Brownian motion of a mass suspended by an anelastic wire

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The theory of elasticity and the fluctuation-dissipation theorem were used to calculate the thermal noise power spectrum of an extended mass suspended by an anelastic wire. The implications for interferometric detectors of gravitational waves are discussed.

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INTRODUCTION

The pendulum is a critical component of interferometric gravitational wave detectors, where it is used to construct test masses that can respond freely to gravitational waves. In this kind of experiment, thermal noise is one of the noise sources limiting the precision of the experiment. For example, in the detectors of the Laser Interferometer Gravitational-wave Observatory (or LIGO), thermal noise is expected to limit the sensitivity at low frequencies, along with seismic noise.[1] The rms amplitude of thermal noise in a simple harmonic oscillator is given by the equipartition theorem, $x^2_{\rm rms}=\frac{1}{2}k_{\rm B}T_0/k$. It only depends on the absolute temperature T_0 times the Boltzmann constant $k_{\rm B}$ and the mechanical spring constant k, but is independent of the sources or amount of energy dissipation. However, what is most significant in an experiment like LIGO is the amplitude of the thermal noise far from the mechanical resonances of the test mass suspension. Therefore, what is relevant is the thermal noise *spectral density*, in the frequency range of interest. The thermal fluctuations' spectral density $x^2(\omega)$ (in units of cm^2/Hz) is related to the complex mechanical admittance of the system, $Y(\omega)$, through the fluctuation-dissipation theorem:[2]

$$x^2(\omega)=\frac{4k_{\rm B}T_0}{\omega^2}\,\text{Re}[Y(\omega)]. \qquad (1)$$

In a pendulum, the admittance acquires a real part when the system has mechanical losses, or "resistive" components. The obvious damping sources, such as air damping or eddy currents, can be made negligible by proper design; the pendulum's oscillations should be damped instead by losses caused by the pendulum fiber's departure from perfect elasticity.

In the description of materials, elastic losses are often characterized by adding an imaginary part to the Young's modulus: $E=E_0(1+i\phi)$.[3] The parameter ϕ is referred to as the material's "internal friction" or "loss factor." In general, E_0 and ϕ are frequency dependent (where it is understood that the description is given in the frequency domain, through appropriate Fourier transforms). For example, viscous friction would be described by a function ϕ linear in frequency. Although $\phi(\omega)$ and $E_0(\omega)$ are not completely independent, since they are related by the Kramers–Kronig relations, as long as $\phi\ll 1$, the frequency dependence of E_0 is very weak. In typical anelastic materials, ϕ can exhibit so-called "Debye peaks," superposed on a background of roughly constant loss.[4] Assuming that $E_0(\omega)$ and $\phi(\omega)$ are known for the fiber's material itself, we will show how the losses in a pendulum design are related to it.

Previous estimates of the thermal noise power spectrum of a pendulum have used a modal expansion.[5] These calculations have been based on an estimate of the loss factor ϕ_p for the pendulum mode, calculated as a ratio of the elastic to the gravitational energy, times the loss factor ϕ of the wire material itself. Similar arguments have been used[5,6] to estimate the thermal noise associated with the higher frequency modes. An analogy with an electrical transmission line has also been used.[7]

In this paper, we eschew modal analysis in favor of a direct solution of the equation of motion of a simple model of a pendulum, as a thin beam under tension. This enables us to avoid some of the approximations used in earlier work. Our results agree at the expected level with earlier approximate methods; they also yield some insights that previous approaches had missed. We will present in Sec. I a description of our model. Next, Sec. II gives an outline of the solution, while in Sec. III we describe the results we obtained for particular parameters representative of the pendulums in some gravitational wave detector prototypes. In Sec. IV, we will discuss our results, compare them with previous estimates and provide approximate formulae and justifications for the loss factors of the different modes. In Sec. V we summarize and present our conclusions.

I. MODEL

We consider the pendulum as an elastic beam (representing a single fiber) with a complex Young's modulus E, an area moment of inertia I, and a mass per unit length ρ. The fiber is perfectly clamped at one end; it has a rigid mass of finite size attached at its other end, a distance L away. The mass M has a moment of inertia J about its center of mass. The fiber is attached, with a perfect clamp, a distance h from the center of mass. The mass provides the fiber with a tension $T=Mg$. The transverse vibrations of the pendulum are described with a function $x(z,t)$, for $0\leqslant z\leqslant L$.

