Complex Analysis[1]

–TW–

2024 年 3 月 24 日

[1]课堂教材：《*Complex Analysis*》— *Elias M. Stein*

序

天道几何，万品流形先自守；

变分无限，孤心测度有同伦。

2024 年 3 月 24 日

长夜伴浪破晓梦，梦晓破浪伴夜长

目录

第零章　课程要求

- **任课教师**：林明华
- **辅导时间**：周一 $9a.m. - 11a.m.$
- **办公室**：数学楼 210
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- **总评成绩组成**：阅读报告及汇报 20% + 期末考试 80%

第一章 *Week* 1

1.1 复数的引入

引入

下面从代数结构 (*Group*, *Ring*, *Field*) 的角度引入复数的概念.

Consider the set $\mathbb{R}^2$. *Define two operations.* $\forall (a,b),(c,d) \in \mathbb{R}^2$,

$$(a,b)+(c,d) \coloneqq (a+c, b+d) \tag{1.1}$$

$$(a,b)\cdot(c,d) \coloneqq (ac-bd, bc+ad) \tag{1.2}$$

$"\cdot"$ *is commutative.*

$"+"$, $"\cdot"$ *satisfy associative and distributive laws.*

$$(0,0): \textit{The additive identity} \tag{1.3}$$

$$(1,0): \textit{The multiplicative identity} \tag{1.4}$$

$\Rightarrow (\mathbb{R}^2, +, \cdot)$ *is a communicative ring.*

$\forall (a,b) \in \mathbb{R}^2,\ (a,b) \neq (0,0),$ *if*

$$(a,b)\cdot(x,y) = (1,0) \tag{1.5}$$

$$\Rightarrow x = \frac{a}{a^2+b^2},\ y = \frac{-b}{a^2+b^2} \tag{1.6}$$

Therefore, $(\mathbb{R}^2, +, \cdot)$ *is a field, renoted as* $\mathbb{C}$.

复数的乘法 在上述对 $\mathbb{C}$ 的定义中，唯一非平凡的点便是乘法运算 $"\cdot"$ 的定义.

下面我们从代数的方法，从另一个角度理解复数的乘法.

We may ask a question : *Can we define a "$\cdot$" and let* $(\mathbb{R}^3, +, \cdot)$ *be a field*?

However, the answer is certainly not!

Consider $M_2 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$ *equipped with the usual matrix addition and multiplication.*

Define a map σ.

$$\sigma : \mathbb{R}^2 \longrightarrow M_2 \tag{1.7}$$

$$(a, b) \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{1.8}$$

Then, σ *is bijective.*

$$\sigma(a, b) \cdot \sigma(c, d) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix} = \sigma((a, b) \cdot (c, d)) \tag{1.9}$$

$\Rightarrow \sigma$ *is an isomorphism*(同构映射).

于是复数乘法可视作复平面上带伸缩的旋转.

1.2 复数的基本性质

Some Facts

$$|Rez| \leq |z|, \ |Imz| \leq |z| \tag{1.10}$$

$$Rez = \frac{z+\bar{z}}{2}, \ Imz = \frac{z-\bar{z}}{2i} \tag{1.11}$$

性质 下面给出一些命题.

1. 三角不等式.

命题 1.2.1 (*Triangle Inequality*)**.** *Let* $z, w \in \mathbb{C}$. *Then*

$$|z+w| \leq |z| + |w| \tag{1.12}$$

证明. *Let* $z = a + bi, \ w = c + di$. *Then*

$$\Leftrightarrow \sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2} \tag{1.13}$$

$$\Leftrightarrow ac + bd \leq \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{(ac)^2 + (bd)^2 + a^2d^2 + b^2c^2} \tag{1.14}$$

□

推论 1.2.1. *If* $z, w \in \mathbb{C}$, *then*

$$||z| - |w|| \leq |z - w| \tag{1.15}$$

证明.

$$|z| = |z - w + w| \leq |z - w| + |w| \tag{1.16}$$

$$|w| = |z - w - z| \leq |z - w| + |z| \tag{1.17}$$

$$\Rightarrow |z - w| \geq max\{|z| - |w|, \ |w| - |z|\} = ||z| - |w|| \tag{1.18}$$

□

2. *Cauchy − Schwarz* 不等式.

命题 1.2.2 (*Cauchy − Schwarz*)**.** *Let* $z_1, \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$. *Then*

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) \tag{1.19}$$

证明. $\forall \lambda \in \mathbb{R},\ \vartheta \in \mathbb{R},$

$$0 \le \sum_{k=1}^{n} \left|z_k - \lambda e^{i\vartheta}\overline{w_k}\right|^2 = \sum_{k=1}^{n} (z_k - \lambda e^{i\vartheta}\overline{w_k})(\overline{z_k} - \lambda e^{-i\vartheta} w_k) \tag{1.20}$$

$$= \sum_{k=1}^{n} |z_k|^2 - 2\left(Re\ e^{-i\vartheta} \sum_{k=1}^{n} z_k w_k\right)\lambda + \lambda^2 \sum_{k=1}^{n} |w_k|^2 \tag{1.21}$$

$$= a\lambda^2 - 2b\lambda + c \tag{1.22}$$

$$\Rightarrow b^2 \le ac \tag{1.23}$$

Then

$$\left(Re\ e^{-i\vartheta} \sum_{k=1}^{n} z_k w_k\right)^2 \le \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) \tag{1.24}$$

Suppose $z = \sum_{k=1}^{n} z_k w_k = |z|\, e^{i\varphi} \in \mathbb{C},$ *let* $\vartheta = \varphi$. *Then*

$$Re\ e^{-i\vartheta} \sum_{k=1}^{n} z_k w_k = \left|\sum_{k=1}^{n} z_k w_k\right| \tag{1.25}$$

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \le \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) \tag{1.26}$$

□

1.3 课堂例题 2024 − 02 − 26

1. *Let* $z_1, z_2 \in \mathbb{C}$, $|z_1| \le 1$, $|z_2| \le 1$. *If* $|z_1 - z_2| \ge 1$, *show that*

$$|z_1 + z_2| \le \sqrt{3} \tag{1.27}$$

证明. (平行四边形对角线的平方和等于四边的平方和.)

$$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - \overline{z_1}z_2 \tag{1.28}$$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2 \tag{1.29}$$

$\Rightarrow$

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) \tag{1.30}$$

$$|z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) - |z_1 - z_2|^2 \le 3 \tag{1.31}$$

□

2. *Let* $z_1, \cdots, z_n \in \mathbb{C}$, *and let* $e_0, e_1, \cdots, e_{n+1} \in \mathbb{C}$ *be the coefficients of* $(z+1)\prod_{k=1}^{n}(z+z_k)$, *i.e.*

$$(z+1)\prod_{k=1}^{n}(z+z_k) = \sum_{k=0}^{n+1} e_k z^{n+1-k} \tag{1.32}$$

Show that $\sum_{k=0}^{n+1}(k+1)e_k z^{n+1-k} = 0$ *has a root of modulus* ≥ 1.

