Complex Analysis[1]

–TW–

2024 年 4 月 23 日

[1]课堂教材：《*Complex Analysis*》— *Elias M. Stein*

序

天道几何，万品流形先自守；

变分无限，孤心测度有同伦。

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长夜伴浪破晓梦，梦晓破浪伴夜长

目录

第零章　课程要求

- **任课教师：** 林明华
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- **总评成绩组成：** 阅读报告及汇报 20% + 期末考试 80%

第一章 *Week* 1

1.1 复数的引入

引入

下面从代数结构 (*Group*, *Ring*, *Field*) 的角度引入复数的概念.

Consider the set $\mathbb{R}^2$. *Define two operations.* $\forall (a,b),(c,d) \in \mathbb{R}^2$,

$$(a,b)+(c,d) \coloneqq (a+c, b+d) \tag{1.1}$$

$$(a,b)\cdot(c,d) \coloneqq (ac-bd, bc+ad) \tag{1.2}$$

$"\cdot"$ *is commutative.*

$"+"$, $"\cdot"$ *satisfy associative and distributive laws.*

$$(0,0) : \textit{The additive identity} \tag{1.3}$$

$$(1,0) : \textit{The multiplicative identity} \tag{1.4}$$

$\Rightarrow (\mathbb{R}^2, +, \cdot)$ *is a communicative ring.*

$\forall (a,b) \in \mathbb{R}^2,\ (a,b) \neq (0,0),$ *if*

$$(a,b)\cdot(x,y) = (1,0) \tag{1.5}$$

$$\Rightarrow x = \frac{a}{a^2+b^2},\ y = \frac{-b}{a^2+b^2} \tag{1.6}$$

Therefore, $(\mathbb{R}^2, +, \cdot)$ *is a field, renoted as* $\mathbb{C}$.

复数的乘法 在上述对 $\mathbb{C}$ 的定义中，唯一非平凡的点便是乘法运算 $"\cdot"$ 的定义.

下面我们从代数的方法，从另一个角度理解复数的乘法.

We may ask a question : *Can we define a "$\cdot$" and let $(\mathbb{R}^3, +, \cdot)$ be a field*?

However, the answer is certainly not!

Consider $M_2 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$ *equipped with the usual matrix addition and multiplication.*

Define a map σ.

$$\sigma : \mathbb{R}^2 \longrightarrow M_2 \tag{1.7}$$

$$(a, b) \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{1.8}$$

Then, σ *is bijective.*

$$\sigma(a, b) \cdot \sigma(c, d) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix} = \sigma((a, b) \cdot (c, d)) \tag{1.9}$$

$\Rightarrow \sigma$ *is an isomorphism*(同构映射).

于是复数乘法可视作复平面上带伸缩的旋转.

1.2 复数的基本性质

Some Facts

$$|Rez| \leq |z|,\ |Imz| \leq |z| \tag{1.10}$$

$$Rez = \frac{z+\bar{z}}{2},\ Imz = \frac{z-\bar{z}}{2i} \tag{1.11}$$

性质 下面给出一些命题.

1. 三角不等式.

命题 1.2.1 (*Triangle Inequality*)**.** *Let* $z, w \in \mathbb{C}$*. Then*

$$|z+w| \leq |z| + |w| \tag{1.12}$$

证明. *Let* $z = a+bi,\ w = c+di$*. Then*

$$\Leftrightarrow \sqrt{(a+c)^2+(b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2} \tag{1.13}$$

$$\Leftrightarrow ac+bd \leq \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{(ac)^2+(bd)^2+a^2d^2+b^2c^2} \tag{1.14}$$

□

推论 1.2.1. *If* $z, w \in \mathbb{C}$*, then*

$$||z|-|w|| \leq |z-w| \tag{1.15}$$

证明.

$$|z| = |z-w+w| \leq |z-w| + |w| \tag{1.16}$$

$$|w| = |z-w-z| \leq |z-w| + |z| \tag{1.17}$$

$$\Rightarrow |z-w| \geq max\{|z|-|w|,\ |w|-|z|\} = ||z|-|w|| \tag{1.18}$$

□

2. *Cauchy − Schwarz* 不等式.

命题 1.2.2 (*Cauchy − Schwarz*)**.** *Let* $z_1, \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$*. Then*

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) \tag{1.19}$$

证明. $\forall \lambda \in \mathbb{R},\ \vartheta \in \mathbb{R}$,

$$
\begin{aligned}
0 \leq \sum_{k=1}^{n} \left| z_k - \lambda e^{i\vartheta} \overline{w_k} \right|^2 &= \sum_{k=1}^{n} (z_k - \lambda e^{i\vartheta} \overline{w_k})(\overline{z_k} - \lambda e^{-i\vartheta} w_k) && (1.20)\\
&= \sum_{k=1}^{n} |z_k|^2 - 2 \left(Re\ e^{-i\vartheta} \sum_{k=1}^{n} z_k w_k \right) \lambda + \lambda^2 \sum_{k=1}^{n} |w_k|^2 && (1.21)\\
&= a\lambda^2 - 2b\lambda + c && (1.22)\\
&\Rightarrow b^2 \leq ac && (1.23)
\end{aligned}
$$

Then

$$
\left(Re\ e^{-i\vartheta} \sum_{k=1}^{n} z_k w_k \right)^2 \leq \left(\sum_{k=1}^{n} |z_k|^2 \right) \left(\sum_{k=1}^{n} |w_k|^2 \right) \tag{1.24}
$$

Suppose $z = \sum_{k=1}^{n} z_k w_k = |z|\, e^{i\varphi} \in \mathbb{C}$, *let* $\vartheta = \varphi$. *Then*

$$
Re\ e^{-i\vartheta} \sum_{k=1}^{n} z_k w_k = \left| \sum_{k=1}^{n} z_k w_k \right| \tag{1.25}
$$

$$
\left| \sum_{k=1}^{n} z_k w_k \right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2 \right) \left(\sum_{k=1}^{n} |w_k|^2 \right) \tag{1.26}
$$

□

1.3　课堂例题 2024 − 02 − 26

1. *Let* $z_1, z_2 \in \mathbb{C}$, $|z_1| \le 1$, $|z_2| \le 1$. *If* $|z_1 - z_2| \ge 1$, *show that*

$$|z_1 + z_2| \le \sqrt{3} \tag{1.27}$$

证明. (平行四边形对角线的平方和等于四边的平方和.)

$$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - \overline{z_1}z_2 \tag{1.28}$$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2 \tag{1.29}$$

$\Rightarrow$

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) \tag{1.30}$$

$$|z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) - |z_1 - z_2|^2 \le 3 \tag{1.31}$$

□

2. *Let* $z_1, \cdots, z_n \in \mathbb{C}$, *and let* $e_0, e_1, \cdots, e_{n+1} \in \mathbb{C}$ *be the coefficients of* $(z+1)\prod_{k=1}^{n}(z + z_k)$, *i.e.*

$$(z+1)\prod_{k=1}^{n}(z + z_k) = \sum_{k=0}^{n+1} e_k z^{n+1-k} \tag{1.32}$$

Show that $\sum_{k=0}^{n+1}(k+1)e_k z^{n+1-k} = 0$ *has a root of modulus* ≥ 1.

Specifically, try to show $n = 1$ *case.*

$\Leftrightarrow$ (*Let* $c \in \mathbb{C}$, *show* $z^2 + 2(1+c)z + 3c = 0$ *has a root of modulus* ≥ 1.)

证明. 下面对方程 $z^2 + 2(1+c)z + 3c = 0$ 的根的情况进行分类 (事实上同时对 $c \in \mathbb{C}$ 的取值进行了分类).

(1) 若方程存在实根 $z_0 \in \mathbb{R}$，下面可以证明，事实上 (1) $\Leftrightarrow c \in \mathbb{R}$.

$$z_0^2 + 2(1+c)z_0 + 3c = 0 \tag{1.33}$$

$$\Rightarrow (2z_0 + 3)c = -z_0^2 - 2z_0 \tag{1.34}$$

$$\Rightarrow c = \frac{-z_0^2 - 2z_0}{2z_0 + 3} \in \mathbb{R} \text{ 或 } z_0 = \frac{3}{2}(\text{此时 } -z_0^2 - 2z_0 \ne 0 \text{ 矛盾}) \tag{1.35}$$

于是 $c \in \mathbb{R}$，$z^2 + 2(1+c)z + 3c = 0$ 为实系数一元二次方程.

$$\Delta = 4(1+c)^2 - 12c = 4(c^2 - c + 1) > 0,\ \forall c \in \mathbb{R} \tag{1.36}$$

$$z = -1 - c \pm \sqrt{c^2 - c + 1} \in \mathbb{R} \tag{1.37}$$

下面再对实数 $c \in \mathbb{R}$ 的范围分类讨论.

i). $c \geq 0$，则其中一根 $z = -1 - c - \sqrt{c^2 - c + 1} < -1$，$|z| > 1$.

ii). $c < 0$，考虑其中一根

$$z = -1 - c - \sqrt{c^2 - c + 1} \tag{1.38}$$

$$= -1 - (\sqrt{c^2 - c + 1} + c) \tag{1.39}$$

由于 $c < 0$，因此 $1 - c > 0$.

$$\sqrt{c^2 - c + 1} = \sqrt{c^2 + (1-c)} > \sqrt{c^2} = |c| \tag{1.40}$$

$$\sqrt{c^2 - c + 1} + c > 0 \tag{1.41}$$

$$z = -1 - (\sqrt{c^2 - c + 1} + c) < -1 \tag{1.42}$$

$$|z| > 1 \tag{1.43}$$

于是对于 $\forall c \in \mathbb{R}$，都有 $|z| > 1$. 从而得证.

事实上，根据上述证明过程可知，若 $c \in \mathbb{R}$，则原方程必有实根，且两根均为实根，从而

$$(1): \text{方程存在实根} \Leftrightarrow c \in \mathbb{R} \Leftrightarrow \text{两根均为实根} \tag{1.44}$$

(2) 若方程无实根，即 $c \in \mathbb{C}$

□

1.4 复数域 $\mathbb{C}$ 上的拓扑概念 & 性质

Let $a \in \mathbb{C}$, *open disc of radius r centered at a*

$$D_r(a) := \{z \in \mathbb{C} \mid |z - a| < r\} \tag{1.45}$$

$$D_r^*(a) := \{z \in \mathbb{C} \mid 0 < |z - a| < r\} \tag{1.46}$$

closed disc of radius r centered at a

$$\overline{D}_r(a) := \{z \in \mathbb{C} \mid |z - a| \leq r\} \tag{1.47}$$

unit disc :

$$\mathbb{D} := D_1(0) \tag{1.48}$$

Let $\Omega \subseteq \mathbb{C}$

定义 1.4.1. $a \in \Omega$ *is an <u>interior point of Ω</u> if* $\exists r > 0,\ \mathrm{s.t.}\ D_r(a) \subseteq \Omega$.

注. *The set of all interior points of Ω is called <u>the interior of Ω</u>, denoted by* $Int(\Omega)$.

定义 1.4.2. Ω *is open if* $\Omega = Int(\Omega)$.

注. $\mathbb{C}$ *is open.* $\varnothing$ *is open. (by convention)*

定义 1.4.3. Ω *is closed if* $\Omega^c := \mathbb{C} \backslash \Omega$ *is open.*

定理 1.4.1. *Every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$. That is, $\mathbb{C}$ is Complete.*

1.5 课堂例题 2024 – 03 – 01

1.

$$\lim_{n\to+\infty} z_n = w \Leftrightarrow \lim_{n\to+\infty} Rez_n = Rew,\ \lim_{n\to+\infty} Imz_n = Imw \tag{1.49}$$

证明.

$\Rightarrow$: $|Rez_n - Rew| = |Re(z_n - w)| \le |z_n - w|$

$\Leftarrow$: $|z_n - w| \le |Re(z_n - w)| + |Im(z_n - w)| = |Rez_n - Rew| + |Imz_n - Imw|$

□

2. *z is a limit point of* Ω $\Leftrightarrow$ *z is an accumulation point of* Ω

证明.

$\Rightarrow$: $\forall r > 0,\ \exists N_r,\ \mathrm{s.t.}\ n > N,\ where\ z_n \in \Omega,\ z_n \neq z.$

$z_n \in D_r^*(z),\ z_n \in \Omega,\ \forall n > N_r.$

Hence $z_n \in D_r^*(z) \cap \Omega \neq \varnothing,\ \forall r > 0,\ n > N_r,$ *i.e.*

z is an accumulation point of Ω

$\Leftarrow$: *Take a point* z_n *from* $D_{\frac{1}{n}}^*(z) \cap \Omega$ *which is not empty.*

Then $\{z_n\}$ *is a Cauchy sequence which converges to z.*

Hence z is a limit point of Ω.

□

注. *A limit point of* Ω *may not belong to* Ω.

3. 课本第一章练习 $T3, T5, T7$.

第二章　*Week 2 – – Functions on* $\mathbb{C}$

2.1　连续函数和极值

定义 2.1.1. *Let* $\Omega \subseteq \mathbb{C}$ *be open. We say* $f : \Omega \longrightarrow \mathbb{C}$ *is continuous at* $z_0 \in \Omega$ *if* $\forall \epsilon > 0, \exists \delta > 0, \mathrm{s.\,t.}$

$$\text{whenever } |z - z_0| < \delta,\ z \in \Omega,\ \text{then } |f(z) - f(z_0)| < \epsilon \tag{2.1}$$

To say it another way, $\forall \epsilon > 0, \exists \delta > 0, \mathrm{s.\,t.}\ f(D_\delta(z_0) \cap \Omega) \subseteq D_\epsilon(f(z_0))$

注. *We say* f *is continuous on* Ω *if* f *is continuous at every point of* Ω.

Here are some facts.

Fact 1. *If* f *is continuous on* Ω, *then so are* $\overline{f}$, $|f|$, $\frac{1}{f}$ *(if* $f(z) \neq 0$ *for all* $z \in \Omega$*).*

证明. *For* $|f|$, *use* $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$ □

Fact 2. f *is continuous iff* Ref *and* Imf *are continuous.*

命题 2.1.1. *Let* $\Omega \subseteq \mathbb{C}$ *and let* f *be continuous on* Ω. *Then*

(1) *For every open set* $S \subseteq \mathbb{C}$, $f^{-1}(S) = \{z \in \Omega \mid f(z) \in S\}$ *is open.*

(2) *For every compact set* $K \subseteq \mathbb{C}$, $f(K)$ *is compact.*

证明.

(1) *If* $f^{-1}(S) = \varnothing$, *true.*

Assume $f^{-1}(S) \neq \varnothing$ *and let* $z_0 \in f^{-1}(S)$. *Write* $w_0 = f(z_0) \in S$.

