Functional Analysis[1]

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2024 年 9 月 4 日

[1]参考书籍：

《Linear and Nonlinear Functional Analysis with Applications》– Philippe G. Ciarlet

《Real Analysis – Modern Techniques and Their Applications》– Gerald B. Folland

《Functional Analysis – Introduction to Further Topics in Analysis》– Elias M. Stein

《泛函分析讲义》– 张恭庆、林源渠

序

天道几何，万品流形先自守；

变分无限，孤心测度有同伦。

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目录

第一章　度量空间

1.1　L^p 空间为赋范向量空间

回顾实分析中对**范数**、**度量**及 L^p **空间**的定义.

1.1.1　范数，度量

下面给出**范数**和**度量**的严格定义.

定义 1.1.1. Let X be a vector space over $\mathbb{F}$, a **norm** is a function:

$$X \longrightarrow \mathbb{R}_{\geq 0} \tag{1.1}$$

$$f \longmapsto \|f\| \tag{1.2}$$

satisfying the following properties:

(i) $\|f\| \geq 0, \forall f \in X.$　　　$(\|f\| = 0 \iff f = 0 \ a.e.)$

(ii) $\|af\| = |a|\,\|f\|, \forall a \in \mathbb{F}, f \in X.$

(iii) $\|f + g\| \leq \|f\| + \|g\|, \forall f, g \in X.$

注.

- (i) 中的“$\|f\| = 0 \iff f = 0 \ a.e.$”的“**a.e.**”是对于 X 取函数空间时的条件，在实分析的取等条件中基本为默认叙述，在后续定义中往往省略. 在对 $\mathcal{L}^p$ 空间的定义 (**定义 1.1.3**) 中可以看到其合理性.

- **范数**实际上是对 $\mathbb{R}^n$ 空间中“**与原点之间的距离**”这一概念的推广. 将函数视作向量，则其范数即为到原点的距离，即**模长**.

- 若一个**线性空间** X 上配备了一个**范数**，则称其为赋范向量空间 (赋范线性空间).

将函数视作向量，就有其**到原点的距离为范数**. 但若是想要衡量**任意两个函数之间的距离**，则需要引入下面**度量**的概念.

定义 1.1.2. A **metric** on X is a map

$$\rho : X \times X \longrightarrow \mathbb{R}_{\geq 0} \tag{1.3}$$

$$(x, y) \longmapsto \rho(x, y) \tag{1.4}$$

satisfying

(i) $\rho(x, y) \geq 0, \forall x, y \in X.$ $\quad (\rho(x, y) = 0 \iff x = y)$

(ii) $\rho(x, y) = \rho(y, x), \forall x, y \in X.$

(iii) $\rho(x, y) + \rho(y, z) \geq \rho(x, z), \forall x, y, z \in X.$

注.

- 若 X 为函数空间，则 (i) 中"$\rho(x, y) = 0$"等价条件默认为"$x = y$ $a.e.$".
- **度量**可看作将两个函数 (向量) 的起点均平移至原点后，其两个终点之间的**距离**.

1.1.2 L^p Space

L^p Space 下面给出 L^p **空间**的定义.

定义 1.1.3. For any measure space $(X, \mathcal{M}, \mu)$, define the *L^p Space $L^p(X)$* on X $(1 \leq p < \infty)$

$$L^p(X) = \left\{ f \in \mathcal{M} \mid \int_X |f|^p \, d\mu < \infty \right\}, \ \forall 1 \leq p < \infty \tag{1.5}$$

$$L^\infty(X) = \left\{ f \in \mathcal{M} \middle| \inf \{ C \geq 0 \mid |f| \leq C \ a.e. \} < \infty \right\} \tag{1.6}$$

注. 与 L^1 空间 (即可积函数构成的空间) 类似, 后面我们考虑的 $L^p(X)$ 空间事实上为**几乎处处相等**意义下的完全剩余系, 即在 $L^p(X)$ 空间上定义一个等价关系 $\sim$:

$$f \sim g \iff f = g \ a.e.$$

此时再令 $L^p(X)$ 空间模去该等价关系 $\sim$, 即

$$L^p(X) := L^p(X) / \sim$$

L^p 范数 在 L^p 空间上, 我们来定义 L^p 范数.

定义 1.1.4. Measure space $(X, \mathcal{M}, \mu)$. For any function $f \in L^p(X)$, define the **L^p norm** of f

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}, \ \forall 1 \leq p < \infty \tag{1.7}$$

$$\|f\|_\infty = \inf\{C \geq 0 \mid |f| \leq C \ a.e.\} \tag{1.8}$$

注.

- 不难得到 L^∞ 范数的**等价定义**为

$$\|f\|_\infty = \sup\{C \geq 0 \mid \mu(|f| > C) > 0\} \tag{1.9}$$

- 为了说明上述定义是 well-defined, 我们需要验证其满足**范数的三条公理 (Def 1.1.1)**. 其中前面两条 (正定性、绝对齐性) 是显然的, 而对于三角不等式, 我们需要用到后续证明的 **Minkowski Inequality (Thm 1.1.4)**.

事实上, 在证明了 **Minkowski Inequality (Thm 1.1.4)** 后, 我们还可得到 $L^p(X)$ 为**线性空间**, 从而证明 $(L^p(X), \|\cdot\|_p)$ 为**赋范向量空间**. 下面我们的证明思路如下:

Young Inequality $\Rightarrow$ **Hölder Inequality** $\Rightarrow$ **Minkowski Inequality**

1.1.3 Young Inequality

为了证明 **Hölder 不等式**, 先来给出 **Young 不等式**, 可视作一条**均值不等式的加权推广**.

定理 1.1.1. Young Inequality.

Suppose $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $\forall a, b \geq 0$, s. t.

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \tag{1.10}$$

注. Young 不等式可视作一条均值不等式 (几何平均数 $\leq$ 平方平均数) 的**加权推广**, 即

$$\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}$$

证明. (利用**指数函数的凸性**及 **Jensen Inequality**).

It's trivial when $a = 0$ or $b = 0$. 不妨设 $a, b \neq 0$, 即 $a, b > 0$.

Since $f(x) = e^x$ is convex, $\frac{1}{p} + \frac{1}{q} = 1$, then by **Jensen Inequality**,

$$e^{\frac{x}{p}+\frac{y}{q}} \leq \frac{1}{p}e^x + \frac{1}{q}e^y, \ \ \forall x, y \in \mathbb{R} \tag{1.11}$$

Let $x = \log a^p$, $y = \log b^q$, $\forall a, b > 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \ \ \forall a, b > 0 \tag{1.12}$$

□

下面给出一条推论, 将用于 **Hölder Inequality** 的证明中.

推论 1.1.2. Suppose $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $\forall f \in L^p, g \in L^q$, s. t.

$$\int |fg| \leq \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q \tag{1.13}$$

证明. By **Young Inequality (Thm 1.1.1)**, 逐点均有

$$|fg|(x) \leq \frac{1}{p}|f|^p(x) + \frac{1}{q}|g|^q(x), \ \ \forall x \in X \tag{1.14}$$

积分, 得

$$\int |fg| \leq \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q \tag{1.15}$$

□

1.1.4 Hölder Inequality

下面给出二元情形下的 **Hölder** 不等式.

定理 1.1.3. Hölder Inequality.

Suppose $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\forall f \in L^p, g \in L^q$, s. t. $fg \in L^1$ and

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \tag{1.16}$$

证明. It's trivial when $\|f\|_p = 0$ or $\|g\|_q = 0$. 不妨设 $\|f\|_p$, $\|g\|_q \neq 0$.

不妨设 $\|f\|_p = \|g\|_q = 1$. (Otherwise we can let $\widetilde{f} = \frac{f}{\|f\|_p}$ and $\widetilde{g} = \frac{g}{\|g\|_q}$)

Then by **Young Inequality (Cor 1.1.2)**,

$$\begin{aligned} \|fg\|_1 = \int |fg| &\leq \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q && (1.17) \\ &= \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q && (1.18) \\ &= \frac{1}{p} + \frac{1}{q} && (1.19) \\ &= 1 = \|f\|_p \cdot \|g\|_q && (1.20) \end{aligned}$$

□

1.1.5 Minkowski Inequality

下面给出 **Minkowski 不等式**的内容, 它说明了我们所定义的 L^p 范数 $\|\cdot\|_p$ (Def 1.1.4) 的合理性, 并且可以推出 L^p 空间为**线性空间**, 从而得到 $(L^p(X), \|\cdot\|_p)$ 为**赋范向量空间**.

定理 1.1.4. Minkowski Inequality.

Suppose $1 \leq p < \infty$. Then for $\forall f, g \in L^p$, s. t.