Specifically, try to show $n = 1$ *case.*

$\Leftrightarrow$ (*Let* $c \in \mathbb{C}$, *show* $z^2 + 2(1+c)z + 3c = 0$ *has a root of modulus* ≥ 1.)

证明. 下面对方程 $z^2 + 2(1+c)z + 3c = 0$ 的根的情况进行分类 (事实上同时对 $c \in \mathbb{C}$ 的取值进行了分类).

(1) 若方程存在实根 $z_0 \in \mathbb{R}$，下面可以证明，事实上 (1) $\Leftrightarrow c \in \mathbb{R}$.

$$z_0^2 + 2(1+c)z_0 + 3c = 0 \tag{1.33}$$

$$\Rightarrow (2z_0 + 3)c = -z_0^2 - 2z_0 \tag{1.34}$$

$$\Rightarrow c = \frac{-z_0^2 - 2z_0}{2z_0 + 3} \in \mathbb{R} \text{ 或 } z_0 = \frac{3}{2}(\text{此时 } -z_0^2 - 2z_0 \neq 0 \text{ 矛盾}) \tag{1.35}$$

于是 $c\in\mathbb{R}$，$z^2+2(1+c)z+3c=0$ 为实系数一元二次方程.

$$\Delta = 4(1+c)^2-12c = 4(c^2-c+1) > 0,\ \forall c\in\mathbb{R} \tag{1.36}$$

$$z = -1-c\pm\sqrt{c^2-c+1}\in\mathbb{R} \tag{1.37}$$

下面再对实数 $c\in\mathbb{R}$ 的范围分类讨论.

i). $c\geq 0$，则其中一根 $z=-1-c-\sqrt{c^2-c+1}<-1$，$|z|>1$.

ii). $c<0$，考虑其中一根

$$z = -1-c-\sqrt{c^2-c+1} \tag{1.38}$$

$$= -1-(\sqrt{c^2-c+1}+c) \tag{1.39}$$

由于 $c<0$，因此 $1-c>0$.

$$\sqrt{c^2-c+1} = \sqrt{c^2+(1-c)} > \sqrt{c^2} = |c| \tag{1.40}$$

$$\sqrt{c^2-c+1}+c > 0 \tag{1.41}$$

$$z = -1-(\sqrt{c^2-c+1}+c) < -1 \tag{1.42}$$

$$|z| > 1 \tag{1.43}$$

于是对于 $\forall c\in\mathbb{R}$，都有 $|z|>1$. 从而得证.

事实上，根据上述证明过程可知，若 $c\in\mathbb{R}$，则原方程必有实根，且两根均为实根，从而

$$(1): \text{方程存在实根} \Leftrightarrow c\in\mathbb{R} \Leftrightarrow \text{两根均为实根} \tag{1.44}$$

(2) 若方程无实根，即 $c\in\mathbb{C}$

□

1.4 复数域 $\mathbb{C}$ 上的拓扑概念 & 性质

Let $a \in \mathbb{C}$, *open disc of radius r centered at a*

$$D_r(a) := \{z \in \mathbb{C} \mid |z-a| < r\} \tag{1.45}$$

$$D_r^*(a) := \{z \in \mathbb{C} \mid 0 < |z-a| < r\} \tag{1.46}$$

closed disc of radius r centered at a

$$\overline{D}_r(a) := \{z \in \mathbb{C} \mid |z-a| \leq r\} \tag{1.47}$$

unit disc :

$$\mathbb{D} := D_1(0) \tag{1.48}$$

Let $\Omega \subseteq \mathbb{C}$

定义 1.4.1. $a \in \Omega$ *is an* *interior point of* Ω *if* $\exists r > 0$, s. t. $D_r(a) \subseteq \Omega$.

注. *The set of all interior points of* Ω *is called* *the interior of* Ω, *denoted by* $Int(\Omega)$.

定义 1.4.2. Ω *is open if* $\Omega = Int(\Omega)$.

注. $\mathbb{C}$ *is open.* $\varnothing$ *is open. (by convention)*

定义 1.4.3. Ω *is closed if* $\Omega^c := \mathbb{C} \backslash \Omega$ *is open.*

定理 1.4.1. *Every Cauchy sequence in* $\mathbb{C}$ *has a limit in* $\mathbb{C}$. *That is,* $\mathbb{C}$ *is Complete.*

1.5 课堂例题 $2024-03-01$

1.

$$\lim_{n\to+\infty} z_n = w \Leftrightarrow \lim_{n\to+\infty} Rez_n = Rew,\ \lim_{n\to+\infty} Imz_n = Imw \tag{1.49}$$

证明.

$\Rightarrow$: $|Rez_n - Rew| = |Re(z_n - w)| \le |z_n - w|$

$\Leftarrow$: $|z_n - w| \le |Re(z_n - w)| + |Im(z_n - w)| = |Rez_n - Rew| + |Imz_n - Imw|$

□

2. *z is a limit point of* $\Omega \Leftrightarrow$ *z is an accumulation point of* Ω

证明.

$\Rightarrow$: $\forall r > 0,\ \exists N_r,\ \mathrm{s.t.}\ n > N,\ where\ z_n \in \Omega,\ z_n \neq z.$

$z_n \in D_r^*(z),\ z_n \in \Omega,\ \forall n > N_r.$

Hence $z_n \in D_r^*(z) \cap \Omega \neq \emptyset,\ \forall r > 0,\ n > N_r,$ *i.e.*

z is an accumulation point of Ω

$\Leftarrow$: *Take a point* z_n *from* $D_{\frac{1}{n}}^*(z) \cap \Omega$ *which is not empty.*

Then $\{z_n\}$ *is a Cauchy sequence which converges to z.*

Hence z is a limit point of Ω.

□

注. *A limit point of* Ω *may not belong to* Ω.

3. 课本第一章练习 $T3, T5, T7$.

第二章 Week 2 – – Functions on $\mathbb{C}$

2.1 连续函数和极值

定义 2.1.1. *Let $\Omega \subseteq \mathbb{C}$ be open. We say $f : \Omega \longrightarrow \mathbb{C}$ is continuous at $z_0 \in \Omega$ if $\forall \epsilon > 0, \exists \delta > 0, \text{s.t.}$*

$$\text{whenever } |z - z_0| < \delta,\ z \in \Omega,\ \text{then } |f(z) - f(z_0)| < \epsilon \tag{2.1}$$

To say it another way, $\forall \epsilon > 0, \exists \delta > 0, \text{s.t. } f(D_\delta(z_0) \cap \Omega) \subseteq D_\epsilon(f(z_0))$

注. *We say f is continuous on Ω if f is continuous at every point of Ω.*

Here are some facts.