We assume no shearing deformation in the fiber, so the position of the mass is $x_M(t)=x(L,t)+h\Phi(t)$, where $\Phi(t)=x'(L,t)$. (An apostrophe indicates a derivative with respect to the longitudinal z coordinate.) The neglect of shearing is justified when the deformation angle that the tension would produce, $T/(k'GA)$, is small (k' is a numerical factor of order unity depending on the fiber's cross section, G is the shear modulus and A is the cross section area).

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Other shearing effects are scaled with the ratio of the fiber's diameter to its length, also assumed small. We also neglect the rotary inertia, which is a valid approximation for frequencies smaller than $\sqrt{T/j}$, with j the fiber's moment of inertia per unit length.

Under these assumptions, the kinetic energy of the system will be given by the sum of the kinetic energies of the spring and the mass,

$$\mathscr{T}=\frac{1}{2}\int_0^L \rho[\dot{x}(z)]^2dz+\frac{1}{2}M\dot{x}_M^2+\frac{1}{2}J\dot{\Phi}^2, \tag{2}$$

where dots indicate time derivatives. The potential energy consists of the gravitational energy, including the work of tension on the string and on the mass (to lift it above its equilibrium position), and the elastic energy:[7]

$$\mathscr{V}=V_{\text{grav}}+V_{\text{el}},$$

where

$$V_{\text{grav}}=\frac{1}{2}\int_0^L T[x'(z)]^2dz+\frac{1}{2}Th\Phi^2, \tag{3}$$

and

$$V_{\text{el}}=\frac{1}{2}\int_0^L EI[x''(z)]^2dz. \tag{4}$$

The tension T is considered constant along the fiber, since we neglect the small effect of the fiber's mass, and assume displacements small enough so that nonlinear increase of tension with displacement can be neglected as well.

By varying the action $S=\int(\mathscr{T}-\mathscr{V})dt$ with respect to $x(z)$, x_M and Φ, we get the equations of motion, which must be supplemented with the appropriate boundary conditions. In our case, the boundary conditions are $x(0)=x'(0)=0$, since we are considering the fiber rigidly clamped at $z=0$. The Euler–Lagrange equations are

$$-EIx^{(iv)}(z)+Tx''(z)=\rho\ddot{x}(z), \tag{5}$$

$$EIx'''(L)-Tx'(L)=M\ddot{x}_M, \tag{6}$$

and

$$-EI[x''(L)+hx'''(L)]=J\ddot{\Phi}. \tag{7}$$

Equation (5) is the continuous beam equation,[8] and Eqs. (6) and (7) are the force and torque equations for the mass, respectively. Eqs. (6) and (7) complete the four boundary conditions needed for the fourth-order differential Eq. (5) for $x(z)$.

II. SOLUTION

To calculate the admittance $Y(\omega)$, we assume there is a force of the form $Fe^{i\omega t}$ applied to the center of mass of the test mass, and therefore a dependence on time of all variables also proportional to $e^{i\omega t}$. The force has to be added as a term to the left side of Eq. (6). The general solution to the beam equation and the $z=0$ boundary conditions is $x(z,t)=x(z)e^{i\omega t}$, with

$$x(z)=A(\cos kz-\cosh k_ez)+B(\sin kz-k/k_e \sinh k_ez), \tag{8}$$

where

$$k=\sqrt{\frac{-T+\sqrt{T^2+4EI\rho\omega^2}}{2EI}}, \tag{9}$$

and

$$k_e=\sqrt{\frac{T+\sqrt{T^2+4EI\rho\omega^2}}{2EI}}. \tag{10}$$

The function k is the wave number of an elastic fiber, which for low frequencies reduces to $k \approx \sqrt{\rho/T}\omega$, the wave number of an ideal string. The function k_e is the wave number associated with the flexural stiffness of the fiber, which at low frequencies is $k_e \approx \sqrt{T/EI}$. The fiber's stiffness causes it to exhibit smooth changes in slope with a distance scale k_e^{-1} at the top and bottom of the wire, instead of the sharp slope changes a pure string would show. Notice that since E is complex, both k and k_e are also complex functions of frequency.

