gaussian modes, the combination of the matrix formulation in terms of Hermite–Gaussian modes and the Jones-matrix formulation can describe any change of a monochromatic light beam apart from a change in colour. However, the original article [1] only lists matrices up to mode order $N = 2$ and gives no indication as to how higher-order matrices might be calculated. This restricts the usefulness and generality of the matrix formulation.

Along with an alternative formulation, which is based on another basis set of orthogonal laser modes, namely the Laguerre–Gaussian modes, the matrix formulation based on Hermite–Gaussian modes is derived in full generality in this paper. This greatly extends the usefulness of the matrix formulation derived in Ref. [1]. What

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is more, when the two formulations are compared, an interesting analogy is found between rotations in one formulation and mode conversions in the other formulation. This is used in the construction of a novel variable-phase mode converter.

A note on notation: In order to facilitate the readability of the equations, mode functions and scalar products are written in the bra-ket notation throughout this paper. The scalar product between the mode functions $u_a(x,y,z)$ and $u_b(x,y,z)$, represented by the kets $|a\rangle$ and $|b\rangle$, respectively, is defined as

$$\langle a|b\rangle \equiv \int\int (u_a(x,y,0))^* u_b(x,y,0)\, \mathrm{d}x\, \mathrm{d}y. \tag{1}$$

2. Hermite–Gaussian (HG) modes and Laguerre–Gaussian (LG) modes

2.1. Decomposition of HG modes in terms of LG modes and vice versa

Hermite–Gaussian (HG) modes form a complete orthogonal set of modes [3], i.e. any monochromatic light beam can be understood as a superposition of HG modes with the same optic axis, waist size and waist position, but different mode indices, n and m. The Laguerre–Gaussian (LG) modes that differ only in their mode indices, l and p, form another complete orthogonal set [3].

In particular, LG modes can be expressed as sums of HG modes and vice versa, i.e.

$$|\mathrm{LG}_{l,p}\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |\mathrm{HG}_{n,m}\rangle\langle \mathrm{HG}_{n,m}|\mathrm{LG}_{l,p}\rangle \tag{2}$$

and

$$|\mathrm{HG}_{n,m}\rangle = \sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty} |\mathrm{LG}_{l,p}\rangle\langle \mathrm{LG}_{l,p}|\mathrm{HG}_{n,m}\rangle. \tag{3}$$

Comparison with Eq. (8) from Ref. [4] reveals that the coefficients in these decompositions take on the values

$$\langle \mathrm{HG}_{n,m}|\mathrm{LG}_{l,p}\rangle = \langle \mathrm{LG}_{l,p}|\mathrm{HG}_{n,m}\rangle^* = \begin{cases} \mathrm{i}^m b\left(\dfrac{N+l}{2},\dfrac{N-l}{2},m\right) & : \quad 2p+|l| = n+m \\ 0 & : \quad 2p+|l| \neq n+m \end{cases} \tag{4}$$

where the coefficients $b(n',m',m)$ are defined as

$$b(n',m',m) = \sqrt{\frac{(n'+m'-m)!m!}{2^{n'+m'}n'!m'!}}\,\frac{1}{m!}\,\frac{\mathrm{d}^{m'}}{\mathrm{d}t^{m}}\left[(1-t)^{n'}(1+t)^{m'}\right]\Big|_{t=0}. \tag{5}$$

The coefficients given by Eq. (4) are the elements of the important matrices that describe a change from a Hermite–Gaussian basis to a Laguerre–Gaussian basis and vice versa (Section 3.2).

2.2. Conversion between modes

Any HG mode can be transformed into a corresponding LG mode by passing it through a cylindrical-lens mode converter [4] (Fig. 1). This mode conversion is an example of a transformation that can be described well with the matrix approach described in this paper. It also establishes a correspondence between individual HG and LG modes, which is important for the order of the components in the matrix formulation with a LG basis.

Table 1
Pairs of mode indices (n,m) and (l,p) of corresponding Hermite–Gaussian and Laguerre–Gaussian modes up to mode order $N=3$. When read from top to bottom, pairs of HG mode indices with the same value of N are ordered according to Eq. (9), pairs of LG mode indices with the same N are ordered according to Eq. (15).