Fact 1. If f is continuous on Ω, then so are $\overline{f}$, $|f|$, $\frac{1}{f}$ (if $f(z) \neq 0$ for all $z \in \Omega$).

证明. *For $|f|$, use $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$* □

Fact 2. f is continuous iff Ref and Imf are continuous.

命题 2.1.1. *Let $\Omega \subseteq \mathbb{C}$ and let f be continuous on Ω. Then*

(1) *For every open set $S \subseteq \mathbb{C}$, $f^{-1}(S) = \{z \in \Omega \mid f(z) \in S\}$ is open.*

(2) *For every compact set $K \subseteq \mathbb{C}$, $f(K)$ is compact.*

证明.

(1) *If $f^{-1}(S) = \varnothing$, true.*

Assume $f^{-1}(S) \neq \varnothing$ and let $z_0 \in f^{-1}(S)$. Write $w_0 = f(z_0) \in S$.

Since S *is open*, $\exists \epsilon > 0, \text{s.t.}$ $D_\epsilon(w_0) \subseteq S$

Since f *is continuous, taking* ϵ *in the definition, we get a* $\delta > 0, \text{s.t.}$

$$D_\delta(z_0) \subseteq \Omega \text{ and } f(D_\delta(z_0)) \subseteq D_\epsilon(f(z_0)) = D_\epsilon(w_0) \subseteq S \tag{2.2}$$

Thus $D_\delta(z_0) \subseteq f^{-1}(S)$, *and so* $f^{-1}(S)$ *is open.*

(2) *Let* $\{\Omega_j\}_{j\in J}$ *be an open cover of* $f(K)$, *i.e.*

$$f(K) \subseteq \bigcup_{j\in J} \Omega_j \tag{2.3}$$

Then

$$K \subseteq f^{-1}(\bigcup_{j\in J} \Omega_j) = \bigcup_{j\in J} f^{-1}(\Omega_j) \tag{2.4}$$

By (1), $f^{-1}(\Omega_j)$ *is open for all* $j \in J$. *Thus* $\{f^{-1}(\Omega_j)\}_{j\in J}$ *is an open cover of* K.

Since K *is compact*, $\exists j_1, \cdots, j_n \in J, \text{s.t.}$

$$k \subseteq \bigcup_{k=1}^{n} f^{-1}(\Omega_{j_k}) = f^{-1}(\bigcup_{k=1}^{n} \Omega_{j_k}) \tag{2.5}$$

$$\Rightarrow f(K) \subseteq \bigcup_{k=1}^{n} \Omega_{j_k} \tag{2.6}$$

□

We say that f *contains a maximum at* $z_0 \in \Omega$ *if*

$$|f(z)| \leq |f(z_0)|, \ \forall z \in \Omega \tag{2.7}$$

命题 2.1.2. *A continuous function on a compact set* Ω *is bounded and attains a maximum and a minimum on* Ω.

证明. *use* $|f|^2 = (Ref)^2 + (Imf)^2$. □

2.2 复变函数的极限，全纯函数

定义 2.2.1. *Assume* $\Omega \subseteq \mathbb{C}$, $\Omega \neq \emptyset$ *and* $a \in Acc(\Omega)$, $f : \Omega \longrightarrow \mathbb{C}$, $\lim_{z \to a,\, z \in \Omega} f(z) = w$ *means*

$$\forall \epsilon > 0, \exists \delta > 0, \text{s. t. } 0 < |z - z_0| < \delta \Rightarrow |f(z) - w| < \epsilon \tag{2.8}$$

注. 容易证明若极限存在，则极限唯一.

定义 2.2.2. *Let* $\Omega \subseteq \mathbb{C}$ *be open,* $f : \Omega \longrightarrow \mathbb{C}$. *We say* $f(z)$ *is Complex differentiable at* $z_0 \in \Omega$ *if* $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ *exists. If* f *is complex differentiableat* z_0, *we denote the limit of the quotient by* $f'(z_0)$. *i.e.*

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{2.9}$$

$f'(z_0)$ *is called the derivative of* f *at* z_0.

注. *If* f *is complex differentiable at every point of* Ω, *then we say* f *is holomorphic on* Ω.

例 2.2.1.

- $f(z) = \frac{1}{z}$ *is holomorphic on* $\mathbb{C} \backslash \{0\}$.
- $f(z) = \bar{z}$ *is not complex differentiable at any point of* $\mathbb{C}$.
- $f(z) = |z|^2$ *is only complex differentiable at* $z = 0$.

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) \Leftrightarrow \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - hf'(z_0)}{h} = 0 \tag{2.10}$$

Let $\circ(h)$ *denote any complexed valued function with the property* $\frac{\circ(h)}{h} \to 0$, *as* $h \to 0$
Then f *is complex differentiable at* z_0 *iff* $\exists a \in \mathbb{C}, \text{s. t.}$

$$f(z_0 + h) - f(z_0) - ha = \circ(h), \text{ where } a = f'(z_0) \tag{2.11}$$

注. *According to equation*(2.11), *holomorphic* $\Rightarrow$ *continuity.*

命题 2.2.1. *If f, g are holomorphic on an open set $\Omega \subseteq \mathbb{C}$, then*

$$(f+g)^{'} = f^{'} + g^{'},\ (fg)^{'} = f^{'}g + fg^{'} \tag{2.12}$$

If $g(z_0) \neq 0$, then $\frac{f}{g}$ is complex differentiable at z_0 and

$$\left(\frac{f}{g}\right)^{'}_{z=z_0} = \frac{f^{'}g - fg^{'}}{g^2}\Big|_{z=z_0} \tag{2.13}$$

If $f : \Omega \longrightarrow U$ and $g : U \longrightarrow \mathbb{C}$ are holomorphic, then the chain rule holds

$$(g \circ f)^{'}(z) = g^{'}(f(z))f^{'}(z),\ \forall z \in \Omega \tag{2.14}$$

2.3 *Cauchy – Riemann Equations*

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{2.15}$$

Assume $\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ *exists, we may let* $h \to 0$ *in whichever manner we please.*
(*let* $z_0 = x_0 + iy_0$)

- *Let* $h = t \in \mathbb{R}$,

$$f'(z_0) = \lim_{t\to 0,\ t\in\mathbb{R}} \frac{f(z_0+h)-f(z_0)}{h} = u_x(x_0,y_0) + iv_x(x_0,y_0) \tag{2.16}$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) \tag{2.17}$$

- *Let* $h = it, t \in \mathbb{R}$,

$$f'(z_0) = \lim_{t\to 0,\ t\in\mathbb{R}} \frac{f(z_0+h)-f(z_0)}{it} = v_y(x_0,y_0) - iu_y(x_0,y_0) \tag{2.18}$$

$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0) \tag{2.19}$$

Thus, we conclude $f = u + iv$ *is holomorphic* $\Rightarrow$ $u,\ v$ *satisfy*

$$\begin{cases} u_x = v_x \\ u_y = -v_y \end{cases} \tag{2.20}$$

The equations(2.20) *is called* <u>*Cauthy – Riemann Equations*</u>.

