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ON THE PHYSICAL SIGNIFICANCE OF THE RIEMANN TENSOR

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Some of the difficulties of interpretation of general relativity theory can be avoided by employing the tetrad formalism. The timelike vector of any tetrad is identified physically with the velocity vector of an observer, the spacelike vectors with the directions of local Cartesian coordinate axes used by him. It is found that the Newtonian concept of nonrotation is represented most closely if the spacelike vectors undergo Fermi propagation along the observer's world-line.

The behaviour of free particles is investigated by referring to tetrads the equation of geodesic deviation, which provides an immediate physical interpretation for the Riemann tensor. A simple model of a gyroscope is constructed by modifying Papapetrou's equations of a spinning test particle. The behaviour of this model supports the interpretation of Fermi propagation.

The interpretation is developed by investigating the discontinuity in the Riemann tensor across the boundary of a world-tube of matter, using Lichnerowicz's conditions.

1. Introduction

A difficulty in general relativity theory is the lack of what might be called a theory of measurement. One learns that all coordinate systems are equivalent to one another, but one does not learn systematically how to choose the appropriate coordinate system in which to calculate this or that quantity to be compared with observation. Coordinate systems are usually chosen for mathematical convenience, not for physical appropriateness. This would not matter if calculations were always carried out in a manner independent of the coordinate system, but this is not the case. The result is fruitless controversy, like that over the harmonic coordinate condition.

This difficulty is not just an accidental one. It is an inherent difficulty of general relativity theory, which arises because the general theory reduces to the special theory in the neighbourhood of any event in space-time, and it happens that this locally Minkowskian property of the general theory can be expressed in terms of a particular class of coordinate systems. This suggests the idea that there might be some class of coordinate systems in the general theory

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which is a satisfactory generalization of the class of Minkowskian systems, not just in an infinitesimal neighbourhood but in an extended region of space-time. It seems to the present writer that this idea must be received with caution, because it ignores the role of the observer, in the sense that it need not relate the choice of a particular coordinate system to the results of measurements made using that coordinate system.

The danger arises from the special theory, where there is a preferred class of coordinate systems, namely that class for which the metric tensor takes the Minkowskian form*

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

There is consequently a preferred class of hypothetical observers, namely those who are at rest (i. e. have world-lines $x^r =$ constant) in one of the preferred coordinate systems.

Consider the observers who are at rest in any one particular Minkowskian system. Each of them will make measurements by referring the positions of neighbouring particles, the directions of incoming null rays, and so forth, to a set of local coordinates which he carries along his world line. To fix ideas, it is simplest to suppose that each employs local Cartesian coordinates. Then at each event on the world-line of one of the observers, his coordinate system may be represented by three spacelike unit vectors, orthogonal to one another and to his 4-velocity vector, which point along the coordinate axes at that event. The peculiar thing about Minkowskian space-time is that it is possible for all the sets of unit vectors to be oriented in the same way, along the whole of each world-line, and also on the world-lines of different observers. The coordinates used by each observer for measurements in his neighbourhood can differ merely by a translation from those of the Minkowskian system at large. It is as if the measuring rods used by different observers fitted together neatly to form the "mollusc of reference".

However, when one goes over into general relativity, with a general Riemannian space-time, there is no longer any obviously preferred coordinate system** and so no preferred class of observers can be defined in the way used above. The timelines $x^r =$ constant (assuming that the coordinate system is such that these lines are timelike) will in any chosen coordinate system define a family of observers, but if one of these observers uses the three spacelike coordinate directions to define his local coordinate axes, then these axes will in general not be orthogonal to one another or to the observer's world-line, and the angles between them will vary from event to event.

This is why it is a mistake to refer measurements, made by a hypothetical observer, to a local coordinate system which is part of the coordinate system

*$c = 1$ here and throughout. The range and summation conventions are assumed for Greek indices over 0, 1, 2, 3 and for Latin indices over 1, 2, 3.

**Cosmological space-times and other space-times admitting groups of motions may have preferred coordinate systems defined by their special symmetry properties, but the present argument is about general space-times which cannot be assumed to have such symmetries.

at large. Because of irregularities in the coordinate system at large, such a description cannot bear any convenient or consistent relation to the actual measurements made by real observers. It is more convenient and natural to assume that each observer uses at each event local Cartesian coordinate axes, defined by three spacelike unit vectors orthogonal to each other and to his 4-velocity. The results of possible measurements, referred to these Cartesian axes by an observer with this 4-velocity, should be compared with the *physical components* of the corresponding tensor, got by contracting that tensor with the 4-velocity vector and the three spacelike vectors. These four unit vectors form an orthonormal *tetrad* (Vierbein, tetrapod, quadruped, 4-nuple) at each event of the world-line.

Having divorced the space axes from the coordinate system, one may as well complete the process by distinguishing the observers' world-lines from the timelines. Then the observers and the space axes they use are characterized completely by tetrads of vectors given at every point of space-time. In general it is not possible to find a coordinate system at large which incorporates the local coordinates used by different observers, because the latter, defined by the tetrads, will not in general be holonomic. It is the dichotomy between congruences of *curves*, to which the tetrad vectors are tangent, and families of coordinate *surfaces*, which, geometrically speaking, lies at the root of the difficulties about measurements in general relativity theory. To avoid these difficulties one may develop parts of the theory in terms of tetrads. One then has the advantage of being able, at every stage in a calculation, to identify results of the calculation with measurements made by a specific observer or observers using specified coordinates.

Of course it is not necessary that the observers should be attached to material particles or anything of that kind. They are introduced only for the easier visualization of ideas of measurement, and only kinematical, not dynamical, properties need be ascribed to them*.