N	(n,m)	(l,p)
0	(0,0)	(0,0)
1	(1,0)	(1,0)
	(0,1)	$(-1,0)$
2	(2,0)	(2,0)
	(1,1)	(0,1)
	(0,2)	$(-2,0)$
3	(3,0)	(3,0)
	(2,1)	(1,1)
	(1,2)	$(-1,1)$
	(0,3)	$(-3,0)$

The indices l and p of the LG mode resulting from mode conversion of a HG mode are related to the indices n and m of the original HG mode through the equations

$$l(n,m) = n - m, \quad p(n,m) = \min(n,m). \tag{6}$$

Please note that the l index can take on negative values. HG and LG modes corresponding in this sense are of the same mode order, which is defined as

$$N = n + m = 2p + |l|. \tag{7}$$

Table 1 lists all corresponding pairs of mode indices (n,m) and (l,p) for mode orders $N = 0,1,2,3$.

The same mode converter transforms any LG mode into its corresponding HG mode. The mode indices n and m of the resulting HG mode can be calculated by inverting Eqs. (6). In terms of the mode index l and the mode order N of the original LG mode, n and m can be expressed in the form

$$n(N,l) = \frac{N+l}{2}, \quad m(N,l) = \frac{N-l}{2}, \tag{8}$$

where N is the mode order of the input LG mode (which is also the mode order of the output HG mode).

3. Matrix formulations of mode transformations

3.1. Vector representation of light beams

A general monochromatic light beam consists of Hermite–Gaussian (or Laguerre–Gaussian) components of all orders. In order to describe every possible monochromatic light beam an infinite number of constituent HG or LG modes are therefore needed. The infinitely many corresponding coefficients do not lend themselves to a vector formulation. Some useful linear mode transformations, on the other hand, preserve mode order. It therefore appears sensible to concentrate on modes of a given order and collect the corresponding amplitude coefficients in vectors.

In this paper the order of the elements of vectors in the HG basis is chosen to be the same as in Ref. [1]: in a vector representing a superposition of HG modes of order N, the jth element is the coefficient corresponding to HG mode with indices n_j and m_j given by

$$n_j = N - j, \quad m_j = j. \tag{9}$$

The pairs of indices (n,m) in Table 1 are ordered according to Eq. (9).

The actual elements of a vector representing a mode $|a\rangle$ of order N in the HG representation are the numbers

$$\langle \mathrm{HG}_{N-j,j}|a\rangle, \quad j \in \{0,\ldots,N\}. \tag{10}$$

These vector elements are, of course, the coefficients in a decomposition in terms of the HG modes of order N,

$$|a\rangle = \sum_{j=0}^{N} |\mathrm{HG}_{n_j,m_j}\rangle\langle \mathrm{HG}_{n_j,m_j}|a\rangle. \tag{11}$$

For example, a mode $|a\rangle$ of order $N=2$ that can be written as a superposition in terms of HG modes in the form

$$|a\rangle = |\mathrm{HG}_{2,0}\rangle\langle \mathrm{HG}_{2,0}|a\rangle + |\mathrm{HG}_{1,1}\rangle\langle \mathrm{HG}_{1,1}|a\rangle + |\mathrm{HG}_{0,2}\rangle\langle \mathrm{HG}_{0,2}|a\rangle \tag{12}$$

is represented by the column vector

$$(\langle \mathrm{HG}_{2,0}|a\rangle, \langle \mathrm{HG}_{1,1}|a\rangle, \langle \mathrm{HG}_{0,2}|a\rangle)^T. \tag{13}$$

Eq. (9) orders the pairs of Hermite–Gaussian mode indices (n_j,m_j) of a given mode order N. With the help of Eq. (6), each one of these ordered pairs of HG mode indices, (n_j,m_j), can be assigned a corresponding pair of LG mode indices, (l_j,p_j):

$$l_j = l(n_j,m_j) = l(N-j,j), \quad p_j = p(n_j,m_j) = p(N-j,j). \tag{14}$$

Substituting the expressions for $l(n,m)$ and $p(n,m)$, the expressions for the indices l_j and p_j become

$$l_j = (N-j) - j = N - 2j, \quad p_j = \min(N-j,j). \tag{15}$$

It turns out to be advantageous to order the coefficients in a vector that uses the LG basis such that the jth $(j=0,1,\ldots,N)$ element corresponds to a LG mode with indices l_j and p_j as given in (15); this is how the pairs of indices (l,p) in Table 1 are ordered.