例 2.3.1. $f(x+iy) = x^2 - y^2 - 2xyi,\ x,y \in \mathbb{R}$ *is not holomorphic on* $\mathbb{C}\backslash\{0\}$.

2.4 全纯条件

Let $f = u + iv : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Then

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad on\ \Omega \tag{2.21}$$

下面给出函数 *holomorphic* 的充分条件.

定理 2.4.1. Let $\Omega \subset \mathbb{C}$ be open, $f = u + iv : \Omega \longrightarrow \mathbb{C}$. If u, v are differentiable on Ω and satisfy the $Cauchy - Riemann\ equations$, then f is holomorphic on Ω.

证明. (Goal : $\forall z_0 = x_0 + iy_0 \in \Omega,\ h = h_1 + ih_2 \in \mathbb{C},\ z_0 + h \in \Omega,\ |h|\ small\ enough,$ $f(z_0 + h) - f(z_0) = ah + \circ(h)$)

Since $u(x, y)$ is differentiable on Ω,

$$u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = h_1 u_x(x_0, y_0) + h_2 u_y(x_0, y_0) + \circ(h_1, h_2) \tag{2.22}$$

Here $\circ(h_1, h_2)$ is any expression with the property that $\frac{\circ(h_1, h_2)}{\sqrt{h_1^2 + h_2^2}} \to 0$, as $(h_1, h_2) \to 0$.
Similarly,

$$v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0) = h_1 v_x(x_0, y_0) + h_2 v_y(x_0, y_0) + \circ(h_1, h_2) \tag{2.23}$$

Then

$$f(z_0 + h) - f(z_0) = h_1 u_x + h_2 u_y + i(h_1 v_x + h_2 v_y) + \circ(h_1, h_2) \tag{2.24}$$
$$= h_1 u_x - h_2 v_x + i(h_1 v_x + h_2 u_x) + \circ(h_1, h_2) \tag{2.25}$$
$$= (u_x + iv_x)(h_1 + ih_2) + \circ(h_1, h_2) \tag{2.26}$$

Note that we may write $\circ(h)$ instead of $\circ(h_1, h_2)$, since

$$(h_1, h_2) \to 0 \Leftrightarrow h \to 0 \Leftrightarrow |h| \to 0 \tag{2.27}$$

Then the previous expression is equal to $f'(z_0)h + \circ(h)$.
Since z_0 is arbitrary, f is holomorphic on Ω. □

$f = u + iv$ can be seen as a mapping

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{2.28}$$

$$(x, y) \longmapsto (u(x, y), v(x, y)) \tag{2.29}$$

F is said to be differentiable at a point $P_0 = (x_0, y_0)$, if $\exists$ a linear transformation $J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, s. t.

$$F(P_0 + H) - F(P_0) = J(H) + |H|\,\psi(H),\ \ with\ \ |\psi(H)| \to 0\ \ as\ \ |H| \to 0 \tag{2.30}$$

命题 2.4.1. If f is complex differentiable at $z_0 = x_0 + iy_0$, then F is differentiable at (x_0, y_0).

证明. Since f is complex differentiable at $z_0 = x_0 + iy_0$, we have

$$f(z_0 + h) - f(z_0) = f^{'}(z_0)h + \circ(h) \tag{2.31}$$

$$= (u_x + iv_x)(h_1 + ih_2) + \circ(h) \tag{2.32}$$

$$= u_x h_1 - v_x h_2 + i(v_x h_1 + u_x h_2) + \circ(h) \tag{2.33}$$

$$= u_x h_1 + u_y h_2 + i(v_x h_1 + v_y h_2) + \circ(h) \tag{2.34}$$

Thus, $F(P_0 + H) - F(P_0) = J(H) + \circ(H)$, where $J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ and $H = (h_1, h_2)$. □

2.5 复变函数微分

$$z = x + iy,\ \overline{z} = x - iy \ \Leftrightarrow \ x = \tfrac{z+\overline{z}}{2},\ y = \tfrac{z-\overline{z}}{2i}$$

A given function $f : \Omega \longrightarrow \mathbb{C}$ can be expressed either in variables x, y or $z, \overline{z}$. That is, for the given f, we may write $f(x, y)$ or $f(z, \overline{z})$.

注. 可视作复平面上可建立两个坐标系 xOy 和 $zO\overline{z}$，即 $\mathbb{C}$ 中存在两组基. 由于将复数 z 转化为 $x + iy$ 后再进行计算常常会产生不便，因此下面通过这两组基之间的转化，探讨不同形式下函数微分的表达方式.

Suppose the relevant derivatives exist.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f \tag{2.35}$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f \tag{2.36}$$

Define two operations.($Wirtinger\ operations,\ 1927$)

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{2.37}$$

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2.38}$$

命题 2.5.1. $Cauchy - Riemann\ equations$ are equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0 \tag{2.39}$$

证明. Let $f = u + iv$. Then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\Big(u_x + v_x + i(u_y + v_y)\Big) = \frac{1}{2}\Big(u_x - v_y + i(u_y + v_x)\Big) \tag{2.40}$$

$$\frac{\partial f}{\partial \overline{z}} = 0 \Leftrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \tag{2.41}$$

□

注. We note that $f^{'}(z) = u_x + iv_x = u_x - iu_y = 2\frac{\partial u}{\partial z}$.

调和算子 / 拉普拉斯算子 Define the *Laplacian*(or the *Laplace operator*).

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.42}$$

注. $C^k(\Omega)$ denotes the set of all k times continuously differentiable functions on Ω.

下面给出调和函数的定义.

定义 2.5.1. Let $\Omega \subset \mathbb{C}$ be an open set. $g : \Omega \longrightarrow \mathbb{C}$ is called *harmonic* if $g \in C^2(\Omega)$ and $\Delta g = 0$.

下面的命题说明了全纯函数的实部和虚部均调和.(全纯的必要条件)

命题 2.5.2. Let $f = u + iv : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Assume $u, v \in C^2(\Omega)$. Then u, v are harmonic.