2. Tetrad Analysis. Deviation Equations

The tetrad formalism has been used before in general relativity, in unified theories based on distant parallelism (cf. Levi-Civita 1929), in the theory of the Dirac electron (Fock 1929), and in cosmological theory (McCrea and Mikhail 1956). Ideas similar to those developed here have been put forward by Band (1942) in discussing Rosen's flat space-time theory. The present formalism is modified from Eisenhart's (1949).

In both Greek and Latin suffix alphabets, only letters from the *second half* $(\mu, \nu, \ldots; m, n, \ldots.)$ are used for *tensor and affinity indices.* Letters from the *first half* $(\alpha, \beta, \ldots; a, b, \ldots)$ are used for *labels for tetrad quantities.* Thus the tetrad vectors are written $\lambda_\alpha{}^\mu$; here μ is the vector index and α the label.

*The idea of "measurement" is just the interaction of two physical systems. The physical components of tensors are objectively characteristic of phenomena and no subjective attributes need be attached to the observers.

This notation makes it unnecessary to put tetrad labels in parenthesis – thus: ${\lambda_{(\alpha)}}^{\mu}$ – as is sometimes done, but introduces ambiguity when indices are given particular values. Since this happens much more often for labels than for coordinate indices, it simplifies the appearance of formulae very much if one uses the convention: *An index having a particular numerical value is to be read as a coordinate index only if a dash is attached to it, and as a tetrad label otherwise*. Thus ${\lambda_1}^{2'}$ means ${\lambda_\alpha}^{\mu}$ with $\alpha = 1$, $\mu = 2$.

It is convenient to assume that the tetrad is so labelled that ${\lambda_0}^{\mu}$, abbreviated λ^μ, is timelike and the three ${\lambda_a}^{\mu}$ spacelike. The possibility that one or more tetrad vectors might be null is specifically excluded, because of the physical interpretation. The coordinate system is assumed to satisfy the continuity conditions of Lichnerowicz (1955, Chapter I). Writing the Minkowski eta

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$$

the orthogonality relations for the tetrad vectors take the form

$${\lambda_a}^{\mu} g_{\mu\nu} {\lambda_\beta}^{\nu} = {\lambda_\alpha}^{\mu} \lambda_{\beta\mu} = \eta_{\alpha\beta} \tag{2.1a}$$

where as usual

$$\lambda_{\beta\mu} = g_{\mu\nu} {\lambda_\beta}^{\nu}. \tag{2.1b}$$

If now[1]

$$\lambda^{\alpha\mu} = \eta^{\alpha\beta} {\lambda_\beta}^{\mu}, \tag{2.1c}$$

(so that $\lambda^{0\mu} = {\lambda_0}^{\mu} = \lambda^\mu$, while $\lambda^{a\mu} = -{\lambda_a}^{\mu}$) then (2.1a) implies

$${\lambda^\alpha}_{\mu} {\lambda_\beta}^{\mu} = {\delta^\alpha}_{\beta}. \tag{2.1d}$$

Further, it follows from (2.1) that $\det|{\lambda_\alpha}^{\mu}| \neq 0$, so that by considering the corresponding matrices or by constructing ${\Lambda^\alpha}_{\mu}$ such that

$${\lambda^\mu}_{\alpha} {\Lambda^\alpha}_{\nu} = {\delta^\mu}_{\nu} \tag{2.2a}$$

and multiplying both sides by ${\lambda^\beta}_{\mu}$, one finds that

$${\lambda_\alpha}^{\mu} {\lambda^\alpha}_{\nu} = {\delta^\mu}_{\nu}, \tag{2.2b}$$

and it follows from (2.1b) and (2.1c) that

$$\lambda_{\alpha\mu} {\lambda^\alpha}_{\nu} = {\lambda^\beta}_{\mu} \eta_{\beta\alpha} {\lambda^\alpha}_{\nu} = g_{\mu\nu} \tag{2.2c}$$

and

$${\lambda_\alpha}^{\mu} \lambda^{\alpha\nu} = g^{\mu\nu} \tag{2.2d}$$

and so forth. Thus $\eta_{\alpha\beta}$ takes the place of an indicator. If now $A_{\mu\nu}$ is any tensor, one may write four vectors $A_{\alpha\nu} = {\lambda_\alpha}^{\mu} A_{\mu\nu}$ and sixteen scalars

$$A_{\alpha\beta} = {\lambda_\alpha}^{\mu} {\lambda_\beta}^{\nu} A_{\mu\nu}. \tag{2.3}$$

[1] The position of the second index μ corrected by the editor.

These last are the physical components of $A_{\mu\nu}$ for the observer with velocity λ^{μ}, using local Cartesian coordinates defined by the $\lambda_a{}^{\mu}$.

It follows trivially from (2.1a) that the physical components of the metric tensor are everywhere, for all observers and all reference frames, the Minkowski eta. The physical meaning of this is that one can get no information from a knowledge of the metric tensor at a single event.

The non-Minkowskian character of space-time, embodied in the change in orientation of the quadrupeds from event to event, is conveniently described by the covariant derivatives of the tetrad vectors, which in effect take the places of the Christoffel symbols. The physical components of these derivatives are just the Ricci rotation coefficients*

$$\gamma_{\alpha\beta\gamma} = -\gamma_{\beta\alpha\gamma} = \lambda_{\alpha\mu;\nu}\lambda_{\beta}{}^{\mu}\lambda_{\gamma}{}^{\nu}. \tag{2.4}$$