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \tag{1.21}$$

证明. $\forall f, g \in L^p$, we have

$$\|f+g\|_p^p = \int |f+g|^p = \int |f+g| \cdot |f+g|^{p-1} \tag{1.22}$$

$$\leq \int |f| \cdot |f+g|^{p-1} + \int |g| \cdot |f+g|^{p-1} \tag{1.23}$$

By **Hölder Inequality (Thm 1.1.3)**,

$$\int |f| \cdot |f+g|^{p-1} = \| \, |f| \cdot |f+g|^{p-1} \, \|_1 \leq \left(\int |f|^p\right)^{\frac{1}{p}} \cdot \left(\int |f+g|^{(p-1)\cdot q}\right)^{\frac{1}{q}} \tag{1.24}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Thus $q = \frac{p}{p-1}$, $(p-1)q = p$, we have

$$\int |f| \cdot |f+g|^{p-1} \leq \left(\int |f|^p\right)^{\frac{1}{p}} \cdot \left(\int |f+g|^p\right)^{\frac{p-1}{p}} = \|f\|_p \cdot \|f+g\|_p^{p-1} \tag{1.25}$$

Similarly, we get

$$\int |g| \cdot |f+g|^{p-1} \leq \left(\int |g|^p\right)^{\frac{1}{p}} \cdot \left(\int |f+g|^p\right)^{\frac{p-1}{p}} = \|g\|_p \cdot \|f+g\|_p^{p-1} \tag{1.26}$$

Therefore,

$$\|f+g\|_p^p \leq \int |f| \cdot |f+g|^{p-1} + \int |g| \cdot |f+g|^{p-1} \tag{1.27}$$

$$\leq (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{p-1} \tag{1.28}$$

i.e.

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p, \ \forall f, g \in L^p \tag{1.29}$$

□

1.2 Completion of a metric space

下面我们来讨论**度量空间的完备化**的内容. 在此之前先给出一些基础概念.

1.2.1 Complete metric spaces

柯西列 先来推广**一般度量空间** (X, ρ) **上的柯西列**的定义.

定义 1.2.1. In a metric space (X, ρ), a sequence $\{x_n\}_{n=1}^{\infty}$ of points $x_n \in X$ is a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$\rho(x_m, x_n) < \epsilon, \ \ \forall m, n > N \tag{1.30}$$

注. **Cauchy sequence** 也有一种**等价定义**, 涉及到**直径 diam** 在一般度量空间 (X, ρ) 上的推广, 即

定义 1.2.2. In a metric space (X, ρ), a sequence $\{x_n\}_{n=1}^{\infty}$ of points $x_n \in X$ is a **Cauchy sequence** if

$$\lim_{n\to\infty} diam(\bigcup_{m=n}^{\infty}\{x_m\}) = 0 \tag{1.31}$$

where

$$diam(\Omega) = \sup_{x,y\in\Omega} \rho(x, y), \ \ \forall \Omega \subset X \tag{1.32}$$

完备性 下面给出一般度量空间**完备性**的定义.

定义 1.2.3. A metric space (X, ρ) is **complete** if every Cauchy sequence of points of X converges in X.

下面给出几个**完备**与**不完备**度量空间的例子.

例 1.2.1.

- $\mathbb{Q}$ 不完备, $\mathbb{R}$ 完备.
- 在 L^∞ 意义下, $P[a, b]$ **不完备** ($[a, b]$ 上的多项式空间), $C[a, b]$ **完备**.

下面给出度量空间**完备的等价表述**.

命题 1.2.1. Suppose (X, ρ) be a metric space, then

$$(X, \rho) \text{ is complete } \Leftrightarrow X \text{ 中闭集套定理成立}$$

i.e. $\forall$ 非空闭集列 $\{F_n\}_{n=1}^\infty \subset \mathcal{P}(X)$

$$\text{If } F_1 \supset F_2 \supset \cdots \text{ and } diam(F_n) \to 0, \text{ then } \bigcap_{n=1}^{\infty} F_n \text{ 为单点集}$$

证明.

(a) 必要性 $\Rightarrow$: Suppose (X, ρ) is complete.

$\forall \{F_n\}_{n=1}^\infty$, $F_n \underset{closed}{\subset} X$, $F_1 \supset F_2 \supset \cdots$ and $diam(F_n) \to 0$. Take $x_n \in F_n$, $\forall n \in \mathbb{N}$.

Then $\{x_n\}_{n=1}^\infty$ is a Cauchy seqeunce in X.

Since (X, ρ) is complete, then $x_n \to x_0 \in X$. 不难证明, $x_0 \in \bigcap_{n=1}^{\infty} F_n$. Thus $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

下用反证法证明 $\bigcap_{n=1}^{\infty} F_n$ 为单点集:

Assume $\exists x' \in \bigcap_{n=1}^{\infty} F_n$, $x' \neq x_0$, then

$$x', x_0 \in F_n, \ \forall n \in N$$

Then

$$diam(F_n) \geq \rho(x', x_0) > 0, \ \forall n \in \mathbb{N}$$
$$diam(F_n) \not\to 0$$

which is a contradiction. Therefore, $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$ 为单点集.

(b) 充分性 $\Leftarrow$: $\forall$ Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subset X$. Let

$$F_j = \overline{\bigcup_{n=j}^{\infty} \{x_n\}}, \ j = 1, 2, \cdots \tag{1.33}$$

Then $\{F_n\}_{n=1}^{\infty}$ 满足闭集套条件, $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$ 为单点集, $x_n \to x_0 \in X$. (X, ρ) complete.

□

1.2.2 Nowhere dense & Category Set

在这一节我们给出**无处稠密 (稀疏)** 以及**纲集**的概念.

Nowhere dense 下面给出**无处稠密 / 稀疏**的定义.

定义 1.2.4. Suppose (X, ρ) be a metric space. We call $A \subset X$ **nowhere dense** if

$$\left(\overline{A}\right)^{\circ} = \varnothing \tag{1.34}$$

注.

- $A \subset X$ **无处稠密 / 稀疏**即 A 的闭包 $\overline{A}$ 无内点.

- **稠密 (dense)** 和**无处稠密 / 稀疏 (nowhere dense)**并不是一对**对偶概念**, 有如下关系:

$$A \text{ dense} \;\Leftarrow\; A^c \text{ nowhere dense}$$
$$A \text{ dense} \;\not\Rightarrow\; A^c \text{ nowhere dense}$$

证明. A^c nowhere dense $\Rightarrow$ $(\overline{A^c})^{\circ} = \varnothing$ $\Rightarrow$ $(A^c)^{\circ} = (\overline{A})^c = \varnothing$ $\Rightarrow$ $\overline{A} = X$ $\Rightarrow$ A dense □

- **单点集**不一定为**无处稠密集 / 稀疏集**. 这取决于度量 ρ 的选取, 下面给出反例.

例 1.2.2. Consider a metric space $(\mathbb{Z}, \rho)$ with

$$\rho : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{R}_{\geq 0} \tag{1.35}$$

$$(x, y) \longmapsto \rho(x, y) = \begin{cases} 0, & \mathit{if}\ x = y \\ 1, & \mathit{if}\ x \neq y \end{cases} \tag{1.36}$$

Then for $\forall \{x\} \subset \mathbb{Z}$, $B(x, \frac{1}{2}) \cap \mathbb{Z} \subset \overline{\{x\}}$, 于是 $\left(\overline{\{x\}}\right)^{\circ} = \{x\}$ 非空, 单点集 $\{x\}$ **不稀疏**.

下面给出更常用的用于判断**无处稠密 / 稀疏**的**等价刻画**.

命题 1.2.2. Suppose (X, ρ) be a metric space. Then

$$A \subset X \text{ nowhere dense } \Leftrightarrow \ \forall B(x,r) \subset X,\ \exists \overline{B(x',r')} \subset B(x,r), \text{ s. t. } \overline{B(x',r')} \cap \overline{A} = \varnothing$$

证明.

(a) 必要性 $\Rightarrow$: 反证法. Assume $\exists B(x,r) \subset X$, s. t. $\forall \overline{B(x',r')} \subset B(x,r)$, $\overline{B(x',r')} \cap \overline{A} \neq \varnothing$.
Then $\forall x' \in B(x,r),\ x' \in \overline{A}$. Thus $x \in \overline{A}$ and $B(x,r) \subset \overline{A} \ \Rightarrow\ x$ 为 $\overline{A}$ 的内点, 矛盾.

(b) 充分性 $\Leftarrow$: 反证法. Suppose $\exists x_0 \in \left(\overline{A}\right)^{\circ}$, then $\exists B(x_0, r_0) \subset \overline{A}$.
$\forall \overline{B(x',r')} \subset B(x_0, r_0)$, $\overline{B(x',r')} \subset \overline{A}$, 矛盾.

□

Category Set 下面我们来给出**纲集**的定义, 这实际上给出了度量空间 (X, ρ) 的**子集的分类**.

定义 1.2.5. Suppose (X, ρ) be a metric space. If $A \subset X$ is a countable union of nowhere dense subsets of X, i.e.

$$A = \bigcup_{n=1}^{\infty} E_n, \ \textit{where } E_n's \textit{ are nowhere dense} \tag{1.37}$$

then we say A is a **First Category Set**. Otherwise we call it a **Second Category Set**.

例 1.2.3. 考虑欧式度量 $(\mathbb{R}^1, d)$, 则有理数集 $\mathbb{Q}$ 为**第一纲集**. 一般地, $(\mathbb{R}^1, d)$ 中的**可数点集**均为**第一纲集**.

下面给出 **Baire** 定理, 它给出了**完备度量空间的刻画**.

定理 1.2.1. Baire's Theorem.

Complete metric spaces are Second Category Sets.

证明. 反证法. Assume complete metric space (X, ρ) is a first category set. Then $\exists \{E_n\}_{n=1}^{\infty}$, $E_n \subset X$ nowhere dense, s. t.

$$X = \bigcup_{n=1}^{\infty} E_n \tag{1.38}$$

Since E_n is nowhere dense, then $\exists \overline{B(x_1, r_1)} \subset X$, s. t. $\overline{B(x_1, r_1)} \cap \overline{E_1} = \varnothing$.