Since S is open, $\exists\epsilon > 0, \text{s. t.}$ $D_\epsilon(w_0) \subseteq S$

Since f *is continuous, taking* ϵ *in the definition, we get a* $\delta > 0, \text{s. t.}$

$$D_\delta(z_0) \subseteq \Omega \text{ and } f(D_\delta(z_0)) \subseteq D_\epsilon(f(z_0)) = D_\epsilon(w_0) \subseteq S \tag{2.2}$$

Thus $D_\delta(z_0) \subseteq f^{-1}(S)$*, and so* $f^{-1}(S)$ *is open.*

(2) *Let* $\{\Omega_j\}_{j\in J}$ *be an open cover of* $f(K)$*, i.e.*

$$f(K) \subseteq \bigcup_{j\in J} \Omega_j \tag{2.3}$$

Then

$$K \subseteq f^{-1}(\bigcup_{j\in J} \Omega_j) = \bigcup_{j\in J} f^{-1}(\Omega_j) \tag{2.4}$$

By (1)*,* $f^{-1}(\Omega_j)$ *is open for all* $j \in J$*. Thus* $\{f^{-1}(\Omega_j)\}_{j\in J}$ *is an open cover of* K*.*

Since K *is compact,* $\exists j_1, \cdots, j_n \in J, \text{s. t.}$

$$k \subseteq \bigcup_{k=1}^{n} f^{-1}(\Omega_{j_k}) = f^{-1}(\bigcup_{k=1}^{n} \Omega_{j_k}) \tag{2.5}$$

$$\Rightarrow f(K) \subseteq \bigcup_{k=1}^{n} \Omega_{j_k} \tag{2.6}$$

□

We say that f *contains a maximum at* $z_0 \in \Omega$ *if*

$$|f(z)| \leq |f(z_0)|, \ \forall z \in \Omega \tag{2.7}$$

命题 2.1.2. *A continuous function on a compact set* Ω *is bounded and attains a maximum and a minimum on* Ω*.*

证明. *use* $|f|^2 = (Ref)^2 + (Imf)^2$. □

2.2 复变函数的极限，全纯函数

定义 2.2.1. *Assume* $\Omega \subseteq \mathbb{C}$, $\Omega \neq \varnothing$ *and* $a \in Acc(\Omega)$, $f : \Omega \longrightarrow \mathbb{C}$, $\lim\limits_{z \to a,\, z \in \Omega} f(z) = w$ *means*

$$\forall \epsilon > 0, \exists \delta > 0, \text{s.t. } 0 < |z - z_0| < \delta \;\Rightarrow\; |f(z) - w| < \epsilon \tag{2.8}$$

注. 容易证明若极限存在，则极限唯一.

定义 2.2.2. *Let* $\Omega \subseteq \mathbb{C}$ *be open,* $f : \Omega \longrightarrow \mathbb{C}$. *We say* $f(z)$ *is* <u>*Complex differentiable at* $z_0 \in \Omega$</u> *if* $\lim\limits_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ *exists. If* f *is complex differentiableat* z_0, *we denote the limit of the quotient by* $f'(z_0)$. *i.e.*

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{2.9}$$

$f'(z_0)$ *is called* <u>*the derivative of* f *at* z_0</u>.

注. *If* f *is complex differentiable at every point of* Ω, *then we say* f *is* <u>*holomorphic*</u> *on* Ω.

例 2.2.1.

- $f(z) = \frac{1}{z}$ *is holomorphic on* $\mathbb{C} \backslash \{0\}$.
- $f(z) = \bar{z}$ *is not complex differentiable at any point of* $\mathbb{C}$.
- $f(z) = |z|^2$ *is only complex differentiable at* $z = 0$.

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) \;\Leftrightarrow\; \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - hf'(z_0)}{h} = 0 \tag{2.10}$$

Let <u>$\circ(h)$</u> *denote* <u>*any complexed valued function*</u> *with the property* $\frac{\circ(h)}{h} \to 0$, *as* $h \to 0$
Then f *is complex differentiable at* z_0 *iff* $\exists a \in \mathbb{C}, \text{s.t.}$

$$f(z_0 + h) - f(z_0) - ha = \circ(h), \text{ where } a = f'(z_0) \tag{2.11}$$

注. *According to equation*(2.11), *holomorphic* $\Rightarrow$ *continuity.*

命题 **2.2.1.** *If f, g are holomorphic on an open set $\Omega \subseteq \mathbb{C}$, then*

$$(f+g)' = f' + g', \ (fg)' = f'g + fg' \tag{2.12}$$

If $g(z_0) \neq 0$, then $\frac{f}{g}$ is complex differentiable at z_0 and

$$\left(\frac{f}{g}\right)'_{z=z_0} = \frac{f'g - fg'}{g^2}\Big|_{z=z_0} \tag{2.13}$$

If $f : \Omega \longrightarrow U$ and $g : U \longrightarrow \mathbb{C}$ are holomorphic, then the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z), \ \forall z \in \Omega \tag{2.14}$$

2.3 *Cauchy – Riemann Equations*

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \tag{2.15}$$

Assume $\lim\limits_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ *exists, we may let* $h \to 0$ *in whichever manner we please.* (*let* $z_0 = x_0 + iy_0$)

- *Let* $h = t \in \mathbb{R}$,

$$f^{'}(z_0) = \lim_{t\to 0,\ t\in\mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{h} = u_x(x_0, y_0) + iv_x(x_0, y_0) \tag{2.16}$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \tag{2.17}$$

- *Let* $h = it, t \in \mathbb{R}$,

$$f^{'}(z_0) = \lim_{t\to 0,\ t\in\mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{it} = v_y(x_0, y_0) - iu_y(x_0, y_0) \tag{2.18}$$

$$= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \tag{2.19}$$

Thus, we conclude $f = u + iv$ *is holomorphic* $\Rightarrow$ $u,\ v$ *satisfy*

$$\begin{cases} u_x = v_x \\ u_y = -v_y \end{cases} \tag{2.20}$$

The equations(2.20) *is called* <u>*Cauthy – Riemann Equations*</u>.

例 2.3.1. $f(x + iy) = x^2 - y^2 - 2xyi,\ x, y \in \mathbb{R}$ *is not holomorphic on* $\mathbb{C}\backslash\{0\}$.

2.4 全纯条件

Let $f = u + iv : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Then

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad on\ \Omega \tag{2.21}$$

下面给出函数 *holomorphic* 的充分条件.

定理 2.4.1. Let $\Omega \subset \mathbb{C}$ be open, $f = u + iv : \Omega \longrightarrow \mathbb{C}$. If u, v are differentiable on Ω and satisfy the $Cauchy - Riemann\ equations$, then f is holomorphic on Ω.

证明. (Goal : $\forall z_0 = x_0 + iy_0 \in \Omega,\ h = h_1 + ih_2 \in \mathbb{C},\ z_0 + h \in \Omega,\ |h|\ small\ enough,$ $f(z_0 + h) - f(z_0) = ah + \circ(h)$)

Since $u(x, y)$ is differentiable on Ω,

$$u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = h_1 u_x(x_0, y_0) + h_2 u_y(x_0, y_0) + \circ(h_1, h_2) \tag{2.22}$$

Here $\circ(h_1, h_2)$ is any expression with the property that $\frac{\circ(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} \to 0$, as $(h_1, h_2) \to 0$.
Similarly,

$$v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0) = h_1 v_x(x_0, y_0) + h_2 v_y(x_0, y_0) + \circ(h_1, h_2) \tag{2.23}$$

Then

$$f(z_0 + h) - f(z_0) = h_1 u_x + h_2 u_y + i(h_1 v_x + h_2 v_y) + \circ(h_1, h_2) \tag{2.24}$$

$$= h_1 u_x - h_2 v_x + i(h_1 v_x + h_2 u_x) + \circ(h_1, h_2) \tag{2.25}$$

$$= (u_x + iv_x)(h_1 + ih_2) + \circ(h_1, h_2) \tag{2.26}$$

Note that we may write $\circ(h)$ instead of $\circ(h_1, h_2)$, since

$$(h_1, h_2) \to 0 \Leftrightarrow h \to 0 \Leftrightarrow |h| \to 0 \tag{2.27}$$

Then the previous expression is equal to $f'(z_0)h + \circ(h)$.
Since z_0 is arbitrary, f is holomorphic on Ω. □

$f = u + iv$ can be seen as a mapping

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{2.28}$$

$$(x, y) \longmapsto (u(x, y), v(x, y)) \tag{2.29}$$

F is said to be differentiable at a point $P_0 = (x_0, y_0)$, if $\exists$ a linear transformation $J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, s. t.

$$F(P_0 + H) - F(P_0) = J(H) + |H|\,\psi(H),\ \ with\ \ |\psi(H)| \to 0\ \ as\ \ |H| \to 0 \tag{2.30}$$

命题 **2.4.1.** If f is complex differentiable at $z_0 = x_0 + iy_0$, then F is differentiable at (x_0, y_0).

证明. Since f is complex differentiable at $z_0 = x_0 + iy_0$, we have

$$f(z_0 + h) - f(z_0) = f^{'}(z_0)h + \circ(h) \tag{2.31}$$

$$= (u_x + iv_x)(h_1 + ih_2) + \circ(h) \tag{2.32}$$

$$= u_x h_1 - v_x h_2 + i(v_x h_1 + u_x h_2) + \circ(h) \tag{2.33}$$

$$= u_x h_1 + u_y h_2 + i(v_x h_1 + v_y h_2) + \circ(h) \tag{2.34}$$

Thus, $F(P_0 + H) - F(P_0) = J(H) + \circ(H)$, where $J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ and $H = (h_1, h_2)$. □

2.5 复变函数微分

$$z = x + iy,\ \overline{z} = x - iy \ \Leftrightarrow\ x = \tfrac{z+\overline{z}}{2},\ y = \tfrac{z-\overline{z}}{2i}$$

A given function $f : \Omega \longrightarrow \mathbb{C}$ can be expressed either in variables x, y or $z, \overline{z}$. That is, for the given f, we may write $f(x, y)$ or $f(z, \overline{z})$.

注. 可视作复平面上可建立两个坐标系 xOy 和 $zO\overline{z}$，即 $\mathbb{C}$ 中存在两组基. 由于将复数 z 转化为 $x + iy$ 后再进行计算常常会产生不便，因此下面通过这两组基之间的转化，探讨不同形式下函数微分的表达方式.

Suppose the relevant derivatives exist.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f \tag{2.35}$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f \tag{2.36}$$

Define two operations.($Wirtinger\ operations,\ 1927$)

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{2.37}$$

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2.38}$$

命题 2.5.1. $Cauchy - Riemann\ equations$ are equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0 \tag{2.39}$$

证明. Let $f = u + iv$. Then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\Big(u_x + v_x + i(u_y + v_y)\Big) = \frac{1}{2}\Big(u_x - v_y + i(u_y + v_x)\Big) \tag{2.40}$$

$$\frac{\partial f}{\partial \overline{z}} = 0 \Leftrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \tag{2.41}$$

□

注. We note that $f'(z) = u_x + iv_x = u_x - iu_y = 2\frac{\partial u}{\partial z}$.

调和算子 / 拉普拉斯算子 Define the *Laplacian*(or the *Laplace operator*).

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.42}$$

注. $C^k(\Omega)$ denotes the set of all k times continuously differentiable functions on Ω.

下面给出调和函数的定义.

定义 2.5.1. Let $\Omega \subset \mathbb{C}$ be an open set. $g : \Omega \longrightarrow \mathbb{C}$ is called *harmonic* if $g \in C^2(\Omega)$ and $\Delta g = 0$.

下面的命题说明了全纯函数的实部和虚部均调和.(全纯的必要条件)

命题 2.5.2. Let $f = u + iv : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Assume $u, v \in C^2(\Omega)$. Then u, v are harmonic.

注. 事实上后面会证明此处无需 $u, v \in C^2(\Omega)$.

证明. The *Cauchy − Riemann equations* tell $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{2.43}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{2.44}$$

Since $v \in C^2(\Omega)$,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \tag{2.45}$$

Therefore, $u_{xx} + u_{yy} = 0$. Similarly we can proof that $v_{xx} + v_{yy} = 0$. □

A holomorphic function is necessarily harmonic, so is $\overline{f}$.

命题 2.5.3. Let $\Omega \subset \mathbb{C}$ be a region, $f : \Omega \longrightarrow \mathbb{C}$. Then

$$f \text{ is consistant iff } f'(z) = 0, \forall z \in \Omega.$$

证明.

$\Rightarrow$: clear

$\Leftarrow$: Let $f = u + iv$, then

$$
\begin{aligned}
f^{'}(z) = 0 \Rightarrow u_x + iv_x = 0 \Rightarrow u_x = 0, v_x = 0 \quad &(2.46)\\
\overset{C-R}{\Rightarrow} v_y = 0, u_y = 0 \quad &(2.47)\\
\Rightarrow u = c_1, v = c_2 \ (by\ mean\ value\ theorem) \quad &(2.48)\\
(\text{区域连通，利用中值定理}) \quad &(2.49)
\end{aligned}
$$

□

2.6 课堂例题 2024 − 03 − 08

1. $f(x + iy) = x^2 - y^2 + 2xyi$ is holomorphic.

2. Is $f(z) = z^2\overline{z} + \frac{1}{z} + \frac{1}{z^2}$ holomorphic on $\mathbb{C}\backslash\{0\}$?

3. Let $f = u + iv$ be holomorphic on a region Ω. Assume $au + bv + c = 0$ for some $a, b, c \in \mathbb{R}$ and a, b are not all zero. Show f is consistant.

4. Find a holomorphic function f on $\mathbb{C}$ s. t.

$$Ref = x^2 - y^2 + xy,\ f(0) = 0 \tag{2.50}$$

5. Let $\Omega = \mathbb{C}\backslash\{0\}$ and $u : \Omega \longrightarrow \mathbb{R}$ be given by $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$.
   Is there a holomorphic function $f : \Omega \longrightarrow \mathbb{C}$, s. t. $Ref = u$?

解. Suppose $f = u + iv$ is holomorphic on Ω. Then

$$\begin{cases} v_x = -u_y = -\frac{y}{x^2+y^2} \\ v_y = u_x = \frac{x}{x^2+y^2} \end{cases} \tag{2.51}$$

By $v_y = \frac{x}{x^2+y^2}$,

$$v = \arctan\frac{y}{x} + c(x) \tag{2.52}$$

Then by $v_x = -\frac{y}{x^2+y^2}$, $c(x) = c$ is constant. $\Rightarrow v = \arctan\frac{y}{x} + c$.
However, $\arctan\frac{y}{x} : \mathbb{R}^2 \longrightarrow (-\pi, \pi]$ is not continuous on $\mathbb{R}_{\leq 0} = \{x \leq 0 \mid x \in \mathbb{R}\}$.

(Let $z = x + iy$, then $\arctan\frac{y}{x}$ is an argument of z.)

Therefore, there is no function satisfying the conditions.

注. If the region $\Omega = \mathbb{C}\backslash\{0\}$ is replaced by $\Omega = \mathbb{C}\backslash\mathbb{R}_{\leq 0}$, then the answer is yes.

$\square$

6. 课本第一章练习 $T8, T9, T10, T13$.