In analogy to the HG mode representation, the actual elements of a vector representing a mode $|a\rangle$ of order N in the LG representation are the numbers

$$\langle \mathrm{LG}_{N-2j,\min(N-j,j)}|a\rangle, \quad j \in \{0,\ldots,N\}. \tag{16}$$

3.2. Change of basis

For the purposes of the derivation of matrices describing rotation and mode conversion, the matrices that turn a vector in a HG basis into the vector in the corresponding LG basis and vice versa, which will be derived in the following, will prove extremely useful.

These matrices can be derived by starting with a vector representing a mode $|a\rangle$ of order N in the HG basis. According to Eq. (10), the jth element of this vector is

$$\langle \mathrm{HG}_{n_j,m_j}|a\rangle. \tag{17}$$

The ith element in the corresponding LG basis, given in Eq. (16), is

$$\langle \mathrm{LG}_{l_i,p_i}|a\rangle. \tag{18}$$

On substitution of the HG decomposition of $|a\rangle$, Eq. (11), this equation becomes

$$\langle \mathrm{LG}_{l_i,p_i}|a\rangle = \sum_{j=0}^{N} \langle \mathrm{LG}_{l_i,p_i}|\mathrm{HG}_{n_j,m_j}\rangle \langle \mathrm{HG}_{n_j,m_j}|a\rangle . \tag{19}$$

This last equation implies that the vector describing the mode $|a\rangle$ in the LG basis, whose elements are the numbers $\langle \mathrm{LG}_{l_i,p_i}|a\rangle$, can be written as the product between a matrix $B_{\mathrm{HG}\to\mathrm{LG},N}$, and the corresponding vector in the HG basis. The elements of the matrix $B_{\mathrm{HG}\to\mathrm{LG},N}$ are

$$(B_{\mathrm{HG}\to\mathrm{LG},N})_{i,j} = \langle \mathrm{LG}_{l_i,p_i}|\mathrm{HG}_{n_j,m_j}\rangle . \tag{20}$$

For example, the matrices $B_{\mathrm{HG}\to\mathrm{LG},N}$ up to mode order $N=2$ are

$$B_{\mathrm{HG}\to\mathrm{LG},0} = 1, \quad B_{\mathrm{HG}\to\mathrm{LG},1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{\mathrm{i}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix}, \quad B_{\mathrm{HG}\to\mathrm{LG},2} = \begin{pmatrix} \frac{1}{2} & -\frac{\mathrm{i}}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{\mathrm{i}}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}. \tag{21}$$

A similar argument yields the elements of the matrix that changes the LG basis for modes of order N to a HG basis, $B_{\mathrm{LG}\to\mathrm{HG},N}$:

$$(B_{\mathrm{LG}\to\mathrm{HG},N})_{i,j} = \langle \mathrm{HG}_{n_i,m_i}|\mathrm{LG}_{l_j,p_j}\rangle . \tag{22}$$

The explicit forms of the matrices up to mode order $N=2$ are

$$B_{\mathrm{LG}\to\mathrm{HG},0} = 1, \quad B_{\mathrm{LG}\to\mathrm{HG},1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\mathrm{i}}{\sqrt{2}} & -\frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix}, \quad B_{\mathrm{LG}\to\mathrm{HG},2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{\mathrm{i}}{\sqrt{2}} & 0 & -\frac{\mathrm{i}}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}. \tag{23}$$

Obviously, the result of a change to a different basis and back is the original vector. Consequently, the corresponding matrices satisfy the relations

$$B_{\mathrm{LG}\to\mathrm{HG},N} \cdot B_{\mathrm{HG}\to\mathrm{LG},N} = I = B_{\mathrm{HG}\to\mathrm{LG},N} \cdot B_{\mathrm{LG}\to\mathrm{HG},N}, \tag{24}$$

where I denotes the identity matrix, $I = \mathrm{diag}(1,1,\ldots,1)$.

3.3. Rotation

The azimuthal dependence of Laguerre–Gaussian mode functions is of the form [5]

$$|\mathrm{LG}_{l,p},\phi\rangle \propto \exp(\mathrm{i}l\phi) . \tag{25}$$

It can be seen from this expression that a LG mode function which has been rotated through an angle $\Delta\phi$, $|\mathrm{LG}_{l,p};\phi-\Delta\phi\rangle$, can be expressed in terms of the non-rotated LG mode function, $|\mathrm{LG}_{l,p};\phi\rangle$, as

$$|\mathrm{LG}_{l,p};\phi-\Delta\phi\rangle = \mathrm{e}^{\mathrm{i}l\Delta\phi}|\mathrm{LG}_{l,p};\phi\rangle . \tag{26}$$

On rotation, a Laguerre–Gaussian mode simply changes phase. The phase change is proportional to the mode's azimuthal mode index, l, and the rotational angle, $\Delta\phi$.