Similarly, for E_2 nowhere dense, $\exists \overline{B(x_2, r_2)} \subset B(x_1, r_1)$, s. t. $\overline{B(x_2, r_2)} \cap \overline{E_2} = \varnothing$

$$\cdots$$

Denote $F_n = \overline{B(x_n, r_n)}$, we can always choose F_k with $diam(F_{k+1}) \leq \frac{diam(F_k)}{2}$. Then F_n's satisfies:

$$F_n \underset{closed}{\subset} X, F_1 \supset F_2 \supset \cdots, diam(F_n) \to 0$$

Since X is complete, then by **Prop 1.2.1 (完备的等价表述)**,

$$\bigcap_{n=1}^{\infty} F_n = \{x_0\} \text{ 为单点集.}$$

Since $(\bigcap_{n=1}^{\infty} F_n) \cap (\bigcup_{n=1}^{\infty} \overline{E_n}) = \varnothing$, 而 $\bigcup_{n=1}^{\infty} \overline{E_n} = X$, then $x_0 \notin X$, 矛盾.

Therefore, (X, ρ) is a Second Category Set. □

1.2.3 保距同构, 完备化空间

这一小节我们来介绍**等距同构 (Isometry)** 和**完备化 (度量) 空间**的概念.

等距同构 (Isometry) 下面给出度量空间之间的**等距 (保距) 同构**的定义.

定义 1.2.6. Suppose $(X_1, \rho_1), (X_2, \rho_2)$ are both metric spaces. Suppose

$$T : (X_1, \rho_1) \to (X_2, \rho_2)$$

If $\rho_2 \circ T = \rho_1$, then we call T an **isometry (等距 / 保距映射)**. 若进一步 T 为满射, 则称 T 为等距 / 保距同构.

注. 事实上, 条件 "$\rho_2 \circ T = \rho_1$" 已经蕴含了 T **为单射**. 从而加上**满射**的条件即为**同构**.

证明. $\forall x, y \in X_1, x \neq y$, we have

$$\rho_1(x, y) = \rho_2(T(x), T(y)) \neq 0 \;\Rightarrow\; T(x) \neq T(y) \;\Rightarrow\; T \text{ injective}$$

□

完备化空间 下面给出一般度量空间的**完备化空间**的定义.

定义 1.2.7. Suppose (X, ρ) be a metric space. If there exists a complete metric space (X_1, ρ_1), s. t.

$$(X, \rho) \text{ 等距同构于 } (X_1, \rho_1) \text{ 的某个稠密子集}$$

则称 X_1 为 X 的完备化空间.

注. 事实上, 不难说明度量空间的**完备化空间**有如下的等价定义[1].

定义 1.2.8. 包含 (X, ρ) 的最小的完备度量空间称为 (X, ρ) 的完备化空间.

[1]详见《泛函分析讲义》– 张恭庆、林源渠, 定义 1.2.2 & 命题 1.2.5

1.2.4 Completion of a metric space

下面给出一般度量空间**完备化的过程**.

定理 1.2.2. Completion of a metric space.

任一度量空间 (X,ρ) 存在完备化空间, 且在保距同构意义下唯一.

证明.

1. Construction of the complete metric space (X_2,ρ_2):

   Let

   $$X_1 = \{\{x_n\}_{n=1}^{\infty} \mid \{x_n\}_{n=1}^{\infty} \subset X \textit{ is a Cauchy squence}\} \tag{1.39}$$

   $\forall \xi = \{x_n\}_{n=1}^{\infty}, \eta = \{y_n\}_{n=1}^{\infty} \in X_1$, we define a equivalence relation $\sim$[2]:

   $$\xi \sim \eta \iff \lim_{n\to\infty} \rho(x_n, y_n) = 0 \tag{1.40}$$

   Then let

   $$X_2 = X_1 / \sim \tag{1.41}$$

   Define the metric ρ_2 on X_2

   $$\rho_2 : X_2 \times X_2 \longrightarrow \mathbb{R}_{\geq 0} \tag{1.42}$$

   $$([\xi],[\eta]) \longmapsto \rho_2([\xi],[\eta]) = \lim_{n\to\infty} \rho(x_n, y_n) \tag{1.43}$$

   where $\xi = \{x_n\}_{n=1}^{\infty}, \eta = \{y_n\}_{n=1}^{\infty} \in X_1$.

   下面说明 ρ_2 is well-defined (与代表元无关 & 极限存在):

   (a) 与代表元无关: $\forall \widetilde{\xi} = \{\widetilde{x_n}\}_{n=1}^{\infty}, \widetilde{\eta} = \{\widetilde{y_n}\}_{n=1}^{\infty}$ with $[\widetilde{\xi}] = [\xi], [\widetilde{\eta}] = [\eta]$. Then

   $$\rho(\widetilde{x_n}, \widetilde{y_n}) \leq \rho(\widetilde{x_n}, x_n) + \rho(x_n, y_n) + \rho(y_n, \widetilde{y_n}), \ \forall n \in \mathbb{N} \tag{1.44}$$

   $$\rho(x_n, y_n) \leq \rho(x_n, \widetilde{x_n}) + \rho(\widetilde{x_n}, \widetilde{y_n}) + \rho(\widetilde{y_n}, y_n), \ \forall n \in \mathbb{N} \tag{1.45}$$

[2]不难证明 well-defined: 自反性、对称性、传递性

Since $[\widetilde{\xi}] = [\xi], [\widetilde{\eta}] = [\eta]$, then

$$\rho(x_n, \widetilde{x_n}) \to 0,\ \ \rho(y_n, \widetilde{y_n}) \to 0 \tag{1.46}$$

Letting $n \to \infty$, we get

$$\rho_2(\widetilde{x_n}, \widetilde{y_n}) = \rho_2(x_n, y_n) \tag{1.47}$$

(b) 极限存在: 即证 $\{\rho(x_n, y_n)\}_{n=1}^{\infty}$ 为 $\mathbb{R}$ 中 Cauchy sequence.

$\forall [\xi], [\eta] \in X_2$, where $\xi = \{x_n\}_{n=1}^{\infty}, \eta = \{y_n\}_{n=1}^{\infty} \subset X$ are Cauchy sequences. Then

$$|\rho(x_n, y_n) - \rho(x_m, y_m)| = |(\rho(x_n, y_n) - \rho(x_m, y_n)) + (\rho(x_m, y_n) - \rho(x_m, y_m))| \tag{1.48}$$

$$\leq |\rho(x_n, y_n) - \rho(x_m, y_n)| + |\rho(x_m, y_n) - \rho(x_m, y_m)| \tag{1.49}$$

$$\leq \rho(x_n, x_m) + \rho(y_n, y_m),\ \ \forall n, m \in \mathbb{N} \tag{1.50}$$

Since $\xi = \{x_n\}_{n=1}^{\infty}, \eta = \{y_n\}_{n=1}^{\infty} \subset X$ are Cauchy sequences, then $\{\rho(x_n, y_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.

2. Construct isometry T:

Consider 嵌入映射

$$T : X \to X_2 \tag{1.51}$$

$$x \longmapsto [\{x\}_{n=1}^{\infty}] \tag{1.52}$$

下面证明 T 为保距映射:

$\forall x, y \in X$, then

$$\rho(T(x), T(y)) = \lim_{n\to\infty} \rho(x, y) = \rho(x, y),\ \ \forall x, y \in X \tag{1.53}$$

Thus T is an isometry (保距映射).

3. $T(X)$ is dense in X_2:

   $\forall[\xi] = [\{x_n\}_{n=1}^{\infty}] \in X_2$, where $\xi = \{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Then

   Consider the sequence $\{T(x_n)\}_{n=1}^{\infty}$ in X_2. We have

$$\rho_2(T(x_n), [\xi]) = \lim_{m\to\infty} \rho(x_n, x_m),\ \forall n \in \mathbb{N} \tag{1.54}$$

   Since $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence in X, then

$$\lim_{n\to\infty} \rho_2(T(x_n), [\xi]) = \lim_{n\to\infty} \lim_{m\to\infty} \rho(x_n, x_m) = 0 \tag{1.55}$$

   i.e.

$$T(x_n) \xrightarrow{\rho_2} [\xi],\ \forall[\xi] = [\{x_n\}_{n=1}^{\infty}] \in X_2 \tag{1.56}$$

   Therefore, $T(X)$ is dense[3] in X_2, and so (X, ρ) 保距同构于 (X_2, ρ_2) 的稠密子集 TX.

4. (X_2, ρ_2) is complete:

   $\forall$ Cauchy sequence $\{[\xi_n]\}_{n=1}^{\infty} \subset X_2$, where $\xi_n = \{x_j^n\}_{j=1}^{\infty} \subset X$ is a Cauchy sequence.