注. 事实上后面会证明此处无需 $u, v \in C^2(\Omega)$.

证明. The *Cauchy – Riemann equations* tell $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{2.43}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{2.44}$$

Since $v \in C^2(\Omega)$,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \tag{2.45}$$

Therefore, $u_{xx} + u_{yy} = 0$. Similarly we can proof that $v_{xx} + v_{yy} = 0$. □

A holomorphic function is necessarily harmonic, so is $\overline{f}$.

命题 2.5.3. Let $\Omega \subset \mathbb{C}$ be a region, $f : \Omega \longrightarrow \mathbb{C}$. Then

$$f \text{ is consistant iff } f'(z) = 0, \forall z \in \Omega.$$

证明.

$\Rightarrow$: clear

$\Leftarrow$: Let $f = u + iv$, then

$$
\begin{aligned}
f^{'}(z) = 0 \Rightarrow u_x + iv_x = 0 &\Rightarrow u_x = 0, v_x = 0 && (2.46)\\
&\overset{C-R}{\Rightarrow} v_y = 0, u_y = 0 && (2.47)\\
&\Rightarrow u = c_1, v = c_2 \; (by\ mean\ value\ theorem) && (2.48)\\
&(\text{区域连通，利用中值定理}) && (2.49)
\end{aligned}
$$

□

2.6 课堂例题 2024 – 03 – 08

1. $f(x+iy) = x^2 - y^2 + 2xyi$ is holomorphic.

2. Is $f(z) = z^2\overline{z} + \frac{1}{z} + \frac{1}{z^2}$ holomorphic on $\mathbb{C}\backslash\{0\}$?

3. Let $f = u + iv$ be holomorphic on a region Ω. Assume $au + bv + c = 0$ for some $a, b, c \in \mathbb{R}$ and a, b are not all zero. Show f is consistant.

4. Find a holomorphic function f on $\mathbb{C}$ s. t.

$$Ref = x^2 - y^2 + xy,\ f(0) = 0 \tag{2.50}$$

5. Let $\Omega = \mathbb{C}\backslash\{0\}$ and $u : \Omega \longrightarrow \mathbb{R}$ be given by $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$.
   Is there a holomorphic function $f : \Omega \longrightarrow \mathbb{C}$, s. t. $Ref = u$?

解. Suppose $f = u + iv$ is holomorphic on Ω. Then

$$\begin{cases} v_x = -u_y = -\frac{y}{x^2+y^2} \\ v_y = u_x = \frac{x}{x^2+y^2} \end{cases} \tag{2.51}$$

By $v_y = \frac{x}{x^2+y^2}$,

$$v = \arctan\frac{y}{x} + c(x) \tag{2.52}$$

Then by $v_x = -\frac{y}{x^2+y^2}$, $c(x) = c$ is constant. $\Rightarrow v = \arctan\frac{y}{x} + c$.
However, $\arctan\frac{y}{x} : \mathbb{R}^2 \longrightarrow (-\pi, \pi]$ is not continuous on $\mathbb{R}_{\le 0} = \{x \le 0 \mid x \in \mathbb{R}\}$.

(Let $z = x + iy$, then $\arctan\frac{y}{x}$ is an argument of z.)

Therefore, there is no function satisfying the conditions.

注. If the region $\Omega = \mathbb{C}\backslash\{0\}$ is replaced by $\Omega = \mathbb{C}\backslash\mathbb{R}_{\le 0}$, then the answer is yes.

□

6. 课本第一章练习 $T8, T9, T10, T13$.

第三章　*Week 3*

3.1　幂级数，解析函数，复对数

与数学分析中的概念一致，下面相当于来复习一下有关**幂级数**的概念.

- 幂级数 $\sum_{n=0}^{\infty} z_n$ converges ⇔ 部分和 $\{S_N = \sum_{n=0}^{N} z_n\}$ converges.
- $\sum_{n=0}^{\infty} |z_n|$ converges ⇒ The series converges absolutely(绝对收敛).
- **Absolutely convergent ⇒ convergent**
- If $\sum_{n=0}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

A power series (with center 0) is an expansion of the form $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ are fixed and z varies in $\mathbb{C}$.(下面通常讨论形式为 $\sum_{n=0}^{\infty} a_n z^n$ 的幂级数)

下面给出复幂级数的**收敛半径**的定义及**收敛圆盘**.

定理 3.1.1. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, define

$$R = \varliminf_{n\to\infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\varlimsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \quad (\textbf{Hardamard's Formula}) \tag{3.1}$$

(Here we use the convertion $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$.) Then

(1) If $|z| < R$, the series converges absolutely.

(2) If $|z| > R$, the series diverges.

注. The number R is called the **radius of convergence** of the power series, and the region $|z| < R$ is called the **disc of convergence**.

例 3.1.1. 下面给出一些用幂级数定义的常见函数的例子.

- Exponential function

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \ z \in \mathbb{C} \tag{3.2}$$

- Trigonometric function

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \ , \ \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{3.3}$$

- 双曲余弦、正弦

$$\cosh z := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \ , \ \sinh z := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \tag{3.4}$$

注. 由定义容易得到，$e^{iz} = \cos z + i \sin z \Rightarrow$ 将 z 限制到 $\mathbb{R}$ 上则有：$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \sin z = \frac{e^{iz} - e^{-iz}}{2} \tag{3.5}$$

下面这个定理说明了幂级数在收敛圆盘内解析. 并给出了幂级数的导数.

定理 3.1.2. The power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. Moreover, $f'(z) = \sum\limits_{n=0}^{\infty} n a_n z^{n-1}$, which has the same radius of convergence.

证明. **Hadamard's formula** tells $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} n a_n z^{n-1}$ have the same R.
Let $g(z) = \sum\limits_{n=0}^{\infty} n a_n z^{n-1}$, $\forall z$ with $|z| < R$, we can find r, s. t. $|z| < r < R$.
For $\forall h \in \mathbb{C}$ s. t. $|h| < r - |z|$, we estimate

$$|f(z+h) - f(z) - hg(z)| = \left| \sum_{n=0}^{\infty} a_n \left((z+h)^n - z^n - nhz^{n-1} \right) \right| \tag{3.6}$$

$$= \left| \sum_{n=2}^{\infty} \left(a_n \sum_{k=2}^{n} \binom{n}{k} h^k z^{n-k} \right) \right| \tag{3.7}$$

$$\leq |h|^2 \sum_{n=2}^{\infty} |a_n| \sum_{k=0}^{n-2} \binom{n}{k+2} \left| h^k z^{n-2-k} \right| \tag{3.8}$$

Since $\binom{n}{k+2} \le n(n-1)\binom{n-2}{k}$, then

$$|f(z+h)-f(z)-hg(z)| \le |h|^2 \sum_{n=2}^{\infty} |a_n|\, n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} |h|^k\, |z|^{n-2-k} \tag{3.9}$$

$$= |h|^2 \sum_{n=2}^{\infty} |a_n|\, n(n-1)\,(|z|+|h|)^{n-2} \tag{3.10}$$

$$< |h|^2 \sum_{n=2}^{\infty} |a_n|\, n(n-1) r^{n-2} = |h|^2 \cdot c \tag{3.11}$$

Thus

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| < |h| \cdot c \tag{3.12}$$

Therefore, the result follows. □

推论 3.1.3. A power series is infinitely differentiable in its disc of convergence.