第三章 *Week 3*

3.1 幂级数，解析函数，复对数

与数学分析中的概念一致，下面相当于来复习一下有关**幂级数**的概念.

- 幂级数 $\sum_{n=0}^{\infty} z_n$ converges $\Leftrightarrow$ 部分和 $\{S_N = \sum_{n=0}^{N} z_n\}$ converges.
- $\sum_{n=0}^{\infty} |z_n|$ converges $\Rightarrow$ The series converges absolutely(绝对收敛).
- **Absolutely convergent $\Rightarrow$ convergent**
- If $\sum_{n=0}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

A power series (with center 0) is an expansion of the form $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ are fixed and z varies in $\mathbb{C}$.(下面通常讨论形式为 $\sum_{n=0}^{\infty} a_n z^n$ 的幂级数)

下面给出复幂级数的**收敛半径**的定义及**收敛圆盘**.

定理 3.1.1. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, define

$$R = \varliminf_{n\to\infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\varlimsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \quad \textbf{(Hardamard's Formula)} \tag{3.1}$$

(Here we use the convertion $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$.) Then

(1) If $|z| < R$, the series converges absolutely.

(2) If $|z| > R$, the series diverges.

注. The number R is called the **radius of convergence** of the power series, and the region $|z| < R$ is called the **disc of convergence**.

例 3.1.1. 下面给出一些用幂级数定义的常见函数的例子.

- Exponential function

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!},\ z \in \mathbb{C} \tag{3.2}$$

- Trigonometric function

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\ ,\ \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{3.3}$$

- 双曲余弦、正弦

$$\cosh z := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\ ,\ \sinh z := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \tag{3.4}$$

注. 由定义容易得到，$e^{iz} = \cos z + i \sin z \Rightarrow$ 将 z 限制到 $\mathbb{R}$ 上则有：$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},\ \sin z = \frac{e^{iz} - e^{-iz}}{2} \tag{3.5}$$

下面这个定理说明了幂级数在收敛圆盘内解析. 并给出了幂级数的导数.

定理 3.1.2. The power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. Moreover, $f'(z) = \sum\limits_{n=0}^{\infty} n a_n z^{n-1}$, which has the same radius of convergence.

证明. **Hadamard's formula** tells $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} n a_n z^{n-1}$ have the same R.
Let $g(z) = \sum\limits_{n=0}^{\infty} n a_n z^{n-1}$, $\forall z$ with $|z| < R$, we can find r, s. t. $|z| < r < R$.
For $\forall h \in \mathbb{C}$ s. t. $|h| < r - |z|$, we estimate

$$|f(z+h) - f(z) - hg(z)| = \left| \sum_{n=0}^{\infty} a_n \left((z+h)^n - z^n - nhz^{n-1} \right) \right| \tag{3.6}$$

$$= \left| \sum_{n=2}^{\infty} \left(a_n \sum_{k=2}^{n} \binom{n}{k} h^k z^{n-k} \right) \right| \tag{3.7}$$

$$\leq |h|^2 \sum_{n=2}^{\infty} |a_n| \sum_{k=0}^{n-2} \binom{n}{k+2} \left| h^k z^{n-2-k} \right| \tag{3.8}$$

Since $\binom{n}{k+2} \le n(n-1)\binom{n-2}{k}$, then

$$|f(z+h)-f(z)-hg(z)| \le |h|^2 \sum_{n=2}^{\infty} |a_n|\, n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} |h|^k\, |z|^{n-2-k} \tag{3.9}$$

$$= |h|^2 \sum_{n=2}^{\infty} |a_n|\, n(n-1)\,(|z|+|h|)^{n-2} \tag{3.10}$$

$$< |h|^2 \sum_{n=2}^{\infty} |a_n|\, n(n-1) r^{n-2} = |h|^2 \cdot c \tag{3.11}$$

Thus

$$\left| \frac{f(z+h)-f(z)}{h} - g(z) \right| < |h| \cdot c \tag{3.12}$$

Therefore, the result follows. □

推论 3.1.3. A power series is infinitely differentiable in its disc of convergence.

注. Thm 3.1.2 即说明了幂级数在收敛圆盘内解析.

下面给出推广到更一般的幂级数的导数，即中心不一定在原点的情形.

A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \tag{3.13}$$

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f(z) = g(w)$, where $w = z - z_0$.

According to the **Chain Rule**(链式法则), $f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$

下面严格地给出**解析**的定义.

定义 3.1.1. A function f defined on an open set Ω is said to be **analytic** at $z_0 \in \mathbb{C}$ if there is a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with positive radius of convergence, s. t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \; \textit{for all } z \textit{ in a neighbourhood of } z_0 \tag{3.14}$$

$$(i.e.\ \forall z \in D_r(z_0),\ \textit{for some } r > 0) \tag{3.15}$$

If f is analytic at every point of Ω, then we say f is **analytic on Ω**.

下面给出有关指数函数 e^z 的一些等式 (命题).
在此之前，先给出 **Cauchy Multiplication Theorem**.

引理 3.1.4. If $\sum a_n$, $\sum b_n$ are absolutely convergent, then

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) = \left(\sum a_n\right)\left(\sum b_n\right) \tag{3.16}$$

命题 3.1.1. For $z_1, z_2 \in \mathbb{C}$, $e^{(z_1+z_2)} = e^{z_1} \cdot e^{z_2}$.

推论 3.1.5. If $z = x + iy$, $x, y \in \mathbb{R}$, then

$$e^z = e^x(\cos y + i \sin y) \tag{3.17}$$

推论 3.1.6. **De Moire's Formula**.
For $\vartheta \in \mathbb{R}$,

$$(\cos\vartheta + i\sin\vartheta)^n = \cos n\vartheta + i \sin n\vartheta \tag{3.18}$$

下面来引入复数域上的**对数函数 (Complex Logarithm)**.
$\forall z \in \mathbb{C}\backslash\{0\}$, write $z = re^{i\vartheta}$. Then $e^w = z$ can be solved.
If $w = u + iv$, $u, v \in \mathbb{R}$, then

$$e^u \cdot e^{iv} = re^{i\vartheta} \;\Rightarrow\; u = \log r,\; v = \vartheta + 2k\pi, k \in \mathbb{Z} \tag{3.19}$$

Let $Log(z)$ be the set of above, then we get Complex Logarithm.

定义 3.1.2. $\forall z \in \mathbb{C}\backslash\{0\}$. Define

$$Log(z) := \log|z| + i(\arg z + 2k\pi), k \in \mathbb{Z} \tag{3.20}$$

Here $\arg z$ is an argument of z satisfying $-\pi < \arg z \leq \pi$.

(We call $\arg z$ the **principal argument** of z.)

下面介绍复对数的**主值支**的概念.

定义 3.1.3. Define the **principal branch** of the logarithm on a ”cut plane”

$$\log : \mathbb{C}\backslash\mathbb{R}_{\leq 0} \longrightarrow \mathbb{C} \tag{3.21}$$

$$z \longmapsto \log|z| + i\arg z,\ \ -\pi < \arg z < \pi \tag{3.22}$$

例 3.1.2.

$$Log(-1) = (2k+1)\pi i \tag{3.23}$$

$$Log(i) = (2k+\frac{1}{2})\pi i \tag{3.24}$$

$$\log i = \frac{\pi}{2} i \tag{3.25}$$

$$\log(1+i) = \frac{1}{2}\log 2 + \frac{\pi}{4} i \tag{3.26}$$

命题 3.1.2.

$$e^{Log(z)} = z,\ \ z \neq 0 \tag{3.27}$$

$$Log(z_1 z_2) = Log(z_1) + Log(z_2) \tag{3.28}$$

$$\log z_1 z_2 \neq \log z_1 + \log z_2\ \ in\ general \tag{3.29}$$

3.2 课堂例题 2024 − 03 − 11

1. Let $z \neq 0$. Then $\exists n$ different $z_0, \cdots, z_{n-1}$, s. t.

$$z_k^n = z,\ \ k = 0, \cdots, n-1 \tag{3.30}$$

解. Let $z = |z|\, e^{i\vartheta}$, $w = re^{it}$, $r > 0$, $t \in \mathbb{R}$. Then

$$w^n = z \ \Rightarrow\ r^n e^{int} = |z|\, e^{i\vartheta} \ \Rightarrow\ \begin{cases} r = |z|^{\frac{1}{n}} \\ nt = \vartheta + 2k\pi, k \in \mathbb{Z} \end{cases} \tag{3.31}$$

□

2. Proof

$$\left|\sum_{k=0}^{n} e^{ikx}\right| \leq \left|\frac{1}{\sin\frac{x}{2}}\right|,\ \ \forall x \in \mathbb{R}\backslash\{2k\pi \mid k \in \mathbb{Z}\} \tag{3.32}$$

3. 课本第一章练习 $T16, T19$

3.3 复对数的性质

Let $a \in \mathbb{C}$. We may define

$$z^a = e^{a \log z}, \ z \neq 0 \tag{3.33}$$

命题 3.3.1. The function $f(z) = \log z$, $z \in \mathbb{C} \backslash \mathbb{R}_{\leq 0}$ is holomorphic.

证明. $\forall z_0 \in \mathbb{C} \backslash \mathbb{R}_{\leq 0}$, let $w = \log z$, $w_0 = \log z_0$. Then

$$\lim_{z \to z_0} \frac{\log z - \log z_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0} \tag{3.34}$$

Therefore, $(\log z)' = \frac{1}{z}$. □

命题 3.3.2. Show

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \ on \ \mathcal{D} \tag{3.35}$$

证明. Let $f(z) = \log(1+z)$, $g(z) = \sum\limits_{n=1}^{\infty} (-1)^n \frac{z^{n-1}}{n}$. Both are holomorphic on $\mathcal{D}$ and

$$f^{'}(z) = \frac{1}{1+z} \ , \ g^{'}(z) = \sum_{n=1}^{\infty} (-1)^n z^{n-1} = \frac{1}{1+z} \tag{3.36}$$

And so $(f-g)^{'} = 0$ on $\mathcal{D}$. Therefore, $f - g = c$. Taking $z = 0, f(0) = g(0) \Rightarrow c = 0$. □

3.4 道路

先给出**道路 (path)** 的定义.

定义 3.4.1. A continuous function $z(t) = x(t) + iy(t)$ from $[a, b] \subset \mathbb{R}$ to $\mathbb{C}$ is called a **path** (or a parametric curve) connecting $z(a)$ and $z(b)$.($z(a)$ is called the starting point, $z(b)$ the end point) The path is **closed** if $z(a) = z(b)$.

The path is **simple** if $z(t) \neq z(s)$ unless $\begin{cases} (1) t = s \\ (2) t = a, s = b \end{cases}$

下面给出道路**光滑性**的描述.

定义 3.4.2. We say that a path $z(t) = x(t)+iy(t)$, $t \in [a, b]$ is **smooth** if $x(t), y(t)$ are continuously differentiable and $z'(t) = x'(t) + iy'(t) \neq 0$, $t \in [a, b]$. Here $z'(a), z'(b)$ are understood as one-sided derivative.

下面给出两条道路**等价**的定义.

定义 3.4.3. Two paths $z : [a, b] \longrightarrow \mathbb{C}$, $\widetilde{z} : [c, d] \longrightarrow \mathbb{C}$ are **equivalent** if $\exists$ bijection and differential

$$t : [c, d] \longrightarrow [a, b] \tag{3.37}$$

$$s \longmapsto t(s) \tag{3.38}$$

s. t. $\widetilde{z}(s) = z(t(s))$ and $t'(s) > 0$.

下面给出**道路反向**的定义.

定义 3.4.4. Given a path z, we can define a path $\widetilde{z}$ obtained from z by reversing the orietation

$$z(t) : [a, b] \longrightarrow \mathbb{C} \tag{3.39}$$

$$\widetilde{z}(t) = z(a + b - t) : [a, b] \longrightarrow \mathbb{C} \tag{3.40}$$

这里我们规定一下道路的**正向 / 逆向** (逆时针为正向).

定义 3.4.5. A path has **positive orientation** if it travels counterclockwisely.

($\cdots$ **negative orientation** $\cdots$ clockwisely.)

下面我们给出**分段光滑**的定义.

定义 3.4.6. A path $z(t) : [a, b] \longrightarrow \mathbb{C}$ is **piecewise smooth** if $\exists$ a partion $a = a_0 < a_1 < \cdots < a_n = b$, s. t. $z(t)$ is smooth in each $[a_k, a_{k+1}]$, $k = 0, \cdots, n-1$.

下面说明两条道路的**连接**.

Paths can be concatenated. If $z : [a, b] \longrightarrow \mathbb{C}$, $\widetilde{z} : [b, c] \longrightarrow \mathbb{C}$ and $z(b) = \widetilde{z}(b)$, we can define $w : [a, c] \longrightarrow \mathbb{C}$ as $w(t) = \begin{cases} z(t), a \leq t \leq b \\ \widetilde{z}(t), b \leq t \leq c \end{cases}$. **Concatenation** of $z, \widetilde{z}$ is denoted as $z \circ \widetilde{z}$.

下面给出 **zig-zag 道路**的定义.

定义 3.4.7. A path is **zig-zag** if it consists of finitely many horizontal or vertical line seqments.

下面的命题说明区域内的任两点可由一条 zig-zag 道路连接.

命题 3.4.1. Let $\Omega \subset \mathbb{C}$ be a region. Then any two points in Ω can be joined by a zig-zag path.

证明.

- Case when $\Omega = D_R(z_0)$, where $z_0 \in \mathbb{C}$, $R > 0$.

  $\forall \alpha, \beta \in \Omega$, we can join them to the horiziontal diameter via a vertical line seqment.

- Now let Ω be an arbitrary region. $\forall \alpha \in \Omega$. Let

$$A := \{\beta \in \Omega \mid \exists\ zig-zag\ path\ \gamma\ connecting\ \beta\ and\ \alpha\} \tag{3.41}$$

  Then 容易证 $\alpha \in A \neq \varnothing$ 既开又闭，从而 $A = \Omega$.

$\square$

3.5 课堂例题 2024 − 03 − 15

1. Calculate 2^i, i^i.

2. Find all possible values of $(1+\sqrt{3}i)^{\frac{1}{8}}$.

3. Let $z_n \in \mathbb{C}$, $Rez_n \geq 0$, $n = 1, 2, \cdots$. If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_n^2$ both converge, show that $\sum_{n=1}^{\infty} |z_n|^2$ converges.

4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be holomorphic on $\mathcal{D}$. Assume $|f(z)| \leq 1$, $\forall z \in \mathcal{D}$. Show $|a_n| \leq 1$, $n = 1, 2, \cdots$.

第四章　*Week* 4

4.1　曲线积分

积分　下面先给出复数域上积分的定义.

定义 4.1.1. Let $z(t) = x(t) + iy(t)$, $t \in [a, b] \subset \mathbb{R}$. If $x(t), y(t)$ are differentiable, we define $z^{'}(t) = x^{'}(t) + iy^{'}(t)$.