The assumption that the shearing due to the tension is negligible can be explained now in a clearer way. Since $T/k'GA\sim(k_er)^2$, where r is the fiber diameter, the neglect of shearing is equivalent to saying that the fiber diameter is small compared to the elastic length scale k_e^{-1}. This condition is considerably more restrictive than the condition for neglect of shearing in beams with no tension, namely, that the beam diameter-to-length ratio be small, but it is also typically satisfied in the thin wires, stressed to a substantial fraction of their breaking stress, typically used in gravitational wave interferometer pendulums.

We substitute Eq. (8)'s expression for $x(z)$ into Eq. (7), and solve for the ratio B/A. We use Eq. (6) to solve formally for the external force needed to establish an arbitrary sinusoidal motion of the mass:

$$F=-M\omega^2x_M+Tx'(L)-EIx'''(L). \tag{11}$$

The impedance $Z(\omega)=Y^{-1}(\omega)$ can then be obtained:

$$Z(\omega)\equiv\frac{F}{i\omega x_M}=\frac{K(\omega)-M\omega^2}{i\omega}, \tag{12}$$

with $K(\omega)$ an effective spring constant, given by

$$K(\omega)=\frac{Tx'(L)-EIx'''(L)}{x(L)+hx'(L)} \tag{13}$$

or, more explicitly,

$$\begin{aligned}K(\omega)=&\,\omega\sqrt{\rho EI}[(hT-J\omega^2)\sqrt{T^2+4EI\rho\omega^2}(k\sin kL+k_e\cos kL\tanh k_eL)+\omega T\sqrt{\rho EI}\sin kL\tanh k_eL\\&+(T^2+2EI\rho\omega^2)\cos kL+2EI\rho\omega^2/\cosh k_eL][EI\sqrt{T^2+4EI\rho\omega^2}\{k[(k_eh)^2-1]\cos kL\tanh k_eL\\&+k_e[(kh)^2+1]\sin kL\}+[h(T^2+4EI\rho\omega^2)-JT\omega^2]\sin kL\tanh k_eL+2\omega^3J\sqrt{\rho EI}(\cos kL-1/\cosh k_eL)]^{-1}.\end{aligned} \tag{14}$$

At low frequencies, $K(\omega)\approx T/(L+h)$, the gravitational spring constant of an pendulum of length $L+h$. In general, however, $K(\omega)$ is a complicated complex function of frequency.

Equations (12) to (14) give an exact analytical expression for the admittance (within the assumptions of our model). We can then calculate the real part of the admittance and thus from Eq. (1) the thermal noise amplitude. Notice that since $Y(\omega)$ depends in a complicated way on E, the only complex parameter, its real part is a cumbersome function of frequency and the rest of the parameters. An analytical expression for $x^2(\omega)$ is therefore neither easily obtainable nor illuminating, but Eqs. (12) to (14) provide a useful expression to numerically calculate $x^2(\omega)$ for any given set of parameters.

The spectrum $x^2(\omega)$ will exhibit peaks at the resonant frequencies ("pendulum," "rocking" and "violin" modes, as illustrated in Fig. 3). The width of each resonance is determined by the mechanical loss of the system due to the fiber's anelasticity. From the numerical results, quality factors Q can be obtained for each peak, by calculating the ratio of each resonance frequency to the width of the resonance at half power. We have made no assumptions concerning the frequency dependence of the material's loss factor, so our results may be used to describe a pendulum made with fibers of any anelastic material.

Note here the difference in method, compared with previous approaches to the problem: We solve most directly for the power spectrum $x^2(\omega)$. Modal Q's are obtained as an easy by-product. In previous methods, modal Q's had to be determined by separate calculations, then plugged into a modal expansion.

III. NUMERICAL RESULTS

We assumed a mass $M=10$ kg, suspended from a single tungsten wire of length $L=30$ cm, with a diameter of 5×10^{-2} cm. The mass therefore provides a tension $T=9.8\times10^{6}$ dyn, about half of the breaking stress. These dimensions yield $\rho=3.78\times10^{-2}$ cm and $E_0I=6.11\times10^{5}$ dyn cm^2. We took $\phi=10^{-3}$, independent of frequency, a rough approximation to the measurements of Kovalik and Saulson.[9] We assumed the mass is a 20-cm-long, 20-cm-diam fused quartz cylinder hung with its axis horizontal, giving $J=5.83\times10^{5}$ g cm^2. The wire is attached at the top of the mass, so $h=10$ cm. [With these values, the parameter indicating the relative importance of shearing to bending is $T/(k'GA)\approx5\times10^{-3}$. The smallness of this number justifies neglecting shear in the rest of the calculation.]