注. Thm 3.1.2 即说明了幂级数在收敛圆盘内解析.

下面给出推广到更一般的幂级数的导数，即中心不一定在原点的情形.

A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \tag{3.13}$$

Let $g(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, then $f(z) = g(w)$, where $w = z - z_0$.

According to the **Chain Rule**(链式法则), $f'(z) = \sum\limits_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$

下面严格地给出**解析**的定义.

定义 3.1.1. A function f defined on an open set Ω is said to be **analytic** at $z_0 \in \mathbb{C}$ if there is a power series $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ with positive radius of convergence, s. t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \ \textit{for all } z \textit{ in a neighbourhood of } z_0 \tag{3.14}$$

$$(\textit{i.e. } \forall z \in D_r(z_0),\ \textit{for some } r > 0) \tag{3.15}$$

If f is analytic at every point of Ω, then we say f is **analytic on Ω**.

下面给出有关指数函数 e^z 的一些等式 (命题).
在此之前，先给出 **Cauchy Multiplication Theorem**.

引理 3.1.4. If $\sum a_n$, $\sum b_n$ are absolutely convergent, then

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) = \left(\sum a_n\right)\left(\sum b_n\right) \tag{3.16}$$

命题 3.1.1. For $z_1, z_2 \in \mathbb{C}$, $e^{(z_1+z_2)} = e^{z_1} \cdot e^{z_2}$.

推论 3.1.5. If $z = x + iy$, $x, y \in \mathbb{R}$, then

$$e^z = e^x(\cos y + i \sin y) \tag{3.17}$$

推论 3.1.6. **De Moire's Formula**.
For $\vartheta \in \mathbb{R}$,

$$(\cos\vartheta + i\sin\vartheta)^n = \cos n\vartheta + i \sin n\vartheta \tag{3.18}$$

下面来引入复数域上的**对数函数 (Complex Logarithm)**.
$\forall z \in \mathbb{C}\backslash\{0\}$, write $z = re^{i\vartheta}$. Then $e^w = z$ can be solved.
If $w = u + iv$, $u, v \in \mathbb{R}$, then

$$e^u \cdot e^{iv} = re^{i\vartheta} \;\;\Rightarrow\;\; u = \log r,\;\; v = \vartheta + 2k\pi, k \in \mathbb{Z} \tag{3.19}$$

Let $Log(z)$ be the set of above, then we get Complex Logarithm.

定义 3.1.2. $\forall z \in \mathbb{C}\backslash\{0\}$. Define

$$Log(z) := \log|z| + i(\arg z + 2k\pi), k \in \mathbb{Z} \tag{3.20}$$

Here $\arg z$ is an argument of z satisfying $-\pi < \arg z \leq \pi$.

(We call $\arg z$ the **principal argument** of z.)

下面介绍复对数的**主值支**的概念.

定义 3.1.3. Define the **principal branch** of the logarithm on a "cut plane"

$$\log : \mathbb{C}\backslash\mathbb{R}_{\leq 0} \longrightarrow \mathbb{C} \tag{3.21}$$

$$z \longmapsto \log|z| + i\arg z,\ \ -\pi < \arg z < \pi \tag{3.22}$$

例 3.1.2.

$$Log(-1) = (2k+1)\pi i \tag{3.23}$$

$$Log(i) = (2k+\frac{1}{2})\pi i \tag{3.24}$$

$$\log i = \frac{\pi}{2}i \tag{3.25}$$

$$\log(1+i) = \frac{1}{2}\log 2 + \frac{\pi}{4}i \tag{3.26}$$

命题 3.1.2.

$$e^{Log(z)} = z,\ \ z \neq 0 \tag{3.27}$$

$$Log(z_1 z_2) = Log(z_1) + Log(z_2) \tag{3.28}$$

$$\log z_1 z_2 \neq \log z_1 + \log z_2\ \ in\ general \tag{3.29}$$

3.2 课堂例题 $2024-03-11$

1. Let $z \neq 0$. Then $\exists n$ different $z_0, \cdots, z_{n-1}$, s. t.

$$z_k^n = z,\ \ k = 0, \cdots, n-1 \tag{3.30}$$

解. Let $z = |z|\, e^{i\vartheta}$, $w = re^{it}$, $r > 0$, $t \in \mathbb{R}$. Then

$$w^n = z \ \Rightarrow\ r^n e^{int} = |z|\, e^{i\vartheta} \ \Rightarrow \begin{cases} r = |z|^{\frac{1}{n}} \\ nt = \vartheta + 2k\pi, k \in \mathbb{Z} \end{cases} \tag{3.31}$$

□

2. Proof

$$\left|\sum_{k=0}^{n} e^{ikx}\right| \leq \left|\frac{1}{\sin\frac{x}{2}}\right|,\ \ \forall x \in \mathbb{R}\backslash\{2k\pi \mid k \in \mathbb{Z}\} \tag{3.32}$$

3. 课本第一章练习 $T16, T19$

3.3 复对数的性质

Let $a \in \mathbb{C}$. We may define

$$z^a = e^{a \log z}, \ z \neq 0 \tag{3.33}$$

命题 3.3.1. The function $f(z) = \log z$, $z \in \mathbb{C} \backslash \mathbb{R}_{\leq 0}$ is holomorphic.

证明. $\forall z_0 \in \mathbb{C} \backslash \mathbb{R}_{\leq 0}$, let $w = \log z$, $w_0 = \log z_0$. Then

$$\lim_{z \to z_0} \frac{\log z - \log z_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0} \tag{3.34}$$

Therefore, $(\log z)' = \frac{1}{z}$. □

命题 3.3.2. Show

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \ on \ \mathcal{D} \tag{3.35}$$

证明. Let $f(z) = \log(1+z)$, $g(z) = \sum\limits_{n=1}^{\infty} (-1)^n \frac{z^{n-1}}{n}$. Both are holomorphic on $\mathcal{D}$ and

$$f'(z) = \frac{1}{1+z}, \ g'(z) = \sum_{n=1}^{\infty} (-1)^n z^{n-1} = \frac{1}{1+z} \tag{3.36}$$

And so $(f-g)' = 0$ on $\mathcal{D}$. Therefore, $f - g = c$. Taking $z = 0, f(0) = g(0) \Rightarrow c = 0$. □

3.4 道路

先给出**道路 (path)** 的定义.