Similarly, if $x(t), y(t)$ are continuous, we define

$$\int_a^b z(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt \tag{4.1}$$

容易证明，复数域上的积分同样具有三角不等式.

命题 4.1.1. Let $f : [a, b] \longrightarrow \mathbb{C}$ be continuous. Then

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt \tag{4.2}$$

证明. Write $\int_a^b f(t)dt = re^{i\vartheta}$, $r \geq 0$. Then

$$r = e^{-i\vartheta}\int_a^b f(t)dt = \int_a^b e^{-i\vartheta}f(t)dt = \left|\int_a^b Re\, e^{-i\vartheta}f(t)dt\right| \tag{4.3}$$

$$\leq \int_a^b \left|Re\, e^{-i\vartheta}f(t)\right| dt \tag{4.4}$$

$$\leq \int_a^b \left|e^{-i\vartheta}f(t)\right| dt = \int_a^b |f(t)|\, dt \tag{4.5}$$

□

曲线积分 下面给出复数域上**连续道路**的**曲线积分**的定义.

定义 4.1.2. Let $\Omega \subset \mathbb{C}$ be open. Given a smooth path γ in Ω parametrized by $z : [a, b] \longrightarrow \Omega$ and a continuous funciton $f : \Omega \longrightarrow \mathbb{C}$. We define the **integral of f along γ** by

$$\int_\gamma f(z)dz := \int_a^b f(z(t))z'(t)dt \tag{4.6}$$

Let $\widetilde{z}(t) : [c, d] \longrightarrow \Omega$ be equivalent to $z(t)$. Then

$$\int_a^b f(z(t))z'(t)dt = \int_c^d f(\widetilde{z}(t))\widetilde{z}'(t)dt \tag{4.7}$$

下面给出**分段连续**道路的曲线积分及**曲线长度**的定义.

定义 4.1.3. If γ is piecewise smooth and $z(t)$ is a piecewise smooth parametrization as before, we define

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt \tag{4.8}$$

The **length** of the smooth curve γ is

$$length(\gamma) = \int_a^b \left|z'(t)\right| dt \tag{4.9}$$

If $f = u + iv$, $z(t) = x(t) + iy(t)$, then

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b (u + iv)(x'(t) + iy'(t))dt \tag{4.10}$$

$$= \int_a^b (ux'(t) - vy'(t))dt + i \int_a^b (vx'(t) + uy'(t))dt \tag{4.11}$$

$$= \int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy) \tag{4.12}$$

下面给出曲线积分的几条性质.

命题 4.1.2. 记 γ^- 为 γ 的反向.

(1) $\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$.

(2) If $f(z), g(z)$ are continuous, and γ is a path, then $\forall \alpha, \beta \in \mathbb{C}$,

$$\int_\gamma (\alpha f + \beta g)dz = \alpha \int_\gamma f dz + \beta \int_\gamma g dz \tag{4.13}$$

(3)

$$\left|\int_\gamma f(z)dz\right| \leq \sup_\gamma |f(z)| \cdot length(\gamma) \tag{4.14}$$

原函数　下面我们给出**原函数**的概念.

定义 4.1.4. If $f : \Omega \longrightarrow \mathbb{C}$. Assume $\exists$ a complex differentiable $F : \Omega \longrightarrow \mathbb{C}$, s. t.

$$F^{'}(z) = f(z), \ for\ every\ z \in \Omega \tag{4.15}$$

Then we say f admits a **primitival** (or an antiderivative) on Ω.

下面的命题说明若函数有原函数，则其曲线积分只与始末点有关，而与路径无关.

命题 4.1.3. If f is a continuous function that admits a primitive F on Ω, and γ is a path in Ω that begins at w_1 and ends at w_2, then

$$\int_\gamma f(z)dz = F(w_2) - F(w_1) \tag{4.16}$$

证明. Let $z(t) : [a, b] \longrightarrow \Omega$ be a parametrization for γ with $z(a) = w_1$, $z(b) = w_2$.

- Assume γ is smooth. Compute

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z^{'}(t)dt = \int_a^b F^{'}(z(t))z^{'}(t)dt = \int_a^b \frac{dF(z(t))}{dt}dt \tag{4.17}$$

  According to **the fundamental theorem of calculus**, we get
  (分别对实部和虚部运用微积分基本定理)

$$\int_\gamma f(z)dz = \int_a^b F^{'}(z(t))z^{'}(t)dt = \int_a^b \frac{dF(z(t))}{dt}dt \tag{4.18}$$

$$= F(z(b)) - F(z(a)) = F(w_2) - F(w_1) \tag{4.19}$$

- γ is piecewise smooth, we can proof similarly.

□

由命题 4.1.3，可得到有原函数的函数 f 在闭曲线上积分为 0.

推论 4.1.1. If γ is a closed path in Ω, f is continuous and admits a primitive on Ω, then

$$\int_\gamma f(z)dz = 0 \tag{4.20}$$

同时，由命题 4.1.3，还可得到区域 Ω 上导数恒为 0 的全纯函数只能为常值函数.

推论 4.1.2. If f is holomorphic on a region Ω and $f' \equiv 0$, then f is constant.

下面给出具有原函数的充要条件.

定理 4.1.3. Let $\Omega \subset \mathbb{C}$ be a region. $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function. Then the following statements are equivalent:

(1) f admits a primitive on Ω.

(2) $\forall \alpha, \beta \in \mathbb{C}$, $\int_\gamma f(z)dz$ is invariant for any path γ in Ω that joins α to β.

(3) $\forall \alpha, \beta \in \mathbb{C}$, $\int_\gamma f(z)dz$ is invariant for any zig-zag path γ in Ω that joins α to β.

注. 我们将在**定理 5.2.5** 中补充一条具有原函数的充要条件. (详见 **Thm 5.2.5 (4)**)

证明. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1): Fix $\alpha \in \Omega$ and define $F : \Omega \longrightarrow \mathbb{C}$ by

$$F(z_0) = \int_\gamma f(z)dz, \ \ z_0 = x_0 + iy_0 \in \Omega \tag{4.21}$$

where γ is any zig-zag path joining α to z_0.

(F **is Well-defined** : Condition (3) tells $F(z_0)$ is independent of the choice of γ.)

Let $F(z) = U + iV$, $f(z) = u + iv$. It suffices to show

$$\begin{cases} U_x(x_0, y_0) &= u(x_0, y_0) \ , \ V_x(x_0, y_0) = v(x_0, y_0) \\ U_y(x_0, y_0) &= -v(x_0, y_0) \ , \ V_y(x_0, y_0) = u(x_0, y_0) \end{cases} \tag{4.22}$$

- $U_x(x_0, y_0) = u(x_0, y_0) \ , \ V_x(x_0, y_0) = v(x_0, y_0)$

  Let $h \in \mathbb{R}$. Let γ be a zig-zag path joining α to z_0,

  $\gamma_H : z_H(t) = z_0 + th, 0 \leq t \leq 1$. $\gamma_H \subset \Omega$. Then

$$F(z_0 + h) = \int_{\gamma \circ \gamma_H} f(z)dz = \int_\gamma f(z)dz + \int_{\gamma_H} f(z)dz \tag{4.23}$$

$$= F(z_0) + \int_{\gamma_H} f(z)dz \tag{4.24}$$

Then we get

$$\frac{F(z_0+h)-F(z_0)}{h} = \int_0^1 f(z_0+th)dt \tag{4.25}$$

Since f is continuous,

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{F(z_0+h)-F(z_0)}{h} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \int_0^1 f(z_0+th)dt \tag{4.26}$$

$$= f(z_0) \tag{4.27}$$

$$= u(x_0,y_0) + iv(x_0,y_0) \tag{4.28}$$

- $U_y(x_0,y_0) = -v(x_0,y_0)\ ,\ V_y(x_0,y_0) = u(x_0,y_0)$

  Similarly.

$\square$

4.2 课堂例题 2024 − 03 − 18

1. Let $f(x+iy)=x$. Consider two paths γ_1, γ_2 joining 0 to 1.

$$z_1(t) = t,\ \ 0 \le t \le 1 \tag{4.29}$$

$$z_2(t) = \begin{cases} t + 2ti,\ \ 0 \le t \le \frac{1}{2} \\ \\ t + 2(1-t)i,\ \ \frac{1}{2} \le t \le 1 \end{cases} \tag{4.30}$$

   Evaluate

$$\int_{\gamma_1} f(z)dz,\ \ \int_{\gamma_2} f(z)dz \tag{4.31}$$

$$(= \frac{1}{2},\ \ = \frac{1-i}{2}) \tag{4.32}$$

2. Let $f(z) = \frac{1}{z}$, $z \in \mathbb{C}\backslash\{0\}$. Let $\gamma : z(t) = Re^{it}$, $R > 0$, $0 \le t \le 2\pi$.
   Evaluate

$$\int_{\gamma} f(z)dz\ (= 2\pi i) \tag{4.33}$$

3. Let $f(z) = z^3$ and let σ be any path joining 1 to $2+i$. Evaluate

$$\int_{\gamma} f(z)dz \tag{4.34}$$

4. 课本第一章练习 $T26$.

4.3 *Cauchy's Theorem*

这节我们来介绍一个重要的定理——**Cauchy's Theorem**.

单连通 下面先给出一个定理并借此给出曲线的**内部**和**外部**的定义.

定理 4.3.1. Jordan Curve Therom.
Let γ be a simple closed curve on $\mathbb{C}$. Then $\mathbb{C}\backslash\gamma$ has two connected components. The bounded component is called the **interior of γ** and the unbounded component is called the **exterior of γ**.

If the simple closed path γ is positively oriented, then Interior(γ) is to the **left** while traversing γ.

证明. 证明见书[1]*Page* 351 □

下面给出**单连通集**的定义.

定义 4.3.1. A region $\Omega \subset \mathbb{C}$ is **simply connected** if for every closed path $\gamma \subset \Omega$, $Interior(\gamma) \subset \Omega$.

例 4.3.1. 下面给出几个常见的单连通集 / 非单连通集的例子.

- $\mathbb{C}$, $D_r(z_0)$, $r > 0$, $z_0 \in \mathbb{C}$ are simply connected.
- $\mathbb{C}\backslash\{0\}$, $D_r^*(z_0)$ are not simply connected.
- $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ is simply connected.

[1]课堂教材:《*Complex Analysis*》— *Elias M. Stein*

Cauchy's Theorem 下面介绍 **Cauchy's Theorem**.

定理 4.3.2. Let $\Omega \subset \mathbb{C}$ be **simply connected**, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Then for any closed path γ,

$$\int_{\gamma} f(z)dz = 0 \tag{4.35}$$

注. 条件中 Ω 单连通**不可省略**. 下面给出反例.

例 4.3.2. Let $\Omega = \mathbb{C}\backslash\{0\}, f(z) = \frac{1}{z}$. Then f is holomorphic on Ω and ($C_1(0)$ 表示单位圆周)

$$\int_{C_1(0)} f(z)dz = 2\pi i \neq 0 \tag{4.36}$$

证明. The result would follow if f has a primitive on Ω.
By Thm 4.1.3, thus it suffices to show that for any two zig-zag paths γ_1, γ_2 having the same starting points and ending points,

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz \tag{4.37}$$

$$i.e. \int_{\gamma_1 \circ \gamma_2^{-1}} f(z)dz = 0 \tag{4.38}$$

Equivalently, we have to show $\int_\gamma f(z)dz = 0$ for any closed zig-zag path γ.
By concatecnating some horizontal or vertical paths, any closed zig-zag path is the union of rectangle paths.
Thus we are done if we can show $\int_R f(z)dz = 0$, where R is a rectangle path.
Note that

$$\int_R f(z)dz = \int_{T_1} f(z)dz + \int_{T_2} f(z)dz \tag{4.39}$$

Then the theorem boils down to showing $\int_T f(z)dz = 0$ for any triangle path T in Ω.

(This is Coursat Theorem on P34.)

□

下面给出 **Cauchy's Theorem** 的另一种叙述，这里并不对 f 的定义域 Ω 做单连通要求.

定理 4.3.3. Let γ be a simple closed path. If f is holomorphic in Interior(γ) and continuous on γ, then

$$\int_\gamma f(z)dz = 0 \tag{4.40}$$

4.4 课堂例题 $2024-03-22$

1. 课本第一章练习 $T25$.

2. 课本第二章练习 $T5, T6$.

第五章　*Week 5*

5.1　闭路变形原理

围道　为了叙述方便，我们将简单闭曲线记作**围道 (contour)**. 并默认其为正向.

定义 5.1.1. A simple closed path is called a **contour**. If nothing is specified, we'll assume the contour is positively oriented.

闭路变形原理　下面来介绍**闭路变形原理**.(实际上可视为 Cauchy's Theorem 的推论)

定理 5.1.1. Principle of contour deformation.
Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Let γ_1 be a contour in Ω, γ_2 be another contour in $\Omega \cap Interior(\gamma_1)$. If $Interior(\gamma_1) \cap Exterior(\gamma_2) \subset \Omega$, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz \tag{5.1}$$

注. 条件 $Interior(\gamma_1) \cap Exterior(\gamma_2) \subset \Omega$ 保证了 γ_1 与 γ_2 之间围成的区域不存在空洞 (亏格)，从而在这片区域上 Cauchy's Theorem 总是奏效的.

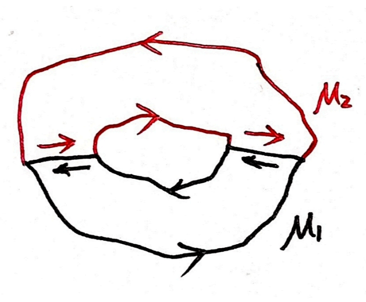

图 5.1: **Principle of contour deformation**

证明.

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2^-} f(z)dz \tag{5.2}$$

$$= \int_{\mu_1} f(z)dz + \int_{\mu_2} f(z)dz = 0 + 0 = 0 \tag{5.3}$$

□

下面给出 Thm 5.1.1 的一种 Special Case. (Interior(γ_1) 仅有一点不在 Ω 内)

推论 5.1.2. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Let γ be a contour in Ω and $\exists a \in Interior(\gamma)$, s. t. $Interior(\gamma)\backslash\{a\} \subset \mathbb{C}$. Then

$$\int_{\gamma} f(z)dz = \int_{C_r(a)} f(z)dz, \ where\ C_r(a) \subset Interior(\gamma) \tag{5.4}$$

下面给出更一般的表述.(Interior(γ) 内有有限个点不在 Ω 内)

推论 5.1.3. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic and γ is a contour in Ω. If $a_1, \cdots, a_n \in Interior(\gamma)$, such $Interior(\gamma)\backslash\{a_1, \cdots, a_n\} \subset \Omega$, then

$$\int_{\gamma} f(z)dz = \sum_{k=1}^{n} \int_{C_{r_k}(a_k)} f(z)dz \tag{5.5}$$

where $C_{r_k}(a_k)$ are disjoint “small” circles.