   By **Step 3**, $T(X)$ is dense in X_2 and $\forall n \in \mathbb{N}$,

$$\rho_2(T(x_j^n), [\xi_n]) \to 0,\ as\ j \to \infty \tag{1.57}$$

   Thus $\exists j_n \in \mathbb{N}$, s. t.

$$\rho_2(T(x_{j_n}^n), [\xi_n]) < \frac{1}{n},\ \forall n \in \mathbb{N} \tag{1.58}$$

   Let $\xi = \{x_{j_n}^n\}_{n=1}^{\infty}$. It suffices to show $[\xi_n] \to [\xi]$, i.e.

$$\rho_2([\xi_n], [\xi]) \to 0,\ as\ n \to \infty \tag{1.59}$$

   而这需要证明两点结论, 即 $[\xi] \in X_2$ & $[\xi_n] \to [\xi]$:

[3]此处实际用到了**度量空间稠密子集的等价刻画**, 具体可见**附录 A - Lemma A.1.1**

(a) $[\xi] \in X_2$, i.e. $\xi = \{x_{j_n}^n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence in X:

Fix $\epsilon > 0$. Since $\{[\xi_n]\}_{n=1}^{\infty}$ is a Cauchy sequence in X_2, and $\rho_2(T(x_{j_n}^n), [\xi_n]) \to 0$, then since T is isometry (by **Step 2**)

$\exists N \in \mathbb{N}$, s. t.

$$\rho(x_{j_k}^k, x_{j_l}^l) = \rho_2(T(x_{j_k}^k), T(x_{j_l}^l)) \tag{1.60}$$

$$\leq \rho_2(T(x_{j_k}^k), [\xi_k]) + \rho_2([\xi_k], [\xi_l]) + \rho_2([\xi_l], T(x_{j_l}^l)) \tag{1.61}$$

$$< \epsilon, \ \forall k, l > N \tag{1.62}$$

Therefore, $\xi = \{x_{j_n}^n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence, thus $[\xi] \in X_2$.

(b) $[\xi_n] \to [\xi]$:

WTS: $\rho_2([\xi_n], [\xi]) \to 0$, i.e.

$$\lim_{n\to\infty} \rho_2([\xi_n], [\xi]) = \lim_{n\to\infty} \lim_{k\to\infty} \rho(x_k^n, x_{j_k}^k) = 0 \tag{1.63}$$

Fix $n \in \mathbb{N}$. Since

$$\lim_{k\to\infty} \rho(x_{j_n}^n, x_k^n) = \rho_2(T(x_{j_n}^n), [\xi_n]) < \frac{1}{n} \tag{1.64}$$

Then $\forall \epsilon > 0$, $\exists k_0 \in \mathbb{N}$, s. t.

$$\rho(x_{j_n}^n, x_k^n) < \frac{1}{n} + \epsilon, \ \forall k > k_0 \tag{1.65}$$

Since $\xi = \{x_{j_n}^n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence in X, then

$$\rho(x_{j_n}^n, x_{j_k}^k) \to 0 \ as \ n, k \to \infty \tag{1.66}$$

Then

$$\rho(x_k^n, x_{j_k}^k) \leq \rho(x_k^n, x_{j_n}^n) + \rho(x_{j_n}^n, x_{j_k}^k) \tag{1.67}$$

$$\leq \frac{1}{n} + \epsilon + \rho(x_{j_n}^n, x_{j_k}^k) \tag{1.68}$$

Letting $\epsilon \to 0^+$ and $n, k \to \infty$, we have

$$\lim_{n\to\infty} \rho_2([\xi_n], [\xi]) = \lim_{n\to\infty} \lim_{k\to\infty} \rho(x_k^n, x_{j_k}^k) = 0 \tag{1.69}$$

5. X_2 在保距同构下的唯一性:

Suppose $(X_2, \rho_2), (X_2^{'}, \rho_2^{'})$ 均为 (X, ρ) 的完备化空间. $i_1 : X \to X_2$, $i_2 : X \to X_2^{'}$ 为保距同构. $\forall [\xi] \in X_2$. By **Step 3**, $\exists \{x_n\}_{n=1}^{\infty} \subset X$, s. t.

$$i_1(x_n) \to [\xi] \ in \ X_2, \ as \ n \to \infty \tag{1.70}$$

$\Rightarrow$ $\{i_1(x_n)\}_{n=1}^{\infty} \subset X_2$ is a Cauchy sequence in X_2.

$\Rightarrow$ $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$\rho_2(i_1(x_n), i_1(x_m)) = \rho(x_n, x_m) < \epsilon, \ \forall n, m > N \tag{1.71}$$

$\Rightarrow$ $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence in X.

$\Rightarrow$ $\{i_2(x_n)\}_{n=1}^{\infty} \subset X_2^{'}$ is a Cauchy sequence in $X_2^{'}$.

$\Rightarrow$ Suppose $i_2(x_n) \to [\xi^{'}] \in X_2^{'}$ as $n \to \infty$. Let

$$T : X_2 \longrightarrow X_2^{'} \tag{1.72}$$

$$[\xi] \longmapsto T([\xi]) = [\xi^{'}] \tag{1.73}$$

不难证明 $T : X_2 \to X_2^{'}$ 为保距映射. Similarly, we can prove T is surjective.

$\Rightarrow$ T is an isometry. i.e. $X_2, X_2^{'}$ 保距同构.

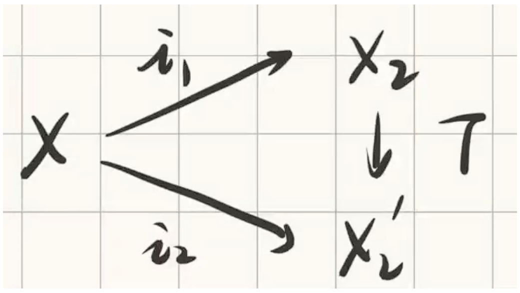

图 1.1: X_2 在保距同构下的唯一性

$\square$

例 **1.2.4.** 下面给出两个完备化空间的例子.

1. $(P[a, b], \rho_\infty) \to (C[a, b], \rho_\infty)$, 即区间 $[a, b]$ 上的多项式全体在度量 ρ_∞ 下的完备化空间为 $[a, b]$ 上的连续函数全体. 其中

$$\rho_\infty(x, y) = \max_{a \le t \le b} |x(t) - y(t)| \tag{1.74}$$

2. $(C[a, b], \rho_1) \to (L^1[a, b], \rho_1)$, 即区间 $[a, b]$ 上的连续函数全体在度量 ρ_1 下的完备化空间为 $[a, b]$ 上 Lebesgue 可积函数全体.

1.3 Sequentially Compact

引入 回顾在拓扑和数学分析中接触过的概念, **列紧性 (sequentially compact)**. 现将其限制于**度量空间**上给出定义.

定义 1.3.1. Suppose (X, ρ) be a metric space, $A \subset X$. If $\forall \{x_n\}_{n=1}^{\infty} \subset A$, $\exists$ subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ convergent in X, then we call A **sequentially compact (列紧的)**.

注. • 将条件 "**metric space** (X, ρ)" 改为 "**拓扑空间** X" 即可得到拓扑中的一般性定义. 回顾一般拓扑空间中 "**紧致**"、"**列紧**"、"**极限点紧**" 的定义与性质, 有如下关系:

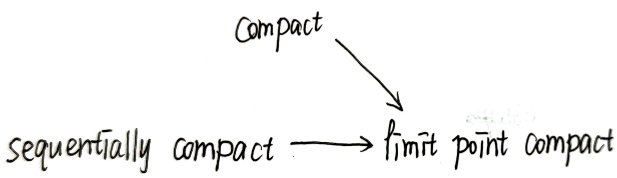

图 1.2: Relations among **compact**, **sequentially compact** and **limit point compact**

• If $\forall \{x_n\}_{n=1}^{\infty} \subset A$, $\exists$ subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ convergent in A, then 称 A 自列紧.

例子 下面给出一个经典的**非列紧空间**的例子.

例 1.3.1. Consider the set

$$l^1 = \left\{ \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} |x_n| < \infty,\ x_n \in \mathbb{R} \right\} \tag{1.75}$$

$\forall \xi = \{x_n\}_{n=1}^{\infty}, \eta = \{y_n\}_{n=1}^{\infty} \in l^1$, define the metric ρ_1 on l^1:

$$\rho_1 : l^1 \times l^1 \longrightarrow \mathbb{R}_{\geq 0} \tag{1.76}$$

$$(\xi, \eta) \longmapsto \rho_1(\xi, \eta) = \sum_{n=1}^{\infty} |x_n - y_n| \tag{1.77}$$

Let

$$A = \left\{ \{\delta_{kj}\}_{j=1}^{\infty} \right\}_{k=1}^{\infty} \tag{1.78}$$

$$= \{(1, 0, \cdots, 0, \cdots), (0, 1, \cdots, 0, \cdots), \cdots, (0, 0, \cdots, 1, \cdots), \cdots\} \subset l^1 \tag{1.79}$$

则 $A \subset l^1$ 中每两个元素之间的距离均为 2, 无收敛子列, 故 (l^1, ρ_1) 非列紧.

性质 对于**度量空间中的列紧集**, 容易得到其必为**完备度量空间**.

命题 1.3.1. 列紧度量空间必完备.

证明. Suppose (X, ρ) be sequentially compact. Then for $\forall$ Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $\exists$ subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ convergent $\Rightarrow$ Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ convergent $\Rightarrow$ (X, ρ) complete □

为了更好地理解一般度量空间中的**列紧性**, 我们可以与欧氏空间 $\mathbb{R}^n$ 中**有界**的概念联系.

$\mathbb{R}^n$	度量空间
有界 (*bounded*)	**列紧**
有界闭	**自列紧**

1.4 完全有界集

ϵ-网 在一般的度量空间中, 我们来引入一个比**有界集**更强的概念. 首先来给出 **ϵ-网**的定义.