There are 24 independent γ's. One may eliminate the Christoffel symbols hidden in (2.4), obtaining

$$\gamma_{\alpha\beta\gamma} = \frac{1}{2}\lambda_{\alpha}{}^{\mu}\lambda_{\beta}{}^{\rho}\lambda_{\gamma}{}^{\nu}\lambda^{\delta}{}_{\sigma} \;\; P^{\sigma\pi\tau}{}_{\mu\rho\nu}\lambda_{\delta\pi,\tau}, \tag{2.5}$$

where

$$P^{\sigma\pi\tau}{}_{\mu\rho\nu} = \delta^{\sigma}_{\mu}\delta^{\pi\tau}_{\rho\nu} + \delta^{\sigma}_{\nu}\delta^{\pi\tau}_{\rho\mu} - \delta^{\sigma}_{\rho}\delta^{\pi\tau}_{\mu\nu}, \quad \delta^{\pi\tau}_{\mu\nu} = \delta^{\pi}_{\mu}\delta^{\tau}_{\nu} - \delta^{\pi}_{\nu}\delta^{\tau}_{\mu}.$$

In what follows, the derivative of any tensor $A^{\mu\cdots}{}_{\nu\ldots}$ along the curve to which $\lambda_{\alpha}{}^{\mu}$ is tangent will be written

$$\frac{\delta A^{\mu\cdots}{}_{\nu\ldots}}{\delta s^{\alpha}} = A^{\mu\cdots}{}_{\nu\ldots;\rho}\lambda_{\alpha}{}^{\rho}\left(\frac{\delta A^{\mu\cdots}{}_{\nu\ldots}}{\delta s} = A^{\mu\cdots}{}_{\nu\ldots;\rho}\lambda^{\rho}\right). \tag{2.6}$$

The physical interpretation of the γ's is seen most directly by writing the equations of a geodesic in terms of them: if $v^{\mu} = \dfrac{dx^{\mu}}{d\tau}$ is the unit tangent vector to a geodesic, then

$$0 = \frac{\delta v^{\mu}}{\delta\tau}\lambda^{\alpha}{}_{\mu} = \frac{dv^{\alpha}}{d\tau} - \gamma^{\alpha}{}_{\beta\gamma}v^{\beta}v^{\gamma}, \tag{2.7}$$

where $v^{\alpha} = \lambda^{\alpha}{}_{\mu}v^{\mu}$ are the physical components of v^{μ}, which is to be interpreted as the 4-velocity of a spherically symmetric test particle**[2]. The proper-time derivatives of the 4-velocity do not vanish, but are given by (2.7). The deviations of the tetrads from the Minkowskian arrangement are represented by the γ's. Part of the deviation is of course unavoidable, but part may arise from a peculiar choice of observers and of reference frames used by

*A semi-colon denotes covariant differentiation, a comma partial differentiation.

**Test particles are particles so small that, as is found by experience, they do not appreciably interact with one another. In general, only spherically symmetric test particles move on geodesics (see § 4 below).

[2]In the preceding footnote, 'as in found by experience' corrected to 'as is found by experience' by the editor

them. In the case of a single observer, at least, the peculiarities can be identified and eliminated, the tetrads providing a ready physical interpretation. In the above example, $\gamma^{\alpha}{}_{\beta\gamma}v^{\beta}v^{\gamma}$ is the effective gravitational 4-force. In the local Minkowskian frame (i.e. for that class of observers) in which the particle is instantaneously at rest, the effective gravitational 3-force is $\gamma^{a}{}_{00}$.

It is necessary to develop more fully the connection between the tetrads and the measurements made by an individual observer, by investigating the description which an observer would give the motion of particles or of other observers in his neighbourhood. It is assumed that an observer can, by the use of light signals or otherwise, determine the coordinates of a neighbouring particle in his local Cartesian coordinate system. These coordinates are defined in the following way:

Consider two world-lines C and C' in space-time. Let Γ be a geodesic cutting C and C' at the events P and P', and drawn orthogonal to C. Let p^{μ} be the unit tangent vector to Γ at P, directed towards P'. Then the displacement vector from P to P' is defined to be

$$\eta^{\mu} = \eta p^{\mu}$$

where η is the length of the interval PP'. In a sufficiently small neighbourhood of P the displacement vector to any other given curve will be unique, and the whole set of displacement vectors from P defines Riemannian coordinates in the instantaneous 3-space orthogonal to C at P (cf. Synge and Schild 1949, p. 60). Moreover, in the neighbourhood of P,

$$\eta^{\mu} = x'^{\mu} - x^{\mu}$$

where x^{μ} and x'^{μ} are the coordinates of P and P' respectively, and $O(\eta^2)$ is neglected (as it will be in what follows). The physical components

$$X^{\alpha} = \lambda^{\alpha}{}_{\mu}\eta^{\mu} \tag{2.8}$$

are to be identified with the Cartesian coordinates of P' as measured at P by an observer with 4-velocity λ^{μ} and local Cartesian coordinate system defined by $\lambda_{\alpha}{}^{\mu}$. Now let C and C' be two world-lines out of a whole congruence defined by a vector field λ^{μ}. Then it is not difficult to show that

$$\frac{\delta\eta^{\mu}}{\delta\tau} = \left(\lambda^{\mu}{}_{;\nu} - \lambda^{\mu}\delta\lambda_{\nu}/\delta\tau\right)\eta^{\nu} \tag{2.9}$$

where τ is proper time along C. If this *deviation equation* is multiplied by $\lambda^{\alpha}{}_{\mu}$ and written in terms of physical components, it becomes

$$\frac{dX^{a}}{d\tau} = \left(\gamma^{ab}{}_{0} + \gamma_{0}{}^{ab}\right)X_{b}, \tag{2.10}$$

where $X_b = \eta_{bc}X^c = -X^b$. This is a set of simultaneous first order ordinary differential equations for the rectangular Cartesian coordinates of an observer (or a particle) with world-line C' referred to axes in the instantaneous space of the observer with world-line C. The axes point in the directions of the $\lambda_a{}^\mu$; the first term $\gamma^{ab}{}_0$ on the right hand side of (2.10) describes the way in which these directions are propagated along C. The second term $\gamma_0{}^{ab}$ describes the relative motion of the observers with world-lines C and C'; it is just the set of physical components of the derivatives of λ^μ.