The calculated thermal noise spectral density $\sqrt{x^2(\omega)}$ is shown in Fig. 1, for frequencies between 0.1 and 10^4 Hz. The low frequency peaks of the pendulum and rocking mode can be observed, together with the violin modes at higher frequencies. The thermal noise amplitude falls like $f^{-5/2}$ from 3 to 150 Hz.

By increasing the frequency resolution in the vicinity of each peak, we determined the frequencies and quality factors for each mode, up to the tenth violin mode, shown in Table I. The Q of the lowest frequency mode is the highest, the pendulum design enhancing the system Q over the value of ϕ^{-1} for the wire itself by a factor of about 340. The Q of the rocking mode is considerably lower than the pendulum mode's, by a factor of 6. The first violin mode's Q is also lower than the pendulum mode by a factor of 6, and it decreases further with mode number, with Q_n^{-1} being a quadratic function of the mode number. The violin frequencies also exhibit some anharmonicity, with ω_n/n being a quadratic function of n. The behavior of the violin modes' frequencies and Q's are shown in Fig. 2.

IV. DISCUSSION

The Q obtained for the pendulum mode agrees with previous estimates.[5] The original argument used to determine Q follows from assuming that the only mechanical loss is the fraction of the mechanical energy associated with flexing of the wire. Therefore, the pendulum loss Q_p^{-1} is given by $\phi E_{\rm el}/E_{\rm grav}$. The energy ratio, in turn, is equal to the ratio of spring constants, $k_{\rm el}/k_{\rm grav}$, which are, respectively, $\sqrt{TEI}/2l^2$ and T/l. Here, we have to take as effective pendulum length l, the distance $L+h$ from the clamped end to the mirror center of mass. Let us define a "Q enhancement factor" as the product of a mode's Q times ϕ. (For a simple mass-spring oscillator, $Q\phi=1$.) When using the parameters of our model pendulum, this argument gives a value for the product $Q_p\phi$ of 320, very close to the value of 342 that we obtained by the method of this paper.

Our solution depends on our assumption that we can treat the tension T as a purely real number. This approximation will hold as long as the tension's fractional change due to stretching of the fiber is small compared to $1/Q\phi$. This will be true so long as the thermal displacements of the fiber are smaller than $1/\sqrt{Q\phi}$ times the fiber diameter. This condition is well satisfied in our model: The rms amplitude for the pendulum mode is the thermal energy k_BT_0 divided by the characteristic spring constant, $Mg/(L+h)$, or $x_{p,\rm rms}\sim 3\times10^{-10}$ cm (at room temperature). For the violin

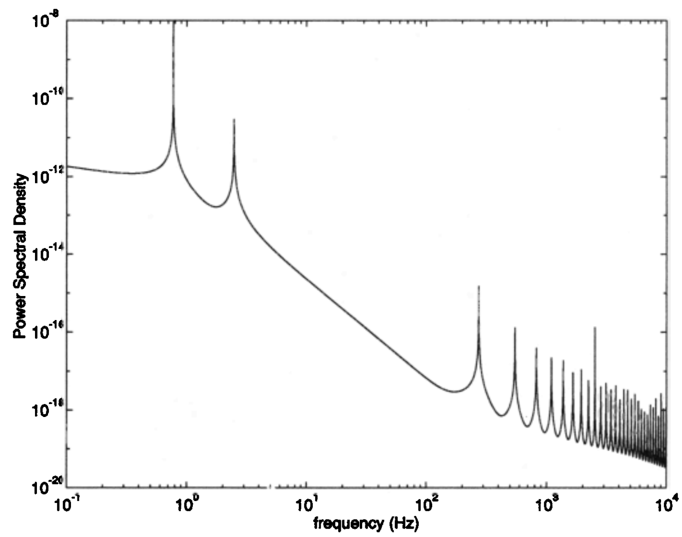

FIG. 1. Thermal noise spectral density $\sqrt{x^2(\omega)}$, in cm/$\sqrt{\rm Hz}$, for $T_0=300$ K.

TABLE I. Frequencies in Hz and quality factors Q obtained for the first 12 modes (pendulum, rocking, and 10 violin modes).