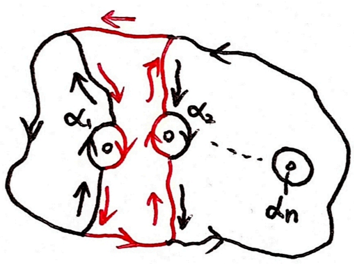

图 5.2: **Principle of contour deformation (Special Case)**

证明. 即证

$$\int_{\gamma} f(z)dz + \sum_{k=1}^{n} \int_{C_{r_k}^-(a_k)} f(z)dz = \sum_{k=1}^{n} \int_{\mu_k} f(z)dz = \sum_{k=1}^{n} 0 = 0 \tag{5.6}$$

□

5.2 *Cauchy Integral Formulas*

接下来我们介绍另一个计算环路积分非常重要的公式——**Cauchy Integral Formulas**.

Cauchy Integral Formulas 首先给出一个命题.

命题 5.2.1. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic, γ be a contour in Ω. Assume $\exists a \in Interior(\gamma)$, s. t.

$$Interior(\gamma)\backslash\{a\} \subset \Omega \quad and \quad \lim_{z\to a}(z-a)f(z) = 0 \tag{5.7}$$

Then

$$\int_\gamma f(z)dz = 0 \tag{5.8}$$

证明. $\forall \epsilon > 0, \exists \delta > 0$, s. t.

$$|z-a| < \delta \;\Rightarrow\; |(z-a)f(z)| < \epsilon \tag{5.9}$$

By the **principle of contour deformation** (Thm 5.1.1),

$$\int_\gamma f(z)dz = \int_{C_r(a)} f(z)dz, \;where\; 0 < r < \delta \tag{5.10}$$

Then

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in C_r(a)} |f(z)| \cdot length(\gamma) = \sup_{z\in C_r(a)} |f(z)| \cdot 2\pi r \tag{5.11}$$

$$< \frac{\epsilon}{r} \cdot 2\pi r = 2\pi\epsilon \tag{5.12}$$

□

下面给出 **Cauchy Integral Formulas**.

定理 5.2.1. CIF.

Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Then for any $z_0 \in \Omega$,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} d\zeta \tag{5.13}$$

where γ is any contour in Ω s. t. $z_0 \in Interior(\gamma) \subset \Omega$.

证明. Let

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \tag{5.14}$$

Then $g(z)$ is holomorphic on $\Omega \backslash \{z_0\}$. Clearly

$$\lim_{z \to z_0} (z - z_0) g(z) = 0, \ \ Interior(\gamma) \backslash \{z_0\} \subset \Omega \backslash \{z_0\} \tag{5.15}$$

By Prop 5.2.1,

$$\int_\gamma g(z) dz = 0 \ \Rightarrow \ \int_\gamma \frac{f(z)}{z - z_0} dz = \int_\gamma \frac{f(z_0)}{z - z_0} dz \tag{5.16}$$

Therefore

$$\int_\gamma \frac{f(z)}{z - z_0} dz = \int_\gamma \frac{f(z_0)}{z - z_0} dz = f(z_0) \int_\gamma \frac{1}{z - z_0} dz = f(z_0) \int_{C_r(z_0)} \frac{1}{z - z_0} dz \tag{5.17}$$

$$= 2\pi i \cdot f(z_0) \tag{5.18}$$

□

Therefore, according to Cauchy Integral Formulas, we get

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \tag{5.19}$$

where f is holomorphic on an open set containing the contour γ and its interior, $z \in Interior(\gamma)$.

高阶 Cauchy Integral Formulas 下面给出高阶的 Cauchy Integral Formulas.

定理 5.2.2. Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Then f is complex differentiable to all orders and moreover, $\forall z \in \Omega$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n = 0, 1, 2, \cdots \tag{5.20}$$

where γ is any contour in Ω s. t. $z \in Interior(\gamma) \subset \Omega$.

Cauchy's Inequalities 作为**柯西积分公式 (CIF)** 的推论，下面可得到 **Cauchy's Inequalities**.

推论 5.2.3. Cauchy's Inequalities.

If f is holomorphic on an open set Ω with $D_R(z_0) \subset \Omega$, $R > 0$, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \, \|f\|_{C_R(z_0)}}{R^n}, \quad where \ \|f\|_{C_R(z_0)} = \sup_{z \in C_R(z_0)} |f(z)| \tag{5.21}$$

Liouville's therom 下面给出 **CIF** 的另一条重要推论. 在此之前，先给出**整函数 (entire)** 的定义.

定义 5.2.1. A holomorphic function defined on the whole $\mathbb{C}$ is called an **entire function**.

下面给出**刘维尔定理 (Liouville's theorem)** 的内容.

推论 5.2.4. Liouville's theorem.

If f is entire and bounded, then f is constant.

证明. $\forall z \in \mathbb{C}$, by **Cauchy's Inequalities (Cor 5.2.3)**

$$\left|f'(z)\right| \leq \frac{\|f\|_{\mathbb{C}}}{R} \leq \frac{M}{R}, \quad where \ |f| \leq M \tag{5.22}$$

Letting $R \to \infty$, we get $f'(z) = 0$, $\forall z \in \mathbb{C} \ \Rightarrow \ f$ is constant. □

具有原函数的充要条件 下面在定理 4.1.3 的基础上，再增加一条具有原函数的充要条件.

定理 5.2.5. Let $\Omega \subset \mathbb{C}$ be a region, $f : \Omega \longrightarrow \mathbb{C}$ be continuous. Then **TFAE** (the followings are equivalent):

(1) f admits a primitive on Ω.

(2) $\forall \alpha, \beta \in \Omega$, $\int_\gamma f(z)dz$ is invariant for any path.

(3) $\forall \alpha, \beta \in \Omega$, $\int_\gamma f(z)dz$ is invariant for any zig-zag path.

(4) $\int_\gamma f(z)dz = 0$ for any closed path $\gamma \subset \Omega$.

证明. (4) $\Rightarrow$ (2): Fix $\alpha, \beta \in \Omega$, $\forall$ two paths $\gamma_1, \gamma_2 \subset \Omega$ joining α to β, by (4)

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \int_{\gamma_1 \circ \gamma_2^-} f(z)dz = 0, \ where\ \gamma_1 \circ \gamma_2^-\ is\ a\ closed\ path\ in\ \Omega \tag{5.23}$$

Therefore

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz, \ \forall \gamma_1, \gamma_2 \subset \Omega \tag{5.24}$$

□

Morera's Theorem 有了具有原函数的充要条件之后 (Thm 5.2.5)，下面我们可以对全纯函数进行充要刻画，此即为 **Morera's Theorem** 的推广.

命题 5.2.2. Let $\Omega \subset \mathbb{C}$ be **simply connected**, $f : \Omega \longrightarrow \mathbb{C}$ be continuous. Then **TFAE**

(1) f is holomorphic on Ω.

(2) $\int_\gamma f(z)dz = 0$ for any closed path $\gamma \subset \Omega$.

注. 命题中 (2) $\Rightarrow$ (1) 的部分即为 **Morera's Theorem**.

证明. (1) $\Rightarrow$ (2): is by **Cauchy's Theorem** (Thm 4.3.2).
(2) $\Rightarrow$ (1): By Thm 5.2.5, f admits a primitive F on Ω, i.e. $F' = f$, then F is holomorphic on Ω.
By **CIF** (Thm 5.2.2), F is infinitely complex differentiable.
In particular, F is twice complex differentiable. $\Rightarrow F'' = f' \Rightarrow f$ is holomorphic on Ω. □

由**命题 5.2.2**可直接得到如下推论.

推论 5.2.6. Every holomorphic function on a simply connected region admits a primitive.

在利用**命题 5.2.2 (Morera's Thereom)** 判定函数全纯时，要注意条件 (2) 中道路 γ 的**任意性**. 下面便给出一个反例.

例 5.2.1. Suppose $\int_{C_r(0)} f(z)dz = 0$ for all $0 < r < 1$, can we conclude f is holomorphic on $\mathbb{D}$?

解. The answer is absolutely No. Take $f(z) = |z|^2$. Then $\int_{C_r(0)} f(z)dz = 0, \forall 0 < r < 1$. Since

$$f(z) = |z|^2 = z \cdot \overline{z}, \ \frac{\partial f}{\partial \overline{z}} = z \neq 0, \ \forall z \in \mathbb{C} \backslash \{0\} \tag{5.25}$$

Therefore, f is not holomorphic on $\mathbb{D}$.

注. 事实上，取道路 γ 为以原点为圆心，$\frac{1}{2}$ 为半径的上半圆，逆时针方向，可证 $\int_\gamma f(z)dz \neq 0$.

□

5.3 课堂例题 2024 − 03 − 25

1. Evaluate

$$\int_\gamma \frac{2z-1}{z^2-z}dz \tag{5.26}$$

where γ is any contour s. t. $\overline{\mathcal{D}} \subset Interior(\gamma)$.

解. By Cor 5.1.3,(闭路变形原理)

$$\int_\gamma f(z)dz = \int_{C_{\frac{1}{3}}(0)} f(z)dz + \int_{C_{\frac{1}{3}}(1)} f(z)dz \tag{5.27}$$

$$= \int_{C_{\frac{1}{3}}(0)} (\frac{1}{z-1}+\frac{1}{z})dz + \int_{C_{\frac{1}{3}}(1)} (\frac{1}{z-1}+\frac{1}{z})dz \tag{5.28}$$

Since $\frac{1}{z-1}$ is holomorphic on $Interior(C_{\frac{1}{3}}(0)) = D_{\frac{1}{3}}(0)$, where $D_{\frac{1}{3}}(0)$ is simply connected, then by Cauchy's Theorem,

$$\int_{C_{\frac{1}{3}}(0)} (\frac{1}{z-1}+\frac{1}{z})dz = \int_{C_{\frac{1}{3}}(0)} \frac{1}{z}dz \tag{5.29}$$

Similarly, since $\frac{1}{z}$ is holomorphic on $D_{\frac{1}{3}}(1)$, then

$$\int_{C_{\frac{1}{3}}(1)} (\frac{1}{z-1}+\frac{1}{z})dz = \int_{C_{\frac{1}{3}}(1)} \frac{1}{z-1}dz \tag{5.30}$$

Therefore

$$\int_\gamma f(z)dz = \int_{C_{\frac{1}{3}}(0)} (\frac{1}{z-1}+\frac{1}{z})dz + \int_{C_{\frac{1}{3}}(1)} (\frac{1}{z-1}+\frac{1}{z})dz \tag{5.31}$$

$$= \int_{C_{\frac{1}{3}}(0)} \frac{1}{z}dz + \int_{C_{\frac{1}{3}}(1)} \frac{1}{z-1}dz \tag{5.32}$$

$$= 2\pi i + 2\pi i \tag{5.33}$$

$$= 4\pi i \tag{5.34}$$

□

2. Evaluate

$$\int_{C_3(0)} \frac{e^{z^2}}{z-2} dz \quad (= 2\pi i f(z_0) = 2\pi i e^4) \tag{5.35}$$

$$\int_{C_1(0)} \frac{e^{z^2}}{z-2} dz \quad (= 0) \tag{5.36}$$

3. Evaluate

$$\int_{C_2(0)} \frac{\sin z}{z^2+1} dz \tag{5.37}$$

解.

$$\int_{C_2(0)} \frac{\sin z}{z^2+1} dz = \int_{C_{\frac{1}{2}}(i)} \frac{\sin z}{z+i} \cdot \frac{1}{z-i} dz + \int_{C_{\frac{1}{2}}(-i)} \frac{\sin z}{z-i} \cdot \frac{1}{z+i} dz \tag{5.38}$$

$$= 2\pi i \cdot \frac{\sin i}{2i} + 2\pi i \cdot \frac{\sin(-i)}{-2i} \tag{5.39}$$

$$= 2\pi \sin i \tag{5.40}$$

□

4. Evaluate

$$\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz \tag{5.41}$$

解. By Thm 5.2.2 (高阶 Cauchy Integral Formulas),

$$\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz = \int_{C_{\frac{1}{3}}(1)} \frac{z}{(z-2)^2} \cdot \frac{1}{z-1} dz + \int_{C_{\frac{1}{3}}(2)} \frac{z}{z-1} \cdot \frac{1}{(z-2)^2} dz \tag{5.42}$$

$$= 2\pi i + 2\pi i \left(\frac{z}{z-1}\right)' \Big|_{z=2} \tag{5.43}$$

$$= 0 \tag{5.44}$$

□

5. 课本第二章练习 $T6$.

5.4 *Sequence of functions*

概念 这一节我们来介绍有关复数域上函数列的相关概念和性质. 先回顾函数列收敛的定义.

定义 5.4.1. Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \longrightarrow \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of function.

We say $\{f_n\}_{n=1}$ **converges** if $\forall z \in \Omega$, $\{f_n(z)\}_{n=1}^{\infty}$ converges.

We say $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$m, n > N \ \Rightarrow \ |f_m(z) - f_n(z)| < \epsilon, \ \forall z \in \Omega$$

We say $\{f_n\}_{n=1}^{\infty}$ **uniformly converges to f** if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$n > N \ \Rightarrow \ |f_n(z) - f(z)| < \epsilon, \ \forall z \in \Omega$$

下面给出一个经典的收敛但不一致收敛的例子.

例 5.4.1. $\{f_n(z) = z^n\}_{n=1}^{\infty}$ on $\mathbb{D}$ is convergent but not uniformly convergent. However, the sequence is uniformly convergent on any compact subset on $\mathbb{D}$.

一致收敛 下面给出函数列一致收敛的性质，即**积分与极限可交换次序**.

定理 5.4.1. Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \longrightarrow \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of continuous functions that converges uniformly to f **on every compact subset of** Ω. Then

(1) f is continuous.

(2) If $\gamma \subset \Omega$ is a path with finite length, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz \tag{5.45}$$

(3) If f_n is holomorphic for all n, then so is f. Moreover,
$\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' **on every compact subset of** Ω.

注. 性质 (2) 即说明了对于一致收敛的函数列，**极限与积分可交换次序**，即

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz = \int_\gamma f(z)dz \tag{5.46}$$

证明.

(1) : Fix $z_0 \in \Omega$, $\forall \epsilon > 0$

Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, there exists $N \in \mathbb{N}$, s. t.

$$|f(z) - f_n(z)| \le \epsilon, \ \ \forall n \ge N, \forall z \in \Omega \tag{5.47}$$

Since f_N is continuous at z_0, there exist $\delta > 0$, s. t.

$$|f_n(z) - f_n(z_0)| \le \epsilon, \ \ \forall z \in D_\delta(z_0) \tag{5.48}$$

Therefore

$$|f(z) - f(z_0)| \le |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \tag{5.49}$$

$$\le 3\epsilon, \ \ \forall z \in D_\delta(z_0) \tag{5.50}$$

(2) : Fix $\epsilon > 0$. For any path γ with finite length, $\gamma \subset \Omega$ is a compact subset of Ω.