Now the manner of propagation of the axes is at the disposal of the observer; there is no prescribed way of relating to one another the directions of the space axes at different events on his world-line. In the Newtonian theory these axes would have directions fixed in absolute space, which is to say that they would be non-rotating relative to a Galilean inertial system. In general relativity there is no appeal to absolute space, but the appropriate generalization of Newtonian fixed directions may be found by investigating the dynamics of particles according to equation (2.10).

For a given world-line, the γ's in (2.10) are functions only of τ. Therefore the skew part of the coefficient of X_b represents a rigid rotation about the observer in question, while the symmetric part represents a pure strain.

The term $\gamma^{ab}{}_0$ is wholly skew, and since it is at the disposal of the observer one is inclined to require it to vanish, because it represents a rotation engendered entirely by the choice of axes. The condition for vanishing is that $\lambda_a{}^\nu$ be propagated along the world-line C according to

$$(\delta^\mu_\nu - \lambda^\mu \lambda_\nu)\delta\lambda_a{}^\nu/\delta\tau = 0. \tag{2.11}$$

This mode of propagation is called *Fermi propagation* by Walker (1935), who reaches a similar interpretation of it by different methods. The second term on the right hand side of (2.10), which depends on the motion of the neighbouring observers or particles, has a skew part which vanishes if the congruence of world-lines is normal.

The connection with Newtonian mechanics may be developed by differentiating (2.9) with respect to τ to obtain a second order deviation equation. This will be used in what follows only in the case that C and C' are geodesics, so that the equation becomes just the equation of geodesic deviation (Synge and Schild 1949, p. 93)

$$\frac{\delta^2\eta^\mu}{\delta\tau^2} + R^\mu{}_{\nu\rho\sigma}\lambda^\nu\eta^\rho\lambda^\sigma = 0. \tag{2.12}$$

Here

$$R^\mu{}_{\nu\rho\sigma} = -\Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\pi_{\nu\rho}\Gamma^\mu_{\pi\sigma} + \Gamma^\pi_{\nu\sigma}\Gamma^\mu_{\pi\rho}$$

is the Riemann curvature tensor. Now when C is a geodesic, Fermi propagation by (2.11) reduces to parallel propagation

$$\frac{\delta\lambda_a{}^\mu}{\delta\tau} = 0. \tag{2.13}$$

Referring (2.12) to axes defined along C by (2.13), one finds the equations

$$\frac{d^2X^a}{d\tau^2} + K^a{}_b X^b = 0 \tag{2.14}$$

where

$$K^a{}_b = K^a{}_b(\tau) = R^a{}_{0b0} \tag{2.15}$$

In another context, Synge (1935) has shown how these equations may be written in Hamiltonian form. If

$$H = \frac{1}{2}\left(P_a P^a + K^a{}_b X_a X^b\right) \tag{2.16}$$

then[3]

$$\frac{dX^a}{d\tau} = \frac{\partial H}{\partial P_a}, \quad \frac{dP_a}{d\tau} = -\frac{\partial H}{\partial X^a} \tag{2.17}$$

and it follows from the ordinary Hamiltonian theory that the circulation $\oint P_a dX^a$ around a closed circuit among the geodesics near C is an integral invariant.

This has an immediate physical interpretation. If C is the geodesic world-line of a freely falling observer and at some event he throws out a cloud of spherically symmetric test particles, then $\oint P_a dX^a = 0$ for this cloud, and so its motion is irrotational.

The essential point about equation (2.14) is that by observing a number of particles in the cloud, the observer can determine the full Riemann tensor in the neighbourhood of his world-line.

In Newtonian language, one would say that (2.14) give the relative acceleration of two particles in a gravitational field. It is instructive to compare them with the corresponding Newtonian equations. Consider a particle of unit mass in the field of a Newtonian potential V. Its acceleration is

$$f^i = -\frac{\partial V}{\partial x^i}.$$

The acceleration of a neighbouring particle in the inertial frame in which the first particle is at rest is

$$f^{i\prime} = -\frac{\partial V}{\partial x^i} - \frac{\partial^2 V}{\partial x^i \partial x^j} X^j$$

where X^j are the coordinates of the second particle relative to the first one. The relative acceleration is therefore

$$\frac{d^2 X^i}{dt^2} = f^{i\prime} - f^i = -\frac{\partial^2 V}{\partial x^i \partial x^j} X^j$$

which may be written

$$\frac{d^2 X^i}{dt^2} + K^i{}_j X^j = 0 \tag{2.18}$$

where

$$K^i{}_j = \frac{\partial^2 V}{\partial x^i \partial x^j}. \tag{2.19}$$

[3] In eq. (2.17) the first X in the denominator was corrected to τ by the editor.

Now equation (2.18) has exactly the same physical meaning as (2.14). Moreover on account of (2.19), Poisson's equation may be written in the form

$$K^j{}_j = 4\pi k\rho. \tag{2.20}$$

But the coefficient $K^a{}_b$ in (2.14) is defined by (2.15), so that

$$K^a{}_a = R^a{}_{0a0} = R^\alpha{}_{0\alpha 0} = -R_{00}, \tag{2.21}$$

that is, just minus the 00 physical component of the Ricci tensor. Comparing (2.20) and (2.21) would suggest that one write

$$R_{00} = -4\pi k\rho$$

as a first attempt at generalizing Poisson's equation to general relativity theory. By considering the R_{00} found by several observers with different velocities one at once obtains the equations $R_{\alpha\beta} = 0$ and hence $R_{\mu\nu}$ for empty space-time, but to get the complete Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu} \tag{2.22}$$

still requires an appeal to conservation considerations.