When a superposition of LG modes is rotated through an angle $\Delta\phi$, its LG components are individually phase-shifted according to Eq. (26), i.e. by the product of their azimuthal mode index and the rotation angle, $l \cdot \Delta\phi$. Consequently a matrix that describes rotation through an angle $\Delta\phi$ in a LG basis simply has to multiply each component $\langle \mathrm{LG}_{l_j,p_j}|a\rangle$ by $\exp(\mathrm{i} l_j \Delta\phi)$. The matrix which describes the rotation of a mode of order N through an angle $\Delta\phi$ in the LG basis is therefore

$$R_{\mathrm{LG},N}(\Delta\phi) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}l_0\Delta\phi}, \mathrm{e}^{\mathrm{i}l_1\Delta\phi}, \ldots, \mathrm{e}^{\mathrm{i}l_N\Delta\phi}). \tag{27}$$

The rotation matrices for mode orders $N = 0,1,2$ can be written explicitly as

$$R_{\mathrm{LG},0}(\Delta\phi) = 1, \quad R_{\mathrm{LG},1}(\Delta\phi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta\phi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\Delta\phi} \end{pmatrix}, \quad R_{\mathrm{LG},2}(\Delta\phi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}2\Delta\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}2\Delta\phi} \end{pmatrix}. \tag{28}$$

A superposition of HG modes can be rotated by first changing basis to the LG modes, rotating in the LG basis, and returning to the HG basis. Mathematically, these three actions can be described in one matrix which is the matrix product of the three matrices that describe the individual actions:

$$R_{\mathrm{HG},N}(\Delta\phi) = B_{\mathrm{LG}\to\mathrm{HG},N} \cdot R_{\mathrm{LG},N}(\Delta\phi) \cdot B_{\mathrm{HG}\to\mathrm{LG},N}. \tag{29}$$

Please note that, being a matrix product, the expression on the right is evaluated from right to left.

The HG rotation matrices for mode orders $N = 0,1,2$ are

$$R_{\mathrm{HG},0}(\Delta\phi) = 1, \quad R_{\mathrm{HG},1}(\Delta\phi) = \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix},$$

$$R_{\mathrm{HG},2}(\Delta\phi) = \begin{pmatrix} \cos^2\Delta\phi & \dfrac{\sin(2\Delta\phi)}{\sqrt{2}} & \sin^2\Delta\phi \\ -\dfrac{\sin(2\Delta\phi)}{\sqrt{2}} & \cos(2\Delta\phi) & \dfrac{\sin(2\Delta\phi)}{\sqrt{2}} \\ \sin^2\Delta\phi & -\dfrac{\sin(2\Delta\phi)}{\sqrt{2}} & \cos^2\Delta\phi \end{pmatrix}. \tag{30}$$

3.4. Mode conversion

Fig. 1 in Section 1 shows a setup that converts Hermite–Gaussian modes into corresponding Laguerre–Gaussian modes and vice versa. This cylindrical-lens mode converter is a special case of a more general mode converter. Neglecting any constant phase offset imposed on the beam as a whole, such a generalised mode converter individually phase-shifts the HG components $\langle \mathrm{HG}_{n_j,m_j}|a\rangle$ of a light beam $|a\rangle$ passing through it by $m_j\theta$, where θ is a constant associated with the converter [4]. Such a device is called a 'θ converter'. In this terminology, the cylindrical-lens mode converter in Fig. 1 is a $\pi/2$ converter.