定义 1.4.1. Suppose (X,ρ) be a metric space, $N \subset M \subset X$ and $\epsilon > 0$. If for $\forall x \in M$, $\exists y \in N$, s. t.

$$\rho(x,y) < \epsilon$$

则称 N 为 M 的一个**ϵ-网**, 即 $M \subset \bigcup\limits_{x \in N} B(x,\epsilon)$. 进一步若 N 为有穷集 (*finite*), 则称 N 为 M 的一个**有穷 ϵ-网**.

完全有界集 下面给出**完全有界集**的概念.

定义 1.4.2. Suppose (X,ρ) be a metric space, $A \subset X$. If $\forall \epsilon > 0$, A 存在有穷 ϵ-网, 则称 A 为**完全有界集**.

注. **完全有界集**的概念比**有界集**要更强, 即

$$\textbf{完全有界集} \Rightarrow \textbf{有界集} \quad , \quad \textbf{完全有界集} \nLeftarrow \textbf{有界集}$$

例1.3.1 中集合 $A = \left\{\left\{\delta_{kj}\right\}_{j=1}^{\infty}\right\}_{k=1}^{\infty}$ 即为有界集但非**完全有界**.

等价表述 下面我们将给出一般度量空间中**完全有界集的等价表述**, 便于我们判断和理解**完全有界集**的概念.

定理 1.4.1. 完全有界集的等价表述.
Suppose (X, ρ) be a metric space and $A \subset X$. Then

$$A \text{ 完全有界 } \Leftrightarrow A \text{ 中任意点列存在 } Cauchy \text{ 子列}$$

注. 根据该定理我们容易得到, 在**一般的度量空间**中, **列紧** $\Rightarrow$ **完全有界**. 而在**完备度量空间**中, **完全有界** $\Leftrightarrow$ **列紧**. 特别地, 在欧氏空间 $\mathbb{R}^n$ 中, **列紧、完全有界、有界**三者等价.

证明.

$\Rightarrow$: 若 A 完全有界. $\forall \{x_n\}_{n=1}^{\infty} \subset A$, 下面证明 $\{x_n\}_{n=1}^{\infty}$ 存在 *Cauchy* 子列:

- For $\epsilon = 1$, $\exists y_1 \in A$, s. t. $B(y_1, 1)$ 中包含 $\{x_n\}_{n=1}^{\infty}$ 无穷多项, 记为 $\{x_n^{(1)}\}_{n=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$.

  (否则若 $\forall y \in A$, $B(y, 1)$ 均至多包含 $\{x_n\}_{n=1}^{\infty}$ 中有穷个点, 则 A 无有穷 1-网)

- For $\epsilon = \frac{1}{2}$, $\exists y_2 \in A$, s. t. $B(y_2, \frac{1}{2})$ 中包含 $\{x_n^{(1)}\}_{n=1}^{\infty}$ 无穷多项, 记为 $\{x_n^{(2)}\}_{n=1}^{\infty} \subset \{x_n^{(1)}\}_{n=1}^{\infty}$.

  $\cdots$

- For $\epsilon = \frac{1}{k}$, $\exists y_k \in A$, s. t. $B(y_k, \frac{1}{k})$ 中包含 $\{x_n^{(k-1)}\}_{n=1}^{\infty}$ 中无穷多项, 记为 $\{x_n^{(k)}\}_{n=1}^{\infty} \subset \{x_n^{(k-1)}\}_{n=1}^{\infty}$.

从而我们得到了 $\{x_n\}_{n=1}^{\infty}$ 的一列子列: $\{x_n^{(1)}\}_{n=1}^{\infty}, \{x_n^{(2)}\}_{n=1}^{\infty}, \cdots, \{x_n^{(k)}\}_{n=1}^{\infty}, \cdots$
取出第 k 个子列 $\{x_n^{(k)}\}_{n=1}^{\infty}$ 的第 k 项 $x_k^{(k)}$, 得到子列 $\{x_n^{(n)}\}_{n=1}^{\infty}$[4].
下面证明: $\{x_n^{(n)}\}_{n=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ 为 *Cauchy* 列.
Since

$$\rho(x_{n+p}^{(n+p)}, x_n^{(n)}) \leq \rho(x_{n+p}^{(n+p)}, y_n) + \rho(y_n, x_n^{(n)}) \leq \frac{2}{n}, \quad \forall n, p \in \mathbb{N} \tag{1.80}$$

Therefore, $\{x_n^{(n)}\}_{n=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A.

[4]这种从一列序列中各取出一个元素构成新序列, 再 (一致) 收敛的方法称为**对角线法则**, 在**实分析** (*Real Analysis*) 中证明**任一可测函数可由简单函数列逼近**时曾使用, 详情可见 *Real Analysis* 笔记**定理 2.2.1**.

$\Leftarrow$: 反证法. Assume A 非完全有界, 即 $\exists\epsilon_0 > 0$, s. t. A 无有穷 ϵ_0-网.

- $\forall x_1 \in A, \exists x_2 \in A \backslash B(x_1, \epsilon_0)$ (Otherwise $A \subset B(x_1, \epsilon_0)$, A 存在有穷 ϵ_0-网)
- Similarly, $\exists x_3 \in A \backslash (B(x_1, \epsilon_0) \cup B(x_2, \epsilon_0))$

$$\cdots$$

- $\exists x_k \in A \backslash \left(\bigcup_{i=1}^{k-1} B(x_i, \epsilon_0) \right)$

从而得到 A 中的一列点 $\{x_n\}_{n=1}^{\infty} \subset A$, 其中

$$\rho(x_i, x_j) > \epsilon_0 > 0,\ \forall i \neq j$$

于是 $\{x_n\}_{n=1}^{\infty} \subset A$ 无 Cauchy 子列, 矛盾.

$\square$

1.5 可分度量空间

作为一类特殊的**拓扑空间**, 下面我们来讨论一些常见的**度量空间**的**可分性**. 首先回顾一下**可分**的定义.

定义 1.5.1. Suppose (X, τ) be a Topological space. If (X, τ) 存在**可数稠密子集**, then (X, τ) is called **separable (可分的)**.

根据**可分空间**的定义, 不难得到上节所介绍的**完全有界空间可分**.

命题 1.5.1. 完全有界空间为可分度量空间.

证明. Suppose (X, ρ) is totally bounded. Then for $\forall k \in \mathbb{N}$, $\exists n_k \in \mathbb{N}$, $y_1, \cdots, y_{n_k} \in X$, s. t.

$$X \subset \bigcup_{i=1}^{n_k} B(y_i^k, \frac{1}{k}) \tag{1.81}$$

Let

$$A = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{n_k} \{y_i^k\} \subset X \ countable \tag{1.82}$$

Then for $\forall x \in X$, $\exists 1 \le l_k \le n_k$, $y_{l_k}^k \in A$, s. t.

$$\rho(x, y_{l_k}^k) < \frac{1}{k}, \ \ \forall k \in \mathbb{N}$$

Thus $\{y_{l_k}^k\}_{k=1}^{\infty} \subset A$ convergent to $x \in X$, i.e. $y_{l_k}^k \xrightarrow{\rho} x$ as $k \to \infty$.

Therefore, by **Lemma A.1.1**, $A \subset X$ is dense in X. X is separable. □

下面来讨论一些常见的**度量空间**的**可分性**.

例 1.5.1. [可分空间].

- $(C[a,b],\rho_\infty)$ 可分.
  - **证明.** 有理系数多项式 $\stackrel{\text{一致逼近}}{\Longrightarrow}$ 多项式 $\stackrel{\text{一致逼近}}{\Longrightarrow}$ 连续函数 □

- (l^p,ρ_p) 可分.
  - **证明.** 此处 (l^p,ρ_p) 定义与**例 1.3.1** 中一致, 即

$$l^p = \left\{\{x_n\}_{n=1}^\infty \mid \sum_{n=1}^\infty |x_n|^p < \infty,\ x_n \in \mathbb{R}\right\} \tag{1.83}$$

$$\rho_p : l^p \times l^p \longrightarrow \mathbb{R}_{\geq 0} \tag{1.84}$$

$$(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \longmapsto \rho_p(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) = \sum_{n=1}^\infty |x_n - y_n|^p \tag{1.85}$$

下面来构造 (l^p,ρ_p) 的可数稠密子集:

* Let

$$A_1 = \{\{x_n\}_{n=1}^\infty \mid x_1 \in \mathbb{Q},\ x_n = 0,\ \forall n > 1\} \subset l^p \tag{1.86}$$

$$A_2 = \{\{x_n\}_{n=1}^\infty \mid x_1, x_2 \in \mathbb{Q},\ x_n = 0,\ \forall n > 2\} \subset l^p \tag{1.87}$$

$$\cdots \tag{1.88}$$

$$A_k = \{\{x_n\}_{n=1}^\infty \mid x_1, \cdots, x_k \in \mathbb{Q},\ x_n = 0,\ \forall n > k\} \subset l^p \tag{1.89}$$

$$A = \bigcup_{k=1}^\infty A_k \subset l^p \tag{1.90}$$

* $A \subset l^p$ 即为 (l^p,ρ_p) 的可数稠密子集:

  $\forall \{x_n\}_{n=1}^\infty \in l^p$. Since

$$\sum_{n=1}^\infty |x_n|^p < \infty$$

  Then for $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$\sum_{n=N+1}^\infty |x_n|^p < \frac{\epsilon}{2}$$

  Thus $\exists \{y_n\}_{n=1}^\infty \in A_N \subset A$, $y_n = 0$, $\forall n > N$ and

$$|y_n - x_n|^p < \frac{\epsilon}{2N},\quad \forall 1 \leq n \leq N$$

  Therefore

$$\rho_p(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) = \sum_{n=1}^\infty |x_n - y_n|^p < \epsilon$$

  $A \subset l^p$ is dense in l^p while it's also countable.