3. Motion of Particles in Schwarzschild Space-Time

As an example, the equations of Fermi propagation will be solved for a "circular" path in Schwarzschild space-time. Consider the Schwarzschild space-time with metric in the form

$$ds^2 = \gamma dt^2 - \left\{\gamma^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\Phi^2)\right\}, \tag{3.1}$$

where*

$$\gamma = 1 - 2mr^{-1}$$

The curve

$$C: \quad r = a, \quad \Phi = \omega t, \quad \theta = \frac{\pi}{2} \quad (a, \omega \text{ consts}). \tag{3.2}$$

is "circular" and "described at a uniform rate" in invariant senses independent of the choice of coordinate system (for example, if the metric is continued into the Schwarzschild interior to the origin world-line $r = 0$, then all points of C are at the same light-signal distance from the origin world-line).

The equations of Fermi propagation along C are linear, with constant coefficients. It is instructive to consider first a single vector α^μ. One finds, for example**

$$\frac{d^2\alpha^{1'}}{d\Phi^2} + n^2\alpha^{1'} = 0, \tag{3.3}$$

*In this section the gravitational constant K is set equal to unity by suitable choice of units.

**The coordinates are numbered $0'$, $1'$, $2'$, $3'$ in the order t, r, θ, Φ.

where $n = \beta^{-1}\mu^2$, $\beta^2 = \gamma(a) - a^2\omega^2$ and $\mu^2 = \gamma(a) - ma^{-1} = 1 - 3ma^{-1}$. The other components of α^μ satisfy the same equation, except for $\alpha^{2'}$, which is constant.

There are three cases of interest:

I). C represents a circular orbit in flat space-time: $m = 0$. Then $\mu = 1$ and $n \sim 1 + \frac{1}{2}a^2\omega^2$.

II) C represents an orbit revolving arbitrarily slowly about the central body: $\omega \to 0$ after the calculation. Then $\beta = \gamma^{1/2}$ and $n \sim 1 - 2ma^{-1}$.

III) C is geodesic: $ma^{-1} = a^2\omega^2$. Then $\beta = \mu$, and $n \sim 1 - \frac{3}{2}ma^{-1}$.

Thus in each case the values of the components of α^μ will change secularly with successive revolutions about the central body in the orbit represented by C. A vector propagated along C through an angle $\Phi = 2\pi$ (as seen from the origin world-line) will not return to its original direction. This change in direction may be given an invariant significance by considering two particles constrained to move so that α^μ is the direction of the displacement vector from one to the other. Then the relative positions of the particles, as seen from the origin world-line, will change secularly. Two effects may be distinguished, for the three cases enumerated above:

I). The change in direction is essentially the Thomas precession, a special relativity effect.

II). The change in direction results from the "inertial drag" of the central body. It is a purely g e n e r a l relativity effect, because of the limiting slow motion.

III). The change in direction is a combination of the previous two effects.

This combined effect is the geodesic precession or de Sitter-Fokker effect (de Sitter 1916; Fokker 1920). This is essentially a gravitational couple ("Gravo-magnetic effect"; Sciama 1953) exerted by the central body which causes a secular rotation in the local inertial frame for a body moving around it. This should not be confused with the rotation of the local inertial frame near an extended body resulting from the axial rotation of the body (Thirring and Lense 1919).

Now consider a tetrad of unit vectors $\lambda^{\alpha\mu}$ under parallel propagation along a geodesic C (case III above). If initially $(\Phi = 0)\lambda^{1\mu}$ and $\lambda^{2\mu}$ point in the r and θ coordinate directions, respectively, then at a general point of C, the tetrad components are

$$\begin{array}{lllll}
\mu \to & 0' & 1' & 2' & 3' \\
\lambda^{0\mu} : & (\mu^{-1} & , 0 & , 0 & , \quad \omega\mu^{-1} \quad) \\
\lambda^{1\mu} : & (-m(\gamma^{1/2}a^2\omega\mu)^{-1}\sin\mu\Phi & , \gamma^{1/2}\cos\mu\Phi & , 0 & , -\gamma^{1/2}(a\mu)^{-1}\sin\mu\Phi) \\
\lambda^{2\mu} : & (0 & , 0 & , a^{-1} & , \quad 0 \quad) \\
\lambda^{3\mu} : & (-m(\gamma^{1/2}a^2\omega\mu)^{-1}\cos\mu\Phi & , \gamma^{1/2}\sin\mu\Phi & , 0 & , -\gamma^{1/2}(a\mu)^{-1}\cos\mu\Phi)
\end{array} \tag{3.4}$$

The equation of geodesic deviation (2.14) can now be used to investigate the behaviour of two neighbouring free particles, referred to this tetrad. The following physical components of the Riemann tensor, referred to the tetrad, are needed:

$$K^a{}_b = R^a{}_{0b0} = \frac{1}{2}ma^{-3}\mu^{-2}\begin{bmatrix} 1+3\gamma\cos 2\mu\Phi & 0 & 3\gamma\sin 2\mu\Phi \\ 0 & -2 & 0 \\ 3\gamma\sin 2\mu\Phi & 0 & 1-3\gamma\cos 2\mu\Phi \end{bmatrix} \tag{3.5}$$

In (3.4) and (3.5), $\mu\Phi = \omega\tau$ where τ is proper time on C. The relative motion in the 2-direction, given by (2.14), which is in Newtonian language orthogonal both to the velocity of the particles and to the direction of the gravitational field, is not coupled to the motion in the other directions, and may be left out of consideration. The motion in the 13-plane given by (2.14) and (3.5) may be written compactly in terms of a complex variable U:

$$U = X^1 + iX^3, \quad \overline{U} = X^1 - iX^3.$$

In terms of U and $\overline{U}$, (2.14) and (3.5) yield

$$\frac{d^2U}{d\Phi^2} = \frac{1}{2}\left(U + 3\gamma\overline{U}e^{2i\mu\Phi}\right). \tag{3.6}$$

This may be integrated by elementary methods after a substitution $U = Ve^{i\mu\Phi}$. The corresponding Newtonian equation is easily found from (2.18); it is

$$\frac{d^2U}{d\Phi^2} = \frac{1}{2}\left(U + 3\overline{U}e^{2i\Phi}\right), \tag{3.7}$$

the only difference being that $\gamma = 1 - 2ma^{-1}$ and $\mu = (1 - 3ma^{-1})^{1/2}$ are replaced by unity. It must be remembered that (3.7) describes a motion relative to axes which are not rotating in a Newtonian inertial frame, while (3.6) describes a motion relative to axes defined by the parallelly propagated tetrad (3.4).