Frequency (Hz)	0.77	2.47	273	547	821	1098	1376	1657	1942	2231	2523	2821
$Q\times10^{-3}$	342	57	58	51	44	36	30	24	20	17	14	12

modes, the spring constant is $Mw_n^2 \sim M^2 g/(\rho L^2)(n\pi)^2$, and $x_{n,\text{rms}} \sim 10^{-12}\,\text{cm}/n$.

If we want to compare our results for the Q's of the higher modes to the prediction of the "energy ratio" argument, we need a method of calculating the appropriate energies. We could (numerically) calculate the mode shapes $x(z)$ for each mode, and then plug these functions in Eqs. (3) and (4), in order to obtain the elastic and gravitational energies. However, for any realistic choice of the parameters involved, this turns out to be an ill-conditioned numerical problem, requiring very high precision, long computing times, and providing little physical insight. We propose instead to use

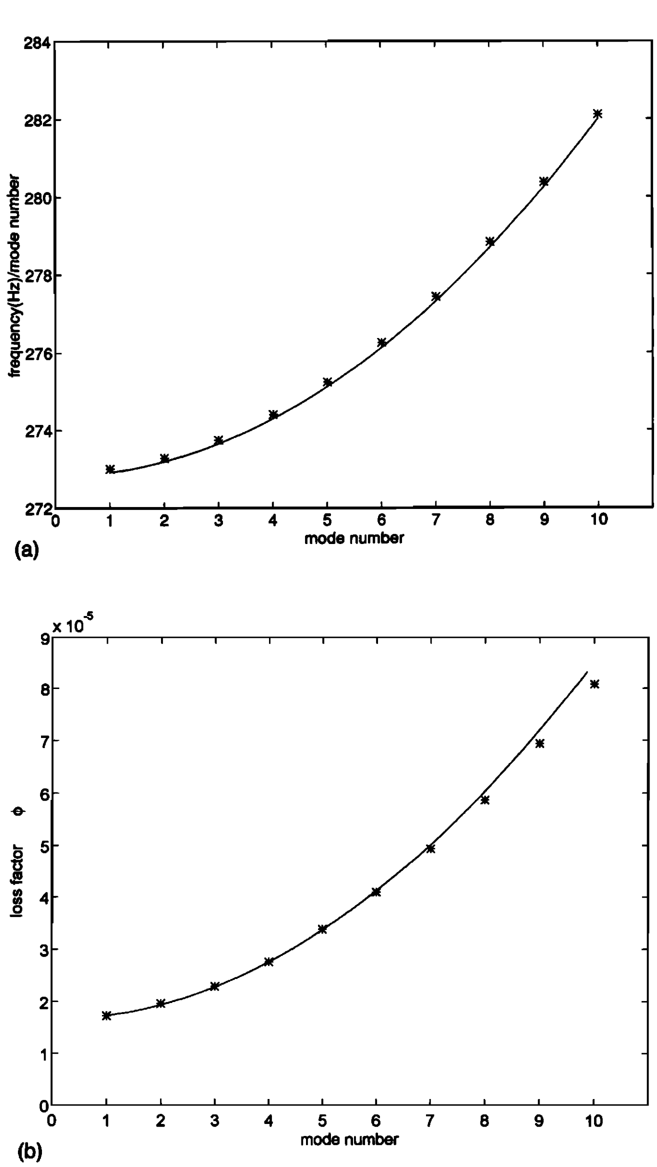

FIG. 2. (a) The anharmonicity of the violin mode frequencies is shown in the graph of f_n/n vs n, with f_n frequencies in Hz and n the mode number. The solid curve is drawn using Eq. (16). (b) Violin mode loss factors $\phi_n = Q_n^{-1}$ vs mode number n. The solid curve is drawn with Eq. (17).

simplified mode shapes that only approximately satisfy the equations and boundary conditions, but for which the computations are tractable. We took as an approximate mode shape

$$x(z)=C\left[\sin kz-\frac{k}{k_e}\,(\cos kz-e^{-k_e z})-\frac{k}{k_e}\cos kL\right.$$
$$\left.\times\left(1+\frac{\tan kL}{kh}-\frac{T}{Mh\omega^2}\right)e^{-k_e(L-z)}\right]. \qquad (15)$$

This expression was obtained assuming that $x(z)$ is the sum of the functions $\sin kz - k/k_e(\cos kz - e^{-k_e z})$, which satisfies the boundary conditions at $z=0$, and $e^{-k_e(L-z)}$, which is negligible at $z=0$. We then looked for the right coefficient between these functions from Eq. (6), keeping only leading terms. We used the low frequency expressions for k and k_e, and neglected all terms of order $EI\rho\omega^2/T^2$. [We also neglected terms with factors of $1/(k_e L)$ and $1/(k_e h)$ in front of unity.] Our approximation is good for frequencies low compared with $T/\sqrt{EI\rho}$, in our model about 10^4 Hz. This is good enough (within 10%) for all the modes we considered.