Let $L = length(\gamma)$. Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly on compact subset γ, $\exists N \in \mathbb{N}$, s. t.

$$|f_n(z) - f(z)| \le \frac{\epsilon}{L}, \ \ \forall z \in \gamma, \forall n > N \tag{5.51}$$

Hence, for all $n > N$,

$$\left| \int_\gamma f_n(z)dz - \int_\gamma f(z)dz \right| \le \int_\gamma |f_n(z) - f(z)| \le \frac{\epsilon}{L} \cdot L = \epsilon, \ \ \forall n > N \tag{5.52}$$

Therefore

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz \tag{5.53}$$

(3) : 下面分两部分证明. (对应书[1]P54 Thm 5.3)

- f is holomorphic.

  $\forall a \in \Omega$, $\exists r > 0$, s. t. $D_r(a) \subset \Omega$. Let γ be any closed path in $D_r(a)$.

  Since f_n is holomorphic on $D_r(a)$, by **Cauchy's Theorem (Thm 4.3.2)**,

$$\int_\gamma f_n(z)dz = 0 \tag{5.54}$$

  By (2) proofed previously,

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0 \tag{5.55}$$

[1]课堂教材：《*Complex Analysis*》— *Elias M. Stein*

By **Morera's Theorem (Prop 5.2.2)**, f is holomorphic on $D_r(a)$.

In particular, f is complex differentiable at a. Since a is arbitrary, f is holomorphic on Ω.

- $\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' on every compact subset of Ω.

  $\forall z_0 \in \Omega$, $\exists r > 0$, s. t. $\overline{D}_r(z_0) \subset \Omega$.

  Since $f_n \longrightarrow f$ converges uniformly on $C_r(z_0)$, we have $\forall z \in D_{\frac{r}{2}}(z_0)$

$$\frac{f_n(\zeta)}{(\zeta - z)^2} \to \frac{f(\zeta)}{(\zeta - z)^2} \textit{ uniformly on } C_r(z_0) \tag{5.56}$$

  i.e. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$\left| \frac{f_n(\zeta)}{(\zeta - z)^2} - \frac{f(\zeta)}{(\zeta - z)^2} \right| < \frac{\epsilon}{r}, \ \forall n > N, \forall \zeta \in C_r(z_0) \tag{5.57}$$

  Therefore

$$\left| f_n'(z) - f'(z) \right| = \left| \frac{1}{2\pi i} \int_{C_r(z_0} \left(\frac{f_n(\zeta)}{(\zeta - z)^2} - \frac{f(\zeta)}{(\zeta - z)^2} \right) d\zeta \right| \tag{5.58}$$

$$< \frac{1}{2\pi i} \cdot \frac{\epsilon}{r} \cdot 2\pi r i \tag{5.59}$$

$$= \epsilon, \ \forall n > N, \forall z \in D_{\frac{r}{2}}(z_0) \tag{5.60}$$

  It tells $f_n' \to f'$ uniformly on $D_{\frac{r}{2}}(z_0)$.

  For any compact subset $K \subset \Omega$, consider the open covering $\{D_{\frac{1}{2}r_x}(x) \subset \Omega\}_{x \in K}$.

  There exists a finite subcovering $K \subset \{D_{r_i}(x_i) \subset \Omega\}_{i=1}^{n}$.

  We can proof $f_n' \to f'$ uniformly on K.

□

5.5 课堂例题 2024 − 03 − 29

1. Evaluate

$$\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz \tag{5.61}$$

解.

$$\begin{align}
\int_{C_3(0)} \frac{z}{(z-1)(z-2)^2} dz &= \int_{C_{\frac{1}{3}}(0)} \frac{z}{(z-2)^2} \cdot \frac{1}{z-1} dz + \int_{C_{\frac{1}{3}}(2)} \frac{z}{z-1} \cdot \frac{1}{(z-2)^2} dz \tag{5.62}\\
&= 2\pi i + 2\pi i \left(\frac{z}{z-1}\right)' \Big|_{z=2} \tag{5.63}\\
&= 0 \tag{5.64}
\end{align}$$

□

第六章　*Week* 6

6.1　课堂例题 $2024-04-01$

本节为习题课. (博士研究生助教代课)

1. 课本第二章练习 $T1, T2, T3, T4$.

6.2 函数项级数，全纯函数解析

回顾 在介绍复数域上函数项级数的性质之前，先回顾一下函数列的性质 (Thm 5.4.1).

定理 6.2.1. 一致收敛 $\Rightarrow$ 积分与极限可交换次序.
Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \longrightarrow \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of continuous functions that converges uniformly to f **on every compact subset of** Ω. Then

(1) f is continuous.

(2) If $\gamma \subset \Omega$ is a path with finite length, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz \tag{6.1}$$

(3) If f_n is holomorphic for all n, then so is f. Moreover,
$\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' **on every compact subset of** Ω.

注.
- 事实上，结论 (3) 可做推广，即当函数列 $\{f_n\}_{n=1}^{\infty}$ 满足上述条件时，有：

  $\{f_n^{(k)}\}_{n=1}^{\infty}$ converges uniformly to $f^{(k)}$ **on every compact subset of** Ω.

- 注意实变函数列与复变函数列的**可微性**的区别，即实变函数列不满足定理中的 (3). 下面给出结论 (3) 在实变函数列下的反例.

  例 6.2.1. Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $x \in [-1, 1]$. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f(x) = |x|$. Though f_n, $n = 1, 2, \cdots$ are differentiable over $[-1, 1]$, the limit function $f(x) = |x|$ is **not differentiable** at $x = 0$.

证明. 详见定理 5.4.1证明. $\square$

函数项级数　下面给出函数项级数收敛的定义.

定义 6.2.1. Let $\Omega \subset \mathbb{C}$ be open, $\{f_n : \Omega \longrightarrow \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of functions.
We say $\sum\limits_{n=1}^{\infty} f_n$ **converges** if $\{S_N = \sum\limits_{n=1}^{N} f_n\}_{N=1}^{\infty}$ converges.
We say $\sum\limits_{n=1}^{\infty} f_n$ **converges uniformly** if $\{S_N = \sum\limits_{n=1}^{N} f_n\}_{N=1}^{\infty}$ converges uniformly).

下面给出判断函数项级数一致收敛性的经典方法 (**Weierstrass M-test**).

命题 6.2.1. Weierstrass M-test.
If $|f_n(z)| \le M_n,\ n = 1, 2, \cdots, \forall z \in \Omega$, and $\sum\limits_{n=1}^{\infty} M_n < \infty$, then $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly.

证明. Let $S_N = \sum\limits_{n=1}^{N} f_n, \forall N \in \mathbb{N}$. Since $\sum\limits_{n=1}^{\infty} M_n < \infty$, then
$\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t. $n > m > N$

$$|S_n - S_m| = |f_n(z) + \cdots + f_{m+1}(z)| \le \sum_{j=m+1}^{\infty} M_j \le \epsilon,\ \ \forall z \in \Omega \tag{6.2}$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ converges uniformly. $\Rightarrow$ $\{f_n\}_{n=1}^{\infty}$ converges uniformly. □

解析与全纯等价　下面证明**全纯**函数均**解析 (可展成幂级数)**.

(该定理与定理 3.1.2共同说明了，解析 $\Leftrightarrow$ 全纯)

定理 6.2.2. Suppose f is holomorphic on an open set $\Omega \subset \mathbb{C}$. Then $\forall z_0 \in \Omega$ with $D_r(z_0) \subset \Omega$ for some $r > 0, f$ has a power series expansion at z_0.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,\ \ \forall z \in D_r(z_0) \tag{6.3}$$

$where\ a_n = \frac{f^{(n)}(z_0)}{n!},\ \ \forall n \ge 0.$

证明. Fix $z \in D_r(z_0)$. By **CIF (Thm 5.2.1)**,

$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{6.4}$$

Then

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \tag{6.5}$$

Since $z \in D_r(z_0)$ is fixed, and $\forall \zeta \in C_r(z_0)$, there exists $0 < r < 1$, s. t.

$$\left|\frac{z-z_0}{\zeta-z_0}\right| < r \tag{6.6}$$

Therefore,

$$\sum_{n=0}^{N}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \ \rightrightarrows \ \frac{1}{1-\frac{z-z_0}{\zeta-z_0}}, \ N\to\infty, \ \forall \zeta \in C_r(z_0) \tag{6.7}$$

converges uniformly **w.r.t. (with respect to , 关于)** $\zeta \in C_r(z_0)$. i.e.

$$\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n = \frac{1}{1-\frac{z-z_0}{\zeta-z_0}}, \ \forall \zeta \in C_r(z_0) \tag{6.8}$$

Let

$$g_N(\zeta) = \sum_{n=0}^{N}\left(f(\zeta)\cdot\frac{1}{\zeta-z_0}\cdot\left(\frac{z-z_0}{\zeta-z_0}\right)^n\right) \tag{6.9}$$

$$= f(\zeta)\cdot\frac{1}{\zeta-z_0}\sum_{n=0}^{N}\left(\frac{z-z_0}{\zeta-z_0}\right)^n, \ \zeta \in C_r(z_0) \tag{6.10}$$

Then we have

$$g_N \ \rightrightarrows \ f(\zeta)\cdot\frac{1}{\zeta-z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} \tag{6.11}$$

$$= \frac{f(\zeta)}{\zeta-z}, \ N\to\infty, \ \zeta \in C_r(z_0) \tag{6.12}$$

converges uniformly w.r.t. $\zeta \in C_r(z_0)$. Therefore, by Thm 6.2.1,

$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{\zeta-z}d\zeta = \frac{1}{2\pi i}\int_{C_r(z_0)}\lim_{N\to\infty} g_N(\zeta)d\zeta \tag{6.13}$$

$$= \lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_r(z_0)} g_N(\zeta)d\zeta \tag{6.14}$$

$$= \lim_{N\to\infty}\frac{1}{2\pi i}\int_{C_r(z_0)}\left(f(\zeta)\cdot\frac{1}{\zeta-z_0}\sum_{n=0}^{N}\left(\frac{z-z_0}{\zeta-z_0}\right)^n\right)d\zeta \tag{6.15}$$

$$= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{2\pi i}\int_{C_r(z_0)}\left(f(\zeta)\cdot\frac{1}{\zeta-z_0}\left(\frac{z-z_0}{\zeta-z_0}\right)^n\right)d\zeta \tag{6.16}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\right)(z-z_0)^n \tag{6.17}$$

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{6.18}$$

By **CIF (Thm 5.2.2)**, we have

$$a_n = \frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta = \frac{f^{(n)}(z_0)}{n!}, \ \forall n \geq 0 \tag{6.19}$$

□

注. 事实上，该定理也提供了 **CIF 高阶形式 (Thm 5.2.2)** 的另一种证明 (比较系数可得).

6.3 解析延拓

定义 下面先给出**解析延拓 (Analytic continuation)** 的定义.

定义 6.3.1. Suppose f and F are holomorphic in nonempty regions Ω and $\hat{\Omega}$ respectively with $\Omega \subset \hat{\Omega}$. If $f(z) = F(z)$ in Ω, then we say F is an **analytic continuation** of f in $\hat{\Omega}$.

解析延拓的唯一性 在说明这之前，先给出一个有关全纯函数的非常重要的结论.

定理 6.3.1. Suppose f is holomorphic in a region Ω that vanishes on a sequence of distinct points with a limit in Ω. Then $f(z) \equiv 0$ for all $z \in \Omega$.

注. 该定理说明了，不恒为零的全纯函数的**零点均为孤立点** (不为聚点).

证明. 反证法. Suppose $\{w_k\}_{k=1}^{\infty} \subset \Omega$ with $\lim\limits_{k\to\infty} w_k = z_0 \in \Omega$ and $f(w_k) = 0$, $k = 1, 2, \cdots$. Since f is holomorphic in Ω, $\forall z \in D_r(z_0)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \textit{for some } r > 0 \tag{6.20}$$

下面分为两步证明.

- We first show $f(z) \equiv 0$ on $D_r(z_0)$.

  Assume $f(z) \neq 0$ for $z \in D_r(z_0)$, then $\exists$ smalledt integer m s. t. $a_m \neq 0$. Now

  $$f(z) = a_m (z - z_0)^m (1 + g(z)), \ \textit{where } g(z) = \sum_{n=m+1}^{\infty} a_n (z - z_0)^{n-m} \tag{6.21}$$

  Since $g(z) \to 0$ as $z \to z_0$, $\forall \epsilon < 1$, there exists $\delta > 0$, s. t.

  $$|g(z)| \leq \epsilon < 1, \ \forall z \in D_\delta^*(z_0) \tag{6.22}$$

  Since $w_k \to z_0 \in \Omega$, $\exists k_0 \in \mathbb{N}$, s. t. $w_{k_0} \in D_\delta^*(z_0)$. Then

  $$f(w_{k_0}) = a_m (z - z_0)^m (1 + g(w_{k_0})) \neq 0 \tag{6.23}$$

  which is a contradiction with that $f(w_{k_0}) = 0$.

- Then we shall show $f \equiv 0$ on Ω.

  Let U be the interior of $\{z \in \Omega \mid f(z) = 0\}$. Since $f(z) \equiv 0, \forall z \in D_r(z_0)$, $U \neq \varnothing$ and U is open. Moreover, 下面我们证明 U is closed.

- For all $\{z_k\}_{k=1}^{\infty} \subset U$ with $z_k \to p \in \Omega$. 与第一步证明相同，可以得到 $\exists r_p > 0$, s. t. $f(z) \equiv 0, \forall z \in D_{r_p}(p)$. 于是 $p \in U$. 即 U 包含了自身序列的所有极限点. which means that U is closed.

Therefore, $U \subset \Omega$ is both open and closed. Since $U \neq \varnothing$ and Ω is connected, then $U = \Omega$, which means $f \equiv 0$ on Ω.

□

通过上述定理可得到，全纯函数的取值只由区域上**可数个点**决定.

推论 6.3.2. Suppose f, g are holomorphic in a region $\hat{\Omega}$ and $f(z) = g(z)$ for all $z \in \Omega$, where Ω is an open subset of $\hat{\Omega}$. Then $f(z) = g(z)$ for $z \in \hat{\Omega}$.

By Cor 6.3.2，我们得到解析延拓若存在，则**必唯一**.

推论 6.3.3. Suppose F and G are both analytic continuation of f into $\hat{\Omega}$. Then

$$F = G \text{ in } \hat{\Omega}$$

6.4 对称原理

引入 在实分析中，我们曾探讨过有关**连续函数的延拓 (Tietze 延拓定理)**.

但在复分析中，对于**全纯函数的延拓**似乎不再那么容易与显然，因为全纯函数不仅要求在复平面上光滑，而且还具有一些 additional characteristically rigid properties.

(书 P67 Problem 1. 给出了无法 (解析) 延拓至 $\overline{\mathbb{D}}$ 的定义在 $\mathbb{D}$ 上的全纯函数.)