To understand these results one must first understand the physical significance of the Schwarzschild space-time. One must recognise that the Minkowskian boundary conditions represent an approximately isotropic distribution of matter, on average at rest or in uniform radial motion, at a great distance from the central body. It is not sensible to interpret this space-time as if the central body were in an otherwise completely empty universe, and if any justification is needed for this remark, it is that predictions based on the Schwarzschild model agree with observations of planets in the solar field, there being in fact a great deal of matter in the universe apart from the Sun.

With this interpretation, the time-lines of the Schwarzschild metric (3.1) may be regarded as the world-lines of observers at rest relative to distant matter. An invariant significance may be given to the coordinate directions by consideration of observations of distant "fixed" stars, that is, by integration of the equations of null geodesics. Two directions may be said to be the same when the corresponding null geodesics intersect asymptotically the same infinitely distant time-line. Now it follows from symmetry that Fermi propagation along any time-line preserves the coordinate directions; therefore an observer with a time-line for world-line is in the exceptional position that he *does* see the distant "fixed" stars in fixed directions relative to a local inertial frame defined by dynamical experiments. This is a consequence of the static character of the space-time: the time-lines are paths of a motion of space-time into itself. It is clear from the example that if one could find a space-time representing a solution of the n-body problem, there would be no observers with this property, and deviations from it would be large for an observer near one of the moving bodies.

To sum up: the calculation of n in equation (3.3) shows that a vector undergoing parallel propagation along a circular geodesic in a Schwarzschild space-time changes direction secularly, relative to the central body, and relative to observers interpreted as being at rest with respect to distant matter. The change in direction may be interpreted as the combined effect of the Thomas precession and the inertial drag of the central body. If the motion of spherically symmetric test particles is referred to parallely propagated axes, then the equations obtained are of the same form as in the Newtonian theory, the numerical constants appearing in them differing from the Newtonian values by small amounts.

It follows that in general, the local inertial frame determined by local dynamical experiments is not exactly fixed relative to the "distant" stars.

4. A Simple Model Gyroscope

It is not difficult to construct a model Foucault pendulum in order to elaborate the results of the preceding sections, but it does not add much to the discussion of free particles already given. However, there is an almost ready-made model gyroscope, which is more interesting. This is the spinning test particle of Papapetrou (1951). By making certain assumptions about the form of the energy-momentum tensor in a small world-tube, Papapetrou derived, in the limit that the tube becomes a line, the equations

$$\frac{\delta}{\delta\tau}\left(mv^\mu + v_\rho \frac{\delta}{\delta\tau}S^{\mu\rho}\right) + \frac{1}{2}R^\mu{}_{\nu\rho\sigma}v^\nu S^{\rho\sigma} = 0, \tag{4.1}$$

$$\frac{\delta}{\delta\tau}S^{\mu\nu} + v^\mu v_\rho \frac{\delta}{\delta\tau}S^{\nu\rho} - v^\nu v_\rho \frac{\delta}{\delta\tau}S^{\mu\rho} = 0. \tag{4.2}$$

for a particle described by an ordinary 4-momentum vector mv^μ together with a skew tensor $S^{\mu\nu}$ which is in effect the internal angular momentum tensor of the particle. In (4.1) and (4.2), τ is proper time along the particle world-line. The derivation is not obviously covariant but could presumably be made so. Papapetrou (with Corinaldesi, 1951) had some difficulty in going further, because[4] (4.1) and (4.2) are three equations too few to determine v^μ and $S^{\mu\nu}$ from given initial conditions. They adopted the additional conditions

$$S^{\mu\nu}\beta_\nu = 0$$

[4]The two equation numbers that follow corrected to (4.1) and (4.2) by the editor.

for the special case of Schwarzschild space-time; here β_ν is the tangent vector field to the time-lines of the metric (3.1). It would seem more satisfactory to employ instead the condition

$$S^{\mu\nu} v_\nu = 0, \tag{4.3}$$

rejected by Corinaldesi and Papapetrou for reasons which they do not make clear, which is a straightforward generalization from the corresponding equation in special relativity theory (cf. Synge 1955, p. 221, Møller 1949). The physical meaning of (4.3), which is essentially an assumption about the nature of the internal forces in the particle, is that angular momentum is conserved (see equations[5] (4.10) and (4.11) below).