In the HG basis, the matrix $C'_{\mathrm{HG},N}(\theta)$ that describes the action of a θ converter is

$$C'_{\mathrm{HG},N}(\theta) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}m_0\theta}, \mathrm{e}^{\mathrm{i}m_1\theta}, \ldots, \mathrm{e}^{\mathrm{i}m_N\theta}) = \mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\theta}, \ldots, \mathrm{e}^{\mathrm{i}N\theta}). \tag{31}$$

It is useful to symmetrise these matrices through multiplication by $-(N\theta)/2$; this multiplication introduces an additional constant phase offset to the beam as a whole and can be neglected. The resulting symmetrised matrices are

$$C_{\mathrm{HG},N}(\theta) = \mathrm{e}^{-\mathrm{i}(N\theta)/2} C'_{\mathrm{HG},N}(\theta) = \mathrm{diag}(\mathrm{e}^{-\mathrm{i}(N/2)\theta}, \mathrm{e}^{-\mathrm{i}(N/2+1)\theta}, \ldots, \mathrm{e}^{\mathrm{i}(N/2)\theta}). \tag{32}$$

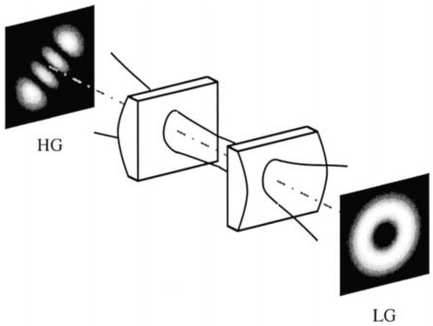

Fig. 1. Schematic of a cylindrical-lens $\pi/2$ mode converter. Hermite–Gaussian modes of all orders can be converted into corresponding Laguerre–Gaussian modes, and vice versa. The mode converter comprises two cylindrical lenses of focal length f, which are separated by a distance $\sqrt{2}f$. Both the input and output modes have a beam waist size of $w_0 = \sqrt{\left(1+1/\sqrt{2}\right) f\lambda/\pi}$, their waists are positioned in the plane half way between the two lenses. The sample intensity cross-sections are those of a HG mode at 45° with indices $n=3$ and $m=0$ and a LG mode with indices $l=3$ and $p=0$.

The mode converter matrix that describes a mode converter in the LG basis can be written as the product of the matrices that describe a change to the HG basis, $B_{\mathrm{LG}\to\mathrm{HG},N}$, the mode conversion in the HG basis, $C_{\mathrm{HG},N}(\theta)$, and a change back to the LG basis, $B_{\mathrm{HG}\to\mathrm{LG},N}$:

$$C_{\mathrm{LG},N}(\theta) = B_{\mathrm{HG}\to\mathrm{LG},N} \cdot C_{\mathrm{HG},N}(\theta) \cdot B_{\mathrm{LG}\to\mathrm{HG},N}. \tag{33}$$

The matrices corresponding to mode converters for orders $N=0,1,2$ are the HG basis

$$C_{\mathrm{HG},0}(\theta)=1,\quad C_{\mathrm{HG},1}(\theta)=\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2}\end{pmatrix},\quad C_{\mathrm{HG},2}(\theta)=\begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta}\end{pmatrix}, \tag{34}$$

and in the LG basis

$$C_{\mathrm{LG},0}(\theta)=1,\quad C_{\mathrm{LG},1}(\theta)=\begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2)\end{pmatrix},$$

$$C_{\mathrm{LG},2}(\theta)=\begin{pmatrix} \cos^2(\theta/2) & -i\dfrac{\sin\theta}{\sqrt{2}} & \left(\cos^2(\theta/2)-1\right) \\ -i\dfrac{\sin\theta}{\sqrt{2}} & \cos\theta & -i\dfrac{\sin\theta}{\sqrt{2}} \\ \left(\cos^2(\theta/2)-1\right) & -i\dfrac{\sin\theta}{\sqrt{2}} & \cos^2(\theta/2)\end{pmatrix}. \tag{35}$$

The most common mode converters are $\pi/2$ converter and π converter. The corresponding matrices in the HG basis are

$$C_{\mathrm{HG},N}(\pi/2) = e^{-iN\pi/4}\,\mathrm{diag}(1,-i,-1,i,1,\dots) \tag{36}$$

and

$$C_{\mathrm{HG},N}(\pi) = (-i)^N\,\mathrm{diag}(1,-1,1,-1,\dots); \tag{37}$$

the factors in front of the diagonal matrices are the symmetrisation factors.

In order to convert a Hermite–Gaussian mode into the corresponding Laguerre–Gaussian mode, a $\pi/2$ converter has to be rotated through an angle 45° with respect to the incoming HG mode (compare Fig. 1).