而本节将给出一种十分有用的情况下全纯函数的延拓，即在**关于实轴对称**的区域上的延拓.

对称原理 为了讨论的方便，接下来的命题将默认以下几个记号：

- Let Ω be an open subset of $\mathbb{C}$ that is **symmetric** w.r.t. the real axis.
- Let Ω^+ denote the part of Ω that lies in the upper half-plane and Ω^- the part that lies in the lower half-plane.

下面给出对称原理.

定理 6.4.1. Symmetric principle.
If f^+ and f^- are holomorphic in Ω^+ and Ω^- respectively that extends continuously to I ($I = \Omega \cap \mathbb{R}$) and $f^+(x) = f^-(x)$ for all $x \in I$. Then

$$f(z) = \begin{cases} f^+(z), z \in \Omega^+ \\ f^+(z) = f^-(z), z \in I \\ f^-(z), z \in \Omega^- \end{cases} \tag{6.24}$$

is holomorphic in Ω.

证明. 详见书 P58 Thm 5.5 证明. □

Schwartz 反射原理 有了上述对称原理的铺垫后，下面给出全纯函数在**关于实轴对称**区域上的延拓定理. (**Schwartz 反射原理**)

定理 6.4.2. Schwartz reflection principle.

Suppose f is holomorphic in Ω^+ that extends continuously to I and f is real-valued on I. Then $\exists F$ holomorphic in all Ω, s. t. $F = f$ in Ω^+.

证明. Define $f^-(z) = \overline{f(\overline{z})}$ for $z \in \Omega^-$. Fix $z_0 \in \Omega^-, \exists r > 0$, s. t. $D_r(z_0) \subset \Omega^-$.

Then $\overline{z_0} \in \Omega^+$ and $D_r(\overline{z_0}) \subset \Omega^+$. $\forall x \in D_r(z_0), \overline{x} \in D_r(\overline{z_0})$

Since f is holomorphic in Ω^+, by Thm 6.2.2 (全纯函数均解析)

$$f(\overline{z}) = \sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z_0})^n,\ \ \forall \overline{z} \in D_r(\overline{z_0}) \tag{6.25}$$

Then we have

$$f^-(z) = \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^n,\ \ \forall z \in D_r(z_0) \tag{6.26}$$

Therefore, f^- is analytic in ω^-. By Thm 3.1.2, f^- is holomorphic in Ω^-. (解析函数均全纯)

Therefore, by **Symmetric principle (对称原理)**,

$$F(z) = \begin{cases} f^+(z), z \in \Omega^+ \\ f^+(z) = f^-(z), z \in I \\ f^-(z), z \in \Omega^- \end{cases} \tag{6.27}$$

is holomorphic in Ω. □

6.5 课堂例题 $2024-04-07$

1. **(课本 P66 Ex10.)**

   Weierstrass's theorem states that a continuous function on $[0,1]$ can be uniformly approxiamted by polynomials. **Can every continuous function on the closed unit disc be approxiated uniformly by polynomials in the variable z?**

   解. Absolutely No. Take $f(z)=|z|$ continuous on $\mathbb{D}$ into consideration.

   If there exists polynomials $\{f_n\}_{n=1}^{\infty}$, s. t. $f_n \rightrightarrows f$, then by Thm 6.2.1,

   f is holomorphic on $\mathbb{D}$, which is a contradiction with that f is not differentiable at $x=0$. □

2. Suppose f is entire and real-valued on the real-axis. If $f(1+i)=2+i$, then what is $f(1-i)$?

   解. Ans : $2-i$. (by **Schwartz reflection principle**) □

3. 课本第二章练习 $T7-10, T15$.

第七章　Week 7

7.1　零点，极点，留数

零点　根据**定理 6.3.1**知，不恒为零的全纯函数只含**孤立零点**. 下面我们将给出非零全纯函数在其孤立零点附近的**局部刻画**.

下面先给出孤立零点的定义.

定义 7.1.1. Let $\Omega \subset \mathbb{C}$ be a region, $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. We say the zero z_0 is **isolated** if $\exists r > 0$, s. t. $f(z) \neq 0$ for all $z \in D_r^*(z_0)$.

注. By Thm 6.3.1, we note that if $f(z) \not\equiv 0$, $z \in \Omega$ (不全为零), then the zeros of $f(z)$ are isolated.

下面给出非零全纯函数在其孤立零点附近的**局部刻画**.

定理 7.1.1. Suppose $f(z) \not\equiv 0$ is holomorphic in a region Ω. z_0 is a zero of f. Then $\exists r > 0$ and **nonvanishing** holomorphic function $g(z)$ in $D_r(z_0)$ and a **unique** integer n, s. t.

$$f(z) = (z - z_0)^n g(z), \ z \in D_r(z_0)$$

注.

- $f(z) \not\equiv 0$ 指的是 f 不恒为零，而 g nonvanishing 指的是 g 恒不为零.
- In this theorem, we say z_0 is a **zero of f of multiplicity of n**.

  If $n = 1$, we say z_0 is a **simple** zero of f.

证明.

- **存在性**: Since f is holomorphic, by Thm 6.2.2, $\exists R > 0$, s. t.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \ \forall z \in D_R(z_0) \tag{7.1}$$

$f(z_0) = 0 \;\Rightarrow\; a_0 = 0$. Since $f(z) \not\equiv 0$, $\exists$ the smallest integer n, s. t. $a_n \neq 0$. Then

$$f(z) = (z - z_0)^n(a_n + a_{n+1}(z - z_0) + \cdots) = (z - z_0)^n g(z) \tag{7.2}$$

Clearly, $\exists 0 < r < R$, s. t. $g(z) \neq 0$ for all $z \in D_r(z_0)$.

- **唯一性**: 详见书 P73 Thm 1.1 证明.

□

极点 下面给出复变函数的**极点**的定义.

定义 7.1.2. We say $f : D_r^*(z_0) \to \mathbb{C}$ has a **pole** at z_0 if $\frac{1}{f}$ is holomorphic in $D_r(z_0)$ and has a zero at z_0.

注. • 牢林此处的定义**并不严谨**，并未对 $1/f$ 在 z_0 处无定义的情况说明 **pole** 的定义. 事实上这会导致后面对**奇点附近性质**的讨论带来不便，下面引用书[1]P74 的定义.

定义 7.1.3. We say $f : D_r^*(z_0) \to \mathbb{C}$ has a **pole** at z_0, if the function $\frac{1}{f}$, defined to be zero at z_0, is holomorphic in a full neighbourhood of z_0.

- 由定义可知，a pole of a function is isolated.

根据非零全纯函数在**孤立零点附近的局部刻画 (Thm 7.1.1)**，可以很容易得到全纯函数在其**极点**附近的**局部刻画**.

定理 7.1.2. If f has a pole at z_0, then $\exists r > 0$ and a **nonvanishing** holomorphic function $h(z)$ in $D_r(z_0)$ and a **unique** positive integer n, s. t.

$$f(z) = (z - z_0)^{-n} h(z), \;\; z \in D_r^*(z_0)$$

证明. 详见书 P74 Thm 1.2 证明. □

[1]课堂教材：《*Complex Analysis*》— *Elias M. Stein*

留数 在定理 7.1.2的基础上，可以更进一步给出更精细的刻画. 对于 n 阶极点，我们存在这样的刻画.

定理 7.1.3. If f has a pole of order n at z_0, then we can write

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z) \tag{7.3}$$

where $G(z)$ is holomorphic in some neighbourhood of z_0.

注. • The sum

$$\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} \tag{7.4}$$

is called **the principal part** (or **singular part**) of f at the pole z_0.

- $G(z)$ is called **the holomorphic part** of f at the pole z_0.
- The coefficient a_{-1} is called the **residue** of f at the pole z_0. We write $Res_{z_0} f = a_{-1}$.

证明. 详见书 P75 Thm 1.3 证明. □

关于**留数**的用途和含义，在于其绕对应极点的环路积分之中.

对于 $f : \Omega \to \mathbb{C}$ with a pole $z_0 \in \Omega$, since $\frac{a_{-k}}{(z-z_0)^k}, k = 2, \cdots, n$ have primitives and G is holomorphic, we have

$$\int_{C_r(z_0)} f(z)dz = \int_{C_r(z_0)} \frac{a_{-1}}{z-z_0} dz = 2\pi i \cdot a_{-1} \tag{7.5}$$

环路积分的值只剩下与 a_{-1} 有关，此即为**"留数"**之意.

下面介绍留数的**计算技巧**. 设 z_0 为 f 的 n 阶极点.

- $n = 1$ 时，$Res_{z_0} f = \lim_{z\to z_0} (z-z_0)f(z)$
- $n > 1$ 时，我们有

$$Res_{z_0} f = \lim_{z\to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z) \tag{7.6}$$

(根据**定理 7.1.3**的公式可轻松得证.)

7.2 *Laurent Series Expansion*

事实上，对于全纯函数 f，其不仅能在定义域内展开为幂级数 (**Thm 6.2.2**)，其同样能在极点周围类似地展开为幂级数的形式，此即为 **Laurent Series Expansion (洛朗级数展开)**.

定理 7.2.1. Let f be holomorphic on a region containing the annulus and its boundary

$$\mathcal{A} = \{z \mid r_1 < |z - z_0| < r_2\},\ where\ 0 \le r_1 < r_2 \tag{7.7}$$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{7.8}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,\ for\ any\ r \in [r_1, r_2] \tag{7.9}$$

注.
- The series in the Theorem is called the **Laurent Series Expansion** of f near z_0 or in the annulus.
- 在同一圆环域内，Laurent 展式唯一；在不同的圆环域内，Laurent 展式可能不同.

证明. Fix $z \in \mathcal{A}$, $\exists \delta > 0$, s. t. $C_\delta(z) \subset \mathcal{A}$.

Consider

$$g(\zeta) = \frac{f(\zeta)}{\zeta - z} \tag{7.10}$$

Then $g(\zeta)$ is holomorphic in a region containing $\mathcal{A} \backslash D_\delta(z)$ and its boundary.

By the **principle of contour deformation (Thm 5.1.1, 闭路变形原理)**

$$\int_{C_{r_2}(z_0)} g(\zeta) d\zeta = \int_{C_\delta(z)} g(\zeta) d\zeta + \int_{C_{r_1}(z_0)} f(\zeta) d\zeta \tag{7.11}$$

By **CIF (Thm 5.2.1)**

$$2\pi i \cdot f(z) = \int_{C_\delta(z)} g(\zeta) d\zeta \tag{7.12}$$

Thus

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{7.13}$$

下面分别计算积分 $\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta$ 和 $-\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta$.

- If $\zeta \in C_{r_2}(z_0)$, then $|\zeta - z_0| > |z - z_0|$.

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z-z_0}{\zeta-z_0}\right)} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n \tag{7.14}$$

converges w.r.t. ζ. Hence

$$\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \tag{7.15}$$

(此处具体证明过程可见**定理 6.2.2** 的证明.)

- If $\zeta \in C_{r_1}(z_0)$, then $|\zeta - z_0| < |z - z_0|$.

$$-\frac{1}{\zeta - z} = \frac{1}{z - \zeta} = \frac{1}{(z - z_0) - (\zeta - z_0)} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{\zeta-z_0}{z-z_0}} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^n \tag{7.16}$$

$$= \sum_{n=0}^{\infty} \frac{(\zeta - z_0)^{n-1}}{(z - z_0)^n} \tag{7.17}$$

$$= \sum_{n=-1}^{-\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \tag{7.18}$$

Hence

$$-\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=-1}^{-\infty} \left(\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \tag{7.19}$$

由于 $\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}$, $\forall n$ 在 $\mathcal{A}$ 上 holomorphic，因此根据**闭路变形原理 (Thm 5.1.1)**, $\forall r \in [r_1, r_2]$

$$\int_{C_{r_2}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \tag{7.20}$$

$$\int_{C_{r_1}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \tag{7.21}$$

Therefore

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{7.22}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n + \sum_{n=-1}^{-\infty} \left(\frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \tag{7.23}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n + \sum_{n=-1}^{-\infty} \left(\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \tag{7.24}$$

$$= \sum_{-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \tag{7.25}$$

$$= \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \ where\ a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \ for\ any\ r \in [r_1, r_2] \tag{7.26}$$

□

7.3 课堂例题 2024 – 04 – 08

1. Find the Laurent Expansion of

$$\frac{1}{z^2(z-i)} \ in \ \frac{1}{4} < |z-i| < \frac{3}{4} \tag{7.27}$$

解. Ans:

$$\sum_{n=-1}^{\infty} (n+2)i^{n+1}(z-i)^n \tag{7.28}$$

□

2. Find the Laurent Expansion of

$$\frac{z^3}{1+z^2} \ in \ 2 < |z| < 4 \tag{7.29}$$

解. Ans:

$$= \frac{z}{1+\frac{1}{z^2}} = z \cdot \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n \tag{7.30}$$

□

3. Find the Laurent Expansion of

$$f(z) = \frac{z^2-2z+5}{(z-2)(z^2+1)} \tag{7.31}$$

in $1 < |z| < 2$, $2 < |z| < +\infty$ respectively.

解. $f(z) = \frac{1}{z-2} - \frac{2}{z^2+1}$.

- In the annulus $1 < |z| < 2$,

$$f(z) = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} - \frac{2}{z^2} \cdot \frac{1}{1+\frac{1}{z^2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{2}{z^2}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n \tag{7.32}$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{z^{2n}} \tag{7.33}$$

- In the annulus $2 < |z| < +\infty$,

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} - \frac{2}{z^2} \cdot \frac{1}{1+\frac{1}{z^2}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \sum_{n=1}^{\infty}\frac{2(-1)^n}{z^{2n}} \tag{7.34}$$

□

7.4 *Residue Formula*

引入 对于**单连通区域**，**Cauchy's Theorem (Thm 4.3.2)** 已经告诉了我们全纯函数的环路积分为 0.

但对于更一般的区域，若其中含有极点，则 **Cauchy's Theorem** 便不再奏效. 此时便需要使用接下来所要介绍的 **Residue Formula** 来进行计算.

Residue Formula 下面先给出单个极点的圆形环路上函数的积分值.

定理 7.4.1. Suppose f is holomorphic in a region containing $\overline{D_r^*(z_0)}$, $r > 0$, and z_0 is a pole of f. Then

$$\int_{C_r(z_0)} f(z)dz = 2\pi i \cdot Res_{z_0} f \tag{7.35}$$

证明. 证明是 trivial 的. Suppose z_0 is a pole of order n. Then by **Thm 7.1.3**,

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z) \tag{7.36}$$

where $G(z)$ is holomorphic in $\overline{D_r(z_0)}$.
Since $\frac{a_{-k}}{(z-z_0)^k}$, $k = 2, 3, \cdots, n$ admit primitives and G is holomorphic in a region containing $\overline{D_r^*(z_0)}$, this yields

$$\int_{C_r(z_0)} f(z)dz = \int_{C_r(z_0)} \frac{a_{-1}}{z-z_0}dz = 2\pi i \cdot a_{-1} = 2\pi i \cdot Res_{z_0} f \tag{7.37}$$

□

下面给出 **Residue Formula**. 它给出了环路内部存在**有限个极点**时的积分计算公式.