定义 3.4.1. A continuous function $z(t) = x(t) + iy(t)$ from $[a, b] \subset \mathbb{R}$ to $\mathbb{C}$ is called a **path** (or a parametric curve) connecting $z(a)$ and $z(b)$.($z(a)$ is called the starting point, $z(b)$ the end point)

The path is **closed** if $z(a) = z(b)$.

The path is **simple** if $z(t) \neq z(s)$ unless $\begin{cases} (1) t = s \\ (2) t = a, s = b \end{cases}$

下面给出道路**光滑性**的描述.

定义 3.4.2. We say that a path $z(t) = x(t)+iy(t)$, $t \in [a, b]$ is **smooth** if $x(t), y(t)$ are continuously differentiable and $z^{'}(t) = x^{'}(t) + iy^{'}(t) \neq 0$, $t \in [a, b]$. Here $z^{'}(a), z^{'}(b)$ are understood as one-sided derivative.

下面给出两条道路**等价**的定义.

定义 3.4.3. Two paths $z : [a, b] \longrightarrow \mathbb{C}$, $\widetilde{z} : [c, d] \longrightarrow \mathbb{C}$ are **equivalent** if $\exists$ bijection and differential

$$t : [c, d] \longrightarrow [a, b] \tag{3.37}$$

$$s \longmapsto t(s) \tag{3.38}$$

s. t. $\widetilde{z}(s) = z(t(s))$ and $t^{'}(s) > 0$.

下面给出**道路反向**的定义.

定义 3.4.4. Given a path z, we can define a path $\widetilde{z}$ obtained from z by reversing the orietation

$$z(t) : [a, b] \longrightarrow \mathbb{C} \tag{3.39}$$

$$\widetilde{z}(t) = z(a + b - t) : [a, b] \longrightarrow \mathbb{C} \tag{3.40}$$

这里我们规定一下道路的**正向 / 逆向** (逆时针为正向).

定义 3.4.5. A path has **positive orientation** if it travels counterclockwisely.
($\cdots$ **negative orientation** $\cdots$ clockwisely.)

下面我们给出**分段光滑**的定义.

定义 3.4.6. A path $z(t) : [a, b] \longrightarrow \mathbb{C}$ is **piecewise smooth** if $\exists$ a partion $a = a_0 < a_1 < \cdots < a_n = b$, s. t. $z(t)$ is smooth in each $[a_k, a_{k+1}]$, $k = 0, \cdots, n-1$.

下面说明两条道路的**连接**.

Paths can be concatenated. If $z : [a, b] \longrightarrow \mathbb{C}$, $\widetilde{z} : [b, c] \longrightarrow \mathbb{C}$ and $z(b) = \widetilde{z}(b)$, we can define $w : [a, c] \longrightarrow \mathbb{C}$ as $w(t) = \begin{cases} z(t), a \leq t \leq b \\ \widetilde{z}(t), b \leq t \leq c \end{cases}$. **Concatenation** of $z, \widetilde{z}$ is denoted as $z \circ \widetilde{z}$.

下面给出 **zig-zag** 道路的定义.

定义 3.4.7. A path is **zig-zag** if it consists of finitely many horizontal or vertical line seqments.

下面的命题说明区域内的任两点可由一条 zig-zag 道路连接.

命题 3.4.1. Let $\Omega \subset \mathbb{C}$ be a region. Then any two points in Ω can be joined by a zig-zag path.

证明.

- Case when $\Omega = D_R(z_0)$, where $z_0 \in \mathbb{C}$, $R > 0$.

  $\forall \alpha, \beta \in \Omega$, we can join them to the horiziontal diameter via a vertical line seqment.

- Now let Ω be an arbitrary region. $\forall \alpha \in \Omega$. Let

$$A := \{\beta \in \Omega \mid \exists\ zig-zag\ path\ \gamma\ connecting\ \beta\ and\ \alpha\} \tag{3.41}$$

  Then 容易证 $\alpha \in A \neq \varnothing$ 既开又闭，从而 $A = \Omega$.

$\square$

3.5 课堂例题 2024 − 03 − 15

1. Calculate 2^i, i^i.

2. Find all possible values of $(1+\sqrt{3}i)^{\frac{1}{8}}$.

3. Let $z_n \in \mathbb{C}$, $Rez_n \geq 0$, $n = 1, 2, \cdots$. If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_n^2$ both converge, show that $\sum_{n=1}^{\infty} |z_n|^2$ converges.

4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be holomorphic on $\mathcal{D}$. Assume $|f(z)| \leq 1$, $\forall z \in \mathcal{D}$. Show $|a_n| \leq 1$, $n = 1, 2, \cdots$.

第四章 *Week 4*

4.1 曲线积分

积分 下面先给出复数域上积分的定义.

定义 4.1.1. Let $z(t) = x(t) + iy(t)$, $t \in [a, b] \subset \mathbb{R}$. If $x(t), y(t)$ are differentiable, we define $z^{'}(t) = x^{'}(t) + iy^{'}(t)$.

Similarly, if $x(t), y(t)$ are continuous, we define

$$\int_a^b z(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt \tag{4.1}$$

容易证明，复数域上的积分同样具有三角不等式.

命题 4.1.1. Let $f : [a, b] \longrightarrow \mathbb{C}$ be continuous. Then

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt \tag{4.2}$$

证明. Write $\int_a^b f(t)dt = re^{i\vartheta}$, $r \geq 0$. Then

$$r = e^{-i\vartheta}\int_a^b f(t)dt = \int_a^b e^{-i\vartheta}f(t)dt = \left|\int_a^b Re\, e^{-i\vartheta}f(t)dt\right| \tag{4.3}$$

$$\leq \int_a^b \left|Re\, e^{-i\vartheta}f(t)\right| dt \tag{4.4}$$

$$\leq \int_a^b \left|e^{-i\vartheta}f(t)\right| dt = \int_a^b |f(t)|\, dt \tag{4.5}$$

□

曲线积分　下面给出复数域上**连续道路**的**曲线积分**的定义.

定义 4.1.2. Let $\Omega \subset \mathbb{C}$ be open. Given a smooth path γ in Ω parametrized by $z : [a, b] \longrightarrow \Omega$ and a continuous funciton $f : \Omega \longrightarrow \mathbb{C}$. We define the **integral of f along γ** by

$$\int_{\gamma} f(z)dz := \int_{a}^{b} f(z(t))z'(t)dt \tag{4.6}$$

Let $\widetilde{z}(t) : [c, d] \longrightarrow \Omega$ be equivalent to $z(t)$. Then

$$\int_{a}^{b} f(z(t))z'(t)dt = \int_{c}^{d} f(\widetilde{z}(t))\widetilde{z}'(t)dt \tag{4.7}$$

下面给出**分段连续**道路的曲线积分及**曲线长度**的定义.