Consequently, in the product of the matrices which describe rotation of a HG mode by $-45°$, passage through a $\pi/2$ converter, rotation through 45° (or, indeed, any other angle), and a change to the corresponding LG basis,

$$B_{\mathrm{HG}\rightarrow\mathrm{LG},N}\cdot R_{\mathrm{HG},N}(45°)\cdot C_{\mathrm{HG},N}(\pi/2)\cdot R_{\mathrm{HG},N}(-45°), \tag{38}$$

all the off-diagonal elements are zero while the modulus of every diagonal element is 1. This implies that a pure jth mode in the HG basis ($j\in\{0,1,\dots,N\}$) is converted into the pure jth mode in the corresponding LG basis.

Please note that, although all the diagonal elements have a modulus of 1, they differ in phase. This implies that, although an in-phase superposition of HG modes will be converted into a superposition of the corresponding LG modes with the same modulus, the LG modes will not necessarily be in phase. However, the LG modes will be in phase if they are considered in a frame of reference that is rotated through an additional $45°\pm180°$. Mathematically, this can be expressed in the form

$$B_{\mathrm{HG}\rightarrow\mathrm{LG},N}\cdot R_{\mathrm{HG},N}(\pm90°)\cdot C_{\mathrm{HG},N}(\pi/2)\cdot R_{\mathrm{HG},N}(-45°)\propto\mathrm{diag}(1,1,\dots,1). \tag{39}$$

A phase-preserving mode conversion of a superposition of Laguerre–Gaussian modes into a superposition of Hermite–Gaussian modes can be performed by reversing the order of the steps described in Eq. (39) and the direction of rotations and phase changes in mode converters, i.e.

$$R_{\mathrm{HG},N}(45°)\cdot C_{\mathrm{HG},N}(-\pi/2)\cdot R_{\mathrm{HG},N}(\pm90°)\cdot B_{\mathrm{LG}\rightarrow\mathrm{HG},N}\propto\mathrm{diag}(1,1,\dots,1). \tag{40}$$

The same can be achieved with a $\pi/2$ converter instead of a $-\pi/2$ converter by following the passage through the converter with a rotation through $-45°$. Mathematically,

$$B_{\mathrm{LG}\rightarrow\mathrm{HG},N}\cdot R_{\mathrm{LG},N}(-45°)\cdot C_{\mathrm{LG},N}(\pi/2)\propto\mathrm{diag}(1,1,\dots,1). \tag{41}$$

An interesting property of π conversion is that it is equivalent to vertical flipping of the beam. This can be shown by individually flipping the beam's Hermite–Gaussian mode components. From the mode function of HG modes [5] and the property of the Hermite polynomials that

$$H_m(-t)=(-1)^m H_m(t), \tag{42}$$

it is easily shown that

$$|\mathrm{HG}_{n,m}(x,-y)\rangle=(-1)^m|\mathrm{HG}_{n,m}(x,y)\rangle. \tag{43}$$

The matrix describing vertical flipping of a beam in a HG basis is therefore

$$F_{v,\mathrm{HG},N}=\mathrm{diag}(1,-1,1,-1,\dots), \tag{44}$$

which, apart from a constant phase offset, is the same as the matrix describing a π conversion in the same HG basis, given by Eq. (37). One optical component that flips a light beam passing through it is a Dove prism. This component therefore can be, and indeed has been [6], used as a π converter.

Just like a Dove prism, a π converter changes the handedness, i.e. the sign of the l index, of LG modes passing through it. This is reflected mathematically as

$$C_{\mathrm{LG},N}(\pi)\propto\begin{pmatrix}0&0&\dots&0&1\\0&0&\dots&1&0\\0&1&\dots&0&0\\1&0&\dots&0&0\end{pmatrix}. \tag{45}$$

A sequence of π converters can also be used to rotate the beam passing through it. Two π converters, rotated with respect to each other through an angle $\Delta\phi/2$, act like a rotation through an angle $\Delta\phi$, i.e.

$$R_{\mathrm{LG},N}(\Delta\phi/2)\cdot C_{\mathrm{LG},N}(\pi)\cdot R_{\mathrm{LG},N}(-\Delta\phi/2)\cdot C_{\mathrm{LG},N}(\pi)=R_{\mathrm{LG},N}(\Delta\phi). \tag{46}$$