定理 7.4.2. Residue Formula.

Suppose f is holomorphic in an open set containing a contour γ and its interior except for poles $z_1, \cdots, z_n \in Interior(\gamma)$. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z_k} f \tag{7.38}$$

证明. By **Principle of Contour Deformation (Cor 5.1.3, 闭路变形原理)**,

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{C_{r_k}(z_k)} f(z)dz \tag{7.39}$$

where $C_{r_k}(z_k)$, $k = 1 \sim n$ are disjoint circles in $Interior(\gamma)$.

Then by **Thm 7.4.1**, the desired result follows. □

7.5 课堂例题 2024 − 04 − 12

1. **(课本 P103 T2.)**

Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx \tag{7.40}$$

解. Consider $f(z) = \frac{1}{1+z^4}$ and the contour $\gamma_1 \circ \gamma_2$.

(γ_1 为 $(-R, 0)$ 到 $(R, 0)$ 的实直线，γ_2 为 $(R, 0)$ 到 $(-R, 0)$ 的上半圆周)

For sufficiently large R, the contour $\gamma_1 \circ \gamma_2$ contains poles $e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}$ of f.

By the **Residue Formula (Thm 7.4.2)**,

$$\int_{\gamma_1 \circ \gamma_2} f(z) dz = 2\pi i \left(Res_{e^{\frac{\pi i}{4}}} f + Res_{e^{\frac{3\pi i}{4}}} f \right) \tag{7.41}$$

By the **L'Hospital's Rule (Thm A.1.1)**, since $e^{\frac{\pi i}{4}}$ is a simple pole of f, we compute

$$Res_{e^{\frac{\pi i}{4}}} f = \lim_{z \to e^{\frac{\pi i}{4}}} (z - e^{\frac{\pi i}{4}}) f(z) = \lim_{z \to e^{\frac{\pi i}{4}}} \frac{z - e^{\frac{\pi i}{4}}}{1+z^4} \overset{L'Hospital}{=} \frac{1}{4e^{\frac{3\pi i}{4}}} = \frac{1}{4} e^{-\frac{3\pi i}{4}} \tag{7.42}$$

Similarly, we have

$$Res_{e^{\frac{3\pi i}{4}}} f = \lim_{z \to e^{\frac{3\pi i}{4}}} (z - e^{\frac{3\pi i}{4}}) f(z) = \frac{1}{4} e^{-\frac{\pi i}{4}} \tag{7.43}$$

Then

$$\int_{\gamma_1 \circ \gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz \tag{7.44}$$

$$= 2\pi i \left(Res_{e^{\frac{\pi i}{4}}} f + Res_{e^{\frac{3\pi i}{4}}} f \right) = \frac{\sqrt{2}}{2} \pi \tag{7.45}$$

Note that

$$\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_{\gamma_2} \frac{1}{1+z^4} dz \right| \leq \sup_{z \in \gamma_2} |f(z)| \cdot length(\gamma_2) \tag{7.46}$$

Since $\left|1 + z^4\right| \geq \left|z^4\right| - 1$, then $\sup_{z \in \gamma_2} \leq \frac{1}{R^4 - 1}$.

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \sup_{z \in \gamma_2} |f(z)| \cdot length(\gamma_2) \leq \frac{\pi R}{R^4 - 1} \to 0, \ as\ R \to \infty \tag{7.47}$$

Thus

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \lim_{R \to \infty} \int_{\gamma_1} f(z) dz = \lim_{R \to \infty} \int_{\gamma_1} f(z) dz + \lim_{R \to \infty} \int_{\gamma_2} f(z) dz \tag{7.48}$$

$$= \lim_{R \to \infty} \int_{\gamma_1 \circ \gamma_2} f(z) dz \tag{7.49}$$

$$= \frac{\sqrt{2}}{2} \pi \tag{7.50}$$

□

2. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^n} dx, \ n \geq 2 \tag{7.51}$$

3. **(课本 P103 T3.)**

   Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx, \ a > 0 \tag{7.52}$$

解. Let $f(z) = \frac{e^{iz}}{z^2+a^2}$. Consider the contour $\gamma_1 \circ \gamma_2$.

(γ_1 为 $(-R, 0)$ 到 $(R, 0)$ 的实直线，γ_2 为 $(R, 0)$ 到 $(-R, 0)$ 的上半圆周)

Since $\frac{\sin x}{x^2+a^2}$ 为奇函数，

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x^2+a^2} dx = 0 \tag{7.53}$$

Then

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+a^2} dx = \lim_{R\to\infty} \int_{\gamma_1} f(z)dz \tag{7.54}$$

By the **Residue Formula (Thm 7.4.2)**,

$$\int_{\gamma_1\circ\gamma_2} f(z)dz = 2\pi i \cdot Res_{ai}f = 2\pi i \lim_{z\to ai} \frac{e^{iz}}{z+ai} = \frac{\pi e^{-a}}{a} \tag{7.55}$$

Since

$$\left| \int_{\gamma_2} f(z)dz \right| \leq \sup_{0\leq\vartheta\leq\pi} \left| \frac{e^{i(R\cos\vartheta + iR\sin\vartheta)}}{R^2 e^{i2\vartheta} + a^2} \right| \cdot \pi R \leq \frac{\pi R}{R^2 - a^2} \to 0, \ as \ R \to \infty \tag{7.56}$$

Therefore

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+a^2} dx = \lim_{R\to\infty} \int_{\gamma_1} f(z)dz = \lim_{R\to\infty} \int_{\gamma_1\circ\gamma_2} f(z)dz \tag{7.57}$$

$$= \frac{\pi e^{-a}}{a} \tag{7.58}$$

□

4. 课本第三章练习 $T1 \sim T8$.

5. 课本 P79 例 2.

第八章　*Week 8*

8.1　均值定理

下面补充一个 **CIF (Thm 5.2.1, 柯西积分公式)** 的推论，即**均值定理**.

命题 8.1.1. (Mean Value Property.)

If f is holomorphic on $D_R(z_0)$, where $z_0 \in \mathbb{C}$, $R > 0$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\vartheta})d\vartheta,\ 0 < r < R \tag{8.1}$$

证明. By **CIF (Thm 5.2.1)**,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z_0}d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\vartheta})}{re^{i\vartheta}} r \cdot i \cdot e^{i\vartheta} d\vartheta \tag{8.2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\vartheta})d\vartheta \tag{8.3}$$

□

8.2 奇点

下面给出**奇点**的定义.

定义 8.2.1. A complex number z_0 is a **singular point** (or a **singularity**) of f if f is not analytic at z_0. We say z_0 is an **isolated singularity** if f is analytic in a deleted neighbourhood of z_0.

注. 大多数情况下我们研究的都是**孤立奇点**，但也存在着**非孤立奇点**.

例 8.2.1.

- 0 is an isolated singularity of $\frac{1}{z}, \frac{1}{\sin z}, \frac{1}{e^z-1}$.
- **Poles are isolated singularities. (极点均为孤立奇点)**
- For

$$\frac{1}{\sin\frac{\pi}{z}} \tag{8.4}$$

0 is not an isolated singularity.

8.2.1 Classification of isolated singularities

下面我们对**奇点**进行分类.

定义 8.2.2. Let $f : D_r^*(z_0) \to \mathbb{C}$ where $r > 0, z_0 \in \mathbb{C}$ be holomorphic with the Laurent expansion

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n \tag{8.5}$$

(1) z_0 is called a **removable singularity (可去奇点)** if $a_{-n} = 0, n = 1, 2, \cdots$.

(2) z_0 is called a **pole (极点)** if $a_{-n} \neq 0, a_{-(n+k)} = 0, k = 1, 2, \cdots$.

(3) z_0 is called a **essential singularity (本性奇点)** if $a_{-n} \neq 0$ for infinitely many $n \geq 1$.

例 8.2.2.

- $f(z) = z^{-n}, n \geq 1$ has a **pole** of order n at $z = 0$.
- $f(z) = e^{\frac{1}{z}}$ has an **essential singularity** at $z = 0$.
- $f(z) = \frac{e^z-1}{z}$ has a **removable singularity** at $z = 0$.

证明.

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} \tag{8.6}$$

□

8.3 孤立奇点的等价刻画

下面主要研究**孤立奇点**附近的性态，并给出各类孤立奇点的等价刻画.

8.3.1 Removable Singularity

首先，介于**可去奇点**的良好性质，我们可在简单的操作后令函数**全纯**，即：

注. If z_0 is a removable singularity of $f : D_r^*(z_0) \to \mathbb{C}$, then we define $f(z_0) = a_0$ so that f is holomorphic in $D_r(z_0)$, where

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,\ \ z \in D_r^*(z_0) \tag{8.7}$$

下面给出**可去奇点**的**等价刻画**.

定理 8.3.1. If z_0 is a singularity of f, then

$$z_0 \in \mathbb{C} \text{ is a removable singularity} \quad \text{iff} \quad f \text{ is bounded near } z_0.$$
$$\text{(i.e. in a deleted neighbourhood of } z_0\text{)}$$

证明.

" $\Rightarrow$ " : We can define the value of f at z_0 s. t.

$$f \text{ is holomorphic in } D_r(z_0) \text{ for some } r > 0.$$

In particular, f is continuous in $\overline{D_{\frac{r}{2}}(z_0)}$ and so f is bounded near z_0.

" $\Leftarrow$ " : Define

$$g : D_r(z_0) \longrightarrow \mathbb{C} \tag{8.8}$$

$$z \longmapsto g(z) = \begin{cases} (z - z_0)^2 f(z), z \neq z_0 \\ 0, z = z_0 \end{cases} \tag{8.9}$$

Then

$$g'(z) = 2(z - z_0)f(z) + (z - z_0)^2 f'(z),\ \ \forall z \in D_r^*(z_0) \tag{8.10}$$

Since f is bounded near z_0,

$$\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}=\lim_{z\to z_0}(z-z_0)f(z)=0 \tag{8.11}$$

Thus g is holomorphic in $D_r(z_0)$ and so g is analytic in $D_r(z_0)$,

$$g(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n,\ \ \forall z\in D_r(z_0) \tag{8.12}$$

Note that $c_0=g(z_0)=0$, $c_1=g^{'}(z_0)=0$. Thus

$$g(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^{n+2},\ \ where\ a_n=c_{n+2}, n=0,1,2,\cdots \tag{8.13}$$

Therefore

$$f(z)=(z-z_0)^{-2}g(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n,\ \ \forall z\in D_r^*(z_0) \tag{8.14}$$

and so z_0 is a removable singularity of f.

□

从上述定理的证明过程中，可以直接得到下面的推论，也是对**可去奇点**的**等价刻画**.

推论 8.3.2. If z_0 is a singularity of f, then

$$z_0 \text{ is a removable singularity} \quad \text{iff} \quad \lim_{z\to z_0}(z-z_0)f(z)=0.$$

8.3.2 Pole

作为**定理 8.3.1**的推论，下面我们给出**极点**的**等价刻画**.

推论 8.3.3. [1] If z_0 is a singularity of f, then

$$z_0 \text{ is a pole} \quad \text{iff} \quad |f(z)| \to \infty \text{ as } z \to z_0.$$

证明. " $\Rightarrow$ ": $\frac{1}{f(z_0)} = 0 \;\Rightarrow\; |f(z)| \to \infty$ as $z \to z_0$.

" $\Leftarrow$ ": Suppose $|f(z)| \to \infty$ as $z \to z_0$, then $\exists r > 0$, s. t.

$$f(z) \neq 0, \;\; \forall z \in D_r^*(z)$$

and so $\frac{1}{f}$ is holomorphic on $D_r^*(z_0)$. Moreover, $\left|\frac{1}{f(z)}\right| \to 0$ as $z \to z_0$.
By **Thm 8.3.1**, $\frac{1}{f}$ is bounded near z_0,

$$\Rightarrow \;\; z_0 \text{ is a removable singularity of } \tfrac{1}{f}.$$

Since $\left|\frac{1}{f(z)}\right| \to 0$ as $z \to z_0$, we have

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \;\; \forall z \in D_r^*(z_0), \;\; where\ a_0 = 0 \tag{8.15}$$

Therefore, if we define $\frac{1}{f(z_0)} = 0$, then $\frac{1}{f}$ is holomorphic on $D_r(z_0)$. By **Def 7.1.3**, z_0 is a pole of f. □

8.3.3 Essential Singularity

在排除了**定理 8.3.1**和 **Cor cor 8.3.3** 的情况后，下面我们给出**本性奇点**的**等价刻画**.

推论 8.3.4. If z_0 is a singularity of f, then

$$z_0 \text{ is an essential singularity} \quad \text{iff} \quad \lim_{z \to z_0} |f(z)| \text{ does not exists.}$$

(Here we allow the limit to be ∞)

例 8.3.1. Consider $f(z) = e^{\frac{1}{z}}$. Since

$$\lim_{\substack{z \in \mathbb{R} \\ z \to 0^+}} e^{\frac{1}{z}} = \infty, \;\; \lim_{\substack{z \in \mathbb{R} \\ z \to 0^-}} e^{\frac{1}{z}} = 0 \tag{8.16}$$

Therefore 0 is an essential singularity of f.

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[1] 对应课本 P85 Cor 3.2

8.4 课堂例题 2024 – 04 – 15

1. **(课前 Question 1.)**

   Let $z_1, z_2 \in \mathbb{C}$ with $Rez_1 \leq 0, Rez_2 \leq 0$. Show

   $$|e^{z_1} - e^{z_2}| \leq |z_1 - z_2|$$

2. **(课前 Question 2.)**

   If f, g are entire functions that agree in infinite number of points, then $f = g$?

   解. Eg: $f(z) = \sin z, g(z) = e^1 \sin z.$ □

3. **(课前 Question 3.)**

   Is there a holomorphic function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ with a simple pole at $z = 0$, s. t.

   $$\int_{C_1(0)} f(z)dz = 0 \ ? \tag{8.17}$$

   解. simple pole $\Rightarrow$ $Res_0 f \neq 0$ $\Rightarrow$ $\int_{C_1(0)} f(z)dz = 2\pi i \cdot Res_0 f \neq 0.$ □

4. 课本第三章练习 $T13$.

附录 A　$L'H\hat{o}spital's\ Rule$

事实上，在复数域 $\mathbb{C}$ 上，**L'Hospital's Rule** 同样成立. 下面便将其推广至 $\mathbb{C}$.

A.1　弱化版本

首先先给出一个常用的弱化版本.

定理 A.1.1. Suppose f, g are holomorphic in a region containing $D_r(z_0)$ for some $r > 0$. If $f(z_0) = g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{A.1}$$

证明. Since $f(z_0) = g(z_0) = 0$, then

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{z \to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)} = \lim_{z \to z_0} \frac{f(z)}{g(z)} \tag{A.2}$$

□