The exponential terms model the bending at the ends of the wire, which occurs over a characteristic distance k_e^{-1}. The factor multiplying $e^{-k_e(L-z)}$ varies for each mode, being very small for the pendulum mode with $\omega_p \approx T/(Ml)$, increasing to $l/h\times[1-(\omega_0/\omega_r)^2]$ for the rocking mode ω_r, and being almost exactly equal to ±1 for the violin modes. Notice that in the pendulum mode, the fiber almost does not bend at all at its bottom end, while in the rocking mode the fiber bends more at the bottom that at the top, by a factor of l/h; the violin modes exhibit symmetrical bending. (See Fig. 3.) We numerically checked the agreement of the shapes given by Eq. (12) with the exact ones for wires of large cross section, where the exact shape could be calculated.

By using Eq. (15) for $x(z)$, we can calculate the expressions for the elastic and gravitational energies, according to Eqs. (3) and (4); the answers are shown in Table II. These expressions give Q's that agree with the results obtained from our direct numerical calculation to substantially better than 10%, so they provide a good estimate of the loss factors. Notice that, although the elastic and gravitational energies both depend on the mode frequency through k^2, this dependence drops out in the result for Q's; these depend only on the "stiffness wave number" k_e times some distance, different for each mode.

The result for the Q of the first violin mode surprised us, since we expected it to be only about one half of the Q of the pendulum mode. Our oversimplified argument was that the wire in the violin mode bends as much at the bottom as at the top, while the pendulum mode loss is produced only at the top. However, even when the effect of the finite size of the

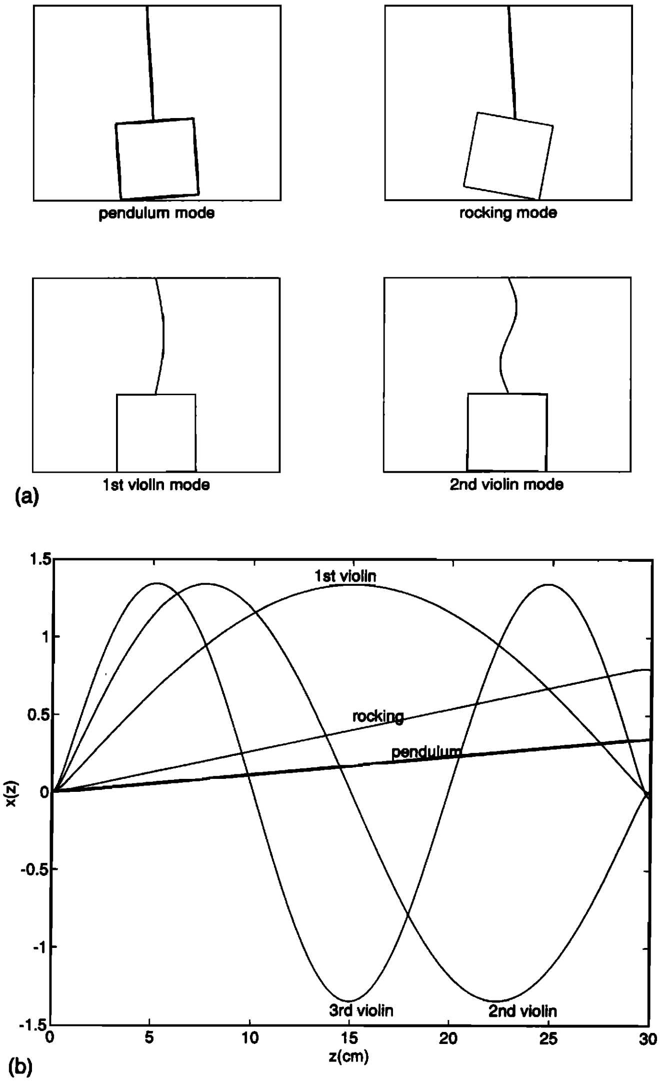

FIG. 3. Mode shapes (wire displacement) $x(z)$ (in $1/\sqrt{g}$), from Eq. (15), using the normalization condition $1=\rho\int_0^L x(z)^2dz+Mx_M^2+Jx'(L)^2$. (The pendulum and rocking mode are amplified 20 times.)

inertial mass (i.e., parameters J and h) are made small, the loss in the violin mode is 4 times more than in the pendulum mode. More surprising still, was the fact that when we solved the problem for a point mass [with vanishing J and h in Eqs. (6) and (7)], the Q of the first violin mode *is* about half of the Q of the pendulum mode, but there is no bending at all in the bottom of the fiber (as seen in the mode shapes graphed in Fig. 3).