4. Varible-phase-shift mode converter

In Ref. [1] it was pointed out that while in the HG basis the mode converter matrices will be relatively simple and the rotation matrices complex, the reverse is true in the LG basis. This is an expression of the analogy between the descriptions of a light beam's phase and intensity structure in terms of HG and LG modes on the one hand, and descriptions of the polarisation of a light beam in terms of linear and circular polarisation components on the other hand. Comparison of the HG rotation matrices, Eq. (30), and the LG mode converter matrices, Eq. (35), moreover reveals striking similarity. Perhaps more importantly, though, comparison of the LG rotation matrices, Eq. (28), and the HG mode converter matrices, Eq. (34), reveals that the LG matrices describing a rotation through an angle ϕ are the same as the HG matrices describing a '(-2ϕ) converter', i.e.

$$C_{\mathrm{HG},N}(-2\phi) = R_{\mathrm{LG},N}(\phi). \tag{47}$$

This fact can be used to construct a variable-phase mode converter.

As rotation through an angle ϕ is for Laguerre–Gaussian modes what passage through a -2ϕ converter is for Hermite–Gaussian modes (Eq. (47)), a general -2ϕ converter can be constructed by placing a ϕ beam rotator between two phase-preserving mode converters, one that converts Hermite–Gaussian into Laguerre–Gaussian modes and another one that converts the other way. Such phase-preserving mode converters are described in Eqs. (39) and (41)). Indeed, the matrix for the combination of all these elements,

$$R_{\mathrm{HG},N}(-45°)\cdot C_{\mathrm{HG},N}(\pi/2)\cdot R_{\mathrm{HG},N}(\phi)\cdot R_{\mathrm{HG},N}(90°)\cdot C_{\mathrm{HG},N}(\pi/2)\cdot R_{\mathrm{HG},N}(-45°) = C_{\mathrm{HG},N}(-2\phi), \tag{48}$$

turns out to be describing a -2ϕ converter.

It transpires that a rotation of the two $\pi/2$ converters through 90° turns this setup into a $+2\phi$ converter. Splitting the rotation through 90° into two rotations through 45°, this can be written in the form

$$\begin{aligned} &R_{\mathrm{HG},N}(45°)\cdot C_{\mathrm{HG},N}(\pi/2)\cdot R_{\mathrm{HG},N}(-45°)\cdot R_{\mathrm{HG},N}(\phi)\cdot R_{\mathrm{HG},N}(-45°)\cdot C_{\mathrm{HG},N}(\pi/2)\cdot R_{\mathrm{HG},N}(45°) \\ &\quad = C_{\mathrm{HG},N}(2\phi). \end{aligned} \tag{49}$$

This equation describes the setup shown in Fig. 2. Figs. 3 and 4 show examples of simulated and experimentally obtained intensity distributions of a Hermite–Gaussian mode passing through such a variable-phase mode converter.

An alternative setup could consist of two $\pi/2$ converters, rotated with respect to each other through an angle $90° + \phi$. The beam behind this setup would then, apart from being mode-converted, also be rotated through ϕ.

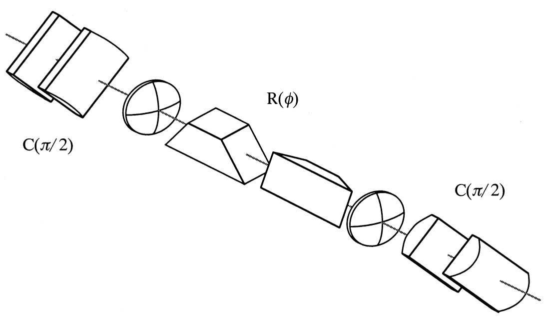

Fig. 2. Schematic of a variable-phase-shift mode converter. The beam between two $\pi/2$ converters at $+45°$ and $-45°$ to the beam, respectively, is rotated through an angle ϕ. This setup acts like a 2ϕ converter. The image rotator $R(\phi)$ is drawn here as comprising two Dove prisms that are rotated with respect to each other through an angle $\phi/2$.