定义 4.1.3. If γ is piecewise smooth and $z(t)$ is a piecewise smooth parametrization as before, we define

$$\int_{\gamma} f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt \tag{4.8}$$

The **length** of the smooth curve γ is

$$length(\gamma) = \int_{a}^{b} \left|z'(t)\right| dt \tag{4.9}$$

If $f = u + iv$, $z(t) = x(t) + iy(t)$, then

$$\begin{aligned} \int_{\gamma} f(z)dz = \int_{a}^{b} f(z(t))z'(t)dt &= \int_{a}^{b} (u + iv)(x'(t) + iy'(t))dt && (4.10) \\ &= \int_{a}^{b} (ux'(t) - vy'(t))dt + i\int_{a}^{b} (vx'(t) + uy'(t))dt && (4.11) \\ &= \int_{\gamma} (udx - vdy) + i\int_{\gamma} (vdx + udy) && (4.12) \end{aligned}$$

下面给出曲线积分的几条性质.

命题 4.1.2. 记 γ^- 为 γ 的反向.

(1) $\int_{\gamma} f(z)dz = -\int_{\gamma^-} f(z)dz$.

(2) If $f(z), g(z)$ are continuous, and γ is a path, then $\forall \alpha, \beta \in \mathbb{C}$,

$$\int_{\gamma} (\alpha f + \beta g)dz = \alpha\int_{\gamma} fdz + \beta\int_{\gamma} gdz \tag{4.13}$$

(3)

$$\left|\int_{\gamma} f(z)dz\right| \leq \sup_{\gamma} |f(z)| \cdot length(\gamma) \tag{4.14}$$

原函数　下面我们给出**原函数**的概念.

定义 4.1.4. If $f : \Omega \longrightarrow \mathbb{C}$. Assume $\exists$ a complex differentiable $F : \Omega \longrightarrow \mathbb{C}$, s. t.

$$F^{'}(z) = f(z), \ for \ every \ z \in \Omega \tag{4.15}$$

Then we say f admits a **primitival** (or an antiderivative) on Ω.

下面的命题说明若函数有原函数，则其曲线积分只与始末点有关，而与路径无关.

命题 4.1.3. If f is a continuous function that admits a primitive F on Ω, and γ is a path in Ω that begins at w_1 and ends at w_2, then

$$\int_\gamma f(z)dz = F(w_2) - F(w_1) \tag{4.16}$$

证明. Let $z(t) : [a, b] \longrightarrow \Omega$ be a parametrization for γ with $z(a) = w_1$, $z(b) = w_2$.

- Assume γ is smooth. Compute

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z^{'}(t)dt = \int_a^b F^{'}(z(t))z^{'}(t)dt = \int_a^b \frac{dF(z(t))}{dt}dt \tag{4.17}$$

  According to **the fundamental theorem of calculus**, we get
  (分别对实部和虚部运用微积分基本定理)

$$\int_\gamma f(z)dz = \int_a^b F^{'}(z(t))z^{'}(t)dt = \int_a^b \frac{dF(z(t))}{dt}dt \tag{4.18}$$

$$= F(z(b)) - F(z(a)) = F(w_2) - F(w_1) \tag{4.19}$$

- γ is piecewise smooth, we can proof similarly.

□

由命题 4.1.3，可得到有原函数的函数 f 在闭曲线上积分为 0.

推论 4.1.1. If γ is a closed path in Ω, f is continuous and admits a primitive on Ω, then

$$\int_\gamma f(z)dz = 0 \tag{4.20}$$

同时，由命题 4.1.3，还可得到区域 Ω 上导数恒为 0 的全纯函数只能为常值函数.

推论 4.1.2. If f is holomorphic on a region Ω and $f' \equiv 0$, then f is constant.

下面给出具有原函数的充要条件.

定理 4.1.3. Let $\Omega \subset \mathbb{C}$ be a region. $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function. Then the following statements are equivalent:

(1) f admits a primitive on Ω.

(2) $\forall \alpha, \beta \in \mathbb{C}$, $\int_\gamma f(z)dz$ is invariant for any path γ in Ω that joins α to β.

(3) $\forall \alpha, \beta \in \mathbb{C}$, $\int_\gamma f(z)dz$ is invariant for any zig-zag path γ in Ω that joins α to β.

证明. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1): Fix $\alpha \in \Omega$ and define $F : \Omega \longrightarrow \mathbb{C}$ by

$$F(z_0) = \int_\gamma f(z)dz, \ \ z_0 = x_0 + iy_0 \in \Omega \tag{4.21}$$

where γ is any zig-zag path joining α to z_0.

(F **is Well-defined** : Condition (3) tells $F(z_0)$ is independent of the choice of γ.)

Let $F(z) = U + iV, f(z) = u + iv$. It suffices to show

$$\begin{cases} U_x(x_0, y_0) & = u(x_0, y_0) \ , \ V_x(x_0, y_0) = v(x_0, y_0) \\ U_y(x_0, y_0) & = -v(x_0, y_0) \ , \ V_y(x_0, y_0) = u(x_0, y_0) \end{cases} \tag{4.22}$$

- $U_x(x_0, y_0) = u(x_0, y_0) \ , \ V_x(x_0, y_0) = v(x_0, y_0)$

  Let $h \in \mathbb{R}$. Let γ be a zig-zag path joining α to z_0,

  $\gamma_H : z_H(t) = z_0 + th, 0 \le t \le 1$. $\gamma_H \subset \Omega$. Then

$$F(z_0 + h) = \int_{\gamma \circ \gamma_H} f(z)dz = \int_\gamma f(z)dz + \int_{\gamma_H} f(z)dz \tag{4.23}$$

$$= F(z_0) + \int_{\gamma_H} f(z)dz \tag{4.24}$$

Then we get

$$\frac{F(z_0+h)-F(z_0)}{h} = \int_0^1 f(z_0+th)dt \tag{4.25}$$

Since f is continuous,

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{F(z_0+h)-F(z_0)}{h} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \int_0^1 f(z_0+th)dt \tag{4.26}$$

$$= f(z_0) \tag{4.27}$$

$$= u(x_0,y_0) + iv(x_0,y_0) \tag{4.28}$$

- $U_y(x_0,y_0) = -v(x_0,y_0)$, $V_y(x_0,y_0) = u(x_0,y_0)$

  Similarly.

□