Introducing (4.3) into (4.1) and (4.2), one may reduce these equations to

$$\frac{\delta}{\delta\tau}\left(mv^\mu - S^{\mu\rho}\frac{\delta v_\rho}{\delta\tau}\right) + \frac{1}{2}R^\mu{}_{\nu\rho\sigma}v^\nu S^{\rho\sigma} = 0 \tag{4.4}$$

$$\frac{\delta}{\delta\tau}S^{\mu\nu} = \left(v^\mu S^{\nu\rho} - v^\nu S^{\mu\rho}\right)\frac{\delta v_\rho}{\delta\tau} \tag{4.5}$$

so that

$$\frac{\delta S^{\mu\nu}}{\delta\tau}\frac{\delta v_\nu}{\delta\tau} = 0. \tag{4.6}$$

Hence (4.4) becomes

$$\frac{\delta}{\delta\tau}\left(mv^\mu\right) - S^{\mu\nu}\frac{\delta^2 v_\nu}{\delta\tau^2} + \frac{1}{2}R^\mu{}_{\nu\rho\sigma}v^\nu S^{\rho\sigma} = 0. \tag{4.7}$$

Contracting this with v_μ leaves only

$$\frac{dm}{d\tau} = 0 \tag{4.8}$$

so that the mass is a constant of the motion. Equation (4.7) becomes

$$m\frac{\delta v^\mu}{\delta\tau} - S^{\mu\nu}\frac{\delta^2 v_\nu}{\delta\tau^2} + R^\mu{}_{\nu\rho\sigma}v^\nu S^{\rho\sigma} = 0. \tag{4.9}$$

Now introduce the angular momentum 4-vector

$$H^\mu = \frac{1}{2}\eta^{\mu\nu\rho\sigma}v_\nu S_{\rho\sigma}, \tag{4.10}$$

where $\eta^{\mu\nu\rho\sigma}$ is the alternating tensor. The physical components of H^μ (which is orthogonal to v^μ) for an observer moving with the particle are just the components of the angular momentum 3-vector. If H^μ is introduced into (4.5), the latter becomes

$$\left(\delta^\mu_\nu - v^\mu v_\nu\right)\frac{\delta H^\nu}{\delta\tau} = 0 \tag{4.11}$$

[5] The equation numbers that follow corrected from (4.11) and (4.12) in the original by the editor.

so that the angular momentum 4-vector is Fermi-propagated along the world-line of the particle. Multiplying (4.11) by H_μ, one sees that the angular momentum invariant is constant.

Thus a spinning test particle will have a fixed angular momentum (and so, for a correctly started particle, a fixed axis of spin) relative to Fermi-propagated axes. This supports the earlier interpretation of Fermi propagation.

5. *Discontinuities in the Riemann Tensor*

The interpretation can be developed further by investigating the discontinuity in the Riemann tensor across the surface of a world-tube of matter. As a basis for such an investigation there is available the mathematical theory developed by Lichnerowicz (1955), which is based on hypotheses about the continuity of the metric tensor and its derivatives. These hypotheses are designed to ensure that the initial value problem posed by Einstein's field equations (2.22) is correctly set. They are equivalent to assumptions arrived at by O'Brien and Synge (1952) by consideration of the energy-momentum tensor in a boundary layer which is shrunk to zero thickness. Here first of all the continuity conditions for the Riemann tensor will be written in an invariant form, starting from Lichnerowicz's hypotheses.

Lichnerowicz assumes that there exists a coordinate system in which the metric tensor $g_{\mu\nu}$ and its first derivatives $g_{\mu\nu,\sigma}$ are continuous and the $g_{\mu\nu,\sigma}$ have piecewise continuous first and second derivatives.

The theory is developed by taking a coordinate system such that, say, the surface $S : x^{1'} = 0$ is a surface of discontinuity (physically, a surface across which density and stress may be discontinuous). In what follows, S is assumed to be timelike. Then according to the hypotheses, this coordinate system can be chosen so that all the $g_{\mu\nu,\rho\sigma}$ are continuous across S with the possible exception of $g_{\mu\nu,1'1'}$. The covariant components of the Riemann tensor are

$$R_{\rho\sigma\mu\nu} = [\sigma\nu,\rho]_{,\mu} - [\sigma\mu,\rho]_{,\nu} + \Gamma^{\pi}_{\sigma\mu}\,[\rho\nu,\pi] - \Gamma^{\pi}_{\sigma\nu}\,[\rho\mu,\pi] \tag{5.1}$$

(where $[\sigma\nu,\rho]$ and $\Gamma^{\pi}_{\sigma\nu}$ are Christoffel symbols of the first and second kinds). By expanding the first two terms of this, one sees at once that the only components of $R_{\rho\sigma\mu\nu}$ which may be discontinuous are those with just two indices $1'$ (but not the first two together, or the last two together, because of the skew-symmetry in these pairs of indices). This is the same as a result of O'Brien and Synge (equation 5.12 of their paper).

This result may be put into covariant form by referring $R_{\rho\sigma\mu\nu}$ to a local Cartesian coordinate system. At any point P of S, only $x^{1'}$ is so far prescribed. By a linear transformation of the* x^r which leaves $x^{1'}$ unaltered, one may introduce at P a local Cartesian coordinate system in which $g_{\mu\nu} = \eta_{\mu\nu}$.

*In this section, the range and summation conventions for Latin indices will be over 0, 2, 3 instead of 1, 2, 3.

Then at P, the unit normal to S has components $n^\mu = \delta^\mu{}_{1'}$, while any vector t^μ tangent to S has $t^1 = 0$. The continuity conditions on $R_{\rho\sigma\mu\nu}$ may be expressed as follows:

A) On $S, R_{\rho\sigma\mu\nu} n^\sigma n^\nu$ may he discontinuous, but $R_{\rho\sigma\mu\nu} t_0{}^\sigma t_2{}^\mu t_3{}^\nu$ must he continuous where n^σ is the normal to S and $t_a{}^\sigma$ are any three vectors tangent to S.

Alternatively, let $\lambda_\alpha{}^\sigma$ be a tetrad at P, with $\lambda_1{}^\sigma = n^\sigma$ and the other vectors of the tetrad chosen arbitrarily. Then

B) Only the physical components R_{a1c1} may be discontinuous across S.