□

- $L^p[a,b]$ 可分.

证明. Review the conclusions in *Real Analysis*. $\forall f \in L^p[a,b]$, then f is measurable. Since **任一可测函数可由简单函数列逼近** (*Real Analysis* **笔记 Thm 2.2.1**)
又根据 **Lusin 定理** (*Real Analysis* **笔记 Thm 3.8.2**), 可得:

$$\text{连续函数} \;\Rightarrow\; \text{简单函数} \;\Rightarrow\; f \in L^p$$

□

- $L^\infty[a,b]$ 不可分.

证明. Let

$$E = \left\{ f \in L^\infty[a,b] \;\middle|\; f(x) = \begin{cases} 0, x \in [a,r] \\ 1, x \in (r,b] \end{cases}, \;\forall r \in (a,b) \right\} \tag{1.91}$$

Then $E \subset L^\infty[a,b]$ is uncountable, and

$$\rho_\infty(f,g) = 1 > 0, \;\forall f,g \in E$$

下面用反证法证明 $L^\infty[a,b]$ 不可分. Assume $L^\infty[a,b]$ is separable.
Then $\exists$ countable $\{f_n\}_{n=1}^\infty \subset L^\infty[a,b]$, s. t.

$$\overline{\{f_n\}_{n=1}^\infty} = L^\infty[a,b]$$

即 $L^\infty[a,b]$ 中点均可由 $\{f_n\}_{n=1}^\infty$ 中的某些点逼近, 从而

$$\bigcup_{n=1}^\infty B(f_n, \frac{1}{3}) = L^\infty[a,b] \supset E$$

于是 $\exists N \in \mathbb{N}$, s. t.

$$B(f_N, \frac{1}{3}) \text{ 中包含 } E \text{ 中至少 2 个点 (事实上可严格地说包含 } E \text{ 中不可数个点)}$$

而 $\rho_\infty(f,g) = 1 > 0, \;\forall f,g \in E$, 这与 $f,g \in B(f_N, \frac{1}{3})$ 矛盾.
综上, $L^\infty[a,b]$ 不可分. □

- l^∞ 不可分.

证明. Similarly. Let

$$E = \{\{x_n\}_{n=1}^\infty \in l^\infty \mid x_n = 0 \ or \ 1, \ \forall n \in \mathbb{N}\} \subset l^\infty \tag{1.92}$$

Then $E \subset l^\infty$ is uncountable (E 与二进制数一一对应, 而二进制数与实数 $\mathbb{R}$ 一一对应), and

$$\rho_\infty(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) = 1 > 0, \ \forall \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \in E$$

后续步骤与上述 $L^\infty[a,b]$ 不可分证明过程一致. □

1.6 Compact

首先来回顾一下**拓扑学**中关于**紧致**的定义, 在度量空间中也是一脉相承的.

定义 1.6.1. Suppose (X, τ) be a topological space. If X 的任意开覆盖都有有限子覆盖, 则称 X 为**紧致的 (compact)**.

注. 事实上我们运用的更多的为**一般拓扑 / 度量空间**的**紧致子集**的概念, 其定义如下:

Suppose $A \subset (X, \tau)$. If A 的任意开覆盖都有有限子覆盖 (在子空间拓扑意义下为紧集), 则称 A 为紧 (子) 集.

1.6.1 度量空间紧集的性质

下面我们给出度量空间中紧集的 4 条性质 (事实上大部分对一般的 Hausdorff 空间成立).

命题 1.6.1. [紧 $\Rightarrow$ 闭, Hausdorff].

若 $A \subset (X, \rho)$ 紧致, 则 A 为闭集.

注. 该结论对于一般拓扑空间不成立, 但对于 **Hausdorff 空间**成立, 故自然**度量空间**成立.

证明. 下面对一般情形进行证明, 即 (X, τ) 为 Hausdorff 空间, 证明: A^c open.

Fix $x \in A^c$. Since X is Hausdorff, then for $\forall y \in A$, $\exists x \in V_y^x, y \in U_y^x$, s. t.

$$U_y^x \cap V_y^x = \varnothing, \ \ U_y^x, V_y^x \underset{open}{\subset} X$$

Then we have

$$A \subset \bigcup_{y \in A} U_y^x$$

Since A is compact, there exists $U_{y_1}^x, \cdots, U_{y_n}^x \in \{U_y^x\}_{y \in A}$, s. t.

$$A \subset \bigcup_{i=1}^{n} U_{y_i}^x$$

Let

$$V^x = \bigcap_{i=1}^{n} V_{y_i}^x \underset{open}{\subset} X$$

Then $x \in V^x$ and $V^x \cap \bigcup_{i=1}^{n} U_{y_i}^x = \varnothing$, so $V^x \cap A = \varnothing$, $V^x \subset A^c$. Therefore, A^c is open. □

命题 1.6.2. [紧集的闭子集必为紧集, 一般拓扑空间].

$$\text{若 } A \subset (X, \rho) \text{ 紧致}, K \subset A \text{ 为闭集, 则 } K \text{ 紧致}.$$

注. 该结论对于**一般的拓扑空间**均成立.

证明. 下面对一般情形进行证明, 且下述证明过程均以子空间 (A, τ_A) 作为全空间.
$\forall \{U_i\}_{i\in I}, U_i \underset{open}{\subset} X, \forall i \in I$, s. t.

$$K \subset \bigcup_{i\in I} U_i$$

Since $K \subset A$ closed, then $K^c \subset A$ open. Thus

$$A = K^c \cup K = K^c \cup \bigcup_{i\in I} U_i$$

Since A is compact, then $\exists i_1, \cdot, i_n \in I$, s. t.

$$A = K^c \cup \bigcup_{k=1}^{n} U_{i_k}$$

Therefore $K \subset \bigcup_{k=1}^{n} U_{i_k}$. $K \subset A$ is compact. □

根据**命题 1.6.2**, 我们可得到另一条直接推论.

推论 1.6.1. [Hausdorff].

紧集和闭集的交为紧集.

证明. 紧集与闭集相交 $\Rightarrow$ 交集为原紧集的闭子集 $\Rightarrow$ 根据**命题 1.6.2**, 该交集为紧集 □

命题 1.6.3. [Hausdorff].

$\{K_\alpha \subset X\}_{\alpha\in\Lambda}$ 为一个紧子集族, 满足任意有限交非空, 则

$$\bigcap_{\alpha\in\Lambda} K_\alpha \neq \varnothing \tag{1.93}$$

证明. 反证法. Assume $\bigcap\limits_{\alpha\in\Lambda} K_\alpha = \varnothing$. Then for $\forall\alpha_0 \in \Lambda$,

$$K_{\alpha_0} \cap \bigcap_{\substack{\alpha\in\Lambda \\ \alpha\neq\alpha_0}} K_\alpha = \varnothing \quad , \quad K_{\alpha_0} \subset \bigcup_{\substack{\alpha\in\Lambda \\ \alpha\neq\alpha_0}} K_\alpha^c$$

Since X is Hausdorff, $K_\alpha \subset X$ is compact, then K_α is closed, $\forall\alpha \in \Lambda$. Thus

$\{K_\alpha^c \mid \alpha \neq \alpha_0\}_{\alpha\in\Lambda}$ is an open covering of K_{α_0}. Since $K_{\alpha_0} \subset X$ is compact, then $\exists\alpha_1, \cdots, \alpha_n \in \Lambda$, s. t.

$$K_{\alpha_0} \subset \bigcup_{i=1}^{n} K_{\alpha_i}^c$$

Therefore,

$$\bigcap_{i=0}^{n} K_{\alpha_i} = \varnothing$$

which is a contradiction to "任意有限交非空". □

命题 1.6.4. [Hausdorff].

$A \subset (X, \rho)$ 为闭集. 若对于 A 中任意闭子集族 $\{F_\alpha\}_{\alpha\in\Lambda}$, 如果满足任意有限交非空, 便有 $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \neq \varnothing$, 则 A 为紧集.

证明. $\forall\{U_i\}_{i\in I}$, $U_i \underset{open}{\subset} X$, $\forall i \in I$, s. t.

$$A \subset \bigcup_{i\in I} U_i$$

Then $\{U_i^c \cap A\}_{i\in I}$ 为 A 的闭子集族. Since

$$\bigcap_{i\in I} U_i^c \cap A = \varnothing$$

Then 根据条件, $\{U_i^c \cap A\}_{i\in I}$ 存在有限交为空集, 即 $\exists i_1, \cdots, i_n \in I$, s. t.

$$\bigcap_{k=1}^{n} U_i^c \cap A = A \cap \left(\bigcup_{k=1}^{n} U_{i_k}\right)^c = \varnothing$$

Thus

$$A \subset \bigcup_{k=1}^{n} U_{i_k}$$

Therefore, $A \subset X$ is compact. □

1.6.2 度量空间紧致的等价刻画

下面我们将说明, 在**度量空间**中, **紧致性**与**自列紧性**等价.

定理 1.6.2. [紧致 ⇔ 自列紧, 度量空间].
在度量空间 (X, ρ) 中,

$$A \subset X \text{ 紧致 } \Leftrightarrow \text{ 自列紧}$$

证明.