The discrepancy can be explained when looking at the expressions for the gravitational and the elastic energies in Table II. We see that our argument is right in predicting that the elastic energy in the violin mode is a factor of 2 greater than in the pendulum mode (up to the frequency-dependent factors of k^2): $V_{\mathrm{el},v}/V_{\mathrm{el},p} = 2(k_v/k_p)^2$. But, in addition, the gravitational energy in the violin mode is *smaller*, by a second factor of 2 (and again a ratio of k^2 factors) compared to the pendulum mode: $V_{\mathrm{grav},v}/V_{\mathrm{grav},p} = 1/2(k_v/k_p)^2$. Comparing ratios of elastic to gravitational energies, we can now see why the Q of the first violin mode is at least four times smaller than in the pendulum mode, and even more if the dimension h of the mass is comparable to the length of the string (as in our example).

When the mass is a point, the violin mode would have the *same* form of the elastic energy as the pendulum mode, since the string does not feel any need to bend at the bottom to "save" the energy of rotating the mass. The point mass limit is only approached for *very* small J and h, satisfying $k_eh\ll 1$ and $Jk_e/(\rho L^2)\ll 1$. But, just as we saw above, the gravitational energy in the violin mode is smaller than in the pendulum mode by a factor of 2. The similarity in this term comes from the fact that it depends on the overall shapes of the modes, which apart from differences right near the mass are almost the same in the point mass case as in the extended mass case. Thus we can see why the violin modes should have Q's a factor of 2 lower than the pendulum mode, when the mass is effectively a point.

This line of reasoning is consistent with the arguments presented by Gillespie and Raab.[6] They consider a mirror hung from two wire loops, and therefore the mass is constrained in its rotation; the wire must bend at both ends in the pendulum mode as well as in the violin modes. Their prediction that the violin modes in such a pendulum should have Q's a factor of 2 lower than that of the pendulum mode is thus easy to understand: The expressions for the elastic energy are the same for either kind of mode, while the expressions for the gravitational energy differ by a factor of 2, just as in the cases discussed above.

The anharmonicity of the violin modes comes from the fact that the wire will bend more with increasing mode number, and therefore the elastic energy will grow with respect to the gravitational energy. This will affect both the frequencies and the loss factors, although the changes in loss factors will be more obvious since they are more sensitive to changes in elastic energy. We can get an approximate expression for the anharmonic behavior by expanding the impedance $Z(\omega)$ [Eq. (12)] near the resonance frequencies. We obtain the corrections to the frequencies from its imaginary part, and to the loss factors from its real part. Proceeding in this way, we find

$$\omega_n=\sqrt{\frac{T}{\rho}}\,\frac{n\pi}{L}\left[1+\frac{2}{k_eL}+\frac{1}{2}\left(\frac{n\pi}{k_eL}\right)^2\right], \tag{16}$$

TABLE II. Gravitational and elastic energies calculated from Eqs. (15), (3), and (4). Expressions and values for the quality factors, calculated from the ratio of gravitational and elastic energies.

Mode	Pendulum	Rocking	Violin
V_{grav}	$\frac{1}{2}C^2Tk^2(L+h)$	$\frac{1}{2}C^2Tk^2L(L+h)/h$	$\frac{1}{2}C^2Tk^2L/2$
V_{el}	$\frac{1}{2}C^2EIk^2k_e/2$	$\frac{1}{2}C^2EIk^2k_e[h^2+(L+h)^2]/(2h^2)$	$\frac{1}{2}C^2EIk^2k_e$
$Q\phi=V_{\mathrm{grav}}/V_{\mathrm{el}}$	$2k_e(L+h)$	$2k_eLh(L+h)/[h^2+(L+h)^2]$	$k_eL/2$
$Q\phi$(our model)	320	56	60