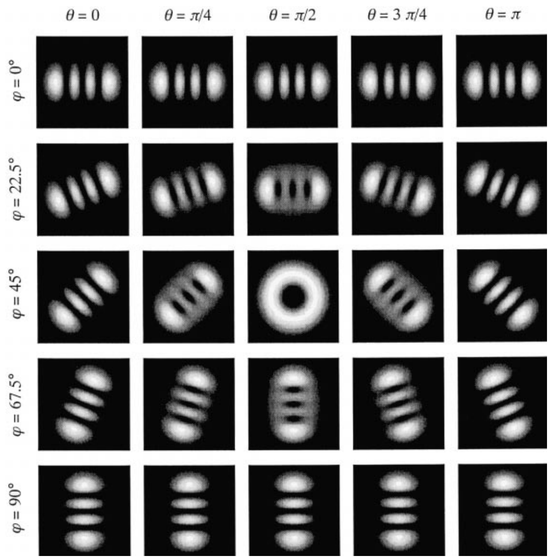

Fig. 3. Modelled intensity cross-sections of a Hermite–Gaussian mode with indices $m=3$ and $n=0$ after rotation through an angle ϕ and passage through a variable-phase-shift mode converter, $\mathrm{C}(\theta)$. For $\phi=0°$ and $\phi=90°$, the mode is simply phase-shifted through θ in the mode converter, leaving its intensity cross-section unaltered. In the case $\phi=45°$ and $\theta=\pi/2$, the input mode is converted into a Laguerre–Gaussian mode with indices $l=-3$ and $p=0$.

A variable-phase shift mode converter is analogous to a wave plate that introduces an arbitrary phase shift between the linear polarisation components of a light beam. In complete analogy to the variable-phase mode converter, such a variable-phase wave plate could first convert the linear polarisation components into corresponding circular polarisation components by passing the beam through a quarter-wave plate, then phase-shift the circular polarisation components with respect to each other by rotating them, for example with

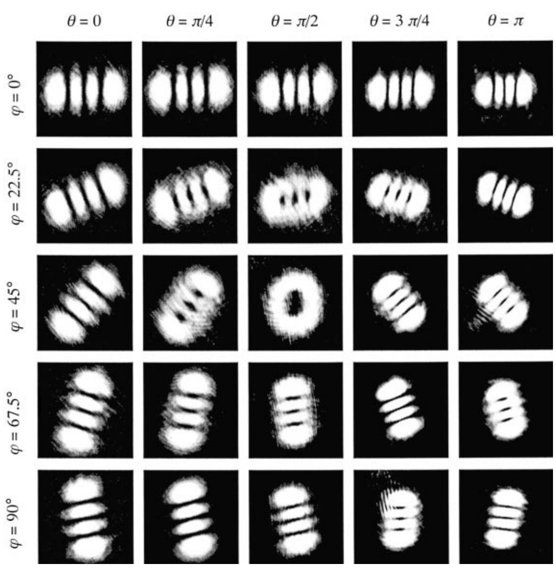

Fig. 4. Experimentally obtained intensity cross-sections corresponding to their modelled counterparts shown in Fig. 3.

the help of two half-wave plates, and then convert the phase-shifted circular polarisation components back to their corresponding linear polarisations. The phase shift between the linear polarisation components would then be proportional to the angle by which the circular polarisation components have been rotated.

5. Conclusions

While the description of the phase and intensity structure of a light beam in terms of Hermite–Gaussian components is analogous to the descriptions of a light beam's polarisation state in terms of the linear polarisation components, the description of the phase and intensity structure in terms of Laguerre–Gaussian components is analogous to the description of the polarisation in terms of the circular polarisation components.

The explicit development of this analogy lead to the design of a novel variable-phase-shift mode converter. Furthermore, just like a description in terms of linear polarisation components provides a more insightful description of a wave plate and a description in terms of circular polarisation components of optical activity is very straightforward, so a formulation of transformations in terms of Laguerre–Gaussian components will be much better suited for certain applications.

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References

[1] L. Allen, J. Courtial, M.J. Padgett, Phys. Rev. E 60 (1999) 7497.
[2] J.M. Bennett, H.E. Bennett, in: W.G. Driscoll, W. Vaughan (Eds.), Handbook of Optics, McGraw-Hill, New York, 1978, Chap. 10.
[3] A.E. Siegman, Lasers, University Science Books, Mill Valley, California, 1968.
[4] M.W. Beijersbergen, L. Allen, H.E.L.O. van der Veen, J.P. Woerdman, Opt. Commun. 96 (1993) 123.
[5] L. Allen, M.W. Beijersbergen, R.J.C. Spreeuw, J.P. Woerdman, Phys. Rev. A 45 (1992) 8185.
[6] J. Courtial et al., Phys. Rev. Lett. 80 (1998) 3217.