$\Rightarrow$: 只需考虑 A 中无穷点集的情形. $\forall$ 无穷点集 $\{x_n\}_{n=1}^{\infty} \subset A$, 不妨设 x_n 互异.
反证法. Assume $\{x_n\}_{n=1}^{\infty} \subset A$ 在 A 中无聚点, 则 $\forall x \in A, \exists U_x \underset{open}{\subset} X$, s. t.

$$U_x \cap \{x_n\}_{n=1}^{\infty} \subset \{x\} \qquad (= \varnothing \text{ or } \{x\})$$

Since $A \subset \bigcup_{x \in A} U_x$, A is compact, then $\exists y_1, \cdots, y_n \in A$, s. t.

$$\{x_n\}_{n=1}^{\infty} \subset A \subset \bigcup_{i=1}^{n} U_{y_i}$$

从而 $\{U_{y_i}\}_{i=1}^{n}$ 中至少有一项中包含 $\{x_n\}_{n=1}^{\infty}$ 中无穷多项, 这与 $U_x \cap \{x_n\}_{n=1}^{\infty} \subset \{x\}$ 矛盾.
综上, $A \subset X$ 紧致 $\Rightarrow$ 自列紧.

$\Leftarrow$: 反证法. 假设 $\exists A$ 的一个开覆盖 $\{U_\alpha\}_{\alpha \in \Lambda}$ 不存在有限子覆盖.
Since A 自列紧 $\Rightarrow$ 列紧 $\Rightarrow$ 完全有界 (**Thm 1.4.1**), then for $\forall$ fixed $n \in \mathbb{N}$,
A 存在有穷 $\frac{1}{n}$-网, i.e. $\exists M_n$ finite, s. t.

$$A \subset \bigcup_{x \in M_n} B(x, \frac{1}{n})$$

Then for the fixed n, $\exists x_n \in M_n$, s. t. $B(x_n, \frac{1}{n})$ 不能被 $\{U_\alpha\}_{\alpha \in \Lambda}$ 有限覆盖.

(否则若 $\forall x \in M_n$, $B(x, \frac{1}{n})$ 均可被 $\{U_\alpha\}_{\alpha \in \Lambda}$ 有限覆盖, 则 $A \subset \bigcup_{x \in M_n} B(x, \frac{1}{n})$ 可被 $\{U_\alpha\}_{\alpha \in \Lambda}$ 有限覆盖. 矛盾)

Then we get a sequence $\{x_n\}_{n=1}^{\infty} \subset A$ in A.

Since A 自列紧, then $\exists$ subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$, s. t.

$$x_{n_k} \to x_0 \in A \ as\ k \to \infty$$

Since $\{U_a\}_{a\in\Lambda}$ covers A, then $\exists a_0 \in \Lambda$, s. t. $x_0 \in U_{a_0}$.

Since U_{a_0} is open, then $\exists \epsilon > 0$, s. t. $B(x_0, \epsilon) \subset U_{a_0}$.

However, since $x_{n_k} \to x_0$, for $\epsilon > 0$, $\exists k_0 \in \mathbb{N}$, s. t.

$$\rho(x_{n_{k_0}}, x_0) < \frac{\epsilon}{3} \text{ and } \frac{1}{n_{k_0}} < \frac{\epsilon}{3}$$

Then $B(x_{n_{k_0}}, \dfrac{1}{n_{k_0}}) \subset U_{a_0}$, 于是 $B(x_{n_{k_0}}, \dfrac{1}{n_{k_0}})$ 被 $\{U_a\}_{a\in\Lambda}$ 有限覆盖, 矛盾.

综上, A 自列紧 $\Rightarrow$ compact.

$\square$

1.7 一致有界, 等度连续

为了给出上 $C[a,b]$ **(列) 紧集的刻画**, 即下节介绍的 **Arzelà-Ascoli 定理**, 本节来给出一些前置概念.

1.7.1 紧致度量空间上的连续函数全体

首先来给出**紧致度量空间** X 上的连续函数全体 $C(X)$, 并赋予相应的**度量 (无穷范数)**, 使其称为**完备度量空间**.

定义 1.7.1. 设 (X,ρ) 为紧致度量空间, 其上的连续函数全体 $C(X)$ 被定义为:

$$C(X) := \{f : X \longrightarrow \mathbb{K} \mid f \text{ continuous}\},\ \ \mathbb{K} \in \{\mathbb{C}, \mathbb{R}\} \tag{1.94}$$

赋予 $C(X)$ 上度量 d, 定义为:

$$d : C(X) \times C(X) \longrightarrow \mathbb{R}_{\geq 0} \tag{1.95}$$

$$(f,g) \longmapsto \max_{x\in X} |f(x) - g(x)| \tag{1.96}$$

注. • 不难证明上述定义的度量 d 满足**度量的三大公理 (Def 1.1.2)**. 关于其定义是否**良好 (Well-defined)** 的问题, 只需说明对于 $\forall f,g \in C(X)$, $d(f,g)$ 的存在性:

– **证明.** 不难证明 $C(X)$ 为线性空间, 于是对于 $\forall f,g \in C(X), f-g \in C(X)$.
根据**绝对值不等式**, 容易得到绝对值保持 $C(X)$ 中函数的连续性.

$$(||\varphi(x)| - |\varphi(y)|| \leq |\varphi(x) - \varphi(y)|\,,\ \ \forall \varphi \in C(X), x, y \in X)$$

因此 $|f-g| \in C(X)$, 即连续.
Since X is compact, then $|f-g|\,(X) \subset \mathbb{R}_{\geq 0}$ compact, 可取到最大值. 故 $d(f,g)$ 存在. □

• 事实上, 上述定义的度量空间 $(C(X),d)$ 是**完备的**, 这在下面的**命题 1.7.1** 中得以证明.

下面的命题证明了, **定义 1.7.1** 中定义的度量空间 $(C(X), d)$ 是完备的.

命题 1.7.1. $(C(X), d)$ **为完备度量空间**.

证明. $\forall$ Cauchy sequence $\{f_n\}_{n=1}^{\infty} \subset C(X)$, i.e. $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$, s. t.

$$d(f_m, f_n) = \max_{x \in X} |f_m(x) - f_n(x)| < \epsilon,\ \ \forall m, n \geq N_\epsilon$$

下面分两步进行证明：

1. $\exists f : X \longrightarrow \mathbb{K}$, s. t. $f(x) = \lim\limits_{n\to\infty} f_n(x),\ \ \forall x \in X$：

   Since $\max\limits_{x \in X} |f_m(x) - f_n(x)| < \epsilon,\ \ \forall m, n \geq N_\epsilon$, then

   $$|f_m(x) - f_n(x)| < \epsilon,\ \ \forall x \in X,\ \ \forall m, n \geq N_\epsilon$$

   Thus $\forall x \in X$, fix x. $\{f_n(x)\}_{n=1}^{\infty} \subset \mathbb{K}$ is a Cauchy sequence in $\mathbb{K}$.

   Since both $\mathbb{C}$ and $\mathbb{R}$ are complete, then $\{f_n(x)\}_{n=1}^{\infty}$ converges, $\forall x \in X$. Let

   $$f : X \longrightarrow \mathbb{K} \tag{1.97}$$

   $$x \longmapsto \lim_{n\to\infty} f_n(x) \tag{1.98}$$

   Then f is a mapping from X to $\mathbb{K}$ with $f(x) = \lim\limits_{n\to\infty} f_n(x),\ \ \forall x \in X$.

2. $f \in C(X)$, i.e. $f : X \longrightarrow \mathbb{K}$ continuous：

   Since

   $$|f_m(x) - f_n(x)| < \epsilon,\ \ \forall x \in X,\ \ \forall m, n \geq N_\epsilon$$

   Then letting $m \to \infty$, we get $f_n \rightrightarrows f$, i.e.

   $$|f(x) - f_n(x)| < \epsilon,\ \ \forall x \in X,\ \ \forall n \geq N_\epsilon$$

   For $f_{N_\epsilon} \in C(X)$, fix $\epsilon > 0$. Since f_{N_ϵ} continuous, $\exists \delta > 0$, s. t.

   $$\left|f_{N_\epsilon}(x) - f_{N_\epsilon}(y)\right| < \epsilon,\ \ \forall \rho(x, y) < \delta$$

   Then for $\epsilon > 0$, $\exists \delta > 0, N_\epsilon \in \mathbb{N}$, s. t.

   $$|f(x) - f(y)| \leq \left|f(x) - f_{N_\epsilon}(x)\right| + \left|f_{N_\epsilon}(x) - f_{N_\epsilon}(y)\right| + \left|f_{N_\epsilon}(y) - f(y)\right| \tag{1.99}$$

   $$< 3\epsilon,\ \ \forall \rho(x, y) < \delta \tag{1.100}$$

   Therefore, $f : X \longrightarrow \mathbb{K}$ is continuous, $f \in C(X)$. In short, $(C(X), d)$ is complete.

□

1.7.2 一致有界, 等度连续

下面我们来给出**一致有界**与**等度连续**的概念, 实际上为数学分析中**一致有界**与**一致连续**在更高维度上的一致性的推广, 即同时对函数和自变量都一致.

定义 1.7.2. [一致有界].
设 (X,ρ) 为紧致度量空间, $A \subset (C(X),d)$. 若

$$\sup_{\substack{x\in X\\ \varphi\in A}} |\varphi(x)| < \infty$$

则称A 一致有界.