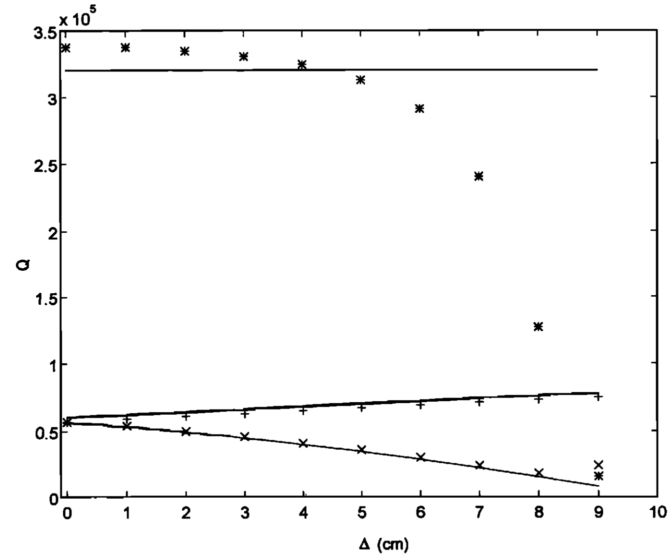

FIG. 4. Quality factors Q vs distance Δ in cm, for the pendulum (*), rocking (+), and first violin mode (×). The curves are drawn using the expressions in Table II.

$$Q_n^{-1}=\phi\,\frac{2}{k_eL}\left(1+\frac{(n\pi)^2}{2k_eL}\right). \tag{17}$$

The corrections due to the mass of the string and the inertia of the mass (not shown above) are smaller and *decrease* with mode number, while the corrections due to the flexural stiffness of the fiber *increase* with mode number. The curves of frequency and Q versus mode number according to Eqs. (16) and (17) are shown in Fig. 2, where it can be seen that the agreement with the exact results obtained is very good.

As a last check on our model and approximations, we worked the numerical solution for quality factors of the different modes when the length of the fiber L was increased and h decreased by the same amount Δ, keeping $L+h$ constant. This represents "drilling" a hole of depth Δ in the mirror substrate to attach the fiber closer to the center of mass, but keeping the mass at the same height. The results are shown in Fig. 4, where we can see that the pendulum mode Q is more or less independent of the parameter Δ for small values (as the approximate formula in Table II predicts), but it drops rather dramatically for larger values. The Q factors of rocking and violin modes do follow the formulae we provided above, with the rocking mode Q decreasing and the violin modes Q's increasing with Δ.

The test masses in gravitational wave interferometers are almost always hung from two or four wires, as opposed to a single wire in our model. Is our model then appropriate to describe such a system? A two wire system behaves the same in the x-z plane as an equivalent one wire pendulum, so it can be modeled directly, with only minor changes to the appropriate values of the various constants defining the wire. In a four wire design, like the one studied by Gillespie and Raab,[6] the rocking mode is prevented, and there would be qualitative differences, as we commented before. In Eq. (15), the factor multiplying the second exponential would be different, approximately ± 1 for all the modes, changing the elastic loss in the pendulum mode. Logan *et al.*[7] assumed the mass is held level by a control system, so their results are like the four wire case.

V. CONCLUSIONS

Our method of calculating thermal noise in a pendulum provides a straightforward numerical way of calculating the thermal noise spectral density, without having to separately calculate the loss factors ϕ_i needed in the modal expansion approach. Instead, we can obtain the effective values of ϕ_i as a by-product of our calculation; we give approximate formulae for these. The anharmonicity of the violin modes can also be derived from our model. When we then plug these results into a modal expansion as in Refs. 5 and 6, equivalent results are obtained. It is also a simple matter, using this method, to determine the spectrum with any form of the function $\phi(\omega)$. We obtain predictions about the magnitude of the thermal noise spectrum for the specific case considered, which has numbers close to the pendulums used in gravitational wave prototypes.

We can also conclude that the spectrum improves with the length $L+h$ of the pendulum, although an excessive ratio L/h (say, greater than 10) would lower significantly the pendulum mode Q. The dependence of Q's with mass comes only through the factor $\sqrt{T/EI}$. If the wire is kept at a fixed ratio of its breaking stress, then $Q \sim M^{-1/2}$. This implies that the thermal noise amplitude spectral density above the pendulum frequency will scale with the mass as $M^{-1/4}$.

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