Now consider an isolated world-tube of matter with S as boundary. Choose $\lambda_\alpha{}^\sigma$ so that $\lambda_1{}^\sigma$ is the normal n^σ everywhere on S. Let Δ denote the difference, at any point of S, between the interior and exterior boundary values of any tensor (labelled$_I$ and$_E$ respectively). Thus

$$\Delta R_{\rho\sigma\mu\nu} = R_{I\rho\sigma\mu\nu} - R_{E\rho\sigma\mu\nu}. \tag{5.2}$$

The physical components of $R_{\rho\sigma\mu\nu}$ must therefore satisfy

$$\Delta R_{abcd} = 0, \quad \Delta R_{abc1} = 0, \quad \Delta R_{\alpha 1\gamma 1} = Z_{\alpha\gamma} \tag{5.3}$$

say, where $Z_{\alpha\gamma} = Z_{\gamma\alpha}$ is a set of scalars defined on S, and $Z_{\alpha 1} = 0$. Then for the physical components of the Ricci tensor,

$$\Delta R_{bc} = \Delta\left(\eta^{\alpha\delta} R_{\alpha bc\delta}\right) = Z_{bc} \tag{5.4}$$

and similarly

$$\delta R_{b1} = 0, \quad \Delta R_{11} = -\eta^{ac} Z_{ac}, \quad \Delta R = 2\eta^{ac} Z_{ac}. \tag{5.5}$$

Hence for the Einstein tensor

$$\Delta G_{bc} = Z_{bc} - \eta_{bc}\eta^{ad} Z_{ad}, \quad \Delta G_{a1} = 0. \tag{5.6}$$

Introducing this into Einstein's field equations (2.22), and taking account of $T_{E\mu\nu} = 0$, one finds

$$\Delta R_{a1c1} = \Delta R_{ac} = -\kappa\left(T_{ac} - \frac{1}{2}\eta_{ac} T\right)_I. \tag{5.7}$$

Equations (5.6) and (5.7) include the condition

$$T_{I\alpha 1} = 0 \tag{5.8}$$

found by Synge and O'Brien and by Lichnerowicz.

Equation (5.7) finds a ready interpretation in terms of the equation of geodesic deviation (2.18). Consider three neighbouring freely falling spherically symmetric particles P_E, P_S, P_I outside, on, and inside S respectively, and lying in a line normal to S. Taking the normal direction again to be the 1-direction of tetrad at P_S which is parallelly propagated along the world-line of P_S, and writing $X_I{}^a$, $X_E{}^a (a = 1, 2, 3)$ for the physical components of the displacement vectors from P_I to P_S and from P_S to P_E respectively, one obtains from (2.18)

$$\begin{aligned} &\frac{d^2}{d\tau^2} X_I{}^1 + K_I{}^1{}_1 X_I{}^1 = 0, \\ &\frac{d^2}{d\tau^2} X_E{}^1 + K_E{}^1{}_1 X_E{}^1 = 0, \end{aligned} \tag{5.9}$$

where $K_I{}^1{}_1 = R_I{}^1{}_{010}$, $K_I{}^1{}_1 = R_E{}^1{}_{010}$ so that if P_I and P_E are at equal distances X from P_S, the difference between the relative accelerations is

$$\Delta \left(\frac{d^2}{d\tau^2} X \right) = -X \Delta K^1{}_1 = \kappa \left(T_{00} - \frac{1}{2} T \right)_I X. \tag{5.10}$$

Now at the surface of a world-tube of matter the normal pressure vanishes, by (5.8), and the timelike vector of the tetrad may be chosen to be the 4-velocity of the surface matter, so that $T_{I00} = \rho_I$, the surface density, and $T_I = \rho_I + p_{I2} + p_{I3}$, where $-p_{I2}$ and $-p_{I3}$ are the principal tangential stresses at the surface. Hence

$$\begin{aligned} \Delta \left(\frac{d^2}{d\tau^2} X \right) &= 8\pi k \left\{ \rho_I - \frac{1}{2} (\rho_I + p_{I2} + p_{I3}) \right\} X \\ &= 4\pi k \{ \rho_I - p_{I2} - p_{I3} \} X, \end{aligned} \tag{5.11}$$

where $k = (8\pi)^{-1} \kappa$ is the Newtonian constant of gravitation. The corresponding Newtonian result is evidently, from § 2, that the second normal derivative $\partial^2 V / \partial n^2$ has a discontinuity $4\pi k \rho$ at the surface of a gravitating body. The appearance of the stresses in (5.11) confirms the interpretation (Whittaker 1935, Synge 1937) that the sum of the density and principal stresses, and not the density alone, is the effective density of gravitational mass.

6. Conclusion

The intention of this paper has been to exhibit the intimate connection between the Riemann tensor and generally relativistic dynamics, and also the value of the tetrad technique in deducing invariant results. I hope to show elsewhere the application of these ideas to the theory of gravitational waves.

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КРАТКОЕ СОДЕЖАНИЕ

Ф. А. П и р а н и, *О физическом знамении тензора Римания*

Можно избежать некоторых затруднений при толковании общей теории относительности, употребляя формализм тетрадов. Вектор

временного рода тетрада физически идентифицирован с вектором скорости наблюдателя, вектор пространственного рода – с направлениями оси локальных картезианских координатов, которыми он пользуется. Оказалось, что ньютоновское понятие об отсутствии ротации лучше всего представлено, когда векторы пространственного рода подвергаются пропагации Фермье вдоль мировой линии наблюдателя.

Поведение свободных частиц исследуется путём подчинения тетрадам уравнений геодезической девиации. Сооружена простая модель жироскопа посредством видоизменения уравнений “Papapetrou” спиновой пробной частицы. Поведение этой модели говорит в пользу вышеуказанного толкования пропагации Фермье.

Толкование это развито на основании изучения прерывности тензора Риманна на ограничений мировой трубки вещества при употреблении условий Лихнаровича.

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