定义 1.7.3. [等度连续].
设 (X,ρ) 为紧致度量空间, $A \subset (C(X),d)$. 若 $\forall \epsilon > 0$, $\exists \delta > 0$, s. t.

$$|\varphi(x) - \varphi(y)| < \epsilon,\ \ \forall \rho(x,y) < \delta,\ \ \forall \varphi \in A$$

则称A 等度连续.

注. 相当于在数学分析**函数一致连续**的概念上增加了对 A 中所有函数的一致性.

1.8 Arzelà-Ascoli 定理

接下来我们将介绍 **Arezelà-Ascoli 定理**, 它给出了**紧致度量空间 X 上连续函数全体** $(C(X), d)$ **的 (列) 紧子集的等价刻画**.

定理 1.8.1. [Arezelà-Ascoli].
设 (X, ρ) 为紧致度量空间, $A \subset C(X)$, 则:

$$A \text{ 列紧 } \Leftrightarrow A \text{ 一致有界且等度连续}$$

证明.

$\Rightarrow$: 下面分两点分别证明一致有界和等度连续.

- A 一致有界: Since A 列紧 $\Rightarrow$ 完全有界, then for $\epsilon = 1$, $\exists \varphi_1, \cdots, \varphi_n \in A$, s. t.

$$A \subset \bigcup_{i=1}^{n} B(\varphi_i, 1)$$

$\forall \varphi \in A$, WTS: $\exists M > 0$, s. t. $|\varphi(x)| \leq M, \ \forall x \in X$.

$\exists 1 \leq i_0 \leq n$, s. t.

$$\varphi \in B(\varphi_{i_0}, 1)$$

Then

$$d(\varphi, \varphi_{i_0}) = \max_{x \in X} \left|\varphi(x) - \varphi_{i_0}(x)\right| < 1$$

Thus

$$\left|\varphi(x) - \varphi_{i_0}(x)\right| < 1, \ \forall x \in X$$

Fix $1 \leq i \leq n$. For $\varphi_i \in A \subset C(X)$.

Since X is compact, φ_i continuous, then $\varphi_i(X) \subset \mathbb{K}$ bounded, i.e. $\exists 0 < k_i < \infty$, s. t.

$$|\varphi_i(x)| \leq k_i, \ \forall x \in X$$

Let $k = \max\limits_{1 \leq i \leq n}\{k_i\}$, then

$$\begin{aligned} |\varphi(x)| &\leq \left|\varphi(x) - \varphi_{i_0}(x)\right| + \left|\varphi_{i_0}(x)\right| && (1.101) \\ &\leq 1 + k, \ \forall x \in X, \ \forall \varphi \in A && (1.102) \end{aligned}$$

Therefore, $A \subset C(X)$ 一致有界.

- A 等度连续：Since A 列紧 $\Rightarrow$ 完全有界, then for $\forall\epsilon > 0$, A 存在有穷 ϵ-网, i.e. $\exists\varphi_1^\epsilon, \cdots, \varphi_{i_\epsilon}^\epsilon \in A$, s. t.

$$A \subset \bigcup_{i=1}^{i_\epsilon} B(\varphi_i^\epsilon, \epsilon)$$

Since $\varphi_i \in C(X)$ continuous, $\forall 1 \le i \le i_\epsilon$, then for $\epsilon > 0$, $\exists\delta_1, \cdots, \delta_{i_\epsilon} > 0$, s. t.

$$\left|\varphi_k^\epsilon(x) - \varphi_k^\epsilon(y)\right| < \epsilon, \ \forall\rho(x, y) < \delta_k, \ \forall 1 \le k \le i_\epsilon$$

Let $\delta_\epsilon = \min\{\delta_1, \cdots, \delta_{i_\epsilon}\} > 0$, then for $\forall 1 \le k \le i_\epsilon$,

$$\left|\varphi_k^\epsilon(x) - \varphi_k^\epsilon(y)\right| < \epsilon, \ \forall\rho(x, y) < \delta_\epsilon$$

$\forall\psi \in A$, $\exists 1 \le i_0 \le i_\epsilon$, s. t.

$$\psi \in B(\varphi_{i_0}^\epsilon, \epsilon) \ \Rightarrow \ d(\psi, \varphi_{i_0}^\epsilon) < \epsilon$$

Thus

$$\left|\psi(x) - \varphi_{i_0}^\epsilon(x)\right| < \epsilon, \ \forall x \in X$$

Then

$$\begin{aligned} |\psi(x) - \psi(y)| &\le \left|\psi(x) - \varphi_{i_0}^\epsilon(x)\right| + \left|\varphi_{i_0}^\epsilon(x) - \varphi_{i_0}^\epsilon(y)\right| + \left|\varphi_{i_0}^\epsilon(y) - \psi(y)\right| && (1.103) \\ &\le 3\epsilon, \ \forall\rho(x, y) < \delta_\epsilon && (1.104) \end{aligned}$$

Therefore, $A \subset C(X)$ 等度连续.

$\Leftarrow$: Since A 等度连续, then for $\forall\epsilon > 0$, $\exists\delta > 0$, s. t.

$$|\varphi(x) - \varphi(y)| < \frac{\epsilon}{3}, \ \forall\rho(x, y) < \delta, \ \forall\varphi \in A$$

Since X is compact, then X 存在有穷 δ-网, i.e. $\exists\{x_1^\delta, \cdots, x_n^\delta\} \subset X$, s. t.

$$X \subset \bigcup_{i=1}^{n} B(x_i^n, \delta)$$

Let

$$\widetilde{A} = \left\{\left(\varphi(x_1^\delta), \cdots, \varphi(x_n^\delta)\right) \in \mathbb{K}^n \mid \varphi \in A\right\}$$

Since A 一致有界, i.e.

$$|\varphi(x)| \leq M, \ \forall x \in X, \ \forall \varphi \in A \text{ for some } M > 0$$

$$\Rightarrow \ \widetilde{A} \subset \mathbb{K}^n \text{ is bounded in } \mathbb{K}^n$$

Since both in $\mathbb{R}^n$ and $\mathbb{C}^n$, 有界 $\Leftrightarrow$ 列紧 $\Leftrightarrow$ 完全有界, then $\widetilde{A}$ 完全有界.

Thus $\widetilde{A}$ 存在有穷 δ-网, i.e. $\exists\{(\varphi_i(x_1^\delta), \cdots, \varphi_i(x_n^\delta)) \in \mathbb{K}^n\}_{i=1}^m \subset \mathbb{K}^n$, s. t.

$$\widetilde{A} \subset \bigcup_{i=1}^m B((\varphi_i(x_1^\delta), \cdots, \varphi_i(x_n^\delta)), \frac{\epsilon}{3})$$

下面证明：$\{\varphi_1, \cdots, \varphi_m\}$ 即为 A 的有穷 ϵ-网：

$\forall \varphi \in A, \exists 1 \leq i_0 \leq m$, s. t.

$$(\varphi(x_1^\delta), \cdots, \varphi(x_n^\delta)) \subset B((\varphi_{i_0}(x_1^\delta), \cdots, \varphi_{i_0}(x_n^\delta)), \frac{\epsilon}{3})$$

$\forall x \in X$, fix x. Since $\{x_1^\delta, \cdots, x_n^\delta\}$ 为 X 的 δ-网, then $\exists 1 \leq j_0 \leq n$, s. t.

$$\rho(x, x_{j_0}^\delta) < \delta$$

Then by A 等度连续,

$$\begin{aligned} \left|\varphi(x) - \varphi_{i_0}(x)\right| &\leq \left|\varphi(x) - \varphi(x_{j_0}^\delta)\right| + \left|\varphi(x_{j_0}^\delta) - \varphi_{i_0}(x_{j_0}^\delta)\right| + \left|\varphi_{i_0}(x_{j_0}^\delta) - \varphi_{i_0}(x)\right| && (1.105) \\ &< \frac{2}{3}\epsilon + \left|(\varphi(x_1^\delta), \cdots, \varphi(x_n^\delta)) - (\varphi_{i_0}(x_1^\delta), \cdots, \varphi_{i_0}(x_n^\delta)\right| && (1.106) \\ &< \epsilon && (1.107) \end{aligned}$$

□

附录 A　*Supplementary Content*

A.1　度量空间稠密子集的等价刻画

引理 **A.1.1.** 度量空间稠密子集的等价刻画.

Suppose (X, ρ) be a metric space. Then for $A \subset X$,

$$A \text{ is dense in } X \iff \forall x \in X, \exists \{x_n\}_{n=1}^{\infty} \subset A, \text{ s.t. } x_n \xrightarrow{\rho} x$$

证明.

$\Rightarrow$: Trivial. $\forall x \in X$, since A is dense in X, then

$$A \cap B(x, r) \cap X \neq \emptyset, \ \forall r > 0 \tag{A.1}$$

不妨设 $B(x, 1) \cap X \backslash \{x\} \neq \emptyset$. Then we take

$$x_n \in A \cap B(x, \frac{1}{n}), \ \forall n \in \mathbb{N} \tag{A.2}$$

where we get $\{x_n\}_{n=1}^{\infty} \subset A$ with $x_n \xrightarrow{\rho} x$.

$\Leftarrow$: $\forall x \in X \backslash A$, WTS: $x \in \overline{A} \backslash A$. Trivial.

□